The atmospheric carbon resilience problem A theoretical analysis*

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1 Introduction

In their well known paper aiming at capturing some essential features of the global carbon cycle, Farzin and Tahvonen (1996) started from a strong conclusion of the Maier-Raimer and Hasselman (1987) study : "Maier-Raimer and Hasselman (1987) (have)[1] show(n) that the development of the carbon concentration and its decay can be approximated by a linear model where the total atmosphere stock is "artificially" divided into different substocks with different decay rates" (F.T., p. 515). Amongst the different substocks of the global atmospheric carbon stock, some have very low self-regeneration capacities giving rise to a long run resilience effect. This resilience effect has been put again recently in the forefront of the atmospheric carbon dynamics and its relationships with other terrestrial carbon stocks in a geologic time perspective by Archer (2005). As far as the short term is concerned, "short" as measured along this geologic scale, Archer concludes that :"A better approximation in the lifetime of fossil fuel CO_2 for public discussion may be "300 years, plus 25% lasts forever"" (A, p. 5)[2].

From the seminal papers of Hoel and Kverndokk (1996), Tahvonen (1997), Ulph and Ulph (1994) and Withagen (1994), all using a one atmospheric carbon stock model self-regenerating at a constant proportional rate[3], it was clearly evident that the linearity of the carbon regeneration process was a strong assumption. That the non linear structure of the regeneration process could have drastic implications on the optimal use of a polluting resource has been pointed out as early as 1975 by Forster (Forster, 1975). Tahvonen and Withagen (1996), using an inverted U shaped self-regeneration function showed that, due to the non-convexity of the optimal extraction problem, there can exist several locally optimal extraction trajectories of the polluting non-renewable resource so that "The choice cannot be made, in general, by the usual marginal analysis but instead requires computation of present values

[1] Our brackets.

[2] See Archer *et alii*, (2009) for further results who conclude that " Nowhere in these model results or in published literature is there any reason to conclude that the effects of CO_2 release will be substantially confined to just a few centuries. In contrast, generally accepted modern understanding of the global carbon cycle indicates that climate effects of CO_2 releases to the atmosphere will persist for tens, if not hundreds, of thousands of years into the future " (A *et al.*, p. 131).

[3] We leave aside the economic growth literature which is not the subject of the present paper, however important may be the Ramsey problem of capital accumulation and saving.

of both paths" (T.W, p. 1777). Similar results were obtained by Tahvonen and Salo (1996) with a concave-convex function. In both studies there is no clean renewable substitute option.

We propose in the present study to go back to the Farzin-Tahvonen setting which does not fit in the two last formulations with one and only one stock, and use their two pollution stocks apparatus (F-T, section 3), one stock being linearly self-regenerating while the other stays permanently into the atmosphere[4]. However we assume that there exists a clean renewable substitute more costly then the polluting non-renewable resource, and we specify a quite different damage function.

To put at the core of the analysis the specific problem of non-regeneration of some part of the atmospheric pollution stock, the costs of the non-renewable and of the renewable are both assumed to be linear. This is inducing some discontinuities in their optimal exploitation paths which would be smoothed in a more sensible or more empirically pertinent model where the non-renewable unitary extraction cost would be increasing with the accumulative extraction and/or the unitary exploitation cost of the renewable resource would be increasing with its instantaneous production rate. This smoothing problem is investigated in a companion paper (Amigues and Moreaux, 2011). Most of the qualitative results of the present paper hold within this more general assumption framework.

Concerning the damages, we assume that they are negligible as far as the atmospheric pollution stock is note overrunning some critical level but that overshooting this benchmark triggers catastrophic events. Models with catastrophic events induced by too high pollution levels have been pioneered by Cropper (1976). However in the Cropper model the catastrophe consists in possibly high but finite damages so that the catastrophe is not, in some sense, too damaging. There we assume that the catastrophe is generating an infinitely large damage, at least as large as the benefits having been accumulated along a path generating the catastrophe up to the time at which the catastrophic events occur[5].

[4]In such a model the decay function cannot be expressed as a function of the sum of the two pollution stocks.

[5]In the Cropper model the society benefits from the current consumption of some good, $c(t)$, according to some utility function $u(c)$, strictly increasing, strictly concave and satisfying the first Inada condition $\lim_{q \downarrow 0} u'(q) = +\infty$. The catastrophe occurs with some

Under these assumptions there exists some definite level of the initial endowment in polluting non-renewable resource under which this resource is rare, more precisely is exhausted in finite time along the optimal path, and over which it is abundant, that is never wholly extracted along the optimal path. In this last case, the ultimate or truly rare resource is the capacity of the atmosphere to receive a limited stock of permanent pollution and not the finite stock of the non renewable resource.

The details of the model and the notations are laid down in Section 2. The social planner problem, maximizing the sum of discounted net surplus under the pollution ceiling constraint, is formulated in Section 3 and some strong characteristics of its solution are then pointed out. In Section 4 we assume that the energy demand is inelastic within the price range of any optimal path. We show that whatever the case of either a rare or an abundant polluting resource, the optimal path consists in consuming first this less costly non-renewable polluting resource up to the time at which the critical level of atmospheric pollution is attained. Then follows a phase at the ceiling of infinite duration when the non-renewable is abundant and of finite duration when the non-renewable resource is rare. During this phase at the ceiling both resources are exploited. In the case of a rare non renewable resource, once the non-renewable is exhausted then the clean renewable resource is evidently the only energy supplier while in the case of an abundant non-renewable resource the substitution process, from the non-renewable to the renewable, is an infinite duration progressive process at the ceiling. Thus in the rare resource case the phase at the ceiling is necessarily a transitory phase since once the non renewable resource is exhausted, the total stock of atmospheric carbon begins to decrease, while in the abundant resource case, the ceiling phase is a permanent phase, necessarily the last one.

The Section 5 is devoted to the case of an elastic energy demand function. Only the abundant non-renewable resource sub-case is investigated. We show that the elasticity of the demand function plays a central role in the characterization of the optimal path. If the elasticity of the inverse demand function, in absolute value, is non-decreasing then either the ceiling is attained at the precise time at which the renewable energy begins to be

probability increasing with the stock of pollution. However $u(0) = 0$, so that the expected utility apparatus holds. In our model the catastrophe is incommensurably more dramatic and $u(0)$ would be rather "equal" to $-\infty$. Clarke and Reed (1994) use a model similar to the Cropper model.

competitive, as in the inelastic demand case, or the ceiling is attained before the time at which the clean renewable energy is competitive and then once the ceiling is attained the ceiling constraint is effective forever. On the contrary if the elasticity is decreasing and if the fourth derivative of gross surplus function is negative then the optimal path many include several disconnected time phases at the ceiling.

We conclude in section 6.

2 The model

We consider a global economy in which the energy can be produced from two primary resources, a polluting non renewable resource, coal, and a clean renewable one, solar.

Gross surplus

The instantaneous gross surplus generated by an instantaneous energy consumption rate q is given by some function $u(q)$ satisfying the following standard assumptions : $u \ : \mathcal{R}_+ \rightarrow \mathcal{R}$ is a C^2 function, strictly increasing, strictly concave and satisfying the first Inada condition $\lim_{q \downarrow 0} u'(q) = +\infty$. We sometime denote by $p(q)$ the inverse demand function $u'(q)$ and by $q^d(p)$ its inverse, that is the usual direct demand function.

We also consider the case in which the demand function $q^d(p)$ is inelastic over the price range within which the energy price must evolve along any optimal path, although such a demand function does not fit the above assumptions upon the gross surplus function u. The lowest and highest benchmarks of this price are determined by the supply side of the model.

The non renewable resource

Let $X(t)$ be the stock of coal available at time t, let $X^0 > 0$ be the initial coal endowment, $X^0 \equiv X(0)$, and $x(t)$ be its instantaneous extraction rate : $\dot{X}(t) = -x(t)$. Coal can be exploited at a constant marginal cost c_x

and absent any fixed cost, c_x is also the average cost. Exploitation costs include not only the extraction cost *stricto sensu* but also the processing and transportation costs to deliver a ready to use energy to the final users.

The pollution stocks

Burning coal implies the relaxation of a pollution flow. Let ζ be the unitary pollution content of coal so that the flow of new pollution relaxed in the atmosphere at time t amounts to $\zeta x(t)$. The atmospheric pollution stock includes two components. The first one, denoted by $Z(t)$, is self regenerating at some proportional rate α assumed to be constant as suggested by Maier-Raimer and Hasselman while the second one, denoted by $S(t)$, is a permanent one as in the Farzin and Tahvonen model.

Let θ be this proportion of the gross flow of pollution $\zeta x(t)$ replenishing the self regenerating stock so that its dynamics is given by :

$$\dot{Z}(t) \quad = \quad \theta\zeta x(t) - \alpha Z(t).$$

We denote by Z^0 the initial level of Z : $Z(0) \equiv Z^0$. As for the non regenerating stock its dynamics is given by :

$$\dot{S}(t) \quad = \quad (1-\theta)\zeta x(t).$$

We denote by S^0 its initial level : $S(0) \equiv S^0$.

The global pollution stock $G(t) \equiv S(t)+Z(t)$ is constrained to stay under some critical level $\bar{G}$ over which catastrophic damages would be triggered as in Chakravorty et al. (2006). At each time t we must have :

$$\bar{G} - (S(t) + Z(t)) = \bar{G} - G(t) \geq 0.$$

Thus we must assume that :

$$\bar{G} - (S^0 + Z^0) = \bar{G} - G^0 > 0.$$

Clearly in this model there exist *de facto* two non renewable primary resources, coal and the *environmental capacity to use coal* (in brief "capacity" thereafter). While X^0 is the initial endowment of coal, $\bar{G} - S^0$ is the initial endowment in capacity. Producing one unit of useful energy from the fossil resource is requiring two inputs : one unit of coal and $\theta\zeta$ units of capacity both of which are non renewables. These two inputs are strict complements. Hence one and only one resource can be rare that is constraining the use the fossil energy in the long run. We denote by $\bar{X}^0$ this critical level of coal endowment under which coal is the rare resource and over which the capacity is the limiting factor for the exploitation of the fossil energy.

$\bar{X}^0$ may be computed as follows. Since $Z(t) + S(t) \leq \bar{G}$, $Z(t)$ must be finite, $t \geq 0$, implying that in the very long run: $\lim_{t\uparrow\infty} Z(t) = 0$ thanks to the self regenerating process. Thus only the permanent pollution stock level will ultimately bind, that is the constraint $\lim_{t\uparrow\infty} S(t) \leq \bar{G}$ has to be satisfied. From the dynamics of the non regenerating stock, this is equivalent to $S^0 + (1-\theta)\zeta \lim_{t\uparrow\infty}(X^0 - X(t)) \leq \bar{G}$. Hence $\bar{X}^0$ is defined as the maximum initial level of coal endowment satisfying this inequality as an equality in a case where $\lim_{t\uparrow\infty} X(t) = 0$, that is[6] :

$$\bar{X}^0 = \frac{\bar{G} - S^0}{\zeta(1-\theta)}.$$

However as we shall show in the next section, fully exploiting the pollution capacity $\bar{G} - S^0$ is necessitating to decrease the instantaneous rate of coal consumption over an infinite duration time period.

The renewable energy

Let $y(t)$ be the instantaneous consumption rate of the renewable energy at time t (e. g. solar energy). To provide a ready to use energy to the users a marginal cost c_y has to be incurred, assumed to be constant and equal to the average cost absent any fixed cost.

We assume that the natural flow of renewable energy is sufficiently large to supply the energy needs of the society when solar energy is the only available primary resource. Let $\tilde{y}$ be solving $u'(y) = c_y$. The natural flow is assumed to be at least as large as $\tilde{y}$ implying that no rent has never to be charged for the exploitation of solar energy.

[6]Initial endowment larger than $\bar{X}^0$ is typically a case of " too much oil" (or coal) in the sense of Gerlagh (2009). However, note that 'too much oil' is not so easy to define with a more general cost structure, as pointed out in Amigues and Moreaux, (2011).

The marginal delivery cost of solar is assumed to be higher than the marginal cost of coal : $c_y > c_x$. Thus as far as possible, that is as far as the cap constraint on the pollution stock is not tight and the coal stock is not exhausted, coal must be exploited rather than solar.

Social rate of discount

All instantaneous surplus and costs are discounted at an instantaneous social rate $\rho > 0$, constant through time.

3 The social planner problem

The social planner problem is to choose the extraction paths of the both resources maximizing the sum of discounted net surplus that is solving the following problem (P) :

$$
\begin{aligned}
(P) \quad \max_{x,y} \quad & \int_0^\infty \{u(x(t)+y(t)) - c_x x(t) - c_y y(t)\} e^{-\rho t} dt && (3.1)\\
\text{s.t} \quad & \dot{X}(t) = -x(t) && (3.2)\\
& X(0) = X^0 > 0 \quad \text{given, and} \quad X(t) \geq 0 && (3.3)\\
& \dot{S}(t) = (1-\theta)\zeta x(t) && (3.4)\\
& \dot{Z}(t) = \theta\zeta x(t) - \alpha Z(t) && (3.5)\\
& S(0) = S^0 \geq 0, Z(0) = Z^0 \geq 0 \quad \text{and} \quad \bar{G} - (S^0 + Z^0) > 0 && (3.6)\\
& \bar{G} - (S(t) + Z(t)) \geq 0 && (3.7)\\
& x(t) \geq 0 \quad \text{and} \quad y(t) \geq 0 && (3.8)
\end{aligned}
$$

Let us denote by λ_X, $-\lambda_S$ and $-\lambda_Z$ the costate variables of X, S and Z respectively, by ν_X the multiplier associated to the non negativity of X and by ν_G the multiplier associated to the cap constraint on the pollution stock, last by γ_x and γ_y the multipliers of the non negativity constraints on the command variables x and y.

The current value Lagrangian of problem (P) reads :

$$\begin{aligned}\mathcal{L} &= u(x(t)+y(t)) - c_x x(t) - c_y y(t) - \lambda_X(t)x(t) + \nu_X(t)X(t) - \lambda_S(t)(1-\theta)\zeta x(t) \\ &\quad -\lambda_Z(t)[\theta\zeta x(t) - \alpha Z(t)] + \nu_G(t)[\bar{G} - (S(t)+Z(t))] + \gamma_x(t)x(t) + \gamma_y(t)y(t).\end{aligned}$$

First order conditions :

$$u'(x(t)+y(t)) = c_x + \lambda_X(t) + (1-\theta)\zeta\lambda_S(t) + \theta\zeta\lambda_Z(t) - \gamma_x(t) \quad (3.9)$$

$$u'(x(t)+y(t)) = c_y - \gamma_y(t) \quad (3.10)$$

$$\gamma_x(t) \geq 0, x(t) \geq 0 \quad \text{and} \quad \gamma_x(t)x(t) = 0 \quad (3.11)$$

$$\gamma_y(t) \geq 0, y(t) \geq 0 \quad \text{and} \quad \gamma_y(t)y(t) = 0 \quad (3.12)$$

Dynamics of the costate variables :

$$\dot{\lambda}_X(t) = \rho\lambda_X(t) - \nu_X(t) \quad (3.13)$$

$$\dot{\lambda}_S(t) = \rho\lambda_S(t) - \nu_G(t) \quad (3.14)$$

$$\dot{\lambda}_Z(t) = (\rho+\alpha)\lambda_Z(t) - \nu_G(t) \quad (3.15)$$

$$\nu_X(t) \geq 0, X(t) \geq 0 \quad \text{and} \quad \nu_X(t)X(t) = 0 \quad (3.16)$$

$$\nu_G(t) \geq 0, \bar{G} - (S(t)+Z(t)) \geq 0 \quad \text{and} \quad \nu_G(t)[\bar{G} - (S(t)+Z(t))] = 0 \quad (3.17)$$

Transversality conditions at infinity :

$$\lim_{t\uparrow\infty} e^{-\rho t}\lambda_X(t)X(t) = 0 \quad (3.18)$$

$$\lim_{t\uparrow\infty} e^{-\rho t}\lambda_S(t)S(t) = 0 \quad (3.19)$$

$$\lim_{t\uparrow\infty} e^{-\rho t}\lambda_Z(t)Z(t) = 0 \quad (3.20)$$

Remark 1 λ_X *is the marginal current value of the available stock of coal.*

In this constant average delivery cost model λ_X is also the current imputed unitary rent of coal, which must grow at the social rate of discount, a well known result. As long as the stock of coal is not exhausted $\nu_X = 0$ so that $\dot{\lambda}_X = \rho\lambda_X$ by (3.13), hence :

$$t < t_X \quad \Rightarrow \quad \lambda_X(t) = \lambda_{X0}e^{\rho t} \ , \quad (3.21)$$

where $t_X = sup\{t \ : X(t) > 0\}$ is either the time at which coal is exhausted or equal to $+\infty$ if coal is never exhausted. In this last case $\lambda_{X0} = 0$ by the transversality condition (3.18) that is the non renewable rent is nil since the resource is abundant.

Remark 2 *λ_S is the current shadow marginal cost of the non regenerating pollution stock.*

Although capacity is a non renewable resource like coal, due care must be given to the fact that the use of this non renewable resource is linked, *via* the ceiling constraint (3.7), to the dynamics of the other pollution stock Z which is self regenerating. Thus the dynamics of λ_S is scanned by the cap constraint on the global pollution stock contrary to the dynamics of λ_X. Given that the ceiling constraint (3.7) can be effective over different disconnected time periods, the equivalent of the above relation (3.21) for λ_X must be formulated as follows for λ_S.

Over any time interval $[t_1, t_2]$ during which the pollution cap constraint (3.7) is not effective so that $\nu_G = 0$, we get from (3.14) :

$$t \in [t_1, t_2] \ \ \Rightarrow \ \ \lambda_S(t) = \lambda_S(t_1) e^{\rho(t-t_1)}. \tag{3.22}$$

Since initially the ceiling constraint is not effective then there exists a first time interval $[0, t_1)$ during which :

$$t \in [0, t_1) \ \ \Rightarrow \ \ \lambda_S(t) = \lambda_{S0} e^{\rho t} \quad \text{where} \quad \lambda_{S0} \equiv \lambda_S(0). \tag{3.23}$$

Last, if the ceiling constraint is no more effective from some time $\bar{t}_G$ onwards and forever, then λ_S must be nil :

$$t > \bar{t}_G \ \ \Rightarrow \ \ \lambda_S(t) = 0. \tag{3.24}$$

Remark 3 *λ_Z is the current shadow marginal cost of the self regenerating pollution stock.*

Over any time period $[t_1, t_2]$ during which (3.7) is not tight, $\nu_G = 0$, so that :

$$t \in [t_1, t_2] \quad \Rightarrow \quad \lambda_Z(t) = \lambda_Z(t_1) e^{(\rho+\alpha)(t-t_1)}. \tag{3.25}$$

hence during some initial period $[0, t_1)$:

$$t \in [0, t_1) \quad \Rightarrow \quad \lambda_Z(t) = \lambda_{Z0} e^{(\rho+\alpha)t} \quad \text{where} \quad \lambda_{Z0} \equiv \lambda_Z(0). \tag{3.26}$$

Last, if there exists some time $\bar{t}_G$ from which the ceiling constraint is definitively ineffective, then λ_Z is nil forever :

$$t > \bar{t}_G \quad \Rightarrow \quad \lambda_Z(t) = 0. \tag{3.27}$$

The following Proposition 1 states that, as suggested in the preceding Section 2, for the non renewable resource use the limiting factor is either its initial endowment X^0 or the pollution capacity initially available $\bar{G} - S^0$. Neglecting the anecdotic case $X^0 = \bar{X}^0$, along any optimal path either the coal stock must be exhausted, coal is rare, or the capacity must be saturated by the non regenerating atmospheric carbon, capacity is rare.

