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"Clustering in communication networks with Different-minded participants"

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Clustering in communication networks with different-minded participants.*

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Abstract

This paper examines how the structure of communication networks influences learning and social welfare when participants have different prior opinions and face uncertainty about an external state. We analyze a game in which players form links to exchange opinions on the state and reduce their uncertainty. The players hold imperfectly correlated subjective priors on the state. Therefore, their opinions transmit their private signals with frictions, termed interpretation noise. Network clustering facilitates learning by eliminating this interpretation noise. Therefore, the egalitarian efficient network is: a complete component if the interpretation noise is sufficiently high, and a flower otherwise. This network constitutes a Nash equilibrium. These findings establish a link between a key feature of social networks (clustering) and the quality of learning through network communication, offering a potential explanation for the prevalence of clustering in real-world social networks.

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1 Introduction

Social networks contribute to diffusion of information and behaviors (Durlauf, 2004; Goyal 2007; Jackson, 2008; Topa and Zenou, 2015). Examples in the literature include the adoption of new product or technology, health-related behaviors, vote, school performance and delinquent behavior. A growing evidence suggests that the speed and distance of diffusion through the network depend on its features (see Jackson 2014 for a systematic discussion).

One key characteristic of social networks is clustering: the tendency of people's contacts to also be connected to each other, forming tightly knit groups (Jackson, 2008; Goyal 2007). Previous studies have shown that this aspect of networks helps explain various phenomena[1] such as altruism (Jackson et al. 2012), cooperative behavior (Coleman 1988, Bloch et al. 2008, Lippert and Spagnolo 2011, Jackson et al. 2012, Ali and Miller 2016), employment (Ruiz-Palazuelos et al. 2023), or the diffusion of certain behaviors (Granovetter 1973, Centola and Macy 2007, Centola 2010, Beaman et al. 2021). Furthermore,[2] variations in clustering across networks can shed light on the inequalities in economic outcomes among their members (Lindenlaub

[1]These findings suggest strategic reasons for clustering alongside the fact that it may be easier to connect with friends of friends than with strangers (Jackson and Rogers 2005; Acemoglu et al. 2014).

[2]Centola and Macy (2007) argue that clustering can facilitate the propagation of behaviors that require social reinforcement to be adopted (e.g., new technologies) but may hinder the spread of phenomena where a single contact with an infected node is enough to induce contagion (e.g., diseases). Empirical evidence supports this theory (see Centola 2010, Chami et al. 2017, or Beaman et al. 2021). In a related context, Alatas et al. (2016) also find that clustering positively affects information aggregation.

and Prummer 2020). However, the reason behind its prevalence and its impact in other contexts remain largely unexplored (Jackson et al. 2017).[3]

This paper examines how network structure in communication networks influences learning and social welfare when participants possess different prior opinions and face uncertainty about the external state, suggesting a potential explanation for the prevalence of clustering in real-world social networks. We consider a game in which the players have subjective imperfectly correlated priors on some relevant state of nature. They build network links, receive private signals on the state and truthfully announce their posterior expectations to direct network neighbours in two successive communication rounds. A player's disutility is measured by subjective posterior variance (his remaining uncertainty about the state after communication).

Network clustering enables learning without frictions (termed *interpretation noise*) which are created by a player's uncertainty about the other players' priors. The reason is that a player can deduce priors by neighbours without distant connections from their reactions to previous announcements by common network neighbours.

We use this insight to analyze networks that are efficient according to Rawlsian criterion: a network is efficient if it minimizes the loss of its least happy member. Since players' disutility is measured by the posterior variance of their beliefs about the state of nature (their uncertainty), a network is efficient if it minimizes the highest variance among the players. We find that the efficient network is composed of completely connected clusters if the interpretation noise is sufficiently high (see Figure 3). Otherwise, it is the

[3] We are extremely grateful to the anonymous referee for suggesting that we add this paragraph, alongside other helpful suggestions.

network commonly termed a *flower* (see Figure 4). We prove that the efficient network constitutes a Nash equilibrium.

Our results establish a relationship between a fundamental aspect of social networks (clustering) and the quality of learning through network communication in uncertain environment with differentiated prior beliefs. This is particularly relevant in contexts such as learning about the relative merits of different public policies or the adoption of innovations (e.g., technology, drugs, or products). In these scenarios, differences in prior beliefs can create barriers to learning, and clustering helps mitigate these barriers. This effect offers a potential explanation for the widespread occurrence of clustering in real-world social networks.

Studying Bayesian learning in networks is recognized as challenging due to complexities involved in belief updating (Jackson 2008).[4] We achieve tractability through two simplifying assumptions. First, we assume that a player holds the following, possibly misspecified, beliefs about his distant network: if his two local neighbours have a common distant connection he attributes this connection to either neighbour (but not both). Second, we focus on arbitrarily small correlation of priors.[5] We show numerically that our results may hold when both assumptions are relaxed, that is, when the players know their local and distant network and the correlation of their priors takes higher (but not too high) values. Furthermore, Rawlsian efficiency criterion

[4]In our framework, learning from neighbours with distant connections (termed *open neighbours*) is complicated, because a player needs to account for various correlations in his neighbours' announcements. First, both successive announcements by some neighbour reflect his prior. Second, announcements by a pair of neighbours reflect their priors which are correlated. Furthermore, their second announcements may both reflect the first announcement by their distant common neighbour.

[5]Notice that it is not equivalent to ignoring correlations in announcements entirely.

may be replaced with (more common) utilitarian efficiency criterion.

Related literature. We study networks, built in a decentralized manner, and their efficiency which connects our work to sizable economic literature on strategic network formation (see surveys in Bloch and Dutta 2010; Goyal 2007; Jackson 2005, 2008). This literature considers a variety of network formation protocols. Bala and Goyal (2000) consider unilateral network formation: a player links to any other player at a cost, and he receives some benefit from connections. Hojman and Szeidl (2008) show that for a class of benefits exhibiting decreasing returns to scale and decaying with network distance (which includes the benefits from information transmission with frictions), the unique Nash equilibrium is a periphery-sponsored "star". We model frictions in information transmission. Clustering helps reduce these frictions, which is specific to our model.

Learning through links following the network formation stage brings us to the literature on rational learning on a given network. Much of this literature has focused on asymptotic learning, showing that Bayesian agents can learn asymptotically if the network is common knowledge (see Gale and Kariv 2003; Mueller-Frank 2013). Li and Tan (2020) consider rational asymptotic learning when communicating agents believe that their local network is the entire network. Such beliefs create double counting of signals coming from distant neighbours, which clustering helps avoid. Asymptotic learning is possible if the network is a tree-like union of clusters. Unlike this literature, we focus on finite-horizon learning and we provide a closed-form expression for the quality of learning depending on the network architecture. Clustering improves learning because interconnected players can deduce each other's (imperfectly correlated) priors from reactions to earlier announcements

This idea is due to Sethi and Yildiz (2012). They attribute public disagreement to differentiated priors and show that there is no scope for disagreement in "integrated societies" where everyone hears each other's opinions.

Endogeneity of our communication network connects us to Sethi and Yildiz (2016). They study how learning with subjective priors shapes observation patterns (represented by an oriented dynamic network). In their setting, however, the priors are independent and better learning of neighbours' priors is achieved through repetitively soliciting opinions by the same individual(s), not through clustering.

The remainder of the paper is organized as follows. Section 2 presents our model. Section 3 relates a player's disutility to the network architecture under two simplifying assumptions and verifies that the expression we find provides a fair approximation for a player's disutility when these assumptions are relaxed. Section 4 studies egalitarian efficient networks. Section 5 proves that the egalitarian efficient network constitutes a Nash equilibrium, and numerically checks the robustness of this result.

2 Basic model.

Set M of m players, indexed with $i \in \{1, ..., m\}$, build a network in order to communicate through its links.

Network formation. We consider simultaneous and unilateral network formation protocol.[6] For simplicity, we model the cost of connection as an opportunity - rather than a direct cost, assuming that each player is endowed with a given connection capacity. Naturally, the players' connection capacities may differ (some people are better at forming friendships than others) and these differences may shape the network they build. We put aside these differences in order to focus on the effect of anticipated learning on the network architecture: each player can connect with at most $n \in \mathbb{N}$ other players,[7] naturally we focus on $n \leqslant m-1$.

Player i chooses a subset of players $L_i \subseteq M$, $|L_i| \leqslant n$ with whom he connects. A pair of players becomes connected if at least one of them links to the other. Each profile of linking choices $(L_1, ..., L_m)$ induces undirected network g. We use common notation $g_{ij} \in \{0, 1\}$ for an indicator of a link between players i and j in network g: $g_{ij} = 1$ indicates that players i and j are connected. We let player i be connected with himself, that is, $g_{ii} = 1$. We use common notations[8]

$$N_i = \{j \in M \mid g_{ij} = 1\}$$

for the local network neighbourhood of player i and $d_i = |N_i|$ for his "degree". We use notations

$$N_i' = \underset{j \in N_i}{\cup} N_j \backslash N_i$$

[6] Appendix I shows that our results may hold under an alternative protocol with investments in links.

[7] Nevertheless, the efficient equilibrium network has unequal degree distribution for a wide range of parameter values.

[8] Here and below, we do not reflect network-dependence of network-dependent variables for notational convenience.

for the distant (distance-2) network neighbourhood of player i,[9] and $d'_i = |N'_i|$ for the number of his distant connections.

Players' priors and signals. When the network is built, the players receive independent private signals

$$s_i = x + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0, \tau^2)$$

on the relevant state of Nature $x \sim N(0, 1)$.

Initially, the players have heterogenous imperfectly correlated priors about the state x. Differentiated priors reflect differentiated manners in which the players process new information. Say, each player i considers a subset of available historical facts to be relevant for understanding the state. His estimator of the state conditional on this subset of facts is his prior p_i (see discussion in Sethi and Yildiz, 2012).

Player i's prior p_i is his private information and he cannot directly communicate this information to the other players (he cannot describe to the others the way in which he thinks). However, it is commonly known that the players' priors are distributed according to a joint normal distribution:

$$p = (p_1, ..., p_m)^T \sim N(0, \sigma^2 \Pi), \tag{1}$$

where Π is m by m variance-covariance matrix with the following elements:[10]

$$\Pi_{jl} = \begin{cases} 1 & \text{if} \quad j = l; \\ \rho & \text{if} \quad j \neq l. \end{cases}$$

[9] We do not introduce any notations for more distant connections because we will consider only two rounds of communication.

[10] We assume that all off-diagonal elements of the variance-covariance matrix Π are the same. The alternative assumption would complicate expressions we use in the numerical part of our analysis, without altering our results qualitatively. The analytical part of our study focuses on arbitrarily weak (but not zero) correlation of priors (see below).

For concreteness, we assume that correlation ρ is positive (following interpretation of differentiated priors by Sethi and Yildiz, 2012, each player i assigns a positive probability to any other player j paying attention to some historical facts which i considers relevant).

Conditional on his prior p_i, player i believes that the law of $(x, (p_j)_{j\in M\setminus i}, (\varepsilon_j)_{j\in M})$ denoted $\mathbb{P}_i$ is a multidimensional normal distribution given by

$$\mathbb{P}_i = \mathcal{N}(p_i, 1) \otimes \mathcal{N}(\rho p_i \mathbf{1}, \widehat{\Pi}) \otimes \mathcal{N}(0, \tau^2 I)$$

where $\mathbf{1} = (1, 1,..., 1)^T$, I denotes m by m identity matrix and $\widehat{\Pi}$ is m by m matrix with the following elements:

$$\widehat{\Pi}_{jl} = \begin{cases} \sigma^2(1-\rho^2) & \text{if} \quad j = l; \\ \sigma^2\rho(1-\rho) & \text{if} \quad j \neq l. \end{cases}$$

Hereafter, $\mathcal{L}_i$ denotes the law (or conditional law) of some variable under $\mathbb{P}_i$, $\mathbb{E}_i$ denotes the expectation under $\mathbb{P}_i$, $\mathbb{V}_i$ denotes the variance under $\mathbb{P}_i$ and $\mathbb{C}_i$ denotes the covariance under $\mathbb{P}_i$.

