Equilibrium Transitions from Non Renewable Energy to Renewable Energy under Capacity Constraints*

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October 2013

*We thank Steve Salant for his precious advice which allowed us to considerably improve the present paper. Jean-Pierre Amigues also thanks all the participants in the INRA-Paris seminar, the Paris I environmental economics seminar and in the EAERE Rome 2011 conference for their fruitful comments. The usual disclaimer applies.

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Abstract

We study the transition between non renewable and renewable energy sources with adjustment costs over the production capacity of renewable energy. Assuming constant variable marginal costs for both energy sources, convex adjustment costs and a more expensive renewable energy, we show the following. With sufficiently abundant non renewable energy endowments, the dynamic equilibrium path is composed of a first time phase of only non renewable energy use followed by a transition phase substituting progressively renewable energy to non renewable energy and a last time phase of only renewable energy use. Before the complete transition towards renewable energy, the energy price follows a Hotelling like path. Depending upon the shape of adjustment costs, investment into renewable energy may either begin before production of renewable energy or be delayed until the energy price achieves a sufficient gap with respect to the renewable energy marginal production cost. In all cases, the renewable energy sector bears negative returns over its investments in its early stage of development. Investment into renewable energy production capacity building first increases before having to decrease strictly before the depletion time of the non renewable resource. Renewable energy capacity continues to expand afterwards but at a forever decreasing rate converging to zero in the very long run. The development path of renewable energy may be largely independent from the non renewable resource scarcity. In particular with initially abundant non renewable energy, the length of the transition phase between non renewable and renewable energy together with the accumulated renewable production capacity at the end of this phase do not depend upon the scarcity rent of the non renewable resource and of the initial size of the resource stock.

JEL classification : D92, Q30, Q40, Q42

Keywords : non renewable resource, renewable energy, adjustment costs, resources transition, capacity constraints.

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1 Introduction

The transition between the use of different natural resources typically takes time. While in use since the sixteenth century in Great Britain, coal mining replaced only very slowly charcoal in iron processing or wood in energy provision until the nineteenth century (Wrigley, 2010, Fouquet, 2008). The same may be said for the use of oil and natural gas which developed over a sixty years range period since the end of the nineteenth century. More recently, the development of new energy sources like solar or biofuel is expected to extend well over the current century (Nakicenovic, 1998). Most policy proposals to develop such alternatives in order to mitigate climate change are explicitly time dependent, the European Union twenty-twenty plan being one prominent example. Current and prospected energy policies thus strongly acknowledge the time lags implied by long run adaptations of the present energy mix. In some sense the climate challenge may be seen as a time to act problem, balancing the speed of possible adaptations to climate change with the speed of such a change.

This time to build issue covers many different problems ranging from the need of a sufficiently rapid technical progress to develop economically relevant energy alternatives to a sufficiently fast investment pace in natural resources services provision. Adaptation, or more generally development of the exploitation of natural resources is a costly process falling under the heading of 'adjustment costs' in investment economics. This issue of adjustment costs is not only of concern for the development of new resources but also for the development of existing ones, a well known feature of resource industries, either for the exploration and exploitation of new oil fields or for mineral resources.

It is also related at a more macroeconomic level to the sustainability debate, the replacement of depletable resources either by renewable ones or by man made capital goods typically requiring time. The classical treatment of this issue (Dasgupta and Heal, 1974, 1980) emphasizes that the speed of replacement of exhaustible resources by man made capital goods is highly dependent upon the ease of substitution between natural and man made inputs, as measured by substitution elasticities between inputs at the aggregate technology frontier level. But at the firm or at the sectoral level, the reference to substitution elasticities is a short cut way to describe complex adjustment processes of inputs combination typically costly and time dependent.

Adjustment costs have received a lot of attention in investment theory, seminal contributions in this strand of literature being Lucas (1967), Gould (1968) and Treadway (1969). However, while fully acknowledged as an important issue in natural resource development problems at least since Hotelling, 1931[1], it has attracted a relatively modest attention from resource economists. The textbook treatments of substitution between natural resources (for example Herfindahl and Kneese, 1974) do not consider explicitly adjustment costs. This results into a description of the history of natural resources use development as a sequence of time phases of exploitation of a dominant resource (the wood age, the coal age, the oil age) separated by quick transitions from a dominant resource to another one, according to their relative cost order.

In the resource literature, reference to adjustment costs has served two main purposes. The first one concerns the validity of the Hotelling rule in the theory of the mine. The well known observation that actual resource prices do not follow the r percent growth rate prescribed by the Hotelling rule may be explained by the presence of adjustment costs in mining utilities operation. That investment costs may result in constant resource prices has been shown by Campbell (1980) extending the previous work of Puu (1977). This problem has also been carefully examined by Gaudet (1983) in the context of the theory of the mine. The strength of the Campbell model is to take explicitly into account the consequences of extracting capacity constraints over the resource price, but its main weakness, as emphasized by Gaudet (1983), is to transform the gradual capacity development process into a static investment problem, the mining industry having to choose initially a given production capacity held constant over the whole mine life duration. The Gaudet analysis describes thoroughly this process but is cast into the microeconomic framework of the individual firm investment theory and thus

[1] "The cases considered in the earlier part of this paper all led to solutions in which the rate of production of a mine always decreases. By considering the influence of fixed investments and the cost of accelerating production at the beginning, we may be led to production curves which rise continuously from zero to a maximum, and then fall more slowly as exhaustion approaches. Certain production curves of this type have been found statistically to exist for whole industries of the extractive type, such as petroleum production." Hotelling H. *The Economics of Exhaustible Resources, 1931, p 164.*

does not take into account the resource price consequences of the firms investment decisions, an aspect examined in a similar context by Lasserre (1986). The second purpose is the study of transitions between different resources, main contributions to this issue being Olsen (1989) and Cairns and Lasserre (1991). The analysis of Olsen and Cairns and Lasserre are somewhat complex and focus upon the transition between different non renewable resources, thus trying to generalize the Herfindahl model to resource transitions under adjustment costs.

Very few parallel effort has been made to describe the transition between a non renewable resource and a renewable one, this last resource being submitted to adjustment costs in its productive capacity. One important contribution in this direction is Tsur and Zemel (2011) which model the capital accumulation process in producing solar energy under competition with existing fossil fuel resources. However Tsur and Zemel do not take into account the exhaustible nature of fossil fuels, assuming a forever constant supply of such resources. The study of transitions between energy sources appears clearly useful in the context of the climate policy debate, the development of 'green' energy alternatives being a major theme in this respect. The perception that green energies develop at a too slow rate is common place in the public debate and it already exist in industrialized countries several policy initiatives aimed at subsidizing renewable energy sources. The rationale for such subsidies has been questioned recently in the so called 'green paradox' puzzle. This is the point raised in conjunction with adjustment costs by Gronwald *et al.* in a recent work (Gronwald, Jus and Zimmer, 2010). But curiously they consider investment costs in the provision of fossil fuels and not in green energy. The issue has also been studied recently by Smulders and Zemel (2011) within the context of macro economic growth theory but without explicit consideration for the exhaustibility of the polluting resource.

The works of Salant and Switzer (1983) and Oren and Powell (1989) are close to our work using a similar model. However they do not derive a complete solution to the problem leading them into incorrect claims, like the necessity to use renewable energy at any time when the energy price would at least cover its marginal cost of production.

The objectives of the paper are two-fold. First we want to stress the importance of investment constraints over the development of renewable energy alternatives. In order to focus upon the investment issue we shall dispense from considering explicitly the pollution problems raised by burning fossil fuels. Hence the main motivation for developing energy alternatives will be the increasing scarcity of non renewable fossil fuels like oil. For the same reason we shall not deal with the important issue of technical progress or learning by doing in the use of new energy sources. This issue has raised significant attention in the macroeconomic endogenous growth literature recently (Acemoglu, Aghion, Burzstynx and Zemmour, 2011) but the precise micro foundations of this analysis, both at the firm level and at the energy sector level remain to be settled carefully. Technical progress should result into the generation of higher quality capital goods, an issue which would require to plug the analysis inside some vintage capital model, a study we deserve for further research.

Second we want to explicitly consider the price implications of the development of renewable energy. One should expect that the gradual increase of renewable energy inside the energy mix will affect both the energy price trajectory and the depletion path of the already in use non renewable resource. Conversely, the time path of investment into renewable alternatives should depend upon the relative scarcity of the non renewable resource. To deal with this issue we shall depart both from the usual investment analysis at the individual firm level and from the aggregate studies at a macro level. We consider a partial equilibrium setting where the energy sector is composed of a population of identical competitive firms either producing energy from a non renewable resource or from a renewable one. Furthermore we assume that the renewable energy industry has to purchase specific equipments, linking at the equilibrium the dynamics of the energy price to the dynamics of the renewable energy capital input price. We assume an upward slopping supply curve of specific equipment of the renewable energy industry or equivalently an increasing marginal cost curve of equipment provision to the renewable industry. Thus the renewable industry faces external adjustment costs in the Lucas (1967) sense rather than internal adjustment costs in the Gould (1968) sense. For simplicity we assume constant average and marginal variable operating costs in the non renewable and renewable energy industries and a lower operating cost of non renewable energy.

Our main findings are the following. With sufficient non renewable resource initial endowments, the equilibrium path is a sequence of three phases, a first phase during which only the cheaper non renewable resource is exploited, followed by a transition phase of simultaneous use of non renewable and renewable energy sources up to some finite time when the non renewable resource reserves become exhausted. This transition phase is followed by a last renewable energy use phase of infinite duration. During the first and second phases, the energy price increases following a Hotelling like path, while it decreases during the last phase because of the continuous expansion of the renewable energy production capacity which occurs all over this phase.

Assuming a strictly positive minimal price of equipment into renewable energy production, the development of the renewable energy alternative may follow two possible scenarios. Under certain conditions, it should start strictly *after* that time when the energy price would be higher than the renewable energy variable marginal cost and strictly *before* the energy price could cover the full marginal cost of the renewable alternative, that is the sum of the variable cost and the minimal rental cost of capital equipment. Under other conditions it is also possible that the energy industry should start to invest into the renewable alternative *before* using it, waiting for the energy price to reach the variable average production cost level of renewable energy. In these two scenarios, the firms face negative returns over their investments in the early stage of the energy transition. This is explained by the permanent energy price increase during the transition combined with increasing investment marginal costs of their equipment efforts.

The investment into renewable energy should first rise and then begin to decrease strictly before the depletion of the non renewable resource. After the exhaustion of the non renewable resource, the renewable energy sector will continue to expand its production capacity permanently up to some long run efficient renewable energy production capacity level. This implies that it is never optimal for the renewable energy sector to hold this efficient long run capacity level at the end of the energy transition.

We also derive a closed form solution for the model at hand. It shows that the characteristics of the equilibrium investment policy into renewable energy may be in fact largely independent from the initial scarcity of the non renewable resource. This will be the case in particular in a scenario where the non renewable resource is sufficiently abundant for a first phase of only non renewable resource exploitation to arise before the beginning of the development of renewable energy. More precisely, we show in this case that a higher availability of the non renewable resource will translate farther in time the same renewable energy investment path, thus resulting into the same level of accumulated capacity in producing renewable energy at the depletion time of the non renewable resource. Thus the expansion path of renewable energy use after the exhaustion of the non renewable resource will be independent from the non renewable resource scarcity. This feature of the investment plan applies both in a scenario of early building of the production capacity before entering the production stage or in a scenario of simultaneous building of the capacity together with the development of the production of renewable energy.

The paper is organized as follows. We describe in the next section a model of transition between a non renewable resource and a renewable resource facing capacity development constraints. Under our constant variable marginal costs assumption it turns out that the non renewable resource will be exhausted in finite time. Thus we proceed in section 3 to the description of the ultimate phase of only renewable energy production. This last time phase may be described using the phase diagram technique developed by Treadway (1969). Section 4 examines the features of the transition phase between non renewable and renewable energy, focusing upon the description of the investment path into the expansion of renewable energy. Section 5 provides a closed form solution to the model and shows that the characteristics of the investment policy into renewable energy are in our model largely independent from the scarcity of the non renewable resource. The last section 6 concludes.

2 The model

We consider an economy with access to two different energy sources. The first one is a non renewable resource, say 'oil', available initially in amount X_0 and we denote by $x(t)$ the instantaneous rate of oil extraction and by $X(t)$ the current oil availability, so that $\dot{X}(t) = -x(t)$. Oil extraction, refinement and processing results into the provision of oil energy services, the amount of energy services being normalized as to be equal to $x(t)$ for the sake of simplicity. The provision of oil energy services to the users incurs a constant unit and marginal cost c_x. We dispense from considering the possible pollution problems raised by burning oil to produce energy.

