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Mixed Markov-Perfect Equilibria in the Continuous-Time War of Attrition*

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Abstract

We prove the existence of a Markov-perfect equilibrium in randomized stopping times for a model of the war of attrition in which the underlying state variable follows a homogenous linear diffusion. We first prove that the space of Markovian randomized stopping times can be topologized as a compact absolute retract. This in turn enables us to use a powerful fixed-point theorem by Eilenberg and Montgomery [16] to prove our existence theorem. We illustrate our results with an example of a war of attrition that admits a mixed-strategy Markov-perfect equilibrium but no pure-strategy Markov-perfect equilibrium.

Keywords: War of Attrition, Markovian Randomized Stopping Time, Markov-Perfect Equilibrium, Fixed-Point Theorem.

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1 Introduction

We consider a two-player nonzero-sum stopping game in continuous time where the payoff for player $i = 1, 2$ is given by (with $j = 3 - i$):

$$J^i(x, \tau^i, \tau^j) = \mathbb{E}_x\Big[\mathbb{1}_{\tau^i \le \tau^j}\, \mathrm{e}^{-r\tau^i} R^i(X_{\tau^i}) + \mathbb{1}_{\tau^j < \tau^i}\, \mathrm{e}^{-r\tau^j} G^i(X_{\tau^j})\Big],$$

where X is a homogenous linear diffusion defined on an interval $\mathcal{I} \subset \mathbb{R}$ and satisfying $X_0 = x$. The reward functions R^i and G^i are sufficiently integrable continuous functions, $r \ge 0$ is a constant discount rate, and τ^1 and τ^2 are stopping times of the filtration of X chosen by players 1 and 2, respectively. Our main assumption is that $R^i \le G^i$ for $i = 1, 2$; this reflects a second-mover advantage and is typical in the timing games referred to as wars of attrition in the economics literature (see [28, 37, 20, 27, 12, 21] for examples of such games under Brownian uncertainty). A pair of stopping times (τ^1, τ^2) is a Nash equilibrium for the continuous-time war of attrition started at x if, for each player $i = 1, 2$, we have $J^i(x, \tau^i, \tau^j) = \sup_\tau J^i(x, \tau, \tau^j)$. A Markov-perfect equilibrium (MPE) in pure strategies is a pair of Markovian stopping times (characterized as hitting times of a closed subset of $\mathcal{I}$) which form a Nash equilibrium for every initial condition x.

The existence of pure-strategy MPEs for games of the war-of-attrition type has been established under a variety of assumptions. (1) Cattiaux and Lepeltier [7] (see also Lepeltier and Etourneau [31]) prove the existence of a pure-strategy MPE under the additional assumption that $(\mathrm{e}^{-rt} G^i(X_t))_{t \ge 0}$ is a supermartingale. (2) More recently, De Angelis, Ferrari, and Moriarty [10] prove the existence of a pure-strategy MPE under geometric conditions on the functions R^i, $i = 1, 2$; similar results have been derived by Attard [1] and Martyr and Moriarty [34]. (3) Existence of an MPE in the zero-sum case is obtained in Ekström and Villeneuve [15] and Ekström and Peskir [14].

In the absence of such additional assumptions, a pure-strategy MPE may not exist—we provide a simple and explicit example in Section 7. Lepeltier and Maingueneau [32] (in the zero-sum case) and Hamadene and Zhang [23] (in the nonzero-sum case) prove that a Nash equilibrium always exists in games of the war-of-attrition type. However, the strategies constructed in, e.g., [23] are not Markovian, and the resulting equilibrium is typically not subgame-perfect in the sense of Riedel and Steg [39], implying that it requires that some player make a noncredible threat.

To recover the existence of equilibria, a classical approach in game theory consists in extending the class of strategies to mixed strategies. In the present context, this amounts to considering randomized stopping times, whereby, loosely speaking, players choose a distribution on the set of stopping times. Randomized stopping times have been considered for a long time in the theory of optimal stopping (see, e.g., Baxter and Chacon [3], Meyer [36], and El Karoui, Lepeltier, and Millet [18]) and in the analysis of stopping games (see, e.g., Touzi and Vieille [43], Riedel and Steg [39], Laraki and Solan [29], Laraki and Solan [30], De Angelis, Merkulov, and Palczewski [11]).

In the case where, as in the present paper, the underlying state variable follows a homogenous linear diffusion process, Décamps, Gensbittel, and Mariotti [12] derive from a representation result for multiplicative functionals due to Sharpe [42] that any Markovian randomized stopping time can be represented by a pair (μ, S), where S is a closed subset of $\mathcal{I}$ and μ is a locally finite measure on $\mathcal{I} \setminus S$ such that the conditional survival function Λ_t (that is, the probability to stop strictly after t conditionally on $(X_s)_{s\in[0,t]}$) writes under the form $\Lambda_t = \mathbb{1}_{t<\tau_S}\, \mathrm{e}^{-\int_{\mathcal{I}\setminus S} L_t^y\, \mu(\mathrm{d}y)}$, where L_t^y is the local time of X at (y,t) and τ_S is the hitting time by X of S. The set S is the region of immediate stopping and μ is a (possibly singular) intensity of stopping outside of S. The pair (μ, S) can alternatively be seen as a nonnegative measure on $\mathcal{I}$ that explodes on S.

In the present paper, we build on this representation theorem to prove the existence of an MPE in randomized stopping times without additional assumptions on the reward functions such as those in Cattiaux and Lepeltier [7] or in De Angelis, Ferrari, and Moriarty [10]. To this end, the natural approach is to use an appropriate fixed-point theorem for the best-reply correspondence. We show that the lack of convexity of the space of Markovian randomized stopping times, reflecting the possibility that a player stops with infinite intensity on some subset of $\mathcal{I}$, can be overcome by invoking a fixed-point theorem due to Eilenberg and Montgomery [16], which applies to (non necessarily convex) compact absolute retracts. In so doing, we also establish another result of interest about the topology of Markovian randomized stopping times. Specifically, we show that the set of nonnegative (but not necessary locally finite) regular measures endowed with a topology that extends vague convergence of locally finite measures is a compact absolute retract.

In contemporaneous independent work, Christensen and Schultz [8] derive an analogous existence theorem using a different method. They first consider a family of auxiliary games in which the players are only allowed to stop over increasingly finer finite subsets of the state space. In these discretized games, the best-reply sets are convex and the existence of an MPE can be directly proved using Kakutani's fixed-point theorem [25]. The existence of an MPE for the primary game is then obtained as the limit of a convergent sequence of MPEs of these auxiliary games. The analysis requires the introduction of two distinct topologies.

The first one allows one to use Kakutani's fixed-point theorem in the auxiliary discretized games. The second, based on the distribution of stopped processes, defines an appropriate notion of convergence allowing one to pass from a sequence of MPEs of the discretized games to an MPE of the primary game.

By using the Eilenberg-Montgomery fixed-point theorem, our approach is more direct, avoids convexity issues, and only requires us to define a natural topology under which the set of Markovian randomized stopping times, which can be identified to the set of nonnegative regular measures on $\mathcal{I}$ taking values in $[0, \infty]$, is a compact absolute retract. Interestingly, when the discretization is locally finite, the topology used in the auxiliary games introduced by Christensen and Schultz [8] actually corresponds to the restriction to a finite or countable subspace of the state space of the topology we define on Markovian randomized stopping times. Another difference is that, in Christensen and Schultz [8], the diffusion X is assumed to live on a compact interval whose endpoints are absorbing points for X, and at which the payoff functions R^i and G^i coincide for every player $i = 1, 2$. In our model, by contrast, the state space for X is a possibly unbounded interval $\mathcal{I}$ whose endpoints are natural boundaries for the diffusion, and the functions R^i and G^i, $i = 1, 2$, may be unbounded, as is often the case in economic applications.

To the best of our knowledge, [8] and the present paper are the only papers proving the existence of an MPE in the continuous-time war of attrition under the weak assumption $R^i \leq G^i$ for $i = 1, 2$. Our approach and that in [8] are complementary in that the method of proof in the latter paper shows that at least some MPEs of the continuous-time game can be obtained as limits of MPEs of suitably discretized games. Whether this is the case of all MPEs remains an open question.

2 Model and Main Results

2.1 A Brownian Model of the War of Attrition

Consider a one-dimensional time-homogeneous diffusion process $X := (X_t)_{t \geq 0}$ defined on the canonical space $(\Omega, \mathcal{F}, \mathbb{P}_x)$ of continuous trajectories with $X_0 = x$ under $\mathbb{P}_x$, that is solution in law to the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t \geq 0, \tag{2.1}$$

driven by some Brownian motion $W := (W_t)_{t \geq 0}$. The state space for X is an interval $\mathcal{I} := (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$, and b and σ are continuous functions, with $\sigma > 0$ on

$\mathcal{I}$. We assume that α and β are natural endpoints for the diffusion. Therefore, X is regular on $\mathcal{I}$ and the SDE (2.1) admits a weak solution that is unique in law.

The process X is defined on the canonical space $(\Omega, \mathcal{F})$ of continuous trajectories endowed with the usual family of shift operators $(\theta_t)_{t\geq 0}$. Let $\mathbb{P}_\mu$ be the law of the process X with initial distribution $\mu \in \Delta(\mathcal{I})$, where $\Delta(\mathcal{I})$ is the space of probability measures on the Borel σ-field $\mathcal{B}(\mathcal{I})$. We denote by $(\mathcal{F}^0_t)_{t\geq 0}$ the natural filtration $(\sigma(X_s; s \leq t))_{t\geq 0}$ generated by X, and we let $\mathcal{F}^0_\infty := \sigma(\bigcup_{t\geq 0} \mathcal{F}^0_t)$. For each μ, we denote by $\mathcal{F}^\mu_\infty$ the completion of $\mathcal{F}^0_\infty$ with respect to $\mathbb{P}_\mu$, and, for each $t \geq 0$, we let $\mathcal{F}^\mu_t$ be the augmentation of $\mathcal{F}^0_t$ by the $\mathbb{P}_\mu$-null, $\mathcal{F}^\mu_\infty$-measurable sets. The usual augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ is then defined by $\mathcal{F}_t := \bigcap_{\mu\in\Delta(\mathcal{I})} \mathcal{F}^\mu_t$ for all $t \geq 0$ and it is right-continuous (see, e.g., [38, Chapter III, §2, Proposition 2.10]). As usual, we say that a property of the trajectories $\omega \in \Omega$ is satisfied a.s. if, for each $x \in \mathcal{I}$, it is satisfied for $\mathbb{P}_x$-a.e. $\omega \in \Omega$.