Communication. Our communication protocol follows Sethi and Yildiz (2012) and earlier literature on public (dis)agreement. However, we reduce communication channels to network links. Furthermore, we assume that the number of communication rounds is finite, to reflect the players' impatience. This makes our analysis technically challenging,[11] so we limit communication to two rounds ($t = 1, 2$). This limitation comports nicely with Mobius et al. (2015) who find that information travels no further than two steps in the conversation network.

[11] The analysis of the case in which the number of communication rounds is unlimited is simplified by the following fact: any player learns all distributed information if the players form a circle or a wheel (details available upon request).

Hence, after receiving their signals on the state, the players communicate with local network neighbours in discrete time periods. In each period they simultaneously announce their true beliefs summarized by an estimator of the state (they cannot transmit the set of "tagged" announcements received from their neighbours or announce their priors).[12]

The first announcement $A_{i,1}$ by player i to his network neighbors is his estimator of the state conditional on his private signal:

$$A_{i,1} = \mathbb{E}_i\left(x \mid s_i\right).$$

After the first round of communication, player i rationally updates his beliefs upon the announcements received from his local network neighbours. His second announcement $A_{i,2}$ is his expectation of the state conditional on his private signal and the first announcements by his local neighbours:

$$A_{i,2} = \mathbb{E}_i\left(x \mid s_i,\ \{A_{j,1} \mid j \in N_i\}\right).$$

Payoffs. After the second round of communication with his neighbours, player i updates, once again, his beliefs about the state and the other players' priors. We assume that his disutility or loss is equal to his remaining uncertainty about the state x, measured by his subjective posterior variance of the state:

$$l_i(g) = \mathbb{V}_i\left(x \mid s_i,\ \{A_{j,t} \mid j \in N_i,\ t = 1, 2\},\ \widehat{g}\right), \tag{2}$$

[12] If strategic communication is allowed for, truthful communication is an equilibrium. That is, when all players truthfully announce their estimate of the state during either round of communication, and believe the others to do the same, no player has the incentives to deviate by sending a message different from his true estimate of the state during some round of communication. Indeed, following such deviation he learns the same information from his neighbours' messages as when he does not deviate, as long as they believe him to tell the truth and react accordingly.

where $\widehat{g}$ refers to the players' beliefs about their network g (see the details in the following paragraph). For example, we could think of player i taking private action resulting in a loss which is equal to the perceived squared distance between his action and the optimal action given by the state x. Notice that a payoff equal to a constant less loss (2) measures a player's confidence in his action, which comports nicely with the psychological literature indicating confidence in private actions as a source of happiness (see Maslow, 1943).

Beliefs about network. A player's updating after the second round of communication is complicated by the need to account for various correlations: Successive announcements by i's neighbour j are correlated because they both reflect j's prior. All announcements by i's neighbours j and l are correlated because they reflect their priors which are correlated. Furthermore, j and l may have common neighbour r on the distance from i which creates an additional source of correlation in their second announcements.

For tractability, we assume that the players hold the following, possibly misspecified, beliefs $\widehat{g}$ about their network g.[13] Any player i knows his local neighbourhood. However, if his neighbours j and l are both linked with player r on distance 2 from i, player i accounts for only one of these links, either between r and j or between r and l (it does not matter for the payoff which of the two links is accounted for). While being purely technical, this assumption may be motivated by sociological evidence of erroneous perception of distant networks (see references in Li and Tan 2020, Dessi et al. 2016). One typical error is overestimation of one's own centrality, including the betweenness

[13] We derive our analytical results under this behavioral assumption. However, in the numerical part of our analysis, we assume that the players perfectly know their network.

centrality (Kumbasar et al. 1994) which comports nicely with ignorance of distant common friendships.[14]

Weakly correlated priors. The above behavioral assumption regarding beliefs about distant network does not suffice to deliver tractability. We achieve tractability by focusing on arbitrarily small correlation of priors (we numerically check robustness of our results for higher values of correlation). Note that this is not equivalent to ignoring this correlation entirely because player i is "undoing" his neighbours' priors using his own prior and so are his neighbours. This allows player i to learn the priors and signals by neighbours without distant connections, as we discuss in the following section.

3 Network architecture and disutility.

This section relates a player's loss (2) to the network architecture. We begin with dividing the set of player i's neighbours into two subsets. Set

$$\overline{N}_i = \{j \in N_i \mid N_j \subseteq N_i\}$$

of ***closed*** neighbours having only common connections with player i and set

$$\overset{\circ}{N}_i = N_i \backslash \overline{N}_i$$

[14]Betweenness centrality of player i is defined as the ratio of the number of shortest paths between a pair of players that pass through player i divided by the total number of such paths, summed over all possible pairs of players different from i. Suppose player i has two neighbors, j and l, who are not directly connected but are both connected to some player r outside i's neighborhood. By ignoring commonality of this connection, player i reduces the number of shortest paths from j to l, which leads to an overestimation of his own betweenness centrality. For example, consider a ring network with four players: i, j, l, and r, connected as described above. The betweenness centrality of player i in this network is $\frac{1}{6}$. However, when player i ignores one of the links - either between players j and r, or between players l and r - his perceived betweenness centrality increases to $\frac{2}{3}$.

of ***open*** neighbours with at least one connection outside i's neighbourhood. It is convenient to introduce notations $\overline{d}_i = \left|\overline{N}_i\right|$ and $\mathring{d}_i = \left|\mathring{N}_i\right|$ for the number of i's closed- and open neighbours.

The above classification of i's neighbours into closed and open is relevant because i can perfectly learn the priors and signals of his closed neighbours by observing their reactions to the first announcements they receive (Sethi and Yildiz, 2012).

Lemma 1 *Any player i learns private signal s_j and prior p_j by any closed neighbour $j \in \overline{N}_i$.*

The argument by Sethi and Yildiz (2012) reproduced in Appendix B goes along the following lines. Consider some player i with at least one closed neighbour j. By standard formula for Gaussian updating (see Appendix A), j's first announcement is a linear combination of j's prior p_j and j's private signal s_j (the higher the variance τ^2 of the signal, the higher weight is put on the prior):

$$A_{j,1} = \mathbb{E}_j[x|s_j] = \tfrac{\tau^2}{1+\tau^2}p_j + \tfrac{1}{1+\tau^2}s_j. \tag{3}$$

After the first round of communication, player j updates his beliefs upon the first announcements received from his neighbours. His second announcement is a linear combination of his own first announcement, the sum of his neighbours' first announcements and his prior p_j (see details in step 2 of Appendix B):

$$\begin{aligned} A_{j,2} = \mathbb{E}_j \left(x \mid s_j, \{A_{r,1}\}_{r \in N_j \setminus \{j\}}\right) = (1 - \lambda_j(d_j - 1))A_{j,1} + \\ \lambda_j \left(1 + \tau^2\right) \sum_{r \in N_j \setminus \{j\}} A_{r,1} - \lambda_j \tau^2 \rho (d_j - 1) p_j, \text{ where} \end{aligned} \tag{4}$$

$$\lambda_j = \frac{1}{(1+\tau^2)(1+\tau^2\sigma^2(1-\rho))+(d_j-1)(1+(1+\tau^2)\tau^2\sigma^2(1-\rho)\rho)}.$$

Because player j is a closed neighbour by player i, player i "hears" the first announcements by all j's neighbours. Therefore, player i can deduce j's prior p_j from j's second announcement, and then deduce j's private signal s_j from j's first announcement. Note that such deduction would be impossible if the correlation of priors was zero (consider parameter ρ in equation (4) being equal to zero).

Let us now describe learning from open neighbours (details are presented in Appendix C). Consider some player i with at least one open neighbour $j \in \overset{\circ}{N}_i$. From the first announcement by player j , which is a linear combination of j's prior and j's signal, player i deduces j's signal with noise associated with i's uncertainty regarding j's priors:

$$\widetilde{s}_{j,1} = (1+\tau^2)\,A_{j,1} - \tau^2\overline{p} = x + \varepsilon_j + \tau^2\,(p_j - \overline{p})\,, \tag{5}$$

$$\text{where } \overline{p} = \mathbb{E}_i\left(p_j \mid \{p_r\}_{r\in\bar{N}_i}\right) = \frac{\rho}{1+\rho(\overline{d}_i-1)}\sum_{r\in\bar{N}_i} p_r. \tag{6}$$

The second announcement by player j is a linear combination of his own first announcement, the sum of his neighbours' first announcements and his prior p_j. From this announcement player i deduces signals by j's neighbours with whom i is not connected, once again with noise associated with uncertainty about their priors and j's priors:

$$\widetilde{s}_{j,2} = x + \frac{1}{d_{j\setminus i}}\left(\sum_{r\in N_j\setminus N_i}\varepsilon_r + \tau^2\sum_{r\in N_j\setminus N_i}(p_r-\overline{p}) - \tau^2\rho(d_j-1)(p_j-\overline{p})\right), \tag{7}$$

$$\text{where } d_{j\setminus i} = |N_j \setminus N_i|\,.$$

Now, consider all open neighbours by player i. Without loss of generality, let us index them with

$$j \in \left\{ \overline{d}_i + 1, ..., d_i \right\}. \tag{8}$$

Let $\widetilde{s}_t = (\widetilde{s}_{\overline{d}_i+1,t}, ..., \widetilde{s}_{d_i,t})$ be the vector of signal deduced from their announcements in round $t = 1, 2$ of communication. By equations (5) and (7),

$$\mathcal{L}_i(x, \widetilde{s}_1, \widetilde{s}_2 | \{s_j, p_j\}_{j \in \bar{N}_i}) = \mathcal{N}\left(\mu \mathbf{1}, \begin{pmatrix} v & v\mathbf{1}^T \\ v\mathbf{1} & \Sigma \end{pmatrix}\right), \text{ where} \tag{9}$$

$$\mu = \mathbb{E}_i \left(x \mid \{s_j\}_{j \in \bar{N}_i} \right) = \tfrac{\tau^2}{\tau^2 + \overline{d}_i} p_i + \tfrac{1}{\tau^2 + \overline{d}_i} \sum_{j \in \bar{N}_i} s_j \tag{10}$$

$$v = \mathbb{V}_i \left(x \mid \{s_j\}_{j \in \bar{N}_i} \right) = \tfrac{\tau^2}{\tau^2 + \overline{d}_i}, \tag{11}$$

and Σ is a square symmetric matrix of size $2\mathring{d}_i$ with elements specified in Appendix C on step 3. By standard formula for Bayesian updating, to find player i's loss, we need to find the sum of elements of the inverted matrix Σ. A combination of our assumption regarding beliefs on distant network and arbitrarily small correlation of priors allows us to represent matrix Σ as a sum of two matrixes of dimension $2\mathring{d}_i$: matrix $vI\mathbf{1}^T$ (with elements equal to v) of rank 1 and a diagonal matrix zG. This decomposition enables us to invert matrix Σ (Miller 1981) and find the loss by player i.[15]

Proposition 1 (network architecture and a player's disutility). *The loss by player i in network g is determined by- and decreasing in- two parameters of network architecture: (i) the total number of his local and distant*

[15] Because the correlation ρ is positive, we treat the signals deduced by player i from the messages by his open neighbors jointly, compute the elements of the variance-covariance matrix Σ and set ρ to be arbitrarily small in the final stage. The outcome is equivalent to treating these signals as independent. That is, player i learns from his open neighbors as if he ignored the correlations in their priors.

neighbours $d_i + d'_i$ and (ii) the number of his closed neighbours $\overline{d}_i$:

$$l_i(g) = \frac{\tau^2\left(1+\sigma^2\tau^2\right)}{\tau^2(1+\sigma^2\tau^2)+\left(d_i+d'_i\right)+\sigma^2\tau^2\overline{d}_i}. \tag{12}$$

Notice that closed neighbours are more valuable than either open or distant neighbors.[16] The reason is that a player learns the priors by his closed neighbours while he remains uncertain about those by his open or distance-2 neighbours. Term $\sigma^2\tau^2$ measuring the noise associated with this uncertainty is called *interpretation noise.*

Performance of approximation (12) in generalized setting. We check numerically whether equation (12) is a good approximation for the value of loss defined by equation (2) if the players know their network and the correlation of their priors takes any value between 0 and 1, termed hereafter the *generalized setting.*

We consider all possible 9589 networks of $m = 8$ players with individual connection capacity $n = 2$.[17] Figure 1 plots the approximation of loss given by equation (12) against the exact value of loss (2) for all players in all these networks, fixing different values of correlation ρ.

