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"Frictionless house-price momentum"

Patrick Fève and Alban Moura

Toulouse School of Economics

FRICTIONLESS HOUSE-PRICE MOMENTUM

PATRICK FÈVE AND ALBAN MOURA

Abstract. This paper establishes that frictionless, rational-expectations models driven by specific ARMA(2,1) forcing processes are consistent with equilibrium asset-price dynamics featuring momentum. To reach this result, we first document that AR(2) models adequately capture the cyclical dynamics found in U.S. house prices, in particular the strong positive first-order autocorrelation in their first difference. Then, we show analytically that ARMA(2,1) exogenous drivers give rise to equilibrium AR(2) asset-price dynamics in a simple present-value model. Our pen-and-paper approach yields a straightforward economic interpretation of the results, emphasizing the contribution of anticipated shocks to generating asset-price momentum. We document the empirical relevance of our theoretical results by estimating the model from house-price data. Our findings suggest that house-price momentum does not necessarily signal irrational exuberance or strong frictions in housing markets.

JEL Codes: C32, E32, G12.

Keywords: house prices; momentum; AR(2) process; rational expectations; news shocks.

November 2023. Patrick Fève: Toulouse School of Economics (TSE) and Université Toulouse 1 - Capitole, 21 allée de Brienne, 31000 Toulouse, France (patrick.feve@tse-fr.eu). Alban Moura: Banque centrale du Luxembourg (BCL), Département Économie et Recherche, 2 boulevard Royal, L-2983 Luxembourg (alban.moura@bcl.lu). This paper has been produced in the context of the partnership agreement between the BCL and TSE. We thank Fabrice Collard, Martial Dupaigne, Stephane Gregoir, Paolo Guarda, Alain Guay, Nour Meddahi, Olivier Pierrard, Morten Ravn, Jean-Pierre Schoder, Vincent Sterk, and BCL colleagues for helpful comments and feedback. Patrick Fève acknowledges funding from the French National Research Agency (ANR) under the Investments for the Future (Investissements d'Avenir) program, grant ANR-17-EURE-0010. This paper should not be reported as representing the views of the BCL or the Eurosystem. The views expressed are those of the authors and may not be shared by other research staff or policymakers in the BCL or the Eurosystem.

Résumé Non Technique

La dynamique des prix immobiliers est souvent caractérisée par un fort degré de momentum, c'est-à-dire que les périodes de hausse (respectivement, de baisse) des prix immobiliers tendent à s'étaler sur plusieurs trimestres consécutifs. Cette régularité est bien établie pour les États-Unis par l'étude fondatrice de Case et Shiller (1989). Formellement, la variation des prix immobiliers à chaque trimestre est positivement corrélée à la variation observée le trimestre précédent. Le même phénomène caractérise l'évolution des prix immobiliers dans les pays européens.

Cette propriété retient l'attention des économistes dans la mesure où elle semble difficilement explicable d'un point de vue théorique. En effet, les forces d'équilibre présentes dans la plupart des modèles économiques tendent à éliminer toute possibilité de momentum dans le prix d'actifs tels que l'immobilier. Dès lors, l'interprétation communément admise du phénomène est qu'il reflète la présence de frictions importantes dans le marché immobilier. Plusieurs auteurs soutiennent notamment que le momentum pourrait signaler une évolution des prix immobiliers partiellement décorrellée des fondamentaux économiques, tels que les niveaux de l'offre, de la demande et des taux d'intérêt.

Dans cet article, nous démontrons que la présence de momentum dans le prix des actifs est en fait compatible avec un environnement économique sans friction, dans lequel les prix reflètent uniquement les fondamentaux économiques. Nous établissons ce résultat dans un cadre délibérément simple, qui nous permet d'identifier le mécanisme à l'origine de l'émergence du momentum : la présence de chocs anticipés sur le prix des actifs. De tels chocs apparaissent lorsque les agents économiques, tels que les ménages et les investisseurs, réagissent immédiatement à des évènements dont ils savent qu'ils affecteront la valeur des actifs dans le futur. Dans le cas du marché immobilier, l'on peut penser à des anticipations relatives au nombre de logements actuellement en construction et qui seront mis sur le marché dans le futur (offre future), à la croissance future de la population (demande future), ou encore à la propagation lente des chocs de politique monétaire dans l'économie (qui peut affecter à la fois l'offre et la demande).

Notre conclusion principale est donc que la présence de momentum dans les prix immobiliers est donc insuffisante pour indiquer un fort degré de frictions dans le marché immobilier. Elle ne permet pas non plus d'affirmer que les prix immobiliers sont décorrélés des fondamentaux économiques. Clarifier la ou les causes du momentum des prix immobiliers reste donc une question importante pour la recherche future, dans la mesure où des mesures de politique macro-prudentielle ou monétaire auront des effets différents selon les frictions auxquelles font face les agents économiques.

1. Introduction

Quarterly changes in U.S. real house prices present strong positive autocorrelation in the short run. This well-known empirical fact, called *momentum*, has been highlighted in a number of studies, including Case and Shiller (1989), Glaeser, Gyourko, Morales, and Nathanson (2014), Head, Lloyd-Ellis, and Sun (2014), and Guren (2018).

The consensus in the literature is that persistence in house-price growth is not consistent with a frictionless environment under rational expectations. For instance, Glaeser, Gyourko, Morales, and Nathanson (2014) report that "no reasonable parameter set" can generate short-run momentum in the frictionless, rational-expectations model they consider. As noted by Guren (2018), models with forward-looking agents embed strong arbitrage forces that tend to eliminate momentum, as optimizing traders react to expected price movements by adjusting supply and demand. Hence, house-price momentum is seen as revealing deep frictions in the housing market and existing explanations for this phenomenon rely on a variety of deviations from the frictionless framework, including non-rational expectations (Gelain, Lansing, and Mendicino, 2013; Gelain and Lansing, 2014; Gelain, Lansing, and Natvik, 2018; Glaeser and Nathanson, 2017; Pancrazi and Pietrunti, 2019), search frictions (Head, Lloyd-Ellis, and Sun, 2014), learning (Anenberg, 2016), belief heterogeneity (Burnside, Eichenbaum, and Rebelo, 2016), and strategic complementarities (Guren, 2018).[1] However, the mechanisms at play are not directly observable and substantial uncertainty remains about the causes of positive serial correlation in U.S. house-price growth.

In this paper, we demonstrate that, contrary to the common wisdom, asset-price momentum can be consistent with frictionless, rational-expectations models. We rely on a simple and transparent analytical approach to identify the central ingredient of our results: anticipated shocks (also known as news shocks) that directly affect the valuation of housing. Our findings highlight yet another difficulty in interpreting house-price momentum: the need to distinguish endogenous drivers (exuberance, frictions) from exogenous drivers (anticipated shocks) of house-price movements.

We proceed in three steps. Our first step is to look for a parsimonious representation of the cyclical behavior in house prices. We find that that simple autoregressive (AR) processes of order 2 adequately reproduce the key properties of detrended U.S. house prices, in particular momentum. This finding holds for a variety of detrending approaches. We complement these empirical results by analyzing the properties of AR(2) models that can generate momentum, identifying the relevant parameter configuration: the autoregressive parameters must lie in the bottom-right corner of the stability triangle associated with the AR(2) process, requiring a large and positive coefficient on the first lag and a negative coefficient on the second lag.

[1] On the other hand, Moura and Pierrard (2022) show that the frictions commonly embedded in quantitative DSGE models, including consumption habits, investment adjustment costs, and collateral constraints, do not generate positive momentum.

Our second step is to establish conditions under which equilibrium AR(2) dynamics for asset prices arise in frictionless environments under rational expectations. We consider the simplest possible model: rational, risk-neutral investors evaluate an asset based on an exogenous stream of dividends. The fundamental solution equates, at each period, the asset price to the expected discounted sum of current and future dividends. Variations of this present-value approach appear in most modern macroeconomic models. The simplicity of the setup makes it possible to obtain analytical results. In particular, we show that equilibrium asset prices have an AR(2) representation in this economy if and only if dividends follow a specific autoregressive moving average (ARMA) process of order (2,1).

Then, we invoke our theoretical results about AR(2) dynamics to characterize parameter restrictions ensuring that equilibrium asset prices display momentum. This allows us to obtain three main results. First, we find parameter configurations under which the frictionless present-value model generates positive autocorrelation in asset-price growth. This is an important finding because it calls into question the common view that house-price momentum provides face-value evidence of strong frictions in real-estate markets.

Second, we identify the key economic ingredient the frictionless model requires to generate momentum: the stochastic process for dividends must feature anticipated shocks. That is, agents must receive advance information about future dividends. This particular information structure is encapsulated by the moving average (MA) component of the ARMA(2,1) dividend process, since rational agents understand that a shock hitting the economy today will also affect dividends tomorrow. We prove that, when investors are patient enough, the MA component of dividends must be non-fundamental for equilibrium asset-price momentum to arise. In this case, the MA coefficient is larger than one in absolute value, so that a shock hitting the economy brings more information about future dividend movements than about current developments, a typical pattern of news shocks.

The idea that changes in expectations about future economic conditions can be important drivers of business cycles has a venerable tradition; see Beaudry and Portier (2004, 2006, 2014), Jaimovich and Rebelo (2009), and Schmitt-Grohe and Uribe (2012) for recent contributions. It has also received attention in the context of the housing market. On the empirical side, Soo (2018) shows that expectations about future house prices forecast up to 70 percent of the variation in house-price growth at the two-year horizon.[2] On the theoretical side, Kanik and Xiao (2014) introduce news shocks in the stochastic process driving housing utility in a DSGE model à la Iacoviello and Neri (2010). They show that positive news shocks lead to faster housing accumulation, which triggers an aggregate economic boom by

[2]We should also mention Lambertini, Mendicino, and Punzi (2013a), who use a structural vector autoregression to show that positive news about U.S. business conditions triggers a rise in house prices, and Gomes and Mendicino (2015), who find that anticipated shocks affecting productivity, markups, and monetary policy explain a substantial share of house-price movements in an estimated DSGE model. These studies are less directly related to ours because they focus on news shocks not directly related to housing demand.

relaxing collateral constraints. If the positive news fails to materialize, a general downturn follows as house prices fall and depress demand.[3] Our approach builds upon these contributions, retaining the key idea that anticipated shocks matter for the housing market and highlighting their potential importance for house-price momentum.