The second energy source is a renewable resource, let say 'solar'. To produce solar energy, the industry has to build a dedicated production capacity, let say a 'solar panels' stock, starting from a zero initial level of equipment. Let $K(t)$ be the installed solar production capacity at time t and assume that $K(0) = 0$. There is no installed solar production capacity initially. We denote by $k(t)$ the purchase of equipment into solar production capacity.

We assume that maintaining the production capacity has a cost c_K per unit of maintained capacity. The firms have thus to decide over a maintenance effort. We assume that any fraction of the capital stock which do not benefit from maintenance is definitively lost. Thus we should introduce the possibility of a negative adjustment of the capital stock by applying maintenance effort to only a fraction of the installed capacity. We shall denote by $K_m(t)$ the capacity benefiting from maintenance at time t, $K_m(t) \leq K(t)$. Since the fraction $K - K_m$ would be definitively lost, the capital dynamics is described by the following equation:

$$\dot{K}(t) = k(t) - (K(t) - K_m(t))$$

Each unit of maintained capacity is assumed to be able to deliver one unit of renewable energy services. Thus denoting by $y(t)$ the flow of such services, $y(t) \leq K_m(t) \leq K(t)$.

The delivery of solar energy services to the users incurs a constant unit and marginal cost c_y. The solar industry has to purchase its equipment over a specific market e.g. the solar panels market. Let $p_K(t)$ be the price of solar panels and denote by $k^s(p_K)$ the supply curve of solar panels. This supply curve would identify to the marginal cost curve of the solar panel industry in a competitive situation. We assume that there exists some positive $p_K^0 > 0$ such that $k^s(p_K^0) = 0$. This is equivalent to assuming that the marginal cost of producing solar panels is positive even for the first produced unit of equipment. We introduce this feature for the sake of realism but it will appear that it allows for a much better understanding of the investment logic. We consider an increasing supply curve (or an increasing marginal cost curve of producing solar panels), that is $k^s(p) : [p_K^0, \infty) \rightarrow R_+$ is a continuous and differentiable function such that $dk^s(p_K)/dp_k > 0$ and $k^s(p_K^0) = 0$.

Two remarks are in order at this stage. Firstly, as pointed out in the introduction, investment theory with adjustment costs (e. g. Lucas, 1967) distinguishes between two kinds of adjustment costs: *external* costs associated to the purchase of new capital equipment and *internal* costs identified to specific costs of putting new equipment into a productive state together with the already existing installations. Our formulation neglects these internal costs, the firms being able first, to run freely any level of available capacity at any time and second, to incorporate new equipment without incurring specific installation costs. We also dispense from considering technical advances in the design of solar panels. A full account of the consequences of such advances over the capital management requires to design a vintage capital model, an interesting research avenue we shall not pursue here.

Secondly, it may be possible that $p_K(t) < c_K$. In such a case the industry should not apply any maintenance effort, scrap entirely the existing production capacity at time t to purchase a new one. We shall rule out such a possibility by assuming that $c_K < p_K^0$. Thus it will always be in the interest of the industry to apply at least some maintenance effort and keep some fraction of the existing production capacity.

The energy services delivered by the oil energy industry and the solar energy industry are perfect substitutes for the users. Let $q(t) = x(t) + y(t)$ be the aggregate energy supply by the energy sector. p denotes the energy price and $p^d(q)$ is the inverse demand function, $p^d(q) : R_+ \to R_+$ is continuous and differentiable with $dp^d(q)/dq < 0$ and $\lim_{q \downarrow 0} p^d(q) = +\infty$.

In the context of capacity investment costs, the both cases of a cheaper solar energy with respect to oil or a cheaper oil than solar appear worth a study. We shall concentrate upon the case of a cheaper oil energy that is assume: $c_x < c_y$. Thus absent any depletion of the oil resource, there should be no development of solar energy. It is the pure logic of resource exhaustion which will motivate the expansion of the solar energy alternative.

We assume that the energy industry is composed of competitive firms having access to the same technologies for energy services provision. Hence it does not matter to assign specialization into oil or solar energy generation for a given firm. Facing the same energy and solar equipment markets conditions, the firms should take identical decisions regarding output and inputs purchase policies. At the equilibrium, their solar energy investment policy will be affected by the levels and dynamics of equipment price. The supply curve of solar equipments having been supposed to be upward sloping, an increased speed of equipment accumulation will result in price increases upon the equipment market. Hence market behavior will mimic at the equilibrium the features of the convex cost structure one finds in the standard investment models with internal adjustment costs.

Assuming perfect competition over both the energy and solar equipment markets, the energy sector has to design supply plans $\{(x(t), y(t)), t \geq 0\}$ and a solar capacity investment and maintenance plan $\{(k(t), K_m(t)), t \geq 0\}$ maximizing the discounted profit stream of the industry. We denote by r the constant level of the interest rate. Formally the energy sector solves:

$$\max_{x(t),y(t),k(t),K_m(t)} \int_0^\infty \{p(t)(x(t)+y(t)) - c_x x(t) - c_y y(t) - p_K(t)k(t) - c_K K_m(t)\} e^{-rt} dt$$

$$s.t. \quad \begin{array}{lll} \dot{X}(t) = -x(t) & & X(0) = X_0 \text{ given} \\ \dot{K}(t) = k(t) + K_m(t) - K(t) & & K(0) = 0 \\ x(t) \geq 0 \ , \ y(t) \geq 0 & , & \\ k(t) \geq 0 \ , \ K_m(t) - y(t) \geq 0 & , & K(t) - K_m(t) \geq 0 \end{array} .$$

The Lagrangian of this problem is (dropping time dependency of the variables for the ease of reading):

$$\begin{aligned} \mathcal{L} \ = \ & p(x+y) - c_x x - c_y y - p_K k - c_K K_m - \lambda_X x + \lambda_K (k + K_m - K) \\ & + \gamma_x x + \gamma_y y + \gamma_k k + \gamma_K (K_m - y) + \gamma_m (K - K_m) \end{aligned}$$

Since the state variable $X(t)$ does not enter inside the expression of the Lagrangian we can infer that in any solution : $\lambda_X(t) = \lambda_X(0)e^{rt}$. Denote : $\lambda_X(0) \equiv \lambda_X$. We obtain the following system of first order conditions:

$$\mathcal{L}_x = 0 \implies p = c_x + \lambda_X e^{rt} - \gamma_x \tag{2.1}$$
$$\mathcal{L}_y = 0 \implies p = c_y + \gamma_K - \gamma_y \tag{2.2}$$
$$\mathcal{L}_k = 0 \implies p_K = \lambda_K + \gamma_k \tag{2.3}$$
$$\mathcal{L}_{K_m} = 0 \implies c_K + \gamma_m = \lambda_K + \gamma_K \tag{2.4}$$
$$\dot{\lambda}_K = r\lambda_K - \mathcal{L}_K \implies \dot{\lambda}_K = r\lambda_K + \lambda_K - \gamma_m \ , \tag{2.5}$$

together with the usual complementary slackness conditions and the transversality conditions:

$$\lim_{t\uparrow\infty} \lambda_X X(t) = 0 \quad , \quad \lim_{t\uparrow\infty} e^{-rt}\lambda_K(t)K(t) = 0 \ . \tag{2.6}$$

We first show that in any equilibrium, the industry should apply maintenance to the whole existing capital stock. Suppose to the contrary that during some time interval Δ, the industry decides both to not fully use the maintained capacity to produce solar energy together with applying maintenance to only a fraction of the capital stock, that is $y(t) < K_m(t) < K(t)$, $t \in \Delta$. Then it results from the complementary slackness conditions that $\gamma_m(t) = \gamma_K(t) = 0$, $t \in \Delta$. But this would imply from (2.4) that $\lambda_K(t) = c_K$ should be constant while (2.5) would imply that $\dot{\lambda}_K = (1+r)\lambda_K(t) > 0$, hence a contradiction. Next, if the industry decides to fully exploit the maintained capacity while not applying maintenance to the whole capital stock, that is if $y(t) = K_m(t) < K(t)$, then $\gamma_m(t) = 0$ while $\gamma_K(t) \geq 0$. Thus (2.4) would imply that $p_K(t) = c_K - \gamma_K(t) \leq c_K$, a possibility that we have previously excluded by assuming that $c_K < p_K^0 \leq p_K(t)$. It will prove convenient to define $\beta_K \equiv \gamma_K - c_K = \gamma_m - \lambda_K$ and rewrite (2.2)-(2.5) as:

$$p = c_y + c_K + \beta_K \tag{2.7}$$

$$\dot{\lambda}_K(t) = r\lambda_K(t) - \beta_K \ . \tag{2.8}$$

β_K stands as the net opportunity cost of the capacity constraint, that is net of the maintenance costs. Note that $\beta_K(t) = -c_K$ in a case where $\gamma_K(t) = 0$, that is if $y(t) < K(t)$, the industry does not fully exploit the existing solar production capacity, and $\beta_K(t) > -c_K$ in the reverse case.

The net value of an investment into solar capacity $\lambda_K(t)$ has to be carefully distinguished from the net opportunity cost of the capacity constraint $\beta_K(t)$. The second only measures the burden of the capacity constraint at some time t, a static measure of the severity of the constraint, while the first is measuring the contribution of an extra solar production capacity over the whole time interval $[t, \infty)$ and thus takes fully into account the whole equilibrium dynamics. The transversality condition provides a simple link between these static and dynamics measures of the value of an investment. Note that (2.8) is equivalent to $(\lambda_K(\dot{t})e^{-rt}) = -\beta_K(t)e^{-rt}$. Under constant average costs, the oil reserves will be exhausted in finite time, implying that the economy will have to rely only upon solar energy in the very long run.

This in turn implies that $\lim_{t\uparrow\infty} K(t) > 0$ and through the transversality condition that $\lim_{t\uparrow\infty} \lambda_K(t)e^{-rt} = 0$. Thus integrating over $[t, \infty)$ results in:

$$\begin{aligned} \lim_{\tau\uparrow\infty} \lambda_K(\tau)e^{-r\tau} - \lambda_K(t)e^{-rt} &= -\int_t^{\infty} \beta_K(\tau)e^{-r\tau}d\tau \implies \\ \lambda_K(t) &= \int_t^{\infty} \beta_K(\tau)e^{-r(\tau-t)}d\tau \end{aligned}$$

The net current value of an investment into capacity building $\lambda_K(t)$ at time t identifies with the discounted sum of the instantaneous net opportunity costs of the capacity constraint from t onwards.

Before turning towards a detailed analysis of the implications of the necessary conditions, let us sketch a reasonable guess solution to the problem.

- Since $c_x < c_y$ and solar development has to be started from scratch, the non renewable resource will be put into exploitation right from the beginning.
- Since the marginal cost of oil energy services has been assumed to be constant, oil should be depleted in finite time.
- There cannot be an abrupt transition from oil to solar energy as in the textbook Herfindahl model since solar capacity building is costly and the supply curve of solar equipment has been assumed to be upward slopping. Hence, contrarily to the Herfindahl model, there should exist a phase of simultaneous exploitation of both solar and oil energy, despite the fact that their respective variable marginal costs are constant and that $c_x < c_y$ by assumption.
- After oil depletion, only the solar energy sector will remain active. Depending upon the previously accumulated production capacity, it may or not be the case that capacity will continue to expand. In the first case, energy supply should increase while the energy price should decline over time.
- Depending upon the cost advantage of oil with respect to solar and the other features of the model, it may or not be the case that the investment into solar production capacity building will be delayed in time, opening the room for a first phase of only oil exploitation before the transition towards solar energy.

- In this last case, it appears possible that the industry should invest into solar production capacity before using it to produce renewable energy.

These features suggest the following study plan. We shall first describe the last phase of only solar energy use. We shall give a necessary condition for a permanent investment into solar capacity building during this phase. We then show that under this condition, investment should slow down while the solar energy production capacity converges asymptotically towards a long run level $\hat{K}$, defined as the solution of : $p^d(K) = rp_K^0 + c_y + c_K$. Both the value of the capacity λ_K and the net opportunity cost of the capacity constraint β_K decrease over time down to the levels p_K^0 and rp_K^0 respectively.

