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Abstract

The theory of Bayesian mechanism design is of interest to economists and computer scientists alike. It has focused on two extreme assumptions on the beliefs of the agents, full-freeness (or independence) and no-freeness (or Beliefs Determine Preferences). We discuss more general conditions that cover intermediate cases between these two extremes and characterize the corresponding set of implementable mechanisms. We also discuss applications of these results to economics and to computer science.

1 Introduction

In recent years, theoretical computer scientists have revisited many of the contributions of the economics mechanism design literature, at the same time proving new results and proposing new interpretations of existing results. In particular, computer scientists have found new applications for Bayesian mechanism design.[1]

In the economic literature, mechanisms implement decision rules, that is, functions from the 'types' of the agents to public decisions. In a quasilinear environment, they must satisfy incentive compatibility constraints, a budget balance constraint, and, sometimes, participation constraints. One of the following questions is generally asked: a) "When is it possible to implement efficient decision rules?', that is, rules that maximize the sum of the welfare of the participating agents or b) "When can we implement any arbitrary decision rule?". In both cases, the 'when' is interpreted as 'under which conditions on the information structure', that is, in practice, under which conditions on the correlation between the types of the agents.

This description of mechanism design, focused on collective efficiency, is also relevant to some applications in computer science; Albert et al. (2022) refer to "new applications of incentive compatible distributed systems, such as federated server farms, where a group shares resources in an efficient manner without any money being transferred out of the system". However, contrary to economists, computer scientists are generally not interested solely in the welfare of the participating agents, but rather in the objective of the mechanism designer, which depends on the public decisions, and sometimes on the types of the agents.[2] Their problem is therefore to find ways to maximize the designer utility under the incentive compatibility, budget balance and participation constraints (Conitzer and Sandholm, 2004) — of course, the collective, 'economic', problem is a special case of this more general problem,

[1] These are mostly internet applications characterized by settings where identical mechanisms are repeated with high frequency providing reliable data about the distribution of types. See for example Hartline and Lucier (2015) who use distributional information to improve the tractability of algorithmic mechanism design (using approximation algorithms) and Albert, Conitzer, Lopomo and Stone (2022), who use such information to extend the class of robust mechanisms

[2] Conitzer and Sandholm (2004) express as follows the limits of the economics literature from a computer science perspective "The most famous and most broadly applicable general mechanisms, VCG [Vickrey, Clarke, Groves] and dAGVA [d'Aspremont, Gérard-Varet, Arrow], only maximize social welfare. If the designer is self-interested, as is the case in many electronic commerce settings, these mechanisms do not maximize the designer's objective."

where the utility of the designer is the aggregate utility of the participants.

With this background, we can simply describe the approach which we take in this paper (this is discussed more formally in Section 2.3). Mechanism designers face very large optimization problems, with an incentive compatibility constraint and a participation constraint for every type of each of the agents. We provide a decomposition of the problem into an easier optimization problem and an implementation problem. The optimization problem is easier because we provide a characterization of implementable decision rules which does not require a fine grained knowledge of the distribution of the types of the agents. The implementation problem is easier as we provide constructive techniques to implement the decision rule identified by the optimization problem.

In the setting in which we work, where agents have fully transferable utilities, economists have answered the questions we asked in the first paragraph of this introduction in two extreme and opposite cases, which depend on the (conditional) beliefs of the agents about the types of the other agents. When the beliefs of any agent about the types of the others are independent of his own type, the mechanism designer can implement efficient decisions. When these beliefs vary sufficiently strongly with the type of agents, so that to different types correspond different beliefs, some conditions, such as Condition $\overline{B}$ that plays an important role in this article and which we generalize, ensure that *any* decision rule can be implemented. These results provide a solution to the decomposition problem: when types are independent, compute the efficient decision rule, and the literature then provides techniques for designing a mechanism which implements it.[3] When Condition $\overline{B}$ holds, the mechanism designer can choose her favorite decision rule, whatever it is, and, as we discuss below, can easily construct a mechanism to implement it.[4]

There are, of course, strong connections with the auction literature, where very different results are obtained with independence and with correlation of types. Indeed, as it will become easy to see in the following, Condition $\overline{B}$ is a stronger version of the Crémer-McLean condition (Crémer and McLean, 1985, 1988))[5]. We discuss more precisely the links between our results with those of the auction literature in the conclusion.

[3]For the use of dAGVA or Arrow-AGV transfers (Arrow, 1979; d'Aspremont and Gérard-Varet, 1979), see below p. 15.

[4]As is well known, a similar dichotomy is found in auction theory. In that case, the main restriction is that independence prevents full extraction of the surplus. See Albert, Conitzer and Lopomo (2015) for further discussion.

[5]See also Lemma A2 in Kosenok and Severinov (2008).

In this paper, we focus on intermediate cases between independence and correlation. These in-between cases are described through "independence classes" of types: Two types of an agent belong to the same independence class if they generate the same beliefs about the types of the other agents. When the types are "independent", each agent has only one independence class, the set of all his types; with no independence, each agent has as many independence classes as types as each independence class contains only one type. In our more general setup, each independence class can contain any number of types. In parallel, we also introduce a restricted notion of efficiency, "Independence Class efficiency" (IC-efficiency), which requires the decision to be efficient within each independence class. IC-efficiency has no special ethical claim, but should be a useful tool for a self-interested mechanism designer.

With this background, we introduce two new conditions on the beliefs of the agents, intermediate between independence and correlation, and show that they are necessary and sufficient for implementing any IC-efficient mechanism. The first of these conditions, the Generalized Condition B, is appropriate when there are no participation constraints. In section 5.1, we show that it is equivalent to the "compatibility condition", Condition C, first introduced by d'Aspremont and Gérard-Varet (1979, 1982) and discussed in d'Aspremont, Crémer and Gérard-Varet (2003). The second condition, the Generalized Condition $\overline{B}$, is stronger than the first and new in the literature. It is appropriate when there are (interim) participation constraints. It requires a new condition on the total surplus to be distributed to the agents, which is presented in Theorem 5.

These results may be seen as generalizing to all in-between cases the results of Makowski and Mezzetti (1994) for the limit case of independent beliefs and of Kosenok and Severinov (2008) for the other limit case where, for every agent, different beliefs are associated with different types.[6]

Some years ago Auriol and Laffont (1992) and Auriol (1993) used the same strategy to study the regulation of a duopoly when knowing the cost of one firm provides some, but not full, information on the cost of the other. A similar procedure is adopted in a recent paper by Albert et al. (2015), using an "automated mechanism design approach" to auctions. They assume that there is a single buyer and an external signal which is correlated with this buyer's valuation. In that sense, their framework is similar to that of Riordan

[6]The famous paper of Myerson and Satterthwaite (1983) proves an impossibility theorem in a framework with two agents and independent types where the equivalent of the generalized surplus condition introduced by Makowski and Mezzetti (1994) does not hold.

and Sappington (1988).

The paper is organized as follows. In Section 2 we introduce the framework and define Bayesian implementation as well as independence classes of beliefs. We also formalize our decomposition approach in 2.3. In Sections 3 and 4, we introduce our conditions on independence classes of beliefs (Generalized Conditions B and $\overline{B}$, respectively) and characterize the set of mechanisms that can be implemented when they hold. In section 5.1, we re-interpret our conditions in terms that are well known in the literature.[7]

2 Framework

2.1 Agents and beliefs

We study the design of a mechanism to which a set $\mathcal{N} = \{1, 2, \dots, n\}$ of agents, with $n \geq 3$, participate. The *payoff* of agent i,

$$u_i(x; \alpha_i) + t_i,$$

is the sum of a) his *utility* u_i, which depends on the public decision x chosen in a set[8] $\mathcal{X}$ and on his type α_i and of b) the monetary transfer t_i (which could be negative) that he receives from the mechanism.[9]

Only agent i knows his type α_i, which belongs to a finite set $\mathcal{A}_i$. This type affects his utility, but may also affect his beliefs about the types of the other agents, which are represented by a probability distribution $p_i(\alpha_{-i} \mid \alpha_i)$ over $\mathcal{A}_{-i} = \prod_{j \in \mathcal{N}_{-i}} \mathcal{A}_j$. The beliefs are "consistent": they are obtained

[7]The older literature on the topic is reviewed in d'Aspremont et al. (2003), pp. 281 to 283. Surveys from a computer science point of view include Hartline (2013) and Conitzer and Sandholm (2004).

[8]Because the agents have a finite number of types, for our implementation results we formally need no assumption over the set of decisions. However, the mechanism designer should be able to maximize her utility over the set of possible mechanisms — this is satisfied, for instance, if $\mathcal{X}$ is compact and her utility continuous. When the mechanism designer is a social welfare maximizer, or, more generally, when she takes into account the utility of the agents in her choices this would require that the utility functions of the agents be also continuous.

[9] Some of our results stand if the utility of agent i is a function from $\mathcal{X} \times \mathcal{A}$ into $\Re$, that is if the types of the other agents also influence the value that agent i attaches to decisions — the so-called "common value" case. Then, the payoff of agent i is written

$$u_i(x; \alpha) + t_i.$$

In economic terms, this generalization may be significant. In many cases, when a public decision is taken agents possess private information that would influence the evaluation of the possible choices by the other agents.

from a common probability distribution p on $\mathcal{A}$ by conditioning with respect to α_i, *i.e.* $p_i(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \alpha_i) = p(\alpha)/p(\alpha_i)$. We let $p(\alpha_i)$ and $p(\alpha_i, \alpha_j)$ denote the corresponding marginal probability distribution of agent i's type and the marginal probability distribution of types of agents i and j, respectively. Without loss of generality, $p(\alpha_i) > 0$ for all i and all α_i.

We call *information structure* an object of the type $(\mathcal{A}, p)$. Information structures, as well as the associated notion of *independence classes* on beliefs, which we will define shortly, play a fundamental role in the following. As Albert et al. (2022) put it "If a mechanism designer intends to maximally exploit a correlated valuations setting, she must use information about the distribution."

We say that agent i has *free beliefs* when

$$p(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \alpha_i') \text{ for all } \alpha_i, \alpha_i' \in \mathcal{A}_i, \text{ and all } \alpha_{-i} \in \mathcal{A}_{-i}. \quad (1)$$

An information structure $(\mathcal{A}, p)$, satisfies *full-freeness* if all agents have free beliefs.

Independence is the standard condition that the types of the agents are independent.

$$p(\alpha) = \prod_{i \in \mathcal{N}} p(\alpha_i).$$

The following lemma, which we have not seen proved in the literature, will be useful.

Lemma 1. *An information structure $(\mathcal{A}, p)$ satisfies full-freeness if and only if it satisfies independence.*

Proof. That independence implies full-freeness is obvious.

If p satisfies full-freeness, there exists a function $k_1(\alpha_{-1})$ such that $p(\alpha_{-1} \mid \alpha_1) = k_1(\alpha_{-1})$ for all α_{-1} and all α_1, and therefore $p(\alpha) = p(\alpha_1)k_1(\alpha_{-1})$ for all α. Summing over all $\alpha_1 \in \mathcal{A}_1$, we obtain $p(\alpha_{-1}) = k_1(\alpha_{-1})$ for all α_{-1} which implies $p(\alpha) = p(\alpha_1)p(\alpha_{-1})$ for all α. Condition (1) implies that p defined over $\mathcal{A}_{-1}$ also satisfies full-freeness, and therefore $p(\alpha) = p(\alpha_1)p(\alpha_2)p(\alpha_{-1-2})$ for all α. The result is proved by induction. $\square$

At the other extreme of independence is *no-freeness*, which holds if two types of an agent always have different beliefs:[10]

For all $i \in \mathcal{N}$ and all $\alpha_i' \neq \alpha_i''$, $p(\alpha_{-i} \mid \alpha_i') \neq p(\alpha_{-i} \mid \alpha_i'')$ for some $\alpha_{-i} \in \mathcal{A}_{-i}$.

[10] The no-freeness assumption is called "belief announcement" (BA) in Johnson, Pratt and Zeckhauser (1990) and "beliefs-determine-preferences" (BDP) in Heifetz and Neeman (2006).

We now introduce the notion of *independence classes*. This will allow us to translate in our setting and to generalize the analysis of Albert et al. (2015), who, as mentioned in the introduction, studied a one seller-one buyer setting and partitioned the buyer's types in such a way that all types in an element of the partition generate the same conditional distributions over an external signal.

Definition (Independence classes)**.** *An independence class for agent i is a subset Q_i of $\mathcal{A}_i$ such that*

$$p(\alpha_{-i} \mid \alpha_i) = p(\alpha_{-i} \mid \alpha_i') \textit{ for all } \alpha_{-i} \in \mathcal{A}_{-i} \textit{ and all } \alpha_i, \alpha_i' \in Q_i,$$

and

$$p(\alpha_{-i} \mid \alpha_i) \neq p(\alpha_{-i} \mid \alpha_i') \textit{ for some } \alpha_{-i} \in \mathcal{A}_{-i} \textit{ if } \alpha_i \in Q_i \textit{ and } \alpha_i' \notin Q_i.$$

This implies

$$p(\alpha_i \mid Q_i) \stackrel{\text{def}}{=} p(\alpha_i' = \alpha_i \mid \alpha_i' \in Q_i) = \begin{cases} 0 & \text{if } \alpha_i \notin Q_i, \\ \dfrac{p(\alpha_i)}{\sum_{\alpha_i' \in Q_i} p(\alpha_i')} & \text{if } \alpha_i \in Q_i. \end{cases}$$

Two types which belong to the same independence class are said to be independent of each other.[11]

The set $\mathcal{Q}_i$ of independence classes of agent i is a partition of $\mathcal{A}_i$ and $Q_i(\alpha_i)$ is the independence class to which α_i belongs. Obviously, $\widetilde{\alpha}_i \in Q_i(\alpha_i)$ if and only if $\alpha_i \in Q_i(\widetilde{\alpha}_i)$. Agent i has free beliefs if and only if the set $\mathcal{Q}_i$ has only one element: $\mathcal{Q}_i = \mathcal{A}_i = Q_i(\alpha_i)$ for all α_i. With no-freeness $\mathcal{Q}_i$ has $|\mathcal{A}_i|$ elements and $Q_i(\alpha_i) = \{\alpha_i\}$.