Third, we show that, in the parameter space associated with asset-price momentum, the ARMA(2,1) dividend can be arbitrarily close to a standard AR(1) process, in the sense that an econometrician would not be able to distinguish the two processes given a finite sample of dividend data. It follows that our momentum-generating mechanism does not require peculiar exogenous dynamics. However, the economic implications of AR(1) and ARMA(2,1) dividends are very different, as the latter generate positive serial correlation in equilibrium asset-price growth while the former do not. We corroborate these observations in our empirical analysis.

Indeed, our third and last step is to document the empirical relevance of our approach. We estimate the asset-pricing model from observations on cyclical house prices, allowing for AR(1), AR(2), and ARMA(2,1) dividends. Gelain and Lansing (2014) and Gelain, Lansing, and Natvik (2018) estimate similar asset-pricing models from house-price data, focusing on deviations from rational expectations instead of on news shocks. We find empirical support for all our results: the ARMA(2,1) setup is largely favored by the data over the alternatives; it is the only one able to reproduce the strong positive autocorrelation of house-price changes; the estimated parameters are in the subspace our theoretical analysis identified as relevant; and the estimated ARMA(2,1) dividend process is very close to a simpler AR(1) representation. Strikingly, the estimated model implies that equilibrium house prices lead realized dividends by more than two years, which signals substantial amplification since news shocks arrive only one quarter in advance when dividends have ARMA(2,1) dynamics.

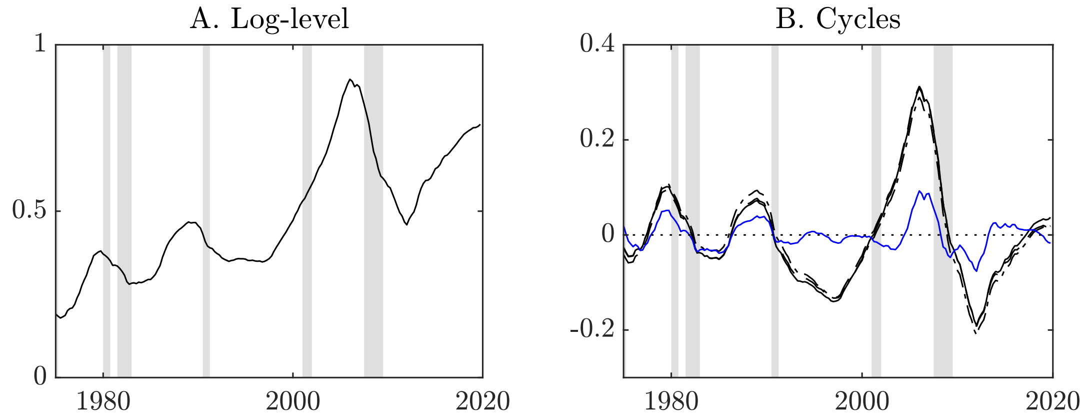
We organize the paper as follows. Section 2 presents the evaluation of simple time-series models of cyclical house prices and highlights the good empirical performance of AR(2) models, especially in terms of momentum. Section 3 provides the theoretical analysis of the frictionless asset pricing model and contains our main results about the relationship between asset-price momentum and anticipated shocks. Finally, Section 4 describes our empirical application, based on a structural estimation of the model, and Section 5 concludes.

2. U.S. House Prices and the AR(2) Model

In this section, we fit various time-series models to U.S. house prices. We find that a parsimonious AR(2) process accurately reproduces the main cyclical properties of the data,

[3] See also Lambertini, Mendicino, and Punzi (2013b) for an analysis of the welfare benefits associated with policies leaning against news-driven cycles in a Iacoviello-type model. Again, the main difference between their study and ours is the fact that they do not include anticipated shocks directly driving housing demand.

Figure 1. Log real U.S. house prices



Notes. Panel A shows the logarithm of real U.S. house prices. Panel B shows the cyclical components extracted by various methods: the dashed, solid, and dash-dotted black lines correspond respectively to linear detrending, quadratic detrending, and band-pass detrending; the solid blue line corresponds to HP detrending. Shaded areas represent NBER recessions.

including momentum. We explain this result by highlighting some persistence properties of AR(2) models.

2.1. **Data.** We measure house prices using the quarterly S&P CoreLogic Case/Shiller nominal price index for U.S. homes, available from Robert Shiller's webpage (`http://www.econ.yale.edu/~shiller/data/Fig3-1.xls`). We convert the series to real terms using the price index for nondurable consumption and services.[4] The sample runs from 1975Q1 to 2019Q4.

We focus on cyclical dynamics. Standard unit-root tests suggest that log house prices are stationary around a deterministic time trend, so we remove either a linear or a quadratic trend from the series.[5] For robustness, we also estimate the cyclical component using filtering methods: a band-pass filter isolating cycles with a duration between 2 and 120 quarters, thus keeping both the short-term and medium-term fluctuations that Drehmann, Borio, and Tsatsaronis (2012) find in house prices, and the standard Hodrick-Prescott (HP) filter with smoothing parameter 1,600.

Figure 1 presents the data. Panel A shows real house prices in log-levels, while Panel B shows the four estimated cyclical components. Visually, it is clear that removing a linear or a quadratic time trend and using the band-pass filter yield very similar cycles (in black): pairwise correlations between these three estimates are above 0.98. These cyclical components are volatile (standard deviation of about 10%) and persistent (first-order autocorrelation of

[4]Our results are robust to deflating nominal house prices by the GDP deflator instead.

[5]Augmented Dickey-Fuller tests reject the null hypothesis of a unit root in log real house prices at conventional levels of significance against the alternative of trend stationarity. See Zhang, de Jong, and Haurin (2016) for more evidence that real U.S. house prices are trend stationary.

Table 1. Estimation results for autoregressive models of U.S. house prices

	Estimates				Evaluation		
Model	ϕ_1	ϕ_2	ϕ_3	ϕ_3	BIC	HQ	$\rho_{\Delta y}(1)$
Quadratic detrending							
AR(1)	0.99	—	—	—	−8.61	−8.63	−0.00
AR(2)	1.87	−0.88	—	—	−10.09	−10.12	0.87
AR(3)	1.77	−0.68	−0.11	—	−10.07	−10.12	0.87
AR(4)	1.74	−0.87	0.41	−0.29	−10.14	−10.19	0.87
HP detrending							
AR(1)	0.96	—	—	—	−9.65	−9.67	−0.02
AR(2)	1.64	−0.70	—	—	−10.28	−10.32	0.67
AR(3)	1.58	−0.56	−0.08	—	−10.26	−10.31	0.66
AR(4)	1.55	−0.74	0.44	−0.33	−10.35	−10.40	0.66

Notes. The estimated processes are of the form $y_t = \phi_0 + \sum_j \phi_j y_{t-j} + u_t$, where y_t is log real house prices and u_t is a residual. The estimation sample is 1975Q1-2019Q4. BIC and HQ refer to the Bayesian and Hannan-Quinn information criteria. $\rho_{\Delta y}(1)$ is the first-order autocorrelation of the first difference of y_t implied by the estimated AR processes, i.e. the theoretical momentum of y_t. The empirical estimate of $\rho_{\Delta y}(1)$ is 0.87 for quadratically-detrended house prices and 0.68 for HP-detrended house prices.

0.99). They tend to rise during expansions and fall during recessions. The HP filter estimates a different cyclical component (in blue), less persistent (autocorrelation of 0.97) and especially less volatile (standard deviation of 3%). This is because the HP filter attributes more of the observed medium-term fluctuations in house prices to the trend, while the other techniques attribute these to the cycle.

Crucially, all estimated cyclical components feature strong momentum: the first-order autocorrelation of their first differences are equal to 0.87 for the linear, quadratic, and band-pass cycles, and 0.68 for the HP cycle. Thus, all detrending methods imply that the serial correlation of cyclical changes in house prices is positive in the short run. This property also holds for undetrended house prices, whose first difference also presents a first-order autocorrelation of 0.87.

2.2. **Time-series models of house prices.** We now look for a time-series model that provides a parsimonious yet reasonable representation of cyclical house-price dynamics. We fit AR processes to the detrended series and evaluate their fit using two consistent information criteria, the Schwarz Bayesian Information Criterion (BIC) and the Hannan-Quinn (HQ) criterion. We allow for a maximum lag order of 4. Table 1 presents the results. In the interest of space, we focus on the cyclical components estimated by the quadratic time trend and the HP filter; results for the linear time trend and the band-pass filter are very similar to the former case.

Table 1 shows that the AR(4) process minimizes the two information criteria, for both quadratically- and HP-detrended series. Therefore, on this basis it is the best model within the subset we consider. However, we see that the AR(2) process also provides a good representation of the cyclical behavior of U.S. house prices: it dominates the AR(1) and AR(3) alternatives, and it reproduces the degree of momentum found in the data as well as the richer AR(4) model.

The ability to generate a strong autocorrelation for house-prices changes seems to be the discriminating factor for model evaluation. The largest drop in information criteria occurs when moving from the AR(1) specification, which fails to reproduce the positive autocorrelation found in the data, to the AR(2) specification, which succeeds. Allowing for extra lags does not improve the momentum statistic and results in only marginal changes in information criteria. In fact, while the BIC and Hannan-Quinn criterion both heavily penalize the AR(1) model, they imply that autoregressive models with 2 to 4 lags fit the data about equally well. We conclude that AR(2) dynamics provide a reasonable representation of detrended house prices.

As further evidence, Figure 2 compares some empirical statistics estimated from the data with the counterparts implied by the estimated AR(2) processes. We focus on the autocorrelation function (ACF), the partial autocorrelation function (PACF), and the ACF of the first difference (FD-ACF). Each panel shows the sample autocorrelations estimated from the data in black, together with heteroskedasticity-robust confidence intervals. For both detrending approaches, we observe strong persistence in house prices, as well as significant momentum.