[16]This effect is due to a positive (although arbitrarily small) correlation of the players' priors. Indeed, if that correlation was equal to zero, player i's loss would depend only on the joint number of his direct and distant connections. Specifically, it would be given by equation (12) with $\overline{d}_i$ being replaced by 1, because the noise created by a player's uncertainty about the other players' priors would be added to all signals deduced by player i from his neighbours' messages and only one signal (his own) would be free from this noise.

[17]The choice $m = 8$, $n = 2$ is dictated by computational feasibility. Indeed, the number of possible networks grows exponentially in the number of players m. Suppose that individual connection capacity n is equal to 2. Then, the number of different networks (up to isomorphism) is: 153 if $m = 6$; 955 if $m = 7$ and 9589 if $m = 8$.

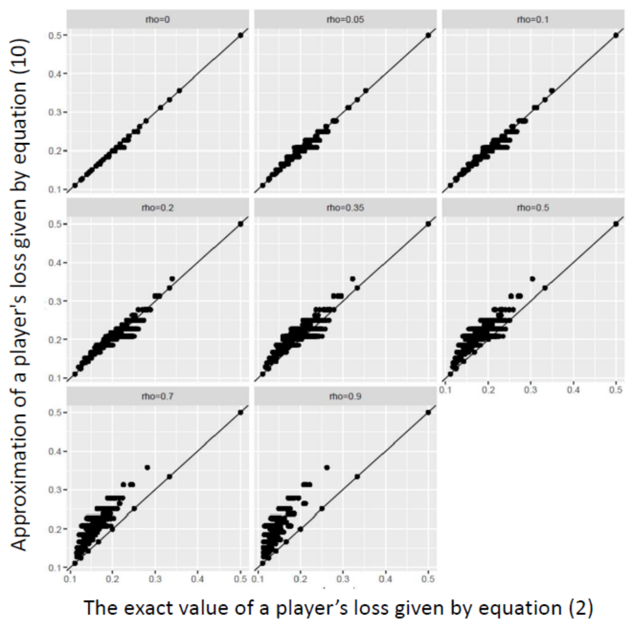

Figure 1: Performance of approximation (12)
($m = 8$, $n = 2$, $\tau^2 = \sigma^2 = 1$).

We note that approximation (12) performs well for sufficiently small values of ρ,[18] and it tends to overestimate the loss for higher values of ρ.

Figure 2 presents statistical measure of the quality of approximation: boxplot of the ratio of approximation (12) over the exact value of (2). We observe, for example, that for $\rho = 0.35$, three quarters of the ratio lies in the interval between 1 and 1.08, and approximation (12) performs even better for smaller values of ρ.

[18]The performance of our approximation for $\rho = 0$ is imperfect because equation (12) computes the loss by a player with misspecified beliefs regarding distant common friendships.

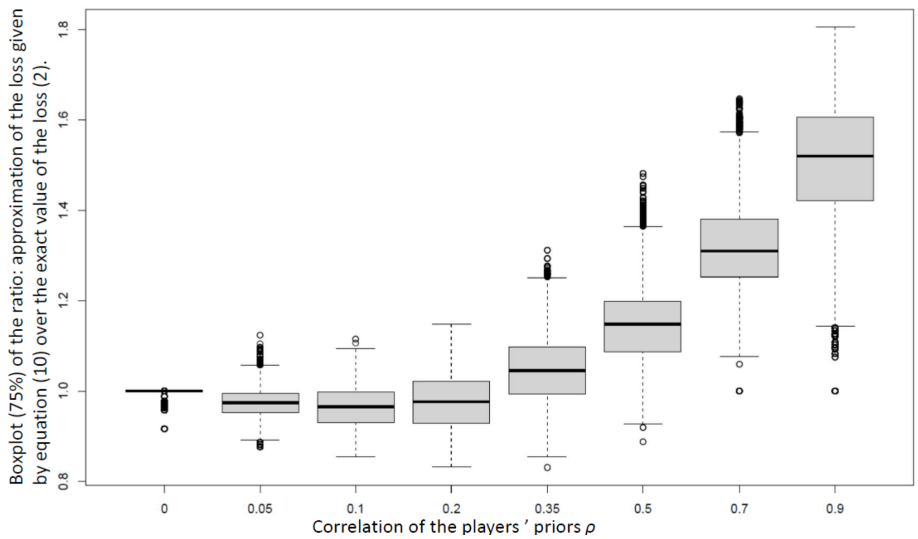

Figure 2: Performance of approximation (12), boxplot: $m = 8$, $n = 2$, $\tau^2 = \sigma^2 = 1$.

4 The efficient network.

This section uses Proposition 1 to study which network a social planner we would like to see build in the network formation stage. For the sake of tractability, we take *egalitarian efficiency criterion* (Rawls' criterion): a network is efficient iff it minimizes the loss by its least happy member.[19] Note that such criterion may be justified by the fact that any player can find himself in the role of the least happy player in the network.

We denote the set of all feasible networks with $\mathcal{G}$, the set of efficient networks with

$$\mathcal{G}^* = \arg\min_{g\in\mathcal{G}} \left(\max_{i\in M} l_i\left(g\right) \right)$$

[19] We will check numerically that our results are likely to hold if we use more common utilitarian efficiency criterion.

and the set of the least happy players or "losers" of network g with

$$L(g) = \arg\max_{i \in M} l_i(g).$$

Candidate efficient networks. Before proceeding with formal analysis, let us try to gain some insight as to which networks are likely to be efficient. By Proposition 1, the efficient network maximizes a combination of the joint number of local and distant neighbours by the least happy player i and the number of his closed neighbours, with the weight of closed neighbour being increasing in the interpretation noise $\sigma^2\tau^2$.

Consider the extreme values of the interpretation noise. Suppose first, that it approaches infinity. Then, it is most important to maximize the number of closed neighbours by the least happy player, suggesting that a network composed of complete components of sizes as equal as possible[20] is efficient.

Suppose, for concreteness, that the number of players m and a player's connection capacity n are such that the players can be divided into completely connected components of equal size:[21]

$$\text{there exists } l \in \mathbb{N} \text{ such that } m = (2n+1)l. \tag{13}$$

An example of such network is depicted in Figure 3.

[20] The difference between the sizes of any pair of components is at most 1.

[21] Analysis without any restrictions on m and n is available upon request; Appendix G provides partial analysis relevant for our numerical results as well as illustrative examples.

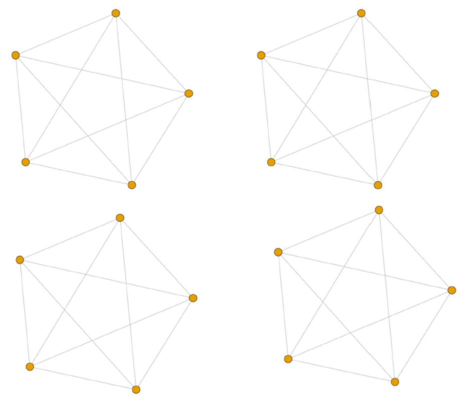

Figure 3: Complete component network ($m = 20$, $n = 2$).

Hereafter, a network composed of completely connected components of size $2n+1$ is called *complete component network* and is denoted with c. The players can build it as follows: divide into groups of size $2n+1$; each group forms a circle; each player in a circle connects to n players on his right.

Now, suppose that the interpretation noise $\sigma^2\tau^2$ approaches zero. Then, it is most important to maximize the total number of neighbours by the least happy player, while maximization of his closed neighbours is the secondary objective. The following network seems a good candidate for being efficient. It is composed of the central "hub" h connected to everyone (hence, the highest possible total number of neighbours m is delivered to any player):

$$f_{ih} = 1 \; \forall \; i \in M,$$

and $m-1$ peripheral players divided into interconnected "petals":

$$N_i = N_j \text{ for any } i \neq h \text{ and for any } j \in N_i \backslash \{h\}.$$

Once again, for concreteness, suppose that m and n are such that the size of one petal (termed the "large petal") is $2n$ and the size of the remaining petals (termed "small petals") is $2n-1$:

$$\text{there exists } k \in \mathbb{N} \text{ such that } m = 2n + 1 + (2n-1)k. \tag{14}$$

Following common terminology, we will call such network a *flower* and denote it with f. It can be built as follows: $2n$ players and the central hub form a circle and each player connects to the next n players on his right. The remaining $(2n-1)k$ players divide into k groups of size $2n-1$. Each group forms a circle. Each player in a circle connects to the central hub and $n-1$ players on his right. An example of flower network is depicted in Figure 4.

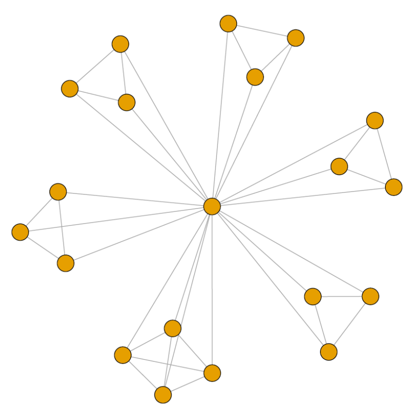

Figure 4: Flower network
($m = 20$, $n = 2$).

Parameter specification. *Hereafter, we focus on the situation in which the number of players m and the individual connection capacity n satisfy equations (13) and (14).*[22]

The following section shows that the efficient network is either the compete component network or the flower, depending on the magnitude of interpretation noise.

Characterization of the efficient network. We show, first, that the flower is the most efficient among all networks in which the least happy player has at least one open neighbour.

Lemma 2 *Suppose that some network is egalitarian efficient and one of its least happy members has an open neighbour. Then, this network is the flower. Formally, if $g \in \mathcal{G}^*$ and $\exists\ i \in L(g)$ such that $d_i > \overline{d}_i$ then $g = f$.*

Constructive proof in Appendix D relies on the observation that any closed neighbour by any least happy i shall be at least as "happy" as i. Using this observation, the fact that a player can build, at most, n links and that network g is efficient, we prove that the closed neighbourhood by player i is a completely connected subgraph of size $2n-1$. Furthermore, i and his closed neighbours share one open neighbour or "hub" who connects them to all other players. Hence, they can be visualized as a small petal. We proceed with considering another least happy player outside i's neighbourhood, would such player exist, to frame another small petal connected to the same hub, and so on until all the least happy players are organized in small petals

[22] Full analysis of complementary cases is available upon request. A part of it is presented in Appendix G.

connected to the central hub. The remaining players, all connected to the central hub, have a closed degree of at least $2n$. It is feasible iff they form a petal of size $2n$. By this construction, g is the flower.