An n-tuple of independence classes is noted $Q = (Q_1, Q_2, \ldots, Q_n)$ and $\mathcal{Q}$ is the set of such n-tuples.[12] Finally, for any $\alpha \in \mathcal{A}$, we have $Q(\alpha) \stackrel{\text{def}}{=} \Big(Q_1(\alpha_1), Q_2(\alpha_2), \ldots, Q_n(\alpha_n)\Big) \in \mathcal{Q}$.

To an information structure $\{\mathcal{A}, p\}$ corresponds therefore another information structure $\{\mathcal{Q}, P\}$, defined on the independence classes of $\{\mathcal{A}, p\}$, with

$$P(Q) \stackrel{\text{def}}{=} \sum_{\{\alpha : Q(\alpha) = Q\}} p(\alpha) \text{ for all } Q \in \mathcal{Q}.$$

[11]In an auction context, Luz (2013) uses a strategy dual to ours: he groups the types of the bidders that yield the same valuation of the object for sale and uses that strategy to provide bounds on the expected profit of the seller.

[12]We will abuse terminology by not distinguishing between Q as elements of the set of independence classes and subsets of $\mathcal{A}$.

For all $i \in \mathcal{N}$ and all $Q_i \in \mathcal{Q}_i$, we have

$$P(Q_i) = \sum_{\alpha_i \in Q_i} p(\alpha_i).$$

The link between the two information structures is clearly seen through the following equality:[13]

$$p(\alpha) = P(Q(\alpha)) \times \prod_{i \in \mathcal{N}} p(\alpha_i \mid Q_i(\alpha_i)) = P(Q(\alpha)) \times \prod_{i \in \mathcal{N}} \frac{p(\alpha_i)}{P(Q_i(\alpha_i))}. \quad (2)$$

In our analysis of participation constraints we need the following Assumption 1. It implies that surplus can be "transferred" from one type of an agent to the other types of the same agent. Theorem 3 shows that this assumption is the weakest regularity condition we need for our results.

Assumption 1. *The undirected graph $G(\mathcal{A}, p)$ defined as follows:*

1. *The set of vertices of $G(\mathcal{A}, p)$ is $\bigcup_{i \in \mathcal{N}} \mathcal{A}_i$.*
2. *The nodes α_i and α_j of $G(\mathcal{A}, p)$ are adjacent if and only if a) $i \neq j$ and b) $p(\alpha_i, \alpha_j) > 0$.*

is connected.

If $G(\mathcal{A}, p)$ is not connected, it has $Q \geq 2$ maximal components. Let us call them G^q with $1 \leq q \leq Q$. The sets $\mathcal{A}_i^q = G^q \cap \mathcal{A}_i$ are not empty,[14] and $p(\alpha) > 0$ if and only if there exist a q such that $\alpha_i \in \mathcal{A}_i^q$ for all i.

Finally we have the following lemma:

Lemma 2. *The information structure $\{\mathcal{A}, p\}$ satisfies Assumption 1 if and only if the associated information structure on independence classes $(\mathcal{Q}, P)$ does.*

Proof. Equation (2) implies $p(\alpha) > 0$ if and only if $p(Q(\alpha)) > 0$. It also implies $p(\alpha_i, \alpha_j) > 0$ if and only if $P(Q_i(\alpha_i), Q_j(\alpha_j)) > 0$. Therefore two nodes α_i and α_j of $G(\mathcal{A}, p)$ are adjacent if and only if the nodes $Q_i(\alpha_i)$ and

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[13] We sketch the two first steps of the recurrence which proves (2).

Defining $P_1(Q_1, \alpha_{-1}) = \sum_{\{\alpha_1 : Q_1(\alpha_1) = Q_1\}} p(\alpha_1, \alpha_{-1})$, we have $p(\alpha_1, \alpha_{-1}) = P_1(Q_1(\alpha_1), \alpha_{-1}) \, p(\alpha_1 \mid Q_1(\alpha_1))$.

Similarly, defining $P_2(Q_1(\alpha_1), Q_2, \alpha_{-1-2}) = \sum_{\{\alpha_2 : Q_2(\alpha_2) = Q_2\}} P_1(Q_1(\alpha_1), \alpha_2, \alpha_{-1-2})$ we have $p(\alpha_1, \alpha_2, \alpha_{-1-2}) = P_2(Q_1(\alpha_1), Q_2(\alpha_2), \alpha_{-1-2}) \, p(\alpha_1 \mid Q_1(\alpha_1)) \, p(\alpha_2 \mid Q_2(\alpha_2))$.

[14] G^p contains at least one element, let us say $\widetilde{\alpha}_j^p$. Because $p(\widetilde{\alpha}_j^p) > 0$, there exist $\widetilde{\alpha}_{-j}^p$ such that $p(\widetilde{\alpha}_j^p, \widetilde{\alpha}_{-j}^p) > 0$; all $\widetilde{\alpha}_i^p$'s are connected to $\widetilde{\alpha}_j^p$ and therefore belong to G^p.

$Q_j(\alpha_j)$ of $G(\mathcal{Q}, P)$ are adjacent. Hence, more generally, to a path from any node α_i to another node α_j of the graph $G(\mathcal{A}, p)$ corresponds a path from the node $Q_i(\alpha_i)$ to the node $Q_j(\alpha_j)$ in the graph $G(Q, P)$. Hence, $G(\mathcal{Q}, P)$ is connected if and only if $G(\mathcal{A}, p)$ is connected. □

2.2 Bayesian Implementation

A decision rule $s : \mathcal{A} \to \mathcal{X}$ associates to any n-tuple α of types the public decision $s(\alpha)$. The mechanism designer cannot impose it directly on the agents, as she does not know their types. As is well known, by the 'revelation principle', she can use a *direct mechanism*, which is a pair (s, t) where s is a decision rule and $t : \mathcal{A} \to \Re^n$ a (monetary) transfer rule, which associates to any α an n-tuple $t(\alpha)$ of payments to agents. She asks the agents to reveal their types and, as a function of their announcements, chooses the decision $s(\alpha)$ and makes monetary transfers $t(\alpha)$. These mechanisms must satisfy a Budget Balance property, truth-telling (also called Bayesian Incentive Compatibility) constraints, and, sometimes, participation (also called Individual Rationality) constraints. In Section 2.3, we discuss the choice by the designer of a mechanism, among all the mechanisms that satisfy the constraints.[15]

The *Budget Balance* constraint requires the mechanism to be self-supporting:

$$\sum_{i \in \mathcal{N}} t_i(\alpha) = 0 \text{ for all } \alpha \in \mathcal{A}. \tag{BB}$$

As we prove below, our results are essentially unchanged if the right hand side of (BB) is any constant β, positive or negative, so that (BB) is replaced by the following β *Budget Balance* condition

$$\sum_{i \in \mathcal{N}} t_i(\alpha) = \beta \text{ for all } \alpha \in \mathcal{A}, \tag{BB-β}$$

When it is negative (resp. positive) the constant β can be interpreted as a lump sum fee levied (resp. a lump sum subsidy provided) by the mechanism designer. We comment on the importance of generalizing from (BB) to (BB-β), as well as on the choice of β by the designer, in 2.3.

Mechanisms must also satisfy the *Bayesian incentive compatibility constraints*, which state that the agents have incentives to announce their true types:

[15] D'Aspremont, Crémer and Gérard-Varet (1999) discuss the problem of unique implementation in this setting.

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i)\Big[u_i(s(\alpha_i,\alpha_{-i});\alpha_i)+t_i(\alpha_i,\alpha_{-i})\Big]$$
$$\geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i)\Big[u_i(s(\widetilde{\alpha}_i,\alpha_{-i});\alpha_i)+t_i(\widetilde{\alpha}_i,\alpha_{-i})\Big]$$
$$\text{for all } i\in\mathcal{N} \text{ and all } (\alpha_i,\widetilde{\alpha}_i)\in\mathcal{A}_i^2. \quad \text{(BIC)}$$

Taking into account the independence classes of the agents, Bayesian incentive compatibility (BIC) can be decomposed into two weaker properties. The first is *Bayesian incentive compatibility within independence classes* (BIC^{wi}) which only requires the BIC constraints to hold when α_i and $\widetilde{\alpha}_i$, belong to the same independence class:

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid Q_i)\Big[u_i(s(\alpha_i,\alpha_{-i});\alpha_i)+t_i(\alpha_i,\alpha_{-i})\Big]$$
$$\geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid Q_i)\Big[u_i(s(\widetilde{\alpha}_i,\alpha_{-i});\alpha_i)+t_i(\widetilde{\alpha}_i,\alpha_{-i})\Big]$$
$$\text{for all } i\in\mathcal{N}, \text{ all } Q_i\in\mathcal{Q}_i, \text{ and all } (\alpha_i,\widetilde{\alpha}_i)\in Q_i^2. \quad (BIC^{\mathrm{wi}})$$

The second weaker property is *Bayesian incentive compatibility across independence classes* (BIC^{ac}) which requires the BIC constraints to hold only when α_i and $\widetilde{\alpha}_i$ belong to different equivalence classes:

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i)\Big[u_i(s(\alpha_i,\alpha_{-i});\alpha_i)+t_i(\alpha_i,\alpha_{-i})\Big]$$
$$\geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i)\Big[u_i(s(\widetilde{\alpha}_i,\alpha_{-i});\alpha_i)+t_i(\widetilde{\alpha}_i,\alpha_{-i})\Big]$$
$$\text{for all } i\in\mathcal{N} \text{ and all } \alpha_i \text{ and } \widetilde{\alpha}_i \text{ such that } \widetilde{\alpha}_i\notin Q_i(\alpha_i). \quad (BIC^{\mathrm{ac}})$$

When only BIC^{ac} holds, the agents will announce their true independence class, but may lie about their types within the independence class.

In the following sections, we consider two kinds of implementation, one in which participation of the agents is imposed and the other in which it is voluntary. In the first case, with imposed participation, there is no participation or individual rationality constraint.[16] A transfer rule $t:\mathcal{A}\to\Re^n$ implements s with imposed participation if it is balanced, and satisfies the

[16]This might be the relevant model, for instance, when governments impose participation on all the agents satisfying certain conditions, such as inhabiting a certain locality and/or being of a certain age group.

Bayesian incentive compatibility constraints. We shall say that an information structure $(\mathcal{A}, p)$ *guarantees implementation with imposed participation* of a class $\mathcal{S}$ of decision rules if, for all utility functions $u_i(x; \alpha_i)$ and all $s \in \mathcal{S}$, there exists a transfer rule t which implements s. When (BB) is replaced by (BB-β) in the definition, the information structure guarantees implementation with budget surplus (or deficit) β.

Participation can also be *voluntary.* Agent i of type α_i participates in the mechanism if and only if it gives him an expected utility at least equal to $\overline{U}_i(\alpha_i)$, his exogenous type-dependent reservation utility. To implement the decision rule s, the transfer rule t must satisfy (BB), (BIC) and the following *participation* or *Individual Rationality* (*IIR*) constraints:[17]

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \Big[u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) + t_i(\alpha_{-i}, \alpha_i) \Big] p(\alpha_{-i} \mid \alpha_i) \geq \overline{U}_i(\alpha_i)$$
$$\text{for all } i \in \mathcal{N}, \text{ and all } \alpha_i \in \mathcal{A}_i. \quad \text{(IIR)}$$

Because of budget balance this is impossible unless the following *global surplus* condition holds:

$$\sum_{\alpha \in \mathcal{A}} p(\alpha) \Big[\sum_{i \in \mathcal{N}} u_i(s(\alpha); \alpha_i) \Big] = \sum_{i \in \mathcal{N}} \Big[\sum_{\alpha \in \mathcal{A}} p(\alpha) u_i(s(\alpha); \alpha_i) \Big]$$
$$\geq \sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \overline{U}_i(\alpha_i) p(\alpha_i). \quad \text{(GS)}$$

If the budget balance constraint is (BB-β), the condition (GS) must be replaced by

$$\sum_{\alpha \in \mathcal{A}} p(\alpha) \Big[\sum_{i \in \mathcal{N}} u_i(s(\alpha); \alpha_i) \Big] = \sum_{i \in \mathcal{N}} \Big[\sum_{\alpha \in \mathcal{A}} p(\alpha) u_i(s(\alpha); \alpha_i) \Big]$$
$$\geq \sum_{i \in \mathcal{N}} \sum_{\alpha_i} \overline{U}_i(\alpha_i) p(\alpha_i) - \beta. \quad \text{(GS-}\beta\text{)}$$

[17] A weaker participation constraint is an "ex-ante" individual rationality condition of the form

$$\sum_{\alpha \in \mathcal{A}} p(\alpha) \left[u_i(s(\alpha); \alpha_i) + t_i(\alpha) \right] \geq \overline{U}_i.$$

Any decision rule that can be implemented by a transfer rule with imposed participation can be implemented by another transfer rule that also satisfies ex-ante individual rationality (see d'Aspremont et al., 2003, Theorem 1) as long as

$$\sum_{i \in \mathcal{N}} \Big[\sum_{\alpha \in \mathcal{A}} p(\alpha) u_i(s(\alpha); \alpha_i) \Big] \geq \sum_{i \in \mathcal{N}} \overline{U}_i.$$

We shall say that an information structure $(\mathcal{A}, p)$ *guarantees implementation with voluntary participation* of a class $\mathcal{S}$ of decision rules if, for all utility functions $u_i(x; \alpha_i)$ and all reservation utilities $\overline{U}_i$ which satisfy (GS) (or, if relevant, (GS-β)), and any $s \in \mathcal{S}$, there exists a transfer rule t that implements s while satisfying (IIR) and (BB) (or (BB-β)).

Efficient decision rules, usually denoted s^*, satisfy

$$\sum_{i \in \mathcal{N}} u_i(s^*(\alpha); \alpha_i) \geq \sum_{i \in \mathcal{N}} u_i(x; \alpha_i) \text{ for all } \alpha \in \mathcal{A} \text{ and all } x \in \mathcal{X}.$$

For simplicity, we assume that such a function s^* exist, although this is not necessary for all our results.