Next we turn towards the transition phase form oil to solar energy. We shall give a necessary condition for this transition phase to be composed of a first phase of only oil exploitation followed by a phase of joint use of both energy sources. During the whole transition process the price of energy should increase in a Hotelling way, that is: $p(t) = c_x + \lambda_X e^{rt}$. The aggregate use of energy should thus decrease while solar energy production, if in use, should increase with the installed solar production capacity, meaning that oil use should decrease in a higher proportion inside the total energy mix. We also show that the investment path into solar equipment has an inverted U shape, characterized by a first phase of increasing investment rates and then by a decrease in the speed of solar capacity expansion. Investment into solar capacity should begin to decrease strictly before the depletion of the oil reserves and will continue to decrease afterwards as noted before. If solar energy development is delayed then investment should start at the minimal level. It will start at some strictly positive level in a case of immediate production of solar energy.

Last, we consider the issue of the optimal timing of solar development in the case of a delayed introduction of this energy source. We identify two possibilities. Either the industry begins to expand its solar production capacity strictly before starting to produce solar energy, either capacity expansion and solar production begin at the same time. In both cases, the solar industry bears profit losses at the beginning of the solar capacity expansion phase. This feature of the profitability plan of solar energy is a consequence of the Hotelling rule which drives up permanently the energy price before the depletion of the oil resource together with an upward slopping curve for solar equipments. We fully characterize the different possible scenarios by means of an algorithmic procedure, thus providing a closed form solution to the present model. It shows that the features of the investment plan into the renewable energy alternative may be completely independent from the oil resource scarcity.

3 The pure solar phase

Let the oil reserves be depleted at some time $t_X > 0$. The sole solar energy industry provides the energy needs over the time interval $[t_X, \infty)$ starting at time t_X with a previously accumulated capacity K_X. From (2.2) we see that a necessary condition for the solar energy industry not to be in excess capacity at time t_X is: $p^d(K_X) > c_y$. Furthermore, the average cost of holding capacity is given by the maintenance cost c_K. Let $\bar{K}$ be the solution of $p^d(K) = c_y + c_K$. Then the solar industry will be in excess capacity at t_X if $K_X > \bar{K}$. Building capacity from a zero initial level being costly for the solar industry, a previous investment policy resulting at time t_X in $K_X > \bar{K}$ cannot be profit maximizing for the industry. Hence we can restrict the attention to initial levels of the installed capacity such that $K_X \leq \bar{K}$.

We now give a condition for an investment into capacity building to occur after t_X. Assume no increase in capacity after t_X. A slight increase of the solar production capacity above K_X would generate a discounted stream of marginal profit $v(K_X)$ in current value at t_X at the equilibrium over the energy market:

$$v(K_X) \quad = \quad \int_{t_X}^{\infty} [p^d(K_X) - c_y - c_K] e^{-r(t-t_X)} dt = \frac{p^d(K_X) - c_y - c_K}{r}$$

Assume that $p_K^0 < v(K_X)$. A slight investment after t_X would bear a cost approximatively equal to p_K^0 while generating a benefit $v(K_X)$. Thus the balance sheet of such an investment arbitrage would be positive, leading to the conclusion that investment should occur after t_X. Hence we can conclude that a sufficient condition for no investment to happen after t_X is $p_K^0 > v(K_X)$. Let $\hat{K}$ be the solution of $v(K_X) = p_K^0$ that is: $p^d(K) = rp_K^0 + c_y + c_K$. $\hat{K}$ is well defined because of the decreasing monotonicity of the inverse demand function. Then we can conclude from the assumption of a decreasing demand that if $\hat{K} < K_X < \bar{K}$, the industry should maintain the solar production capacity at the level K_X, performing no further investment into capacity expansion. Since increasing the capacity bears a full marginal cost, including the resulting extra maintenance cost, at least equal to $rp_K^0+c_K$ in annual rental terms, the industry will proceed to such an investment only in a case where the operating marginal profit $p-c_y$ covers this minimal cost. Since $K_X < \bar{K}$ the solar industry is constrained by the installed production capacity K_X, but since $K_X > \hat{K}$, trying to relax the capacity constraint by investing more would lower profits. Hence a necessary condition for a solar production capacity expansion above K_X after t_X is $K_X < \hat{K}$.

At this stage some remarks are in order. We have modeled capacity expansion as resulting from an irreversible investment process with the possibility of scraping the capacity in a costless way by not applying maintenance to the whole capital stock. Alternatively we could have assumed that dismantling the capacity could be feasible only through some costly readjustment process. But such a costly process can be avoided just by using only a fraction of the existing capacity, living idle the rest while not paying the full maintenance cost. Thus we are not in the standard irreversible investment framework as studied by Arrow (1968), the constraint $k(t) \geq 0$ does not exclude the possibility of a temporary underuse of the existing capacity. In such a situation the industry faces the choice of maintaining the existing productive capacity by paying the maintenance cost over the whole installed capacity or cut these costs and loose some fraction of the capacity.

In the context of the present perfect foresight model where the demand for energy has been assumed to be stationary, such an issue cannot not arise at the equilibrium. Either $K_X > \bar{K}$ and the industry use only the fraction $\bar{K}$ of K_X and pay the maintenance cost only over this capacity level held forever constant. Either $K(t) < \bar{K}$ and the capacity constraint binds at $K(t)$. The installed capacity is in full operation and the maintenance effort applies to the whole capital stock. Furthermore, the investment profitability argument is also valid before t_X, meaning that starting from a null installed capacity, the profit maximizing solar industry should not accumulate before t_X a capacity K_X above $\hat{K}$. We can thus dispense from considering the possibility of dismantling some fraction of the production capacity after t_X. Of course this way of reasoning is only valid because the production capacity is used to provide some flow of renewable resource energy. In an exhaustible resource exploitation problem under adjustment costs, the possibility of excess capacity accumulation during the mine life is an important issue[2].

Consider now the case $K_X < \hat{K}$. We are going to show that the solar industry should perform a permanent investment effort into solar equipment expansion though at a decreasing rate, the installed capacity converging asymptotically towards $\hat{K}$. Consider an investment plan into capacity expansion followed over a time interval $\Delta_k \equiv [t_X, \bar{t})$. Since $k(t) > 0$, $t \in \Delta_k$, $\gamma_k(t) = 0$, $t \in \Delta_k$, and thus $p_K(t) = \lambda_K(t)$. Thus (2.3) and (2.5) result at the equilibrium over the solar equipment market and the energy market into the following differential system in (K, λ_K):

$$\begin{aligned} \dot{K}(t) &= k^s(\lambda_K(t)) \\ \dot{\lambda}_K(t) &= r\lambda_K(t) - p^d(K(t)) + c_y + c_K \ . \end{aligned} \tag{3.1}$$

(3.1) is a simple non linear differential system which can be studied with the phase diagram technique. For exposition convenience expand Δ_k to $[t_X, \infty)$. Since $k(t) > 0$ by assumption, $\dot{K}(t) > 0$ and $\dot{\lambda}_K > / = / < 0$ depending upon $\lambda_K > / = / < v(K(t))$ where $v(K) \equiv [p^d(K) - c_y - c_K]/r$. $v(K)$ is the total net marginal surplus in current value from t onwards if the solar production capacity and hence the solar energy production level would be kept constant after t. Thus $v(K)$ measures the capacity rent resulting from a constant capacity level K. Since $dp^d(q)/dq < 0$ and $\lim_{q\downarrow 0} p^d(q) = +\infty$ under our demand assumption, we get immediately $\lim_{K\downarrow 0} v(K) = +\infty$ and $dv(K)/dK < 0$. Last, there exists $\bar{K}$ solution of $v(K) = 0$ and $v(K) < 0$ for $K > \bar{K}$. The following Figure 1 illustrates the geometry of the phase diagram in the (K, λ_K) plane.

It is easily checked that the saddle branch graphed as a solid bold line on Figure 1 is the only equilibrium solar capacity path starting from $K_X < \hat{K}$. There exist two other main types of trajectories solution of (3.1). A first kind of trajectories initiate under the locus $\dot{\lambda}_K = 0$ and then move in finite time above this curve inside a region where $\dot{\lambda}_K > 0$ and thus $\dot{k} > 0$ since $k = k^s(\lambda_K)$ and $dk^s(p_K)/dp_K > 0$. Furthermore in this region: $p_K(k) =$

[2]See Gaudet (1983) for a careful treatment of this problem in the context of the theory of the mine.

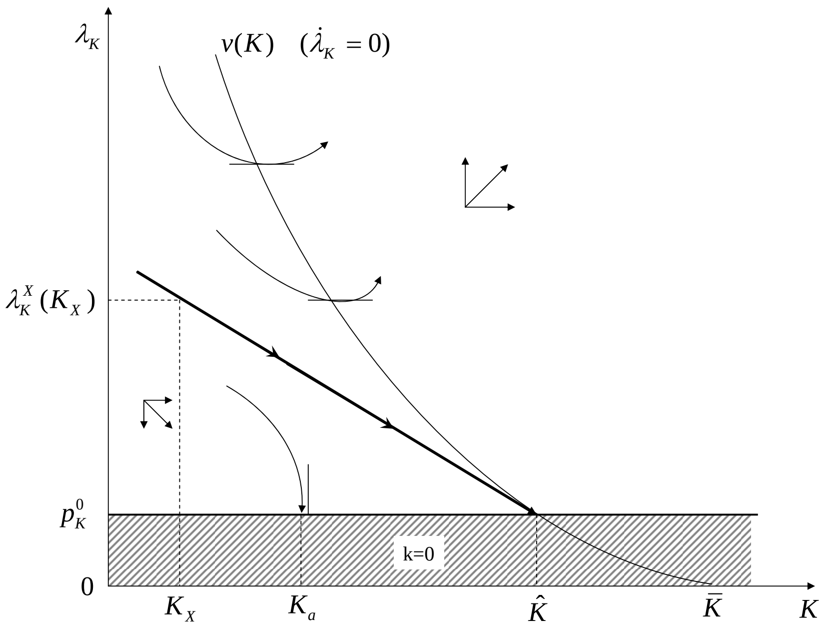

Figure 1: **The Solar Production Capacity Expansion Path after Oil Depletion.**

$\lambda_K > v(K)$. Since $y(t) = K(t)$ increases permanently, $p(t)$ decreases thus:

$$\begin{aligned}\int_t^\infty e^{-r(\tau-t)}[p(\tau) - c_y - c_K]d\tau &< \int_t^\infty e^{-r(\tau-t)}[p(t) - c_y - c_K]d\tau \\ &= \frac{p^d(K(t)) - c_y - c_K}{r} = v(K(t))\end{aligned}$$

Thus above the curve $v(K)$:

$$p_K(k(t)) \quad > \quad v(K(t)) > \int_t^\infty e^{-r(\tau-t)}[p(\tau) - c_y - c_K]d\tau$$

The marginal cost of an investment into an increase of the capacity would be higher than the total marginal gain from such an investment which cannot be profit maximizing.

The other kind of trajectories starts below the saddle branch and then moves towards the horizontal p_K^0 in finite time. Consider such a trajectory ending at some capacity level K_a as illustrated upon Figure 1. Let t_a be that time when $K(t_a) = K_a$. Since along such trajectories $\lambda_K(t) < v(K(t))$ we get at t_a:

$$\lambda_K(t_a) = p_K^0 < v(K_a) = \int_t^\infty e^{-r(\tau - t_a)}[p^d(K_a) - c_y - c_K]d\tau$$

A slight investment effort $dk > 0$ above 0 would generate a surplus gain higher than its cost, showing that such a choice of an investment policy into solar production capacity building could not be efficient for the solar industry.

Hence only remains the trajectory converging in infinite time towards $(\hat{K}, p_K^0)$ corresponding to the saddle branch upon Figure 1. Observe that the long run level of solar capacity, and hence solar energy production, is the solution of $p(K) = c_y + c_K + rp_K^0$. Thus even in the very long run the gross marginal surplus from solar energy consumption will be higher than the marginal cost of solar production. Only in a case where the minimal marginal adjustment cost p_K^0 would be zero together with the maintenance costs, the equalization in the long run of the gross marginal surplus to the variable marginal cost c_y would be profit maximizing for the solar industry. We observe also that since λ_K decreases along the saddle branch and $\lambda_K(t) = p_K(t)$ through (2.3), $k(t) = k^s(p_K(t))$ should also decrease over time. $\lambda_K(t)$ converging asymptotically towards p_K^0, $k(t)$ converges towards $k^s(p_K^0) = 0$. Hence investment into capacity building follows a smooth decreasing pattern towards zero. Since $k(t) > 0$, $t \geq t_X$ along the saddle branch, we also conclude that the unique profit maximizing policy after t_X, starting from $K_X < \hat{K}$ is to perform a permanent investment effort $k(t) > 0$ into solar capacity expansion. At the equilibrium over the energy market, the provision of solar energy will rise while the energy price will decrease, while both at decreasing rates since $K(t)$ decelerates. The following proposition summarizes our findings.