By Lemma 2, the most efficient network is either flower f or some network g in which any looser i has only closed neighbours. Network c is such a network maximizing the number of closed neighbours by its least happy member (see details in Appendix E). By Proposition 1, network c is more efficient than the flower f iff

$$\sigma^2\tau^2 \geqslant \tfrac{m-(2n+1)}{2}. \tag{15}$$

Proposition 2. *The egalitarian efficient network is either the flower or complete component network depending on the magnitude of the interpretation noise:*

$$\mathcal{G}^* = \begin{cases} \{c\} & \textit{if } \sigma^2\tau^2 > \tfrac{m-(2n+1)}{2}, \\ \{c, f\} & \textit{if } \sigma^2\tau^2 = \tfrac{m-(2n+1)}{2}, \\ \{f\} & \textit{otherwise.} \end{cases}$$

Numerical robustness check. We continue to consider $m = 8$ players with connection capacity $n = 2$. Note that Proposition 3 does not apply directly to this example because parameter restriction (13) fails. It is feasible to build the flower network depicted in Figure 5:

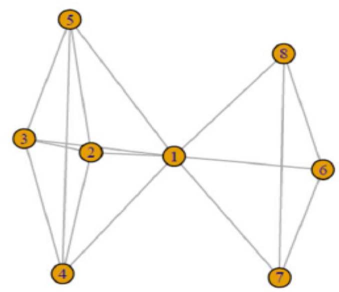

Figure 5: flower network
($m = 8$, $n = 2$).

At the same time, it is impossible to divide the players into completely connected components of size $2n + 1 = 5$. The network maximizing the number of closed neighbours by the least happy player is composed of two complete components of size 4 (termed hereafter in this section as *clustered network*).

The extension of Proposition 2 (in Appendix G.2) shows that the efficient egalitarian network is: the flower network depicted in Figure 5 if $\sigma^2\tau^2 < 4$, clustered network if $\sigma^2\tau^2 > 4$, or both these networks when $\sigma^2\tau^2 = 4$. We check the robustness of this prediction in a setting where the players know their network, and the correlation of their priors is not arbitrarily weak. We consider either egalitarian- or (more common) utilitarian-efficiency criterion.

We normalize signal's variance as $\tau^2 = 1$ (hence, the interpretation noise is measured with σ^2) and let ρ take values in set $\{0.1, 0.2, 0.35, 0.5, 0.7, 0.9\}$ and compare losses by the least happy player across all possible 9589 networks of size 8.[23]

[23] Recall that we count different networks up to isomorphism.

Figure 6, left image depicts losses by the least happy player across all possible networks for $\rho = 0.35$ (the figures for smaller values of ρ are similar). The loss by the least happy player in the flower network is marked with red dotted line. The loss by the least happy player (any player) in the clustered network is marked with dashed horizontal line. We observe that the most efficient egalitarian network is the flower if the interpretation noise σ^2 lies below some threshold and clustered network otherwise suggesting the robustness of our theoretical prediction. Figure 6, right image depicts average losses suggesting that our results may extend to utilitarian efficiency criterion.

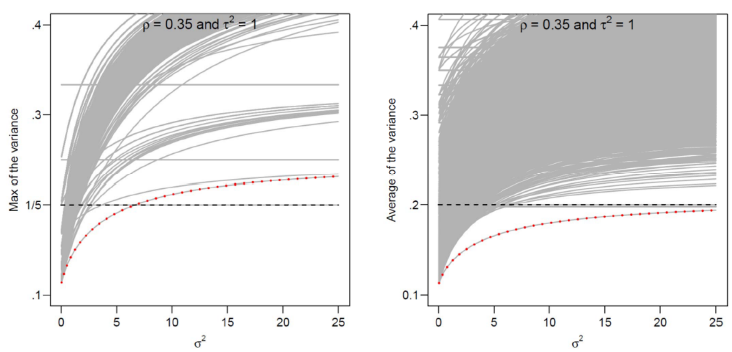

Figure 6: Loss in different networks of size 8: by the least happy player (left); average normalized to component's size (right).

Figure 7, shows that our results fail for sufficiently high values of ρ,[24] for either egalitarian efficiency criterion (three upper figures) or utilitarian efficiency criterion (three lower figures).

[24] It seems that the threshold for our calibration lies somewhere in between 0.35 and 0.4.

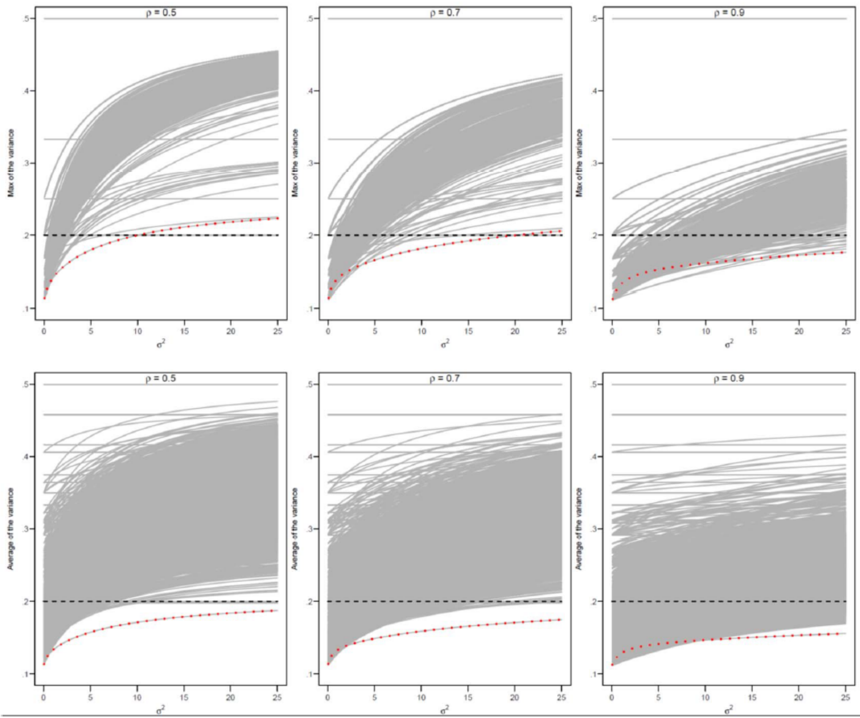

Figure 7: Performance of different networks for relatively high values of correlation ρ.

Indeed, when the correlation ρ is relatively high and the interpretation noise σ^2 is relatively low it becomes most important for a player to have as many local neighbours as possible.[25] Therefore, the efficient network (according to either egalitarian- and utilitarian efficiency criteria) is such as depicted in Figure 8:

[25] Recall that we consider the generalized setting in which the players know their network. In this setting local open- and distant connections are equivalent for a player's disutility.

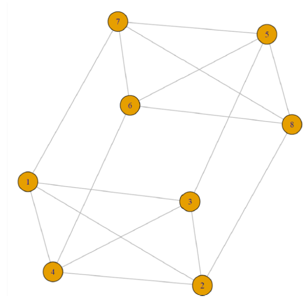

Figure 8: Most efficient (according to either egalitarian- or utilitarian-criterion) network when $\rho \in \{0.5, 0.7, 0.9\}$ and σ^2 is sufficiently low.

5 The efficient Nash equilibrium network.

Proposition 2 tells us which network the social planner would like to see built depending on the magnitude of the interpretation noise. However, it is not guaranteed that the players are going to build the efficient network in Nash equilibrium.

We verify first that flower f is a Nash equilibrium (see Appendix F for details). Indeed, by Proposition 1, a player's deviation from the strategy profile leading to formation of flower f is profitable only if it increases either the number of his closed neighbours or the total number of his local and distant neighbours. In the flower network, any unilateral deviation (weakly) decreases either of these numbers.

Furthermore, complete component network c is an equilibrium whenever it is efficient. The reason is that when the interpretation noise lies above threshold (15), a player in one component of network c does not want to replace a link with a player in his component, (loosing thereby $2n$ closed neighbours), by a link with a player in a different component (gaining thereby one open- and $2n$ distant neighbours).

Proposition 3. *The network in Proposition 2 constitutes the most efficient Nash equilibrium.*

Appendix H provides numerical robustness check of this result, showing that proposition 3 may still hold when our simplifying assumptions are relaxed, that is, the correlation of players' priors is not arbitrarily small and they perfectly know their network.

Naturally, the game may have other Nash equilibria. However, the efficiency may be used as a refinement.

6 Conclusion.

We have modeled formation of a communication network by players with incomplete information about the relevant state of nature and heterogenous weakly correlated prior beliefs about it. We have found that clustering delivers a signal-extraction benefit, and therefore the egalitarian efficient network exhibits a high degree of clustering. We have shown, furthermore, that this network constitutes a Nash equilibrium.

Our results suggest that in situations characterized by uncertainty and diversity of prior beliefs, such as adoption of new agricultural technology, network clustering enables individuals to acquire more robust and precise knowledge about uncertain events. They also provide some insight into the prevalence of clustering in real-world social networks.

There are several natural extensions of our model, such as a longer communication horizon, which we leave beyond the scope of this paper. On the applied side, we hope that the established relationship between the quality of information diffused through the network and its architecture may help network-based targeting, at least when the quality of learning through network is important for adoption.

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Appendix A: Technical review.

The notations in this section are independent from the rest of the paper.

Mean and Variance of a linear combination of Gaussian variables

Consider K random variables $x_k \sim N(\mu_k, \sigma_k^2)$, $k = 1, ..., K$ and a set of constants $\{\alpha_k\}_{k=1...K}$.

$$\sum_{k=1}^{K} \alpha_k x_k \sim N(\mu, \sigma^2), \text{ where } \mu = \sum_{k=1}^{K} \alpha_k \mu_k \text{ and } \sigma^2 = \sum_{k=1}^{K} \alpha_k^2 \sigma_k^2.$$

Conditional multivariate normal distribution Consider n-dimensional colomn-vector of random variables x distributed normally with mean μ and n-by-n variance-covariance matrix Σ: $x \sim N(\mu, \Sigma)$. Consider the following partition of x, μ and Σ:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where x_1 is k-dimensional colomn-vector, x_2 is $(n-k)$-dimensional colomn-vector, μ_1 is k-dimensional colomn-vector, μ_2 is $(n-k)$-dimensional colomn-vector, Σ_{11} is k-by-k matrix, Σ_{12} is k-by-$(n-k)$ matrix, Σ_{21} is $(n-k)$-by-k matrix, and Σ_{22} is $(n-k)$-by-$(n-k)$ matrix. Suppose that realization of the latter $(n-k)$ components of vector x is known: $x_2 = a$. Then, $(x_1 \mid x_2 = a) \sim N\left(\widehat{\mu}, \widehat{\Sigma}\right)$, where

$$\widehat{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}\left(a - \mu_2\right), \tag{16}$$

$$\widehat{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \tag{17}$$

Matrix inversion Consider n-by-n matrix A. The inverse matrix is

$$A^{-1} = \left[(-1)^{i+j}\frac{A_{i,j}}{\det A}\right],$$

where $A_{i,j}$ is the (i,j)-adjunct of matrix A, that is, the determinant of a matrix received from A by removing row i and column j. In particular,

$$\begin{pmatrix} a & b & ... & b \\ b & a & & ... \\ ... & & ... & b \\ b & ... & b & a \end{pmatrix}^{-1} =$$

$$= \tfrac{1}{(a-b)(a+b(n-1))}\begin{pmatrix} a+(n-2)b & -b & ... & -b \\ -b & a+(n-2)b & & ... \\ ... & & ... & -b \\ -b & ... & -b & a+(n-2)b \end{pmatrix}. \tag{18}$$

We introduce the following notation for the sum of elements of matrix A:

$$Sum(A) = e^t Ae.$$

Note that the sum of elements of matrix (18) is equal to:

$$Sum \begin{pmatrix} a & b & ... & b \\ b & a & & ... \\ ... & & ... & b \\ b & ... & b & a \end{pmatrix}^{-1} = \frac{n}{a+b(n-1)}. \quad (19)$$

Furthermore, by Miller (1981),

$$(H+G)^{-1} = G^{-1} - \frac{1}{1+tr(HG^{-1})} G^{-1} H G^{-1}, \quad (20)$$

where matrices G and H have the same dimension, matrix $G+H$ is nonsingular and $rk(H) = 1$.