Proposition 1 *Either coal is rare or the pollution capacity is rare that is, denoting by $\{x^*(t), y^*(t)\}_{t=0}^{\infty}$ the optimal path :*

$$\int_0^\infty x^*(t)dt = \begin{cases} X^0 & \text{if} \quad X^0 < \bar{X}^0 \quad , \quad \textit{coal is rare} \\ \bar{X}^0 & \text{if} \quad X^0 > \bar{X}^0 \quad , \quad \textit{capacity is rare} \end{cases}$$

Proof: Assume that $X^0 > \bar{X}^0$, denote by X^* the cumulated extraction, $X^* \equiv \int_0^\infty x^*(t)dt$, and assume that $X^* < \bar{X}^0$ so that $\lim_{t\uparrow\infty} X(t) = X^0 - X^* > \bar{X}^0 - X^* > 0$, implying that $\lambda_{X0} = 0$ by the transversality condition, hence $\lambda_X(t) = 0, t \geq 0$.

Since X^* is finite then $\lim_{t\uparrow\infty} x^*(t) = 0$ excepted maybe for some infinite set of time intervals the measures of which tend to 0 hence also the cumulated extraction over each one.

Because $X^* < \bar{X}^0$, then $S(t) < \bar{G}, t \geq 0$, and due to the self regeneration $\lim_{t\uparrow\infty} Z(t) = 0$. Hence $\lim_{t\uparrow\infty} Z(t) + S(t) < \bar{G}$ so that, as pointed out in the above Remarks 2 and 3 (cf. (3.24) and (3.27)), there exists some time $\bar{t}_G$ after which $\lambda_S(t) = 0$ and $\lambda_Z(t) = 0$.

Thus the f.o.c's (3.9) - (3.10) reduce to :

$$u'(x^*(t) + y^*(t)) = c_x - \gamma_x(t) \quad \text{and} \quad u'(x^*(t) + y^*(t)) = c_Y - \gamma_Y(t)$$

together with $x^*(t) \to 0$.

If $x^*(t) > 0$, then $\gamma_x = 0 > 0$ so that $u'(x^*(t) + y^*(t)) = c_x$ hence $x^*(t) + y^*(t) > \tilde{y}$ since $c_x < c_y$. But $u' = c_x < c_y$ implies that $y^*(t) = 0$ and $\gamma_y(t) = c_y - c_x$, hence $x^*(t) > \tilde{y}$ together with $x^*(t)$ arbitrary small for t sufficiently high, a contradiction.

If $x^*(t) = 0$ we should have $u'(y^*(t)) = c_y$ that is $y^*(t) = \tilde{y}$ together with $u'(y^*(t)) = c_x - \gamma_x(t), c_x < c_y$ and $\gamma_x(t) \geq 0$, again a contradiction.

The same kind of argument applies in the case $X^0 < \bar{X}^0$. ■

Corollary 1 *Assume that the capacity is rare, then along any optimal path the coal extraction is never closed.*

Proof Note that once Z is positive it is positive forever. Hence assume that there exists some time $\bar{t}_X$ at which coal extraction is terminated : $x^*(t) = 0, t > \bar{t}_X$. At $\bar{t}_X, Z(\bar{t}_X) > 0$ hence $S(t) = S(\bar{t}_X) < \bar{G}, t > \bar{t}_X$. But according to Proposition 1 we should have $\lim_{t\uparrow\infty} S(t) = \bar{G}$, hence a contradiction. ■

Remark 4 *Consumption rates of the primary resources during a phase at the ceiling.*

Consider a time interval during which the ceiling constraint is tight :

$$\bar{G} - (S(t) + Z(t)) = 0 \quad \Rightarrow \quad \dot{S}(t) + \dot{Z}(t) = 0$$

Substituting for $\dot{S}(t)$ and $\dot{Z}(t)$ given respectively by (3.4) and (3.5) we obtain :

$$x(t) = \frac{\alpha}{\zeta} Z(t). \tag{3.28}$$

Next substituting for $x(t)$ in the expression of $\dot{Z}(t)$ results in :

$$\dot{Z}(t) = -(1-\theta)\alpha Z(t). \tag{3.29}$$

Thus denoting by t_G the time at which the phase begins, then :

$$Z(t) = Z(t_G)e^{-(1-\theta)\alpha(t-t_G)} \quad , \quad t \geq t_G \tag{3.30}$$

$$x(t) = \frac{\alpha}{\zeta} Z(t_G)e^{-(1-\theta)\alpha(t-t_G)} \quad , \quad t \geq t_G \tag{3.31}$$

The following Proposition 2 is a straightforward implication of the postulated energy demand function and cost structure of the model and the above remarks.

Proposition 2 *Along an optimal path :*

1. *Joint use of the two energy resources can only occur during a phase at the ceiling.*
2. *Within any time phase during which the ceiling constraint is not binding and coal is not exhausted, only the coal resource is exploited.*
3. *The coal extraction rate is a non increasing function of time. It is strictly decreasing through time if $p(q)$ is strictly decreasing (the elastic demand case). It is either constant or strictly decreasing in the inelastic demand case.*

Proof: See Appendix A.1

Depending upon the levels of $\tilde{y}, S(t_G)$ and $Z(t_G)$ we may have two kinds of phases at the ceiling according to $x(t)$ given by (3.31) is larger or smaller than $\tilde{y}$. The point is that, when $x(t)$ is lower than $\tilde{y}$, the phase at the ceiling is a phase during which the solar energy sector must be active. If not, $u'(x(t)) > c_y$ and the f.o.c (3.10) relative to y cannot be satisfied. In such a situation, a ceiling phase corresponds to a simultaneous use of the two resources phase. On the contrary when $x(t)$ is larger than $\tilde{y}$, the solar energy sector must be inactive : the only way to satisfy (3.10) is that $y(t) = 0$ and $\gamma_y(t) = u'(x(t)) - c_y$. In this case, the ceiling phase is a phase of exploitation of the sole coal resource.

It results that an optimal path is a sequence of four possible kinds of temporary time phases: ceiling phases with exploitation of both energy sources, ceiling phases with only coal exploitation, third, below the ceiling phases with only coal exploitation and last, below the ceiling phase with only renewable energy use, coal being exhausted. During the first and four kinds of phases, the energy price stays constant at the level c_y in view of (3.10). During the third kind of phases, Proposition 2 establishes that the energy price should increase strictly over time. Last, during the second kind of phases, coal being the sole used resource and coal use having to decrease according to the Remark 4, the energy price must also increase strictly over time. We can thus state the following corollary to the Proposition 2 and the Remark 4.

Corollary 2 *Along an optimal path, either the energy price is constant at the level c_y, or is strictly increasing over time.*

The following Proposition states important implications of the previous Proposition 2 and the Corollaries 1 and 2.

Proposition 3 *Along an optimal path, whatever be the demand elesticty:*

1. *In the rare capacity case, the last phase of exploitation is a phase at the ceiling during which the both primary resources are exploited.*
2. *In the rare coal case, coal exploitation is terminated in finite time.*

Proof : Since renewable energy can provide the whole energy demand when $p(t) \geq c_y$, the energy price cannot be higher than c_y along an optimal path.

It results that a ceiling phase with joint use of both energy source cannot be followed by any other possible kind of phases while the coal resource is not exhausted, the energy price being necessarily strictly increasing during such time phases in view of the Corollary 2. Next, in the rare capacity case we know from the Corollary 1 that coal extraction must extend over an infinite time duration. We thus conclude that in this case, the last phase of infinite duration must be a time of joint use of the two energy resources, which is the claim 1 of the proposition.

Assume that coal extraction extends over an infinite duration in the rare coal case. Since coal is rare, $X^0 < \bar{X}^0$ implies that $\lim_{t\uparrow\infty} S(t) < \bar{G}$. Thus there should exist some time $\bar{t}_G$ such that $S(t) + Z(t) < \bar{G}$, $t > \bar{t}_G$. In other words, the ceiling constraint should stop binding in finite time and remain no more active afterwards. But in such a case, we know from Proposition 2, that only the coal sector should satisfy the energy demand. It results from the Corollary 2 that the energy price should increase after $\bar{t}_G$ thus reaching in finite time the level c_y at some time t_y. But after t_y, renewable energy should be introduced which is only possible whence at the ceiling, hence a contradiction. We conclude that in the rare coal case, the coal extraction phase must extend over a finite time interval, that is claim 2 of the Proposition. ■

For $\bar{G}$ sufficiently low the phase at the ceiling is necessarily a phase with an active solar sector. The critical value of $\bar{G}$ is determined as follows. At time $t_G, x(t_G) = (\alpha/\zeta)Z(t_G)$, hence :

$$x(t_G) < \tilde{y} \;\;\Rightarrow\;\; Z(t_G) < \frac{\zeta}{\alpha}\,\tilde{y}.$$

Thus for $\bar{G} < (\zeta/\alpha)\tilde{y}$, then $Z(t_G) < (\zeta/\alpha)\tilde{y}$ and when at the ceiling $x(t) < \tilde{y}$ so that the energy price must be equal to c_y. For $\bar{G} > (\zeta/\alpha)\tilde{y}$, the type of the ceiling phase is depending upon the mix of S and Z at the beginning of the phase. We conclude as follows.

Proposition 4 *Assume that $\bar{G} < \frac{\zeta}{\alpha}\,\tilde{y}$ then a phase at the ceiling is necessarily a phase during which the solar energy sector is active and the price of energy y is equal to c_y. For $\bar{G} > \frac{\zeta}{\alpha}\,\tilde{y}$ a phase at the ceiling is :*

- *either a phase during which the coal is supplying the whole energy consumption, when at the beginning of the phase* $Z(t_G) \in (\zeta\tilde{y}/\alpha, \bar{G})$ *;*

- *or a phase during which the both energy sectors are active, when at the beginning of the phase* $Z(t_G) < \frac{\zeta}{\alpha}\,\tilde{y}$*.*

Last we can show that in the rare capacity case, the costate variables levels λ_S and λ_Z are constant during the last ceiling phase of joint use of the two energy sources, together with $\lambda_X(t) = 0$.

Proposition 5 *In the rare capacity case, during the last phase at the ceiling with joint use of both primary energy sources, the costate variables* λ_S *and* λ_Z *are constant and given by :*

$$\lambda_S(t) = \frac{(\rho+\alpha)(c_y - c_x)}{\zeta[\rho + (1-\theta)\alpha]} \quad \textit{and} \quad \lambda_Z(t) = \frac{\rho(c_y - c_x)}{\zeta[\rho + (1-\theta)\alpha]}\ ,\ t \geq t_G \tag{3.32}$$

Proof: see Appendix A.2

The main conclusion of the above results is that when coal is abundant the last phase of the optimal path is a phase of indefinitely decreasing use of the coal resource and simultaneous increasing use of the solar energy. This process is a rare case of a smooth and progressive substitution process under constant marginal cost. However as we shall see in the next section some brutal switch in the relative uses of the primary resources generally occurs at the beginning of this last phase.

Before trying to solve the general case, strong intuitions can be gained from the investigation of the inelastic demand case.

4 The inelastic demand case

We assume in this section that the energy demand is inelastic within the price range $[c_x, c_y]$, the price range within which the energy price must stay along any optimal path.

Let us denote by $\bar{q}$ the energy flow having to be delivered within this price range. Given that the energy production is fixed at any time the optimality problem may be reduced to the following cost minimization problem (ID) :

$$
\begin{aligned}
(ID) \qquad & \max_{x,y} - \int_0^{\infty} \{c_x x(t) + c_y y(t))\} e^{-\rho t} dt && (4.1)\\
& \text{s.t} \quad (3.2) \text{ to } (3.8) \text{ and :} \\
& x(t) + y(t) - \bar{q} \geq 0 && (4.2)
\end{aligned}
$$

Taking the same notations for the dual variable associated to the constraints (3.2) to (3.8) as in section 3 and denoting by γ_q the Lagrange multiplier of (4.2), the present value Lagrangian of the problem (ID) reads :

$$
\begin{aligned}
L &= -[c_x x(t) + c_y y(t)] - \lambda_X(t)x(t) + \nu_X(t)X(t) - \lambda_S(t)(1-\theta)\zeta x(t) \\
&\quad -\lambda_Z(t)[\theta\zeta x(t) - \alpha Z(t)] + \nu_G(t)[\bar{G} - (S(t) + Z(t))] \\
&\quad +\gamma_q(t)[x(t) + y(t) - \bar{q}] + \gamma_x(t)x(t) + \gamma_y(t)y(t)
\end{aligned}
$$

First order conditions :

$$
\begin{aligned}
\gamma_q(t) &= c_x + \lambda_X(t) + (1-\theta)\zeta\lambda_S(t) + \theta\zeta\lambda_Z(t) - \gamma_x(t) && (4.3)\\
\gamma_q(t) &= c_y - \gamma_y(t) && (4.4)\\
\gamma_q(t) &\geq 0\ , x(t) + y(t) - \bar{q} \geq 0 \quad \text{and} \quad \gamma_q(t)[x(t) + y(t) - \bar{q}] = 0 && (4.5)
\end{aligned}
$$

together with (3.11) and (3.12).

The equations determining the dynamics of the costate variables are those of the preceding section 3 that is (3.13) to (3.17). The same applies to the transversality conditions given by (3.18) to (3.20).

The multiplier $\gamma_q(t)$ must be interpreted as the energy price, the equivalent of u' in the elastic demand case. The other multipliers and costate variables must be interpreted as in the elastic demand case. Let us investigate successively the cases of abundant coal and rare coal.

4.1 The abundant coal case

When coal is abundant or equivalently the capacity is rare and the demand is elastic, we know from Proposition 3 that the last phase is a phase at the ceiling during which the both resources are consumed. Let us show that the same holds in the inelastic demand case and that the solution is a two phases path, the second phase being the last phase of Proposition 3 and the first one a phase of exclusive use of coal up to the time t_G at which begins the second one. From this time onwards the energy price $\gamma_q(t)$ must be equal to c_y.

Dynamics of the pollution stocks and resource uses

During the first phase $[0, t_G)$ the ceiling constraint is inactive and the maximum pollution stock $\bar{G}$ is attained at the end of the phase.

Since $x(t) = \bar{q}$, then the stock of self-regenerating pollution evolves as follows :

$$\begin{aligned} Z(t) &= Z^0 e^{-\alpha t} + \theta\zeta\bar{q}\int_0^t e^{-\alpha(t-\tau)}d\tau \\ &= Z^0 e^{-\alpha t} + \frac{\theta\zeta}{\alpha}\,\bar{q}[1 - e^{-\alpha t}]\ ,\ t \leq t_G, \end{aligned} \tag{4.6}$$

hence :

$$\dot{Z}(t) = e^{-\alpha t}[-\alpha Z^0 + \theta\zeta\bar{q}] \quad \text{and} \quad \ddot{Z}(t) = -\alpha e^{-\alpha t}[-\alpha Z^0 + \theta\zeta\bar{q}]. \tag{4.7}$$

After t_G, $Z(t)$ is given by (3.30) so that :

$$Z(t) = \{Z^0 e^{-\alpha t_G} + \theta\zeta\bar{q}[1 - e^{-\alpha t_G}]\} e^{-(1-\theta)\alpha(t-t_G)}, t \geq t_G. \tag{4.8}$$

Thus we may have the three following cases.

If first $\bar{q} > \frac{\alpha}{\theta\zeta} Z^0$, then $\dot{Z}(t) > 0$ and $\ddot{Z}(t) < 0$ for $t < t_G$ while $\dot{Z}(t) < 0$ and $\ddot{Z}(t) > 0$ for $t > t_G$ so that the time profile of $Z(t)$ is this profile illustrated in the below Figure 1.

Figure 1 here

If $\bar{q} = \frac{\alpha}{\theta\zeta} Z^0$ then the path is first constant up to time t_G and next decreasing.

If last $\bar{q} < \frac{\alpha}{\theta\zeta} Z^0$ then $\dot{Z}(t) < 0$ and $\ddot{Z}(t) > 0$ both before and after t_G. The self regenerating pollution stock is permanently decreasing at a decreasing rate (in absolute value). Note than at $t = t_G$, $Z(t)$ is not differentiable as in the preceding cases.

Concerning the stock of permanent pollution, clearly :

$$S(t) = S^0 + (1-\theta)\zeta\bar{q}t \ , \ t \leq t_G. \tag{4.9}$$

From t_G onwards the ceiling constraints is tight so that $S(t) = \bar{G} - Z(t)$ where $Z(t)$ is given by (3.30), hence :

$$S(t) = \bar{G} - [\bar{G} - (S^0 + (1-\theta))\zeta\bar{q}t_G]e^{-(1-\theta)\alpha(t-t_G)}, t \geq t_G \tag{4.10}$$

The path of $S(t)$ is illustrated in the below Figure 2.

Figure 2 here

Let us examine now the time path of the total atmospheric pollution $G(t)$.

From (4.7) and (4.9), we obtain :

$$\dot{G}(t) = \dot{S}(t) + \dot{Z}(t) = (1-\theta)\zeta\bar{q} + [-\alpha Z^0 + \theta\zeta\bar{q}]e^{-\alpha t} \quad , \quad t < t_G \tag{4.11}$$

$$\ddot{G}(t) = \ddot{S}(t) + \ddot{Z}(t) = -\alpha[\alpha Z^0 + \theta\zeta\bar{q}]e^{-\alpha t} \quad , \quad t < t_G \tag{4.12}$$

In the case $\bar{q} > \frac{\alpha}{\theta\zeta} Z^0$, $\dot{G}(t) > 0$ and $\ddot{G}(t) < 0$: the atmospheric pollution increases at a decreasing rate before attaining its ceiling as illustrated in the below Figure 3.

Figure 3 here

In the case $\bar{q} = \frac{\alpha}{\theta\zeta} Z^0, \dot{G}(t) = \frac{\alpha(1-\theta)}{\theta} Z^0$ and $\ddot{G}(t) = 0$ so that the atmospheric pollution increases linearly during the initial phase $[0, t_G)$.

In the case $\frac{\alpha}{\theta\zeta} Z^0 > \bar{q} - \alpha Z^0$, at time $t = 0$

$$\dot{G}(0) = (1-\theta)\zeta\bar{q} - \alpha Z^0 + \theta\zeta\bar{q} = \zeta\bar{q} - \alpha Z^0 > 0.$$

Since $\ddot{G}(0) > 0$ then the atmospheric pollution stock increases at an increasing rate as illustrated in the below Figure 4.