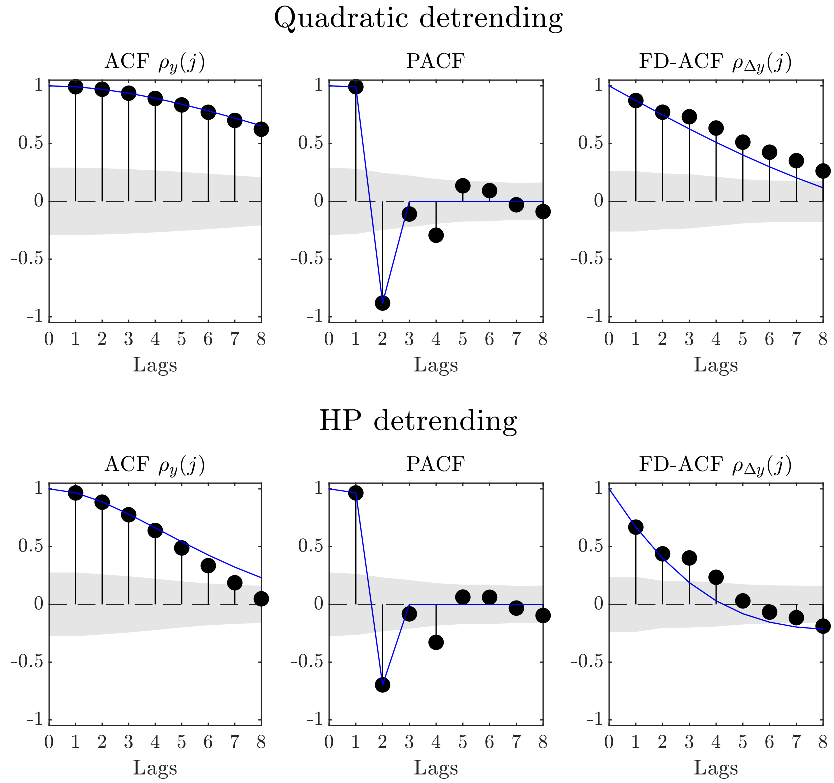
The blue lines show the theoretical values implied by the AR(2) model. The fit is very good: in spite of its simplicity, the model reproduces the slow and concave decay in autocorrelations, the high partial autocorrelations at lags 1 and 2, the sharp decline found starting lag 3 on, and the slow decay in autocorrelations for the first difference of the data. While richer AR(3) and AR(4) processes may be better at reproducing the properties of the series at higher lag orders, we are confident from this analysis that the AR(2) process provides a sensible representation of the short-run behavior of U.S. house prices.

Another notable point is that the estimated AR(2) models present complex roots, suggesting the presence of an oscillatory component. For instance, the estimated roots are equal to $0.93 \pm 0.10i$ for quadratically-detrended house prices, and to $0.82 \pm 0.16i$ for HP-detrended house prices. The estimated AR(3) and AR(4) models also present complex roots, suggesting that cyclical U.S. house prices display oscillatory patterns.

2.3. **Key properties of AR(2) models.** To shed light on these empirical findings, we review some properties of stationary AR(2) processes and establish new results related to momentum.[6] We will build upon this analysis in the next section.

[6]Hamilton (1994) and Box, Jenkins, Reinsel, and Ljung (2015), among others, provide treatments of AR(2) models.

Figure 2. Persistence properties of detrended house prices



Notes. For both quadratically- and HP-detrended log real house prices, the left-hand panel shows the autocorrelation function (ACF), the middle panel shows the partial autocorrelation function (PACF), and the right-hand panel shows the autocorrelation function of the first difference (FD-ACF). Dark bars represent sample statistics and shaded areas show heteroskedasticity-robust 95% confidence intervals centered around zero. Blue lines show the values implied by the AR(2) processes estimated from the data. The sample is 1975Q1-2019Q4.

Consider the following AR(2) representation for an economic variable y_t:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t, \tag{1}$$

where ε_t is a white-noise innovation with zero mean and variance $\sigma^2 > 0$. We omit a constant term without loss of generality. As is well known, the dynamic properties of y_t depend on the roots λ_1 and λ_2 of the characteristic polynomial associated with equation (1),

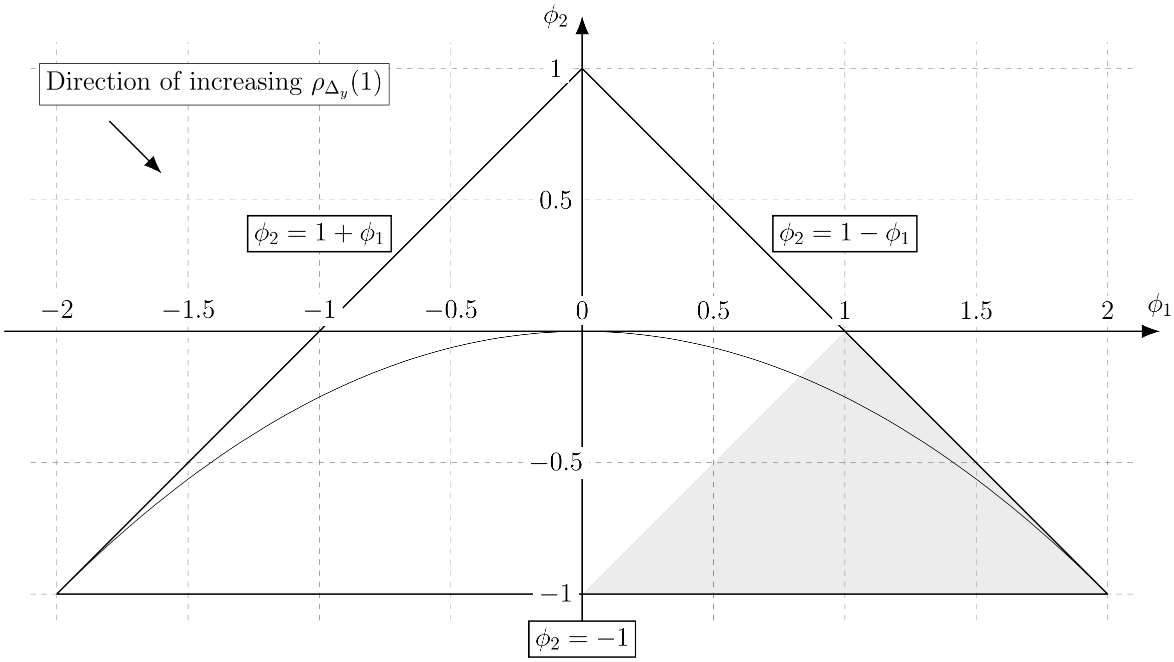
$$(1 - \phi_1 B - \phi_2 B^2) = (1 - \lambda_1 B)(1 - \lambda_2 B),$$

where B denotes the backshift operator, i.e. $By_t = y_{t-1}$. The roots of this polynomial are

$$(\lambda_1, \lambda_2) = \frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2}. \tag{2}$$

They are real if $\phi_1^2 + 4\phi_2 \geq 0$, complex otherwise.

Figure 3. Stability triangle for the AR(2) process



Notes. The stability triangle corresponds to the (ϕ_1, ϕ_2) pairs lying between the $\phi_2 = 1 + \phi_1$, $\phi_2 = 1 - \phi_1$, and $\phi_2 = -1$ lines. The gray triangle in the bottom-right corner identifies the space $\mathcal{M}$ of parameters producing a positive first-order autocorrelation for Δy_t. The curved line separates parameters associated with real roots (above the curve) from parameters associated with complex roots (below the curve).

We focus on stationary processes, so that λ_1 and λ_2 must be inside the unit circle: $|\lambda_1|, |\lambda_2| < 1$. Equivalently, the parameters ϕ_1 and ϕ_2 must be in the triangular region defined by

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad |\phi_2| < 1. \tag{3}$$

Figure 3, inspired from Stralkowski (1968) and Sargent (1979), presents this stability triangle.

The ACF of the AR(2) process provides a direct characterization of the dynamics of y_t. As shown in, e.g., Hamilton (1994), it is given by

$$\rho_y(0) = 1, \qquad \rho_y(1) = \frac{\phi_1}{1 - \phi_2}, \qquad \rho_y(j) = \phi_1 \rho_y(j-1) + \phi_2 \rho_y(j-2) \text{ for } j \geq 2. \tag{4}$$

The first-order autocorrelation is increasing in ϕ_1 and can increase or decrease with ϕ_2 depending on the sign of ϕ_1. We also see that $\rho_y(1) \to 1$ when $\phi_2 \to 1 - \phi_1$, so that the AR(2) process can display strong persistence in levels. More generally, it follows from the expression of $\rho_y(1)$ that y_t presents a high first-order autocorrelation when ϕ_2 is close to, but below, $1 - \phi_1$, that is when the (ϕ_1, ϕ_2) pair lies next to the right border of the stability triangle.

Next, we turn to the first difference of y_t. This allows us to develop analytical insights for the empirical analysis in Section 2.2. Letting Δ denote the first difference operator, we have $\Delta y_t = y_t - y_{t-1}$. This difference represents the change in y_t between two consecutive periods, but it can be interpreted as a growth rate if y_t is in logarithms. Equation (1) implies

$$\Delta y_t = \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \Delta \varepsilon_t,$$

from which we can compute the ACF of Δy_t:

Proposition 1. *The ACF of the first difference of the AR(2) process* (1) *is given by*

$$\rho_{\Delta y}(0) = 1, \quad \rho_{\Delta y}(1) = \frac{\phi_1 - \phi_2 - 1}{2}, \quad \rho_{\Delta y}(j) = \phi_1 \rho_{\Delta y}(j-1) + \phi_2 \rho_{\Delta y}(j-2) \text{ for } j \geq 2.$$

Proof. See Appendix A. □

To the best of our knowledge, the results in Proposition 1 are new to the literature. They show that the first-order autocorrelation of Δy_t is linearly increasing in ϕ_1 and linearly decreasing in ϕ_2, with zero cross-derivatives. Thus, short-run momentum is maximized within the stability triangle when $\phi_1 \to 2$ and $\phi_2 \to -1$: in this limit case, $\rho_{\Delta y}(1) \to 1$.

On the other hand, it is not possible to generate momentum with a stationary AR(1) model, which corresponds to the special case $\phi_2 = 0$. With this restriction,

$$\rho_{\Delta y}(1) = \frac{\phi_1 - 1}{2},$$

which is always negative when $\phi_1 \in (-1, 1)$. At the limit, $\rho_{\Delta y}(1) \to 0$ when $\phi_1 \to 1$, so that the maximum possible first-order autocorrelation for Δy_t in the AR(1) case is zero. This property echoes the estimates reported in Table 1.

In addition, $\rho_y(1) > \rho_{\Delta y}(1)$ for all the (ϕ_1, ϕ_2) pairs verifying the stationarity restrictions. Therefore, short-run persistence is always larger for the level of the variable than for its first difference.