Appendix B: proof of Lemma 1 (Sethi and Yildiz, 2012).

Consider some player i with at least one closed neighbour j.

Step 1 specifies j's first announcement. The vector (x, s_j) is distribued according to the following law

$$\mathcal{N}\left(\begin{pmatrix} p_j \\ p_j \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1+\tau^2 \end{pmatrix}\right).$$

By equations (16) and (17), conditional law of x given s_j under $\mathbb{P}_j$ is

$$\mathcal{L}_i(x|s_j) = \mathcal{N}\left(\frac{\tau^2}{1+\tau^2} p_j + \frac{1}{1+\tau^2} s_j, \frac{\tau^2}{1+\tau^2}\right),$$

hence, $A_{j,1}$ is given by equation (3). Player i deduces j's signal s_j from announcement $A_{j,1}$ with noise $\tau^2\left(p_j - \mathbb{E}_i(p_j \mid p_i)\right)$ associated with i's uncertainty regarding j's priors:

$$\left(1+\tau^2\right) A_{j,1} - \tau^2 \mathbb{E}_i(p_j \mid p_i) = s_j + \tau^2\left(p_j - \mathbb{E}_i(p_j \mid p_i)\right), \tag{21}$$

$$\text{where } \mathbb{E}_i(p_j \mid p_i) = \rho p_i.$$

Step 2 specifies j's second announcement. Let us index player j's neighbours but himself by $r \in \{1, ..., d_j - 1\}$. Consider the second period of communication. Recall equation (21). The state x and player j's signals are distribued by the following law:

$$\mathcal{L}_j \begin{pmatrix} x \\ x+\varepsilon_j \\ x+\varepsilon_1+\tau^2\left(p_1-\rho p_j\right) \\ \vdots \\ x+\varepsilon_{d_j-1}+\tau^2\left(p_{d_j-1}-\rho p_j\right) \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} p_j \\ \vdots \\ \vdots \\ \vdots \\ p_j \end{pmatrix}, \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 1+\tau^2 & 1 & \cdots & 1 \\ \vdots & 1 & & & \\ \vdots & \vdots & & \Sigma^{(1)} & \\ 1 & 1 & & & \end{pmatrix}\right)$$

where $\Sigma^{(1)}$ is $d_j - 1$ by $d_j - 1$ matrix with elements

$$(\Sigma^{(1)})_{r,l} = \begin{cases} 1+\tau^2+\tau^4\sigma^2\left(1-\rho^2\right) & \text{if } r = l, \\ 1+\tau^4\sigma^2\rho\left(1-\rho\right) & \text{if } r \neq l, \end{cases}$$

$r = 1,..., d_i - 1$, and $l = 1,..., d_i - 1$.

By equations (16) and (17),

$$\mathcal{L}_j((x, x+\varepsilon_1+\tau^2\left(p_1-\rho p_j\right), ..., x+\varepsilon_{d_j-1}+\tau^2\left(p_{d_j-1}-\rho p_j\right))^t | s_j) = \mathcal{N}\left(\frac{s_j+\tau^2 p_j}{1+\tau^2}\mathbf{1}, \begin{pmatrix} \frac{\tau^2}{1+\tau^2} & \frac{\tau^2}{1+\tau^2}\mathbf{1}^T \\ \frac{\tau^2}{1+\tau^2}\mathbf{1} & \widetilde{\Sigma}^{(1)} \end{pmatrix}\right),$$

where $\widetilde{\Sigma}^{(1)}$ is $d_j - 1$ by $d_j - 1$ matrix with elements

$$\left(\widetilde{\Sigma}^{(1)}\right)_{r,l} = \begin{cases} \frac{\tau^2}{1+\tau^2} + \tau^2 + \tau^4\sigma^2\left(1-\rho^2\right) & \text{if} \quad r = l, \\ \frac{\tau^2}{1+\tau^2} + \tau^4\sigma^2\rho\left(1-\rho\right) & \text{if} \quad r \neq l, \end{cases} \tag{22}$$

where $r = 1,..., d_j - 1$ and $l = 1,..., d_j - 1$.

By equations (18), (19), (20) and (22),

$$Sum\left(\widetilde{\Sigma}^{(1)}\right)^{-1} = \frac{d_j - 1}{\tau^2 + \tau^4\sigma^2(1-\rho) + (d_j - 1)\left(\frac{\tau^2}{1+\tau^2} + \tau^4\sigma^2(1-\rho)\rho\right)}. \tag{23}$$

By equations (21), (23) and (16), $A_{j,2}$ is given by equation (4).

Step 3 completes the proof. By definition of set $\overline{N}_i$, player i "hears" the first announcements $\{A_{r,1}\}_{r \in N_j}$ by all j's neighbours. Therefore, player i can deduce j's prior p_j from j's second announcement (**??**), and then j's private signal s_j from his first announcement (21).

Appendix C: proof of Proposition 1.

Step 1 specifies player i's beliefs about the state conditional on the signals and priors by his closed neighbours. Let us index i's closed neighbours but himself with $j \in \left\{1, ..., \overline{d}_i - 1\right\}$.

$$\mathcal{L}_i\begin{pmatrix} x \\ x+\varepsilon_i \\ x+\varepsilon_1 \\ \vdots \\ x+\varepsilon_{\overline{d}_i-1}\end{pmatrix} = \mathcal{N}\left(\left(\begin{pmatrix} p_i \\ \vdots \\ \vdots \\ \vdots \\ p_i \end{pmatrix}, \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 1+\tau^2 & 1 & \cdots & 1 \\ \vdots & 1 & 1+\tau^2 & & 1 \\ \vdots & \vdots & & & \vdots \\ 1 & 1 & 1 & \cdots & 1+\tau^2 \end{pmatrix}\right)\right) \tag{24}$$

By equation (18),

$$\begin{pmatrix} 1+\tau^2 & 1 & \cdots & 1 \\ 1 & 1+\tau^2 & & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1+\tau^2 \end{pmatrix}^{-1} = \frac{1}{\tau^2\left(\tau^2+\overline{d}_i\right)}\begin{pmatrix} \tau^2+\overline{d}_i-1 & -1 & \cdots & -1 \\ -1 & \tau^2+\overline{d}_i-1 & & -1 \\ \vdots & & & \vdots \\ -1 & -1 & \cdots & \tau^2+\overline{d}_i-1 \end{pmatrix}. \tag{25}$$

By equations (24), (25), (16) and ((17)

$$\mathcal{L}_i(x|\left\{s_j\right\}_{j\in\bar{N}_i}) = \mathcal{N}\left(\mu, v\right), \tag{26}$$

where μ is given by equation (10) and v is given by equation (11).

Step 2 specifies player i's beliefs about the priors by players outside his closed neighbourhood conditional on the signals and priors by his closed neighbours. Recall that the vector of priors p is distributed according to distribution (1). Let us order the players' priors so that the subvector of priors

$$p|_{\overline{N}_i} = (p_i, p_1, ..., p_{\overline{d}_i-1})$$

comes the last. Let us denote the variance-covariance matrix of vector $p|_{\overline{N}_i}$ by $\Pi|_{\overline{N}_i}$ (this is, $\overline{d}_i$ by $\overline{d}_i$ matrix with elements 1 on the main diagonal and ρ elsewhere). By equation (18),

$$\left(\Pi|_{\overline{N}_i}\right)^{-1} = \frac{1}{(1-\rho)\left(1+\rho\left(\overline{d}_i-1\right)\right)} \begin{pmatrix} 1+\rho\left(\overline{d}_i-2\right) & -\rho & \cdots & -\rho \\ -\rho & 1+\rho\left(\overline{d}_i-2\right) & & -\rho \\ \vdots & & & \vdots \\ -\rho & -\rho & \cdots & 1+\rho\left(\overline{d}_i-2\right) \end{pmatrix}. \tag{27}$$

Let $p|_{g\setminus\overline{N}_i}$ be the vector of priors of all players outside $\overline{N}_i$. Equation (6) follows from equations (1), (27) and (16). By equations (1), (27) and (17),

$$w = \mathbb{V}_i\left(p_r \mid \{p_j\}_{j\in\bar{N}_i}\right) = \frac{\sigma^2(1-\rho)(\rho\overline{d}_i+1)}{1+\rho(\overline{d}_i-1)}, \tag{28}$$

$$\beta = \mathbb{C}_i\left(p_r, p_l \mid \{p_j\}_{j\in\bar{N}_i}\right) = \frac{\sigma^2\rho(1-\rho)}{1+\rho(\overline{d}_i-1)}, \tag{29}$$

where $r \in M\setminus\overline{N}_i$ and $l \in M\setminus\left(\overline{N}_i \cup \{k\}\right)$.

Step 3 specifies the elements of variance-covariance matrix Σ in equation (9). Recall indexation (8). We denote conditional variance (11) by v and conditional expectation (10) by μ. We also introduce notations

$$d_{r\setminus i} = |(N_r \setminus N_i)| \text{ and } d_{(r\cap l)\setminus i} = |(N_r \setminus N_i) \cap (N_l \setminus N_i)|,\ r \neq l.$$