The game is played as follows. Player 1 chooses a stopping time τ^1 and player 2 chooses a stopping time τ^2 in the set $\mathcal{T}$ of all stopping times of $(\mathcal{F}_t)_{t\geq 0}$. Both players discount future payoffs at a constant rate $r \geq 0$. For each $i = 1, 2$, the expected payoff of player i is[1]

$$J^i(x, \tau^i, \tau^j) := \mathbb{E}_x\Big[\mathbb{1}_{\tau^i\leq\tau^j}\, \mathrm{e}^{-r\tau^i} R^i(X_{\tau^i}) + \mathbb{1}_{\tau^i>\tau^j}\, \mathrm{e}^{-r\tau^j} G^i(X_{\tau^j})\Big]. \tag{2.2}$$

For each $i = 1, 2$, we assume

A0 The functions R^i and G^i in (2.2) are continuous on $\mathcal{I}$ and $R^i \leq G^i$.

For each $i = 1, 2$ and every function $f = R^i, G^i$, we assume

A1 $\mathbb{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}|f(X_t)|] < \infty$ for all $x \in \mathcal{I}$.

A2 $\lim_{t\to\infty} \mathrm{e}^{-rt} f(X_t) = 0$ a.s.

Assumption A1 guarantees that the family $(\mathrm{e}^{-r\tau} f(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable, that is, the process $(\mathrm{e}^{-rt} f(X_t))_{t\geq 0}$ is of class (D).

A game satisfying the above assumptions is hereafter generically referred to as a BWoA.

2.2 Randomized Stopping Times

In this section, we briefly recall some definitions and results that are standard in the literature; we refer to [12] for the missing proofs. For every player $i = 1, 2$, consider the enlarged probability space $(\Omega^i, \mathcal{F}^i) := (\Omega \times [0, 1], \mathcal{F} \otimes \mathcal{B}([0, 1]))$, endowed with the product probability $\mathbb{P}^i_x := \mathbb{P}_x \otimes \lambda$, where λ denotes Lebesgue measure.

[1] By convention, we let $\mathrm{e}^{-r\tau} f(X_\tau) := 0$ on $\{\tau = \infty\}$ for any Borel function f and any random time τ, see Assumption A2 below.

Definition 2.1. *For each $i = 1, 2$, a randomized stopping time for player i is an $\mathcal{F} \otimes \mathcal{B}([0,1])$-measurable function $\gamma^i : \Omega^i \to \mathbb{R}_+$ such that, for λ-a.e. $u^i \in [0,1]$, $\gamma^i(\cdot, u^i) \in \mathcal{T}$. The process $\Gamma^i := (\Gamma^i_t)_{t \geq 0}$ defined by*

$$\Gamma^i_t(\omega) := \int_{[0,1]} \mathbb{1}_{\gamma^i(\omega,u^i) \leq t} \, \mathrm{d}u^i, \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{2.3}$$

is the conditional cumulative distribution function (ccdf) of the randomized stopping time γ^i. Likewise, the process $\Lambda^i := (\Lambda^i_t)_{t \geq 0}$ defined by

$$\Lambda^i_t(\omega) := 1 - \Gamma^i_t(\omega), \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{2.4}$$

is the conditional survival function (csf) of the randomized stopping time γ^i.

We denote by $\mathcal{T}_r$ the set of randomized stopping times. The process Γ^i defined by (2.3) takes values in $[0,1]$ and has nondecreasing and right-continuous trajectories.

Lemma 2.2 ([12, Lemma 2])**.** *The ccdf process Γ^i is $(\mathcal{F}_t)_{t \geq 0}$-adapted and, for $\mathbb{P}_x$-a.e. $\omega \in \Omega$,*

$$\Gamma^i_t(\omega) = \mathbb{P}^i_x[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega) \tag{2.5}$$

for all $x \in \mathcal{I}$ and $t \geq 0$.

By convention, we let $\Gamma^i_{0-} := 0$ and thus $\Lambda^i_{0-} := 1$. This allows us in what follows to interpret integrals of the form $\int_{[0,\tau)} \cdot \, \mathrm{d}\Gamma^i_t$ or $\int_{[0,\tau)} \cdot \, \mathrm{d}\Lambda^i_t$ in the Stieltjes sense for any ccdf Γ^i and any csf Λ^i. Notice for further reference that, for any sufficiently integrable process Z,

$$\int_{[0,\tau)} Z_s \, \mathrm{d}\Gamma^i_s = \Gamma^i_0 Z_0 + \int_{(0,\tau)} Z_s \, \mathrm{d}\Gamma^i_s. \tag{2.6}$$

If the players use randomized stopping times γ^1 and γ^2, then their expected payoffs are defined on the product probability space $\Omega \times [0,1] \times [0,1]$ with canonical element (ω, u^1, u^2), endowed with the product probability $\overline{\mathbb{P}}_x := \mathbb{P}_x \otimes \lambda \otimes \lambda$. Specifically, we have

$$J^i(x, \gamma^1, \gamma^2) := \overline{\mathbb{E}}_x\Big[\mathbb{1}_{\gamma^i \leq \gamma^j} \, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i}) + \mathbb{1}_{\gamma^i > \gamma^j} \, \mathrm{e}^{-r\gamma^j} G^i(X_{\gamma^j})\Big], \tag{2.7}$$

where $\gamma^1 := \gamma^1(\omega, u^1)$ and $\gamma^2 := \gamma^2(\omega, u^2)$, reflecting that player 1 and player 2 use the independent randomization devices u^1 and u^2, respectively. The next lemma shows that we may equivalently work with the family of ccdf processes Γ^i.

Lemma 2.3 ([12, Lemma 3])**.** *If the players use randomized stopping times with ccdfs Γ^1 and Γ^2, then their expected payoffs write as*

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbb{E}_x\bigg[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-} \, \mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t) \Lambda^i_t \, \mathrm{d}\Gamma^j_t\bigg]. \tag{2.8}$$

Moreover, any nondecreasing, right-continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted, $[0,1]$-valued process Γ^i is the ccdf of the randomized stopping time $\hat{\gamma}^i$ defined by

$$\hat{\gamma}^i(u^i) := \inf\{t \geq 0 : \Gamma^i_t > u^i\}. \tag{2.9}$$

2.3 Markovian Randomized Stopping Times

We now recall the definition of a Markovian randomized stopping time used in [12].

Definition 2.4. *A randomized stopping time for player $i = 1, 2$ with csf $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0,1]$ is Markovian if, for all $x \in \mathcal{I}$, $\tau \in \mathcal{T}$, and $s \geq 0$,*

$$\Lambda^i_{\tau+s} = \Lambda^i_\tau(\Lambda^i_s \circ \theta_\tau) \text{ on } \{\tau < \infty\} \ \mathbb{P}_x\text{-a.s.} \tag{2.10}$$

Processes satisfying (2.10) are known as multiplicative functionals of the Markov process X, see, e.g., [5]. Combining a result in [42] with the classical representation result for additive functionals of regular diffusions [6, Part I, Chapter II, Section 4, §23] yields the following representation result.

Theorem 2.5 ([12, Theorem 1])**.** *For each $i = 1, 2$, $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0,1]$ is the csf of a Markovian randomized stopping time for player i if and only if there exists a closed set $S^i \subset \mathcal{I}$ and a Radon measure[2] μ^i on $\mathcal{I} \setminus S^i$ such that, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Lambda^i_t = \mathbb{1}_{t<\tau_{S^i}} \, \mathrm{e}^{-\int_{\mathcal{I}\setminus S^i} L^y_t \, \mu^i(\mathrm{d}y)} \ \mathbb{P}_x\text{-a.s.}, \tag{2.11}$$

where $L^y_t := \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \mathbb{1}_{(y-\varepsilon,y+\varepsilon)}(X_s)\sigma^2(X_s)\,\mathrm{d}s$ is the local time of X at (y,t), and $\tau_{S^i} := \inf\{t \geq 0 : X_t \in S^i\}$ is the hitting time by X of S^i. In particular, the mapping $t \mapsto \Lambda^i_t$ is continuous on $[0, \tau_{S^i})$ $\mathbb{P}_x$-a.s.

In the following, we refer to a Markov strategy as a pair (μ^i, S^i), a ccdf Γ^i, or a csf Λ^i, based on the relations established in Theorem 2.5. Three special cases discussed in [12] are worth mentioning.

1. The pure stopping case: If $\mu^i = 0$, then the Markov strategy $(0, S^i)$ is just the hitting time τ_{S^i} by X of S^i.

2. The absolutely continuous case: If $\mu^i = g^i \cdot \lambda$, then, from the occupation time formula

[2] A measure on the Borel sets of a Hausdorff topological space is Radon if it is inner regular with respect to compact sets and locally finite in the sense that every point has a neighborhood of finite measure. When the underlying space is an open subset of $\mathbb{R}$, local finiteness alone implies regularity, i.e., inner regularity with respect to compact sets and outer regularity with respect to open sets [40, Chapter 2, Theorem 2.18].

[38, Chapter VI, §1, Corollary 1.6],

$$\Lambda_t^i = \mathbb{1}_{t<\tau_{S^i}}\, \mathrm{e}^{-\int_{\mathcal{I}\setminus S^i} L_t^y g^i(y)\,\mathrm{d}y} = \mathbb{1}_{t<\tau_{S^i}}\, \mathrm{e}^{-\int_0^t g^i(X_s)\sigma^2(X_s)\,\mathrm{d}s}. \tag{2.12}$$

This absolutely continuous strategy therefore amounts for player i to conceding with intensity $\lambda^i(X_t) := g^i(X_t)\sigma^2(X_t)$ outside S^i.

3. The singular case: If, e.g., $\mu^i = a^i\delta_{x^i}$, where $a^i > 0$ and δ_{x^i} is the Dirac mass at $x^i \in \mathcal{I}\setminus S^i$, then

$$\Lambda_t^i = \mathbb{1}_{t<\tau_{S^i}}\, \mathrm{e}^{-a^i L_t^{x^i}}. \tag{2.13}$$

Such discrete singular strategies are the building blocks of all mixed-strategy MPEs in the model studied in [12, Theorems 2–3] when players are asymmetric.

2.4 Markov-Perfect Equilibrium and Properties of Best Replies

We recall the definition and some properties of best replies.