Proposition P. 1 *Consider a permanent equilibrium phase of only solar energy production, $[t_X, \infty)$, $t_X > 0$, oil being exhausted, then:*

1. *The solar production capacity cannot extend above the level $\hat{K}$ solution of: $p^d(K) = rp_K^0 + c_y + c_K$.*
2. *For an available initial solar production capacity $K_X < \hat{K}$ at the beginning of the phase t_X, the solar energy sector should permanently expand*

its production capacity $K(t)$ towards $\hat{K}$, a level which will be attained in infinite time.

3. *During this capacity expansion phase, the investment level $k(t)$ into new solar equipment will permanently decrease, converging down to zero in the very long run.*

4. *The equilibrium price of solar equipment $p_K(t)$ decreases and converges towards p_K^0.*

5. *The production of solar energy is given by $y(t) = K(t)$, the capacity constraint binding all along the phase. Solar energy provision increases permanently towards $\hat{y} = \hat{K}$. The energy price decreases and converges down to $\hat{c} \equiv rp_K^0 + c_y + c_K > c_y$. $y(t)$ increases at a decreasing rate while $p(t)$ decreases also at a decreasing rate.*

4 Energy transition from oil to solar

Before t_X, the economy consumes the oil resource and we get from (2.1): $p(t) = c_x + \lambda_X e^{rt}$. The equilibrium price of energy should increase over time in a Hotelling way. Since $\gamma_K(t) \geq 0$, inspection of (2.2) reveals that there should be no use of solar energy whence $p(t) < c_y$, that is if the energy price is too low to cover the variable cost of production of the solar alternative.

This does not imply that the solar industry should not accumulate any solar production capacity if $p(t) < c_y$. With sufficiently low interest rate and maintenance costs and a sufficiently steep supply curve of solar panels, it may be optimal for the industry to start investing even before the energy price can cover the solar variable production cost. On the other hand, a high level of the minimal cost of investment p_K^0 and of the maintenance costs c_K may delay the development of the solar energy alternative until a sufficient positive gap between the energy price and the variable marginal production cost of solar energy has been attained.

In all cases, the presence of convex capacity adjustment costs prevents an instantaneous transition from oil to solar energy, implying the existence of some time phase of joint use of the two energy sources. Denote by t_y the beginning of the phase of joint use of oil and solar energy and by t_K the beginning of the investment phase into solar production capacity. Since it has been assumed that $K(0) = 0$, solar energy use requires some productive capacity and hence $t_K \leq t_y$. Depending upon the shapes of the energy demand function and the supply function of solar equipment together with the other model parameters, the equilibrium transition from oil to solar energy may follow four possible scenarios.

- A three phases scenario where $0 < t_K < t_y$ composed of a first phase $[0, t_k)$ of only oil consumption without investment in the solar alternative. This phase is followed by a phase $[t_K, t_y)$ of solar investment without solar energy production before a phase $[t_y, t_X)$ of joint use of both energies extending until the depletion of the oil reserves.
- A two phases scenario where $0 < t_K = t_y$ composed of a first phase $[0, t_y)$ of only oil consumption without solar production capacity development followed by a phase $[t_y, t_X)$ of joint use of oil and solar energy.
- A two phases scenario where $0 = t_K < t_y$ with immediate development of the solar production capacity without use of solar energy before the the joint energy use phase $[t_y, t_X)$.
- A one phase scenario $[0, t_X)$ where $0 = t_K = t_y$ during which the economy uses permanently the two available energy sources until the exhaustion of oil.

Before proceeding to the description of the solar energy development plan in these various scenarios, let us sketch the main features of the price and quantity dynamics during the energy transition. As shown before, the energy price permanently increases implying that the total energy consumption should decline over time. Once solar energy is introduced, the production capacity accumulation will induce an increased use of solar energy inside the energy mix, oil consumption decreasing at a higher rate than total energy consumption. Contrasting with the Herfindahl textbook model, there will not be a downward jump in the use of oil at the depletion time t_X. Under convex adjustment costs, or equivalently an upward bending supply curve of solar equipment, oil consumption will fade over time in a continuous way and $x(t_X) = 0$ while $q(t_X) = y(t_X) = K(t_X) = K_X$ following our previous notations. Note that in the case $0 < t_K < t_y$, the use of solar energy jumps up from zero to the available capacity level at time t_y, $K(t_y)$. Thus, oil consumption should make a parallel downward jump at t_y, total energy consumption having to be time continuous. Furthermore $p(t_y) = c_y$ implies that $\gamma_K(t) = 0$ before t_y.

During the phase of joint use of both energy sources, $p(t) = c_y + \gamma_K(t)$ through (2.2) implies that $\gamma_K(t)$ increases over time and thus that $\beta_K(t)$ should increase. Such an increase of the static opportunity cost of the solar production capacity does not imply an increase of the marginal benefit of an investment into solar production capacity $\lambda_K(t)$ since $\dot{\lambda}_K(t) = r\lambda_K(t) - \beta_K(t)$ can be positive only if $\lambda_K(t) > \beta_K(t)/r$. A monotonous increase of the static opportunity cost does not translate into a monotonous evolution of the value of an investment into solar energy, a feature we are going to examine now in more detail.

Consider first the dynamics of the investment plan after t_y. Making use of (2.4), (2.5) together with the expression of $\gamma_K(t)$ resulting from (2.2), we obtain for $t \geq t_y$:

$$\dot{\lambda}_K(t) \quad = \quad r\lambda_K(t) + c_y + c_K - c_x - \lambda_X e^{rt} \ .$$

Integrating this equation over a time interval $[t_0, t)$, $t_0 < t$, we get:

$$\lambda_K(t) \quad = \quad \lambda_K(t_0)e^{r(t-t_0)} + \frac{c_y + c_K - c_x}{r}\left(e^{r(t-t_0)} - 1\right) - \lambda_X e^{rt}(t - t_0) \ . \tag{4.1}$$

Differentiating with respect to time results in:

$$\dot{\lambda}_K(t) \quad = \quad \pi(t_0)e^{r(t-t_0)} - r\lambda_X e^{rt}(t - t_0) \ ,$$

where $\pi(t_0) \equiv r\lambda_K(t_0) + c_y + c_K - c_x - \lambda_X e^{rt_0}$. This defines a unique $\bar{t}$ solution of $\dot{\lambda}_K(t) = 0$ that is of:

$$t \quad = \quad t_0 + \frac{\pi(t_0)}{\lambda_X e^{rt_0}} \ .$$

$t_0 \leq t$ implies that $\pi(t_0) \geq 0$ that is: $p(t_0) \leq c_y + c_K + r\lambda_K(t_0)$. In the reverse case, $\dot{\lambda}_K(t) < 0$, for any $t > t_0$. Furthermore, differentiating once again with respect to time gets: $\ddot{\lambda}_K(t) = r(\dot{\lambda}_K(t) - \lambda_X e^{rt})$ thus $\ddot{\lambda}_K(\bar{t}) < 0$, which implies that $\bar{t}$ is unique if it exists.

Remark first that investment into solar energy should take place before that time $\underline{t}$ at which $p(t) = rp_K^0 + c_y + c_K$. If $\underline{t} < t_K$, $p(t_K) > c_y + c_K + rp_K^0$ and $\lambda_K(t_k) = p_K^0$ both imply that $\pi(t_K) < 0$ and thus $\dot{\lambda}_K(t) < 0$, $t \geq t_K$. But positive investment after t_K requires that $\lambda_K(t) > p_K^0$, hence a contradiction. $\underline{t}$ is the upper bound over possible investment starting time t_K. We have to consider two possibilities:

- Either $t_K = t_y$ and investment begins with solar energy production, in which case: $t_K = t_y < \underline{t}$ and $c_y \leq c_x + \lambda_X e^{rt_K}$.
- Either $t_K < t_y$ and investment begins strictly before the phase of joint use of solar and oil energy, t_y being the solution of $c_y = c_x + \lambda_X e^{rt}$.

We have also to consider the possibility of an immediate start of the solar development plan from $t = 0$, that is either $0 = t_K < t_y$ or either $0 = t_K = t_y$. Consider first the case $0 < t_K \leq t_y$. Then the investment plan should experience a smooth start at t_K and $\lambda_K(t_K) = p_K^0$. If $t_K < t_y$, $p(t_K) < c_y$ and the industry invests into solar production capacity before beginning to produce solar energy. At time t_y, $p(t_y) = c_y < c_y + c_K + rp_K^0 < c_y + c_K + r\lambda_K(t_y)$. Thus $\pi(t_y) > 0$ and $\lambda_K(t)$ increases over time during the interval $[t_y, \bar{t})$ before decreasing after $\bar{t}$. If $t_K = t_y$, then $t_y < \bar{t}$ implies that $p(t_y) < c_y + c_K + rp_K^0$ but $c_y \leq p(t_y)$ since $\gamma_K(t)$ has to be non negative at t_y. As in the preceding case, the investment plan should have an inverted U shape after t_y being composed of a first increasing phase before $\bar{t}$ followed by a decreasing phase after $\bar{t}$. Since $p_K(t) = \lambda_K(t)$, $k^s(p_K) = k^s(\lambda_K)$ implies that at the equilibrium, the investment level $k(t)$ increases before $\bar{t}$, corresponding to an accelerated accumulation of solar production capacity, $K(t)$. The investment level attains a maximum at $\bar{t}$ before decreasing, corresponding to a decelerating solar production capacity development pace after $\bar{t}$.

This scenario is only valid in a case where the oil resource would not be exhausted before $\bar{t}$. But since the energy price is a continuous time function at the equilibrium, we get also at time t_X: $(p(t_X) - c_y - c_K)/r = v(K_X)$. This implies that $\dot{\lambda}_K(t)$ should be continuous at time t_X, showing that $\lambda_K(t)$ is a continuous and time differentiable function around t_X. We have shown in section 3 that, from t_X onwards, $\lambda_K(t)$ should follow a decreasing time pattern converging asymptotically towards p_K^0 provided that $K_X < \hat{K}$, a point we check below. This implies that $\dot{\lambda}_K(t_X) < 0$. Hence we can conclude that $\bar{t} < t_X$. The oil resource has to be depleted only after the investment policy turning point $\bar{t}$.

If $t_K < t_y$, $K_m(t) = K(t)$, $t \in [t_K, t_y)$ implies that $\gamma_m(t) > 0$ while $y(t) = 0$ implies that $K(t) = K_m(t) > y(t)$ and hence $\gamma_K(t) = 0$. Thus (2.4) implies that $c_K + \gamma_m(t) = \lambda_K(t)$ that is: $\lambda_K(t) - \gamma_m(t) = c_K$. Inserted into (2.5), this results in $\dot{\lambda}_K(t) = r\lambda_K(t) + c_K$. Thus $\lambda_K(t)$ increases with $k(t)$ during the first solar investment phase before solar production, corresponding to an accelerating trend into solar production capacity building.

In the case $0 = t_k \leq t_y$, the initial investment level $k(0)$ is adjusted in a bang bang way to some strictly positive level. Depending upon the size of this initial jump, it may be the case that the solar investment plan exhibits the inverted U shape previously described. But if $\lambda_K(0)$ is such that $\pi(0) < 0$, that is if $c_y + c_K + r\lambda_K(0) < c_x + \lambda_X$, the rate of investment into solar capacity building will permanently decrease before the complete transition towards solar energy.