By equation (26),

$$\mathcal{L}_i\left(\begin{array}{c} x \\ x+\varepsilon_{\overline{d}_i+1}+\left(p_{\overline{d}_i+1}-\overline{p}\right)\tau^2 \\ \vdots \\ x+\varepsilon_{d_i}+\left(p_{d_i}-\overline{p}\right)\tau^2 \\ x+\frac{1}{d_{\overline{d}_i+1\setminus i}}\sum\limits_{r\in N_{\overline{d}_i+1}\setminus N_i}\left(\varepsilon_r+\left(p_r-\overline{p}\right)\tau^2\right)-\frac{\rho\left(d_{\overline{d}_i+1}-1\right)}{d_{\overline{d}_i+1\setminus i}}\tau^2\left(p_{\overline{d}_i+1}-\overline{p}\right) \\ \vdots \\ x+\frac{1}{d_{d_i\setminus i}}\sum\limits_{r\in N_{d_i}\setminus N_i}\left(\varepsilon_r+\left(p_r-\overline{p}\right)\tau^2\right)-\frac{\rho\left(d_{d_i}-1\right)}{d_{d_i\setminus i}}\tau^2\left(p_{d_i}-\overline{p}\right)\end{array}\right)=$$

$$N\left(\left(\begin{array}{c}\mu\\ \vdots\\ \mu\end{array}\right),\left(\begin{array}{ccc} v & \dots & v\\ \vdots & \Sigma_{1,1} & \Sigma_{1,2}\\ v & \left(\Sigma_{1,2}\right)^T & \Sigma_{2,2}\end{array}\right)\right), \tag{30}$$

where: μ is given by equation (10), v is given by equation (11),

$$(\Sigma_{11})_{r,l}=\begin{cases} v+\tau^2+\tau^4 w & \text{if} \quad r=l,\\ v+\tau^4\beta & \text{if} \quad r\neq l,\end{cases}$$

$$(\Sigma_{12})_{r,l}=\begin{cases} v+\tau^4\left(\beta-\frac{\rho(d_r-1)}{d_{r\setminus i}}w\right) & \text{if } r=l,\\ v+\tau^4\beta\left(1-\frac{\rho(d_l-1)}{d_{l\setminus i}}\right) & \text{if } r\neq l,\end{cases}$$

$$(\Sigma_{22})_{r,r}=v+\frac{\tau^2}{d_{r\setminus i}}+\frac{\tau^4}{d_{r\setminus i}}\left[w\left(1+\frac{\rho^2(d_r-1)^2}{d_{r\setminus i}}\right)+\beta\left(d_{r\setminus i}-1-2\rho(d_r-1)\right)\right],$$

$$(\Sigma_{22})_{r,l}=v+\tau^2\frac{d_{(r\cap l)\setminus i}}{d_{r\setminus i}d_{l\setminus i}}+w\tau^4\frac{d_{(r\cap l)\setminus i}}{d_{r\setminus i}d_{l\setminus i}}$$
$$+\beta\frac{\tau^4}{d_{r\setminus i}d_{l\setminus i}}\left(d_{r\setminus i}d_{l\setminus i}-d_{(r\cap l)\setminus i}-\rho(d_{r\setminus i}(d_l-1)+d_{l\setminus i}(d_r-1))+\rho^2(d_r-1)(d_l-1)\right),$$
$$r,l\in\left\{\overline{d}_i+1,...,d_i\right\},\ r\neq l.$$

Step 4 specifies the elements of the variance-covariance matrix Σ in equation (9) in the simplified setting. Recall that we focus on $\rho \to 0$, which implies that $w \to \sigma^2$ and $\beta \to 0$. Recall, furthermore, that player i believes that $d_{(r \cap l)\backslash i} = 0$. By Step 3, the elements of matrix Σ are given by the following set of equations:

$$\Sigma_{r,l} = \begin{cases} v + z, & \text{if } \; r = l \leqslant d_i; \\ v + \frac{z}{d_{r\backslash i}}, & \text{if } \; r = l > d_i; \\ v & \text{if } \; r \neq l, \end{cases} \tag{31}$$

where, according to indexation (8), indices r and l take values in set

$$\left\{ \overline{d}_i + 1, ..., d_i, ..., \overline{d}_i + 2\overset{\circ}{d}_i \right\},$$

v is given by equation (11), and

$$z = \tau^2 \left(1 + \tau^2 \sigma^2\right). \tag{32}$$

Step 4 completes the proof. By set of equations (31),

$$\Sigma = H + zG,$$

$$\text{where } H = vI\mathbf{1}^T \; ,$$

$$G_{r,l} = \begin{cases} 1 & \text{if } \; r = l \leqslant d_i; \\ \frac{1}{d_{r\backslash i}} & \text{if } \; r = l > d_i; \\ 0, & \text{if } \; r \neq l \end{cases}$$

and z is given by equation (32). Note that $rk(H) = 1$. In order to use equation (20), we find:

$$\left(H\left(zG\right)^{-1}\right)_{r,l} = \begin{cases} \frac{v}{z} & \text{if } \; l \leqslant d_i, \\ \frac{v}{z} d_{l\backslash i} & \text{if } \; l > d_i, \end{cases} \tag{33}$$

$$1 + tr(H\,(zG)^{-1}) = 1 + \varphi v, \tag{34}$$

$$\text{where } \varphi = \frac{\overset{\circ}{d}_i + d'_i}{z},\ d'_i = |N'_i| \tag{35}$$

$$\left((zG)^{-1} H\,(zG)^{-1}\right)_{r,l} = \begin{cases} \frac{v}{z^2} & \text{if } \ r \leqslant d_i \text{ and } l \leqslant d_i; \\ \frac{v}{z^2} d_{l\backslash i} & \text{if } \ r \leqslant d_i \text{ and } l > d_i; \\ \frac{v}{z^2} d_{r\backslash i} & \text{if } \ l \leqslant d_i \text{ and } r > d_i; \\ \frac{v}{z^2} d_{r\backslash i} d_{l\backslash i} & \text{if } \ r > d_i \text{ and } l > d_i. \end{cases} \tag{36}$$

By equations (20) and (33)-(36),

$$Sum\,(H + zG)^{-1} = \varphi - \frac{\varphi^2 v}{1+\varphi v} = \frac{\varphi}{1+\varphi v}. \tag{37}$$

By construction (standard properties of conditional independence) and equations (17) and (37),

$$\begin{gathered} \mathbb{V}_i(x \mid s_i,\ \{A_{j,t} \mid j \in N_i,\ t = 1, 2\}) = \\ \mathbb{V}_i(x \mid \{p_j, s_j\}_{j \in \overline{N}_i},\ \{\widetilde{s}_{j,1}, \widetilde{s}_{j,2}\}_{j \in \overset{\circ}{N}_i}) = v\left(1 - v\mathbf{1}^T(\Sigma)^{-1}\mathbf{1}\right) = \\ v\left(1 - v\frac{\varphi}{1+\varphi v}\right) = \frac{v}{1+\varphi v}. \end{gathered}$$

Appendix D: proof of Lemma 2.

Step 1 shows that if $f \in \mathcal{G}$ and $g \in \mathcal{G}^*$, then $d_i \geqslant 2n-1$ for any $i \in M$. Indeed, suppose (by contradiction) that exist $i \in M$ with $d_i < 2n-1$, then, $\overline{d}_i < 2n-1$. By true inequality $n_i \leqslant m$ and Proposition 1, $l_i\,(g) < \max_{i \in M} l_j\,(f)$, hence $g \notin \mathcal{G}^*$.

Step 2 proves that closed neighbourhood by any least happy player i is a completely connected subgraph of size $2n-1$, that is,

$$N_r = N_i \text{ for any } i \in L\,(g) \text{ any } r \in \overline{N}_i \text{ and any } g \in \mathcal{G}. \tag{38}$$

Consider $i \in L(g)$ and $r \in \overline{N}_i$. By definition of $\overline{N}_i$, $N_i \subseteq N_r$, hence, $d_r + d'_r \leqslant d_i + d'_i$ and $\overline{d}_r \leqslant \overline{d}_i$. However, $N_i \subset N_r$, means that there exist $j \in N_i \backslash N_r$. If $j \in N_i$, then $\overline{d}_r < \overline{d}_i$. Otherwise, $d_r + d'_r < d_i + d'_i$. In either case, by Proposition 1, $l_r(g) < l_i(g)$, which contradicts to $i \in L(g)$.

Step 3 proves that for any $g \in \mathcal{G}^*$, if there exist $i \in L(g)$ such that $\mathring{d}_i > 0$ then $\overline{d}_i \leqslant 2n - 1$. Suppose, by contradiction, that there exist $i \in L(g)$ such that $\mathring{d}_i > 0$ and $\overline{d}_i \geqslant 2n$. By statement (38), $N_r = N_i$ for any $r \in \overline{N}_i$. It takes $\frac{\overline{d}_i(\overline{d}_i - 1)}{2} \geqslant n\left(\overline{d}_i - 1\right)$ links to interconnect all players in $\overline{N}_i$ and $2n\mathring{d}_i$ links to connect them to i's open neighbour(s). At the same time, players in N_i can build nd_i links. Therefore,

$$n\left(\overline{d}_i - 1\right) + 2n\mathring{d}_i \leqslant nd_i, \text{ which implies } \mathring{d}_i \leqslant 1.$$

Suppose that $\mathring{d}_i = 1$. Call h the unique open neighbour by i common with i's closed neighbours. Consider $M \backslash N_i$. Let $R = |M \backslash N_i|$. Players in $M \backslash N_i$ can build nR links of which at least one link goes to player h. Hence, their average degree is $\frac{2nR-1}{R} < 2n$, which implies that there exist player $j \in M \backslash N_i$ such that $d_j \leqslant 2n-1$, and so either $\overline{d}_j < 2n-1$ or $d_j + d'_j \leqslant 2n-1$. In either case, by Proposition 1, $l_j(g) > \max_{i \in M} l_i(f)$, hence $g \notin \mathcal{G}^*$ (a contradiction).

Step 4 proves that for any $g \in \mathcal{G}^*$, and for all $i \in L(g)$ such that $\mathring{d}_i > 0$,

$\overline{d}_i = 2n - 1$ and

$$d_i + d_i' = m. \tag{39}$$

Consider $i \in L(g)$ such that $\mathring{d}_i > 0$. By Step 3, $\overline{d}_i \leqslant 2n - 1$. By Proposition 1, $\overline{d}_i = 2n - 1$ and $d_i + d_i' = m$ (if $\overline{d}_i < 2n - 1$ or $d_i + d_i' < m$, then $l_i(g) > \max_{i \in M} l_j(f)$, which contradicts $g \in \mathcal{G}^*$).

Step 5 shows that if $g \in \mathcal{G}^*$ and there exist $i \in L(g)$ such that $\mathring{d}_i > 0$ then $\mathring{d}_i = 1$. Indeed, by step 3 and statement (38), all $2n - 1$ players in $\overline{N}_i$ are interconnected, which leaves capacity to build at most one 1 link per player and $2n - 1$ links with other players overall. At the same time, by statement (38), all $2n-1$ players in $\overline{N}_i$ are connected to each of the players in $\mathring{N}_i$, which requires $\mathring{d}_i (2n - 1)$ links. Hence, players in $\mathring{N}_i$ build at least

$$\mathring{d}_i (2n - 1) - (2n - 1) = \left(\mathring{d}_i - 1\right)(2n - 1)$$

links to players in $\overline{N}_i$, which leaves them with a possibility to build at most

$$\mathring{d}_i n - \left(\mathring{d}_i - 1\right)(2n - 1) \leqslant 2n - 1 - \mathring{d}_i (n - 1) \tag{40}$$

links to $m - 2n$ players in $M \backslash N_i$. Suppose that $\mathring{d}_i \geqslant 2$. Then the right-hand side of inequality (40) is weakly below 1. That is, players in $\mathring{N}_i$ build at most one link to players in $M \backslash N_i$. At the same time, by equation (39), each player in $M\backslash N_i$ is connected to at least one player in $\overset{\circ}{N}_i$, which requires $m - 2n$ links. By inequality (40), all but one player in $M\backslash N_i$ use at least one of their hands to connect to at least one of the players in N_i. Hence, their average degree is at most

$$\tfrac{2(n(m-2n)-(m-2n-1))+m-2n}{m-2n} = 2n - 1 + \tfrac{1}{m-2n}.$$

Therefore, there exist player $j \in M\backslash N_i$ such that $d_j \leqslant 2n - 1$ and $\overset{\circ}{d}_j > 0$, so $\overline{d}_j \leqslant 2n - 2$. By Step 1, $g \notin \mathcal{G}^*$ (a contradiction).