Figure 4 here

In the last case $\frac{\alpha}{\zeta} Z^0 > \bar{q} > 0$, then $\dot{G}(t) < 0$ and $\ddot{G}(t) > 0$. Thus initially the atmospheric pollution stock decreases. Because $\ddot{G} > 0$, the minimum of $G(t)$ over $[0, t_G)$ is unique. The time profile of $G(t)$ is illustrated in the below Figure 5.

Figure 5 here

It should be clear from the above discussion of the time profile of $G(t)$, that there exists a unique time $t = t_G$ solving :

$$G(t) = S^0 + (1-\theta)\zeta\bar{q}t + Z^0 e^{-\alpha t} + \frac{\theta\zeta}{\alpha} \bar{q}(1 - e^{-\alpha t}) = \bar{G}. \quad (4.13)$$

The dynamics of the resource consumptions is illustrated in the below Figure 6. The coal consumption is discontinuous at the time t_G at which the ceiling $\bar{G}$ is attained. The reason is that, for $x(t)$ given by (3.31) :

$$\frac{\alpha}{\zeta} Z(t_G) < \bar{q} \;\; \Rightarrow \;\; lim_{t \downarrow t_G} x(t) < \bar{q}.$$

The first above inequality is an immediate implication of $\dot{Z}(t_G^-) > 0$, where $\dot{Z}(t_G^-) = lim_{t \uparrow t_G} \dot{Z}(t)$. From

$$\dot{Z}(t_G^-) = \theta\zeta\bar{q} - \alpha Z(t_G^-) > 0$$

and from $\theta < 1$, we get :

$$\bar{q} > \frac{\alpha}{\theta\zeta} Z(t_G^-) > \frac{\alpha}{\zeta} Z(t_G).$$

Figure 6 here

What remains to be checked is the existence of values of the costate variables λ_S and λ_Z and the Lagrange multipliers sustaining the above two phases path suggested as the optimal path.

Shadow prices sustaining the optimal path

During the last phase at the ceiling the both resources are exploited so that the full marginal cost of coal must be equal to the full marginal cost of renewable energy. Since $\lambda_X(t) = 0$ because coal is abundant, we deduce from (4.3) and (4.5) :

$$c_x + (1-\theta)\zeta\lambda_S(t) + \theta\zeta\lambda_Z(t) = c_y \quad , \quad t \geq t_G$$

hence a first relationship between $\lambda_S(t)$ and $\lambda_Z(t)$:

$$\lambda_S(t) = \frac{c_y - c_x}{(1-\theta)\zeta} - \frac{\theta}{1-\theta}\lambda_Z(t) \quad , \quad t \geq t_G \tag{4.14}$$

Let us look for constant values of λ_S, λ_Z and γ_q.
From (3.14), we get :

$$\dot{\lambda}_S(t) = 0 = \rho\lambda_S(t) - \nu_G(t) \quad \Rightarrow \quad \nu_G(t) = \rho\lambda_S(t).$$

Substituting for $\nu_G(t)$ in (3.15) results in :

$$\dot{\lambda}_Z(t) = 0 = (\rho+\alpha)\lambda_Z(t) - \rho\lambda_S(t) \quad \Rightarrow \quad \lambda_S(t) = \frac{\rho+\alpha}{\rho}\,\lambda_Z(t) \tag{4.15}$$

Solving (4.14) and (4.15) for λ_S and λ_Z, we obtain :

$$\lambda_S(t) = \frac{(\rho+\alpha)(c_y - c_x)}{\zeta[\rho + (1-\theta)\alpha]} \quad \text{and} \quad \lambda_Z(t) = \frac{\rho(c_y - c_x)}{\zeta[\rho + (1-\theta)\alpha]} \ , \ t \geq t_G \tag{4.16}$$

The above values of the costate variables λ_S and λ_Z are those constant values we have obtained in the elastic demand case (cf Proposition 5). The reasons are the following ones. First, once the both resources are exploited, then the energy price is constant and equal to c_y. This is implying that the marginal cost discrepancy $c_y - c_x$ is constant. Because the mining rent λ_X is nil when coal is abundant, this in turn implies that the full shadow marginal polluting cost, $\zeta[(1-\theta)\lambda_S(t) + \theta\lambda_Z(t)]$ must be constant and equal to $c_y - c_x$. The argument does not depend upon the elasticity of the energy demand, neither upon the quantity having to be delivered at the price c_y. This is explaining why we get the same shadow values, whatever the energy demand $q^d(c_y)$ at the price c_y in the elastic demand case (cf Proposition 5) and the quantity $\bar{q}$ having to be delivered in the inelastic demand case.

Before the time t_G at which the ceiling $\bar{G}$ is attained, the dynamics of the shadow marginal costs λ_S and λ_Z are evolving respectively at the proportional rate ρ for λ_S (cf (3.22)) and the proportional rate $\rho + \alpha$ for λ_Z (cf (3.26)). Thus denoting by $\bar{\lambda}_S$ and $\bar{\lambda}_Z$ the constant values of λ_S and λ_Z once at the ceiling as given by (4.16) (equivalently (3.32)) and given that t_G has been determined by (4.13), the values fo λ_S and λ_Z during the first phase of the path, before t_G, are given by :

$$\lambda_S(t) = \bar{\lambda}_S e^{-\rho(t_G - t)} \quad > \quad \lambda_Z(t) = \bar{\lambda}_Z e^{-(\rho+\alpha)(t_G - t)} \ , \ 0 \le t \le t_G \quad (4.17)$$

The optimal paths of the energy price and its components are illustrated in the below Figure 7. What is worth to be pointed out is that λ_S, the shadow marginal cost of the permanent pollution stock S, is higher than λ_Z, the shadow marginal cost of the self-regenerating stock Z. This is evident from (4.16) for the second phase of the path. Because λ_Z is increasing at a higher proportional rate than λ_S during the first phase, then λ_S is also higher initially over the first phase $[0, t_G)$. The reason is that as far as the pollution ceiling is constraining the use of the pollution resource, then when coal is abundant the ultimate rare resource is the *capacity*. The self-regenerating stock is a lagging device permitting to postpone the date at which the ultimate resource, here the capacity, will be exhausted.

Figure 7 here

It is easy to check that all the optimality conditions are satisfied along the above two phases path, using the expression of these values of the costate variables λ_S and λ_Z, and $\lambda_X = 0$.

4.2 The rare coal case

The intuition is the same as the one sustaining the solution in the abundant coal case. Because coal energy is less costly than solar energy, the best is to exploit it as much as possible. But now, when at the ceiling, the coal resource cannot be exploited indefinitely. This is suggesting a three phases path. First coal is exploited exclusively up the time $\underline{t}_G$ at which the atmospheric pollution ceiling $\bar{G}$ is attained. Next begins a phase at the ceiling, $(\underline{t}_G, \bar{t}_G]$, during which the both resources are exploited. The phase is ending at the time $\bar{t}_G$ at which the coal endowment is exhausted. The third and last phase $(\bar{t}_G, \infty)$ is a phase of exclusive use of clean solar energy. Because the coal consumption is nil during this last phase, the self-regenerating component of the pollution stock decreases over time so that the ceiling constraint is no more active from $\bar{t}_G$ onwards.

Dynamics of the pollution stocks and resource uses

Concerning first the permanent pollution stock, its time profile is the time profile of the abundant coal case up to the time $\bar{t}_G$ at which coal extraction is closed. After $\bar{t}_G$, $S(t)$ remains constant at the $S(\bar{t}_G)$ level. The scheme is the same concerning the self-regenerating stock Z and the global stock G, that is the dynamics of these stocks are the dynamics of the abundant coal case up to time $\bar{t}_G$.

Concerning Z after $\bar{t}_G$, its dynamics is given by:

$$Z(t) = Z(\bar{t}_G)e^{-\alpha(t-\bar{t}_G)} \quad , \quad t \geq \bar{t}_G \quad ,$$

which is lower than in the abundant coal case since now the stock is no more fed by new emissions. Thus at time $t = \bar{t}_G$, $Z(t)$ is not differentiable its r.h.s time derivative being larger in absolute value than its l.h.s derivative. Note that in the both cases of rare and abundant coal, $Z(t)$ decreases down to zero at infinity, the difference being that in the abundant coal case, the gap $\bar{G} - Z(t)$ is filled up by permanent pollution while in the rare coal case, $\bar{G} - Z(t)$ is no more fed and increasing through time from $\bar{t}_G$ onwards.

Last concerning the global pollution stock after $\bar{t}_G$, it begins to decrease since $S(t)$ is constant while $Z(t)$ decreases down to zero, hence: $\lim_{t\uparrow\infty} G(t) = S(\bar{t}_G)$. The precise time profile of $G(t)$ is depending upon the size of $\bar{q}$ relative to the size of Z^0 as in the abundant coal case. A possible time profile of G is illustrated in the below Figure 8, corresponding to the case $\bar{q} < \alpha Z^0/\zeta$.

Figure 8 here

The time profile of the resources uses is illustrated in Figure 9.

Figure 9 here

Shadow costs and mining rents sustaining the optimal path

We first show that the characteristics of the optimal path are determined by $\underline{t}_G$ and $\bar{t}_G$, respectively the date of arrival at the ceiling and the date of exhaustion of coal. We then show that $\underline{t}_G$ and $\bar{t}_G$ are themselves the unique solutions of a system of two equations, the equations (4.12) and (4.13), corresponding respectively to the coal stock constraint and the global ceiling constraint.

During the last post coal phase $[\bar{t}_G, \infty)$, $\lambda_S(t) = \lambda_Z(t) = \nu_G(t) = 0$, the cap constraint being no more active forever. In order to determine the dual variables paths during the phases $[0, \underline{t}_G)$ and $[\underline{t}_G, \bar{t}_G)$ it is convenient to introduce the following auxiliary variables:

$$\mu(t) \equiv (1-\theta)\lambda_S(t) + \theta\lambda_Z(t) \qquad (4.1)$$

$$\delta(t) \equiv \lambda_S(t) - \lambda_Z(t) \ , \qquad (4.2)$$

where $\mu(t)$ is the carbon tax having to be levied per unit of pollution emissions so that $\zeta\mu(t)$ is the tax having to be levied per unit of burnt coal. It is immediately checked that λ_S and λ_Z can be expressed as the following functions of μ and δ:

$$\lambda_S = \mu + \theta\delta \tag{4.3}$$
$$\lambda_Z = \mu - (1-\theta)\delta \ . \tag{4.4}$$

Characterizing the time paths of $\mu(t)$ and $\delta(t)$ allows to determine the dual variables paths of λ_S and λ_Z.

Firstly, Let us determine the time paths of $\mu(t)$ and $\delta(t)$ over the ceiling phase $[\underline{t}_G, \bar{t}_G)$. These time paths are implicitly defined as functions of $\underline{t}_G$ and $\bar{t}_G$. During the ceiling phase:

$$c_y = c_x + \lambda_X e^{\rho t} + \zeta\mu(t) \ , \quad t \in [\underline{t}_G, \bar{t}_G) \ .$$

Time differentiating results in:

$$\dot{\mu}(t) = -\frac{\rho}{\zeta}\lambda_X e^{\rho t} = \rho\mu(t) - \frac{\rho}{\zeta}(c_y - c_x) \ , \quad t \in [\underline{t}_G, \bar{t}_G) \ .$$

Since $\lambda_Z(\bar{t}_G) = \lambda_S(\bar{t}_G) = 0$, $\mu(\bar{t}_G) = 0$ is a particular solution of the above linear differential equation. Integrating over $[t, \bar{t}_G)$, $t \geq \underline{t}_G$, we get:

$$\mu(t) = \frac{c_y - c_x}{\zeta}\left[1 - e^{-\rho(\bar{t}_G - t)}\right] \ , \quad t \in [\underline{t}_G, \bar{t}_G) \ . \tag{4.5}$$

Remarking that since $\lambda_Z(t)$ and $\lambda_S(t)$ have to be continuous time functions, $\mu(t)$ has also to be a continuous time function at $t = \underline{t}_G$, we get the following expression of $\mu(\underline{t}_G)$:

$$\mu(\underline{t}_G) = \frac{c_y - c_x}{\zeta}\left[1 - e^{-\rho(\bar{t}_G - \underline{t}_G)}\right] \ .$$

Turning to the dynamics of $\delta(t)$, we get making use of (4.3):

$$\begin{aligned}\dot{\delta}(t) &= \dot{\lambda}_S(t) - \dot{\lambda}_Z(t) = \rho\lambda_S(t) - (\rho + \alpha)\lambda_Z(t) \\ &= -\alpha\mu(t) + (\rho + \alpha(1-\theta))\delta(t)\end{aligned} \tag{4.6}$$

Observe that this last differential equation describing the motion of $\delta(t)$ has to apply during both the pre ceiling phase $[0, \underline{t}_G)$ and the ceiling phase $[\underline{t}_G, \bar{t}_G)$. $\lambda_S(\bar{t}_G) = \lambda_Z(\bar{t}_G) = 0$ give the particular solution $\delta(\bar{t}_G) = 0$. Thus integrating over $[t, \bar{t}_G)$, $t \geq \underline{t}_G$, while making use of the expression (4.5) of $\mu(t)$ previously computed, we obtain:

$$\begin{aligned}
\delta(t) &= \alpha \int_t^{\bar{t}_G} \mu(\tau) e^{-(\rho+\alpha(1-\theta))(\tau-t)} d\tau \\
&= \frac{\alpha(c_y - c_x)}{\zeta} \left\{ \frac{1 - e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - t)}}{\rho + \alpha(1-\theta)} \right. \\
&\qquad \left. - \frac{e^{-\rho(\bar{t}_G - t)} - e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - t)}}{\alpha(1-\theta)} \right\} \quad , \quad t \in [\underline{t}_G, \bar{t}_G) \ . \qquad (4.7)
\end{aligned}$$

Taking into account (4.3), (4.4), straightforward computations show that $\lambda_S(t)$ and $\lambda_Z(t)$ are defined by the following relations during the ceiling phase:

$$\lambda_S(t) = \frac{c_y - c_x}{\zeta} \left[\frac{\alpha + \rho}{\rho + \alpha(1-\theta)} - \frac{e^{-\rho(\bar{t}_G - t)}}{1-\theta} + \frac{\theta \rho e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - t)}}{(1-\theta)(\rho + \alpha(1-\theta))} \right] \qquad (4.8)$$

$$\lambda_Z(t) = \frac{\rho(c_y - c_x)}{\zeta(\rho + \alpha(1-\theta))} \left[1 - e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - t)} \right] \ , \quad t \in [\underline{t}_G, \bar{t}_G) \ . \qquad (4.9)$$

Some remarks are in order at this stage. First it is easily checked by differentiating through time the expressions (4.9) and (4.8) of $\lambda_S(t)$ and $\lambda_Z(t)$ during the ceiling phase, that both $\lambda_S(t)$ and $\lambda_Z(t)$ decrease during this time phase. Moreover, differentiating through time the expression (4.7) of $\delta(t)$ during the ceiling phase, it is also easily verified that $\dot{\delta}(t) < 0$, that is not only the opportunity costs of pollution in the two atmospheric reservoirs S and Z decrease through time but also the difference between λ_S and λ_Z along the optimal path. Since $\delta(t) \to 0$ during the ceiling phase, we also conclude that $\delta(t) > 0$, $t < \bar{t}_G$, that is $\lambda_S(t) > \lambda_Z(t)$.As in the abundant coal case with an inelastic demand, the opportunity cost of accumulating pollution in the non renewable reservoir S is higher than the opportunity cost of accumulating carbon in the renewable reservoir Z.

Second, observe that for $\bar{t}_G \to \infty$, $\lambda_Z(\underline{t}_G) \to \rho(c_y - c_x)/\zeta[\rho + \alpha(1-\theta)]$ and $\lambda_S(\underline{t}_G) \to (\rho + \alpha)(c_y - c_x)/\zeta[\rho + \alpha(1-\theta)]$, that is towards their constant levels in the abundant coal case already computed, a case in which $\bar{t}_G$ would be equal to ∞.

Now consider the determination of the dual variables path during the first phase before the ceiling, $[0, \underline{t}_G)$. $\lambda_S(\underline{t}_G)$ and $\lambda_Z(\underline{t}_G)$ may be computed as functions of $(\underline{t}_G, \bar{t}_G)$ by evaluating (4.9) and (4.8) at $t = \underline{t}_G$. Since $\lambda_Z(t) = \lambda_{Z0}e^{(\alpha+\rho)t}$ and $\lambda_S(t) = \lambda_{S0}e^{\rho t}$ during the pre ceiling phase $t \in [0, \underline{t}_G)$, we obtain the following expressions of λ_{S0} and λ_{Z0}, thus characterizing the dual variables path before $\underline{t}_G$ as a function of $(\underline{t}_G, \bar{t}_G)$.

$$\lambda_{S0} = \frac{c_y - c_x}{\zeta}\left[\frac{\alpha+\rho}{\rho+\alpha(1-\theta)}e^{-\rho\underline{t}_G} - \frac{e^{-\rho\bar{t}_G}}{1-\theta} + \frac{\theta\rho e^{\alpha(1-\theta)\underline{t}_G-(\rho+\alpha(1-\theta))\bar{t}_G}}{(1-\theta)(\rho+\alpha(1-\theta))}\right] \tag{4.10}$$

$$\lambda_{Z0} = \frac{\rho(c_y - c_x)}{\zeta(\rho+\alpha(1-\theta))}\left[e^{-(\rho+\alpha)\underline{t}_G} - e^{-\alpha\theta\underline{t}_G-(\rho+\alpha(1-\theta))\bar{t}_G}\right]\ . \tag{4.11}$$

Observe that during the first phase $[0, \underline{t}_G)$, $\delta(t)$ increases iff $\lambda_S(t)/\lambda_Z(t) > (\rho+\alpha)/\rho$. It may be shown that: $\lambda_S(\underline{t}_G)/\lambda_Z(\underline{t}_G) > (\rho+\alpha)/\rho$. Thus we can conclude that $\delta(t)$ must increase before the economy reaches the ceiling. In the contrary case, $\lambda_{S0}/\lambda_{Z0} < (\rho+\alpha)/\rho$ would imply that $\lambda_S(\underline{t}_G)/\lambda_Z(\underline{t}_G) < (\rho+\alpha)/\rho$, a contradiction. The growth of the difference between $\lambda_S(t)$ and $\lambda_Z(t)$ before the ceiling phase reflects the increasing relative burden of the accumulation of pollution inside the non renewable reservoir with respect to the renewable one. Note that this qualitative feature does not depend upon θ, the relative pollution flow share accruing to the two reservoirs.