The formula for $\rho_{\Delta y}(1)$ implies that the (ϕ_1, ϕ_2) pairs associated with momentum in y_t lie below the straight line $\phi_2 = \phi_1 - 1$. Given the stability restrictions (3), the parameter space associated with stationary AR(2) processes displaying momentum is defined by

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad |\phi_2| < 1, \qquad \phi_2 < \phi_1 - 1. \tag{5}$$

We will use the symbol $\mathcal{M}$ to refer to the (ϕ_1, ϕ_2) pairs verifying the restrictions in (5). Visually, $\mathcal{M}$ is located in the bottom-right corner of the stability triangle, as shown in gray in Figure 3.

We summarize this discussion in the following:

Corollary 1. *Let $(\phi_1, \phi_2) \in \mathcal{M}$. Then, the AR(2) process defined by equation* (1) *is stationary and displays momentum, i.e. $\rho_{\Delta y}(1) > 0$. In addition, the process is persistent in levels: $\rho_y(1) > 0$.*

The virtue of this result is twofold. First, it explains why the AR(2) process is able to reproduce the strong momentum found in real house prices. Second, it identifies the most plausible parameter configuration for a time series whose level and first difference both exhibit high first-order autocorrelations: such dynamics occur for (ϕ_1, ϕ_2) pairs located in the bottom-right corner of the stability triangle, i.e. for relatively high values of ϕ_1 and relatively low values of ϕ_2. The estimates reported in Section 2.2 all lie in this area.

3. AR(2) Dynamics in a Frictionless Asset-Pricing Model

In this section, we establish conditions under which AR(2) processes arise as equilibrium solutions of a standard frictionless asset-pricing model under rational expectations. Then, we build upon our theoretical analysis of AR(2) models to identify parameter configurations associated with asset-price momentum. Our analytical approach allows us to isolate the economic forces at play, namely anticipated shocks. Finally, we discuss some equilibrium properties of the model.

3.1. **Economic environment.** We consider the following environment. Risk-neutral investors trade identical Lucas trees, which each pay an exogenous dividend x_t at date t (Lucas, 1978). Letting y_t denote the price of a tree, a simple present-value formula implies

$$y_t = \beta E_t y_{t+1} + x_t, \tag{6}$$

where $\beta \in (0, 1)$ is the investor discount factor and E_t denotes the expectation operator conditional on period-t information. Blanchard (1979) and Gourieroux, Laffont, and Monfort (1982) discuss similar models.[7] We impose a transversality condition to isolate the no-bubble solution.

From a modeling perspective, the tree is a very durable asset that resembles a house. Under this interpretation, the dividend is the yield associated with owning a house, for instance the utility service from living in the house or the payoff from renting the house to other agents. Gelain and Lansing (2014) and Gelain, Lansing, and Natvik (2018) use similar interpretations to study house-price dynamics with asset-pricing models.

Usual practice is to close the model by postulating an AR(1) process for x_t,

$$x_t = \varphi x_{t-1} + \varepsilon_t,$$

[7] The present-value formula (6) characterizes a pre-dividend asset price y_t, that is the price of a tree that is going to pay x_t today. Alternatively, following Cochrane (2005, Section 1.4, pp. 24-25), one can compute a post-dividend asset price $\widetilde{y}_t$, that is the price of a tree that has already paid x_t today and is going to pay x_{t+1} tomorrow. This price verifies $\widetilde{y}_t = \beta E_t (\widetilde{y}_{t+1} + x_{t+1})$. It is straightforward to extend our analysis to this alternative valuation model with very similar results.

with $\varphi \in (0,1)$ and ε_t a white noise, and to deduce the asset price y_t consistent with equation (6). The solution is

$$y_t = \frac{1}{1-\beta\varphi} x_t,$$

implying that the equilibrium tree price y_t is proportional to the exogenous dividend. Then, the results from Section 2.3, in particular Proposition 1, imply that the first-order autocorrelation of Δy_t is negative and converges to a maximum of zero as $\varphi \to 1$. Thus, the model with AR(1) dividends cannot generate momentum in asset prices. This is a simple illustration of the widespread idea that frictionless, rational-expectations environments are not consistent with momentum.

Several variations on equation (6) have been proposed to create momentum. For instance, Gelain, Lansing, and Mendicino (2013), Gelain and Lansing (2014), Gelain, Lansing, and Natvik (2018), Glaeser and Nathanson (2017), and Pancrazi and Pietrunti (2019) replace rational expectations by extrapolative or natural expectations. Head, Lloyd-Ellis, and Sun (2014) and Guren (2018) replace the frictionless market by a search equilibrium. Anenberg (2016) allows for gradual learning about market conditions. Burnside, Eichenbaum, and Rebelo (2016) introduce belief heterogeneity among traders. A very simple representation of these variations modifies equation (6) into

$$y_t = \frac{b}{1+\beta b} y_{t-1} + \frac{\beta}{1+\beta b} E_t y_{t+1} + \frac{1}{1+\beta b} x_t, \tag{7}$$

with $\beta \in (0,1)$, $b \in (0,1)$, and x_t the same AR(1) process as above. Now, the valuation equation involves a backward-looking term, which captures the impact of past events on current asset prices. The parameter b measures the strength of this effect, and equation (7) reduces to the frictionless model (6) when $b = 0$.

It is straightforward to show that, when x_t follows the above AR(1) process, this extended model implies an AR(2) representation for y_t:

$$y_t = (b+\varphi) y_{t-1} - b\varphi y_{t-2} + \frac{1}{1-\beta\varphi} \varepsilon_t. \tag{8}$$

The results from from Section 2.3, in particular Corollary 1, then imply that a sufficient condition for asset-price momentum in the extended model is $b + \varphi > 1$. Economically, this configuration occurs when the backward-looking friction is sufficiently strong (high b) and/or when dividends are persistent enough (high φ). Furthermore, the extended model can only generate an AR(2) process with real roots, which rules out the oscillatory dynamics found in the AR(2) estimates reported in Table 1 and isolates a tiny part of the parameter space associated with momentum in Figure 3. Nevertheless, this example highlights how introducing backward-looking frictions in the structural model can relax the predictions of theory and generate richer dynamics.

3.2. **Frictionless AR(2) dynamics.** Having established that the model with AR(1) dividends does not generate asset-price momentum, we now show that allowing for a richer dividend process can overturn this implication. To do so, we maintain the frictionless structure of the model and we look for dividend processes that yield AR(2) dynamics for equilibrium asset prices.

Our main result is:

Proposition 2. *Let x_t be a stationary stochastic process and y_t be the solution to equation* (6) *given x_t. Then, y_t has the stationary AR(2) representation given by equation* (1)*,*

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t,$$

with (ϕ_1, ϕ_2) verifying the system of restrictions (3) *and ε_t a white noise innovation with zero mean and variance $\sigma^2 > 0$, if and only if x_t evolves according to*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + (1 - \beta\phi_1)\varepsilon_t - \beta\phi_2\varepsilon_{t-1}. \tag{9}$$

Proof. See Appendix B. □

Proposition 2 establishes that a frictionless environment can generate endogenous AR(2) asset-price dynamics under rational expectations, provided that we allow for richer dividend dynamics. This result implied that asset-price momentum is perfectly consistent with frictionless economies populated with rational, forward-looking agents:

Corollary 2. *Let $(\phi_1, \phi_2) \in \mathcal{M}$. Then the rational-expectations equilibrium asset price y_t features momentum, i.e. $\rho_{\Delta y} > 0$.*

Proof. Immediate from Corollary 1 and Proposition 2. □

Corollary 2 shows that the arbitrage force created by rational expectations is not enough to rule out persistent asset-price growth in a frictionless environment. To the best of our knowledge, we are the first to demonstrate this result, which questions the common wisdom that asset-price momentum necessarily reveals the presence of strong frictions.[8]

3.3. **Momentum and news shocks.** To understand how the model generates asset-price momentum, we start by studying the dividend process.

[8]This result also echoes a recurring theme of the literature on economic modeling: the near-equivalence of endogenous and exogenous dynamics. Our engineering of equilibrium AR(2) dynamics through a richer dividend process is the flip side of the engineering of AR(2) dynamics through the addition of a backward-looking component to the valuation equation in Section 3.1. As discussed in, e.g., An and Schorfheide (2007), structural frictions (the backward-looking term) and exogenous forces (the dividend process) can often generate similar or identical equilibrium properties for the endogenous variables in dynamic economic models.

Corollary 3. *If $\beta\phi_1 \neq 1$, then x_t admits the stationary ARMA(2,1) representation*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \tilde{\varepsilon}_t - \theta\tilde{\varepsilon}_{t-1}, \tag{10}$$

where

$$\tilde{\varepsilon}_t := (1 - \beta\phi_1)\varepsilon_t, \qquad \theta := \frac{\beta\phi_2}{1 - \beta\phi_1}. \tag{11}$$

If $\beta\phi_1 = 1$, then x_t admits the stationary AR(2) representation

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} - \beta\phi_2\varepsilon_{t-1}. \tag{12}$$

Proof. Immediate from equation (9). □

Corollary 3 shows that, in general, equilibrium AR(2) dynamics in asset prices require ARMA(2,1) dividends in the frictionless model. A key implication of the moving-average component is that dividends must be driven at least partly by anticipated shocks. This is especially clear in the limit case $\beta\phi_1 = 1$, in which dividends follow an AR(2) process driven by pure news shocks: agents learn about future dividend movements one period in advance and the equilibrium asset price immediately adjusts to the news. This is also true in the generic ARMA(2,1) case, as dividend movements at each period originate from both contemporaneous and lagged shocks. As discussed in Beaudry and Portier (2014), in MA models, the parameter θ measures the relative importance of anticipated and contemporaneous shocks. In the specific context of the house market, the important role of news shocks has been previously documented by Lambertini, Mendicino, and Punzi (2013a), Kanik and Xiao (2014), Gomes and Mendicino (2015), and Soo (2018).