Step 6 shows (by construction) that if $g \in \mathcal{G}^*$ and there exist $i \in L(g)$ such that $\overset{\circ}{d}_i > 0$ then $g = f$. By statement (38), for any player $i \in L(g)$, we can visualize i and his closed neighbours as a "petal". By Steps 4 and 5, player i and his closed neighbours have one common open neighbour, say, h connected with all players in $M\backslash N_i$. Consider $m - 2n$ players in $M\backslash N_i$. If there exist player $i_1 \in (M\backslash N_i) \cap L(g)$, then, $d_{i_1} = 2n - 1$ and by Steps 3 and 4, $d_{i_1} = 2n$, $d_{i_1} + d'_{i_1} = m$ and $N_j = N_{i_1}$ for all $j \in \overline{N}_{i_1}$. We can therefore visualize i_1 and his closed neighbours as the second petal connected to the first petal through h. Applying this argument repetitively, we end up with a the situation in which any player not organized in a petal yet has closed degree of at least $2n$ and is connected to h.

We denote the set of these remaining players with $\mathcal{R}$. Note that by construction, none of the players in set $\mathcal{R}$ receives links from the players organized in the above petals. Therefore, relatively high degree by each of them is achieved through their own linking capacity plus possibly that of the central hub h. Players in set $\mathcal{R}$ and hub h together can build $n(|\mathcal{R}|+1)$ links, increasing their own sum of degrees by $2n(|\mathcal{R}|+1)$.[26] The hub h receives $|\mathcal{R}|$ links. The average degree by the players in set $\mathcal{R}$ is therefore equal to

$$\frac{\sum\limits_{i\in\mathcal{R}} d_i}{|\mathcal{R}|} = \frac{2n(|\mathcal{R}|+1)}{|\mathcal{R}|} + 1. \tag{41}$$

Recall that it shall exceed $2n+1$, which implies

$$|\mathcal{R}| \leqslant 2n. \tag{42}$$

However, by construction of small petals, true equation $|g| = m$ and equation (14), we find

$$|\mathcal{R}| \geqslant 2n + (2n-1)l, \tag{43}$$

where $0 \leqslant l \leqslant m-1$. By equations (42) and (43), $|\mathcal{R}| = 2n$. This means that the hub h and $2n$ players in set $\mathcal{R}$ are interconnected, forming the large petal.

[26] $|\mathcal{R}|$ denotes the cardinality of set $\mathcal{R}$, that is, the number of players in set $\mathcal{R}$.

Appendix E: proof of Proposition 2.

Step 1 shows that if $g \in \mathcal{G}^*$ and $g \neq f$, then any player $i \in L(g)$ belongs to a completely connected component of size at least $2n$. By Lemma 2, if $g \in \mathcal{G}^*$ and $g \neq f$, then $\mathring{d}_i = 0$ for any $i \in L(g)$. Hence, $N_i' = \varnothing$, and

$$n_i = \overline{d}_i < m. \tag{44}$$

Because $g \in \mathcal{G}^*$,

$$l_i(g) \leqslant \min_{j \in M} l_j(f). \tag{45}$$

By equation (44), inequality (45), and proposition 1, $\overline{d}_i \geqslant 2n$. Hence,

$$\text{if } g \in \mathcal{G}^* \text{ and } g \neq f \text{ then } \mathring{d}_i = 0 \text{ and } \overline{d}_i \geqslant 2n \text{ for any } i \in L(g). \tag{46}$$

By Step 2 in Appendix D, $N_j = N_i$ for any $j \in \overline{N}_i$.

Step 2 shows that if $i \in M$, then $\mathring{d}_i = 0$, $N_j = N_i$ for any $j \in \overline{N}_i$ and $\overline{d}_i \geqslant 2n$. By Step 1, the statement is true for any $i \in L(g)$. Consider $M \backslash L(g)$, that is set M without the least happy players. If $M \backslash L(g) = \varnothing$, the statement of Step 2 holds. Suppose $M \backslash L(g) \neq \varnothing$. By Step 1, $|L(g)| \geqslant 2n$. Therefore, $|M \backslash L(g)| \leqslant m - 2n$. Therefore, $d_k + d_k' \leqslant m - 2n < m$ for any $r \in M \backslash L(g)$. Because $g \in \mathcal{G}^*$ and $g \neq f$, $l_r(g) \leqslant \min_{j \in M} l_j(f)$ for any $r \in M \backslash L(g)$. By Proposition 1, $\overline{d}_r \geqslant 2n$ for any $r \in M \backslash L(g)$. At the same time, the average degree by players in $M\backslash L(g)$ is $2n$ (each player can build n links, each link increases sum of degrees by 2). Therefore,

$$d_r = \overline{d}_r = 2n \text{ for any } r \in M\backslash L(g). \tag{47}$$

Suppose there exist $r \in M\backslash L(g)$ and $j \in \overline{N}_r$ such that $N_j \subset N_r$. Then, r is an open neighbour by j, which contradicts to statement (47).

Step 3 proves that if $g \in \mathcal{G}^*$ and $g \neq f$, then $\overline{d}_i \leqslant 2n+1$ for any player i in network g. Consider $g \in \mathcal{G}^*$, $g \neq f$ and some player i in network g. By Step 2, all players in i's closed neighbourhood are interconnected, which takes $\frac{\overline{d}_i(\overline{d}_i-1)}{2}$ links. These players can build only $\overline{d}_i n$ links. Therefore,

$$\frac{\overline{d}_i(\overline{d}_i-1)}{2} \leqslant \overline{d}_i n,$$

which is equivalent to $\overline{d}_i \leqslant 2n+1$.

Step 4 shows that

$$\mathcal{G}^* = \begin{cases} \{c\}, \text{ if } \sigma^2\tau^2 > \frac{m-(2n+1)}{2} \\ \{c, f\}, \text{ if } \sigma^2\tau^2 = \frac{m-(2n+1)}{2} \\ \{f\}, \text{ otherwise.} \end{cases}$$

Recall that it is feasible to build network c as follows: players divide into groups of size $2n+1$, $2n+1$ players in a group form a circle and each player connects to n next players on his right. By Steps 2 to 4, network c is the most efficient network in set $\mathcal{G}\backslash f$. By Proposition 1, c is weakly more efficient than f iff inequality (15) is true.

Appendix F: proof of Proposition 3.

Step 1 shows that flower network is a Nash equilibrium. Indeed, by equation (14), flower network may be build as follows: The players divide into groups of which one has size $2n+1$ and other k have size $2n-1$. Players in the group of size $2n+1$ interconnect, say, they form a circle and each player connects to n players on his right. One player in this group is marked with index h. Players in each group of size $2n-1$ form a circle. Each player connects to $n-1$ players on his right and to player h.

Consider a unilateral deviation by player i from the above strategy profile. By this deviation player i establishes a link with a player in a different "petal", scarifying a link with either one of the players in his petal or the central hub. As a result, his total degree does not increases while his closed degree decreases by $2n-2$ (he looses all closed neighbours but himself). By Proposition 1, his loss goes up. Hence, flower network is an equilibrium.

Step 2 by equation (13), network c may be built as follows: The players divide into groups of size $2n+1$. Players in each group of size $2n+1$ interconnect, say, they form a circle and each player connects to n players on his right.

2.1. Consider a unilateral deviation by player i form the above strategy profile. By this deviation player i establishes a link with a player in a different component sacrificing a link with a player in his component. Thereby, he increases his total degree by $2n+1$ and he decreases his closed degree by $2n$ (he looses all closed neighbours but himself). By proposition 1, the deviation is unprofitable iff

$$\sigma^2\tau^2 \geqslant 1 + \tfrac{1}{2n}. \tag{48}$$

Hence, network c is an equilibrium iff inequality (48) holds.

2.2. Let us show that equations (14) and (13) imply

$$1 + \tfrac{1}{2n} < \tfrac{m-(2n+1)}{2}. \tag{49}$$

By equation (14), inequality (49) is equivalent to

$$(2n-1)k \geqslant 2 + \tfrac{1}{n},$$

which holds for any $k \geqslant 2$. By equations (14) and (13)

$$(2n+1)\,(l-1) = (2n-1)k > 0,$$

which implies $l \geqslant 2$, hence, $k \geqslant 2$.

Appendix G: different number of players and connection capacity.

Propositions 2 and 3 were obtained under restrictions (14) and (13) on the number of players m and a player's connection capacity n. We can show that if we relax these restrictions then either the flower network or flower-like network(s) is/are efficient and constitute(s) a Nash equilibrium provided that the interpretation noise is sufficiently low (full analysis is available upon request). The following examples illustrate this claim. In all examples a player's connection capacity n is equal to 2.

G.1 Illustrative examples.

Examples G.1 and G.2 illustrate that when parameter restriction (14) holds while parameter restriction (13) fails, flower network f is the most efficient network constituting a Nash equilibrium (except if $n = m = 1$).

Example G.1. Suppose that $m = 11$. Note that parameter restriction (14) holds, and it is, therefore, feasible to build flower network with 2 petals of size $2n - 1 = 3$ and one petal of size $2n + 1 = 5$. At the same time, parameter restriction (13) fails. The size of the smallest component in any network composed of completely connected components is, at most, 3. Therefore, any such network is less efficient than the flower. Hence, the flower is the unique equilibrium network.

Example G.2. Suppose that $m = 29$. Then, parameter restriction (14) holds and it is, therefore, feasible to build flower network with 6 petals of size 3 and one petal of size 5. At the same time, parameter restriction (13) fails and it is, therefore, impossible to divide the players into completely connected components of size 5 each. There are at least two networks composed of completely connected components of size 4: network c_1 composed of 6 components of size 4 and one of size 5 and network c_2 composed of 5 components of size 5 and one of size 4. Flower network f is efficient, outperforming either network c_1 or c_2, iff the interpretation noise is weakly below threshold $m - 2n = 25$ (note that this threshold lies above that in Propositions 2 and 3). Furthermore, while the flower network f constitutes a Nash equilibrium, this is not true for either network c_1 or c_2, because any player with an excess connection capacity benefits from deviation.

Examples G.3 and G.4 illustrate that flower-like network(s) is/are efficient and constitute a Nash equilibrium the interpretation noise is sufficiently low.

Example G.3. Suppose that $m = 10$. Parameter restriction (13) holds and it is therefore possible to build network c composed of two complete components of size 5. At the same time, parameter restriction (14) fails and we cannot build network f. However, we can build network $\widetilde{f}$ termed hereafter *symmetric flower* which is depicted in Figure G.1. Propositions 2 and 3 hold for flower network f being replaced for symmetric flower network $\widetilde{f}$.

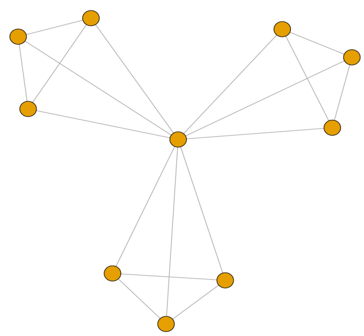

Figure G.1: symmetric flower $\widetilde{f}$ ($m = 10$, $n = 2$).

Example G.4. Suppose that $m = 9$. Then, it is possible to build flower-like network termed *generalized flower*, see Figure G.2. Note that alternatively we could build flower-like network with one petal of size 4 and two petals of size 2. Either of these networks is efficient[27] and it constitutes a Nash equilibrium when the interpretation noise lies below threshold 0.43.

[27] We could refine efficiency criterion by requiring the number of losers to be minimal.

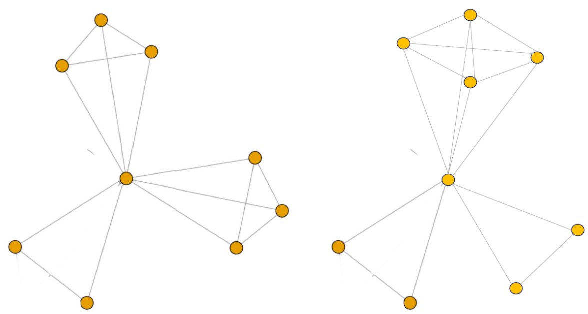

Figure G.2: generalized flowers of level 2
($m = 9$, $n = 2$).