We also introduce a weaker notion of efficiency, which does not have much ethical appeal from a social welfare point of view, but which will prove useful to describe larger sets of implementable decisions: A decision rule s is *Independence Class efficient* (IC-efficient) if

$$\sum_{i \in \mathcal{N}} u_i(s(\alpha); \alpha_i) \geq \sum_{i \in \mathcal{N}} u_i(x; \alpha_i) \text{ for all } x \in X(Q(\alpha)),$$

where, for any Q,

$$X(Q) = \{x \mid x = s(\alpha) \text{ for some } \alpha \in Q\}.$$

Decision rules which are constant on independence classes are, trivially, IC-efficient. Under no-freeness all independence classes are singletons, hence all decisions rules are IC-efficient.

Part of the literature has focused on the case of independence. Notice that in this case, efficiency and IC-efficiency are not equivalent: for instance, constant decision rules which associate to every α the same decision $s(\alpha)$ are IC-efficient. More generally, efficiency plays two roles in mechanism design: first, together with budget balance, it ensures Pareto-optimality, and, second, it may be used to design incentives. With the introduction of IC-efficiency we separate these two roles. We will discuss the important consequences of this distinction after presenting Theorem 1.

We skip the definitions of guaranteed implementability of efficient or IC-efficient decision rules, with imposed or with voluntary participation, since they are similar to the definitions of guaranteed implementability of all decision rules.

2.3 Maximizing the objective function of the mechanism designer

For the sake of consistency with the literature, we present our main results under the form "An information structure guarantees implementation of all decision rules satisfying this or that property if and only if it satisfies the following conditions". However, as we explain in this section, the reader can, and maybe should, think of them as providing recipes for the mechanism designer.

Consider a mechanism designer who would like to maximize her objective function over all implementable decision rules under the budget balance condition. This is a very large and complicated problem. With 10 agents each with 15 types, there are $10^{15} \times (1+10)$ variables — for each α: one public decision and one transfer to each agent. There are 10^{15} budget balance constraints (one for each n-tuple of types), $10 \times 15 \times 14 = 2100$ incentive compatibility constraints (one for each pair of types for each agent), and, if relevant, 150 participation constraints (one for each type of each agent).

Our results allow for simplification of this problem: we identify an easily characterizable subset of the implementable decisions and propose that the designer maximize her objective function over this subset. When she does this maximization, she need not take into account the incentive compatibility constraints. It is only in a second step that she builds a mechanism which satisfy these constraints.

For instance, in Theorem 1 below, we will present conditions under which all IC-efficient decision rules can be implemented. Therefore, the mechanism designer can proceed as follows:

a) check that the distribution of the types satisfy the conditions of the theorem;

b) maximize her objective over all IC-efficient decision rules — the identification of IC-efficient decision rules relies on very coarse information about the information structure;

c) construct a mechanism that implements the optimal decision rules — because our proofs are constructive we provide techniques for so doing.

Of course, because our theorems, such as Theorem 1, only provide sufficient conditions for implementability, they do not allow for the identification of all the implementable decision rules, and this could yield suboptimal solutions. We conjecture that this loss would not be very severe in expected value terms, but this would need to be checked. On the other hand, this problem would not arise under no-freeness: per the discussion following the presentation of Condition B below, generically all decision rules can be implemented. Similar considerations would apply in the presence of participation constraints, using the results of Section 4.

Our framework allows for many different types of objective functions for the mechanism designer. She could maximize an expected utility function of the type $\sum_{\alpha \in \mathcal{A}} p(\alpha)U(s(\alpha);\alpha)$. But she could also prefer that the public decision not vary too much with α and, for instance be of the type

$$\sum_{\alpha \in \mathcal{A}} p(\alpha)U(s(\alpha);\alpha) - \max_{\{\alpha,\alpha'\} \in \mathcal{A}^2} |s(\alpha) - s(\alpha')|,$$

with $|\cdot|$ denoting a norm over the set $\mathcal{X}$.

Before turning to the presentation of our results, it may be worthwhile to explain the difference between our approach and Automated Mechanism Design (Conitzer and Sandholm, 2003, 2004; Conitzer, 2006; Albert et al., 2015). Automated Mechanism Design develops computational methods for solving problems for which general theoretical implementation results do not exist. For each problem, using constrained optimization techniques, the optimal mechanism is automatically designed for a particular instance. In our approach we develop general algorithms that help reduce the difficulty of computations, at the cost of being sub-optimal in some cases. It would be interesting to explore ways in which these two approaches could complement each other.

3 Implementation without participation constraints

In this section, we study the simpler model in which participation is imposed, that is when the mechanism designer only faces the BIC and Budget Balance constraints. Exploiting the notion of independence classes, we prove new results and provide new interpretations of results already in the literature.

As shown by d'Aspremont and Gérard-Varet (1982) and by d'Aspremont et al. (2003), information structures guarantee implementation of all decision rules if and only if they satisfy the following Condition B.

Condition B. *The information structure $\{\mathcal{A}, p\}$ satisfies Condition B if there exists a balanced transfer rule t^B such that for all $i \in \mathcal{N}$, for all $\widetilde{\alpha}_i \neq \alpha_i$,*

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \alpha_i)\, p(\alpha_{-i} \mid \alpha_i) > \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^B(\alpha_{-i}, \widetilde{\alpha}_i)\, p(\alpha_{-i} \mid \alpha_i).$$

Proving that all decision rules can be implemented when Condition B holds is easy: just use transfers equal to t^B multiplied by a large enough constant to overwhelm the incentives to lie to modify the decision. Proving the converse is more difficult; see d'Aspremont et al. (2003), who also prove that Condition B holds generically and use scoring rules to show how the transfer t^B can be constructed (their Theorem 3).[18]

Because Condition B trivially implies no-freeness, we want to weaken it to allow for some independence of types. This yields the following Generalized Condition B on which we rely to prove the results of this section.

Generalized Condition B. *The information structure $\{\mathcal{A}, p\}$ satisfies the Generalized Condition B if the associated information structure on independence classes $\{\mathcal{Q}, P\}$ satisfies Condition B, i.e. if there exists a balanced transfer rule $\tau^B : \mathcal{Q} \to \Re^n$ such that for all $i \in \mathcal{N}$ and all $\widetilde{Q}_i \neq Q_i$*

$$\sum_{Q_{-i} \in \mathcal{Q}_{-i}} \tau_i^B(Q_{-i}, Q_i)\, P(Q_{-i} \mid Q_i) > \sum_{Q_{-i} \in \mathcal{Q}_{-i}} \tau_i^B(Q_{-i}, \widetilde{Q}_i)\, P(Q_{-i} \mid Q_i).$$

Under no-freeness, the information structures $\{\mathcal{A}, p\}$ and $\{\mathcal{Q}, P\}$ coincide: Condition B and the Generalized Condition B are equivalent. More generally, the Generalized Condition only ensures (strict) incentive compatibility across independence classes (BIC^{ac}). At the other extreme, with independent types, there is only one independence class per agent; the Generalized Condition B holds trivially. In this case, as shown by d'Aspremont and Gérard-Varet (1979), efficient decision rules can be implemented.

Our main interest will be intermediate cases where $\{\mathcal{Q}, P\}$ satisfies Condition B but $\{\mathcal{A}, p\}$ does not. We show constructively that all IC-efficient decision rules can be implemented. This is done in two stages. First, we adapt the classical techniques found in the literature for the independent case and construct transfers that ensure Bayesian incentives compatibility within independence classes, BIC^{wi}. Then, we multiply the transfers τ^B of the definition of the Generalized Condition B by a constant large enough to ensure Bayesian incentive compatibility across independence classes, BIC^{ac}. Finally, we add these two transfers.

More precisely, let us define for all i and all α_i,

$$g_i(\alpha_i) \stackrel{\text{def}}{=} \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \left[\sum_{j \in \mathcal{N}_{-i}} u_j(s(\alpha_{-i}, \alpha_i), \alpha_j) \right] p(\alpha_{-i} \mid \alpha_i), \tag{3}$$

[18] Generically, in the sense that the set of probabilities that satisfy Condition B contains an open and dense subset of all priors in the topology generated by the Euclidean metric. Note, however, that B is not generic for other definitions of genericity for which even no-freeness is not generic (see Heifetz and Neeman, 2006).

the expected aggregate welfare of the agents other than i when agent i is of type α_i and s is implemented. A Arrow-AGV mechanism is defined by the following balanced monetary transfers:

$$\theta_i\left(\alpha_{-i}, \alpha_i\right) \stackrel{\text{def}}{=} g_i\left(\alpha_i\right) - \frac{1}{n-1} \sum_{j \in \mathcal{N}_{-i}} g_j\left(\alpha_j\right). \tag{4}$$

The following lemma has the same proof as Theorem 6 in d'Aspremont and Gérard-Varet (1979), which it generalizes. It relies on the fact that θ makes the agents internalize the (expected) welfare of the other agents.

Lemma 3. *For any IC-efficient decision rule s, the Arrow-AGV mechanism (s, θ) is BIC^{wi}.*

Combining BIC^{ac} from the Generalized Condition B and BIC^{wi} from Arrow-AGV mechanisms, we obtain the following theorem.

Theorem 1. *The information structure $(\mathcal{A}, p)$ guarantees implementation of all IC-efficient decision rules with imposed participation if and only if it satisfies the Generalized Condition B.*

Proof. To prove sufficiency, for any IC-efficient s consider the following Arrow-AGV transfers

$$t_i(\alpha_{-i}, \alpha_i) \stackrel{\text{def}}{=} \theta_i(\alpha_{-i}, \alpha_i) + K\tau_i^B(Q(\alpha_{-i}), Q(\alpha_i)).$$

By Lemma 3, for any K the mechanism (s, t) is BIC^{wi} as τ is constant on Independence classes. Choosing K large enough ensures that the agents do not lie across independence class; this ensures BIC^{ac} and therefore proves sufficiency.

To prove necessity, assume that the information structure implements all IC-efficient rules. In particular it implements all decision rules which are constant on independence classes: $s(\alpha) = s(\widetilde{\alpha})$, whenever $Q(\alpha) = Q(\widetilde{\alpha})$. Then, the argument of the 'only if' part of Theorem 2 in d'Aspremont et al. (2003) can be used to prove the result (see also the 'only if' part of Theorem 4 in the next section). □

Observe that this proof is constructive. It relies on Arrow-AGV transfers which are directly obtained from the utility functions and on the Generalized Condition B. As mentioned earlier, the balanced transfer rule τ^B can "nearly always" be built from proper scoring rules (see d'Aspremont et al. (2003) and Appendix C of the present paper).

As mentioned above, when types are independent, Theorem 1 implies that implementation of efficient decision rules is guaranteed. If the mechanism designer works for the benefit of the agents, this is what she should implement. If the incentives of the mechanism designer are different from those of the agents, the theorem opens many more possibilities. Trivially, she can simply choose one decision and impose it, but she can also do much more: choose any subset of the decisions, and restrict herself to efficiency within this subset. When the Generalized Condition B holds, she can choose a different subset for each independence class. [19] Finally, the mechanism designer may want to raise a fee (or grant a subsidy) β and choose a mechanism guaranteeing implementation with imposed participation and β Budget Balance, replacing the budget constraint (BB) by the more general constraint (BB-β). We can prove the following theorem.

Theorem 2. *Whatever $\beta \in \Re$, an information structure guarantees implementation of IC-efficient decision rules with imposed participation and β Budget Balance if and only if it guarantees implementation of IC-decision rules with imposed participation and Budget Balance.*

Proof. Assume that the information structure guarantees implementation of IC-efficient decision rules with Budget Balance (BB). For any set of utility functions u_i, let

$$v_i(x;\alpha_i) = u_i(x;\alpha_i) - \beta/n \text{ for all } i,\ \alpha_i, \text{ and } s.$$

Any decision rule which is IC-efficient for the u_is is also IC-efficient for the v_is. Therefore, there exists transfers $\widetilde{t}$ such that

$$\sum_{i\in\mathcal{N}} \widetilde{t}_i(\alpha) = 0 \text{ for all } \alpha \in \mathcal{A},$$

and

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)\Big[v_i(s(\alpha_i,\alpha_{-i});\alpha_i) + \widetilde{t}_i(\alpha_i,\alpha_{-i})\Big] \geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)\Big[v_i(s(\widetilde{\alpha}_i,\alpha_{-i});\alpha_i) + \widetilde{t}_i(\widetilde{\alpha}_i,\alpha_{-i})\Big]$$
$$\text{for all } i\in\mathcal{N}, \text{ and all } (\alpha_i,\widetilde{\alpha}_i)\in\mathcal{A}_i^2.$$

[19] In section 5.1, we show that the Generalized Condition B is equivalent to Condition C, known to be sufficient to guarantee implementation of efficient decision rules.

Let $t_i(\alpha) = \widetilde{t}_i(\alpha) + \beta/n$. It is straightforward that the t_is satisfy (BB-β) as well as (BIC), and therefore that the implementation of IC-efficient decision rules of the u_is is possible. $\square$

Notice that we could define a Condition B-β similar to Condition B, except that the right-hand side of the budget balance condition is β. It is straightforward to show that, for any β an information structure satisfies Condition B-β if and only if it satisfies Condition B. We will not use this result directly but it shows that the important part of equation (BB) is the fact that the sum of the transfers is constant in α, not that it is equal to 0.

4 Implementation with participation constraints

The main focus of this paper is on the limits that incentives constraints put on the set of decision rules that can be implemented under different information structures. When we introduce participation constraints, a natural preliminary question is what limits these constraints, by themselves, put on the set of implementable decision rules. In Section 4.1 we show that the answer to this question is "they do not put any constraints"! We then turn to the way in which incentives and participation constraints interact.

4.1 Participation constraints alone do not bite

As discussed in 2.2, the participation constraints can only be met if the global surplus (GS) condition is satisfied. In a remarkable result, Matsushima (2007, Proposition 1, p. 6) showed that, for any decision rule, the GS condition is not only necessary but also sufficient for the existence of a balanced transfer function which satisfies all the IIR constraints. Therefore, as long as we have enough expected aggregate surplus to distribute, it is *only* the incentive compatibility constraints which prevent the mechanism designer from allocating the surplus arbitrarily among the agents and types of agents. Matsushima assumes $p(\alpha) > 0$ for all α. Theorem 3 shows a) that the result holds under our much weaker Assumption 1 and also b) that Assumption 1 is necessary for the result to hold.

We have also added a second part to the theorem, which shows that the surplus generated by the mechanism can be arbitrarily allocated among the agents.