To assess the role of news shocks in generating asset-price momentum, we use the following result:

Corollary 4. *Let $(\phi_1, \phi_2) \in \mathcal{M}$, $\beta \in (0, 1)$, and $\beta \neq 1/\phi_1$. If, in addition, $\beta > 1/(\phi_1 - \phi_2)$, then $|\theta| > 1$.*

Proof. See Appendix C. □

Corollary 4 shows that, if agents are patient enough (i.e., if β is large enough), asset-price momentum occurs only if the MA parameter θ in the dividend process is larger than one in absolute value. In this case, we say that the ARMA(2,1) process for x_t defined by equation (10) is *non-fundamental*.[9] From an economic perspective, this property means that the dividend process features a strong anticipated component because the news shock $\tilde{\varepsilon}_{t-1}$

[9]Intuitively, non-fundamentalness means that observing present and past realizations of dividends x_t is not enough to recover the history of shocks $\tilde{\varepsilon}_t$. More formally, non-fundamentalness implies that $\tilde{\varepsilon}_t$ cannot be expressed as a stationary function of past observations of x_t; instead, it is a stationary function of *future* realizations of x_t (see, e.g, Hamilton, 1994, Section 3.7, pp. 64-68). These econometric issues do not matter for our analysis.

explains a larger share of the dividend variance than the surprise shock $\tilde{\varepsilon}_t$. Indeed, Beaudry and Portier (2014) refer to such an economy as "news rich."

Corollaries 3 and 4 emphasize that the frictionless model needs anticipation effects to generate momentum in asset prices. The results also highlight the role of the discount factor: forward-looking agents value the trees according to the expected discounted sum of current and future dividends, and more patient agents put more weight on still unrealized but anticipated dividend movements. Therefore, higher values of β are associated with a larger role for anticipated shocks in the model.

A last interesting property is that our results can hold even when dividend dynamics are close to an AR(1) process:

Corollary 5. *If $\beta \to 1$ and $\phi_2 \to 1 - \phi_1$, then $\theta \to 1$.*

Proof. Immediate from equation (11). □

Corollary 5 shows that a specific parameter configuration within the stability triangle pushes the MA parameter θ arbitrarily close to one. At the same time, one of the roots of the AR component converges to one, as can be seen from equation (2). These AR and MA roots exactly cancel out at the limit and the ARMA(2,1) process for x_t simplifies to an unrestricted AR(1) process with autoregressive parameter $\phi_1 - 1$. Since the stability conditions do not hold at the limit, the corresponding asset price y_t converges to an ARIMA(1,1,0) process. Nevertheless, close to the limit *within* the stability triangle, y_t remains a stationary AR(2) process and the dividend x_t, while still ARMA(2,1) in population, becomes indistinguishable in finite sample from a simpler AR(1) process. If, in addition, $\phi_1 \in (0, 2)$, then y_t retains its momentum properties because the (ϕ_1, ϕ_2) pair lies in $\mathcal{M}$.

This result establishes that, while our analytical results exploit richer dividend dynamics, in practice equilibrium asset prices may exhibit strong momentum with dividend dynamics very close to a standard AR(1) process. This property is especially relevant in light of the empirical estimates of (ϕ_1, ϕ_2) reported in Section 2, which approximately verify the restrictions in Corollary 5.[10] We show below that structural estimates based on the asset-pricing model share this property.

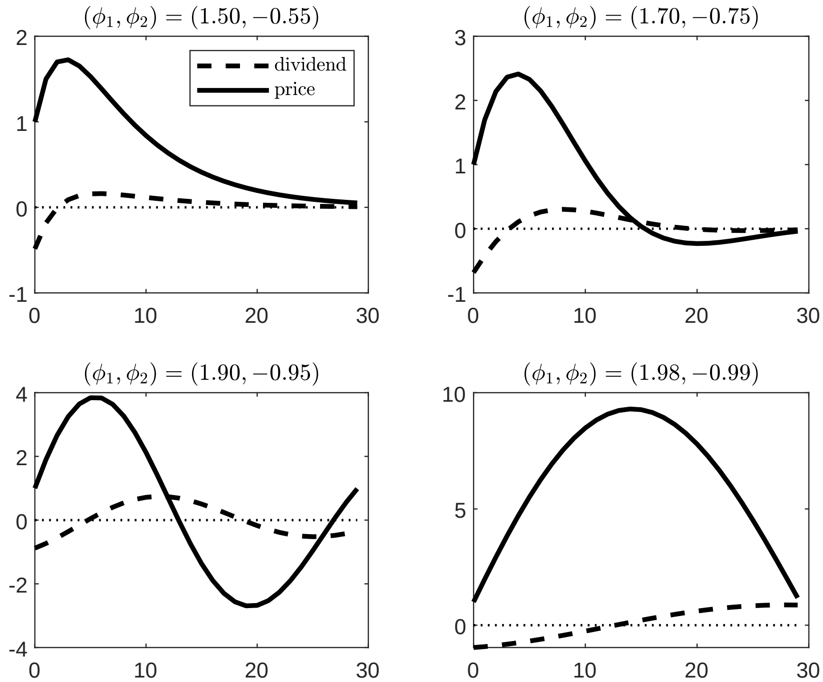
3.4. **Price-dividend dynamics.** Finally, we study the equilibrium price-dividend comovements, which help understand the economic mechanisms at play in the model.

Corollary 6. *The correlation between x_t and y_t is*

$$\operatorname{corr}(x_t, y_t) = [1 - \beta\rho_y(1)] \sqrt{\frac{1-\phi_2}{(1-\phi_2)[(1-\beta\phi_1)^2 + \beta^2\phi_2^2] - 2\beta\phi_1\phi_2(1-\beta\phi_1)}},$$

[10]For instance, the estimate of (ϕ_1, ϕ_2) is equal to $(1.87, -0.88)$ for quadratically-detrended house prices, and to $(1.64, -0.70)$ for HP-detrended house prices. Thus, in both cases, $\widehat{\phi}_2 \approx 1 - \widehat{\phi}_1$, as required by Corollary 5.

FIGURE 4. Impulse-response functions in the asset-pricing model with AR(2) dynamics



Notes. In each chart, the solid and dashed black lines show the responses of the equilibrium asset price y_t and the dividend x_t to a unitary ε_t shock. We use $\beta = 0.99$ in the computations. The contemporaneous correlation between asset prices and dividends are equal to 0.29 when $(\phi_1, \phi_2) = (1.50, -0.55)$, 0.21 when $(\phi_1, \phi_2) = (1.70, -0.75)$, 0.16 when $(\phi_1, \phi_2) = (1.90, -0.95)$, and 0.15 when $(\phi_1, \phi_2) = (1.98, -0.99)$.

with $\rho_y(1) = \phi_1/(1 - \phi_2)$.

Proof. See Appendix D. The value of $\rho_y(1)$ comes from equation (4). □

The term in brackets is positive, implying that $\text{corr}(x_t, y_t) > 0$ within the stability triangle. This is not surprising: the valuation equation (6) creates a positive long-run link between asset prices and dividends, by which higher dividends imply higher asset prices on average. At the same time, the formula shows that very different correlation patterns between x_t and y_t are possible, depending on parameter values. On the one hand, there is a unit price-dividend correlation when $\beta = 0$ (zero present value of future dividends) and when $\phi_2 = 0$ (AR(1) process for dividends). On the other hand, there are cases in which asset prices and dividends are almost uncorrelated: in particular, $\text{corr}(x_t, y_t) \to 0$ when $\beta \to 1$ and $\phi_2 \to 1 - \phi_1$.

From an economic perspective, such a decoupling between dividends and asset prices highlights once more the role of anticipated shocks, discounting, and rational expectations. Low correlations are possible only when $\beta \to 1$, that is when very patient investors put strong weight on expected future dividends when computing asset values. In this case, equilibrium asset prices are largely driven by anticipated dividend movements, which loosens the contemporaneous link between asset prices and dividends. Therefore, the same mechanism that creates momentum in the model works to lower the contemporaneous correlation between dividends and asset prices. A confirmation of this property follows from varying the value of the discount factor β while keeping the coefficients (ϕ_1, ϕ_2) fixed. Larger values of β amplify the role of anticipated shocks by making future dividends more relevant to investors, which in turn lowers the contemporaneous correlation between dividends and asset prices.[11]

An equivalent interpretation is that decoupling occurs when negative *short-run* comovements between dividends and asset prices offset the positive *long-run* relationship embedded in equation (6). The unconditional correlation averages comovements across different frequencies, so that opposite short-run and long-run dynamics can combine into low values for $\text{corr}(x_t, y_t)$. Indeed, equations (1) and (9) imply that a positive ε_t innovation raises y_t immediately, but lowers x_t when $\beta\phi_1 > 1$, a condition likely to hold in the parameter space $\mathcal{M}$ associated with momentum for reasonable values of the discount factor β. Asset prices rise on impact in spite of the drop in dividends because forward-looking investors correctly anticipate that a long string of above-average dividend realizations will follow.

Impulse responses of asset prices and dividends to an ε_t shock support this interpretation, as shown in Figure 4.[12] We focus on four pairs of parameters (ϕ_1, ϕ_2) that are consistent with momentum in y_t and we calibrate the discount factor $\beta = 0.99$. For all parametrizations we consider, the shock triggers *negative* short-run comovements between asset prices, which rise on impact, and dividends, which fall on impact. These negative short-run comovements lead to small price-dividend correlations in the four parametrizations considered, with values below 0.30.

We also see that asset-price momentum is consistent with impulse-responses of various shapes. If $(\phi_1, \phi_2) = (1.50, -0.55)$, the roots are real and the response of asset prices returns to zero monotonically after a few periods. The other parametrizations considered in Figure 4 correspond to complex roots and give rise to oscillatory dynamics. Stronger oscillations occur as (ϕ_1, ϕ_2) gets closer to the bottom-right corner of the stability triangle.

We close this section with two comments. First, our focus on AR(2) dynamics arises both from their empirical ability to generate momentum documented in Section 2 and from their analytical tractability. Certainly, richer univariate time-series models could provide better representations of house prices, but they would come at the expense of intractable computations that would prevent us from reaching the same understanding of the economic

[11] These results are available upon request.

[12] Equations (1) and (9) would allow us to derive analytical responses, but the computations become tedious after a few periods and the expressions are difficult to interpret. Therefore, we resort to numerical illustrations for clarity.

forces at play. Second, the model we consider is highly stylized: there is no friction, investors are risk neutral, and asset prices and dividends are explained by a single shock. We view this simplicity as double advantage: it allows us to obtain closed-form results with straightforward interpretation, which would not be possible in more complicated setups, and it enables us to demonstrate that equilibrium price momentum can be consistent with the simplest frictionless, rational expectations environment.