G.2. Relaxing parameter restriction (13).

Let us keep parameter restriction (14) and relax parameter restriction (13). Without loss of generality there exist $l \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N} \cup \{0\}$, $q \leqslant 2n$ such that

$$m = l(2n + 1) + q. \tag{50}$$

Note that by equation (14), $l > 0$. The main text focuses on $q = 0$. Suppose that $q > 0$.

Definition G.1. *The set of networks composed of completely connected components of which the smallest has size* $2n$ *is denoted*

$$\mathcal{C} = \{g \in \mathcal{C} \mid \textit{for any } i \in M: \ d_i \geqslant 2n \textit{ and } N_i = N_j \textit{ for any } j \in N_i\}. \tag{51}$$

Proposition G.1. *Suppose that m and n are such that parameter restriction (14) holds. Suppose, furthermore, that q defined by equation (50) is positive. Then,*

$$\mathcal{G}^* = \begin{cases} \mathcal{C} \text{ if } q + l \geqslant 2n \text{ and } \sigma^2\tau^2 > m - 2n, \\ \mathcal{C} \cup \{f\} \text{ if } q + l \geqslant 2n \text{ and } \sigma^2\tau^2 = m - 2n, \\ f \text{ otherwise.} \end{cases}$$

Proof.

Step 1 shows that if $q > 0$ and

$$q + l \geqslant 2n, \tag{52}$$

$$\mathcal{G}^* = \begin{cases} \mathcal{C}, \text{ if } \sigma^2\tau^2 > m - 2n \\ \mathcal{C} \cup \{f\}, \text{ if } \sigma^2\tau^2 = m - 2n \\ \{f\} \text{ otherwise.} \end{cases}$$

By Step 2 in Appendix E, any efficient network different from f lies in set $\mathcal{C}$. Let us build a network in $\mathcal{C}$ maximizing the size of its smallest cluster. To this goal, let us divide the total number of players m into groups of sizes as equal as possible in the following way: Start with l groups of size $2n + 1$ and one "residual" group of size q and repetitively move one player from the largest existing group to the smallest one. After $2n - q \leqslant l$ steps, the size of the residual group is $2n$, hence, the difference between the sizes of any pair of groups becomes no higher than one. Once the procedure is over, let the players in each group interconnect. Thereby, we form a network in $\mathcal{C}$. Potentially, we could build other networks in $\mathcal{C}$ by continuing the above procedure as long as the distribution of the clusters' sizes remains constant. The size of the smallest cluster is $2n$. By proposition 1, this network is weakly more efficient than f iff

$$\sigma^2\tau^2 \geqslant m - 2n. \tag{53}$$

Step 2 shows that when $q > 0$ and $q + l < 2n$, $\mathcal{G}^* = f$. Consider the procedure described in Step 1. After l steps the size of the residual group is still below $2n$. If we continue the procedure until the distribution of groups' sizes becomes constant, at least one group will have size $2n - 1$.[28] Hence, set $\mathcal{C}$ is empty. By Step 2 in Appendix E the unique efficient network is f.

Proposition G.2. *If $n = k = 1$, the most efficient equilibrium network is that in proposition G.1. Otherwise, the most efficient Nash equilibrium is network f.*

Proof.

Step 1. Suppose $n = k = 1$, so that $m = 4$. By Step 1 in Appendix F, network f (a "star" with three peripheral players connected to the central hub) is a Nash equilibrium. Set $\mathcal{C}$ is a singleton. Its unique element is complete network connecting 4 players. Trivially, it is a Nash equilibrium. Hence, Proposition G.1 describes not only the most efficient network but also an equilibrium

[28] Note that by equation (14) the size of the smallest cluster is weakly above $2n - 1$.

network.

Step 2. Suppose, from now on, that $n + k \geqslant 2$. Suppose first, that inequality (52) holds. By Proposition G.1, the efficient network is either f or a network is set $\mathcal{C}$, depending on the magnitude of the interpretation noise. However, no network in set $\mathcal{C}$ is a Nash equilibrium. Indeed, inequality (52) is equivalent to $q \geqslant 2n - l$. Therefore,

$$m = l(2n+1) + q \geqslant 2n(l+1), \text{ hence, } m > 2n,$$

which implies that any network in set $\mathcal{C}$ has at least two components. Consider the smallest component of network in set $\mathcal{C}$. Its size is $2n$. It takes $n(2n-1)$ links to build it. At the same time, the players in this component can build $2n^2$ links. Therefore, at least one of them has unused connection capacity, which he can use to establish a link with a player in a different component increasing thereby his total degree by at least $2n$. By Proposition 1, this deviation is profitable. By Step 1 in Appendix F network f is a Nash equilibrium. By Step 2 in Appendix F, network f is the more efficient than any network in set $\mathcal{G} \backslash \mathcal{C}$. Hence, network f is the most efficient Nash equilibrium.

Step 3. Suppose finally that inequality (52) does not hold. Then, by Proposition G.1, the most efficient network is f. By step 1 in Appendix F network f is a Nash equilibrium. Hence, network f is the most efficient Nash equilibrium.

Appendix H: numerical robustness check of proposition 3.

Proposition 3 (as Proposition 2) does not apply when the number of players is $m = 8$ and the individual connection capacity is $n = 2$ because parameter restriction (13) is not satisfied. Appendix G.2 shows that the most efficient Nash equilibrium is the flower depicted in Figure 5, regardless of the magnitude of transmission noise. The reason is that the network composed of two compete components with 4 players each is not an equilibrium because there is unused connection capacity, hence, the incentives to deviate.

We check numerically whether this result may extend to the generalized setting. We have seen that Proposition 2 may extend to the generalized setting for sufficiently small values of ρ namely, ρ in set $\{0.1, 0.2, 0.35\}$. We therefore focus on these values of ρ. As in the previous section, we keep $\tau^2 = 1$, and vary the interpretation noise σ^2. We consider a profile of strategies leading to formation of flower f (proposed equilibrium) and show that no player can benefit from unilateral deviation.

The central hub cannot deviate in a profitable way, as in the proposed equilibrium he learns the signals of all the players and receives payoff $\frac{1}{9}$. Figure H.1 depicts possible deviations by a peripheral player. Deviations 1 and 2 refer to a peripheral player from the small petal (for concreteness, player 8). Deviations 3 and 4 refer to a peripheral player from the large petal (for concreteness, player 5).

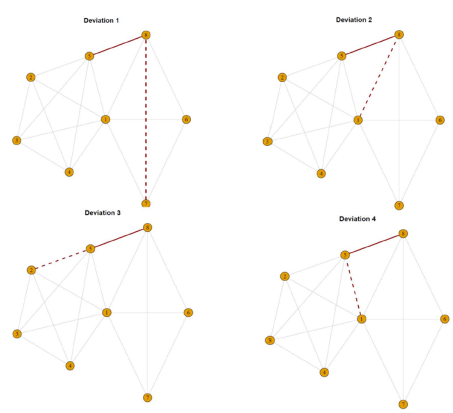

Figure H.1: Deviations by a peripheral player from the proposed equilibrium.

Figure H.2 depicts the losses given by equation (2) by peripheral players 5 and 8 in the proposed equilibrium and under the above deviations, depending on interpretation noise σ^2 for $\rho = 0.35$ (the figures for smaller values ρ of are similar). We observe that both player 5 and player 8 have strong incentives to comply with the proposed equilibrium strategy. Hence, the flower network is a Nash equilibrium.

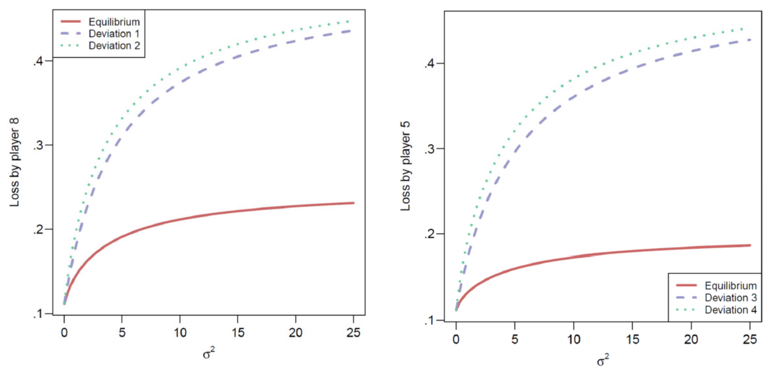

Figure H.2: Losses by player 5 (right) and player 8 (left) in the proposed equilibrium (red solid curve) and following possible deviations (blue dashed and green dotted curves).

Appendix I: alternative network formation protocol.

Unilateral link formation may be viewed as an extremely asymmetric investments in links. This section illustrates that our results do not hinge on this asymmetry. Following Hojman and Szeidl (2008), we modify network formation protocol as follows. The players simultaneously choose their investments in links. Player i invests t_i^j in link with player j. The link between i and j is formed iff the joint investment by players i and j lies above a given threshold

I:

$$t_i^j + t_j^i \geqslant I.$$

Investments in links by player i are added to his disutility, hence his objective is to minimize

$$\widetilde{l}_i(g) = l_i(g) + \sum_{j \in M} t_i^j. \tag{54}$$

We can prove that the efficient network can be formed in equilibrium via a profile of strategies involving strictly positive contributions from the hub (available upon request). We illustrate our argument taking Example G.3 in Section G.1 ($m = 10$ and $n = 2$). Suppose that $\tau^2 = 1$, $\sigma^2 = 4$. Propositions 2 and 3 hold with flower f being replaced with the symmetric flower $\widetilde{f}$ depicted in Figure G.1 (proof is available upon request). Let us prove that when I lies in the interval [0.03, 0.1], symmetric flower may be formed in equilibrium via fully symmetric investments: any player invests $\frac{I}{2}$ in any of his links.

Note that loss (12) is convex in either the number of closed neighbours and the joint number of local and distant neighbours:

$$\frac{\partial l_i(g)}{\partial \bar{d}_i} < 0,\ \frac{\partial^2 l_i(g)}{\partial \bar{d}_i^2} < 0,\ \frac{\partial l_i(g)}{\partial \left(d_i + d_i'\right)} < 0,\ \frac{\partial^2 l_i(g)}{\partial \left(d_i + d_i'\right)^2} < 0.$$

Consider the central hub h. He can deviate from the proposed equilibrium strategy by sacrificing any subset of links, saving thereby sum $\frac{I}{2}$ multiplied by the cardinality of this subset. However, by the above convexity of loss (12) in the number of closed neighbours $\overline{d}_h$ it suffices to check that h is not willing to deviate at the margin. That is, he does not want to save sum $\frac{I}{2}$ loosing a link with one of the peripheral players and thereby 3 closed neighbours. By equation (12), it is true for any $I \leqslant 0.1$.

Now consider a peripheral player i. He can deviate from the proposed equilibrium strategy in three ways. First, he can save $\frac{I}{2}$ by sacrificing a link with one of his closed neighbours, loosing, thereby, all closed neighbours but himself. Second, he can save $\frac{I}{2}$ by sacrificing the link with the central hub and losing thereby all closed neighbours but himself and all distant neighbours. By equation (12), these deviations are unprofitable when $I \leqslant 0.1$. The third possible deviation is to link with distant player from a different petal so as to gain one closed neighbour (recall that i perceives this player to be linked only with the central hub). This is unprofitable for any I above 0.03.