We have shown that λ_{Z0} and λ_{S0} may be expressed as functions of $(\underline{t}_G, \bar{t}_G)$. Furthermore since $\mu(\bar{t}_G) = 0$, we get also: $c_y = c_x + \lambda_{X0}e^{\rho\bar{t}_G}$, and thus λ_{X0} is a function of $\bar{t}_G$: $\lambda_{X0} = (c_y - c_x)e^{-\rho\bar{t}_G}$. Let $\lambda_Z(\underline{t}_G, \bar{t}_G)$, $\lambda_S(\underline{t}_G, \bar{t}_G)$ and $\lambda_X(\bar{t}_G)$ be that functions. Differentiating we get easily:

$$
\begin{aligned}
\frac{\partial \lambda_Z(\underline{t}_G, \bar{t}_G)}{\partial \underline{t}_G} &= -\frac{\partial \lambda_Z(\underline{t}_G, \bar{t}_G)}{\partial \bar{t}_G} = -\frac{\rho(c_y - c_x)}{\zeta} e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - \underline{t}_G)} < 0 \\
&\Longrightarrow \begin{cases} \dfrac{\partial \lambda_{Z0}(\underline{t}_G, \bar{t}_G)}{\partial \underline{t}_G} = e^{-(\alpha+\rho)\underline{t}_G} \left(\dfrac{\partial \lambda_Z(\underline{t}_G)}{\partial \underline{t}_G} - (\alpha + \rho) \right) & < \ 0 \\ \dfrac{\partial \lambda_{Z0}(\underline{t}_G, \bar{t}_G)}{\partial \bar{t}_G} = e^{-(\alpha+\rho)\underline{t}_G} \dfrac{\partial \lambda_Z(\underline{t}_G)}{\partial \bar{t}_G} & > \ 0 \end{cases} \\
\frac{\partial \lambda_S(\underline{t}_G, \bar{t}_G)}{\partial \underline{t}_G} &= -\frac{\partial \lambda_S(\underline{t}_G, \bar{t}_G)}{\partial \bar{t}_G} \\
&= -\frac{\rho(c_y - c_x)}{\zeta(1-\theta)} \left[e^{-\rho(\bar{t}_G - \underline{t}_G)} - \theta e^{-(\rho+\alpha(1-\theta))(\bar{t}_G - \underline{t}_G)} \right] < 0 \\
&\Longrightarrow \begin{cases} \dfrac{\partial \lambda_{S0}(\underline{t}_G, \bar{t}_G)}{\partial \underline{t}_G} = e^{-\rho \underline{t}_G} \left(\dfrac{\partial \lambda_S(\underline{t}_G)}{\partial \underline{t}_G} - \rho \right) & < \ 0 \\ \dfrac{\partial \lambda_{S0}(\underline{t}_G, \bar{t}_G)}{\partial \bar{t}_G} = e^{-\rho \underline{t}_G} \dfrac{\partial \lambda_S(\underline{t}_G)}{\partial \bar{t}_G} & > \ 0 \end{cases} .
\end{aligned}
$$

λ_{Z0} and λ_{S0} are increasing functions of $\bar{t}_G$, the coal reserves depletion time and decreasing functions of $\underline{t}_G$, the time of introduction of renewable energy and also the beginning of the ceiling period. Last, $\lambda_{X0}(\bar{t}_G)$ is trivially a decreasing function of $\bar{t}_G$.

We conclude that the dual variables paths are completely determined once $\underline{t}_G$ and $\bar{t}_G$ are determined, and that $\underline{t}_G$ and $\bar{t}_G$ may themselves be determined by making use of the coal stock constraint and the global ceiling constraint:

$$
\begin{aligned}
X^0 &= \int_0^{\bar{t}_G} x(t) dt & (4.12) \\
\bar{G} &= S(\underline{t}_G) + Z(\underline{t}_G) \ . & (4.13)
\end{aligned}
$$

Note first that $x(t) = \bar{q}$, $t < \underline{t}_G$ while $x(t) = (\alpha Z(\underline{t}_G)/\zeta) e^{-\alpha(1-\theta)(t - \underline{t}_G)}$, during the ceiling phase $[\underline{t}_G, \infty)$. Thus the above set of constraints may be expressed as:

$$
X^0 = \bar{q} \underline{t}_G + \frac{\alpha Z(\underline{t}_G)}{\zeta} \frac{1 - e^{-\alpha(1-\theta)(\bar{t}_G - \underline{t}_G)}}{\alpha(1-\theta)} \tag{4.14}
$$

$$
\bar{G} = S^0 + \zeta(1-\theta)\bar{q}\underline{t}_G + Z(\underline{t}_G), \tag{4.15}
$$

while $Z(\underline{t}_G)$ is defined by:

$$Z(\underline{t}_G) \;=\; e^{-\alpha \underline{t}_G} Z^0 + \frac{\theta\zeta\bar{q}}{\alpha}\left(1 - e^{-\alpha \underline{t}_G}\right) .$$

Appendix A.3 shows that the above system has a unique solution $(\underline{t}_G, \bar{t}_G)$ and thus defines unique dual variables trajectories $\{\lambda_X(t), \lambda_S(t), \lambda_Z(t), t \geq 0\}$.

It also provides the details of the comparative dynamics of the system with respect to the vector $(X^0, S^0, Z^0, \bar{G}, \bar{q})$. Concerning first the variations of $\underline{t}_G$ and $\bar{t}_G$, it is shown that the length of the ceiling phase $\bar{t}_G - \underline{t}_G$ is increased by higher levels of the initial coal endowment, X^0, energy consumption $\bar{q}$, and by lower pollution capacity levels either in the form of an increase of S^0, Z^0, or a stricter cap constraint $\bar{G}$. The initial coal stock size has no effect upon $\underline{t}_G$, that time at which the economy hits the pollution cap constraint but a higher amount of coal reserves will naturally delay the depletion time $\bar{t}_G$. Stricter pollution capacity constraints will induce a sooner arrival at the ceiling together with a longer stay under the ceiling constraint. A higher level of energy demand accelerates the attainment of the ceiling while reducing the total length of the coal use phase.

Concerning last the costate variables, as expected, a higher coal availability results in an increase of the opportunity costs of pollution $(\lambda_Z(t), \lambda_S(t))$ before coal depletion and a lower level of the resource scarcity rent λ_{X0}. A reduction of the pollution capacity, either in the form of a higher S_0, a higher Z^0 or either a stricter cap $\bar{G}$ has quite intuitively the same consequence over the dual variables. The effect of a higher energy demand $\bar{q}$ has a more intricate effect since it means both a sooner arrival at the ceiling and a sooner depletion of the coal reserves. Only the intuitive positive effect of a higher energy demand upon the shadow value of coal may be identified.

5 The elastic demand case

We now examine the characteristics of the optimal policy when the energy demand is an elastic function. It will appear that some features of the optimal resources use plan described in the inelastic case will translate qualitatively to more general situations. But new possibilities will also arise. In order to keep matters simple, we shall limit the study to the abundant coal case. Thus $\lambda_X(t) = 0$ and only a fraction $\bar{X}^0$ of the initial coal endowments will be consumed along the optimal resources use path.

Some features of the optimal policy have already been identified in Section 3. In particular Proposition 2 shows that the introduction of renewable energy inside the energy mix will be delayed until some finite time we denote by t_y. Whence introduced, renewable energy will be used permanently in conjunction with coal. The global ceiling constraint will bind all along this phase of simultaneous exploitation of the two natural resources, a phase of infinite duration $[t_y, \infty)$. This was already the case with an inelastic demand function and comes at no surprise. Since during this last simultaneous phase, the gross marginal surplus is constant and equal to c_y, the marginal cost of renewable energy provision, the form of the demand function plays no role in determining the dynamics of energy supply from the two possible energy sources. The only difference with the inelastic demand case is that now the energy consumption level q is endogenously determined by the condition $u'(q) = c_y$, that is by $\tilde{y}$, and no more exogenously given by $\bar{q}$ as was the case with an inelastic demand.

What happens before t_y shows striking differences with the inelastic case. From Proposition 2, we know that this first phase is a phase of only coal exploitation. Firstly, it is now possible that the global ceiling is attained before the introduction of renewable energy. In such a scenario, the renewable energy marginal cost is high compared to the opportunity cost of the ceiling constraint and the economy prefers to rely only upon coal use even at the ceiling rather than satisfying the energy demand through a mix of non renewable and renewable energy provision.

Secondly, there is no reason for the ceiling constraint to bind only once before the introduction or renewable energy. Depending upon the shape of the demand function, it is possible that the economy experiences a sequence of temporary phases at the ceiling and below the ceiling. This stands both in sharp contrast with the inelastic demand case and also with the conclusions of the seminal work of Chakravorty *et al.* 2006, in the one pollution reservoir case. In this last case, it may be shown that the ceiling constraint may bind only once with an elastic demand function. We thus consider the optimal policy during this first phase in more detail.

5.1 The optimal coal exploitation policy before the introduction of renewable energy

We are going to show that the first pure coal phase $[0, t_y)$ may be composed of a sequence of temporary phases at the ceiling separated by phases below the ceiling. Since the economy relies only upon coal exploitation before t_y we must have $u'(x) < c_y$, that is $x(t) > \tilde{y}$, $t < t_y$.

Denote by $\Delta_G^i \equiv [\underline{t}_G^i, \bar{t}_G^i)$ a ceiling phase indexed by $i \geq 1$. $G(t) = \bar{G}$, $t \in \Delta_G^i$ within a ceiling phase. Denote also by $D_G^i \equiv [\bar{t}_G^i, \underline{t}_G^{i+1})$ a below the ceiling phase, that is $G(t) < \bar{G}$, $t \in D_G^i$. Since $G(0) < \bar{G}$ by assumption, the optimal path begins below the ceiling, that is there exists $D_G^0 \equiv [0, \underline{t}_G^1)$, a first time phase during which the economy has not yet reached the ceiling for the first time and $\underline{t}_G^1 \leq t_y$.

Time differentiating $\delta(t)$ we obtain:

$$\dot{\delta}(t) = -\alpha\mu(t) + (\rho + \alpha(1-\theta))\delta(t) \quad t \leq t_y \tag{5.1}$$

Since $\mu(t) = (u'(x) - c_x)/\zeta$, $t \leq t_y$, and the gross surplus must be a continuous time function along any optimal path, $\mu(t)$ is also a continuous time function together with $\delta(t)$. Thus $\delta(t)$ is a differentiable time function in view of (5.1). Note also that the general form of the equation giving the dynamics of $\delta(t)$ does not depend upon the possible type of phase before t_y, either a Δ_G^i phase or either a D_G^i phase.

Next, let us compute the dynamics of $x(t)$, the coal extraction rate before t_y. These dynamics depend upon the type of possible phases before the introduction of renewable energy. During a D_G^i phase below the ceiling:

$$\begin{aligned}\dot{\mu}(t) &= \theta\dot{\lambda}_Z(t) + (1-\theta)\dot{\lambda}_S(t) = \theta\alpha\lambda_Z(t) + \rho\mu(t) \\ &= (\theta\alpha + \rho)\mu(t) - \theta\alpha(1-\theta)\delta(t) \ .\end{aligned}$$

This implies since $u'(x) = c_x + \zeta\mu$ that:

$$u''(x)\dot{x} = \zeta\dot{\mu}(t) = \zeta(\theta\alpha + \rho)\mu - \zeta\theta\alpha(1-\theta)\delta \ ,$$

and thus:

$$\dot{x}(t) = \frac{1}{u''(x(t))}\left\{(\theta\alpha + \rho)[u'(x(t)) - c_x] - \zeta\theta\alpha(1-\theta)\delta(t)\right\} \ , \ t \in D_G^i \tag{5.2}$$

On the other hand, remember that $\dot{x}(t) = -\alpha(1-\theta)x(t) < 0$, $t \in \Delta_G^i$.

We adopt the following solving strategy. First we portrait the dynamics of $x(t)$ and $\delta(t)$ in an unconstrained situation with a phase diagram in the (x, δ) plane. We illustrate the corresponding optimal trajectory. Second, we show that the ceiling constraint may be represented as some critical function in this same plane. The issue of the existence of several ceiling phase is thus equivalent to the study of possible intersection points between the optimal trajectory and this critical border.

First consider a possible unconstrained trajectory. Remember that $x(t) > \tilde{y}$, $t < t_y$. Second since $u''(x) < 0$, during a below the ceiling phase, D_G^i, (5.2) implies that:

$$\dot{x}(t) \leq 0 \quad t \in D_G^i \iff \delta \leq \frac{\theta\alpha + \rho}{\zeta\theta\alpha(1-\theta)}\left[u'(x) - c_x\right] \equiv \varphi_x(x) \qquad (5.3)$$

$\lim_{x\downarrow 0} u'(x) = +\infty$ implies that $\lim_{x\downarrow 0} \varphi_x(x) = +\infty$ and $u''(x) < 0$ implies that $\varphi_x'(x) < 0$. Last note that $u'(\tilde{y}) - c_x = c_y - c_x > 0$ implies that $\varphi_x(\tilde{y}) > 0$. Furthermore $\dot{\lambda}_Z > 0$ and $\dot{\lambda}_S > 0$ during a below the ceiling phase both imply that $\dot{\mu} = \theta\dot{\lambda}_Z + (1-\theta)\dot{\lambda}_S > 0$. This implies in turn that necessarily $\dot{x}(t) < 0$, $t \in D_G^i$. Thus the optimal trajectory must be located below the $\dot{x} = 0$ border.

Next, turning towards the dynamics of $\delta(t)$, we get:

$$\dot{\delta}(t) \geq 0 \iff \delta \geq \frac{\alpha}{\zeta(\rho + \alpha(1-\theta))}(u'(x) - c_x) \equiv \varphi_\delta(x) \ . \qquad (5.4)$$

Under our assumptions, we get immediately: $\lim_{x\downarrow 0} \varphi_\delta(x) = +\infty$ and $\varphi_\delta'(x) < 0$. It is also easily checked that:

$$\frac{\alpha}{\rho + \alpha(1-\theta)} < \frac{\theta\alpha + \rho}{\theta\alpha(1-\theta)} \ ,$$

implying that $\varphi_\delta(x) < \varphi_x(x)$. Since $u'(q)$ has to be a continuous time function along an optimal path, it results from our previous discussion that:

$$\delta(t_y) = \frac{\alpha(c_y - c_x)}{\zeta(\rho + \alpha(1-\theta))} = \frac{\alpha}{\zeta(\rho + \alpha(1-\theta))}(u'(\tilde{y}) - c_x) = \varphi_\delta(\tilde{y}) \ .$$

Hence the locus $\dot{\delta} = 0$, that is the curve $\varphi_\delta(x)$, cuts the vertical line $x = \tilde{y}$ at the level $\delta(t_y)$ corresponding to the constant level of δ throughout the last phase $[t_y, \infty)$. Since $\delta(t)$ has to be a continuous time function along an optimal path, we thus conclude that the optimal trajectory $\mathcal{O} \equiv \{x^*(t), \delta^*(t),\ 0 \leq t \leq t_y\}$ and the $\dot{\delta} = 0$ locus intersect themselves at $x = \tilde{y}$.

Since we have already established that $\dot{x}(t) < 0$, $t \in \Delta_G^i$, we know that $\dot{x} < 0$ in all possible time phases before t_y. Consider a possible sequence composed of a below the ceiling phase D_G^i followed by a phase at the ceiling Δ_G^{i+1}. During this last phase, $x(t)$ decreases but it could be possible that the $\{x(t), \delta(t), t \in \Delta_G^{i+1}\}$ trajectory moves above the $\dot{x} = 0$ locus of the unconstrained case. If the ceiling phase is followed by a below the ceiling phase while the system stays above the $\dot{x} = 0$ locus, we get a contradiction since $x(t)$ and $\delta(t)$ are continuous time functions and $\dot{x}(t) > 0$ is never optimal during a below the ceiling phase. If such a ceiling phase is followed until the introduction of renewable energy and the trajectory remains above the $\dot{x} = 0$ locus, we obtain another contradiction since $\delta(t_y) < \varphi_x(\tilde{y})$ would prevent such a trajectory to connect to the point $(\tilde{y}, \delta(t_y))$. Hence only remains the possibility of a ceiling phase during which the $\{x(t), \delta(t)\}$ trajectory stays temporarily above the $\varphi_x(x)$ curve before moving below this frontier. However we shall show below that such an outcome is never optimal.

The following figure 10 illustrates the phase diagram in the (x, δ) plane.

Figure 10 here

Since $\dot{x}(t) < 0$, $t \leq t_y$ for any possible time phase, trajectories in the (x, δ) plane either initiated above the locus $\dot{x} = 0$ or either initiated from below this locus and cutting it in finite time cannot be optimal. Trajectories remaining below the locus $\dot{x} = 0$ and above the locus $\dot{\delta} = 0$ but reaching the frontier $x = \tilde{y}$ above $\delta(t_y)$ are not optimal since $\delta(t)$ has to be continuous at time t_y. The trajectories initiated below the locus $\dot{\delta} = 0$ or cutting this locus from above in finite time at a level $x > \tilde{y}$ move in the south west direction and end converge towards to some level of δ strictly lower than $\delta(t_y)$. Since δ is a continuous time function, all these trajectories cannot be optimal. Hence only remains the trajectory labeled by $\mathcal{O}$ on figure 10 and converging in finite time towards the point $(\tilde{y}, \delta(t_y))$ in the (x, δ) plane. We observe that not only $x(t)$ decreases along the optimal path before t_y but also that δ should increase. This means that in all possible phases the difference between

$\lambda_S(t)$ and $\lambda_Z(t)$ must increase, reflecting the fact that the opportunity cost of accumulating pollution in the non renewable reservoir with respect to the renewable reservoir should rise over time.

Now consider the possibility of a ceiling phase before t_y. During such a phase:

$$\begin{aligned} u'(x(t)) &= c_x + \zeta\mu(t) \\ \dot{x}(t) &= -\alpha(1-\theta)x(t) \\ \dot{\mu}(t) &= (\theta\alpha+\rho)\mu(t) - \theta\alpha(1-\theta)\delta(t) - \nu_G(t) \quad t \in \Delta_G^i \end{aligned}$$

Time differentiating the first relation while using the others gets:

$$\begin{aligned} -\alpha(1-\theta)u''(x)x &= \zeta\dot{\mu} = \zeta(\theta\alpha+\rho)\mu - \zeta\theta\alpha(1-\theta)\delta - \zeta\nu_G \\ &= (\theta\alpha+\rho)[u'(x)-c_x] - \zeta\theta\alpha(1-\theta)\delta - \zeta\nu_G \end{aligned}$$

It results that during a ceiling phase, $\nu_G(t)$, $t \in \Delta_G^i$, is defined by:

$$\nu_G(t) = \frac{\theta\alpha+\rho}{\zeta}[u'(x(t))-c_x] - \theta\alpha(1-\theta)\delta(t) + \frac{\alpha(1-\theta)}{\zeta}u''(x(t))x(t) \qquad t \in \Delta_G^i\ . \tag{5.5}$$

Thus the condition $\nu_G(t) \geq 0$ having to apply during any ceiling phase, it defines the following critical border in the (x,δ) plane:

$$\begin{aligned} \nu_G \geq 0 \iff \delta &\leq \frac{\theta\alpha+\rho}{\zeta\theta\alpha(1-\theta)}[u'(x)-c_x] + \frac{u''(x)x}{\zeta\theta} \\ \iff \delta &\leq \varphi_x(x) + \frac{u''(x)x}{\zeta\theta} \equiv \Phi(x) \end{aligned} \tag{5.6}$$

Since $u''(x) < 0$ we observe that the curve $\Phi(x)$ must be located below the curve $\varphi_x(x)$, that is the $\dot{x} = 0$ locus in the unconstrained case. Thus we can conclude that during a ceiling phase, the optimal trajectory $\{x(t), \delta(t)\}$ must remain strictly below the $\varphi_x(x)$ border.