It remains to check that $K_X < \hat{K}$, that is the solar production capacity development during the transition phase between oil and solar energy should not end at t_X with an available capacity K_X higher than $\hat{K}$. This requires that $\lambda_K(t_X) \leq p_K^0$. Since we have shown that during the transition phase, $\lambda_K(t)$ first increase before $\bar{t}$ and then decreases after $\bar{t}$, this in turn implies the existence of a point of time $\tilde{t}$, $\bar{t} < \tilde{t} \leq t_X$ such that $\lambda_K(\tilde{t}) = p_K^0$ and $\lambda_K(t) > p_K^0$ for $\bar{t} < t < \tilde{t}$. Since $\lambda_K(t)$ should decrease before $\tilde{t}$, $p_K^0 < (p(\tilde{t}) - c_y - c_K)/r$. But since the energy priceshould increase up to t_X and remain constant afterwards, the solar energy production capacity being no more expanded, $p(t) > p(\tilde{t})$, $t > \tilde{t}$. Hence a slight investment above zero at time $\tilde{t}$ would generate a discounted benefit at this time higher than its cost, contradicting the optimality of such a capacity investment plan. The following figure 2 illustrates the dynamics of investment into capacity building in terms of the dynamics of the dual variables $(\lambda_K(t), \beta_K(t))$. We can summarize as follows our findings so far:

Proposition P. 2 *During the transition from oil to solar energy:*

- *The energy price increases over time in a Hotelling way and total energy*

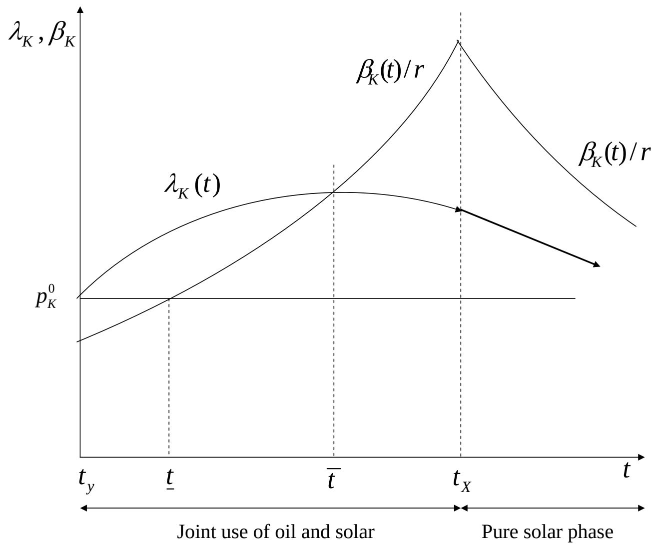

Figure 2: **Dynamics of λ_K and $\beta_K(t)$ if $0 < t_K = t_y$.**

consumption declines.

- *The complete transition towards solar energy is preceded by a time phase of simultaneous exploitation of both oil and solar energy.*

- *Solar energy use rises during this time phase with the solar production capacity and the solar energy share increases inside the energy mix.*

- *At t_X, the depletion time of the oil reserves, $x(t_X) = 0$ and $q(t_X) = y(t_X) = K(t_X) \equiv K_X$.*

- *Depending upon the model parameters, investment into solar production capacity may start before the beginning of the use of solar energy. In this case, solar energy is introduced at that time t_y when the energy price reaches the variable cost level of producing solar energy, that is $p(t_y) = c_y$.*

- *It may also be the case that solar production and investment begin at the same time t_y in which case $p(t_y) \geq c_y$.*

- *In both cases, solar energy development must occur before the energy price has reached the level $c_y + c_K + rp_K^0$, implying that the energy* *industry should experience negative returns over their investments in the early stage of the energy transition.*

Furthermore, the investment plan into solar production capacity has the following characteristics:

Proposition P. 3 *During the solar production capacity development phase* $[t_K, t_X)$*:*

- *If* $t_K > 0$*, the marginal benefit of investing into solar energy has an inverted U shape, being first increasing and then decreasing. The investment rate into solar equipment begins to decrease strictly before the depletion of the oil stock.*
- *The price of the solar equipment* $p_K(t)$ *first rises at the beginning of the solar capacity building phase and then strictly decreases before the complete transition towards solar energy.*
- *If solar investment starts immediately at a sufficiently high strictly positive level, that is if* $t_K = 0$*, it is possible that this shape reduces to a decreasing pattern of investment together with a declining trend of the equipment price.*
- *The solar capacity constraint binds all along the phase of joint use of both energies. The net instantaneous opportunity cost of the capacity constraint in rental terms,* $\beta_K(t)/r$*, permanently increases during this phase, being first lower than the equilibrium equipment price* $p_K(t)$ *at the beginning of solar production capacity development and then higher than* $p_K(t)$ *at the end of the energy transition.*
- *The capacity building process during the transition phase results into an available capacity at time* t_X*,* K_X *being strictly lower than the economically maximum capacity level* $\hat{K}$*.*

It remains to examine several issues. We have to check the domain of validity of the various scenarios. Secondly, we have to show that the policies previously described are indeed profit maximizing before the complete transition towards solar energy. We are going to describe an algorithmic argument able to address these issues and provide a closed form solution to the present model. The solving procedure will make appear that the characteristics of the solar energy investment plan may be independent from the size of the oil reserves. More precisely, we show that with a sufficiently high initial level of oil reserves, the length of the solar development phase, $T \equiv t_X - t_K$, and the accumulated capacity at the end of the transition phase, K_X, do not depend upon X_0, the initial stock of the non renewable resource.

5 Characterizing the profit maximizing scenario at the equilibrium

Consider the most complex case of a four phases scenario. Considering first the phase $[t_K, t_y)$, we know that $\dot{\lambda}_K = r\lambda_K + c_K$. Integrating this equation over $[t_K, t)$, $t \leq t_y$, we obtain:

$$\lambda_K(t) = e^{r(t-t_K)} p_K^0 + \frac{c_K}{r}\left(e^{r(t-t_K)} - 1\right) \quad t \in [t_K, t_y) \; . \tag{5.1}$$

During the transition phase between oil and solar energy $[t_y, t_X)$, $\lambda_K(t)$ is given by:

$$\lambda_K(t) = e^{r(t-t_y)}\lambda_K(t_y) + \frac{c_y + c_K - c_x}{r}\left(e^{r(t-t_y)} - 1\right) - \lambda_X e^{rt}(t - t_y) \; .$$

Since $\lambda_K(t)$ is a continuous time function at $t = t_y$:

$$\lambda_K(t_y) = e^{r(t_y - t_K)} p_K^0 + \frac{c_K}{r}\left(e^{r(t_y - t_K)} - 1\right) \; ,$$

thus:

$$\begin{aligned} \lambda_K(t) = \; & p_K^0 e^{r(t-t_K)} + \frac{c_K}{r}\left(e^{r(t-t_K)} - 1\right) + \frac{c_y - c_x}{r}\left(e^{r(t-t_y)} - 1\right) \\ & - \lambda_X e^{rt}(t - t_y) \; . \end{aligned} \tag{5.2}$$

Next $k(t) = k^s(\lambda_K(t)))$ during the two phases $[t_K, t_y)$ and $[t_y, t_X)$ considering (2.3). Since $K(t_K) = 0$:

$$K_X = K(t_X) \quad = \quad \int_{t_K}^{t_X} k(t)dt = \int_{t_K}^{t_X} k^s(\lambda_K(t))dt \ .$$

In this scenario, solar energy is introduced once the energy price has reached the level c_y, hence:

$$c_y \quad = \quad c_x + \lambda_X e^{rt_y} \ .$$

At the equilibrium, the energy price path and hence the energy consumption path must be continuous time functions at t_X. This requires that $x(t_X) = 0$ and $q(t_X) = y(t_X)$. Since the capacity constraint binds at t_X, $y(t_X) = K(t_X) \equiv K_X$ and at the equilibrium:

$$p(t_X) = p^d(K_X) \quad = \quad c_x + \lambda_X e^{rt_X} \ .$$

Denote by $\lambda_K^X(K)$ the implicit equation of the saddle branch in the space (K, λ_K) linking λ_K to the installed capacity K during the last pure solar phase $[t_X, \infty)$. Since $\lambda_K(t)$ is a time continuous function at t_X and $K(t)$ is also a time continuous function at t_X, $\lambda_K(t_X) = \lambda_K^X(K(t_X)) = \lambda_K^X(K_X)$. Thus making use of (5.2) evaluated at t_X:

$$\lambda_K^X(K_X) \quad = \quad \lambda_K(t_X) \ .$$

Last, the oil reserves have to be exhausted during the time interval $[0, t_X)$, that is:

$$X_0 \quad = \quad \int_0^{t_X} x(t)dt = \int_0^{t_X} q(t)dt - \int_{t_y}^{t_X} y(t)dt = \int_0^{t_X} q^d(p(t))dt - \int_{t_y}^{t_X} K(t)dt \ .$$

Since $\dot{K}(t) = k(t)$ and $K(t_y) \equiv K_y$, we obtain for $t \in (t_y, t_X)$:

$$K(t) \quad = \quad K_y + \int_{t_y}^{t} k(\tau)d\tau$$

Hence:

$$\int_{t_y}^{t_X} K(t)dt \quad = \quad K_y(t_X - t_y) + \int_{t_y}^{t_X} \int_{t_y}^{t} k(\tau)d\tau dt$$

Inverting the integration order:

$$\int_{t_y}^{t_X} K(t)dt = (t_X - t_y) \int_{t_K}^{t_y} k(t)dt + \int_{t_y}^{t_X} k(t)(t_X - t)dt \ .$$

It appears that in a four phases scenario, the vector of variables $(t_K, t_y, t_X, K_X, \lambda_X)$ must be solution of the following system of five conditions:

$$\begin{aligned}
K_X &= \int_{t_K}^{t_y} k^s(\lambda_K(t)dt + \int_{t_y}^{t_X} k^s(\lambda_K(t))dt && (5.3)\\
c_y &= c_x + \lambda_X e^{rt_y} && (5.4)\\
p^d(K_X) &= c_x + \lambda_X e^{rt_X} && (5.5)\\
\lambda_K^X(K_X) &= \lambda_K(t_X) && (5.6)\\
X_0 &= \int_0^{t_X} q^d(c_x + \lambda_X e^{rt})dt - (t_X - t_y)\int_{t_K}^{t_y} k^s(\lambda_K(t))dt \\
&\quad - \int_{t_y}^{t_X} k^s(\lambda_K(t))(t_X - t)dt && (5.7)
\end{aligned}$$

Note that once this vector of variables has been computed, the characterization of the scenario is complete. λ_X determines the energy price trajectory and hence the oil exploitation plan during the first phase $[0, t_K)$. t_X and K_X both determine the whole characteristics of the last pure solar phase $[t_X, \infty)$ in terms of solar energy consumption, energy price and capacity development path.

Nevertheless, it may be checked that the system (5.3)-(5.7) is not of full rank by computing the determinant of the corresponding linearized system. Considering the sub-system (5.3)-(5.6), Appendix A.1 shows that t_K and t_X are functions of λ_X such that: $dt_K/d\lambda_X = dt_X/d\lambda_X = -1/(r\lambda_X) < 0$, hence the length $t_X - t_K$ is independent of λ_X while K_X is also independent from λ_X. The same holds in a three phases scenario where $0 < t_K = t_y$.

This suggests the following solving strategy. Assume $t_K = 0$, that is take the lower bound for possible values of t_K and set $\lambda_K(0) = p_K^0$, that is consider a smooth start of the investment process from $t = 0$. Denote by $T \equiv t_X - t_K$, the length of the development phase of solar production capacity before the depletion of the oil reserves. In the case $t_K = 0$, $T = t_X$. Then solve the following subsystem in (T, λ_X, K_X):

$$\begin{aligned}
K_X &= \int_0^T k^s(\lambda_K(t))dt && (5.8)\\
p^d(K_X) &= c_x + \lambda_X e^{rT} && (5.9)\\
\lambda_K^X(K_X) &= \lambda_K(T) && (5.10)
\end{aligned}$$

We check in Appendix A.1 that the above system has a unique solution (T, K_X, λ_X^0). We face two possibilities:

- Either $c_x + \lambda_X^0 < c_y$ and in this case, $t_K < t_y$. The optimal path of solar energy development before t_X is split in two successive phases, a phase $[t_K, t_y)$ with solar capacity building without production of solar energy followed by a phase $[t_y, t_X)$ of joint investment and use of solar energy.
- Either $c_x + \lambda_X^0 \geq c_y$ and in this case $t_K = t_y$, investment into solar production capacity occurs simultaneously with the rise of solar energy inside the energy mix and t_y is defined by $p(t_y) = c_x + \lambda_X^0$.

Note that this feature of the profit maximizing development plan of solar energy does not depend upon the oil resource scarcity and results from the shapes of the energy demand function, the supply function of solar panels, the variable energy production costs and the interest rate.

We check also in Appendix A.1 that the oil stock constraint defines in a unique way λ_X as a decreasing function of X_0, a function we denote by $\lambda_X(X_0)$. If $\lambda_X < \lambda_X^0$ then $t_K > 0$ and $\lambda_X e^{rt_K} = \lambda_X^0$ defines t_K and hence $t_X = T + t_K$. Taking into account our previous description of the solar energy development plan, we conclude that if $c_x + \lambda_X^0 < c_y$, the optimal scenario is a four phases scenario, with a first phase $[0, t_K)$ of only oil energy without solar development followed by two phases $[t_K, t_y)$, $[t_y, t_X)$ of solar development before the depletion of the oil reserves and concluded by the last phase $[t_X, \infty)$ of pure solar energy. Conversely, if $c_x + \lambda_X^0 > c_y$, then the phase $[t_K, t_y)$ vanishes and the optimal path reduces to a three phases scenario with a common development of solar production infrastructure and solar energy production. Note that in this situation, solar development will take place after that time when the energy price $p(t)$ reaches the level c_y of the variable cost of solar energy production. Let X_0^0 be the unique solution of $\lambda_X(X_0) = \lambda_X^0$. Since $\lambda_X(X_0)$ is a decreasing function we conclude that the above scenarios will be valid iff $X_0^0 < X_0$.