The proof is presented in Appendix A.[20,21]

Theorem 3. *There exists a balanced transfer rule t which satisfies* (IIR) *for all u_is, all $\overline{U}_i$s, and all decision rules that satisfy the aggregate surplus Condition* (GS) *if and only if Assumption 1 holds.*

Furthermore, for any set of interim utilities $V_i(\alpha_i) \geq \overline{U}_i(\alpha_i)$ such that

$$\sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{N}_i} V_i(\alpha_i) p(\alpha_i) = \sum_{\alpha \in \mathcal{N}} \sum_{i \in \mathcal{N}} u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) p(\alpha)$$

there exists a balanced transfer rule t which satisfies (IIR) *and*

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) + t_i(\alpha_{-i}, \alpha_i)] = V_i(\alpha_i)$$

$$\text{for all } i \in \mathcal{N} \text{ and } \alpha_i \in \mathcal{A}_i.$$

The crucial step in the proof of Theorem 3 is lemma A.1, which is of independent interest. It can be found in Appendix A, along with its proof. We also use it in the proof of Theorem 5.

4.2 Implementing all decision rules with voluntary participation: the no-freeness case

We now show how Condition B can be reinforced to allow for implementation with voluntary participation under no freeness. Many of the results in this section are present in Kosenok and Severinov (2008). Our presentation and our proofs are more direct, and prepare for the new results of 4.3.

Condition $\overline{\mathbf{B}}$. *An information structure $\{\mathcal{A}, p\}$ satisfies condition $\overline{B}$ if and only if there exists a balanced transfer rule $t^{\overline{B}}$ which satisfies both*

- *the following strict incentive compatibility condition:*

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{B}}(\alpha_{-i}, \alpha_i)\, p(\alpha_{-i} \mid \alpha_i) > \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{B}}(\alpha_{-i}, \widetilde{\alpha}_i)\, p(\alpha_{-i} \mid \alpha_i)$$

$$\text{for all } i \in \mathcal{N}, \text{ and any } (\alpha_i, \widetilde{\alpha}_i) \in \mathcal{A}_i^2, \text{ with } \alpha_i \neq \widetilde{\alpha}_i, \quad (5)$$

[20] Kosenok and Severinov (2008, Lemma A.3, p. 147-148) independently prove Matsushima's proposition with an assumption weaker than his (but stronger than ours!): $p(\alpha_i, \alpha_j) > 0$, for all i, all $j \in \mathcal{N}_{-i}$, all α_i and all α_j.

[21] Theorem 3 also holds if the utility functions exhibit common value (see footnote 9). This strengthens the if statement and weakens the only if statement.

- *and the following zero surplus condition:*

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} t_i^{\overline{B}}(\alpha_{-i},\alpha_i)p(\alpha_{-i}\mid\alpha_i) = 0 \textit{ for all } i\in\mathcal{N} \textit{ and all } \alpha_i\in\mathcal{A}_i. \tag{6}$$

Written under its primal form, the weaker of the two Crémer-McLean conditions states that there exist transfers that satisfy (5) and (6), but not necessarily budget balance — see the two conditions in the proof of the 'if part' of Theorem 2 in (Crémer and McLean, 1988, p. 1253); it is therefore weaker than Condition $\overline{B}$. Kosenok and Severinov (2008) show that an information structure satisfies Condition $\overline{B}$ if and only if it satisfies Crémer-McLean and an additional "identifiability" condition.

In Appendix C, we prove that Condition $\overline{B}$ holds generically (Theorem C.1).[22] We do this constructively: we show how the transfers $t^{\overline{B}}$ can be constructed for "nearly all" information structures.

Theorem 4 is similar to Theorem 3, but allows for voluntary participation. It shows that $\overline{B}$ is necessary and sufficient for any decision rule to be implemented.

Theorem 4. *An information structure* $\{\mathcal{A},p\}$ *guarantees implementation with voluntary participation of all decision rules which satisfy the global surplus condition* (GS) *if and only it satisfies Condition* $\overline{B}$.

Furthermore, if Condition $\overline{B}$ *holds, for any decision rule which satisfy* (GS) *and any utility levels* $V_i(\alpha_i)\geq\overline{U_i}(\alpha_i)$ *such that*

$$\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i} V_i(\alpha_i)\,p(\alpha_i) = \sum_{\alpha\in\mathcal{A}}\sum_{i\in\mathcal{N}}\Big[u_i\left(s\left(\alpha_{-i},\alpha_i\right);\alpha_i\right)\Big]p(\alpha)$$

there exist transfers t *which implement* s *and satisfy*

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\Big[u_i(s(\alpha_{-i},\alpha_i);\alpha_i)+t_i(\alpha_{-i},\alpha_i)\Big]\,p(\alpha_{-i}\mid\alpha_i) = V_i(\alpha_i).$$

for $i\in\mathcal{N}$ *and all* $\alpha_i\in\mathcal{A}_i$.

The proof is adapted for the case of voluntary participation from the proof with imposed participation presented in Theorem 2 in d'Aspremont et al. (2003). It would be valid in the common value case, which is the one considered by Kosenok and Severinov (2008, see their Theorem 1, Corollary 1 and Lemma A.2). Restricting ourselves to private values makes our proof of the "only if" statement significantly more challenging.

[22] Matsushima (2007) also shows that his *Conditions 1&2* are generic, which implies that $\overline{B}$ is generic, but adds further conditions on the type spaces.

Proof. We prove the second part of the theorem, of which the "if" statement of the first part is a straightforward consequence. By Theorem 3, there exists a balanced IIR transfer function t that satisfies

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} \Big[u_i(s(\alpha_{-i},\alpha_i);\alpha_i) + t_i(\alpha_{-i},\alpha_i)\Big] p(\alpha_{-i} \mid \alpha_i) = V_i(\alpha_i)$$

$$\text{for all } i \in \mathcal{N} \text{ and all } \alpha_i \in \mathcal{A}_i.$$

For all $\alpha \in \mathcal{A}$, let $t(\alpha) = t'(\alpha) + kt^{\overline{B}}(\alpha)$ with k "large enough". Because t is the sum of two budget balanced functions, it is budget balanced. By (5), BIC is satisfied. By (6), IIR is also satisfied.

It is more difficult to show that condition $\overline{B}$ holds for every information structure $\{\mathcal{A}, p\}$ for which any decision rule can be implemented. The proof is presented in Appendix B. □

4.3 Implementing IC-efficient decision rules with voluntary participation: allowing for freeness

Using the same strategy as in section 3 we introduce a generalized version of Condition $\overline{B}$, which allows for freeness as long as the information structure $\{\mathcal{Q}, P\}$ satisfies Condition $\overline{B}$.

Generalized Condition $\overline{\mathbf{B}}$. *The information structure $\{\mathcal{A}, p\}$ satisfies the Generalized Condition $\overline{B}$ if the associated information structure on independence classes $\{\mathcal{Q}, P\}$ satisfies Condition $\overline{B}$, i.e., if there there exists a balanced transfer rule $\tau^{\overline{B}}$ such that*

$$\sum_{Q_{-i}\in\mathcal{Q}_{-i}} \tau_i^{\overline{B}}(Q_{-i}, Q_i)\, P(Q_{-i} \mid Q_i) > \sum_{Q_{-i}\in\mathcal{Q}_{-i}} \tau_i^{\overline{B}}(Q_{-i}, \widetilde{Q}_i)\, P(Q_{-i} \mid Q_i)$$

$$\text{for all } i \in \mathcal{N} \text{ and all } (Q_i, \widehat{Q}_i) \in \mathcal{Q}_i^2 \text{ with } \widetilde{Q}_i \neq Q_i\ , \quad (7)$$

as well as

$$\sum_{Q_{-i}\in\mathcal{Q}_{-i}} \tau_i^{\overline{B}}(Q_{-i}, Q_i) P(Q_{-i} \mid Q_i) = 0 \text{ for all } i \in \mathcal{N} \text{ and all } Q_i \in \mathcal{Q}_i. \quad (8)$$

Assuming the Generalized Condition $\overline{B}$ and that the public decision rule s is IC-efficient, we derive a result analogous to Theorem 1. Allowing for freeness reduces the set of implementable decision rules to IC-efficient decision ones. However, as first noticed by Makowski and Mezzetti (1994), we need a stronger surplus condition than the global surplus condition (GS).

To understand why we need to reinforce the global surplus condition, consider the following example of a mechanism designer who wants to implement an efficient decision rule s^* with independent types. We keep the discussion at an informal level and consider a situation where the participation of all the agents is necessary for implementation of a project which increases aggregate welfare by a small amount. The agents derive the same utility from every decision: there exists an $\eta > 0$, such that $u_i(x; \alpha_i) = \eta$ for all $i \in \mathcal{N}$ and all $x \in \mathcal{X}$. On the other hand, the reservation utility of the agents vary with their types.

We assume that the expected value of $\overline{U}_i(\alpha_i)$ is equal to 0 for all i; therefore, the aggregate surplus condition holds. For every agent i we denote α_i^0 the type which corresponds to his largest reservation utility. When agent i is of type α_i^0, he participates only if he receives an expected transfer of at least $\overline{U}_i(\alpha_i^0) - \eta$. If he systematically announces that he is of type α_i^0, he will therefore obtain an expected utility at least equal to $\overline{U}_i(\alpha_i^0)$. Given that the project generates an aggregate net expected gain in utility of $n\eta$, the project is not feasible if $n\eta < \sum_{i \in \mathcal{N}} \overline{U}_i(\alpha_i^0)$. On the other hand, it is easy to implement if $n\eta \geq \sum_i \overline{U}_i(\alpha_i^0)$: for all i, $t_i(\alpha_i)$ is a constant smaller than or equal to $\overline{U}_i(\alpha_i^0) - \eta$.

Theorem 5 below shows how to generalize the insights from this example: the surplus generated by the decision must not vary "too much" as a function of the types of the agents. Because we study not only the implementation of efficient decision rules, but that of all IC-efficient rules, this opens up new possibilities for designers who work for the benefits of the participants, as we discuss after the proof of the theorem.

We turn to a formal statement. Let us first define

$$S_i(\alpha_i) \stackrel{\text{def}}{=} \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \Big[u_i(s(\alpha_i, \alpha_{-i}); \alpha_i) - \overline{U}_i(\alpha_i) + \sum_{j \in \mathcal{N}_{-i}} [u_j(s(\alpha_i, \alpha_{-i}); \alpha_j) - \overline{U}_j(\alpha_j)] \Big] p(\alpha_{-i} \mid \alpha_i),$$

the expected social surplus conditional on agent i being of type α_i. The expected value of $S_i(\alpha_i)$ is independent of i as

$$\sum_{\alpha_i \in \mathcal{A}_i} S_i(\alpha_i) p(\alpha_i) = \sum_{\alpha \in \mathcal{A}} \left[\sum_{i \in \mathcal{N}} \Big[u_i(\alpha; \alpha_i) - \overline{U}_i(\alpha_i) \Big] p(\alpha) \right] \stackrel{\text{def}}{=} S, \tag{9}$$

where S is the expected social surplus derived from the decision rule, *i.e.,* the difference between the two sides of equation (GS) (which can therefore be written $S \geq 0$). We will also need the notation

$$S_i^{\min}(Q_i) \stackrel{\text{def}}{=} \min_{\alpha_i \in Q_i} S_i(\alpha_i),$$

the minimum expected social surplus conditional on $\alpha_i \in Q_i$. The quantity

$$\begin{aligned}\mathcal{V}_i(Q_i) \stackrel{\text{def}}{=} \Big[\sum_{\alpha_i \in Q_i} S_i(\alpha_i) p(\alpha_i \mid Q_i)\Big] - S_i^{\min}(Q_i) \\ = \sum_{\alpha_i \in Q_i} \Big[S_i(\alpha_i) - S_i^{\min}(Q_i)\Big] p(\alpha_i \mid Q_i)\end{aligned}$$

is a measure of the variability of S_i within the independence class Q_i. When Q_i has only one element, $\mathcal{V}_i(Q_i)$ is equal to zero. More generally, it is small when the utility function of the agent and his beliefs on the types of the other agents vary more or less in parallel. To use the terminology of Neeman (2004), the $\mathcal{V}_i$s are smaller when beliefs are closer to determining preferences.

Theorem 5. *If it satisfies the Generalized Condition $\overline{B}$, the information structure $\{\mathcal{A}, p\}$ guarantees implementation with voluntary participation of IC-efficient decision rules which satisfy*

$$S \geq \sum_{i \in \mathcal{N}} \sum_{Q_i \in \mathcal{Q}_i} \mathcal{V}_i(Q_i) P(Q_i). \tag{GS*}$$

Equation (GS*) states that the total expected surplus generated by the decision rule is larger than the sum of its variabilities within independence classes (note that it implies that the global surplus condition (GS) holds). Under no-freeness, $\mathcal{V}_i(Q_i) = 0$ for all $i \in \mathcal{N}$ and all $Q_i \in \mathcal{Q}_i$, and (GS*) is equivalent to (GS). Under full-freeness, all agents belong to the same independence class, and equation (GS*) is equivalent to the condition under which Theorem 3.1 of Makowski and Mezzetti (1994) holds.[23]

Note that, in some loose sense, there is a trade-off between the stringency of the restriction on the surplus imposed by (GS*) and the stringency of the

[23] Assuming full-freeness, let

$$K_i = \sum_{\alpha_i \in \mathcal{A}_i} \min_{\alpha_i' \in \mathcal{A}_i} \Big\{ \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \Big[u_i\left(s\left(\alpha_{-i}, \alpha_i'\right); \alpha_i\right) - \overline{U}_i(\alpha_i') + \sum_{j \in \mathcal{N}_{-i}} u_j\left(s\left(\alpha_{-i}, \alpha_i'\right); \alpha_j\right) \Big] p\left(\alpha_{-i}\right) \Big\}.$$

Because

$$(n-1) \sum_{\alpha \in \mathcal{A}} \left[\sum_{j \in \mathcal{N}} u_j(s(\alpha); \alpha_j) \right] p(\alpha) = \sum_{j \in \mathcal{N}} \sum_{\alpha \in \mathcal{A}} \Big[\sum_{k \in \mathcal{N}_{-j}} u_k(s(\alpha); \alpha_k) p(\alpha) \Big],$$

restriction on the distribution of the independence classes imposed by the Generalized Condition $\overline{B}$. With full freeness, there is only one independence class and the Generalized Condition $\overline{B}$ imposes no restriction. As the number of independence classes increases, there are fewer types within each class, the $\mathcal{V}(Q_i)$s decrease, and therefore (GS*) become less stringent; at the same time the Generalized Condition $\overline{B}$ bites and becomes more stringent. Eventually, we reach no-freeness, where the number of independence classes is equal to the number of types and each class is a singleton. At that point, (GS*) is trivially met as long as (GS) holds, but the the Generalized Condition $\overline{B}$ (which then coincides with Condition $\overline{B}$) does impose a strong restriction on the distribution P.