Secondly, straightforward computations show that:

$$\Phi(\tilde{y}) \gtreqless \delta(t_y) \iff \frac{\rho(\rho+\alpha)(c_y-c_x)}{\alpha(1-\theta)(\rho+\alpha(1-\theta))} \gtreqless |u''(\tilde{y})|\tilde{y}$$

Denote by $\eta(q) \equiv |u''(q)|q/u'(q)$, the inverse demand function elasticity in absolute value. Since $u'(\tilde{y}) = c_y$, the above is thus equivalent to:

$$\begin{aligned} \Phi(\tilde{y}) \gtreqless \delta(t_y) &\iff \frac{\rho(\rho+\alpha)}{\alpha(1-\theta)(\rho+\alpha(1-\theta))}\frac{c_y - c_x}{c_y} \gtreqless \eta(\tilde{y}) \\ &\iff \eta_y \gtreqless \eta(\tilde{y}) \end{aligned}$$

Thus we can conclude that at least locally, the economy will be constrained by the ceiling just before t_y in a case where $\eta(\tilde{y}) < \eta_y$ and will stay below the ceiling just before t_y in a case where $\eta_y < \eta(\tilde{y})$.

Depending upon the curvature properties of the demand function, it is also possible that $\Phi(x)$ be not a monotonous function of x. Let $\Phi_0 \equiv (\theta\alpha + \rho)/\alpha(1-\theta)$, then:

$$\begin{aligned} \Phi(x) &= \frac{1}{\zeta\theta}\left[\Phi_0(u'(x) - c_x) + u''(x)x\right] \\ &= \frac{1}{\zeta\theta}\left[u'(x)(\Phi_0 - \eta(x)) - \Phi_0 c_x\right] \end{aligned}$$

We have to assume that $\eta(x) < \Phi_0$ to give content to the problem. In the reverse case Φ would be negative and $\delta > 0 > \Phi(x)$ would imply that the economy is not constrained by the ceiling. Thus assume the existence of some interval $I_x \equiv [\underline{x}, \bar{x}]$ such that $\tilde{y} \leq \underline{x} < \bar{x}$ and $\Phi_0 < \eta(x)$, $x \in I_x$. Differentiating $\Phi(x)$ we obtain:

$$\frac{d\Phi(x)}{dx} = \frac{1}{\zeta\theta}\left[u''(x)(\Phi_0 - \eta(x)) - u'(x)\eta'(x)\right] . \tag{5.7}$$

Since $u''(x) < 0$ and $\eta(x) < \Phi_0$ if $x \in I_x$, then $d\Phi(x)/dx < 0$ if $\eta'(x) \geq 0$. Let us consider this first case.

5.2 The non decreasing demand elasticity case

Assume that the demand elasticity is either constant or either increasing with q, that is with x before t_y. Since $x(t)$ permanently decreases before t_y in any sequence of temporary phases, either at the ceiling or either below the ceiling, we remark that the demand elasticity level should decline over time along the optimal coal consumption path. Since it has been shown that

$\Phi(x)$ is a decreasing function of x in this case, we have to consider two main possibilities : either $\Phi(\tilde{y}) < \delta(t_y)$ or either $\Phi(\tilde{y}) > \delta(t_y)$. In the first case we are going to prove that the ceiling is only attained for the first time at t_y, that is when renewable energy is introduced inside the energy mix. In the second case, we show that the ceiling will be attained for the first time strictly before t_y and this ceiling phase will last until the introduction of clean energy. In other words, the first phase of only coal exploitation is composed of an initial phase below the ceiling $D_G^0 = [0, \underline{t}_G^1)$ followed by a ceiling phase with only coal exploitation $\Delta_G^1 = [\underline{t}_G, t_y)$.

The case $\eta_y < \eta(q_y)$. This situation corresponds to $\Phi(\tilde{y}) < \delta(t_y)$. Let us show that $\Phi(\tilde{y}) < \delta(t_y)$ and $\eta'(x) > 0$ both imply that $\Phi(x) < \varphi_\delta(x)$, $\forall x \geq \tilde{y}$. Since $\Phi(x) = \varphi_x(x) - u'(x)\eta(x)/\zeta\theta$:

$$\begin{aligned}\Phi(x) < \varphi_\delta(x) &\iff \varphi_x(x) - \varphi_\delta(x) < \frac{u'(x)\eta(x)}{\zeta\theta} \\ &\iff \frac{\rho(\rho+\alpha)}{\alpha(1-\theta)(\rho+\alpha(1-\theta))}\left[1 - \frac{c_x}{u'(x)}\right] < \eta(x)\end{aligned}$$

Since $u''(x) < 0$, the l.h.s. of the previous inequality is a strictly decreasing function of x while the r.h.s. is an increasing function of x since $\eta'(x) \geq 0$ by assumption. Thus:

$$\Phi(\tilde{y}) < \delta(t_y) = \varphi_\delta(\tilde{y}) \implies \Phi(x) < \varphi_\delta(x) \ , \ \forall x \geq \tilde{y}$$

Since the $\Phi(x)$ critical border is located below the $\dot{\delta} = 0$ locus and thus below the optimal trajectory $\mathcal{O}$, we can conclude that along this optimal trajectory $\delta^*(t) > \Phi(x^*(t))$, $t \in [0, t_y)$. Thus the ceiling constraint never binds along the optimal trajectory before t_y. The case $\eta_y < \eta(\tilde{y})$ corresponds to a scenario analogous to the inelastic demand case with abundant coal composed of two phases, a first phase below the ceiling and a second phase at the ceiling with joint use of the two energy sources.

The case $\eta_y > \eta(\tilde{y})$. This situation corresponds to $\Phi(\tilde{y}) > \delta(t_y)$. By a similar argument as given in the previous case, it is easily verified that the vertical distance $\Phi(x) - \varphi_\delta(x)$ decreases with x in a case where $\Phi(x) > \varphi_\delta(x)$. This proves the existence of a unique value of x, $\bar{x}_\delta$, such that $\Phi(x) \gtreqless \varphi_\delta(x) \iff x \gtreqless \bar{x}_\delta$. Since the optimal trajectory must connect at $\tilde{y}$ to $\delta(t_y)$, we thus conclude that the last phase of coal exploitation must be a ceiling phase before the introduction of renewable energy.

In order to prove that this ceiling phase is unique, we have to prove in addition that the slopes of the unconstrained trajectories are lower in absolute value that the slope of $\Phi(x)$ in absolute value at any point along the graph of $\Phi(x)$ within the range $x \in (\tilde{y}, \bar{x}_\delta)$. In the contrary case, it could be possible that the optimal trajectory crosses several times the graph of the function $\Phi(x)$. We show in Appendix A.4 that such a possibility may be excluded provided that the demand elasticity at $q = \tilde{y}$ is sufficiently high. We conclude as follows:

Proposition 6 *Denote by $\eta(q)$ the elasticity of the inverse demand function in absolute value and let η_y be the following critical value:*

$$\eta_y \equiv \frac{\rho(\rho+\alpha)}{\alpha(1-\theta)(\rho+\alpha(1-\theta))} \frac{c_y - c_x}{c_y} .$$

Then if $\eta(q)$ is a non decreasing function of q:

1. *Either $\eta_y < \eta(\tilde{y})$, the economy reaches the ceiling when the clean energy alternative is introduced within the energy mix ;*
2. *Or $\eta(\tilde{y}) < \eta_y$, provided that the demand elasticity evaluated at $q = \tilde{y}$ be not too low, the economy reaches the ceiling strictly before the introduction of renewable energy at a date $\underline{t}_G < t_y$ and then stays forever at the ceiling for $t \geq \underline{t}_G$.*

Note that the non decreasing elasticity case covers both the class of the constant elasticity inverse demand functions and the class of the linear demand functions of the form $p = a - bq$ within the range $q \in [0, a/b]$ and $a > c_y$. We have already shown in the inelastic demand case that the ceiling is attained only when renewable energy is introduced. Thus the case $\eta_y < \eta(\tilde{y})$ with an increasing elasticity generalizes this property to the elastic demand case. Last, us now consider the case of a strictly decreasing elasticity.

5.3 The decreasing inverse demand elasticity case

We are going to show that if $\eta'(q) < 0$, it is possible that the economy experiences several phases at the ceiling before the introduction of renewable energy. It will appear that such a feature depends critically from the properties of higher order derivatives of the gross surplus function, more precisely from its four order derivative. First remark that since:

$$\begin{aligned}\eta'(q) < 0 &\iff -\frac{u'''(q)q}{u'(q)} - \frac{u''(q)}{u'(q)} + \frac{u''(q)q}{u'(q)}\frac{u''(q)}{u'(q)} < 0 \\ &\iff -u'''(q)q + (1+\eta(q))|u''(q)| < 0 \ ,\end{aligned}$$

$u'''(q) > 0$ appears as a necessary condition to get $\eta'(q) < 0$. Furthermore:

$$\begin{aligned}q\eta'(q) &= -\frac{u'''(q)q^2}{u'(q)} - \frac{u''(q)q}{u'(q)} + \frac{u''(q)q}{u'(q)}\frac{u''(q)q}{u'(q)} \\ &= -\frac{u'''(q)q^2}{u'(q)} + \eta(q) + \eta^2(q) \qquad (5.8)\end{aligned}$$

The existence of several ceiling phases requires that the function $\Phi(x)$ admits several inflexion points, being locally increasing and decreasing in the admissible domain $x \in [\tilde{y}, \infty)$. Next, from our previous computations:

$$\begin{aligned}\frac{d\Phi(x)}{dx} \geq 0 &\iff |u''(x)|(\Phi_0 - \eta(x)) \leq -u'(x)\eta'(x) \\ &\iff \frac{|u''(x)|x}{u'(x)}(\Phi_0 - \eta(x)) \leq -x\eta'(x) \\ &\iff \eta(x)(\Phi_0 - \eta(x)) + x\eta'(x) \leq 0\end{aligned}$$

Taking (5.8) into account, this is equivalent to:

$$\begin{aligned}\frac{d\Phi(x)}{dx} \geq 0 &\iff \eta(x)(\Phi_0 - \eta(x)) - \frac{u'''(x)x^2}{u'(x)} + \eta(x) + \eta^2(x) \leq 0 \\ &\iff \eta(x)(1+\Phi_0) - \frac{u'''(x)x^2}{u'(x)} \leq 0 \\ &\iff \frac{|u''(x)|x}{u'(x)}(1+\Phi_0) - \frac{u'''(x)x^2}{u'(x)} \leq 0 \\ &\iff \sigma(x) \equiv \frac{u'''(x)x}{|u''(x)|} \geq 1 + \Phi_0 = \frac{\rho+\alpha}{\alpha(1-\theta)} \ .\end{aligned}$$

Thus we conclude that: $\Phi'(x) \geq 0 \iff \sigma(x) \geq (\rho+\alpha)/[\alpha(1-\theta)]$. It appears that the possibility of several ceiling phases before the introduction of renewable energy depends upon the behavior of the function $\sigma(x)$, that is of the elasticity of the derivative of the energy price with respect to the energy quantity, $dp(q)/dq$, $p = u'(q) - c_x$. Differentiating $\sigma(x)$ results in:

$$\frac{d\sigma(x)}{dx} = \frac{u''''(x)x}{|u''(x)|} + (1+\sigma(x))\frac{u'''(x)}{|u''(x)|} \ .$$

Thus if the fourth order derivative of $u(x)$ is positive, $\sigma'(x) > 0$ and $\Phi(x)$ admits at most one inflexion point. This implies the existence of at most one ceiling phase before the introduction of renewable energy. But if the fourth order derivative $u''''(x) < 0$, the function $\sigma(x)$ may be non monotonous, introducing the possibility of several inflexion points of the $\Phi(x)$ function and hence of several ceiling phases before t_y.

6 Conclusion

We have shown how should evolve through time the optimal extraction of a polluting non renewable resource and the production of a clean renewable substitute when the problem of the atmospheric CO_2 resilience is seriously taken into account and when overshooting some critical level of the atmospheric carbon stock would trigger catastrophic damages. The production paths, the mining rent trajectory of the non renewable resource, and the path of the carbon tax having to be levied to decentralize the optimum, are all relatively easy to characterize as far as the extraction and production average costs are constant and the energy demand not too elastic within the pertinent price range. In particular in this case there exists one and only one time period at the ceiling of either finite or infinite time duration according to the size of the non renewable endowment is either small or large. On the contrary, if the demand is sufficiently elastic there may exist paths with several disconnected phases at the ceiling, at least for sufficiently large endowments.

The model can be developed in different directions. We have pointed out that most discontinuities in the production paths are eliminated with increasing marginal costs. But three problems are probably the most urging to be investigated. The first one is the problem of optimal use of several kinds of polluting non renewable resources having different pollution powers. Early investigations by Chakravorty *et alii* with a model without permanent atmospheric CO_2, have shown that along some optimal paths the use of the most and the least polluting resources must alternate even if the both resources have the same exploitation costs. The second problem is the optimality of the carbon capture and sequestration policy. As shown in Amigues *et alii* (2010) a sensible model should explicitly take into account that the emissions of the same resource are more or less easy to capture according to its use. The last problem is the well known problem of the allocation of the research efforts much more acute in the present context.

References

Amigues J. P. and M. Moreaux, (2011), The atmospheric carbon resilience problem under increasing marginal costs of the polluting non renewable resource and of its clean substitute, Work in progress.

Amigues J. P., Lafforgue G. and M. Moreaux, (2010), Optimal capture and sequestration from the carbon emission flow and from the atmospheric carbon stock with heterogenous energy consuming sectors, *Toulouse School of Economics Working paper* 10-163.

Archer, D. (2005), Fate of fossil fuel CO_2 in geologic time, *Journal of Geophysical Research*, Vol 110, C09S05, doi: 10.1029/2004JC0002625, 2005.

Archer, D., Eby M., Brovkin V. Ridgwell A., Cao L., Mikolajewicz U., Caldeira K., Matsumoto K., Munhoven G., Montenegro A. and K. Tokos, (2009), Atmospheric lifetime of fossil carbon dioxide, *Annual Review of Earth and Planetary Sciences*, 37, 117-134.

Chakravorty, U, Magne, B. and M. Moreaux, (2006) A Hotelling model with a ceiling on the stock of pollution, *Journal of Economic Dynamics and Control*, Vol 30(12), 2875-2904.

Chakravorty U, Moreaux M. and M. Tidball, (2008), Ordering the extraction of polluting nonrenewable resources, *American Economic Review*, Vol 98(3), 1128-1144.

Clarke, H. R. and Reed, (1994), Consumption/ pollution tradeoffs in an environment vulnerable to pollution-related catastrophic collapse, *Journal of Economic Dynamics and Control*, 18, 991-1010.

Cropper, M. L. (1976), Regulating activities with catastrophic environmental effects, *Journal of Environmental Economics and Management*, 3, 1-15.

Farzin, Y. H. and O. Tahvonen, (1996), Global carbon cycle and the optimal time path of a carbon tax, *Oxford Economic papers, New Series*, 48, 515-536.

Forster, B. A., (1975), Optimal pollution with a non constant exponential rate of decay, *Journal of Environmental Economics and Management*, 2, 1-6.

Gerlagh, R., (2009), Too much oil, *mimeo, Tilburg University*, also available as *Nota di Lavoro 14-2010, Fondazione Eni Enrico Mattei.*

Hoel , M. and S. Kverndokk, (1996), Depletion of fossil fuels and the impacts of global warming, *Resources and Energy Economics*, 18, 115-136.

Maier-Raimer, E. and K. Hasselman, (1987), Transport and storage of CO_2 in the Ocean - An inorganic ocean-circulation carbon cycle model, *Climate Dynamics*, 2, 63-90.

Tahvonen, O., (1997), Fossil fuel, stock externalities and the backstop technology, *Canadian Journal of Economics*, 30, 855-874.

Tahvonen, O and S. Salo, (1996), Non convexities in optimal pollution accumulation, *Journal of Environmental Economics and Management*, 31, 160-177.

Tahvonen, O. and C. Withagen, (1996), Optimality of irreversible pollution accumulation, *Journal of Economic Dynamics and Control*, 20, 1775-1795.

Ulph, A. and D. Ulph, (1994), The optimal time path of a carbon tax, *Oxford Economic Papers, New Series*, 46, 857-869.

Withagen, C., (1994), Pollution and exhaustibility of fossil fuels, *Resources and Energy Economics*, 16, 235-242.

APPENDIX

A.1 Appendix 1: Proof of Proposition 2

Assume that both energy sources are used during a time phase when the ceiling constraint does not bind, $\underline{\Delta} \equiv [t_1, t_2]$, $t_1 < t_2$ and $S(t) + Z(t) < \bar{G}$, $t \in \underline{\Delta}$. Note that this requires that $X(t) > 0$, $t \in [t_1, t_2)$. Then $x(t) > 0$ and $y(t) > 0$, $t \in \underline{\Delta}$ imply that $\gamma_x(t) = 0$ and $\gamma_y(t) = 0$, $t \in \underline{\Delta}$ and we get from (3.9) and (3.10):

$$c_y - c_x \;=\; \lambda_X(t) + \zeta(1-\theta)\lambda_S(t) + \zeta\theta\lambda_Z(t) \;\;,\; t \in \underline{\Delta}$$

Making use of the above remarks 1-3, this is equivalent to:

$$c_y - c_x \;=\; \lambda_X(t_1)e^{\rho(t-t_1)} + \zeta(1-\theta)\lambda_S(t_1)e^{\rho(t-t_1)} + \zeta\theta\lambda_Z(t_1)e^{(\rho+\alpha)(t-t_1)} \;,\; t \in \underline{\Delta},$$

which cannot hold within any non degenerate time interval and proves claim 1 of the Proposition. Note that this conclusion applies both in the rare coal case (where $\lambda_X(t)$ would be strictly positive) and in the rare capacity case (where $\lambda_X(t)$ would be zero).