4. Empirical Application

Finally, we provide an empirical application of our analysis. We estimate the asset-pricing model studied in Section 3 using observations on U.S. house prices. In doing so, we build upon earlier work by Gelain and Lansing (2014) and Gelain, Lansing, and Natvik (2018), who also estimated simple asset-pricing models to understand house-price dynamics. Our goal is to illustrate the quantitative performance of the stylized setup and to link the various conditions we imposed on the parameters β, ϕ_1, and ϕ_2 to data-coherent values. We also obtain smoothed estimates of dividends and study their dynamic relationship with house prices.

We estimate four versions of the model. All versions share the basic valuation equation (6), reproduced here for convenience:

$$y_t = \beta E_t y_{t+1} + x_t.$$

The versions differ in the specification of dividends. First, we consider three *unrestricted* AR(1), AR(2), and ARMA(2,1) processes:

$$\begin{aligned}
x_t &= \phi_1 x_{t-1} + \varepsilon_t, \\
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t, \\
x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1}.
\end{aligned}$$

Then, we consider a *restricted* dividend process, in which x_t has the ARMA(2,1) representation defined by equation (9) that generates exact AR(2) dynamics for y_t:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + (1 - \beta\phi_1)\varepsilon_t - \beta\phi_2 \varepsilon_{t-1}.$$

We refer to this restricted process as R-ARMA(2,1) dividends. In all versions, ε_t is a white noise with standard deviation σ.

We consider AR(1) dividends because of the ubiquity of the first-order autoregressive process in macroeconomics, even though we know that equilibrium asset prices will inherit the same AR(1) structure in this case. We consider the AR(2) process to evaluate whether allowing for richer autoregressive dynamics in dividends while omitting anticipated shocks improves on the AR(1) case. Finally, the unrestricted ARMA(2,1) process provides a slight generalization of the restricted model, which allows us to test whether the restrictions associated with AR(2) asset-price dynamics are supported by the data.

Table 2. Estimation results for asset-pricing models of U.S. house prices

	Estimates				Evaluation	
Version	ϕ_1	ϕ_2	θ	100σ	$\log \mathcal{L}$	$\rho_{\Delta y}(1)$
AR(1)	0.99 (0.01)	—	—	0.02 (0.01)	511.16	−0.00
AR(2)	1.97 (0.01)	−0.98 (0.01)	—	0.01 (0.01)	566.78	0.16
ARMA(2,1)	1.92 (0.03)	−0.93 (0.03)	−1.01 (0.01)	0.48 (0.06)	646.18	0.87
R-ARMA(2,1)	1.87 (0.04)	−0.88 (0.04)	—	0.62 (0.03)	643.82	0.87

Notes. The estimation sample is 1976Q1-2019Q4, conditional on observations 1975Q1-1975Q4, and the observable is quadratically-detrended log house prices. The discount factor is $\beta = 0.99$. Log $\mathcal{L}$ denotes the maximized log-likelihood function and $\rho_{\Delta y}(1)$ is the first-order autocorrelation of the first difference of y_t implied by the estimated models, i.e. the theoretical momentum of y_t. The empirical estimate of $\rho_{\Delta y}(1)$ is 0.87. Parentheses report asymptotic standard errors.

We estimate the four model versions by maximum likelihood. We use detrended log house prices as a direct observation of y_t and we treat dividends as unobservable. We report results for the cyclical component estimated by the quadratic time trend; results for the other detrending approaches are very similar. Estimating the model using the log-difference of house prices as a direct observation of Δy_t also provides similar results, which confirms our insight that momentum plays a crucial role for identification.[13] Throughout, we condition estimation on the first four observations (as in Table 1) and keep the discount factor fixed at $\beta = 0.99$, the standard quarterly value.

Table 2 presents the results. We emphasize four points.

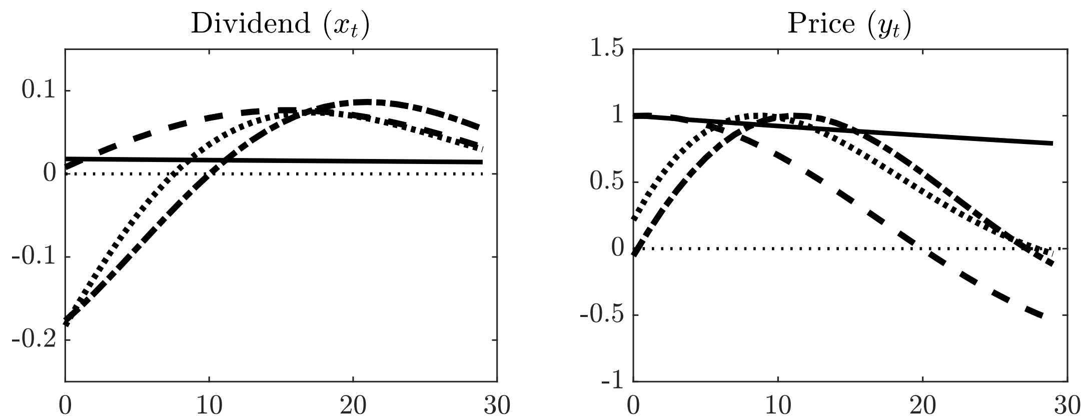
First, specifying AR(1) or AR(2) dividends prevents the model from reproducing the momentum found in house prices. This is not surprising in the AR(1) case in light of our analysis in Section 3. More interestingly, we see that including an additional autoregressive term in the dividend process improves the momentum properties of the model only marginally: the model-implied value of $\rho_{\Delta y}(1)$ rises from zero in the AR(1) case to 0.16 in the AR(2) case, still far from the value of 0.87 found in the data.

Second, allowing for ARMA(2,1) dividends significantly increases the likelihood, formally rejecting the AR(1) and AR(2) versions. The last column also indicates that this extension reproduces the persistence of Δy_t. Comparison with the AR(2) version clearly reveals that introducing anticipated shocks through the MA component has significant impact on the momentum properties of the model.

The estimated AR coefficients (ϕ_1, ϕ_2) belong to $\mathcal{M}$, the parameter space associated with momentum highlighted in Section 3. The associated roots, $0.96 \pm 0.11i$, are close to those estimated in Section 2.2. In addition, the estimated MA coefficient θ is slightly above one, so

[13] All these results are available upon request.

Figure 5. Impulse-response functions in estimated asset-pricing models



Notes. In each chart, the solid line is the response estimated in the AR(1) version, the dashed line is the response estimated in the AR(2) version, the dash-dot line is the response estimated in the unrestricted ARMA(2,1) version, and the dotted line is the response estimated in the restricted ARMA(2,1) version. The estimation sample is 1976Q1-2019Q4, conditional on observations 1975Q1-1975Q4, and the observable is quadratically-detrended log house prices. The discount factor is $\beta = 0.99$.

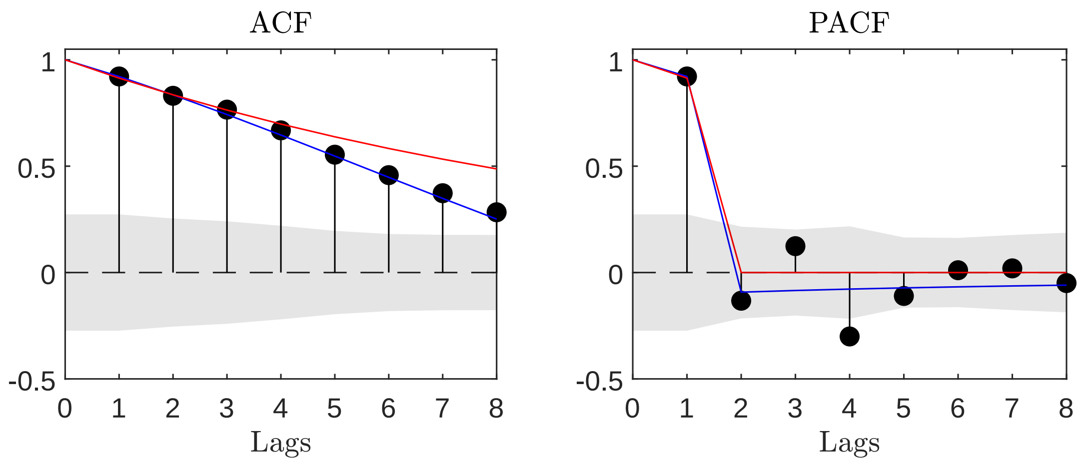
that the MA component of dividends is non-fundamental.[14] These two findings highlight the interest of our theoretical analysis: using only the constraints associated with momentum, we were able to characterize *a priori* the parameter space most likely to reproduce the data correctly. From an identification perspective, this means that momentum is a very informative property to discriminate among various models of house prices.

Third, the version with restricted ARMA(2,1) dividends produces results that closely match those obtained with unrestricted ARMA dividends in terms of point estimates and momentum. On the other hand, a formal likelihood comparison would reject the restricted version at the 5% level, which amounts to rejecting an exact AR(2) representation for house prices in favor of a closely related, albeit slightly more general, process. This outcome echoes the reduced-form estimates reported in Section 2.2: although the AR(2) model may not fit as well as richer alternatives, it provides a parsimonious representation of detrended house prices that captures the most important cyclical properties of the series.

Using equation (11), the estimate $(\phi_1, \phi_2) = (1.87, -0.88)$ and the calibrated value $\beta = 0.99$ imply a restricted MA coefficient of $\theta = -\beta\phi_2/(1 - \beta\phi_1) = -1.03$. This value is close to the unrestricted estimate $\theta = -1.01$ and lies in the non-fundamental space associated with news-rich processes. These observations are confirmed by the IRFs reported in Figure 5: the estimated responses in the unrestricted and restricted ARMA(2,1) versions closely resemble each other and present the same hump-shaped behavior. They also look like the IRFs in the

[14]This is not an econometric issue because we estimate a structural model using observations on the fundamental process y_t.