Lemma 2.6 ([12, Lemma 4]). *For each $x \in \mathcal{I}$ and for any pair of randomized stopping times with ccdfs (Γ^1, Γ^2), $J^i(x, \Gamma^i, \Gamma^j) \le \sup_{\tau^i\in\mathcal{T}} J^i(x, \tau^i, \Gamma^j)$.*

Definition 2.7. *For each $i = 1, 2$, (μ^i, S^i) is a perfect best reply (pbr) for player i to (μ^j, S^j) and $\bar{J}^i(\cdot, (\mu^j, S^j))$ is player i's best-reply value function (brvf) to (μ^j, S^j) if*

$$\forall x \in \mathcal{I},\ J^i(x, (\mu^i, S^i), (\mu^j, S^j)) = \bar{J}^i(x, (\mu^j, S^j)) := \sup_{\tau^i\in\mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j)).$$

The set of pbrs of player i against (μ^j, S^j) is

$$PBR^i(\mu^j, S^j) := \{(\mu^i, S^i) : \forall x \in \mathcal{I},\ J^i(x, (\mu^i, S^i), (\mu^j, S^j)) = \bar{J}^i(x, (\mu^j, S^j))\},$$

and the pbr correspondence is defined by

$$PBR((\mu^1, S^1), (\mu^2, S^2)) := PBR^1(\mu^2, S^2) \times PBR^2(\mu^1, S^1). \tag{2.14}$$

A Markov-perfect equilibrium (MPE) is a profile $((\mu^1, S^1), (\mu^2, S^2))$ of Markov strategies such that, for each $i = 1, 2$, (μ^i, S^i) is a pbr for player i to (μ^j, S^j).

When no confusion can arise as to the strategy of player j, we write $\bar{J}^i$ instead of $\bar{J}^i(\cdot, (\mu^j, S^j))$. It follows from Definition 2.7 that a pair of Markovian randomized stopping times is an MPE if and only if it is a fixed point of the pbr correspondence, i.e.,

$$((\mu^1, S^1), (\mu^2, S^2)) \in PBR((\mu^1, S^1), (\mu^2, S^2))).$$

The following proposition provides useful general properties of pbrs and brvfs.

Proposition 2.8. *Given (μ^j, S^j), the corresponding brvf $\bar{J}^i$ satisfies*

(a) $R^i \leq \bar{J}^i$ *on* $\mathcal{I}$;

(b) $\bar{J}^i = G^i$ *on* S^j;

(c) for each $x \in S^j$, *if* $G^i(x) > R^i(x)$, *then* $\bar{J}^i > R^i$ *on a neighborhood of* x.

Furthermore, if (μ^i, S^i) *is a pbr to* (μ^j, S^j), *then*

(i) $S^i \cap S^j \cap \{G^i > R^i\} = \emptyset$;

(ii) $S^i \subset \overline{S}^i := \{\bar{J}^i = R^i\}$;

(iii) $\operatorname{supp} \mu^i \setminus S^j \subset \overline{S}^i$ *and* $\operatorname{supp} \mu^i \cap S^j \subset \{\bar{J}^i = G^i\}$;

(iv) $(0, S^i)$ *is also a pbr to* (μ^j, S^j); *more generally,* $(\tilde{\mu}^i, S^i)$ *is a pbr to* (μ^j, S^j) *for any* $\tilde{\mu}^i$ *such that* $\operatorname{supp} \tilde{\mu}^i \subset S^i \cup S^j$.

Except for point (c), the proof of Proposition 2.8 essentially follows along the lines of [12, Proposition 1], and is therefore postponed to the Appendix. Notice that points (i)–(iv) assume that a pbr to (μ^j, S^j) exists.

2.5 Main Results

We are now ready to state our two main results, Theorems 2.9 and 2.10.

Let $\mathcal{M}(\mathcal{I})$ be the set of nonnegative, regular, but non necessarily finite measures $m : \mathcal{B}(\mathcal{I}) \to [0, \infty]$. Our first main result, which may be of independent interest, introduces a convenient topological structure on $\mathcal{M}(\mathcal{I})$.

Theorem 2.9. *Let* ϑ *be the coarsest topology on* $\mathcal{M}(\mathcal{I})$ *such that*

1. *for all* $a, b \in \mathcal{I} \cap \mathbb{Q}$ *such that* $a < b$, *the mapping* $\mathcal{M}(\mathcal{I}) \to [0, \infty] : m \mapsto m((a, b))$ *is lower semicontinuous (lsc)*;

2. *for all* $a, b \in \mathcal{I} \cap \mathbb{Q}$ *such that* $a \leq b$, *the mapping* $\mathcal{M}(\mathcal{I}) \to [0, \infty] : m \mapsto m([a, b])$ *is upper semicontinuous (usc).*

Then $(\mathcal{M}(\mathcal{I}), \vartheta)$ *is a compact absolute retract.*

Our second main result encapsulates our central existence claim.

Theorem 2.10. *Any BWoA admits an MPE.*

Notice that the MPE whose existence is asserted in Theorem 2.10 may well have to involve randomized stopping times. Indeed, we provide in Section 7 an example of a BWoA that admits no pure-strategy MPE.

2.6 An Overview of the Argument

The proof of Theorem 2.10 is based on a fixed-point theorem for correspondences applied to a slightly modified version of the pbr correspondence (2.14). In this section, we outline the main steps of the proof, emphasizing the central role played by Theorem 2.9.

An Alternative Representation of Markov Strategies First, it is useful to identify a pair (μ, S), where $S \subset \mathcal{I}$ is a closed set and μ is a Radon measure on $\mathcal{I} \setminus S$, with a measure in $\mathcal{M}(\mathcal{I})$ that is identically ∞ on S. Precisely, given such a pair (μ, S), let $m : \mathcal{B}(\mathcal{I}) \to [0, \infty]$ be the measure defined by

$$m(A) := \begin{cases} \mu(A) & \text{if } A \cap S = \emptyset \\ \infty & \text{if } A \cap S \neq \emptyset \end{cases}, \quad A \in \mathcal{B}(\mathcal{I}). \tag{2.15}$$

That m is regular and, hence, belongs to $\mathcal{M}(\mathcal{I})$, follows directly from (2.15) and from μ being a Radon measure on $\mathcal{I} \setminus S$. Conversely, given $m \in \mathcal{M}(\mathcal{I})$, let $e(m)$ be the explosion set of m, defined as

$$e(m) := \{x \in \mathcal{I} : \forall \varepsilon > 0,\, m(N_\varepsilon(x)) = \infty\},$$

where $N_\varepsilon(x) := (x - \varepsilon, x + \varepsilon) \cap \mathcal{I}$.

Lemma 2.11. *For each $m \in \mathcal{M}(\mathcal{I})$, the set $e(m)$ is closed and $m|_{\mathcal{I} \setminus e(m)}$ is a Radon measure on $\mathcal{I} \setminus e(m)$. Moreover, if $A \in \mathcal{B}(\mathcal{I})$ is such that $A \cap e(m) \neq \emptyset$, then $m(A) = \infty$.*

Proof. First, if $x_n \to x$ with $x_n \in e(m)$ for all $n \geq 0$, then, for each $\varepsilon > 0$, $|x_n - x| < \frac{\varepsilon}{2}$ for any sufficiently large n, so that

$$m(N_\varepsilon(x)) \geq m(N_{\frac{\varepsilon}{2}}(x_n)) = \infty,$$

proving that $e(m)$ is closed. Next, by definition, every point $x \in \mathcal{I} \setminus e(m)$ has a neighborhood with finite m-measure, which implies the second assertion. Finally, the last assertion is a direct consequence of the regularity of m and of the definition of $e(m)$. The result follows. □

Using Lemma 2.11, we can define a mapping

$$m \mapsto (\mu, S) := (m|_{\mathcal{I} \setminus e(m)}, e(m)) \tag{2.16}$$

that associates to each $m \in \mathcal{M}(\mathcal{I})$ a pair (μ, S) such that S is a closed subset of $\mathcal{I}$ and μ is a Radon measure on $\mathcal{I} \setminus S$. By (2.15), this mapping is one-to-one and onto, which allows us to identify a pair (μ, S) with the corresponding measure m, and thus the set of Markovian randomized stopping times with $\mathcal{M}(\mathcal{I})$. With some abuse of notation, we will accordingly write $(\mu, S) \in \mathcal{M}(\mathcal{I})$.

A Fixed-Point Theorem Proving that an MPE exists in any BWoA requires applying an appropriate fixed-point theorem to the pbr correspondence. The main difficulty is that the domain $\mathcal{M}(\mathcal{I})$ of this correspondence is not convex for the two natural vector-space structures we can think of. First, the set of csfs (or ccdfs) associated to Markovian randomized strategies is not convex.[3] Second, because we allow the measures in $\mathcal{M}(\mathcal{I})$ to take the value ∞ on compact sets, $\mathcal{M}(\mathcal{I})$ is not a subset of the vector space of signed locally finite measures. Therefore, we cannot easily apply standard results such as Glicksberg's [22] infinite-dimensional generalization of Kakutani's [25] fixed-point theorem, which requires a convex structure. Our proof is instead based on a more general fixed-point theorem due to Eilenberg and Montgomery [16].[4]

Let us first recall the definition of an absolute retract appearing in Theorem 2.10 as well as the definition of a contractible space appearing in the fixed-point theorem we will use.

Definition 2.12. *A metric space (E, d) is an absolute retract (AR) if, for any continuous map $f : E \to E'$ into a metric space (E', d') such that f is an homeomorphism between E and $f(E)$ and $f(E)$ is closed in E', there exists a continuous map $g : E' \to f(E)$ such that for all $x \in f(E)$, $g(x) = x$ (i.e., $f(E)$ is a retract of E').*

Definition 2.13. *A metric space (E, d) is contractible if there exists a continuous map $H : E \times [0, 1] \to E$ and $x_0 \in E$ such that $H(\cdot, 0) = \mathrm{Id}_E$ and $H(\cdot, 1) = x_0$ (i.e., the identity map is homotopic to a constant map).*

The following result is a corollary of Eilenberg–Montgomery's fixed-point theorem.

Theorem 2.14 ([35, Theorem 14.3])**.** *If (E, d) is a compact AR and $\Phi : E \twoheadrightarrow E$ is a correspondence with a closed graph and nonempty contractible values, then Φ admits a fixed point, i.e., there exists $e^* \in E$ such that $e^* \in \Phi(e^*)$.*

The importance of Theorem 2.9 is now clear. Theorem 2.5 and Lemma 2.11 enable us to identify the set of Markovian randomized stopping times with $\mathcal{M}(\mathcal{I})$, and Theorem 2.9 shows that $\mathcal{M}(\mathcal{I})$ is a compact AR. This provides in turn the required foundation for applying Theorem 2.14.