It is both useful in the proof of the theorem and informative to re-write equation (GS*) as follows. Since

$$
\begin{aligned}
&\sum_{i\in\mathcal{N}}\sum_{Q_i\in\mathcal{Q}_i}\mathcal{V}(Q_i)P(Q_i)\\
&\qquad=\sum_{i\in\mathcal{N}}\sum_{Q_i\in\mathcal{Q}_i}\left[\sum_{\alpha_i\in Q_i}\Big[S_i(\alpha_i)-S_i^{\min}(Q_i)\Big]p(\alpha_i\mid Q_i)\right]p(Q_i)\\
&\qquad=\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}\Big[S_i(\alpha_i)-S_i^{\min}(Q_i(\alpha_i))\Big]p(\alpha_i)\\
&\qquad=nS-\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}S_i^{\min}(Q_i(\alpha_i))\,p(\alpha_i), \qquad\qquad \text{(by (9))}
\end{aligned}
$$

Equation (GS*) is equivalent to

$$
\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}S_i^{\min}(Q_i(\alpha_i))p(\alpha_i)\geq(n-1)S=\frac{n-1}{n}\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}S_i(\alpha_i)p(\alpha_i). \quad (10)
$$

(Obviously, of course, $\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}S_i^{\min}(Q_i(\alpha_i))p(\alpha_i)\leq\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}S_i(\alpha_i)p(\alpha_i)$). Equation (10) provides another interpretation of condition (GS*): it holds if, in expectation, the $S_i^{\min}$'s are not "too much smaller" than the S_i's.

(GS*) is equivalent to

$$
\sum_{i\in\mathcal{N}}K_i\geq(n-1)\sum_{\alpha\in\mathcal{A}}\left[\sum_{j\in\mathcal{N}}u_j(s(\alpha);\alpha_j)\right]p(\alpha),
$$

which is the formula in Theorem 3.1 of Makowski and Mezzetti (1994) adapted to our setup with a finite set of types.

Proof of Theorem 5. We begin by building on the BIC^{wi} mechanism (s, θ) defined by (3) and (4) to construct a mechanism which is BIC^{wi} and also satisfies the participation constraints. We start by translating condition (GS*) in terms of the mechanism (s, θ). To do so, develop (10) to obtain

$$\sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \min_{\alpha_i' \in \mathcal{Q}_i(\alpha_i)} \Big\{ \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \Big[u_i(s(\alpha_{-i}, \alpha_i'); \alpha_i') - \overline{U}_i(\alpha_i') + \sum_{j \in \mathcal{N}_{-i}} (u_j(s(\alpha_{-i}, \alpha_i'); \alpha_j) - \overline{U}_j(\alpha_j)) \Big] p_i(\alpha_{-i} \mid \alpha_i') \Big\} p(\alpha_i) \geq (n-1) \sum_{\alpha \in \mathcal{A}} \Big[\sum_{j \in \mathcal{N}} (u_j(s(\alpha); \alpha_j) - \overline{U_j}(\alpha_j)) \Big] p(\alpha),$$

which, eliminating some of the reservation utilities is equivalent to

$$\sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \min_{\alpha_i' \in \mathcal{Q}_i(\alpha_i)} \left\{ \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \Big[u_i(s(\alpha_i, \alpha_i'); \alpha_i') - \overline{U}_i(\alpha_i') + \sum_{j \in \mathcal{N}_{-i}} (u_j(s(\alpha_{-i}, \alpha_i'); \alpha_j) \Big] p(\alpha_{-i} \mid \alpha_i') \right\} p(\alpha_i) \geq (n-1) \sum_{\alpha \in \mathcal{A}} \left[\sum_{j \in \mathcal{N}} u_j(s(\alpha); \alpha_j) \right] p(\alpha). \quad (11)$$

Because

$$\begin{aligned}
(n-1) \sum_{\alpha \in \mathcal{A}} \left[\sum_{j \in \mathcal{N}} u_j(s(\alpha); \alpha_j) \right] p(\alpha) &= \sum_{j \in \mathcal{N}} \sum_{\alpha \in \mathcal{A}} \Big[\sum_{k \in \mathcal{N}_{-j}} u_k(s(\alpha); \alpha_k) p(\alpha) \Big] \\
&= \sum_{j \in \mathcal{N}} \sum_{\alpha_j \in \mathcal{A}_j} \Big\{ \sum_{\alpha_{-j} \in \mathcal{A}_{-j}} \Big[\sum_{k \in \mathcal{N}_{-j}} u_k(s(\alpha_{-j}, \alpha_j), \alpha_k) \Big] p(\alpha_{-j} \mid \alpha_j) \Big\} p(\alpha_j) \\
&= \sum_{j \in \mathcal{N}} \sum_{\alpha_j \in \mathcal{A}_j} g_j(\alpha_j) p(\alpha_j) = \sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} g_i(\alpha_i) p(\alpha_i) \qquad \text{(by (3))} \\
&= \sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \Big[\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} g_i(\alpha_i) p(\alpha_{-i} \mid \alpha_i) \Big] p(\alpha_i) \\
&= \frac{1}{n-1} \sum_{i \in \mathcal{N}} \Big\{ \sum_{\alpha_i \in \mathcal{A}_i} \Big[\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \sum_{j \in \mathcal{N}_{-i}} g_j(\alpha_j) p(\alpha_{-i} \mid \alpha_i) \Big] p(\alpha_i) \Big\},
\end{aligned}$$

we can rewrite (GS*) as follows:

$$\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}\min_{\alpha_i'\in\mathcal{Q}_i(\alpha_i)}\Bigg\{\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\Big[u_i(s(\alpha_i,\alpha_i');\alpha_i')-\overline{U}_i(\alpha_i')$$
$$+\sum_{j\in\mathcal{N}_{-i}}u_j(s(\alpha_{-i},\alpha_i');\alpha_j)-\frac{1}{n-1}\,g_j(\alpha_j)\Big]p(\alpha_{-i}\mid\alpha_i')\Bigg\}p(\alpha_i)\geq 0,$$

which, by (3) and (4) (the definitions of the g_i's and of θ), is equivalent to

$$\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}\min_{\alpha_i'\in\mathcal{Q}_i(\alpha_i)}$$
$$\Bigg\{\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\Big[u_i\left(s\left(\alpha_{-i},\alpha_i'\right);\alpha_i'\right)+\theta_i\left(\alpha_{-i},\alpha_i'\right)\Big]p(\alpha_{-i}\mid\mathcal{Q}_i(\alpha_i))-\overline{U}_i(\alpha_i')\Bigg\}p(\alpha_i)$$
$$\geq 0. \quad (12)$$

where

$$p(\alpha_{-i}\mid Q_i)=p(\alpha_{-i}\mid\alpha_i)\text{ for all }\alpha_i\in Q_i.$$

Choose any profile V_i of reservation utilities such that $V_i(\alpha_i)\geq\overline{U}(\alpha_i)$ for all $i\in\mathcal{N}$ and all $\alpha_i\in\mathcal{A}_i$ and such that replacing $\overline{U}_i$ by V_i equation (12) holds at equality, and let

$$L_i(Q_i)$$
$$=\min_{\alpha_i'\in Q_i}\quad\Bigg\{\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\Big[u_i\left(s\left(\alpha_{-i},\alpha_i'\right);\alpha_i'\right)+\theta_i\left(\alpha_{-i},\alpha_i'\right)\Big]p(\alpha_{-i}\mid Q_i)-V_i(\alpha_i')\Bigg\}$$
$$\text{for all } i\in\mathcal{N}\text{ and all }Q_i\in\mathcal{Q}_i,$$

By Lemmas 2 and A.1, as applied to the associated information structure on independence classes $\{\mathcal{Q},P\}$, there exists a budget-balanced transfer rule $\tau(Q)$ (constant within equivalence classes) such that

$$\sum_{Q_i\in\mathcal{Q}_i}\tau_i(Q)=0\text{ for all }Q\in\mathcal{Q},$$

and

$$\sum_{Q_{-i}\in\mathcal{Q}_{-i}}\tau_i(Q_{-i},Q_i)p(Q_{-i}\mid\alpha_i)=L_i(Q_i)\text{ for all }i\text{ and all }Q_i\in\mathcal{Q}_i.$$

Abusing slightly the notation, let $\tau(\alpha)=\tau(Q(\alpha))$. The mechanism $(s,\theta+\tau)$ satisfies budget balance, BIC^{wi}, and Individual Rationality because

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} \tau_i(\alpha,\alpha_{-i})p_i(\alpha_i \mid \alpha_{-i}) \geq$$
$$- \min_{\alpha_i'\in Q_i(\alpha_i)} \sum_{\alpha_{-i}} \left[u_i(s(\alpha_i',\alpha_{-i});\alpha_i') + \theta_i\left(\alpha_i',\alpha_{-i}\right) - \overline{U}_i(\alpha_i')\right] p(\alpha_{-i} \mid \alpha_i).$$

The mechanism (s,t) with

$$t_i(\alpha) = \theta_i(\alpha) + \bar{\tau}_i(Q(\alpha)) + k\tau_i^{\overline{B}}(Q(\alpha)),$$

where $\tau^{\overline{B}}$ is the transfer rule that satisfies (7) and (8), with k large enough to overwhelm the incentives to lie "across" equivalence classes satisfies budget balance, individual rationality and incentive compatibility. □

As in Section 3, we conclude by studying mechanisms which allow for implementation with β-budget balance. The global surplus condition (GS) must be modified to take into account the monetary transfer to (or from) the agents.

We prove the following theorem which is analogous to Theorem 5, but where implementation requires that (BB) be replaced by (BB-β) and (GS*) replaced by (GS*-β).

Theorem 6. *If it satisfies the Generalized Condition $\overline{B}$, the information structure $\{\mathcal{A},p\}$ guarantees the implementation with voluntary participation and β-budget balance of IC-efficient decision rules that satisfy*

$$S + \beta \geq \sum_{i\in\mathcal{N}} \sum_{Q_i\in\mathcal{Q}_i} \mathcal{V}_i(Q_i)P(Q_i). \qquad \text{(GS}^*\text{-}\beta)$$

Proof. In order to prove the theorem we only need to prove that whenever an information structure guarantees implementation with voluntary participation and 0-budget balance of IC-efficient decision rules which satisfy (GS*), it also guarantees implementation with voluntary participation and β-budget balance of any IC-efficient decision rules which satisfy (GS*-β).

Consider any utility functions u_i and a decision rule s which together satisfy (GS*-β). Let $v_i(s;\alpha_i) = u_i(s;\alpha_i) - \beta/n$ for all s and all α_i. The v_is and s satisfy (GS*), as the change from the u_i's to the v_i's decreases the surplus S by β, but does not affect the right-hand side. Therefore, if $\{\mathcal{A},p\}$ guarantees implementation with budget balance 0, there exists $\widetilde{t}$ such that $\sum_i \widetilde{t}(\alpha) = 0$ for all α and

$$\sum_{\alpha_{-i}} \Big[v_i(s(\alpha_{-i},\alpha_i);\alpha_i) + \widetilde{t}_i(\alpha_{-i},\alpha_i)\Big] p(\alpha_{-i} \mid \alpha_i) \geq \overline{U}_i(\alpha_i)$$

for all $i \in \mathcal{N}$, and all $\alpha_i \in \mathcal{A}_i$.

Let $t_i(s;\alpha) = \widetilde{t}_i(s;\alpha) + \beta/n$; it satisfies (BB-$\beta$) and (BIC), which proves the result. □

Note the asymmetry between the two sides of equation (GS*-β). When the mechanism designer transfers some of the numeraire to the agents, the effective surplus they derive from the mechanism increases by β, whereas the variability of the S_i's does not change. This enlarges the set of implementable decision rules.

5 Other interpretations and relationship with the literature

5.1 Imposed participation, Condition C and VCG mechanisms

Since d'Aspremont and Gérard-Varet (1979), it is well known that Condition C, which we present under the form proposed by d'Aspremont et al. (2003), guarantees implementation of efficient decision rules:

Condition C**.** *For all* $R : \mathcal{A} \to \mathbb{R}$*, there exists transfers* t^C *such that*

$$\sum_{i\in\mathcal{N}} t_i^C(\alpha) = R(\alpha) \text{ for all } \alpha \in \mathcal{A}, \tag{13}$$

and

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} t_i^C(\alpha_{-i},\alpha_i)p(\alpha_{-i} \mid \alpha_i) \geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} t_i^C(\alpha_{-i},\widetilde{\alpha}_i)p(\alpha_{-i} \mid \alpha_i) \quad \text{for all } i \in \mathcal{N} \text{ and all } (\alpha_i,\widetilde{\alpha}_i) \in \mathcal{A}_i^2. \tag{14}$$

With Condition C written under this primal form, it is easy to show that it guarantees implementation of efficient decision rules with imposed participation: build a VCG mechanism, which implements an efficient decision; thanks to Condition C find transfers t^C built from (13) with $R(\alpha)$ equal to the deficit induced by the VCG transfers; add these transfers to the VCG transfers. For more details see d'Aspremont et al. (2003).

As we assume $n \geq 3$, Condition C holds whenever Condition B holds.[24] It therefore holds generically, as shown by d'Aspremont, Crémer and Gérard-Varet (2004) who also show a) that Condition C holds whenever one agent

[24] If $n = 2$, Condition C is equivalent to independence of types and Condition B never holds (d'Aspremont and Gérard-Varet, 1982).

has free beliefs and b) that it is the most general of the known conditions that guarantee implementation of efficient decision rules.[25] But Condition C allows for implementation of many more decision rules than efficient ones, as the following results, which links Condition C to the results of this paper, show.