Figure 6. Persistence properties of smoothed ARMA dividends



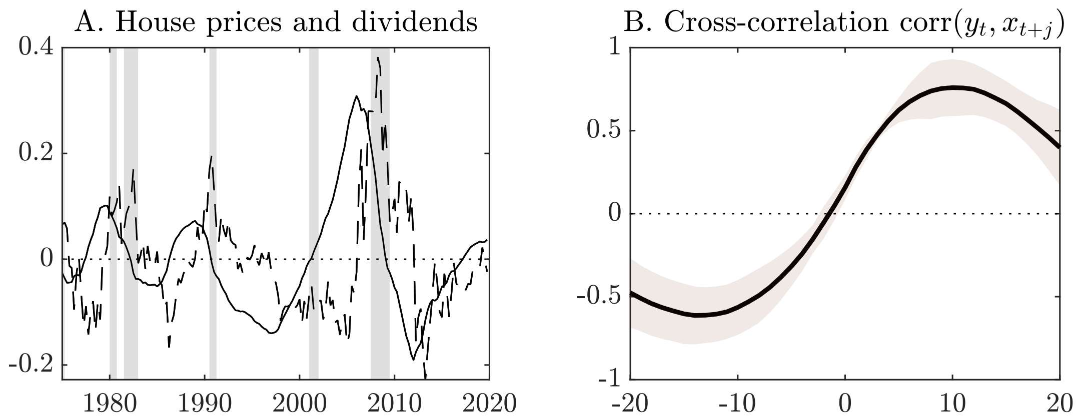
Notes. The left-hand panel shows the autocorrelation function (ACF) of smoothed dividends in the unrestricted ARMA(2,1) model version, while the right-hand panel shows the partial autocorrelation function (PACF). Dark bars represent sample statistics and shaded areas show heteroskedasticity-robust 95% confidence intervals centered around zero. Blue lines show the values implied by the unrestricted ARMA(2,1) process estimated from the data and red lines show the values implied by the AR(1) model with persistence parameter $\varphi = \widehat{\phi}_1 - 1 = 0.92$. The estimation sample is 1976Q1-2019Q4, conditional on observations 1975Q1-1975Q4, and the observable is quadratically-detrended log house prices. The discount factor is $\beta = 0.99$.

lower-left panel of Figure 4, highlighting again the link between our theoretical discussion and the empirical estimates.

Fourth, in the two ARMA versions, estimated dividend dynamics are close to an AR(1) process. Indeed, the structural estimates approximately verify the conditions of Corollary 5: the discount factor β is close to one, the estimates of $1 - \phi_1$ and ϕ_2 are very similar, and the estimates of θ are just above one in absolute value. It follows that the ARMA representations of dividends almost simplify to AR(1) processes in this case.

We can evaluate the quality of this approximation by examining the dividend series smoothed from the estimated models. This is what we do in Figure 6, which reports the ACF and PACF of the smoothed dividends recovered from the unrestricted ARMA(2,1) version of the model. (Results are similar for the restricted ARMA version.) Each panel shows the sample statistics estimated from the dividends in black, together with robust confidence intervals. The blue lines show the theoretical values implied by the ARMA(2,1) parameter estimates. The red lines show the theoretical values implied by the AR(1) process with autoregressive parameter $\varphi = \widehat{\phi}_1 - 1$, which corresponds to the limit process in Corollary 5. The AR(1) process provides a very good approximation of the first four autocorrelations of dividends, whether estimated from the smoothed series or implied by the ARMA estimates. It also generates a PACF very close to that implied by the ARMA estimates.

Figure 7. U.S. house prices and smoothed dividends



Notes. Panel A shows quadratically-detrended log real house prices (solid line) and the demeaned dividend process smoothed from the restricted ARMA(2,1) version of the model (dashed line). Dividends are normalized to have the same standard deviation as house prices. Shaded areas represent the NBER recession dates. Panel B shows the sample cross-correlation between current house prices y_t and past and future dividends x_{t+j} for $j = -20, \ldots, 20$, with the shaded band reporting a 95% GMM confidence interval. The estimation sample is 1976Q1-2019Q4, conditional on observations 1975Q1-1975Q4, and the discount factor is $\beta = 0.99$.

We also see that the partial autocorrelations estimated from the smoothed dividends are small from lag 2 on, with most of them lying well inside the confidence band centered around zero. Clearly it would be difficult for an economist observing these dynamic properties to select an ARMA(2,1) representation for dividends over the simpler AR(1) alternative. The main point of our analysis is that this choice would have crucial implications for the implied momentum properties of house prices in the theoretical model, as ARMA(2,1) dividends generate strong positive serial correlation in equilibrium asset-price growth while AR(1) dividends do not.

Finally, Panel A in Figure 7 reports the cyclical house price variable used in estimation, together with the dividends smoothed from the restricted ARMA(2,1) model. To make the chart easier to read, we rescale dividends so that the two series have the same variance. Visually, we see that dividends tend to rise during expansions and to fall during and after recessions, which is plausible for a variable driving housing demand.

It is also striking that house prices *lead* smoothed dividends by several quarters. This pattern reflects the presence of news shocks in the model, as asset prices adjust instantaneously to information about future dividend movements. The lead is quantified by the cross-correlogram reported in Panel B. While the contemporaneous correlation between house prices and dividends is as low as 0.16, the correlation between current house prices and future dividends increases significantly over a number of periods: 0.56 at the one-year horizon, 0.74 at the two-year horizon, and up to 0.76 at an horizon of 10 quarters. This observation indicates that the model delivers strong amplification: a frictionless economy, driven only in part by shocks anticipated one quarter in advance, implies a lead-lag pattern of more than two years between equilibrium asset prices and fundamentals.

5. Conclusion

Frictionless economic models populated by rational, forward-looking agents and driven by specific ARMA(2,1) forcing processes are consistent with equilibrium asset-price momentum. To establish this result, we start by documenting that simple AR(2) models provide a good approximation to the cyclical behavior of U.S. house prices, in particular as regards the positive autocorrelation of their first difference. Then, we show analytically that ARMA(2,1) exogenous forces give rise to equilibrium AR(2) asset-price dynamics in a frictionless present-value model. Our pen-and-paper strategy identifies the specific parameter configuration associated with momentum and allows us to provide an economic interpretation of the results, emphasizing the role of anticipated shocks.

These results are important because they question the common wisdom that frictionless, rational-expectations environments rule out asset-price momentum. Our analysis suggests that rational expectations about future price developments can be consistent with asset-price momentum if agents in the economy can anticipate future returns. Of course, we do not claim that news shocks account for all momentum in house prices, every time and everywhere. Rather, our claim is that the mechanism we emphasize may coexist with other frictions to explain the strong autocorrelation of house-price growth. Another possibility is that the smoothing induced by seasonal adjustment amplifies the amount of momentum found in the data. Therefore, an important question for future research is to evaluate the respective roles of these alternative explanations, for instance by letting them compete to explain the data within an encompassing setup.

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Appendix A. Proof of Proposition 1

We start from the first-difference representation of the AR(2) process

$$\Delta y_t = \phi_1 \Delta y_{t-1} + \phi_2 \Delta y_{t-2} + \Delta \varepsilon_t, \tag{A.1}$$

where Δ is the difference operator. Then, multiply equation (A.1) by Δy_t, Δy_{t-1}, and Δy_{t-2}, and take expectations. Since y_t has zero mean, this amounts to computing covariances. We get

$$\begin{aligned}
\gamma_{\Delta y}(0) &= \phi_1 \gamma_{\Delta y}(1) + \phi_2 \gamma_{\Delta y}(2) + \mathrm{cov}(\Delta\varepsilon_t, \Delta y_t), \\
\gamma_{\Delta y}(1) &= \phi_1 \gamma_{\Delta y}(0) + \phi_2 \gamma_{\Delta y}(1) + \mathrm{cov}(\Delta\varepsilon_t, \Delta y_{t-1}), \\
\gamma_{\Delta y}(2) &= \phi_1 \gamma_{\Delta y}(1) + \phi_2 \gamma_{\Delta y}(0) + \mathrm{cov}(\Delta\varepsilon_t, \Delta y_{t-2}),
\end{aligned}$$

where $\gamma_{\Delta y}(j)$ denotes the jth autocovariance of Δy_t for $j = 0, 1, 2$.

To solve this system, we use the MA(∞) representation associated with the AR(2) process for y_t. It is given by

$$y_t = \sum_{j=0}^{\infty} \Psi_j \varepsilon_{t-j}, \tag{A.2}$$

where $\Psi_0 = 1$, $\Psi_1 = \phi_1$, $\Psi_2 = \phi_1^2 + \phi_2$, $\Psi_3 = \phi_1^3 + 2\phi_1\phi_2$, and $\Psi_j = \phi_1 \Psi_{j-1} + \phi_2 \Psi_{j-2}$ for $j > 3$. Taking first differences, we get

$$\Delta y_t = \sum_{j=0}^{\infty} \Psi_j \Delta\varepsilon_{t-j}.$$

Using that $\mathrm{cov}(\varepsilon_t, \varepsilon_\tau) = 0$ if $t \neq \tau$, we deduce

$$\begin{aligned}
\mathrm{cov}(\Delta\varepsilon_t, \Delta y_t) &= (2 - \phi_1)\sigma^2, \\
\mathrm{cov}(\Delta\varepsilon_t, \Delta y_{t-1}) &= -\sigma_2, \\
\mathrm{cov}(\Delta\varepsilon_t, \Delta y_{t-2}) &= 0.
\end{aligned}$$

Inserting these covariances in the above system and dividing by $\gamma_{\Delta y}(0)$ yields

$$\begin{aligned}
1 &= \phi_1 \rho_{\Delta y}(1) + \phi_2 \rho_{\Delta y}(2) + \frac{(2 - \phi_1)\sigma^2}{\gamma_{\Delta y}(0)}, \\
\rho_{\Delta y}(1) &= \phi_1 + \phi_2 \rho_{\Delta y}(1) - \frac{\sigma^2}{\gamma_{\Delta x}(0)}, \\
\rho_{\Delta y}(2) &= \phi_1 \rho_{\Delta y}(1) + \phi_2,
\end{aligned}$$

where $\rho_{\Delta y}(0) = 1$ and $\rho_{\Delta y}(j) = \gamma_{\Delta y}(j)/\gamma_{\Delta y}(0)$ for $j = 1, 2$. We deduce that

$$\frac{\sigma^2}{\gamma_{\Delta y}(0)} = \phi_1 - (1 - \phi_2)\rho_{\Delta x}(1).$$

Using this result and the equality $\rho_{\Delta y}(2) = \phi_1 \rho_{\Delta y}(1) + \phi_2$, we can rewrite the first equation of the system as

$$\rho_{\Delta y}(1) = \frac{1 + \phi_1^2 - 2\phi_1 - \phi_2^2}{\phi_1 + \phi_1\phi_2 - (2 - \phi_1)(1 - \phi_2)}.$$

We simplify the numerator to

$$1 + \phi_1^2 - 2\phi_1 - \phi_2^2 = (\phi_1 - \phi_2 - 1)(\phi_1 + \phi_2 - 1)$$

and the denominator to

$$\phi_1 + \phi_1\phi_2 - (2 - \phi_1)(1 - \phi_2) = 2(\phi_1 + \phi_2 - 1).$$

It follows that

$$\rho_{\Delta y}(1) = \frac{\phi_1 - 1 - \phi_2}{2}.$$

Finally, we can determine the full autocorrelation function from the recurrence equation

$$\rho_{\Delta y}(j) = \phi_1 \rho_{\Delta y}(j-1) + \phi_2 \rho_{\Delta y}(j-2), \text{ for } j \geq 2,$$

with $\rho_{\Delta y}(0) = 1$ and $\rho_{\Delta y}(1)$ obtained above.