The Main Steps of the Proof The remainder of the paper is organized as follows:

1. In Section 3, we show that there exists a pbr to any Markovian strategy (Proposition

[3] This can be seen by considering the average of the csfs associated to the hitting times τ_x and τ_y for two points $x \neq y$ in $\mathcal{I}$: the average csf jumps from 1 to $\frac{1}{2}$ at $\tau = \tau_x \wedge \tau_y$, which contradicts (2.10) applied at τ with $s = 0$.

[4] Debreu [9] uses this theorem to prove the existence of a social equilibrium in an abstract economy.

3.1) and we provide a characterization of pbrs (Proposition 3.3). We also introduce a correspondence Φ in (3.12)–(3.13) whose values are subsets of the pbr correspondence (2.14), and to which we will eventually apply Theorem 2.14.

2. In Section 4, we show that the topology ϑ on $\mathcal{M}(\mathcal{I})$ is compact and metrizable and extends the classical vague topology for Radon measures to the whole set $\mathcal{M}(\mathcal{I})$ (Proposition 4.1). We also show that convergence for this topology implies almost sure weak convergence of the associated csfs (Proposition 4.3).

3. In Section 5, we show that the correspondence Φ has a closed graph.

4. In Section 6, we prove Theorem 2.9 and we show that the correspondence Φ has contractible values. Except for the tools from general topology we use, the proof is relatively elementary, and is based on classical convolutions and orthogonal projections. Applying Theorem 2.14 to Φ finally concludes the proof of Theorem 2.10.

5. Finally, in Section 7, we present an example of game which does not admit any MPE in pure stopping times, but admits an MPE in randomized stopping times that has a similar structure as the MPEs identified in [12] in a more specific framework.

3 Existence and Characterization of Pbrs

3.1 Existence of Pure Pbrs

Let us fix $(\mu^j, S^j) \in \mathcal{M}(\mathcal{I})$ and consider the following optimal stopping problem:

$$\bar{J}^i(x) := \sup_{\gamma^i \in \mathcal{T}_r} J^i(x, \gamma^i, (\mu^j, S^j)) = \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j)), \quad x \in \mathcal{I}, \tag{3.1}$$

where the second equality follows from Lemma 2.6. By Definition 2.7, the Markovian randomized stopping times that are optimal in (3.1) for all x are the pbrs to (μ^j, S^j). By (2.8), we have

$$\forall \tau^i \in \mathcal{T}, \; J^i(x, \tau^i, (\mu^j, S^j)) = \mathbb{E}_x[Y_{\tau^i}],$$

where

$$Y_t := \int_{[0,t)} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s + \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-}, \quad t \geq 0. \tag{3.2}$$

Notice that $Y_\infty = Y_{\infty-} = \int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s$. Therefore, the problem

$$\bar{J}^i(x) = \sup_{\tau^i \in \mathcal{T}} \mathbb{E}_x[Y_{\tau^i}] \tag{3.3}$$

falls into the general theory of optimal stopping, from which we will borrow several results below. It follows from Proposition 2.8 and $\overline{S}^i = \{\bar{J}^i = R^i\}$ that

$$S^j \cap \overline{S}^i \subset \{G^i = R^i\} \text{ and } S^j \setminus \overline{S}^i \subset \{G^i > R^i\}. \tag{3.4}$$

Notice that the set $\overline{S}^i$ may be empty. We now prove that a pure pbr exists.

Proposition 3.1. *$\overline{S}^i$ is closed and $(0, \overline{S}^i)$ is a pbr to (μ^j, S^j).*

Proof. The proof consists of five steps.

Step 1 Observe first from $G^i \geq R^i$ that the process $Y := (Y_t)_{t\geq 0}$ defined in (3.2) has càglàd and lsc trajectories; specifically, the only potential discontinuity is at τ_{S^j} whenever $\tau_{S^j} < \infty$. Observe also that the value function of problem (3.3) is not modified if we replace Y with its right-continuous modification $\bar{Y}$, defined by

$$\bar{Y}_t := Y_{t+} = \int_{[0,t]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s + \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_t, \quad t \geq 0. \tag{3.5}$$

Indeed, $\bar{Y} \geq Y$ and thus

$$\bar{J}^i(x) \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[\bar{Y}_\tau], \tag{3.6}$$

and the reverse inequality follows from the fact that, by dominated convergence,

$$\mathbb{E}_x[\bar{Y}_\tau] = \lim_{n\to\infty} \mathbb{E}_x[Y_{\tau+\frac{1}{n}}]. \tag{3.7}$$

Notice that $\bar{Y}$ has càdlàg and usc trajectories. These remarks lead us to consider the problem

$$\bar{J}^i(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x[\bar{Y}_\tau]. \tag{3.8}$$

By Assumption A1, the processes Y and $\bar{Y}$ are of class (D). Therefore, $\bar{J}^i$ is analytically measurable [18, Proposition 2.4], and thus universally measurable. In particular, for each $\tau \in \mathcal{T}$, $\bar{J}^i(X_\tau)$ defines a random variable on Ω. For each $x \in \mathcal{I}$, let Z^x denote the Snell envelope of $\bar{Y}$ on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$. It is known (see [13, Appendix 1, §22] and [17, Theorem 2.28 and Proposition 2.29]) that Z^x is a strong optional supermartingale of class (D) with almost surely càdlàg paths and that

$$\forall \tau \in \mathcal{T},\ Z^x_\tau = \operatorname*{ess\,sup}_{\rho\geq\tau,\, \rho\in\mathcal{T}} \mathbb{E}_x[\bar{Y}_\rho \,|\, \mathcal{F}_\tau].$$

Using the same argument as in (3.6)–(3.7) with conditional expectations, one can check that Z^x is also the Snell envelope of Y.

Step 2 We first claim that the Snell envelope Z^x is indistinguishable under $\mathbb{P}_x$ from the process $\widehat{Z}$ defined by

$$\widehat{Z}_t := \int_{[0,t]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_t\,\mathrm{e}^{-rt}\bar{J}^i(X_t), \quad t \geq 0. \tag{3.9}$$

First, it follows from [18] that, for every stopping time τ of the canonical filtration $(\mathcal{F}^0_t)_{t\geq 0}$,

$$Z^x_\tau = \widehat{Z}_\tau \ \mathbb{P}_x\text{-a.s.} \tag{3.10}$$

The proof of (3.10) is detailed in the Appendix for the sake of completeness. To prove that Z^x and $\widehat{Z}$ are indistinguishable, it is sufficient to show that $\bar{J}^i$ is continuous on $\mathcal{I}\setminus S^j$ and that $\widehat{Z}$ has càdlàg trajectories.

As for the continuity of $\bar{J}^i$ on $\mathcal{I}\setminus S^j$, let $x\notin S^j$ and, for each $n\in\mathbb{N}$, consider the $(\mathcal{F}^0_t)_{t\geq 0}$-stopping time τ_n defined as the first exit time by X of an interval $(x-\delta, x+\varepsilon_n)$ for an arbitrary sequence $\varepsilon_n\to 0$ in $\mathbb{R}_+$ and a fixed $\delta>0$. We have $Z^x_0 = \bar{J}^i(x)$ $\mathbb{P}_x$-a.s., and, as Z^x is $\mathbb{P}_x$-a.s. right-continuous and of class (D),

$$\begin{aligned}
\bar{J}^i(x) &= Z^x_0 \\
&= \lim_{n\to\infty} \mathbb{E}_x[Z^x_{\tau_n}] \\
&= \lim_{n\to\infty} \mathbb{E}_x[\widehat{Z}_{\tau_n}] \\
&= \lim_{n\to\infty} \mathbb{E}_x\left[\int_{[0,\tau_n]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_n}\mathrm{e}^{-r\tau_n}\bar{J}^i(X_{\tau_n})\right],
\end{aligned}$$

where the third equality follows from (3.10). Using that $\bar{J}^i(X_{\tau_n}) = \bar{J}^i(x+\varepsilon_n)$ on $\{X_{\tau_n} = x+\varepsilon_n\}$, we have, for some constant $C>0$,

$$\left|\mathbb{E}_x\left[\int_{[0,\tau_n]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_n}\mathrm{e}^{-r\tau_n}\bar{J}^i(X_{\tau_n})\right] - \bar{J}^i(x+\varepsilon_n)\right| \leq C\mathbb{E}_x[\Gamma^j_{\tau_n} + \mathbb{1}_{X_{\tau_n}=x-\delta}] \to 0.$$

This implies that $\lim_{n\to\infty}\bar{J}^i(x+\varepsilon_n) = \bar{J}^i(x)$, and thus that $\bar{J}^i$ is right-continuous at x as the sequence $(\varepsilon_n)_{n\geq 0}$ in $\mathbb{R}_+$ is arbitrary. The proof of the left-continuity of $\bar{J}^i$ on $\mathcal{I}\setminus S^j$ is similar and thus omitted.

The continuity of $\bar{J}^i$ on $\mathcal{I}\setminus S^j$ implies that $\widehat{Z}$ has càdlàg trajectories, with a single potential discontinuity at τ_{S^j}. Therefore, the processes Z^x and $\widehat{Z}$ are indistinguishable. The claim follows. As the Snell envelope is defined up to an evanescent set, for all $x\in\mathcal{I}$, $\widehat{Z}$ is the Snell envelope of $\bar{Y}$ on the stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P}_x)$. We will use this fact in the subsequent steps.

Step 3 We now prove that $\overline{S}^i$ is closed. Consider a sequence $(x_n)_{n\geq 0}$ in $\overline{S}^i$ converging to $x \in \mathcal{I}$. We need to show that $x \in \overline{S}^i$. If $x \notin S^j$, then $x \in \overline{S}^i$ as $\bar{J}^i$ is continuous on $\mathcal{I} \setminus S^j$ by Step 2. If $x \in S^j$, then it must be that $G^i(x) = R^i(x)$; otherwise, by Proposition 2.8(c), there would exist a neighborhood of x which does not intersect $\overline{S}^i$, a contradiction. By Proposition 2.8(b), $\bar{J}^i(x) = G^i(x)$ as $x \in S^j$. Thus $\bar{J}^i(x) = G^i(x) = R^i(x)$, which implies that $x \in \overline{S}^i$. This concludes the proof that $\overline{S}^i$ is closed.

Step 4 We next claim that $\tau_{\overline{S}^i} \wedge \tau_{S^j}$ is optimal for (3.8). As $\bar{Y}$ has usc trajectories and is of class (D), it follows from [17, Theorem 2.41] that

$$\tau^* := \inf\{t \geq 0 : \widehat{Z}_t = \bar{Y}_t\}$$

is the smallest optimal stopping time for (3.8). In turn [17, Theorem 2.31], the optimality of τ^* in (3.8) is equivalent to the facts that:

- $\widehat{Z}$ is a martingale up to τ^*;
- $\widehat{Z}_{\tau^*} = \bar{Y}_{\tau^*}$.