In the reverse case $X_0 < X_0^0$, $\lambda_X^0 < \lambda_X$ and $t_K = 0$. A scarcer oil resource induces an immediate development of the solar energy alternative.

If $c_x + \lambda_X < c_y$ then $t_y > 0$ and the optimal path is a three phases scenario. During the first phase $[0, t_y)$, the solar industry invests into capacity building without producing solar energy. The second phase $[t_y, t_X)$ is a phase of joint use of oil and solar energy together with a continuous investment into solar production capacity. During the last phase $[t_X, \infty)$ the economy uses only solar energy. In the reverse case $\lambda_X + c_x > c_y$, both t_K and t_y are reduced to zero. The optimal path is a two phases scenario with a first phase of joint use of oil and solar energy sources before the transition towards pure solar energy. In both scenarios $k(0) > 0$, meaning the the solar industry starts immediately to invest at a strictly positive level, the equilibrium price of solar panel being strictly higher than p_K^0. Setting $t_K = 0$ and $\lambda_K(0) > p_K^0$, the system (5.3)-(5.7) becomes of full rank and determines the vector $(t_y, t_X, K_X, \lambda_X, \lambda_K(0))$ and hence $p_K(0)$ in the case $c_x + \lambda_X < c_y$. In the reverse case, t_y is reduced to zero and the system:

$$K_X = \int_0^{t_X} k^s(\lambda_K(t))dt \tag{5.11}$$

$$p^d(K_X) = c_x + \lambda_X e^{rt_X} \tag{5.12}$$

$$\lambda_K^X(K_X) = \lambda_K(t_X) \tag{5.13}$$

$$X^0 = \int_0^{t_X} q^d(c_x + \lambda_X e^{rt})dt - \int_0^{t_X} k^s(\lambda_K(t))(t_X - t)dt \tag{5.14}$$

defines the solution vector $(t_X, K_X, \lambda_X, \lambda_K(0))$.

The following proposition summarizes these findings.

Proposition P. 4 *Let (T, K_X, λ_X^0) be the unique solution of the system (5.8)-(5.10) and let X_0^0 be the unique solution of $\lambda_X(X_0) = \lambda_X^0$. Then:*

1. *If $X_0^0 < X_0$, the optimal path begins with a first phase $[0, t_K)$ of only oil production without investment into the solar energy alternative and:*
   - *If $\lambda_X^0 < c_y - c_x$, then $t_K < t_y$ and the energy transition is composed of a first phase $[t_K, t_y)$ of investment into solar capacity without solar energy production followed by a phase $[t_y, t_X)$ of joint use of both energy sources. t_K is defined by $p(t_K) = c_x + \lambda_X^0$, t_y by $p(t_y) = c_y$ and $t_X = T - t_K$.*

- *If $\lambda_X^0 > c_y - c_x$, then $t_K = t_y$ and the energy transition is reduced to a single phase of joint production from both energy sources until oil depletion. t_y solves $p(t_y) = c_x + \lambda_X^0 > c_y$ and $t_X = T - t_y$.*

2. *If $X_0^0 > X_0$, the development of the solar energy alternative starts immediately from $t = 0$ meaning that t_K is reduced to zero and:*

   - *If $\lambda_X^0 < c_y - c_x$, then $t_y > 0$ and the energy transition is composed of a first phase $[0, t_y)$ of investment into solar capacity without solar energy production followed by a phase $[t_y, t_X)$ of joint use of both energy sources. t_y is defined by $p(t_y) = c_y$ and $(t_X, K_X, \lambda_X, \lambda_K(0))$ is the unique solution of the system (5.3)-(5.7) when $t_K = 0$.*
   - *If $\lambda_X^0 > c_y - c_x$, then $t_K = t_y = 0$ and the energy transition is reduced to a single phase of joint production from both energy sources until oil depletion. $(t_X, K_X, \lambda_X, \lambda_K(0))$ is the unique solution of the system (5.11)-(5.14).*

The following Figure 3 illustrates the energy price dynamics and the solar equipment price dynamics in a case where $X_0^0 < X_0$ and $t_K = t_y$.

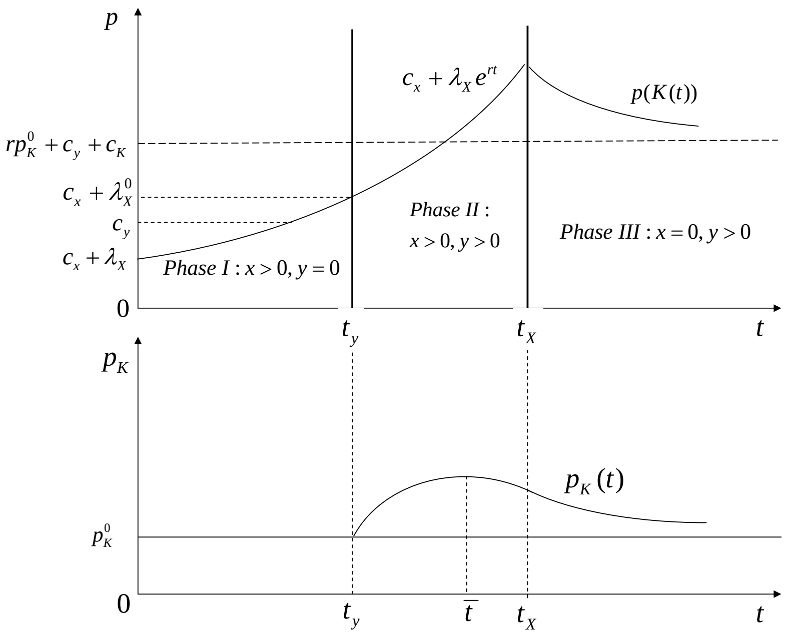

Figure 3: **Energy price and solar equipment price dynamics.**

The peak price of energy is attained at the depletion time of oil. The energy price decreases continuously after oil depletion. It is noteworthy to remark that such a price shape may be observed in learning by doing models of solar energy use. Models of investment into alternative energy sources under adjustment costs can thus generate analogous shapes of the energy price path after oil depletion than learning models.

The Proposition P.4 shows that the features of the solar development policy may be largely independent from the availability of the non renewable resource with sufficiently high initial oil resource endowments. Since a smooth start of investment into solar capacity results from the assumption of an upward sloping supply curve, both $K(t_y)$ and $k(t_y)$ are set to zero in any scenario of this kind. A change of λ_X has no effect over $\lambda_K(t)$. Thus the investment path resulting from the time path of $\lambda_K(t)$ does not change with λ_X. In other words, the investment policy into solar energy is independent from the scarcity rent of the resource, only the timing of introduction of solar energy inside the energy mix being affected. Investment into the solar substitute is delayed by a higher initial oil reserves amount, a quite natural conclusion.

Less straightforward, the accumulated capacity at the end of the transition phase between oil and solar is independent from the oil reserves. If one think of the transition phase as the replacement of some natural capital (the oil stock) by some man made capital (the installed capacity in producing solar energy), our model is an example of a situation where the size of the natural capital stock has no effect over the size of the accumulated man made capital stock before the depletion of the exhaustible resource. Since K_X is unaffected by X_0, we can also conclude that the post oil phase characteristics do not depend upon the previous scarcity of the oil resource.

Some other features of the energy transition are also noteworthy. Depending upon the characteristics of the adjustment cost function with respect to the other model structural elements, it may be the case that either investment into solar energy starts before the energy price can cover the variable marginal cost of solar energy, c_y, or only when a higher level of the price than c_y has been attained. In the first situation, the energy industry begins to develop the solar alternative before using it to produce energy. This is a common feature of R&D models where the industry has to invest into costly research efforts in order to attain a sufficiently productive technological stage. There is no explicit R&D process in this model, and this phenomenon stands as a distinctive feature of the Hotelling price dynamics combined with the convexity of the adjustment cost function.

During the only invest phase, the net opportunity cost of the capacity constraint, β_K, remains at its minimal constant level $-c_K$, which is higher than $-p_K^0$, as we have assumed that $c_K < p_K$. β_K grows over time once solar energy is introduced inside the energy mix. Thus investing early allows to reduce the cost of the capacity constraint. On the other hand, the returns from solar capacity investments are negative at least until the time $\underline{t}$ at which $p(t) = rp_K^0 + c_y + c_K > c_y$, thus later than t_y, the time at which $p(t) = c_y$, because of the Hotelling dynamics of the energy price. The industry should try to minimize the length of this negative returns period by delaying the beginning of its investments into the solar alternative. The trade-off between these two opposite incentives may result either in a early beginning of the solar investment if the first incentive dominates or conversely in a delayed beginning if the second dominates. In all cases, the solar industry will begin to produce even if the energy price is too low to cover the full minimal marginal cost level, that is $rp_K^0 + c_y + c_K$ and hence will experience negative returns over its investments at the early stage of the energy transition.

6 Conclusion

The present study characterizes the economic logic of the transition between oil and solar energy as a mix of investment patterns features under adjustment costs in the Gould or Treadway tradition and Hotelling dynamics. The last phase of only solar energy use is a good illustration of the pure logic of adjustment costs in the line of Treadway. The solar production capacity expansion process should decelerate over time together with a permanent decrease of the capacity rent of solar equipment, or in other terms, of the opportunity cost of the capacity constraint in producing solar energy. In parallel, the value of an investment into further solar energy development should decrease over time down to some constant long run level together with the solar production level.

The rise of renewable energy under the rule of Hotelling follows a rather different logic. Because of the permanent growth of the mining rent in current terms, the gap between the current energy price and the variable marginal cost of producing solar energy rises exponentially up to the exhaustion of the oil reserves. This results into a progressive increase of the capacity rent of solar energy equipments boosting investment in this alternative energy source. But since the energy price follows a declining trend after the depletion of fossil fuels, time passing reduces the incentive to accelerate investment into the solar energy alternative. The combination of these two opposite effects result in a non monotonous investment pattern, the investment rate into solar energy being increasing at the beginning of the transition phase before being decreasing at the end of this phase.

The features of the investment policy into solar energy may be more or less dependent from oil scarcity. Our constant variable marginal cost model stands as an extreme case where the solar investment path is almost independent from the oil initial endowments. Introducing more complex cost structures, for example oil extraction costs depending upon past extraction, would alter this conclusion of course. But this would already be the case in a simple resource transition model without adjustment costs, thus blurring the proper effect of costly investments into the description of energy sources transition. It appears from our analysis that an endogenous production capacity investment process into renewable energy generally soften the link between relative costs dynamics and their effects over production levels dynamics.

When considering the issue of subsidizing 'green 'energy alternatives, our approach emphasizes that one should consider two main types of subsidies : subsidies at the production stage aimed at compensating the gap between the cost of solar energy and the production cost of fossil fuels, and subsidies at the investment stage intended to lower the investment costs. Our analysis suggests that these two types of subsidies should have rather different impacts over the development path of renewable energy with ambiguous consequences over the energy price trends and the depletion of fossil fuels. We deserve this problem for further research.

One strong motivation for developing renewable energy alternatives is climate change mitigation. An explicit account of the polluting nature of fossil fuels inside our model should impact in more or less complex ways both the beginning time and the speed of development of the green energy alternatives, also an issue which appears worth a specific study.