Appendix B. Proof of Proposition 2

We start with the “if” part. Suppose that x_t evolves according to

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + (1 - \beta\phi_1)\varepsilon_t - \beta\phi_2\varepsilon_{t-1},$$

where ε_t is a white-noise innovation with zero mean and variance σ^2, and define y_t as the solution of the valuation equation

$$y_t = \beta E_t y_{t+1} + x_t,$$

where $\beta \in (0, 1)$ is the discount factor and E_t denotes the expectation operator conditional on period-t information.

We solve the model using the method of undetermined coefficients. We suppose that the solution for y_t verifies

$$y_t = \mu_1 x_t + \mu_2 x_{t-1} + \mu_3 \varepsilon_t,$$

where the μ's are unknown parameters. Inserting this guess into the valuation equation and using term-by-term identification yields

$$\mu_1 = \frac{1}{1 - \beta\phi_1 - \beta^2\phi_2}, \qquad \mu_2 = \frac{\beta\phi_2}{1 - \beta\phi_1 - \beta^2\phi_2}, \qquad \mu_3 = \frac{-\beta^2\phi_2}{1 - \beta\phi_1 - \beta^2\phi_2}.$$

Therefore, we have

$$y_t = (\mu_1 + \mu_2 B)x_t + \mu_3\varepsilon_t = \left[\frac{(\mu_1 + \mu_2 B)(1 - \beta\phi_1 - \beta\phi_2 B)}{1 - \phi_1 B - \phi_2 B^2} + \mu_3\right]\varepsilon_t,$$

or equivalently

$$(1 - \phi_1 B - \phi_2 B^2)y_t = \left[(\mu_1 + \mu_2 B)(1 - \beta\phi_1 - \beta\phi_2 B) + \mu_3(1 - \phi_1 B - \phi_2 B^2)\right]\varepsilon_t.$$

Using the expressions for μ_1, μ_2, and μ_3, it is easy to show that the bracket on the right-hand side of the equation simplifies to one. It follows that

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t,$$

which proves the "if" part.

Second, we establish the "only if" part. Suppose that y_t has the stationary AR(2) representation

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \varepsilon_t,$$

where ε_t is a white-noise innovation with zero mean and variance $\widetilde{\sigma}^2$. Suppose also that y_t solves the valuation equation

$$y_t = \beta E_t y_{t+1} + x_t,$$

where $\beta \in (0, 1)$ is the discount factor and E_t denotes the expectation operator conditional on period-t information.

Then, we can substitute out y_{t+1} in the valuation equation to get

$$y_t = \beta E_t \left[\phi_1 y_t + \phi_2 y_{t-1} + \varepsilon_{t+1}\right] + x_t.$$

This equation simplifies to

$$y_t = \beta\phi_1 y_t + \beta\phi_2 y_{t-1} + x_t,$$

or

$$x_t = (1 - \beta\phi_1) y_t - \beta\phi_2 y_{t-1}.$$

Using the AR(2) representation of y_t, this equation implies that x_t evolves according to

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + (1 - \beta\phi_1)\varepsilon_t - \beta\phi_2 \varepsilon_{t-1}.$$

This establishes the "only if" part.

Appendix C. Proof of Corollary 4

Recall that

$$\theta = \frac{\beta\phi_2}{1 - \beta\phi_1}.$$

Viewed as a function of the discount factor, there is a discontinuity in $\theta(\beta)$ at $\beta = 1/\phi_1$.

Consider first small values of β, i.e. $\beta < 1/\phi_1$. Then, $1 - \beta\phi_1 > 0$ and $\theta(\beta) < 0$. Therefore, non-fundamentalness arises in this area when $\theta(\beta) < -1$, which is equivalent to $\beta > 1/(\phi_1 - \phi_2)$. Note that $\phi_2 < 0$ implies that $1/(\phi_1 - \phi_2) < 1/\phi_1$, so that there is a non-empty set of β values generating a non-fundamental ARMA process for x_t in this case.

Second, consider large values of β, i.e. $\beta > 1/\phi_1$. Then, $1 - \beta\phi_1 < 0$ and $\theta(\beta) > 0$. Therefore, non-fundamentalness arises in this area when $\theta(\beta) > 1$, which is equivalent to $\beta(\phi_1 + \phi_2) < 1$. This condition is always true within the stability triangle because $\beta \in (0, 1)$ and $\phi_1 + \phi_2 < 1$.

Therefore, we conclude that, within the subset $\mathcal{M}$ of the stability triangle associated with momentum in y_t, $|\theta(\beta)| > 1$ if $\beta > 1/(\phi_1 - \phi_2)$.

Appendix D. Proof of Corollary 6

We show in Appendix B that the solution of the model verifies

$$x_t = (1 - \beta\phi_1)y_t - \beta\phi_2 y_{t-1}.$$

It follows that

$$\mathrm{cov}(x_t, y_t) = (1 - \beta\phi_1)\gamma_y(0) - \beta\phi_2\gamma_y(1) = \gamma_y(0)[1 - \beta\phi_1 - \beta\phi_2\rho_y(1)].$$

Using that $\rho_y(1) = \phi_1/(1 - \phi_2)$, this expression simplifies to

$$\mathrm{cov}(x_t, y_t) = \gamma_y(0)[1 - \beta\rho_y(1)].$$

It is clear that $\mathrm{cov}(x_t, y_t) > 0$ since $|\beta\rho_y(1)| < 1$. Then, the correlation between x_t and y_t is given by

$$\mathrm{corr}(x_t, y_t) = \frac{\mathrm{cov}(x_t, y_t)}{\sqrt{\gamma_y(0)\gamma_x(0)}} = [1 - \beta\rho_y(1)]\sqrt{\frac{\gamma_y(0)}{\gamma_x(0)}}.$$

To compute $\gamma_y(0)/\gamma_x(0)$, note that

$$\begin{aligned}
\gamma_x(0) &= \phi_1\gamma_x(1) + \phi_2\gamma_x(2) + (1 - \beta\phi_1)^2 - \beta\phi_1\phi_2(1 - \beta\phi_1) + \beta^2\phi_2^2,\\
\gamma_x(1) &= \frac{\phi_1\gamma_x(0)}{1 - \phi_2} - \frac{\beta\phi_2(1 - \beta\phi_1)}{1 - \phi_2}.
\end{aligned}$$

It is also easy to show that

$$\gamma_x(2) = \phi_1\gamma_x(1) + \phi_2\gamma_x(0).$$

Dividing all three equations by $\gamma_x(0)$ yields

$$\begin{aligned}
1 &= \phi_1\rho_x(1) + \phi_2\rho_x(2) + \frac{(1 - \beta\phi_1)^2 - \beta\phi_1\phi_2(1 - \beta\phi_1) + \beta^2\phi_2^2}{\gamma_x(0)},\\
\rho_x(1) &= \frac{\phi_1}{1 - \phi_2} - \frac{\beta\phi_2(1 - \beta\phi_1)}{(1 - \phi_2)\gamma_x(0)},\\
\rho_x(2) &= \phi_1\rho_x(1) + \phi_2.
\end{aligned}$$

Combining these conditions, we get

$$1 = \frac{\phi_1^2}{1 - \phi_2} - \frac{\beta\phi_1\phi_2(1 - \beta\phi_1)}{(1 - \phi_2)\gamma_x(0)} + \frac{\phi_1^2\phi_2}{1 - \phi_2} - \frac{\beta\phi_1\phi_2^2(1 - \beta\phi_1)}{(1 - \phi_2)\gamma_x(0)} + \phi_2^2 + \frac{(1 - \beta\phi_1)^2}{\gamma_x(0)} - \frac{\beta\phi_1\phi_2(1 - \beta\phi_1) - \beta^2\phi_2^2}{\gamma_x(0)}.$$

This can be rearranged into

$$1 - \phi_2^2 - \frac{\phi_2^2(1 + \phi_2)}{1 - \phi_2} = -\frac{\beta\phi_1\phi_2(1 - \beta\phi_1) + \beta\phi_1\phi_2^2(1 - \beta\phi_1)}{(1 - \phi_2)\gamma_x(0)} + \frac{(1 - \beta\phi_1)^2 - \beta\phi_1\phi_2(1 - \beta\phi_1) + \beta^2\phi_2^2}{\gamma_x(0)}.$$

Now, we use the fact that the variance of y_t is

$$\gamma_y(0) = \frac{(1+\phi_2)[(1-\phi_2)^2 - \phi_1^2]}{1-\phi_2},$$

which is just the inverse of the left-hand side of the previous equation. After some algebra, it follows that

$$\frac{\gamma_x(0)}{\gamma_y(0)} = \frac{(1-\phi_2)[(1-\beta\phi_1)^2 + \beta^2\phi_2^2] - 2\beta\phi_1\phi_2(1-\beta\phi_1)}{1-\phi_2}.$$

Plugging this expression in the equation for the correlation between x_t and y_t, we obtain

$$\operatorname{corr}(x_t, y_t) = \left(1 - \frac{\beta\phi_1}{1-\phi_2}\right)\sqrt{\frac{1-\phi_2}{(1-\phi_2)[(1-\beta\phi_1)^2 + \beta^2\phi_2^2] - 2\beta\phi_1\phi_2(1-\beta\phi_1)}}.$$

BANQUE CENTRALE DU LUXEMBOURG

EUROSYSTÈME

2, boulevard Royal
L-2983 Luxembourg

Tél.: +352 4774-1
Fax: +352 4774 4910

www.bcl.lu • info@bcl.lu