Let us focus on the second condition. From (3.5) and (3.9), $\widehat{Z}_t = \bar{Y}_t$ is equivalent to

$$\Lambda^j_t[\bar{J}^i(X_t) - R^i(X_t)] = 0.$$

Recalling that $\inf\{t \geq 0 : \Lambda^j_t = 0\} = \tau_{S^j}$, we deduce that

$$\tau^* = \inf\{t \geq 0 : \bar{J}^i(X_t) = R^i(X_t)\} \wedge \tau_{S^j} = \tau_{\overline{S}^i} \wedge \tau_{S^j} \text{ a.s.}$$

The claim follows.

Step 5 We finally prove that $\tau_{\overline{S}^i}$ is optimal for (3.3). For each $x \in \mathcal{I}$,

$$\begin{aligned}
\bar{J}^i(x) &= \mathbb{E}_x[\bar{Y}_{\tau^*}] \\
&= \mathbb{E}_x[\widehat{Z}_{\tau^*}] \\
&= \mathbb{E}_x\left[\int_{[0,\tau^*]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau^*}\,\mathrm{e}^{-r\tau^*}\bar{J}^i(X_{\tau^*})\right] \\
&= \mathbb{E}_x\left[\mathbb{1}_{\tau_{\overline{S}^i} < \tau_{S^j}}\left[\int_{[0,\tau_{\overline{S}^i}]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}}\mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}})\right]\right. \\
&\qquad + \mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}}\left[\int_{[0,\tau_{\overline{S}^i}]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}}\mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}})\right] \\
&\qquad \left. + \mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}}\left[\int_{[0,\tau_{S^j}]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{S^j}}\mathrm{e}^{-r\tau_{S^j}}\bar{J}^i(X_{\tau_{S^j}})\right]\right],
\end{aligned}$$

where the first and second equalities follow from the optimality of τ^* in (3.8), and the fourth equality follows from $\bar{J}^i(X_{\tau_{\overline{S}^i}}) = R^i(X_{\tau_{\overline{S}^i}})$ as $\overline{S}^i$ is closed.

Let us examine the three terms in the last expression separately. For the first one, using that Λ^j is continuous on $[0, \tau_{S_j})$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\tau_{\overline{S}^i} < \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i}]} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} < \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i})} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}-} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} < \tau_{S^j}} Y_{\tau_{\overline{S}^i}}.
\end{aligned}$$

For the second one, using that $G^i = R^i$ on $S^j \cap \overline{S}^i$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i}]} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i})} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \mathrm{e}^{-r\tau_{\overline{S}^i}} (\Lambda^j_{\tau_{\overline{S}^i}-} - \Lambda^j_{\tau_{\overline{S}^i}}) \, G^i(X_{\tau_{\overline{S}^i}}) + \Lambda^j_{\tau_{\overline{S}^i}} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i})} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}-} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{\overline{S}^i}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}} Y_{\tau_{\overline{S}^i}}.
\end{aligned}$$

For the third one, using that $\Lambda^j = 0$ on $[\tau_{S^j}, \infty)$, we obtain

$$\begin{aligned}
&\mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}} \left[\int_{[0,\tau_{S^j}]} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{S^j}} \mathrm{e}^{-r\tau_{S^j}} R^i(X_{\tau_{S^j}}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}} \int_{[0,\tau_{S^j}]} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s \\
&= \mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}} \left[\int_{[0,\tau_{\overline{S}^i})} \mathrm{e}^{-rs} G^i(X_s) \, \mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{\overline{S}^i}-} \mathrm{e}^{-r\tau_{\overline{S}^i}} R^i(X_{\tau_{S^i}+}) \right] \\
&= \mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}} Y_{\tau_{\overline{S}^i}}.
\end{aligned}$$

Gathering these three equalities, we obtain:

$$\bar{J}^i(x) = \mathbb{E}_x \big[\big(\mathbb{1}_{\tau_{\overline{S}^i} < \tau_{S^j}} + \mathbb{1}_{\tau_{\overline{S}^i} = \tau_{S^j}} + \mathbb{1}_{\tau_{\overline{S}^i} > \tau_{S^j}} \big) Y_{\tau_{\overline{S}^i}} \big] = \mathbb{E}_x \big[Y_{\tau_{\overline{S}^i}} \big],$$

from which it follows that $\tau_{\overline{S}^i}$ is optimal in problem (3.3) and thus that $(0, \overline{S}^i)$ is a pbr to (μ^j, S^j). Hence the result. □

It follows from this result and the definition of $\overline{S}^i$ that $\overline{S}^i$ is the largest set over which it is optimal for player i to stop in a pbr to (μ^j, S^j). Equilibrium may however require that player i stop on a smaller set, and possibly mix on the complement of this set. Thus we need to characterize all pbrs to (μ^j, S^j), a task to which we now turn.

3.2 Characterization of Pbrs

Define the set

$$\Sigma^i := \{S^i \subset \mathcal{I} \text{ closed} : (0, S^i) \text{ is a pbr to } (\mu^j, S^j)\}.$$

The characterization of pbrs relies on the following lemma.

Lemma 3.2. *The set Σ^i is nonempty, stable by intersection, and has a smallest element $\underline{S}^i$. Moreover,*

$$\Sigma^i = \{S^i \subset \mathcal{I} \textit{ closed} : \underline{S}^i \subset S^i \subset \overline{S}^i\}.$$

Proof. By Proposition 3.1, $\overline{S}^i \in \Sigma^i$, so that Σ^i is nonempty. Moreover, by Proposition 2.8(ii), every element of Σ^i is a subset of $\overline{S}^i$. Now, let $S^i \subset \overline{S}^i$ be closed and recall that $R^i = G^i$ on $S^j \cap \overline{S}^i$ by (3.4). We have a.s.

$$\begin{aligned}
\widehat{Z}_{\tau_{S^i}} &= \int_{[0,\tau_{S^i}]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{S^i}} \mathrm{e}^{-r\tau_{S^i}} \bar{J}^i(X_{\tau_{S^i}}) \\
&= \int_{[0,\tau_{S^i}]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{S^i}} \mathrm{e}^{-r\tau_{S^i}} R^i(X_{\tau_{S^i}}) \\
&= \int_{[0,\tau_{S^i})} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau_{S^i}-} \mathrm{e}^{-r\tau_{S^i}} R^i(X_{\tau_{S^i}}) \\
&= Y_{\tau_{S^i}},
\end{aligned} \tag{3.11}$$

where the first equality follows from (3.9)–(3.10), the second equality follows from $X_{\tau_{S^i}} \in \overline{S}^i$, and the third equality follows from the fact that either $\Delta\Gamma^j_{\tau_{S^i}} = 0$ or $\tau_{S^i} = \tau_{S^j}$, the latter implying that $R^i(X_{\tau_{S^i}}) = G^i(X_{\tau_{S^i}})$ as in the proof of Proposition 3.1. The remainder of the proof consists of two steps.

Step 1 We first show that Σ^i is stable by intersection, i.e., that, given two subsets S^i and $\widehat{S}^i$ of Σ^i, the stopping time $\tau_{S^i \cap \widehat{S}^i} = \tau_{S^i} \vee \tau_{\widehat{S}^i}$ is optimal in (3.3). By [17, Theorem 2.31], it is sufficient to prove that $Y_{\tau_{S^i \cap \widehat{S}^i}} = \widehat{Z}_{\tau_{S^i \cap \widehat{S}^i}}$ and that $\widehat{Z}$ is a martingale up to $\tau_{S^i \cap \widehat{S}^i}$. The first property follows from (3.11). As for the martingale property, because $\widehat{Z}$ is a supermartingale, we only need to verify that for each $x \in \mathcal{I}$, $\mathbb{E}_x\left[\widehat{Z}_{\tau_{S^i \cap \widehat{S}^i}}\right] = \mathbb{E}_x[\widehat{Z}_0]$. We have

$$\begin{aligned}
\mathbb{E}_x\left[\widehat{Z}_{\tau_{S^i \cap \widehat{S}^i}}\right] &= \mathbb{E}_x\left[\widehat{Z}_{\tau_{S^i}} \mathbb{1}_{\tau_{\widehat{S}^i} \leq \tau_{S^i}} + \widehat{Z}_{\tau_{\widehat{S}^i}} \mathbb{1}_{\tau_{\widehat{S}^i} > \tau_{S^i}}\right] \\
&= \mathbb{E}_x\left[\widehat{Z}_{\tau_{\widehat{S}^i}} \mathbb{1}_{\tau_{\widehat{S}^i} \leq \tau_{S^i}} + \widehat{Z}_{\tau_{\widehat{S}^i}} \mathbb{1}_{\tau_{\widehat{S}^i} > \tau_{S^i}}\right] \\
&= \mathbb{E}_x\left[\widehat{Z}_{\tau_{\widehat{S}^i}}\right] \\
&= \mathbb{E}_x[\widehat{Z}_0]
\end{aligned}$$

where the second and fourth equalities follow from the fact that $\widehat{Z}$ is a martingale up to τ_{S^i} and $\tau_{\widehat{S}^i}$, respectively. We conclude that $\tau_{S^i \cap \widehat{S}^i} = \tau_{S^i} \vee \tau_{\widehat{S}^i}$ is indeed optimal in (3.3) and thus that $S^i \cap \widehat{S}^i \in \Sigma^i$.