Assume the existence of some non degenerate time interval $\Delta_y \equiv [t_1, t_2]$, $t_1 < t_2$, during which the ceiling constraint does not bind, $S(t) + Z(t) < \bar{G}$, $t \in \Delta_y$ and $x(t) = 0$ while $y(t) > 0$, $t \in \Delta_y$. Then $\gamma_y(t) = 0$ and $\gamma_x(t) \geq 0$ is given by, making use once again of (3.9), (3.10):

$$\gamma_x(t) \;=\; c_x - c_y + \lambda_X(t_1)e^{\rho(t-t_1)} + \zeta(1-\theta)\lambda_S(t_1)e^{\rho(t-t_1)} + \zeta\theta\lambda_Z(t_1)e^{(\rho+\alpha)(t-t_1)} \;,\; t \in \Delta_y \;,$$

which is clearly a strictly increasing time function. Since coal extraction has been interrupted, the atmospheric pollution stock decreases through time and $S(t_2) + Z(t_2) < \bar{G}$. Since joint exploitation of the resources is not optimal when the ceiling constraint does not bind, the time phase Δ_y can only expand to infinity or be followed by a time phase of only coal extraction. But in such a case, either the energy price $p(q)$ must make an upward jump at time t_2 to the level $\lambda_X(t_1)e^{\rho(t_2-t_1)} + \zeta(1-\theta)\lambda_S(t_1)e^{\rho(t_2-t_1)} + \zeta\theta\lambda_Z(t_1)e^{(\rho+\alpha)(t_2-t_1)}$, either $\lambda_X(t)$, $\lambda_S(t)$ and $\lambda_Z(t)$ have to make downward jumps to the level $c_y - c_x$, both jumps which cannot happen along an optimal path. But if the use of only renewable energy extends towards infinity while coal is not exhausted, the ceiling constraint will never bind anymore implying through the remarks 2 and 3, that $\lambda_S(t) = \lambda_Z(t) = 0$, $t \in \Delta_y$ and thus $\lambda_X(t_1) > 0$ since $c_x < c_y$ and $\gamma_x(t) \geq 0$, hence contradicting the transversality condition (3.18). Hence it is never optimal to use only renewable energy when the ceiling constraint does not bind. Since joint exploitation is excluded through claim 1, we conclude that with positive coal endowments, only coal will be used to provide the energy needs when the ceiling constraint is not tight, proving claim 2 of the Proposition.

From claims 1 and 2, either the ceiling constraint binds and in such a case the remark 4 has shown that the use of coal should strictly decrease through time, or either the ceiling constraint is lax and only coal use satisfies the energy demand. Since $q(t) = x(t)$ during such a time phase, (3.9) is equivalent to:

$$p(x(t)) \quad = \quad c_x + \lambda_X(t_1)e^{\rho(t-t_1)} + \zeta(1-\theta)\lambda_S(t_1)e^{\rho(t-t_1)} + \zeta\theta\lambda_Z(t_1)e^{(\rho+\alpha)(t-t_1)} \ .$$

This shows that $p(q)$ should strictly increase over time and thus $x(t)$ should decrease under our demand assumptions. In the particular case of an inelastic demand where the energy sector should supply a constant energy consumption rate $\bar{q}$, coal use rate would decrease during a ceiling phase or fulfill the whole energy demand, that is be given by the constant level $\bar{q}$ during any phase of only coal exploitation. This proves claim 3 and completes the proof of the Proposition 2.

A.2 Appendix 2: Proof of Proposition 5

Consider the case of Appendix 1 with the same notations.
In order that the F.o.c's (3.9)-(3.11) have to be satisfied, we must have :

$$u'(x^*(t) + y^*(t)) = c_y = c_x + (1-\theta)\zeta\lambda_S(t) + \theta\zeta\lambda_Z(t) \qquad \text{(A.2.1)}$$

together with :

$$\gamma_x(t) = 0 \quad \text{and} \quad \gamma_y(t) = 0.$$

Hence $(1-\theta)\lambda_S(t)+\theta\lambda_Z(t)$ must be constant from t_G onwards. Let us denote by $\mu(t)$ this weighted sum of λ_S and λ_Z :

$$(1-\theta)\lambda_S(t)+\theta\lambda_Z(t)\equiv\mu(t)=\frac{c_y-c_x}{\zeta}\ . \tag{A.2.2}$$

Let us define $\delta(t)$ as the difference between λ_Z and λ_S :

$$\delta(t)\equiv\lambda_S(t)-\lambda_Z(t),$$

so that :

$$\lambda_S(t)=\mu(t)+\theta\delta(t)\quad\text{and}\quad\lambda_Z(t)=\mu(t)-(1-\theta)\delta(t). \tag{A.2.3}$$

From (3.14) and (3.15) determining the dynamics of respectively λ_S and λ_Z, and the above expression of λ_Z, we get :

$$\dot{\delta}(t)=\dot{\lambda}_S(t)-\dot{\lambda}_Z(t)=-\alpha\mu(t)+[\rho+(1-\theta)\alpha]\delta(t). \tag{A.2.4}$$

Integrating (A.2.4) over $[t_G,t)$, while taking into account that $\mu(t)$ is constant and given by (A.2.2), we obtain, assuming that $\dot{\delta}(t)\neq 0$:

$$\delta(t)=A(t)\{\delta(t_G)-\frac{\alpha(c_y-c_x)}{\zeta[\rho+(1-\theta)\alpha]}\,[1-B(t)]\ ,\ t\geq t_G \tag{A.2.5}$$

where

$$\begin{aligned}A(t)&=exp\,\{\rho+(1-\theta)\alpha(t-t_G)\}\\ B(t)&=exp\,\{-(\rho+(1-\theta)\alpha(t-t_G))\}.\end{aligned}$$

This is implying that either λ_S or λ_Z must be negative from some time onwards. Thus we must have $\dot{\delta}(t)=0, t\geq t_G$, that is :

$$\delta(t)\ =\ \frac{\alpha(c_y-c_x)}{\zeta[\rho+(1-\theta)\alpha]}\ ,$$

which in turn implies that :

$$\lambda_S(t)=\frac{(\rho+\alpha)(c_y-c_x)}{\zeta[\rho+(1-\theta)\alpha]}\quad\text{and}\quad\lambda_Z(t)=\frac{\rho(c_y-c_x)}{\zeta[\rho+(1-\theta)\alpha]}\ ,t\geq t_G.$$

A.3 Appendix 3 : Comparative dynamics in the inelastic demand case with rare coal

A.3.1 Linearization of the system (4.12), (4.13)

First denote by Z_G the amount of pollution inside the renewable reservoir evaluated at $t = \underline{t}_G$, that is $Z_G = Z(\underline{t}_G)$. Differentiating with respect to $\underline{t}_G$, we get:

$$\begin{aligned}
\frac{\partial Z_G}{\partial \underline{t}_G} &= -\alpha e^{-\alpha \underline{t}_G} Z^0 + \frac{\theta\zeta\bar{q}}{\alpha}\alpha e^{-\alpha \underline{t}_G} \\
&= -\alpha e^{-\alpha \underline{t}_G} Z^0 + \frac{\theta\zeta\bar{q}}{\alpha}\alpha e^{-\alpha \underline{t}_G} + \theta\zeta\bar{q} - \frac{\theta\zeta\bar{q}}{\alpha}\alpha \\
&= -\alpha e^{-\alpha \underline{t}_G} Z^0 - \alpha\frac{\theta\zeta\bar{q}}{\alpha}(1 - e^{-\alpha \underline{t}_G}) + \theta\zeta\bar{q} \\
&= -\alpha Z_G + \theta\zeta\bar{q} \ . \qquad \text{(A.3.1)}
\end{aligned}$$

Furthermore it is immediately checked that:

$$\begin{aligned}
\frac{\partial Z_G}{\partial Z^0} &= e^{-\alpha \underline{t}_G} > 0 \qquad \text{(A.3.2)} \\
\frac{\partial Z_G}{\partial \bar{q}} &= \frac{\theta\zeta}{\alpha}(1 - e^{-\alpha \underline{t}_G}) > 0 \qquad \text{(A.3.3)}
\end{aligned}$$

Thus differentiating the stock condition (4.12) we obtain:

$$dX^0 - A_X^q d\bar{q} - A_X^Z dZ^0 = I_X d\underline{t}_G + J_X d\bar{t}_G$$

where:

$$
\begin{aligned}
A_X^q &\equiv \underline{t}_G + \frac{\partial Z_G}{\partial \bar{q}} \frac{1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} \\
&= \underline{t}_G + \frac{\theta}{\alpha(1-\theta)}(1-e^{-\alpha \underline{t}_G})(1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}) > 0 \\
A_X^Z &\equiv \frac{\partial Z_G}{\partial Z^0} \frac{1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} \\
&= e^{-\alpha \underline{t}_G} \frac{1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} > 0 \\
I_X &\equiv \bar{q} + \frac{\partial Z_G}{\partial \underline{t}_G} \frac{1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} - \frac{\alpha Z_G}{\zeta} e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)} \\
&= \bar{q} + (\theta\zeta\bar{q} - \alpha Z_G) \frac{1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} - \frac{\alpha Z_G}{\zeta} e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)} \\
&= \frac{\bar{q}}{\zeta(1-\theta)} \left[\zeta(1-\theta) + \theta\zeta(1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)})\right] \\
&\qquad - \frac{\alpha Z_G}{\zeta(1-\theta)} \left[1-e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)} + (1-\theta)e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}\right] \\
&= \frac{1-\theta e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)}}{\zeta(1-\theta)} \left[\zeta\bar{q} - \alpha Z_G\right] \equiv k\left[\zeta\bar{q} - \alpha Z_G\right] \\
J_X &= \frac{\alpha Z_G}{\zeta} e^{-\alpha(1-\theta)(\bar{t}_G-\underline{t}_G)} = x(\bar{t}_G) > 0
\end{aligned}
$$

Next, differentiating the ceiling condition (4.13) results in:

$$
\begin{aligned}
d\bar{G} - dS^0 - A_G^q d\bar{q} - e^{\alpha \underline{t}_G} dZ^0 &= \left[\zeta(1-\theta)\bar{q} + \frac{\partial Z_G}{\partial \underline{t}_G}\right] d\underline{t}_G \\
&= \left[\zeta(1-\theta)\bar{q} + \zeta\theta\bar{q} - \alpha Z_G\right] d\underline{t}_G = \left[\zeta\bar{q} - \alpha Z_G\right] d\underline{t}_G
\end{aligned}
$$

where:

$$
A_G^q \equiv \zeta(1-\theta)\underline{t}_G + \frac{\theta\zeta}{\alpha}(1-e^{-\alpha \underline{t}_G}) > 0
$$

Let $Q \equiv \zeta\bar{q} - \alpha Z_G$. Since $\bar{q} > \alpha Z_G/\zeta$ by construction we get $Q > 0$. In matrix form, the linearized system is thus:

$$
\begin{bmatrix} kQ & x(\bar{t}_G) \\ Q & 0 \end{bmatrix} \begin{bmatrix} d\underline{t}_G \\ d\bar{t}_G \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} dX^0 + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dS^0 - \begin{bmatrix} A_X^Z \\ e^{-\alpha \underline{t}_G} \end{bmatrix} dZ^0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d\bar{G} - \begin{bmatrix} A_X^q \\ A_G^q \end{bmatrix} d\bar{q}
$$

A.3.2 Comparative dynamics

The determinant of the system is $\Delta = -Qx(\bar{t}_G) < 0$. Thus $(\underline{t}_G, \bar{t}_G)$ is uniquely determined through the system of conditions (4.12)-(4.13). Next applying Cramer rule we obtain:

	$d\underline{t}_G$	$d\bar{t}_G$
dX^0	0	$\frac{1}{x(\bar{t}_G)} > 0$
dS^0	$-\frac{1}{Q} < 0$	$\frac{k}{x(\bar{t}_G)} > 0$
dZ^0	$-\frac{A_G^Z}{Q} < 0$	$\frac{1}{\zeta x(\bar{t}_G)} e^{-\alpha(\theta \underline{t}_G + (1-\theta)\bar{t}_G)} > 0$
$d\bar{G}$	$\frac{1}{Q} > 0$	$-\frac{k}{x(\bar{t}_G)} < 0$
$d\bar{q}$	$-\frac{A_G^q}{Q} < 0$	$-\frac{\theta\zeta}{\alpha Z(\underline{t}_G)} \left[\underline{t}_G - \frac{1 - e^{-\alpha \underline{t}_G}}{\alpha} \right] < 0$

Furthermore for $H \in \{X^0, S^0, Z^0, \bar{G}, \bar{q}\}$:

$$\begin{aligned}
\frac{\partial \lambda_{Z0}}{\partial H} &= \frac{\partial \lambda_{Z0}}{\partial \underline{t}_G} \frac{\partial \underline{t}_G}{\partial H} + \frac{\partial \lambda_{Z0}}{\partial \bar{t}_G} \frac{\partial \bar{t}_G}{\partial H} \\
\frac{\partial \lambda_{S0}}{\partial H} &= \frac{\partial \lambda_{S0}}{\partial \underline{t}_G} \frac{\partial \underline{t}_G}{\partial H} + \frac{\partial \lambda_{S0}}{\partial \bar{t}_G} \frac{\partial \bar{t}_G}{\partial H} \\
\frac{\partial \lambda_Z(\underline{t}_G)}{\partial H} &= \frac{\partial \lambda_Z(\underline{t}_G)}{\partial \underline{t}_G} \left[\frac{\partial \underline{t}_G}{\partial H} - \frac{\partial \bar{t}_G}{\partial H} \right] \\
\frac{\partial \lambda_S(\underline{t}_G)}{\partial H} &= \frac{\partial \lambda_S(\underline{t}_G)}{\partial \underline{t}_G} \left[\frac{\partial \underline{t}_G}{\partial H} - \frac{\partial \bar{t}_G}{\partial H} \right] \\
\frac{\partial \lambda_{X0}}{\partial H} &= -\rho \lambda_{X0} \frac{\partial \bar{t}_G}{\partial H} \, .
\end{aligned}$$

Thus the qualitative effects of variations of $(X^0, S^0, Z^0, \bar{G}, \bar{q})$ over the critical levels of the dual variables are given by:

	$d\lambda_{Z0}$	$d\lambda_Z(\underline{t}_G)$	$d\lambda_{S0}$	$d\lambda_S(\underline{t}_G)$	$d\lambda_{X0}$
dX^0	+	+	+	+	-
dS^0	+	+	+	+	-
dZ^0	+	+	+	+	-
$d\bar{G}$	-	-	-	-	+
$d\bar{q}$	?	?	?	?	+

A.4 Appendix 4: Proof of Proposition 6

From the expression of $\Phi'(x)$ the slope of the function Φ is given in absolute value by:

$$
\begin{aligned}
|\Phi'| &= \frac{1}{\zeta\theta}\left[|u''(x)|(\Phi_0 - \eta(x)) + u'(x)\eta'(x)\right] \\
&= \frac{|u''(x)|}{\zeta\theta}\left[\Phi_0 - \eta(x) + \frac{x\eta'(x)}{\eta(x)}\right] \qquad \text{(A.4.1)}
\end{aligned}
$$

Let $D(x)$ be the point derivative in absolute value of a trajectory solution of the differential system in (x, δ) evaluated along the curve $\Phi(x)$, that is when $\delta = \Phi(x)$.

$$
\begin{aligned}
D(x) &= \frac{\dot{\delta}}{|\dot{x}|}|_{\delta=\Phi(x)} = \frac{-\frac{\alpha}{\zeta}(u'(x) - c_x) + (\rho + (1-\theta)\alpha)\Phi(x)}{\frac{1}{|u''(x)|}\left[(\theta\alpha + \rho)(u'(x) - c_x) - \zeta\theta\alpha(1-\theta)\Phi(x)\right]} \\
&\equiv |u''(x)|(\rho + (1-\theta)\alpha)\left\{\frac{\Phi(x) - \varphi_\delta(x)}{N(x)}\right\}
\end{aligned}
$$

Making use of the definition of $\Phi(x)$, the denominator is equivalent to:

$$
\begin{aligned}
N(x) &= (\theta\alpha + \rho)(u'(x) - c_x) - \alpha(1-\theta)\left[\frac{\theta\alpha + \rho}{\alpha(1-\theta)}[u'(x) - c_x) + u''(x)x\right] \\
&= (1-\theta)\alpha|u''(x)|x \ . \qquad \text{(A.4.2)}
\end{aligned}
$$

Thus:

$$
D(x) = (\rho + (1-\theta)\alpha)\frac{(\Phi(x) - \varphi_\delta(x))}{(1-\theta)\alpha x} \qquad \text{(A.4.3)}
$$

Observe that since $\Phi(x) - \varphi_\delta(x)$ is a decreasing function of x within the range $x \in [\tilde{y}, \bar{x}_\delta]$, as shown before, $D(x)$ is also a decreasing function of x and $D(\bar{x}_\delta) = 0$. This implies that $\Phi'(x) - \varphi'_\delta(x) < 0$ and thus since $\phi'(x)$ and $\varphi'_\delta(x)$ are both strictly negative that $|\Phi'(x)| > |\varphi_\delta(x)|$. Thus a sufficient condition for the slope of the function $\phi(x)$ to dominate the slopes of the unconstrained trajectories evaluated at the $\Phi(x)$ border is:

$$
x|\Phi'(x)| > x|\varphi'_\delta(x)| \geq \frac{\rho + (1-\theta)\alpha}{(1-\theta)\alpha}(\Phi(x) - \varphi_\delta(x))
$$

The last inequality is equivalent to:

$$|u''(x)|x\frac{\alpha^2\theta(1-\theta)}{\zeta\theta[\rho+(1-\theta)\alpha]^2} \geq \frac{1}{\zeta\theta}\left[\Phi_0(u'(x)-c_x)-|u''(x)|x\right]$$
$$-\frac{\alpha\theta}{\zeta\theta[\rho+(1-\theta)\alpha]}(u'(x)-c_x)$$

Simplifying and rearranging we get:

$$|u''(x)|x\frac{\alpha^2\theta(1-\theta)+[\rho+(1-\theta)\alpha]^2}{\rho+(1-\theta)\alpha} \geq \frac{\rho(\rho+\alpha)}{\alpha(1-\theta)}(u'(x)-c_x) \ ,$$

which after dividing both sides by $u'(x)$ is equivalent to:

$$\eta(x) \geq \frac{\rho(\rho+\alpha)(\rho+(1-\theta)\alpha)}{\alpha^2\theta(1-\theta)+[\rho+(1-\theta)\alpha]^2}\left(1-\frac{c_x}{u'(x)}\right) \ .$$

Since $u'x) < c_y$ if $x > \tilde{y}$ and $\eta(x)$ is an increasing function of x, a sufficient condition for this inequality to be satisfied is:

$$\eta(x) > \eta(\tilde{y}) \geq \frac{\rho(\rho+\alpha)(\rho+(1-\theta)\alpha)}{\alpha^2\theta(1-\theta)+[\rho+(1-\theta)\alpha]^2}\left(1-\frac{c_x}{c_y}\right)$$