Theorem 7. *An information structure $(\mathcal{A}, p)$ satisfies Condition C if and only if it satisfies the Generalised Condition B.*

The proof is presented in Appendix D. It requires the results of subsection 5.3 below, where we define the operations of merging and splitting of types and state Lemma 4.

With Theorem 1, this implies the following corollary.

Corollary 1. *An information structure satisfies Condition C if and only if it guarantees implementation of IC-efficient decision rules under imposed participation.*

5.2 Voluntary participation and Condition $\overline{C}$

Inspired by the results of 5.1, we show that the Generalized Condition $\overline{B}$ is equivalent to a new condition, Condition $\overline{C}$, more restrictive than Condition C. In so doing, we provide new insights on the implementation of efficient decision rules under participation constraints.

Condition $\overline{C}$. *For all $R : \mathcal{A} \to \Re$ and all $\{r_i : \mathcal{Q}_i \to \Re\}_{i \in \mathcal{N}}$ such that*

$$\sum_{\alpha \in \mathcal{A}} R(\alpha) p(\alpha) \geq \sum_{Q \in \mathcal{Q}} \left[\sum_{i \in \mathcal{N}} r_i(Q_i) \right] P(Q), \tag{15}$$

that is, such that the expected value of R is greater than or equal to the expected value of $\sum_{i \in \mathcal{N}} r_i(Q_i)$, there exists a transfer rule $t^{\overline{C}}$ that satisfies the following three conditions:

$$\sum_{i \in \mathcal{N}} t_i^{\overline{C}}(\alpha) = R(\alpha) \text{ for all } \alpha \in \mathcal{A}, \tag{16}$$

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{C}}(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i) \geq \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{C}}(\alpha_{-i}, \widetilde{\alpha}_i) p(\alpha_{-i} \mid \alpha_i) \quad \text{for all } i \in \mathcal{N} \text{ and all } (\alpha_i, \widetilde{\alpha}_i) \in \mathcal{A}_i^2, \tag{17}$$

[25] See also d'Aspremont and Gérard-Varet (1982). However, d'Aspremont et al. (2003) showed (by exhibiting an example) that there exist information structures that do not satisfy Condition C and still guarantee efficient implementation under imposed participation.

and

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} t_i^{\overline{C}}(\alpha_{-i},\alpha_i)p(\alpha_{-i}\mid\alpha_i) \geq r_i(Q_i(\alpha_i)) \text{ for all } i\in\mathcal{N} \text{ and } \alpha_i\in\mathcal{A}_i. \quad (18)$$

We will show (Theorem 10) that Condition $\overline{C}$ is equivalent to the Generalized Condition $\overline{B}$, which proves the following theorem. However, we think that the direct proof which we present below is instructive,

Theorem 8. *If Condition $\overline{C}$ holds, $\{\mathcal{A},p\}$ guarantees implementation of IC-efficient decision rules with voluntary participation whenever* (GS*) *hold.*

Proof. To make the proof easier to read, we assume $\overline{U}_i(\alpha_i)=0$ for all i and all α_i. The results and the reasoning are unchanged if we lift this assumption.

Let s be an IC-efficient decision rule and consider the Vickrey-Clark-Groves transfer rule

$$t_i(\alpha)=\sum_{j\in\mathcal{N}_{-i}} u_j(s(\alpha_{-i},\alpha_i);\alpha_j) \text{ for all } i\in\mathcal{N} \text{ and all } \alpha\in\mathcal{A}.$$

Let

$$R(\alpha)=-\sum_i t_i(\alpha)$$

and

$$r_i(Q_i)= -\min_{\alpha_i\in Q_i}\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\left[u_i(s(\alpha_{-i},\alpha_i);\alpha_i)+\sum_{j\in\mathcal{N}_{-i}}u_j(s(\alpha_{-i},\alpha_i);\alpha_j)\right]p(\alpha_i\mid Q_i),$$

so that (15) holds as

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{A}} R(\alpha)p(\alpha)\\
&= -\sum_{\alpha\in\mathcal{A}}\sum_{i\in\mathcal{N}}\left[\sum_{j\in\mathcal{N}_{-i}} u_j(s(\alpha_{-i},\alpha_i);\alpha_j)\right]p(\alpha)\\
&= -(n-1)\sum_{\alpha\in\mathcal{A}}\left[\sum_{j\in\mathcal{N}}u_j(s(\alpha_{-i},\alpha_i);\alpha_j)\right]p(\alpha)\\
&\geq -\sum_{i\in\mathcal{N}}\sum_{\alpha_i\in\mathcal{A}_i}\min_{\alpha_i'\in Q_i(\alpha_i)}\sum_{\alpha_{-i}\in\mathcal{A}_{-i}}\left[u_i(s(\alpha_{-i},\alpha_i');\alpha_i')+\sum_{j\in\mathcal{N}_{-i}}u_j(s(\alpha_{-i},\alpha_i');\alpha_j)\right]\\
&\qquad p\left(\alpha_{-i}\mid\alpha_i'\right)p_i\left(\alpha_i\right) \quad \text{(by (GS*) and (11))}
\end{aligned}$$

$$= \sum_{i \in \mathcal{N}} \sum_{Q_i \in \mathcal{Q}_i} r_i(Q_i) p(Q_i).$$

Therefore, there exists a transfer rule $t^{\overline{C}}$ which satisfy equations (16) to (18). The balanced transfer rule $\tau = t + t^{\overline{C}}$ clearly satisfies incentive compatibility. It also satisfies the participation constraints as the following inequality holds for any $i \in \mathcal{N}$ and any $\alpha_i \in \mathcal{A}_i$:

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} [u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) + \tau_i(\alpha_{-i}, \alpha_i)] p(\alpha_{-i} \mid \alpha_i)$$

$$= \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} [u_i(s(\alpha_i, \alpha_{-i}); \alpha_i) + t_i(\alpha_{-i}, \alpha_i)] p(\alpha_{-i} \mid \alpha_i)$$

$$+ \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{C}}(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i)$$

$$\geq \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} [u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) + t_i(\alpha_{-i}, \alpha_i)] p(\alpha_{-i} \mid \alpha_i) + r_i(Q_i(\alpha_i))$$

$$= \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} [u_i(s(\alpha_{-i}, \alpha_i); \alpha_i) + t_i(\alpha_{-i}, \alpha_i)] p(\alpha_{-i} \mid \alpha_i)$$

$$- \min_{\alpha_i' \in Q_i(\alpha_i)} \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} [u_i(s(\alpha_{-i}, \alpha' i); \alpha_i') + t_i(\alpha_{-i}, \alpha' i)] p(\alpha_{-i} \mid \alpha_i') \geq 0. \quad \square$$

5.3 Merging and splitting of types

In order to prove Theorem 7, as well as the similar statement for Condition $\overline{C}$, namely that an information structure satisfies Condition $\overline{C}$ if and only it satisfies the Generalized Condition $\overline{B}$, we need to define operations of *merging* and *splitting* of types. We do this in this section as we believe it to be of independent interest. Detailed proofs are given in Appendices D and E.

The information structure $(\mathcal{A}, p)$ is built from the information structure $\{\widehat{\mathcal{A}}, \widehat{p}\}$ by splitting type $\widehat{\alpha}_k \in \widehat{\mathcal{A}}_k$ if type $\widehat{\alpha}_k$ of agent k is replaced in $\mathcal{A}_k$ by two types α_k^1 and α_k^2 which are independent of each other and induce the same conditional probabilities on the types of the other agents. Formally,

$$\mathcal{A}_k = \left(\widehat{\mathcal{A}}_k \setminus \{\widehat{\alpha}_k\}\right) \bigcup \{\alpha_k^1, \alpha_k^2\},$$

$$\mathcal{A}_i = \widehat{\mathcal{A}}_i \text{ for all } i \in \mathcal{N}_{-k},$$

$$p(\alpha) = \widehat{p}(\alpha) \text{ if } \alpha_k \in \mathcal{A}_k \setminus \{\alpha_k^1, \alpha_k^2\},$$

$$p(\alpha_k^1) + p(\alpha_k^2) = \widehat{p}(\widehat{\alpha}_k),$$

$$p(\alpha_k^\ell, \alpha_{-k}) = \widehat{p}(\widehat{\alpha}_k, \alpha_{-k}) \times \frac{p(\alpha_k^\ell)}{\widehat{p}(\widehat{\alpha}_k)}, \quad \text{for } \ell \in \{1,2\} \text{ and all } \alpha_k \in \mathcal{A}_{-k}. \tag{19}$$

The inverse operation is called *merging*. The information structure $\{\widehat{\mathcal{A}}, \widehat{p}\}$ is built from the information structure $(\mathcal{A}, p)$ by merging two independent types α_k^1 and α_k^2 in $\mathcal{A}_k$ if these two types are replaced by a single type $\widehat{\alpha}_k$ in $\widehat{\mathcal{A}}_k$. Formally:

$$\begin{aligned} &\widehat{\mathcal{A}}_k = \left(\mathcal{A}_k \setminus \{\alpha_k^1, \alpha_k^2\}\right) \bigcup \{\widehat{\alpha}_k\}, \\ &\widehat{\mathcal{A}}_i = \mathcal{A}_i \text{ for all } i \in \mathcal{N}_{-k}, \\ &\widehat{p}(\alpha) = p(\alpha) \text{ if } \alpha_k \in \widehat{\mathcal{A}}_k \setminus \{\widehat{\alpha}_k\}, \\ &\widehat{p}(\widehat{\alpha}_k, \alpha_{-k}) = p(\alpha_k^1, \alpha_{-k}) + p(\alpha_k^2, \alpha_{-k}) \text{ for all } \alpha_{-k} \in \mathcal{A}_{-k}. \end{aligned} \tag{20}$$

The information structure $(\mathcal{A}, p)$ is built from $\{\widehat{\mathcal{A}}, \widehat{p}\}$ by splitting $\widehat{\alpha}_k$ into α_k^1 and α_k^2 if and only if $\{\widehat{\mathcal{A}}, \widehat{p}\}$ is built from $\{\mathcal{A}, p\}$ by merging α_k^1 and α_k^2 into $\widehat{\alpha}_k$.

Lemma 4. *If an information structure $\{\mathcal{A}, p\}$ satisfies Condition C (resp. Condition $\overline{C}$) any information structure obtained by merging two of its types or splitting one of them also satisfies Condition C (resp. Condition $\overline{C}$).*

It is trivial, but notationally heavy enough that we will spare the reader, that the information structures on equivalence classes are essentially unchanged by the operations of merging and splitting.

Lemma 4 is proved in Appendices D and E.

In the case of imposed participation, Lemma 4 implies that an information structure satisfies Condition C if and only if iterative elimination through merging of all independent types yields an information structure that also satisfies Condition C. Then, Theorem 7 is a consequence of Theorem 4 of d'Aspremont et al. (2004), which states that under no-freeness Condition B and Condition C are equivalent.

In Appendix E, we follow the same strategy in the case of voluntary participation. We first prove the following theorem, which does for voluntary participation what Theorem 4 of d'Aspremont et al. (2004) does for imposed participation.

Theorem 9. *If an information structure satisfies the no-freeness property Condition $\overline{C}$ is equivalent to Condition $\overline{B}$.*

Along with Lemma 4, this proves Theorem 10.

Theorem 10. *An information structure $(\mathcal{A}, p)$ satisfies Condition $\overline{C}$ if and only it satisfies the Generalized Condition $\overline{B}$.*

6 Conclusion

Introducing the notion of independence classes associated to the agents' conditional beliefs has lead us to define general assumptions covering all cases between the two traditional extremes, full-freeness (independence) or no-freeness (BDP). It has also allowed us to introduce a weaker notion of efficiency, IC-efficiency, which preserves the incentive contribution of efficiency when freeness holds (within independence classes), while enlarging considerably the set of implementable mechanisms available to the mechanism designer.

With participation constraints implementation requires not only that the total surplus to be distributed to the agents be large enough, but also that it varies "not too much" within independence classes. As we discussed on page 22, there is a form of trade-off between the stringency of that condition and the stringency of the restrictions imposed on the distribution of the types.

The techniques we have proposed are immediately applicable to setups such as the federated server farms discussed by Albert et al. (2022), which we already mentioned in the introduction. We believe that they can also be adapted to auctions and help generalize the results of Albert et al. (2015) to auctions with more than one buyer.

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APPENDIX

A Proofs of results linked to Assumption 1

Lemma A.1. *The system of equations*

$$\sum_{i \in \mathcal{N}} t_i(\alpha) = 0 \text{ for all } \alpha \in \mathcal{A}; \tag{A.1}$$

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i) = L_i(\alpha_i) \text{ for all } i \in \mathcal{N} \text{ and all } \alpha_i \in \mathcal{A}_i; \tag{A.2}$$

has a solution for all families of functions $\{L_i : \mathcal{A}_i \to \Re\}_{i \in \mathcal{N}}$, *which satisfy*

$$\sum_{i \in \mathcal{N}, \, \alpha_i \in \mathcal{A}_i} p(\alpha_i) L_i(\alpha_i) = 0. \tag{A.3}$$

if and only if Assumption 1 holds.

The lemma is stated for the information structure $\{\mathcal{A}, p\}$ but, clearly, by Lemma 2, it also holds for $(\mathcal{Q}, P)$, the associated information structure on independence classes.

Proof of Lemma A.1. With the dual variables $\lambda(\alpha)$ for (A.1) and $\mu_i(\alpha_i)$ for (A.2), the system $\{(A.1), (A.2)\}$ has a solution if and only if the following system does not (see Gale's theorem of the alternative in Mangasarian (1969, pp. 33 and 34)):

$$\lambda(\alpha) + p(\alpha_{-i} \mid \alpha_i) \mu_i(\alpha_i) = 0 \text{ for all } \alpha \text{ and all } i; \tag{A.4}$$

$$\sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \mu_i(\alpha_i) L_i(\alpha_i) = 1. \tag{A.5}$$

Lemma A.1 is therefore equivalent to the following statement: "Assumption 1 holds if and only if for all L_is satisfying (A.3), the system of equations $\{(A.4), (A.5)\}$ does not have a solution".

We first assume that Assumption 1 holds and show that the system $\{(A.4), (A.5)\}$ does not have a solution when (A.3) holds.