Step 2 Now, let us define $\underline{S}^i := \bigcap_{S^i \in \Sigma^i} S^i$. Because $\mathcal{I} \setminus \underline{S}^i$ is the union of the open sets $\mathcal{I} \setminus S^i$ for $S^i \in \Sigma^i$, which admits a countable subcover as any open subset of the real line is a Lindelöf space, there exists a sequence $(S^i_n)_{n \geq 0}$ in Σ^i such that $\underline{S}^i = \bigcap_{n \geq 0} S^i_n$. As a result, $\underline{S}^i$ is the intersection of the nonincreasing sequence of closed sets $(\widehat{S}^i_n)_{n \geq 0} := (\bigcap_{p=1}^n S^i_p)_{n \geq 0}$ in Σ^i. The sequence $(\tau_{\widehat{S}^i_n})_{n \geq 0}$ is nondecreasing, and thus $\lim_{n \to \infty} \tau_{\widehat{S}^i_n}$ is well-defined. We claim that $\tau_{\underline{S}^i} = \lim_{n \to \infty} \tau_{\widehat{S}^i_n}$. It is clear that $\tau_{\underline{S}^i} \geq \lim_{n \to \infty} \tau_{\widehat{S}^i_n}$. If this limit is infinite, then the equality holds. If this limit is finite, then $X_{\lim_{n \to \infty} \tau_{\widehat{S}^i_n}}$ belongs to $\underline{S}^i$ and thus $\tau_{\underline{S}^i} \leq \lim_{n \to \infty} \tau_{\widehat{S}^i_n}$, so that the equality again holds. The claim follows. Because $\tau_{\widehat{S}^i_n}$ is optimal in (3.3), it follows from (3.11) that $\bar{J}^i(x) = \mathbb{E}_x\big[\widehat{Z}_{\tau_{\widehat{S}^i_n}}\big] = \mathbb{E}_x\big[Y_{\widehat{S}^i_n}\big]$ for all $n \geq 0$. On the other hand, (3.11) implies $\widehat{Z}_{\tau_{\underline{S}^i}} = Y_{\tau_{\underline{S}^i}}$ a.s. and we conclude that $\bar{J}^i(x) = \mathbb{E}_x\big[\widehat{Z}_{\tau_{\underline{S}^i}}\big] = \mathbb{E}_x\big[Y_{\underline{S}^i}\big]$ by dominated convergence using that Y is left-continuous. In particular, $\underline{S}^i \in \Sigma^i$ and $\underline{S}^i$ is the smallest element of Σ^i. It follows that $\Sigma^i \subset \{S^i \subset \mathcal{I} \text{ closed} : \underline{S}^i \subset S^i \subset \overline{S}^i\}$. To prove the reverse inclusion, it suffices to notice that, if $\underline{S}^i \subset S^i \subset \overline{S}^i$, then (3.11) implies $\widehat{Z}_{\tau_{S^i}} = Y_{\tau_{S^i}}$ and that $\widehat{Z}$ is a martingale up to τ_{S^i} as it is a martingale up to $\tau_{\underline{S}^i} \geq \tau_{S^i}$. The result follows. □

We are now ready to characterize the set of pbrs to (μ^j, S^j).

Proposition 3.3. *(μ^i, S^i) is a pbr to (μ^j, S^j) if and only if $\underline{S}^i \subset S^i \subset \overline{S}^i$ and μ^i is a Radon measure on $\mathcal{I} \setminus S^i$ that is concentrated on $(\overline{S}^i \setminus S^i) \cup S^j$.*

Proof. If (μ^i, S^i) is a pbr to (μ^j, S^j), then, by Proposition 2.8(iv), $(0, S^i)$ is also a pbr to (μ^j, S^j) and thus $S^i \in \Sigma^i$, which proves the inclusions by Lemma 3.2. The second point follows directly from Proposition 2.8-(iii–iv). Conversely, if $\underline{S}^i \subset S^i \subset \overline{S}^i$ and μ^i is a Radon measure on $\mathcal{I} \setminus S^i$ that is concentrated on $(\overline{S}^i \setminus S^i) \cup S^j$, then $(0, S^i)$ is a pbr to (μ^j, S^j) by Lemma 3.2, and (μ^i, S^i) is a pbr to (μ^j, S^j) by Proposition 2.8-(iv). Hence the result. □

3.3 The Correspondence Φ

We now consider a correspondence Φ^i whose values are nonempty subsets of the values of PBR^i. Specifically, for each $(\mu^j, S^j) \in \mathcal{M}(\mathcal{I})$, let

$$\Phi^i(\mu^j, S^j) := \{(\mu^i, S^i) \in \mathcal{M}(\mathcal{I}) : \underline{S}^i \subset S^i \subset \overline{S}^i \text{ and } \mu^i \text{ is concentrated on } \overline{S}^i \setminus S^i\}. \tag{3.12}$$

We will apply the fixed-point Theorem 2.14 to the correspondence $\Phi : \mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I}) \twoheadrightarrow$

$\mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I})$ defined by

$$\Phi((\mu^1, S^1), (\mu^2, S^2)) := \Phi^1(\mu^2, S^2) \times \Phi^2(\mu^1, S^1). \tag{3.13}$$

Our approach is justified by the fact that Φ^i takes values in the set of pbrs of player i, as the following result shows.

Lemma 3.4. *For all $(\mu^j, S^j) \in \mathcal{M}(\mathcal{I})$,*

$$\Phi^i(\mu^j, S^j) = \{(\mu^i, S^i) \in PBR^i(\mu^j, S^j) : \mu^i(S^j \cap \{G^i > R^i\}) = 0\}. \tag{3.14}$$

Proof. Let $(\mu^i, S^i) \in \Phi^i(\mu^j, S^j)$. By Proposition 3.3, $(0, S^i) \in PBR^i(\mu^j, S^j)$. By Proposition 2.8(iv), because μ^i is concentrated on $\overline{S}^i \setminus S^i \subset \overline{S}^i \cup S^j$, $(\mu^i, S^i) \in PBR^i(\mu^j, S^j)$. By (3.4), $S^j \cap \overline{S}^i \subset \{G^i = R^i\}$, and thus

$$\mu^i(S^j \cap \{G^i > R^i\}) = \mu^i(S^j \cap \overline{S}^i \cap \{G^i > R^i\}) = 0.$$

Conversely, let $(\mu^i, S^i) \in PBR^i(\mu^j, S^j)$ such that $\mu^i(S^j \cap \{G^i > R^i\}) = 0$. By Proposition 3.3, $\underline{S}^i \subset S^i \subset \overline{S}^i$ and μ^i is concentrated on $(\overline{S}^i \setminus S^i) \cup S^j$. By (3.4), $S^j \setminus \overline{S}^i \subset S^j \cap \{G^i > R^i\}$, and thus

$$\mu^i(S^j \setminus \overline{S}^i) \leq \mu^i(S^j \cap \{G^i > R^i\}) = 0.$$

Hence, μ^i is concentrated on $\overline{S}^i \setminus S^i$, so that $(\mu^i, S^i) \in \Phi^i(\mu^j, S^j)$. The result follows. □

4 A Compact Topology on $\mathcal{M}(\mathcal{I})$

Recall that $\mathcal{M}(\mathcal{I})$ denotes the set of nonnegative regular measures $m : \mathcal{B}(\mathcal{I}) \to [0, \infty]$, i.e., such that[5]

$$m(A) = \inf\{m(O) : A \subset O,\ O \text{ open}\} = \sup\{m(K) : K \subset A, K \text{ compact}\}, \quad A \in \mathcal{B}(\mathcal{I}).$$

The proof of the following result follows along more or less standard lines (see, e.g., [26, chapter 4]) and is therefore postponed to the Appendix.

Proposition 4.1. *The topology ϑ on $\mathcal{M}(\mathcal{I})$ defined in Theorem 2.9 is metrizable and compact. Moreover,*

1. *for every open set $O \subset \mathcal{I}$, the mapping $\mathcal{M}(\mathcal{I}) \to [0, \infty] : m \mapsto m(O)$ is lsc;*

[5] It is well-known that Radon measures on $\mathcal{I}$ are regular, see Footnote 2. However, this is not true for arbitrary nonnegative measures on $\mathcal{I}$, as, e.g., for the measure defined by $m(A) = \infty$ if $A \cap \mathbb{Q} \neq \emptyset$ and 0 otherwise, which is not outer regular.

2. for every compact set $K \subset \mathcal{I}$, the mapping $\mathcal{M}(\mathcal{I}) \to [0,\infty] : m \mapsto m(K)$ is usc;

3. a sequence $(m_n)_{n\geq 0}$ converges to m if and only if

- *for every open set O such that $O \cap \mathrm{e}(m) \neq \emptyset$, $m_n(O) \to \infty$;*
- *letting $L_\phi(m) := \int_{\mathcal{I}} \phi \, \mathrm{d}m$, $L_\phi(m_n) \to L_\phi(m)$ for all $\phi \in \mathcal{C}_c^+(\mathcal{I} \setminus \mathrm{e}(m))$.*

Remark 4.2. *The second part of point (3) implies that the topology induced by ϑ on the subset $\mathcal{M}_{\mathrm{loc}}(\mathcal{I})$ of locally finite, i.e., Radon measures coincide with the usual vague topology, as a sequence $(m_n)_{n\geq 0}$ in $\mathcal{M}_{\mathrm{loc}}(\mathcal{I})$ converges vaguely to $m \in \mathcal{M}_{\mathrm{loc}}(\mathcal{I})$ if $L_\phi(m_n) \to L_\phi(m)$ for all $\phi \in \mathcal{C}_c^+(\mathcal{I})$ (see, e.g., [26, Chapter 4]).*

The next result is very important as it will allow us to prove convergence of expected payoffs under appropriate assumptions.

Proposition 4.3. *Suppose that $(\mu_n, S_n) \to (\mu, S)$ in $\mathcal{M}(\mathcal{I})$, and let Λ^n, Λ denote the csfs associated with (μ_n, S_n) and (μ, S), respectively. Then*

$$\forall t \neq \tau_S, \; \Lambda_t^n \to \Lambda_t \; a.s.$$

Proof. For each $t \geq 0$, we have

$$\Lambda_t^n = \mathbb{1}_{t<\tau_{S_n}} \mathrm{e}^{-\int_{\mathcal{I}\setminus S_n} L_t^y \, \mu_n(\mathrm{d}y)} \text{ and } \Lambda_t = \mathbb{1}_{t<\tau_S} \mathrm{e}^{-\int_{\mathcal{I}\setminus S} L_t^y \, \mu(\mathrm{d}y)}. \tag{4.1}$$

Recall that by convention $\Lambda_\infty^n = \Lambda_\infty := 0$, so that Λ^n, Λ define for each $\omega \in \Omega$ the survival function of a probability measure on $[0,\infty]$. Let $x \in \mathcal{I}$. We distinguish two cases.