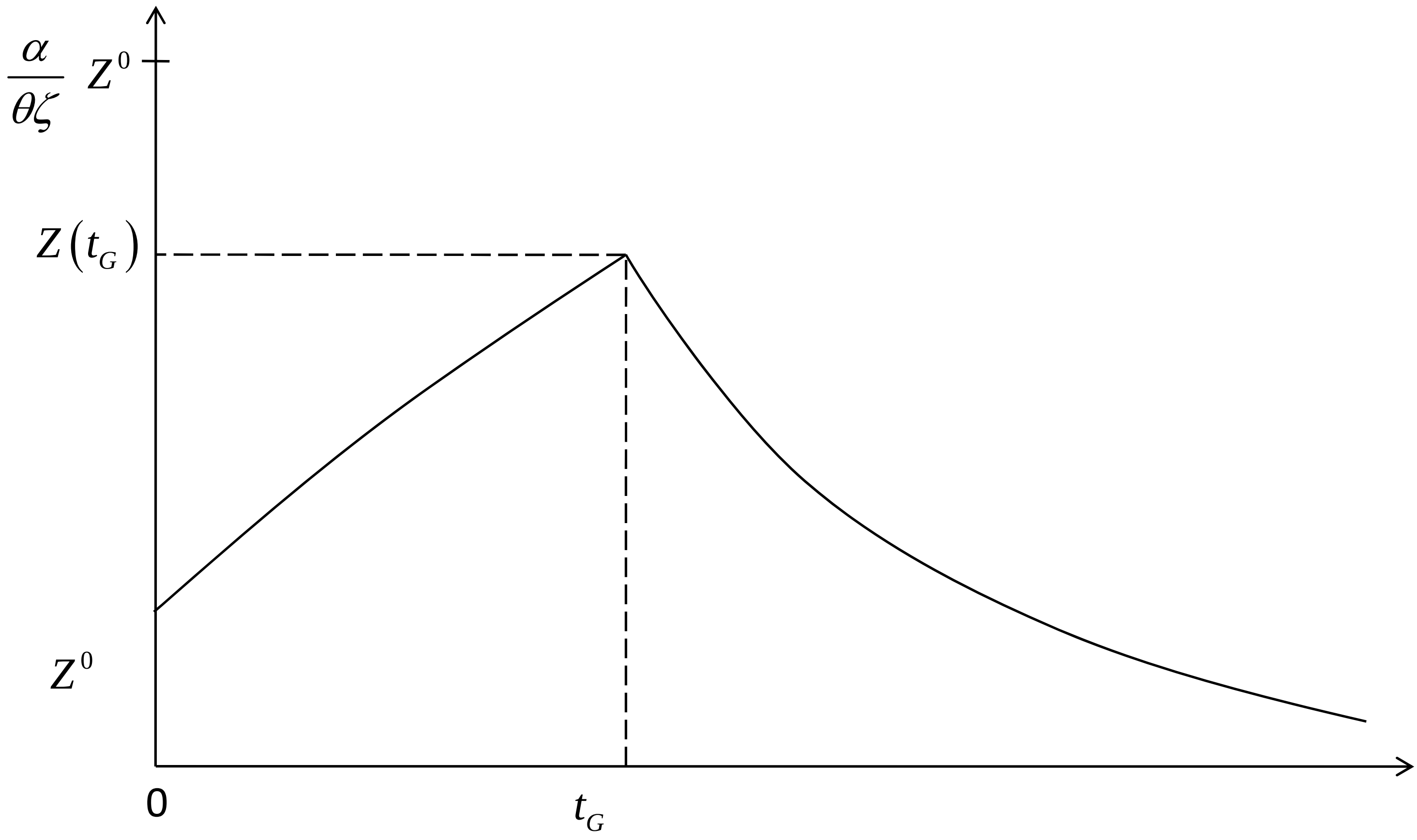


Figure 1 : Time profile of self-regenerating pollution stock. Case $\bar{q} > \frac{\alpha}{\theta\zeta} Z^0$

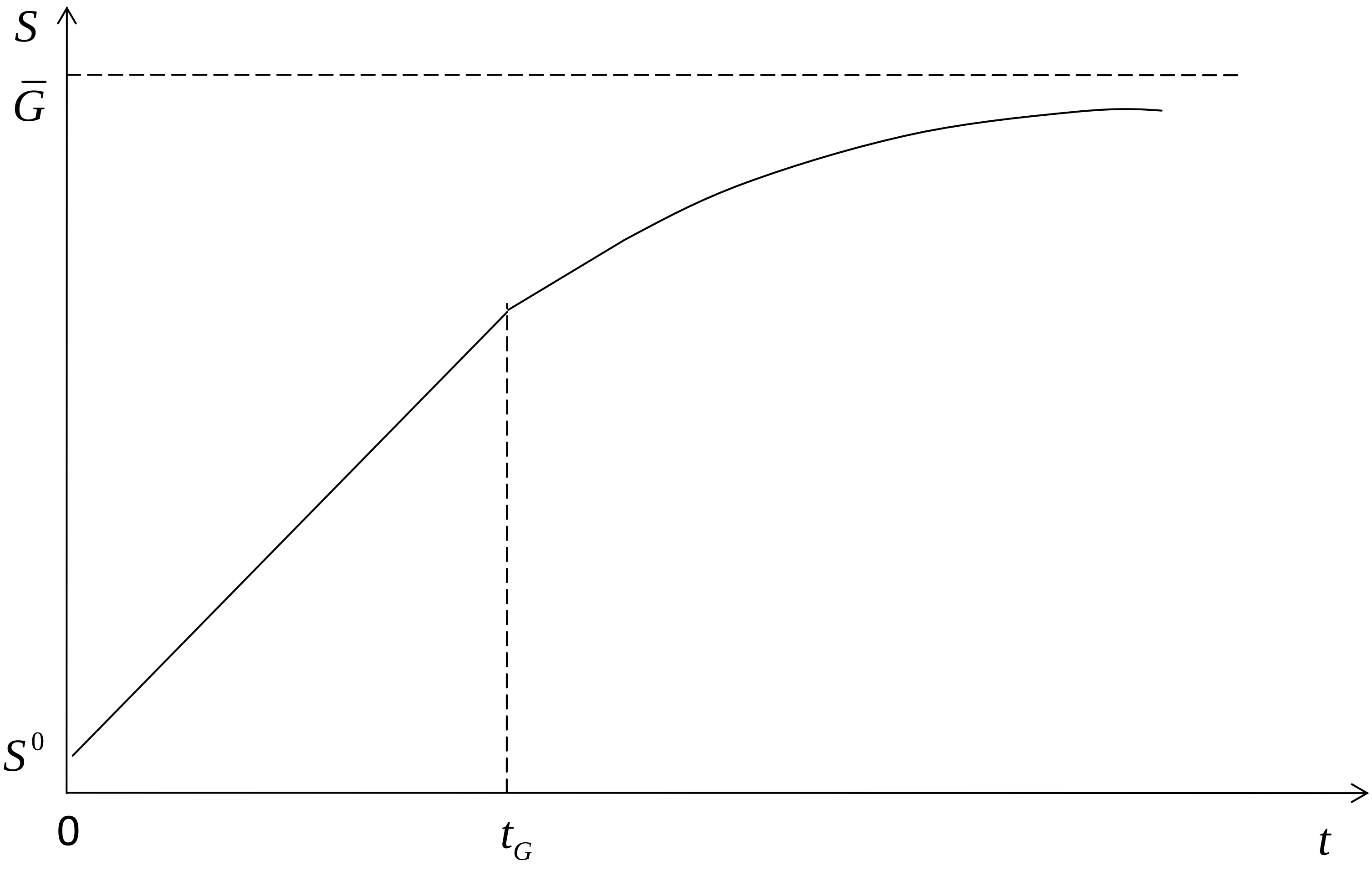


Figure 2 : Time profile of the permanent pollution stock

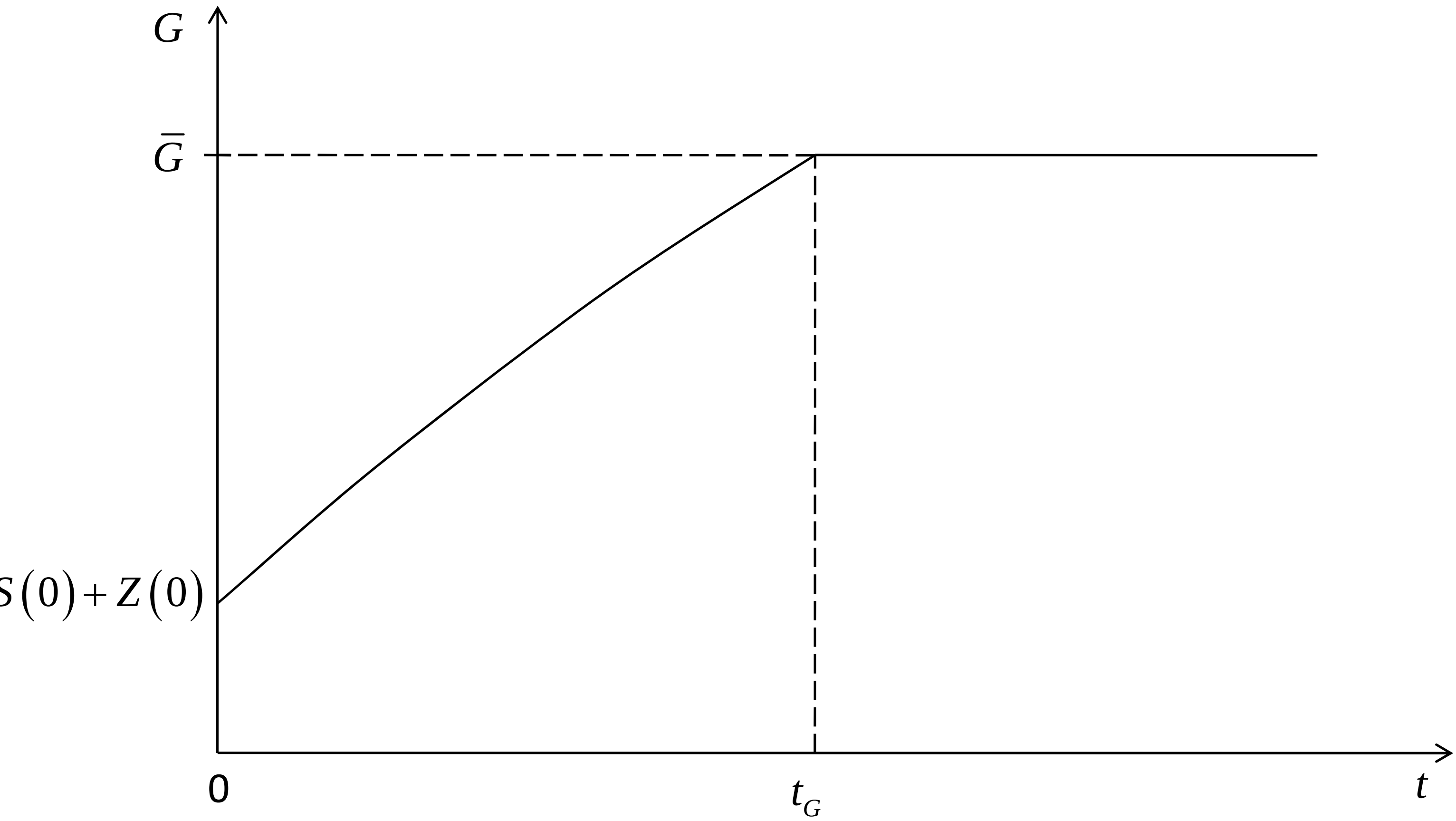


Figure 3 : Time profile of the atmospheric pollution

Case $\bar{q} > \frac{\alpha}{\theta\zeta} Z^0$

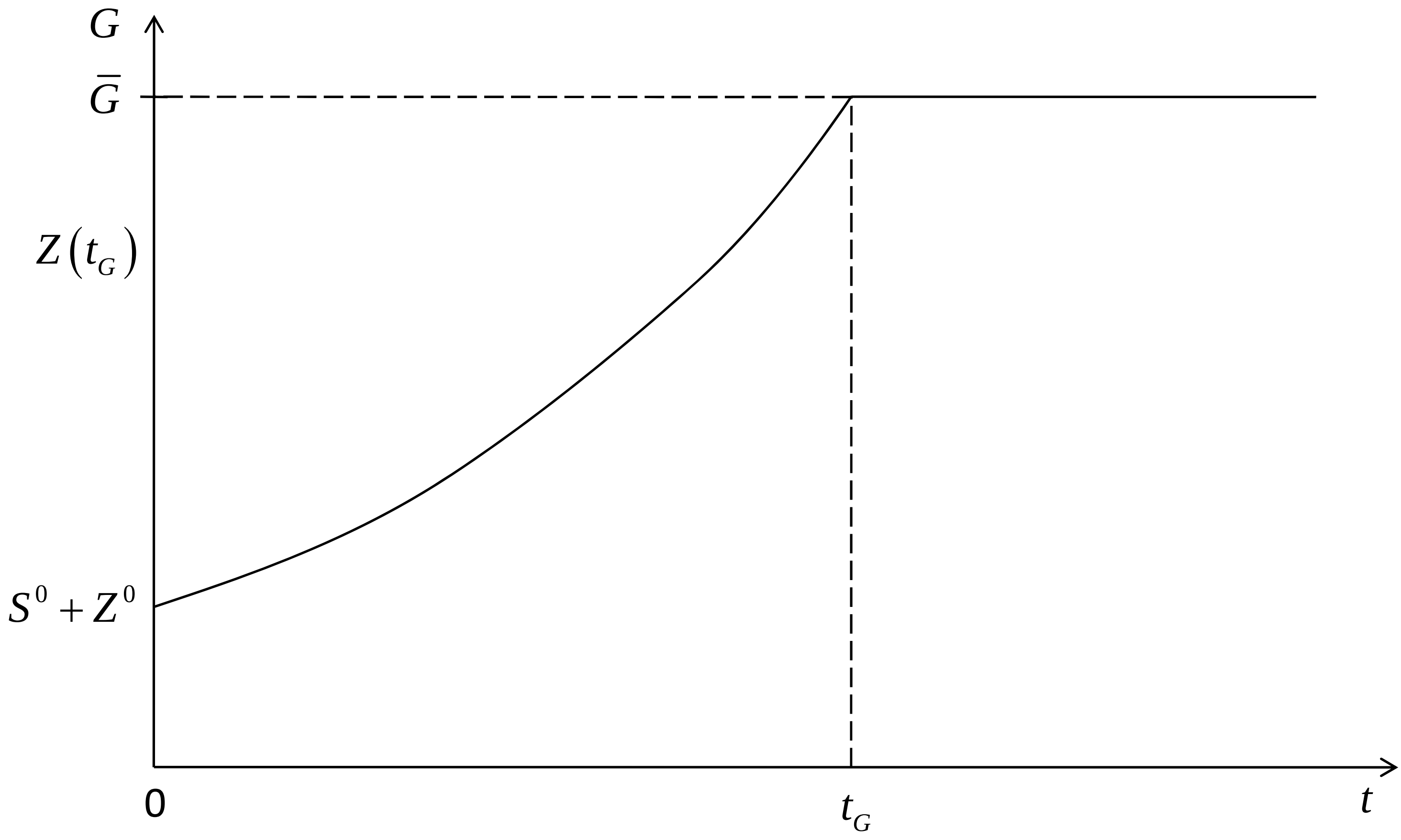


Figure 4 : Time profile of the atmospheric pollution

Case $\frac{\alpha}{\theta\zeta} \geq \bar{q} \geq \frac{\alpha}{\zeta} Z^0$

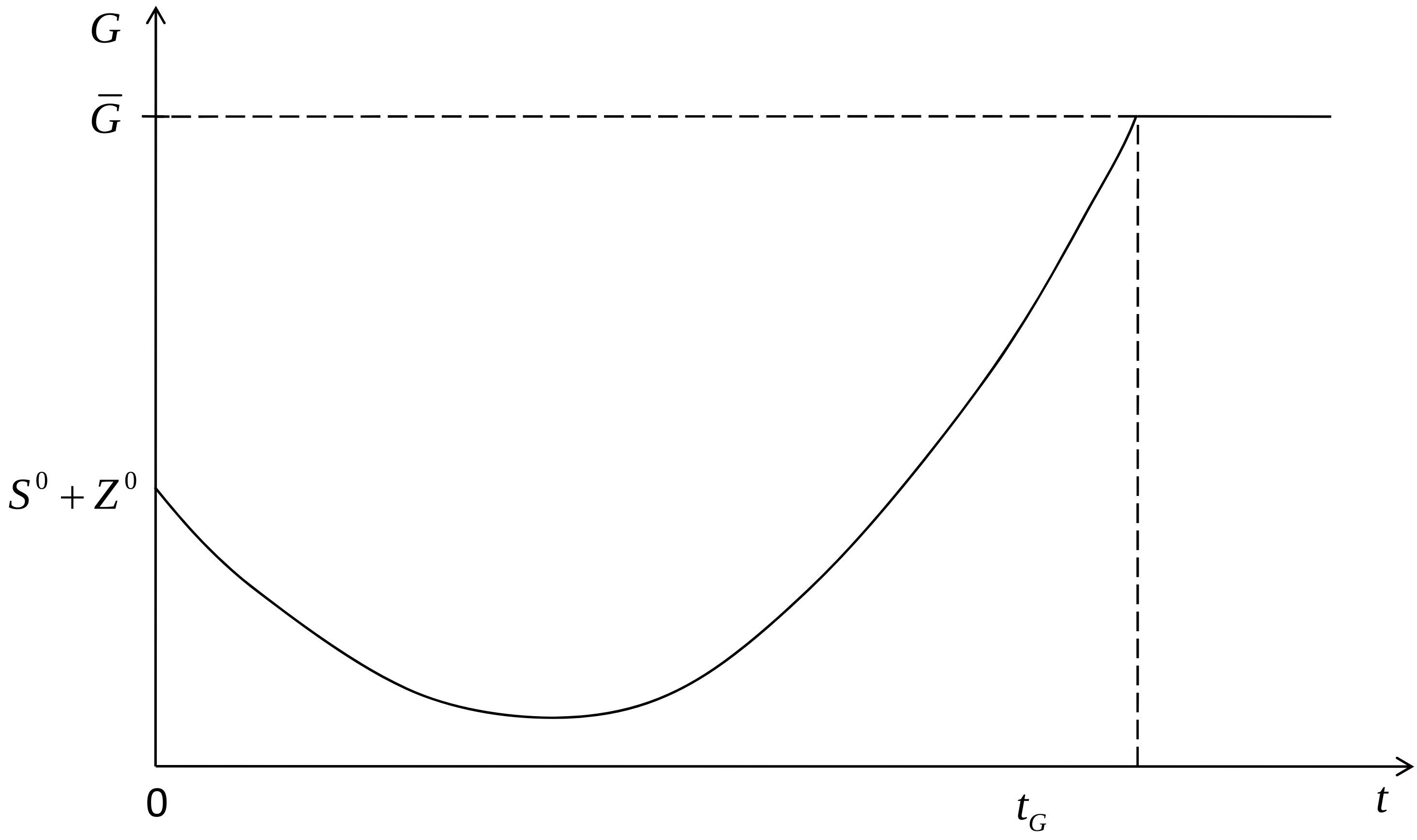
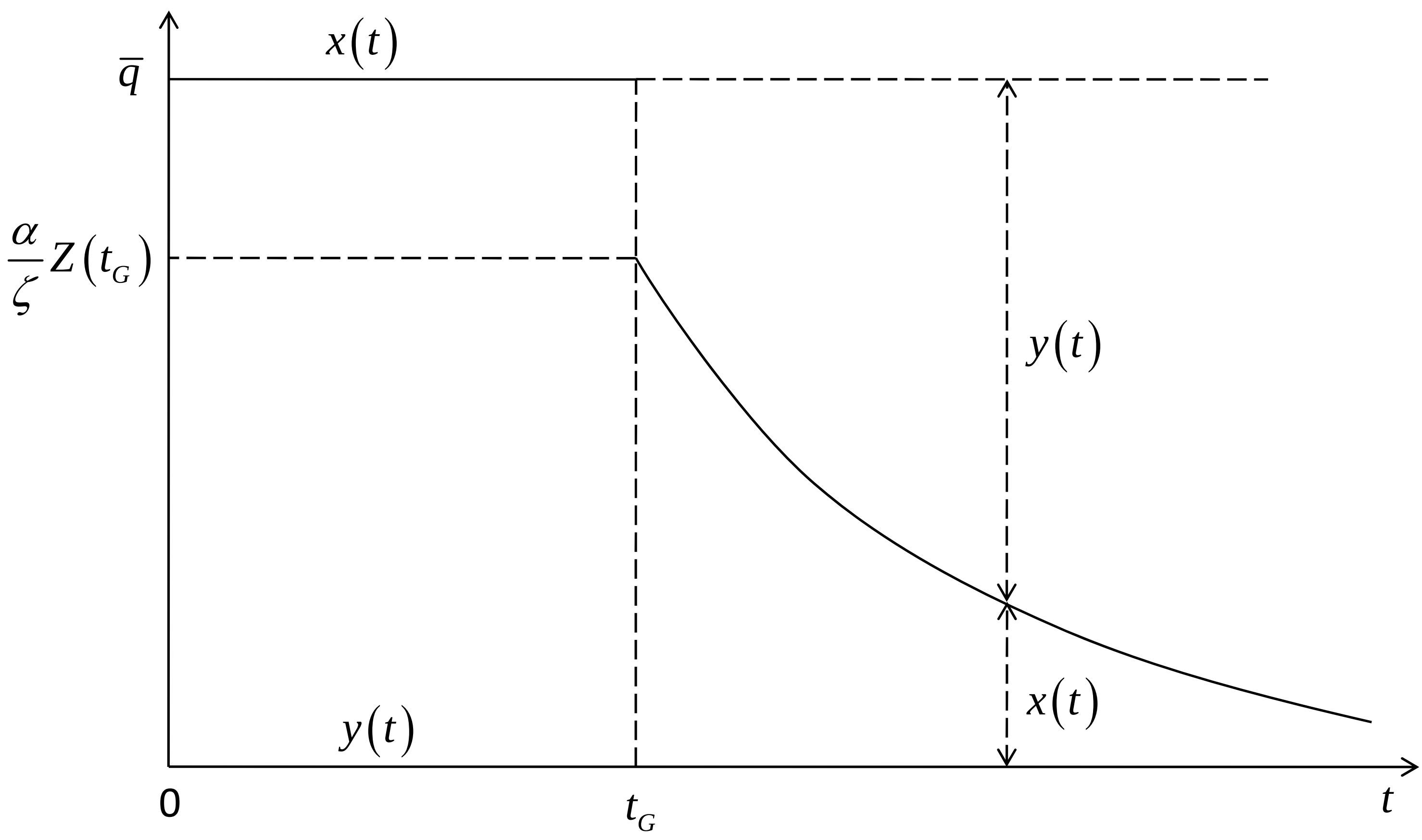