From (A.4) if two edges α_i and α_j are adjacent, then $\mu_i(\alpha_i)/p(\alpha_i) = \mu_j(\alpha_j)/p(\alpha_j)$.[26] By a simple induction argument, if G is connected, then all

[26] Indeed, there exist α_{-i-j} such that $p(\alpha_{-i-j}, \alpha_j, \alpha_i) > 0$. If $\lambda(\alpha_{-i-j}, \alpha_i, \alpha_j) \neq 0$, then

$$\mu_i(\alpha_i)/p(\alpha_i) = \mu(\alpha_j)/p(\alpha_j) = \lambda(\alpha_{-i-j}, \alpha_i, \alpha_j)/p(\alpha_{-i-j}, \alpha_i, \alpha_j).$$

If $\lambda(\alpha_{-i-j}, \alpha_i, \alpha_j) = 0$, then $\mu_i(\alpha_i) = \mu(\alpha_j) = 0$.

the ratios $\mu_i(\alpha_i)/p(\alpha_i)$ are equal and (A.3) and (A.5) cannot simultaneously hold.

Assume now that Assumption 1 does not hold; to prove our result we only need to exhibit a family of L_is which satisfies (A.3) such that the system $\{(A.4), (A.5)\}$ has a solution. Because G is not connected, by the remark which follows the statement of Assumption 1 it has at least two components. Let $|G^q|$ be the number of elements of G^q, $L_i(\alpha_i) = -1/\left(2\left|G^1\right|\right)$ for $\alpha_i \in G^1$, $L_i(\alpha_i) = 1/\left(2\left|G^2\right|\right)$ for $\alpha_i \in G^2$, and $L_i(\alpha_i) = 0$ if α_i belongs to neither G^1 or G^2. Then (A.3) holds. Let $\mu_i(\alpha_i)$ be equal to $-p(\alpha_i)$ if $\alpha_i \in G^1$, to $p(\alpha_i)$ if $\alpha_i \in G^2$, and to 0 otherwise. Finally, let $\lambda(\alpha)$ be equal to $p(\alpha)$ if $\alpha_i \in G^1$ for all i, to $-p(\alpha)$ if $\alpha_i \in G^2$ for all i, and to 0 in all other cases. It is straightforward, if a little tedious, to show that equations (A.4) and (A.5) are satisfied, which proves the result. □

Proof of Theorem 3. Choose some $\overline{V}_i(\alpha_i) \geq \overline{U}_i(\alpha_i)$ such that, replacing the $\overline{U}_i$s by the $\overline{V}_i$s, equation (GS) holds at equality, and take $L_i(\alpha_i) = \sum_{\alpha_{-i}} u_i(s(\alpha_{-i}, \alpha_i); \alpha_i)\, p(\alpha_{-i}, \alpha_i) - \overline{V}_i(\alpha_i)p(\alpha_i)$. This proves at the same time the first and the second part of the theorem. □

B Proof of the only if part of Theorem 4: necessity of Condition $\overline{B}$

Proof. Since any decision rule can be implemented with voluntary participation, we may choose any $i \in \mathcal{N}$ and any $\alpha_i^0 \in \mathcal{A}_i$ and assume $u_j(x; \alpha_j) = 0$ for all j, x and α_j, except that $u_i(\widehat{x}; \alpha_i^0) = -1$ for some $\widehat{x} \in \mathcal{X}$. We may also choose a decision rule that satisfies $s(\alpha) = \widehat{x}$ if and only if $\alpha_i = \alpha_i^0$. Finally, we let $\overline{U}_j(\alpha_j) = 0$ for all j and all α_j except that $\overline{U}_i(\alpha_i^0) = -1$.

Notice that the decision rule minimizes the sum of the utilities of the agents, that these utilities have private values and that the global surplus condition (GS) holds as

$$\sum_{\alpha \in \mathcal{A}} \sum_{i \in \mathcal{N}} u_i(s(\alpha_{-i}, \alpha_i); \alpha)p(\alpha) = \sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} \overline{U}_i(\alpha_i)p(\alpha_i)$$

Because, by hypothesis, s can be implemented, there exists a balanced transfer function $t^{(i,\alpha_i^0)}$ that satisfies the incentive compatibility and participation constraints of the agents. For instance, the IC constraint for agent i of type α_i^0 yields

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)\left[t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i^0)-t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i)\right]$$
$$\geq \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)\left[u_i(s(\alpha_{-i},\alpha_i);\alpha_i^0)-u_i(s(\alpha_{-i},\alpha_i^0);\alpha_i^0)\right]=1$$
$$\Longrightarrow \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i^0) > \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i).$$
$$\text{for all } \alpha_i\neq\alpha_i^0.$$

Using the same reasoning it is easy to show

$$\sum_{\alpha_{-j}\in\mathcal{A}_{-j}} p(\alpha_{-j}\mid\alpha_j)\left[t_j^{(i,\alpha_i^0)}(\alpha_{-j},\alpha_j)-t_j^{(i,\alpha_i^0)}(\alpha_{-j},\widetilde{\alpha}_j)\right]\geq 0$$
$$\text{whenever } (j,\alpha_j)\neq(i,\alpha_i^0).$$

Voluntary participation by agent i of type $\alpha_i\neq\alpha_i^0$ implies

$$\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)\left[u_i(s(\alpha_{-i},\alpha_i^0);\alpha_i^0)+t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i^0)\right]\geq \overline{U}_i(\alpha_i^0)=-1$$
$$\Longrightarrow \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i}\mid\alpha_i^0)t_i^{(i,\alpha_i^0)}(\alpha_{-i},\alpha_i^0)\geq 0$$

Using the same reasoning, it is easy to show that

$$\sum_{\alpha_{-j}\in\mathcal{A}_{-j}} p(\alpha_{-j}\mid\alpha_j)\, t_j^{(i,\alpha_i^0)}(\alpha_{-j},\alpha_j)\geq 0 \text{ for all } j \text{ and all } \alpha_j\in\mathcal{A}_j. \tag{B.1}$$

By budget balance, none of the inequalities (B.1) can be strict. Therefore,

$$\sum_{\alpha_{-j}\in\mathcal{A}_{-j}} p(\alpha_{-j}\mid\alpha_j)t_j^{(i,\alpha^0)}(\alpha_{-j},\alpha_j)=0 \text{ for all } j\in\mathcal{N} \text{ and all } \alpha_j\in\mathcal{A}_j,$$

We can repeat this construction for every i and every α_i^0, obtaining a set of transfer rules t^{i,α_i^0}. Summing over all i and all α_i^0, we obtain the transfer rule

$$t^{\overline{B}}(\alpha)=\sum_{i\in\mathcal{N},\,\alpha_i^0\in\mathcal{A}_i} t^{i,\alpha_i^0}(\alpha)$$

which satisfies Condition $\overline{B}$. □

C Genericity of Condition $\overline{B}$

We present a constructive proof of the genericity of Condition $\overline{B}$, based on the notion of "scoring rule" introduced by Good (1952), discussed by Savage (1974), and applied to Bayesian implementation by Johnson et al. (1990) and by d'Aspremont et al. (2003). We use the same transfer scheme as in d'Aspremont et al. (2003), and will therefore rely on some of the arguments in that paper.

We need the following notation. For $i \neq j$, $\mathcal{A}_{-\{i,j\}} \stackrel{\text{def}}{=} \prod_{k \notin \{i,j\}} \mathcal{A}_k$ is the set of possible types of agents other than i and j. For $\alpha_{-\{i,j\}} \in \mathcal{A}_{-\{i,j\}}$, $p(\alpha_{-\{i,j\}} \mid \alpha_i)$ represents the beliefs of agent i on the types of agents other than himself and other than j, while $p\left(\alpha_j \mid \alpha_i\right)$ represents the beliefs of agent i on the type of agent j.

Theorem C.1. *Condition $\overline{B}$ holds for nearly all information structures.*

Proof. For this proof, addition and subtraction on the indices of agents are defined modulo n so that, for instance, $n+1=1$ and $1-1=n$. Generically, for all $\alpha_i \neq \widetilde{\alpha}_i$ there exists a) an $\alpha_{-\{i,i+1\}}$ such that $p(\alpha_{-\{i,i+1)\}} \mid \widetilde{\alpha}_i) \neq p(\alpha_{-\{i,i+1\}} \mid \alpha_i)$ and b) an $\alpha_{-\{i,i-1\}}$ such that $p(\alpha_{-\{i,i-1\}} \mid \widetilde{\alpha}_i) \neq p(\alpha_{-\{i,i-1\}} \mid \alpha_i)$.

Define the following *transfer scoring rule*:

$$
\begin{aligned}
t_i^{\overline{B}}(\alpha) \stackrel{\text{def}}{=} & \log p\left(\alpha_{-i-(i-1)} \mid \alpha_i\right) - \log p\left(\alpha_{-i-(i-1)} \mid \alpha_{i-1}\right) \\
& + \log p\left(\alpha_{-i-(i+1)} \mid \alpha_i\right) - \log p\left(\alpha_{-i-(i+1)} \mid \alpha_{i+1}\right),
\end{aligned}
$$

which is the same transfer rule as the t^B defined in the proof of Theorem 3 of d'Aspremont et al. (2003). As proved in that article, it satisfies budget balance and incentive compatibility. It also satisfies the zero surplus condition (6) as

$$
-\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} t_i^{\overline{B}}(\alpha_{-i}, \alpha_i) p(\alpha_{-i} \mid \alpha_i)
$$

$$
\begin{aligned}
&= \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} \left[\log p\left(\alpha_{-\{i,i-1\}} \mid \alpha_{i-1}\right) - \log p\left(\alpha_{-\{i,i-1\}} \mid \alpha_i\right)\right] p_i\left(\alpha_{-i} \mid \alpha_i\right) \\
&\quad + \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} \left[\log p\left(\alpha_{-\{i,i+1\}} \mid \alpha_{i+1}\right) - \log p\left(\alpha_{-\{i,i+1\}} \mid \alpha_i\right)\right] p_i\left(\alpha_{-i} \mid \alpha_i\right) \\
&= \sum_{\alpha_{-\{i,i-1\}}\in\mathcal{A}_{-\{i,i-1\}}} \left[\log \frac{p\left(\alpha_{-\{i,i-1\}} \mid \alpha_{i-1}\right)}{p\left(\alpha_{-\{i,i-1\}} \mid \alpha_i\right)}\right] p_i\left(\alpha_{-\{i,i-1\}} \mid \alpha_i\right) \\
&\qquad + \sum_{\alpha_{-\{i,i-1\}}\in\mathcal{A}_{-\{i,i-1\}}} \left[\log \frac{p\left(\alpha_{-\{i,i+1\}} \mid \alpha_{i+1}\right)}{p\left(\alpha_{-\{i,i+1\}} \mid \alpha_i\right)}\right] p_i\left(\alpha_{-\{i,i+1\}} \mid \alpha_i\right) \\
&\leq \log \sum_{\alpha_{-\{i,i-1\}}\in\mathcal{A}_{-\{i,i-1\}}} \left[\frac{p\left(\alpha_{-\{i,i-1\}} \mid \alpha_{i-1}\right)}{p\left(\alpha_{-\{i,i-1\}} \mid \alpha_i\right)} \times p_i\left(\alpha_{-\{i,i-1\}} \mid \alpha_i\right)\right] \\
&\quad + \log \sum_{\alpha_{-\{i,i-1\}}\in\mathcal{A}_{-\{i,i-1\}}} \left[\frac{p\left(\alpha_{-\{i,i+1\}} \mid \alpha_{i+1}\right)}{p\left(\alpha_{-\{i,i+1\}} \mid \alpha_i\right)} \times p_i\left(\alpha_{-\{i,i+1\}} \mid \alpha_i\right)\right] \\
&\quad \text{(by concavity of the function log)} \\
&= 0.
\end{aligned}
$$

This implies $\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} t_i^{\overline{B}}(\alpha_{-i},\alpha_i)p(\alpha_{-i} \mid \alpha_i) \geq 0$ for all i and all α_i; because of budget balance, the inequality must be an equality and this proves that (6) holds. □

D Proof of Theorem 7: Condition C and the Generalized Condition B are equivalent

As discussed in the main text, to prove Theorem 7 we only need to prove Lemma 4 in the case of imposed participation. We do this through the following two lemmas.

Lemma D.1 (Inheriting Condition C through splitting)**.** *If an information structure satisfies Condition C, splitting one of its types into two independent types yields an information structure that also satisfies Condition C.*

Proof. Let $\{\widehat{\mathcal{A}},\widehat{p}\}$ satisfy Condition C, and $\{\mathcal{A},p\}$ be built from $\{\widehat{\mathcal{A}},\widehat{p}\}$ by splitting $\widehat{\alpha}_k$ into α_k^1 and α_k^2. We show that $\{\mathcal{A},p\}$ also satisfies Condition C. To do so, consider any $R:\mathcal{A}\to\Re$. We show that there exist transfers t^C that satisfy equations (13) and (14).