Case 1: $t < \tau_S$ Notice that there exists a set Ω_1 of $\mathbb{P}_x$-probability 1 such that, for each $\omega \in \Omega_1$, the mapping $(t,y) \mapsto L_t^y(\omega)$ is continuous (see, e.g., [38, Chapter VI, §1, Theorem 1.7]). Define $M_t = \max_{0\leq s\leq t} X_s$ and $m_t = \min_{0\leq s\leq t} X_s$. Using the occupation time formula [38, Chapter VI, §1, Corollary 1.6]), for every interval $A \subset \mathcal{I}$,

$$\int_0^t \mathbb{1}_A(X_s)\sigma^2(X_s)\,\mathrm{d}s = \int_{\mathcal{I}} \mathbb{1}_A(y) L_t^y \,\mathrm{d}y \;\; \mathbb{P}_x\text{-a.s.}$$

Therefore, there exists a set Ω_2 of $\mathbb{P}_x$-probability 1 such that the above equality holds for all A with rational endpoints. Now, notice that $\{t < \tau_S\} = \{[m_t, M_t] \subset \mathcal{I} \setminus S\}$. Fix $\omega \in \Omega_1 \cap \Omega_2$ such that $\tau_S(\omega) > 0$. Then, for $t < \tau_S(\omega)$, we have, for every interval $A \subset \mathcal{I} \setminus [m_t(\omega), M_t(\omega)]$ with rational endpoints,

$$0 = \int_0^t \mathbb{1}_A(X_s(\omega))\sigma^2(X_s(\omega))\,\mathrm{d}s = \int_{\mathcal{I}} \mathbb{1}_A(y) L_t^y(\omega)\,\mathrm{d}y$$

as $X_s \in [m_t, M_t]$ for all $s \in [0,t]$. Using the continuity of $y \mapsto L_t^y(\omega)$, we deduce that $L_t^y(\omega) = 0$ for all $y \in \mathcal{I} \setminus [m_t(\omega), M_t(\omega)]$, and therefore that $y \mapsto L_t^y(\omega)$ is a continuous function with compact support $K_t(\omega) \subset \mathcal{I} \setminus S$. Now, for each $n \geq 0$, let m_n denote the measure in $\mathcal{M}(\mathcal{I})$ associated to (μ_n, S_n). Because the restriction of m_n to $\mathcal{I} \setminus S$ converges vaguely to μ by Theorem 4.1(3), $K_t(\omega) \cap S_n = \emptyset$ for any sufficiently large n, and thus the restrictions of μ_n and m_n to $K_t(\omega)$ coincide. As a result, for each $\omega \in \Omega_1 \cap \Omega_2 \cap \{t < \tau_S\}$, $\int_{\mathcal{I}\setminus S^n} L_t^y(\omega)\,\mu_n(\mathrm{d}y) = \int_{K_t(\omega)} L_t^y(\omega)\,\mu_n(\mathrm{d}y) \to \int_{K_t(\omega)} L_t^y(\omega)\,\mu(\mathrm{d}y) = \int_{\mathcal{I}\setminus S} L_t^y(\omega)\,\mu(\mathrm{d}y)$ and thus $\Lambda_t^n(\omega) \to \Lambda_t(\omega)$ by (4.1).

Case 2: $t > \tau_S$ We first claim that, for each $z \in \mathcal{I}$, letting τ_z denote the hitting time of z, we have, for every event $A \in \mathcal{F}_{\tau_z}$

$$\mathbb{P}_x[L_t^z > 0, t > \tau_z, A] = \mathbb{P}_x[t > \tau_z, A]. \tag{4.2}$$

First, because $t \mapsto L_t^z$ is a strongly additive functional of the diffusion process X [6, Part I, Chapter II, Section 2, §13, and Section 4, §21], we have, with $\mathbb{P}_x$-probability 1 on $\{\tau_z < t\}$,

$$L_t^z(\omega) = L_{\tau_z(\omega)}^z(\omega) + L_{t-\tau_z(\omega)}^z(\theta_{\tau_z(\omega)}(\omega)) = L_{t-\tau_z(\omega)}^z(\theta_{\tau_z(\omega)}(\omega)).$$

Then, denoting by $\tilde{\Omega}$ a copy of the canonical space endowed with the probabilities $\tilde{\mathbb{P}}_y = \mathbb{P}_y$ for $y \in \mathcal{I}$,

$$\mathbb{P}_x[L_t^z > 0, t > \tau_z, A] = \mathbb{E}_x[\tilde{\mathbb{P}}_z[L_{t-\tau_z(\omega)}^z(\tilde{\omega}) > 0]\mathbb{1}_{t>\tau_z(\omega)}\mathbb{1}_A(\omega)] = \mathbb{P}_x[t > \tau_z, A],$$

where the first equality follows from the Markov property, and the second equality follows from the fact that $\mathbb{P}_y[L_t^y > 0] = 1$ for all $y \in \mathcal{I}$ and $t > 0$. The claim follows. Now, if $t > \tau_S$, then it must be that either $X_{\tau_S} = x$ if $x \in S$ or $X_{\tau_S} \in \{a,b\} \subset S$ if $x \notin S$, where (a,b) denotes the largest open interval containing x in $\mathcal{I} \setminus S$. We claim that

$$\mathbb{P}_x\left[L_t^{X_{\tau_S}} > 0, t > \tau_S\right] = \mathbb{P}_x[t > \tau_S]. \tag{4.3}$$

If $x \in S$, $X_{\tau_S} = x$ and $\tau_S = 0$, so both sides are equal by the same reasoning as above. If $x \notin S$, we have, again by the same reasoning,

$$\begin{aligned}
\mathbb{P}_x\left[L_t^{X_{\tau_S}} > 0, t > \tau_S\right] &= \mathbb{P}_x[L_t^a > 0, t > \tau_a, X_{\tau_S} = a] + \mathbb{P}_x[L_t^b > 0, t > \tau_b, X_{\tau_S} = b] \\
&= \mathbb{P}_x[t > \tau_a, X_{\tau_S} = a] + \mathbb{P}_x[t > \tau_b, X_{\tau_S} = b] \\
&= \mathbb{P}_x[t > \tau_S].
\end{aligned}$$

The claim follows. Let Ω_3 be a set of $\mathbb{P}_x$-probability 1 such that $L_t^{X_{\tau_S(\omega)}}(\omega) > 0$ for all $\omega \in \Omega_3 \cap \{t > \tau_S\}$. For $\omega \in \Omega_1 \cap \Omega_2 \cap \Omega_3 \cap \{t > \tau_S\}$, the mapping $y \mapsto L_t^y(\omega)$ is continuous, vanishes outside of $[m_t(\omega), M_t(\omega)]$, and $L_t^{X_{\tau_S}(\omega)}(\omega) > 0$. By continuity, it must be that $m_t(\omega) < X_{\tau_S}(\omega) < M_t(\omega)$, and there exist $\varepsilon(\omega), \eta(\omega) > 0$ such that $L_t^y(\omega) \geq \eta(\omega)$ for all $y \in (X_{\tau_S}(\omega) - \varepsilon(\omega), X_{\tau_S}(\omega) + \varepsilon(\omega)) \subset [m_t(\omega), M_t(\omega)]$. Because $(\mu_n, S_n) \to (\mu, S)$ and $X_{\tau_S}(\omega) \in S$, it must be that $m_n((X_{\tau_S}(\omega) - \varepsilon(\omega), X_{\tau_S}(\omega) + \varepsilon(\omega))) \to \infty$, where m_n denotes the measure in $\mathcal{M}(\mathcal{I})$ associated to (μ_n, S_n). Notice that $\mathbb{1}_{t<\tau_{S_n}(\omega)} \neq 0$ if and only if $S_n \cap [m_t(\omega), M_t(\omega)] = \emptyset$, which implies $(X_{\tau_S}(\omega) - \varepsilon(\omega), X_{\tau_S}(\omega) + \varepsilon(\omega)) \subset \mathcal{I} \setminus S_n$ and $m_n((X_{\tau_S}(\omega) - \varepsilon(\omega), X_{\tau_S}(\omega) + \varepsilon(\omega))) = \mu_n((X_{\tau_S}(\omega) - \varepsilon(\omega), X_{\tau_S}(\omega) + \varepsilon(\omega)))$. We deduce that

$$0 \leq \Lambda_t^n(\omega) = \mathbb{1}_{t<\tau_{S_n}(\omega)} \, \mathrm{e}^{-\int_{\mathcal{I}\setminus S_n} L_t^y(\omega)\, \mu_n(\mathrm{d}y)} \leq \mathbb{1}_{t<\tau_{S_n}(\omega)} \, \mathrm{e}^{-\eta(\omega)\mu_n((X_{\tau_S}(\omega)-\varepsilon(\omega), X_{\tau_S}(\omega)+\varepsilon(\omega)))} \to 0,$$

which concludes the proof because $\Lambda_t(\omega) = 0$ by (4.1) as $t > \tau_S(\omega)$. Hence the result. □

5 Closedness of the Graph of Φ

We will use the following classical result.

Theorem 5.1 ([4, Theorems 5.1 and 5.4])**.** *Let E be a Polish space and $(\nu_n)_{n\geq 0}$ a sequence of probability measures on E that converges weakly to ν. Suppose that $f : E \to \mathbb{R}$ is a measurable function such that $\nu(D) = 0$, where D is the set of discontinuity points of f, and that the variables of law $\nu_n \circ f^{-1}$ are uniformly integrable, i.e.,*

$$\lim_{M\to\infty} \sup_{n\geq 0} \int_E |f(x)| \mathbb{1}_{|f(x)|\geq M} \, \nu_n(\mathrm{d}x) = 0.$$

Then

$$\int_E f \, \mathrm{d}\nu_n \to \int_E f \, \mathrm{d}\nu.$$

The following result then holds.

Proposition 5.2. *The correspondence Φ defined by (3.13) has a closed graph.*

Proof. Because Φ is defined as a cartesian product, it is sufficient to prove that, for each $i = 1, 2$, Φ^i has a closed graph in $\mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I})$. Because $\mathcal{M}(\mathcal{I})$ is metrizable, it is sufficient to prove that the graph of Φ^i is sequentially closed. Let us therefore consider a sequence $((\mu_n^i, S_n^i), (\mu_n^j, S_n^j))_{n\geq 0}$ in $\mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I})$ such that, for each $n \geq 0$, $(\mu_n^i, S_n^i) \in \Phi^i(\mu_n^j, S_n^j)$. Assume further that this sequence converges to a limit $((\mu^1, S^1), (\mu^2, S^2))$. We need to prove that $(\mu^i, S^i) \in \Phi^i(\mu^j, S^j)$ or, equivalently, by Lemma 3.4, that (μ^i, S^i) is a pbr to (μ^j, S^j) such that $\mu^i(S^j \cap \{G^i > R^i\}) = 0$.