Figure 5 : Time profile of the atmospheric pollution

Case $\frac{\alpha}{\zeta} Z^0 > \bar{q}$



$y(t)$

Figure 6 : Time profile of the resource uses

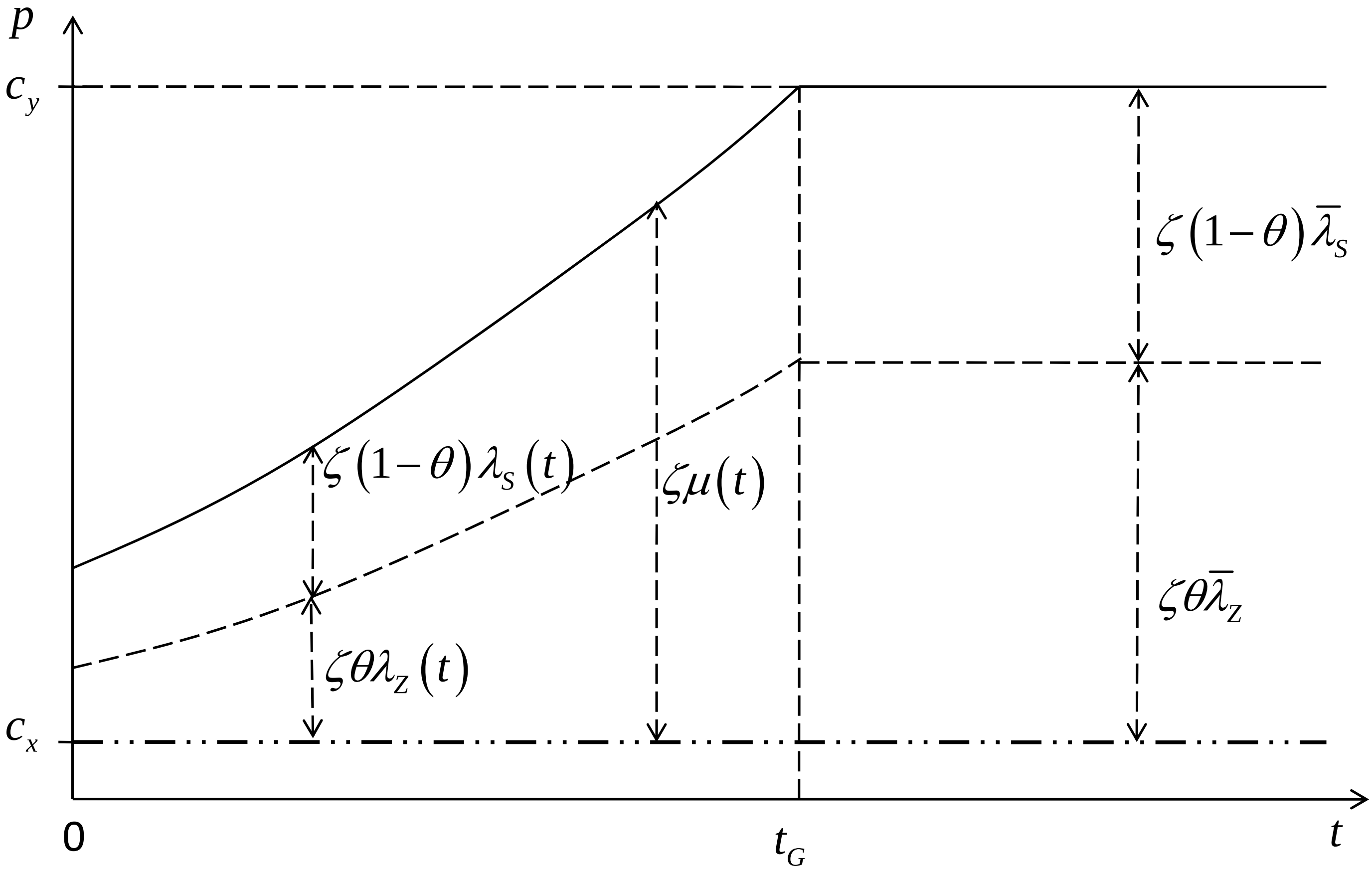


Figure 7 : Optimal time profile of the energy price and its components. Abundant coal case

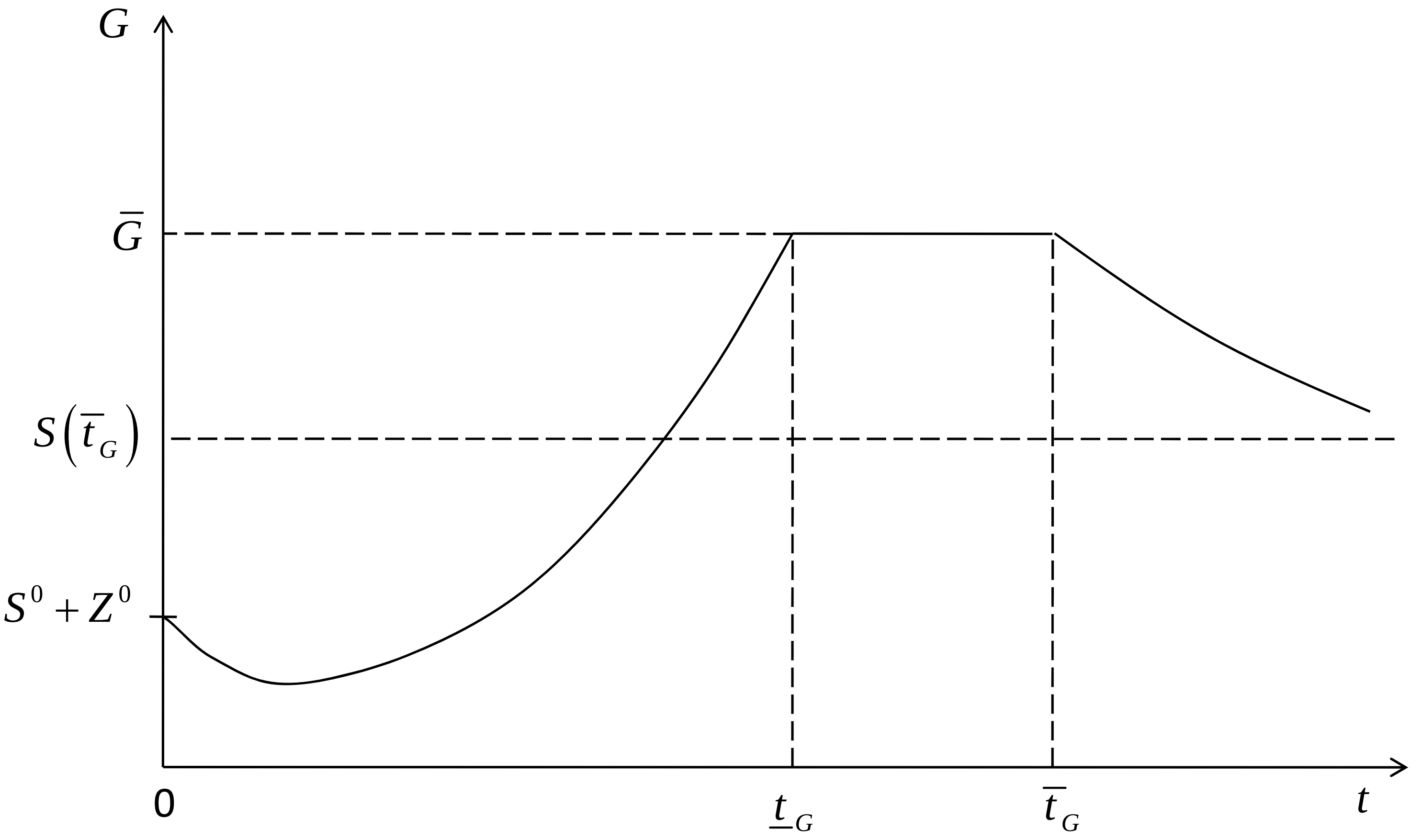


Figure 8 : Time profile of the atmospheric pollution stock.
Case of rare coal and $\frac{\alpha}{\zeta} Z^0 > \overline{q} > 0$

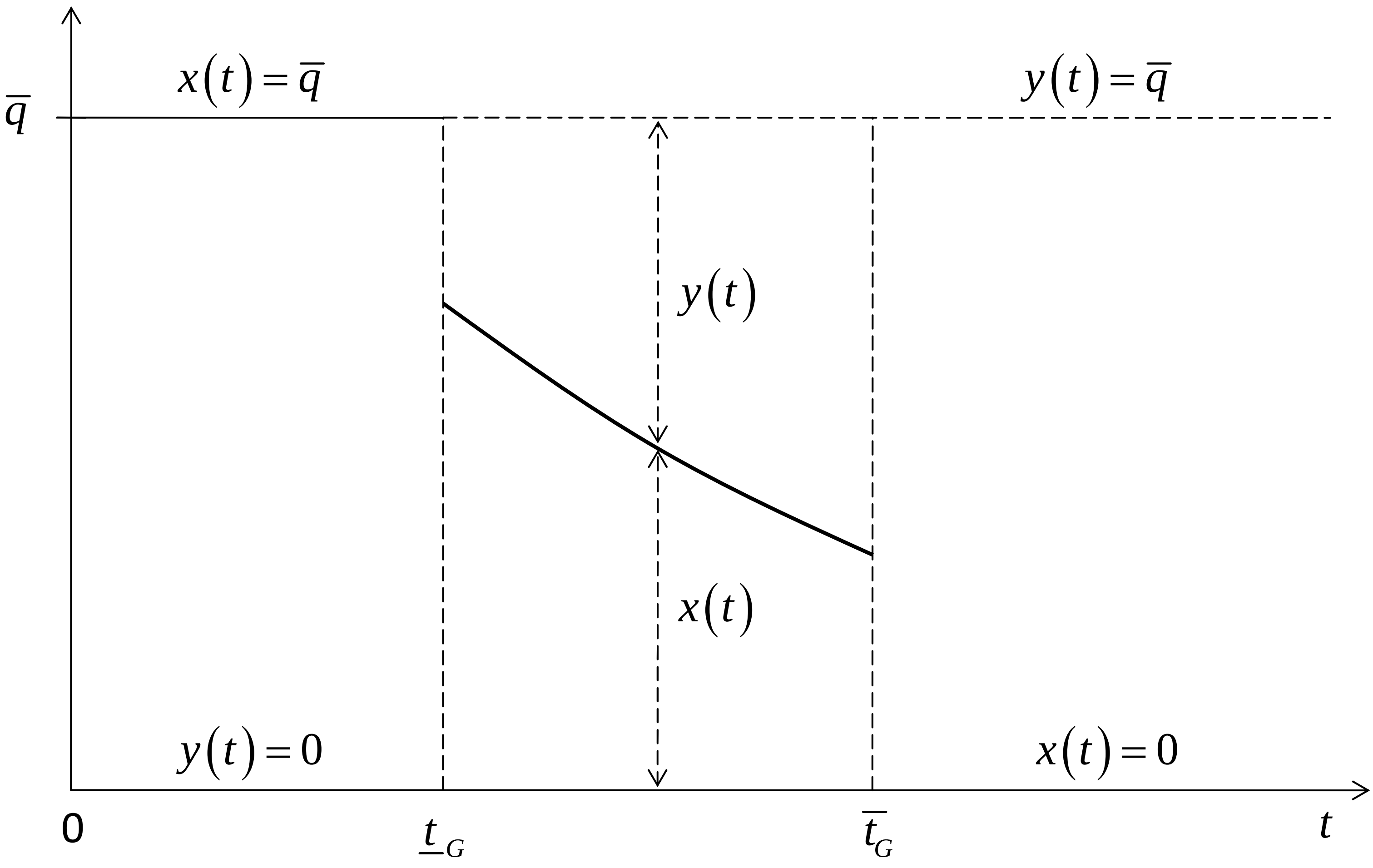


Figure 9 : Resources extraction trajectories.
Rare coal case.

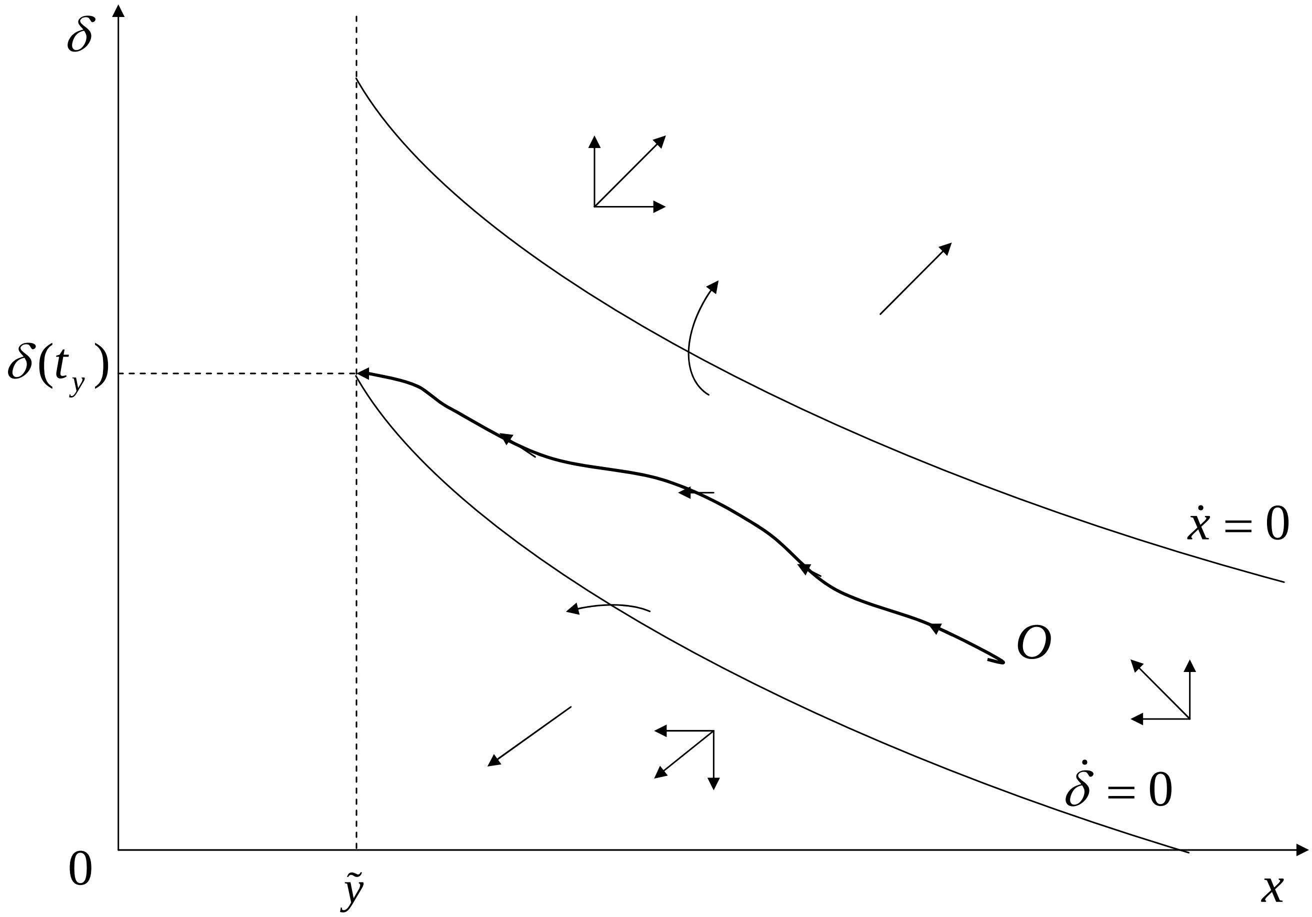


Figure 10 : Optimal dynamics in the (x, δ) plane.
Elastic demand case