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Appendix

A.1 Appendix A.1

A.1.1 Properties of $\lambda_K(t)$

During the time interval $[t_K, t_y)$, $\lambda_K(t)$ is defined by (5.1) as a function of t_K and t, $\lambda_K(t, t_K)$, and:

$$\frac{\partial \lambda_K(t, t_K)}{\partial t} = -\frac{\partial \lambda_K(t, t_K)}{\partial t_K} = (rp_K^0 + c_K)e^{r(t-t_K)} > 0 \ .$$

Furthermore, (5.2) defines $\lambda_K(t)$ as a function $\lambda_K(t; t_K, t_y, \lambda_X)$ during the time interval (t_y, t_X) and:

$$\begin{aligned} \frac{\partial \lambda_K(t, t_y, \lambda_X)}{\partial t} &= (rp_K^0 + c_k)e^{r(t-t_K)} + (c_y - c_x)e^{r(t-t_y)} - r\lambda_X e^{rt}(t - t_y) - \lambda_X e^{rt} \\ &= e^{r(t-t_y)}\left[(rp_K^0 + c_K)e^{r(t_y-t_K)} + c_y - c_x - \lambda_X e^{rt_y}\right] - r\lambda_X e^{rt}(t - t_y) \ . \end{aligned}$$

Denote by $\pi \equiv (rp_K^0 + c_K)e^{r(t_y-t_K)} + c_y - c_x - \lambda_X e^{rt_y} = (rp_K^0 + c_K)e^{r(t_y-t_K)}$ since $c_y = c_x + \lambda_X e^{rt_y}$. Then $\pi > 0$ and:

$$\frac{\partial \lambda_K; t_y, \lambda_X)}{\partial t} = \pi e^{r(t-t_y)} - r\lambda_X e^{rt}(t - t_y) \ . \qquad \text{(A.1.1)}$$

Furthermore:

$$\frac{\partial \lambda_K}{\partial t_y} = e^{r(t-t_y)}\left[\lambda_X e^{rt_y} + c_x - c_y\right] = 0 \ , \qquad \text{(A.1.2)}$$

$$\frac{\partial \lambda_K}{\partial t_K} = -(rp_K^0 + c_K)e^{r(t-t_K)} = -\pi e^{r(t-t_y)} < 0 \ , \qquad \text{(A.1.3)}$$

$$\frac{\partial \lambda_K}{\partial \lambda_X} = -e^{rt}(t - t_y) < 0 \ . \qquad \text{(A.1.4)}$$

A.1.2 Proof that $t_X - t_K$ and K_X are independent from λ_X

Consider the subsystem (5.3), (5.5), (5.6). To prove our claim we differentiate this system with respect to $(t_K, t_y, t_X, \lambda_X)$. Note first that $\lambda_K(t_K) = p_K^0$ implies that $k(t_K) = 0$. Denote by: $k^s(t_X) \equiv k_X > 0$ and $k^{s'} \equiv dk^s(\lambda_K)/d\lambda_K > 0$. Differentiating the condition (5.3) while taking (A.1.2)-(A.1.4) into account results in:

$$\begin{aligned} dK_X &= \int_{t_K}^{t_y} k^{s'} \frac{\partial \lambda_K}{\partial t_K} dt dt_K + dt_X k_X \\ &\quad + \int_{t_y}^{t_X} k^{s'} \left[\frac{\partial \lambda_K}{\partial t_K} dt_K + \frac{\partial \lambda_K}{\partial t_y} dt_y + \frac{\partial \lambda_K}{\partial \lambda_X} d\lambda_X \right] dt \\ &= dt_X k_X - \left[\int_{t_y}^{t_X} k^{s'} e^{rt}(t - t_y) dt \right] d\lambda_X - \left[\int_{t_K}^{t_X} k^{s'} (r p_K^0 + c_K) e^{r(t-t_K)} dt \right] dt_K \ , \end{aligned}$$

Denote by:

$$\begin{aligned} I_K^\lambda &\equiv \int_{t_y}^{t_X} k^{s'} e^{rt}(t - t_y) dt > 0 && \text{(A.1.5)} \\ I_K^K &\equiv \int_{t_K}^{t_X} k^{s'} (r p_K^0 + c_K) e^{r(t-t_K)} dt \\ &= \int_{t_K}^{t_X} k^{s'} \pi e^{r(t-t_y)} dt > 0 \ . && \text{(A.1.6)} \end{aligned}$$

Then:

$$dK_X = k_X dt_X - I_K^\lambda d\lambda_X - I_K^K dt_K \qquad \text{(A.1.7)}$$

Next differentiating (5.5) we obtain:

$$\frac{dp^d(q)}{dq}|_{q=K_X} dK_X = d\lambda_X e^{rt_X} + r\lambda_X e^{rt_X} dt_X \ .$$

Denote by $p^{d'} \equiv dp^d(q)/dq|_{q=K_X} < 0$. Then taking the absolute value and rearranging results in:

$$-|p^{d'}| dK_X e^{-rt_X} = d\lambda_X + r\lambda_X dt_X \ . \qquad \text{(A.1.8)}$$

Inserting the expression (A.1.7) of dK_X inside (A.1.8) we get:

$$\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right] dt_X = \left[|p^{d'}|e^{-rt_X}I_K^{\lambda} - 1\right] d\lambda_X + |p^{d'}|e^{-rt_X}I_K^K dt_K \ . \tag{A.1.9}$$

(A.1.9) expresses dt_X as a function of $d\lambda_X$ and dt_K.

Then differentiating the condition (5.6) taking into account that $\partial\lambda_K/\partial t_y = 0$ results in:

$$\frac{d\lambda_K^X(K)}{dK}|_{K=K_X} dK_X = \frac{\partial\lambda_K(t_X)}{\partial t_X}dt_X + \frac{\partial\lambda_K(t_X)}{\partial t_K}dt_K + \frac{\partial\lambda_K(t_X)}{\partial\lambda_X}d\lambda_X \ .$$

Denote by $\lambda_K^{X'} \equiv d\lambda_K^X(K)/dK|_{K=K_X}$. We have shown that $\lambda_K^{X'} < 0$. Thus the above expression is equivalent to, making use of (A.1.1), (A.1.3) and (A.1.4):

$$-|\lambda_K^{X'}|dK_X = [\pi e^{r(t_X - t_y)} - r\lambda_X e^{rt_X}(t_X - t_y)]dt_X - \pi e^{r(t_X-t_y)}dt_K - e^{rt_X}(t_X - t_y)d\lambda_X \ .$$

Inserting the expression (A.1.7) of dK_X into the above relation results in:

$$\begin{aligned} &\left[|\lambda_K^{X'}|k_X + \pi e^{r(t_X-t_y)} - r\lambda_X e^{rt_X}(t_X - t_y)\right] dt_X \\ - &\left[|\lambda_K^{X'}|I_K^{\lambda} + e^{rt_X}(t_X - t_y)\right] d\lambda_X - \left[|\lambda_K^{X'}|I_K^K + \pi e^{r(t_X-t_y)}\right] dt_K = 0 \end{aligned} \tag{A.1.10}$$

Multiplying both sides of (A.1.10) by $[r\lambda_X + |p^{d'}|e^{-rt_X}k_X]$ and making use of (A.1.9), we obtain an expression of the form:

$$A_\lambda d\lambda_X + A_K dt_K = 0 \tag{A.1.11}$$

where:

$$\begin{aligned} A_\lambda \equiv &\left[|\lambda_K^{X'}|k_X + \pi e^{r(t_X-t_y)} - r\lambda_X e^{rt_X}(t_X - t_y)\right]\left[|p^{d'}|e^{-rt_X}I_K^{\lambda} - 1\right] \\ &-[r\lambda_X + |p^{d'}|e^{-rt_X}k_X]\left[|\lambda_K^{X'}|I_K^{\lambda} + e^{rt_X}(t_X - t_y)\right] \\ A_K \equiv &\left[|\lambda_K^{X'}|k_X + \pi e^{r(t_X-t_y)} - r\lambda_X e^{rt_X}(t_X - t_y)\right]\left[|p^{d'}|e^{-rt_X}I_K^K\right] \\ &-\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]\left[|\lambda_K^{X'}|I_K^K + \pi e^{r(t_X-t_y)}\right] \end{aligned}$$

Straightforward computations show that:

$$\begin{aligned} A_\lambda = &|p^{d'}|\left\{\pi I_K^{\lambda}e^{-rt_y} - (k_X + r\lambda_X I_K^{\lambda})(t_X - t_y)\right\} \\ &-|\lambda_K^{X'}|\left\{k_X + r\lambda_X I_K^{\lambda}\right\} - \pi e^{r(t_X-t_y)} \end{aligned}$$

Now note that since $k(t_K) = 0$:

$$\begin{aligned} k(t_X) - k(t_K) = k_X &= \int_{t_K}^{t_X} \dot{k}(t)dt = \int_{t_K}^{t_X} k^{s'}\dot{\lambda}_K(t)dt \\ &= \int_{t_K}^{t_y} k^{s'}\pi e^{r(t-t_y)}dt + \int_{t_y}^{t_X} k^{s'}\left[\pi e^{r(t-t_y)} - r\lambda_X e^{rt}(t-t_y)\right] dt \ , \\ &= \int_{t_K}^{t_X} k^{s'}\pi e^{r(t-t_y)}dt - r\lambda_X \int_{t_y}^{t_X} k^{s'} e^{rt}(t-t_y)dt \end{aligned}$$

implies together with the expression (A.1.5) of I_K^λ and (A.1.6) of I_K^K that:

$$I_K^K = k_X + r\lambda_X I_K^\lambda \ . \qquad \text{(A.1.12)}$$

Thus A_λ is given by:

$$A_\lambda = -\left\{ I_K^K \left[|\lambda_K^{X'}| + |p^{d'}|(t_X - t_y) \right] + \pi e^{-rt_y} \left[e^{rt_X} - |p^{d'}| I_K^\lambda \right] \right\} \ .$$

Next, A_K is equivalent to :

$$\begin{aligned} A_K &= |p^{d'}| \left[I_K^K(\pi e^{-rt_y} - r\lambda_X(t_X - t_y)) - k_X \pi e^{-rt_y} \right] \\ &\quad -|\lambda_K^{X'}| r\lambda_X I_K^K - r\lambda_X \pi e^{r(t_X - t_y)} \end{aligned}$$

Substituting for k_X its expression given by (A.1.12), we obtain:

$$\begin{aligned} A_K &= |p^{d'}| \left[-r\lambda_X I_K^K(t_X - t_y) + r\lambda_X I_K^\lambda \pi e^{-rt_y} \right] \\ &\quad -|\lambda_K^{X'}| r\lambda_X I_K^K - r\lambda_X \pi e^{r(t_X - t_y)} \\ &= -r\lambda_X \left\{ I_K^K \left[|\lambda_K^{X'}| + |p^{d'}|(t_X - t_y) \right] + \pi e^{-rt_y} \left[e^{rt_X} - |p^{d'}| I_K^\lambda \right] \right\} \\ &= r\lambda_X A_\lambda \end{aligned}$$

Thus we conclude that (A.1.11) is in fact equivalent to:

$$d\lambda_X = -r\lambda_X dt_K \qquad \text{(A.1.13)}$$

Inserting the expression (A.1.13) of $d\lambda_X$ inside (A.1.9) while making use of the expression of k_X resulting from (A.1.12), we then conclude that:

$$\begin{aligned}
\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]dt_X &= \left[-r\lambda_X\left(|p^{d'}|e^{-rt_X}I_K^\lambda - 1\right) + |p^{d'}|e^{-rt_X}I_K^K\right]dt_K \\
&\Longleftrightarrow \\
\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]dt_X &= \left[r\lambda_X + |p^{d'}|e^{-rt_X}\left(I_K^K - r\lambda_X I_K^\lambda\right)\right]dt_K \\
&\Longleftrightarrow \\
\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]dt_X &= \left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]dt_K \\
&\Longleftrightarrow \\
dt_X &= dt_K
\end{aligned}$$

Thus the length of the solar development phase, $T \equiv t_X - t_K$ is independent from λ_X. Making use of the fact that first: $dt_X = dt_K$, second $d\lambda_X = -r\lambda_X dt_K$, and taking (A.1.12) into account, (A.1.7) is equivalent to:

$$dK_X = \left(k_X + r\lambda_X I_K^\lambda - I_K^K\right)dt_K \implies dK_X = 0 \ .$$

Hence K_X is also independent from λ_X.

In a three phases scenario where $0 < t_K = t_y$, a slight adaptation of the above proof leads to the same conclusion. If $t_K = t_y$, $\lambda_K(t)$, $t \in [t_y, t_X)$ is given by:

$$\lambda_K(t) = e^{r(t-t_y)}p_K^0 + \frac{c_y + c_K - c_x}{r}\left(e^{r(t-t_y)} - 1\right) - \lambda_X e^{rt}(t - t_y) \ .$$

Differentiating the capital accumulation equation results in:

$$dK_X = k_X dt_X - I_K^\lambda d\lambda_X - I_K^y dt_y \ ,$$

where:

$$I_K^\lambda \equiv \int_{t_y}^{t_X} k^{s'} e^{rt}(t - t_y)dt > 0 \qquad \text{(A.1.14)}$$

$$I_K^y \equiv \int_{t_y}^{t_X} k^{s'} \pi_y e^{r(t-t_y)}dt > 0 \ , \qquad \text{(A.1.15)}$$

and $\pi_y \equiv rp_K^0 + c_y + c_K - c_x - \lambda_X e^{rt_y}$.