Let $\widehat{R} : \widehat{\mathcal{A}} \to \Re$ be defined as follows:

$$\widehat{R}(\alpha) = R(\alpha) \text{ if } \alpha_k \neq \widehat{\alpha}_k, \tag{D.1}$$

and for all $\alpha_{-k} \in \mathcal{A}_{-k}$:

$$\begin{aligned}\widehat{R}(\widehat{\alpha}_k, \alpha_{-k}) &= \frac{p(\alpha_k^1)}{\widehat{p}(\widehat{\alpha}_k)} R(\alpha_k^1, \alpha_{-k}) + \frac{p(\alpha_k^2)}{\widehat{p}(\widehat{\alpha}_k)} R(\alpha_k^1, \alpha_{-k}), \qquad \text{(D.2)}\\ &= \frac{p(\alpha_k^1, \alpha_{-k})}{\widehat{p}(\widehat{\alpha}_k, \alpha_{-k})} R(\alpha_k^1, \alpha_{-k}) + \frac{p(\alpha_k^2, \alpha_{-k})}{\widehat{p}(\widehat{\alpha}_k, \alpha_{-k})} R(\alpha_k^1, \alpha_{-k})\\ &\qquad \text{by (19).}\end{aligned}$$

Because $\{\widehat{\mathcal{A}}, \widehat{p}\}$ satisfies C, there exists $\widehat{t}^C : \widehat{\mathcal{A}} \to \Re^{\mathcal{N}}$ which satisfies the properly rewritten equations (13) and (14). Define $t^C : \mathcal{A} \to \Re^{\mathcal{N}}$ as follows:

$$t^C(\alpha) = \widehat{t}^C(\alpha) \text{ if } \alpha_k \notin \{\alpha_k^1, \alpha_k^2\}, \tag{D.3}$$

$$t_k^C(\alpha_{-k}, \alpha_k^1) = t_k^C(\alpha_{-k}, \alpha_k^2) = \widehat{t}_k^C(\alpha_{-k}, \widehat{\alpha}_k) \text{ for all } \alpha_{-k} \in \mathcal{A}_{-k}, \tag{D.4}$$

$$t_i^C(\alpha_{-k}, \alpha_k^\ell) = \widehat{t}_i^C(\alpha_{-k}, \widehat{\alpha}_k) + \frac{R(\alpha_{-k}, \alpha_k^\ell) - \widehat{R}(\alpha_{-k}, \widehat{\alpha}_k)}{n-1}$$
$$\text{for all } i \in \mathcal{N}_{-k}, \text{ all } \ell \in \{1, 2\}, \text{ and all } \alpha_{-k} \in \mathcal{A}_{-k}.$$

It is straightforward, if somewhat tedious to show that t^C satisfies (13) and (14). □

Lemma D.2 (Inheriting Condition C through merging)**.** *If an information structure satisfies Condition C, merging two independent types yields an information structure that also satisfies Condition C.*

Proof. Let $\{\mathcal{A}, p\}$ satisfy Condition C, and $\{\widehat{\mathcal{A}}, \widehat{p}\}$ be built from $\{\mathcal{A}, p\}$ by merging the two independent types α_k^1 and α_k^2 into $\widehat{\alpha}_k$. We show that $\{\widehat{\mathcal{A}}, \widehat{p}\}$ also satisfies Condition C. To do so, consider any $\widehat{R} : \widehat{\mathcal{A}} \to \Re$. We show that there exist transfers $\widehat{t}^C$ that satisfy (13) and (14).

Define $R : \mathcal{A} \to \Re$ as follows

$$R(\alpha) = \widehat{R}(\alpha) \text{ if } \alpha_k \notin \{\alpha_k^1, \alpha_k^2\}, \tag{D.5}$$

$$R(\alpha_k^1, \alpha_{-k}) = R(\alpha_k^2, \alpha_{-k}) = \widehat{R}(\widehat{\alpha}_k, \alpha_{-k}) \text{ for all } \alpha_{-k} \in \mathcal{A}_{-k}. \tag{D.6}$$

Because $\{\mathcal{N}, \mathcal{A}, p\}$ satisfies Condition C, there exists transfers t^C which satisfy (13) and (14). We construct the transfers $\widehat{t}^C$ as follows:

$$\widehat{t}_i^C(\alpha) = t_i^C(\alpha) \text{ for all } i \text{ if } \alpha_k \neq \widehat{\alpha}_k; \tag{D.7}$$

$$\widehat{t}_i^C(\widehat{\alpha}_k, \alpha_{-k}) = t_i^C(\alpha_k^1, \alpha_{-k})\frac{p(\alpha_k^1)}{\widehat{p}(\widehat{\alpha}_k)} + t_i^C(\alpha_k^2, \alpha_{-k})\frac{p(\alpha_k^2)}{\widehat{p}(\widehat{\alpha}_k)} \quad \text{for all } \alpha_{-k} \in \mathcal{A}_{-k}. \tag{D.8}$$

Equations (D.5) and (D.7) and the fact that the transfers t^C satisfy (13) imply that $\widehat{t}^C$ also satisfy (13) when $\alpha_k \neq \widehat{\alpha}_k$. By (D.8), for all α_{-k}, we have

$$\sum_i \widehat{t}_i^C(\widehat{\alpha}_k, \alpha_{-k}) = R_i(\alpha_k^1, \alpha_{-k})\frac{p(\alpha_k^1)}{\widehat{p}(\widehat{\alpha}_k)} + R_i(\alpha_k^2, \alpha_{-k})\frac{p(\alpha_k^2)}{\widehat{p}(\widehat{\alpha}_k)} = \widehat{R}(\widehat{\alpha}_k, \alpha_{-k}),$$

which establishes (13).

Proving that $\widehat{t}^C$ satisfies the incentive compatibility constraints (14) is straightforward if tedious. □

E Proof of Theorems 9 and 10

Proof of Theorem 9. Remember that we assume no-freeness, and therefore that independence classes have only one element.

If $\{\mathcal{A}, p\}$ satisfies $\overline{B}$, it also satisfies $\overline{C}$. Take $R : \mathcal{A} \to \Re$ and, for all $i \in \mathcal{N}$, $r_i : \mathcal{A}_i \to \Re$ such that

$$\sum_{\alpha \in \mathcal{A}} R(\alpha)p(\alpha) \geq \sum_{i \in \mathcal{N}} \sum_{\alpha_i \in \mathcal{A}_i} r_i(\alpha_i)\, p(\alpha_i).$$

Let $u_i(s(\alpha_i, \alpha_{-i}); \alpha_i') = R(\alpha)/n$ and $\overline{U}_i(\alpha_i) = r_i(\alpha_i)$ for all α, all α_i' and all i. By Theorem 3 there is a balanced transfer rule t such that

$$\sum_{\alpha_{-i} \in \alpha_{-i}} [R(\alpha_i, \alpha_{-i})/n + t_i(\alpha_i, \alpha_{-i})]\, p_i(\alpha_{-i} \mid \alpha_i) \geq r_i(\alpha_i) \text{ for all } i \text{ and } \alpha_i. \tag{E.1}$$

Let $t^{\overline{C}} = R/n + t + Mt^{\overline{B}}$. Because both t and $t^{\overline{B}}$ are balanced, $t^{\overline{C}}$ satisfies (16). By (6), equation (E.1) implies that it also satisfies (18). Finally, because $t^{\overline{B}}$ satisfies the strict incentive compatibility condition (5), $t^{\overline{C}}$ also satisfies (17) for M large enough.

If $\{\mathcal{A}, p\}$ satisfies $\overline{C}$, it also satisfies $\overline{B}$. Let $\theta_i(\alpha) = \ln p(\alpha_{-i} \mid \alpha_i)$. As the theory of scoring rules has systematically exploited, the strict concavity of the log function implies that θ_i satisfies strict incentive compatibility.

Let $R(\alpha) = -\sum_i \theta_i(\alpha)$ and $r_i(\alpha) = -\theta_i(\alpha)$; condition (15) is satisfied (as an equality), and therefore, there exists $t^{\overline{C}}$ that satisfies conditions (16) to (18). Because (15) is satisfied as an equality, so must (18) for all i and all α_i.

Let $t^{\overline{B}} = \theta + t^{\overline{C}}$. It satisfies the strict incentive compatibility constraint (5). It also satisfies (6) as

$$\begin{aligned}\sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i) t_i^{\overline{B}}(\alpha_{-i}, \alpha_i) &= \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)[\theta_i(\alpha) + t_i^{\overline{C}}(\alpha)] \\ &= \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i)\theta_i(\alpha) + \sum_{\alpha_{-i}\in\mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i) r_i(\alpha_i) \\ &= 0 \quad \text{(because (18) is an equality).}\end{aligned}$$

□

Lemma 4 and, as a consequence, Theorem 10 are derived from the following two lemmas, whose proofs are nearly exactly the same as those of Lemmas D.1 and D.2.

Lemma E.1 (Inheriting Condition $\overline{C}$ through splitting)**.** *If an information structure satisfies Condition $\overline{C}$, splitting one of its types into two independent types yields an information structure that also satisfies Condition $\overline{C}$.*

Proof of Lemma E.1. The proof is a straightforward adaptation of the proof of Lemma D.1. As in that proof, let $\{\widehat{\mathcal{A}}, \widehat{p}\}$ satisfy Condition $\overline{C}$, and $(\mathcal{A}, p)$ be built from $\{\widehat{\mathcal{A}}, \widehat{p}\}$ by splitting $\widehat{\alpha}_k$ into α_k^1 and α_k^2. We show that $(\mathcal{A}, p)$ also satisfies Condition $\overline{C}$.

Notice first that there is an obvious bijection between the independence classes of $(\mathcal{A}, p)$ and $\{\widehat{\mathcal{A}}, \widehat{p}\}$, with

$$\begin{aligned}&\mathcal{Q}_i = \widehat{\mathcal{Q}}_i \text{ if } i \neq k, \\ &\mathcal{Q}_k = \widehat{\mathcal{Q}}_k / \widehat{Q}_k(\widehat{\alpha}_k) \bigcup Q'_k \\ &\qquad \text{where } Q'_k \stackrel{\text{def}}{=} Q_k(\alpha_k^1) = Q_k(\alpha_k^2) = \widehat{Q}_k(\widehat{\alpha}_k)/\{\widehat{\alpha}_k\} \bigcup \{\alpha_k^1, \alpha_k^2\},\end{aligned} \tag{E.2}$$

which implies

$$\begin{aligned}&P(Q) = \widehat{P}(Q) \text{ if } Q_k \neq Q'_k; \\ &P(Q_{-k}, Q'_k) = \widehat{P}(Q_{-k}, Q_k(\widehat{\alpha}_k)) \text{ for all } Q_{-k}.\end{aligned}$$

Consider any $R : \mathcal{A} \to \Re$ and functions $r_i : \mathcal{Q}_i \to \Re$ which satisfy (15). We show that there exist transfers $t^{\overline{C}}$ that satisfy (16), (17) and (18).

Define $\widehat{R}$ as in the proof of Lemma D.1 by (D.1) and (D.2) and $\widehat{r}$ as follows: $\widehat{r}_(Q_i) = r_i(Q_i)$ if $i \neq k$, $\widehat{r}_k(Q_k) = r_k(Q_k)$ if $Q_k \neq Q'_k$, and $\widehat{r}_k(Q'_k) = r_k(Q_k(\widehat{\alpha}_k))$. We have $\sum_{\alpha \in \mathcal{A}} \widehat{R}(\alpha)\widehat{p}(\alpha) = \sum_{\alpha \in \widehat{\mathcal{A}}} R(\alpha)p(\alpha)$ and

$$\sum_{Q \in \widehat{\mathcal{Q}}} \left[\sum_{i \in \mathcal{N}} \widehat{r}_i(Q_i) \right] \widehat{P}(Q) = \sum_{Q \in \mathcal{Q}} \left[\sum_{i \in \mathcal{N}} r_i(Q_i) \right] P(Q).$$

Therefore $\widehat{R}$ and $\widehat{r}$ satisfy (15). This implies that, replacing $\widehat{t}^C$ by $\widehat{t}^{\overline{C}}$, we can construct $t^{\overline{C}}$ in the same way as the t^C in the proof of Lemma D.1. All the proof of the lemma goes through and this implies that t satisfies (16) and (17).

As in the proof of Lemma D.1 one can show

$$\sum_{\alpha_{-i} \in \mathcal{A}_{-i}} p(\alpha_{-i} \mid \alpha_i) t_i^C(\alpha_{-i}, \alpha_i) = \sum_{\alpha_{-i} \in \mathcal{A}_{-i}} \widehat{p}(\alpha_{-i} \mid \alpha_i) \widehat{t}_i^C(\alpha_{-i}, \alpha_i)$$

and therefore (18) holds if $i \neq k$. Similarly (D.4) implies that it holds for $i = k$. □

Lemma E.2 (Inheriting Condition $\overline{C}$ through merging)**.** *If an information structure satisfies Condition $\overline{C}$, merging two independent types yields an information structure that also satisfies Condition $\overline{C}$.*

Proof of Lemma E.2. The proof is a straightforward adaptation of the proof of Lemma D.2. As in that proof, let $(\mathcal{A}, p)$ satisfy Condition $\overline{C}$, and $\{\widehat{\mathcal{A}}, \widehat{p}\}$ be built from $(\mathcal{A}, p)$ by merging the two independent types α_k^1 and α_k^2 to create $\widehat{\alpha}_k$. We show that $\{\widehat{\mathcal{A}}, \widehat{p}\}$ also satisfies Condition C.

As in the proof of Lemma E.1, there is an obvious bijection between the independence classes of $\{\widehat{\mathcal{A}}, \widehat{p}\}$ and $(\mathcal{A}, p)$. Let $Q'_k = Q_k(\alpha_k^1) = Q_k(\alpha_k^2)$. Then

$$\widehat{\mathcal{Q}}_i = \mathcal{Q}_i \text{ if } i \neq k,$$
$$\widehat{\mathcal{Q}}_k = \mathcal{Q}_k / Q'_k \bigcup \widehat{Q}'_k \text{ with } \widehat{Q}'_k \stackrel{\text{def}}{=} Q'_k / \{\alpha_k^1, \alpha_k^2\} \bigcup \{\widehat{\alpha}_k\}.$$

This implies

$$\widehat{P}(Q) = P(Q) \text{ if } Q_k \neq \widehat{Q}'_k;$$
$$\widehat{P}(Q_{-k}, \widehat{Q}'_k) = P(Q_{-k}, Q'_k) \text{ for all } Q_{-k}.$$

For any $\widehat{R} : \widehat{\mathcal{A}} \to \Re$ and functions $\widehat{r}_i : \widehat{\mathcal{Q}}_i \to \Re$ which satisfy (15), we will show that there exist transfers $\widehat{t}^{\overline{C}}$ that satisfy (16), (17) and (18).

Define R as in the proof of Lemma D.2 by (D.5) and (D.6) and r as follows:

$$\begin{aligned} &r_(Q_i) = \widehat{r}_i(Q_i) && \text{if } i \in \mathcal{N}_{-k}; \\ &r_k(Q_k) = \widehat{r}_k(Q_k) && \text{if } Q_k \neq Q_k(\alpha_k^1, \alpha_k^2); \\ &r_k(Q_k') = \widehat{r}_k(Q_k(\widehat{\alpha}_k)). \end{aligned}$$

The functions R and r satisfy(15).

Replacing $\widehat{t}^C$ by $\widehat{t}^{\overline{C}}$, we can construct $\widehat{t}^{\overline{C}}$ in the same way as $\widehat{t}^C$ in the proof of Lemma D.2. All the proof of the lemma goes through and this implies that $\widehat{t}^{\overline{C}}$ satisfies (16) and (17).

We also must show that $\widehat{t}^{\overline{C}}$ also satisfies (18). This is done as for lemma E.2. □