Performing the same kind of computation as for the four phases scenario, it may be shown that:

$$A_\lambda d\lambda_X + A_y dt_y = 0 \ ,$$

where:

$$
\begin{aligned}
A_\lambda &\equiv \left[|\lambda_K^{X'}|k_X + \pi_y e^{r(t_X-t_y)} - r\lambda_X e^{rt_X}(t_X-t_y)\right]\left[|p^{d'}|e^{-rt_X}I_K^\lambda - 1\right] \\
&\quad -[r\lambda_X + |p^{d'}|e^{-rt_X}k_X]\left[|\lambda_K^{X'}|I_K^\lambda + e^{rt_X}(t_X-t_y)\right] \\
A_y &\equiv \left[|\lambda_K^{X'}|k_X + \pi_y e^{r(t_X-t_y)} - r\lambda_X e^{rt_X}(t_X-t_y)\right]\left[|p^{d'}|e^{-rt_X}I_K^y\right] \\
&\quad -\left[r\lambda_X + |p^{d'}|e^{-rt_X}k_X\right]\left[|\lambda_K^{X'}|I_K^y + \pi_y e^{r(t_X-t_y)}\right] .
\end{aligned}
$$

Then the analog of (A.1.12) may be derived. Since $k(t_y) = 0$ in this scenario:

$$
\begin{aligned}
k(t_X) - k(t_y) = k_X &= \int_{t_y}^{t_X} \dot{k}(t)dt = \int_{t_y}^{t_X} k^{s'}\dot{\lambda}_K(t)dt \\
&= \int_{t_y}^{t_X} k^{s'}\left[\pi_y e^{r(t-t_y)} - r\lambda_X e^{rt}(t-t_y)\right]dt \quad ,
\end{aligned}
$$

implies together with the expression (A.1.14) of I_K^λ that:

$$
\begin{aligned}
&\pi_y I_K^\lambda e^{-rt_y} - r\lambda_X(t_X-t_y)I_K^\lambda - k_X(t_X-t_y) = \\
&\int_{t_y}^{t_X} k^{s'}\left\{e^{rt}(t-t_y)\pi_y e^{-rt_y} - r\lambda_X e^{rt}(t_X-t_y)(t-t_y)\right. \\
&\quad \left. -\pi_y e^{r(t-t_y)}(t_X-t_y) + r\lambda_X e^{rt}(t_X-t_y)(t-t_y)\right\}dt \\
&\quad = -\int_{t_y}^{t_X} k^{s'}\pi_y e^{r(t-t_y)}(t_X-t)dt \quad ,
\end{aligned}
$$

and:

$$
\begin{aligned}
k_X + r\lambda_X I_K^\lambda &= \int_{t_y}^{t_X} k^{s'}\left\{\pi_y e^{r(t-t_y)} - r\lambda_X e^{rt}(t-t_y) + r\lambda_X e^{rt}(t-t_y)\right\}dt \\
&= \int_{t_y}^{t_X} k^{s'}\pi_y e^{r(t-t_y)}dt = I_K^y . \qquad \text{(A.1.16)}
\end{aligned}
$$

Making use of (A.1.16), straightforward computations then show that $A_\lambda = r\lambda_X A_y$ which lead to $d\lambda_X = -r\lambda_X dt_y$, implying in turn that $dt_X = dt_y$ and $dK_X = 0$.

A.1.3 Proof that the system (5.8)-(5.10) has a unique solution

We now check that for $t_K = 0$ and $\lambda_K(0) = p_K^0$, the system (5.8)-(5.10) determines in a unique way a vector (T, λ_X^0, K_X) thus defining the critical level λ_X^0 of λ_X such that $t_K = 0$ together with the duration of the development phase T and K_X, the accumulated capacity level at the end of the development phase. Differentiating, we obtain:

$$k_X dT - J_K^\lambda d\lambda_X - dK_X = 0 \quad \text{(A.1.17)}$$

$$r\lambda_X dT + d\lambda_X + |p^{d'}| e^{-rT} dK_X = 0 \quad \text{(A.1.18)}$$

$$\pi dT - (T - t_y) d\lambda_X + |\lambda_K^{X'}| e^{-rT} dK_X = 0 \ , \quad \text{(A.1.19)}$$

where we denote:

$$J_K^\lambda \equiv \int_{t_y}^T k^{s'} e^{rt}(t - t_y) dt \quad \text{and} \quad \pi \equiv r p_K^0 + c_K - r\lambda_X (T - t_y) \ .$$

Straightforward computations show that the determinant Δ of the system (A.1.17)-(A.1.19) is given by:

$$\Delta = (r p_K^0 + c_K) \left\{ 1 + e^{-rT} |\lambda_K^{X'}| \int_0^T k^{s'} e^{rt} dt \right. \\ \left. + e^{-rT} |p^{d'}| \left[(T - t_y) \int_0^T k^{s'} e^{rt} dt - \int_{t_y}^T k^{s'} e^{rt} (t - t_y) dt \right] \right\} \ .$$

Since $t \leq T$ and $t_y \geq 0$ implies that:

$$\int_{t_y}^T k^{s'} e^{rt} (t - t_y) dt < (T - t_y) \int_{t_y}^T k^{s'} e^{rt} dt \leq (T - t_y) \int_0^T k^{s'} e^{rt} dt \ ,$$

we conclude that the term into brackets is positive and thus $\Delta > 0$.

Hence the system (5.8)-(5.10) evaluated at $t_K = 0$ defines a unique vector (T, λ_X^0, K_X).

In the case $t_K = t_y$, the linearized system reduces to:

$$k_X dT - J_K^\lambda d\lambda_X - dK_X = 0$$

$$r\lambda_X dT + d\lambda_X + |p^{d'}| e^{-rT} dK_X = 0$$

$$[\pi_y - r\lambda_X T] \, dT - T d\lambda_X + |\lambda_K^{X'}| e^{-rT} dK_X = 0 \ ,$$

where we denote:

$$J_K^{\lambda} \equiv \int_0^T k^{s'} e^{rt} t dt \quad \text{and} \quad \pi_y \equiv r p_K^0 + c_y + c_K - c_x - \lambda_X \ .$$

Then the determinant Δ of this system is given by:

$$\Delta = \pi_y e^{-rT} \left\{ e^{rT} + \int_0^T k^{s'} e^{rt} \left[|p^{d'}|(T-t) + |\lambda_K^{X'}| \right] dt \right\} > 0 \ .$$

A.1.4 Implications of the oil stock constraint

Next differentiating the oil stock constraint (5.7) we obtain:

$$\begin{aligned} dX_0 &= q(t_X) dt_X + \int_0^{t_X} \frac{dq^d(p(t))}{dp(t)} e^{rt} dt d\lambda_X \\ &\quad -(dt_X - dt_y) \int_{t_K}^{t_y} k^s(t) dt - (t_X - t_y) \int_{t_K}^{t_y} k^{s'} \frac{\partial \lambda_K}{\partial t_K} dt dt_K \\ &\quad - \int_{t_y}^{t_X} k^{s'} \left[\frac{\partial \lambda_K(t)}{\partial t_K} dt_K + \frac{\partial \lambda_K(t)}{\partial \lambda_X} d\lambda_X \right] (t_X - t) dt \\ &\quad - \int_{t_y}^{t_X} k^s(t) dt dt_X \end{aligned}$$

Denote by $q^{d'}(t) \equiv dq^d(p(t))/dp(t)$. Remember that $q(t_X) = K_X$. Since $\partial \lambda_K / \partial t = -\partial \lambda_K / \partial t_K$ for $t \in (t_K, t_y)$, the above is equivalent to:

$$\begin{aligned} dX_0 &= K_X dt_X + \int_0^{t_X} q^{d'}(t) e^{rt} dt d\lambda_X \\ &\quad - \left[\int_{t_K}^{t_y} k^s(t) dt + \int_{t_y}^{t_X} k^s(t) dt \right] dt_X + \int_{t_K}^{t_y} k^s(t) dt dt_y \\ &\quad + (t_X - t_y) \int_{t_K}^{t_y} \dot{k}^s(t) dt dt_K \\ &\quad - \int_{t_y}^{t_X} k^{s'} \left[\frac{\partial \lambda_K(t)}{\partial t_K} dt_K + \frac{\partial \lambda_K(t)}{\partial \lambda_X} d\lambda_X \right] (t_X - t) dt \end{aligned}$$

Simplifying the dt_X terms and rearranging we get:

$$\begin{aligned} dX_0 &= \int_0^{t_X} q^{d'}(t)e^{rt}dtd\lambda_X + \int_{t_K}^{t_y} k^s(t)dtdt_y + (t_X - t_y)k(t_y)dt_K \\ &\quad - \int_{t_y}^{t_X} k^{s'}\left[\frac{\partial\lambda_K(t)}{\partial t_K}dt_K + \frac{\partial\lambda_K(t)}{\partial\lambda_X}d\lambda_X\right](t_X - t)dt \end{aligned}$$

Since $dt_y = -d\lambda_X/(r\lambda_X)$ and $dt_K = -d\lambda_X/(r\lambda_X)$, the above is equivalent to:

$$\begin{aligned} dX_0 &= \frac{d\lambda_X}{r\lambda_X}\left\{r\lambda_X \int_0^{t_X} q^{d'}(t)e^{rt}dt - \int_{t_K}^{t_y} k^s(t)dt - (t_X - t_y)k(t_y)\right. \\ &\quad \left. + \int_{t_y}^{t_X} k^{s'}\left[\frac{\partial\lambda_K(t)}{\partial t_K} - r\lambda_X\frac{\partial\lambda_K(t)}{\partial\lambda_X}\right](t_X - t)dt\right\} \end{aligned}$$

Next taking (A.1.1)-(A.1.4) into account:

$$\begin{aligned} \frac{\partial\lambda_K(t)}{\partial t_K} - r\lambda_X\frac{\partial\lambda_K(t)}{\partial\lambda_X} &= \pi e^{r(t-t_y)} - r\lambda_X e^{rt}(t - t_y) \\ &= -\dot{\lambda}_K(t) \ . \end{aligned}$$

Thus:

$$\begin{aligned} J &\equiv \int_{t_y}^{t_X} k^{s'}\left[\frac{\partial\lambda_K(t)}{\partial t_K} - r\lambda_X\frac{\partial\lambda_K(t)}{\partial\lambda_X}\right](t_X - t)dt \\ &= -\int_{t_y}^{t_X} k^{s'}\dot{\lambda}_K(t)(t_X - t)dt = -\int_{t_y}^{t_X} \dot{k}^s(t)(t_X - t)dt \end{aligned}$$

Integrating by parts:

$$J = -\left\{k^s(t)(t_X - t)|_{t_y}^{t_X} + \int_{t_y}^{t_X} k^s(t)dt\right\} = k(t_y)(t_X - t_y) - \int_{t_y}^{t_X} k^s(t)dt \ .$$

Thus we obtain:

$$\begin{aligned} dX_0 &= \frac{d\lambda_X}{r\lambda_X}\left\{r\lambda_X \int_0^{t_X} q^{d'}(t)e^{rt}dt - \int_{t_K}^{t_y} k^s(t)dt - (t_X - t_y)k(t_y) + J\right\} \\ &= \frac{d\lambda_X}{r\lambda_X}\left\{r\lambda_X \int_0^{t_X} q^{d'}(t)e^{rt}dt - \int_{t_K}^{t_y} k^s(t)dt - \int_{t_y}^{t_X} k^s(t)dt\right\} \\ &= \frac{d\lambda_X}{r\lambda_X}\left\{r\lambda_X \int_0^{t_X} q^{d'}(t)e^{rt}dt - K_X\right\} \end{aligned}$$

Since $\dot{p}(t) = r\lambda_X e^{rt}$ over the time interval $[0, t_X)$, the integral of the RHS is equivalent to:

$$\begin{aligned}\int_0^{t_X} q^{d'} e^{rt} dt &= \frac{1}{r\lambda_X} \int_0^{t_X} q^{d'}(p(t))\dot{p}(t) dt = \frac{1}{r\lambda_X} \int_0^{t_X} \dot{q}^d(t) dt \\ &= \frac{1}{r\lambda_X} \left(q(t_X) - q(0)\right) \ .\end{aligned}$$

Thus:

$$dX_0 = \frac{d\lambda_X}{r\lambda_X} \left[q(t_X) - q(0) - K_X\right] = -\frac{q(0)}{r\lambda_X} d\lambda_X$$

Hence we conclude that λ_X is defined implicitly as a function of X_0, $\lambda_X(X_0)$, and $d\lambda_X(X_0)/dX_0 = -r\lambda_X/q(0) < 0$.