By Proposition 4.3, for each $t \neq \tau_{S^i}$, $\Lambda_{n,t}^i \to \Lambda_t^i$ a.s., where the processes Λ^i and Λ_n^i are the csfs associated to (μ^i, S^i) and (μ^i_n, S^i_n), respectively. Recall also that

$$J^i(x, \Gamma^i, \Gamma^j) = \mathbb{E}_x\bigg[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\bigg], \tag{5.1}$$

$$J^i(x, \Gamma^i_n, \Gamma^j_n) = \mathbb{E}_x\bigg[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{n,t-}\,\mathrm{d}\Gamma^i_{n,t} + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_{n,t}\,\mathrm{d}\Gamma^j_{n,t}\bigg]. \tag{5.2}$$

The remainder of the proof consists of three steps.

Step 1 We first prove that $S^i \cap S^j \cap \{R^i < G^i\} = \emptyset$. Hence suppose, by way of contradiction, that $x \in S^i \cap S^j \cap \{R^i < G^i\}$. For each $n \geq 0$, because $(\mu^i_n, S^i_n) \in \Phi^i(\mu^j_n, S^j_n)$, we have $\bar{J}^i(y, (\mu^j_n, S^j_n)) = R^i(y)$ for all $y \in S^i_n \cup \operatorname{supp}\mu^i_n$, and therefore stopping immediately gives a weakly larger payoff than never stopping, i.e.,

$$R^i(y) \geq \mathbb{E}_y\bigg[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\bigg] =: \kappa_n(y). \tag{5.3}$$

Now, because $x \in S^i$ and $m^i_n := (\mu_n, S_n) \to (\mu^i, S^i)$, for each $\varepsilon > 0$ we have $m_n((x - \varepsilon, x + \varepsilon)) \to \infty$, and there exists $N(\varepsilon) > 0$ such that $m_n((x - \varepsilon, x + \varepsilon)) > 0$ and thus $(x - \varepsilon, x + \varepsilon) \cap \operatorname{supp} m_n \neq \emptyset$ for all $n \geq N(\varepsilon)$. Hence, because $\operatorname{supp} m_n = S^i_n \cup \operatorname{supp}\mu_n$ for all $n \geq 0$, there exists a sequence $x_n \to x$ such that for each $n \geq 0$, $x_n \in S^i_n \cup \operatorname{supp}\mu^i_n$ and thus satisfies (5.3). Fix some $\varepsilon > 0$ such that $G^i > R^i(x) + \varepsilon$ on $[x - \varepsilon, x + \varepsilon]$. By the above reasoning, we can assume that $x_n \in (x - \varepsilon, x + \varepsilon)$ for n large enough.

Next, observe that the mapping $[0, \infty] \to \mathbb{R} : t \to \mathrm{e}^{-rt} G^i(X_t)$, which is a.s. equal to 0 at ∞ by Assumption A2, is a.s. continuous and bounded, and that the sequence of probabilities over $[0, \infty]$ with csfs $(\Lambda^j_n)_{n\geq 0}$ converges weakly to the probability ν with csf Λ^j by Proposition 4.3. Therefore, we can apply Theorem 5.1 with $E = [0, \infty]$ to obtain

$$\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s} \to \int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s \text{ a.s.}$$

Using Assumption A1 and $x \in S^j$, we conclude by dominated convergence that

$$\kappa_n(x) = \mathbb{E}_x\bigg[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\bigg] \to \mathbb{E}_x\bigg[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s\bigg] = G^i(x).$$

It follows that $\kappa_n(x) \geq R^i(x) + \varepsilon$ for n large enough. Let τ denote the exit time from $(x - \varepsilon, x + \varepsilon)$ and τ_x the hitting time of x. Using the Markov property and (2.10) as in the proof of Proposition 2.8(c), we deduce that

$$\begin{aligned}
&\mathbb{E}_{x_n}\bigg[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\bigg] \\
&= \mathbb{E}_{x_n}\bigg[\int_{[0,\tau_x\wedge\tau)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s} + \mathbb{1}_{\tau_x<\tau}\,\Lambda^j_{n,\tau_x-}\,\mathrm{e}^{-r\tau_x}\kappa_n(x) + \mathbb{1}_{\tau<\tau_x}\int_{[\tau,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\bigg]
\end{aligned}$$

$$\geq [R^i(x)+\varepsilon]\,\mathbb{E}_{x_n}[\mathbb{1}_{\tau_x<\tau}\,\mathrm{e}^{-r\tau_x}] + \mathbb{E}_{x_n}\left[\mathbb{1}_{\tau<\tau_x}\int_{[\tau,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\right]. \tag{5.4}$$

Again, as in the proof of Proposition 2.8(c), there exists a constant $C' > 0$ such that

$$\left|\mathbb{E}_{x_n}\left[\mathbb{1}_{\tau<\tau_x}\int_{[\tau,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\right]\right| \leq C'\mathbb{P}_{x_n}[\tau<\tau_x]. \tag{5.5}$$

We deduce from (5.3) applied to x_n and from (5.4)–(5.5) that

$$R^i(x_n) \geq \mathbb{E}_{x_n}\left[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_{n,s}\right] \geq [R^i(x)+\varepsilon]\,\mathbb{E}_{x_n}\left[\mathbb{1}_{\tau_x<\tau}\mathrm{e}^{-r\tau_x}\right] - C'\mathbb{P}_{x_n}[\tau<\tau_x].$$

The right-hand side of this inequality converges to $R^i(x)+\varepsilon$ as $n\to\infty$, whereas the left-hand side converges to $R^i(x)$, a contradiction. We conclude that $S^i\cap S^j\cap\{R^i<G^i\}=\emptyset$.

Step 2 We now prove that $\mu^i(S^j\cap\{R^i<G^i\})=0$. Suppose, by way of contradiction, that $\mu^i(S^j\cap\{R^i<G^i\})>0$. Then there exists $x\in S^j\cap\{R^i<G^i\}$ such that every neighborhood O of x is such that $\mu^i(O)>0$. Because $x\notin S^i$ by Step 1, it must be that $[x-\varepsilon,x+\varepsilon]\cap S^i=\emptyset$ for any sufficiently small $\varepsilon>0$, and thus $\mu^i([x-\varepsilon,x+\varepsilon])<\infty$. Because $(\mu^i_n,S^i_n)=m^i_n\to m^i=(\mu^i,S^i)$, it follows that

$$\limsup_{n\to\infty} m^i_n([x-\varepsilon,x+\varepsilon]) \leq m^i([x-\varepsilon,x+\varepsilon]) = \mu^i([x-\varepsilon,x+\varepsilon])<\infty,$$

so that $S^i_n\cap[x-\varepsilon,x+\varepsilon]=\emptyset$ for n large enough. We deduce that

$$\begin{aligned}\liminf_{n\to\infty}\mu^i_n((x-\varepsilon,x+\varepsilon)) &= \liminf_{n\to\infty} m^i_n((x-\varepsilon,x+\varepsilon))\\ &\geq m^i((x-\varepsilon,x+\varepsilon))\\ &= \mu^i((x-\varepsilon,x+\varepsilon))\\ &>0.\end{aligned}$$

As this is true for any sufficiently small $\varepsilon>0$, there exists a sequence $x_n\to x$ such that for all n, $x_n\in\operatorname{supp}\mu^i_n$. Because $(\mu^i_n,S^i_n)\in\Phi^i(\mu^j_n,S^j_n)$, we have $R^i(x_n)=\bar{J}^i(x_n,(\mu^j_n,S^j_n))$. Thus, for any large enough n, inequality (5.3) holds with $y=x_n$, which leads to a contradiction as in Step 2. We conclude that $\mu^i(S^j\cap\{R^i<G^i\})=0$.

Step 3 We finally prove that (μ^i,S^i) is a pbr to (μ^j,S^j). By Assumptions A1–A2, the random function $f:[0,\infty]^2\to\mathbb{R}$ defined by

$$f(t,t') := \mathbb{1}_{t\leq t'}\,\mathrm{e}^{-rt}R^1(X_t) + \mathbb{1}_{t'<t}\,\mathrm{e}^{-rt'}G^1(X_{t'}),$$

with $f(\infty,\infty):=0$, is a.s. bounded, and the set of discontinuities of f is the set

$$\{(t,t)\in[0,\infty)^2 : R^1(X_t)<G^1(X_t)\}.$$

Letting ν^i_n and ν^i denote the probabilities over $[0,\infty]$ with csfs Λ^i_n and Λ^i respectively, and similarly for player j, we have

$$\int_{[0,\infty]^2} f(t,t')\,\nu^i \otimes \nu^j(\mathrm{d}t,\mathrm{d}t') = \int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t. \tag{5.6}$$

Because $S^i \cap S^j \cap \{R^i < G^i\} = \emptyset$, the probability $\nu^1 \otimes \nu^2$ does not charge the set of discontinuities of f; indeed, the conditional probability that $t' = t$ given t is 0 unless $t = \tau_{S^j}$, the probability that $t = \tau_{S^j}$ is 0 unless $\tau_{S^i} = \tau_{S^j}$, and it cannot be that $\tau_{S^i} = \tau_{S^j}$ and $R^i(X_{\tau_{S^i}}) < G^i(X_{\tau_{S^i}})$. Proposition 4.3 thus implies that the sequence $(\nu^i_n \otimes \nu^j_n)_{n\geq 0}$ converges weakly to $\nu^i \otimes \nu^j$. We can thus apply Theorem 5.1 to obtain

$$\int_{[0,\infty]^2} f(t,t')\,\nu^i_n \otimes \nu^j_n(\mathrm{d}t,\mathrm{d}t') \to \int_{[0,\infty]^2} f(t,t')\,\nu^i \otimes \nu^j(\mathrm{d}t,\mathrm{d}t') \text{ a.s.}$$

By Assumption A1, the random variables $(f(t,t'))_{(t,t')\in[0,\infty]^2}$ are uniformly integrable. As a result, the sequence of random variables $(\int_{[0,\infty]^2} f(t,t')\,\nu^i_n \otimes \nu^j_n(\mathrm{d}t,\mathrm{d}t'))_{n\geq 0}$ is also uniformly integrable. Therefore, the above convergence also holds in expectation, which leads by (5.1)–(5.2) and (5.6) to

$$J^i(x,\Gamma^i_n,\Gamma^j_n) \to J^i(x,\Gamma^i,\Gamma^j).$$

Let now $\tau \in \mathcal{T}$ be an arbitrary stopping time for player i such that $\tau \neq \tau_{S^j}$ on $R^i(X_\tau) < G^i(X_\tau)$ a.s. Replacing Λ^i_n and Λ^i in the preceding proof by $\Lambda_t = \mathbb{1}_{t<\tau}$, we obtain

$$J^i(x,\tau,\Gamma^j_n) \to J^i(x,\tau,\Gamma^j).$$

Because, for each $n \geq 0$, $(\mu^i_n, S^i_n) \in PBR^i(\mu^j_n, S^j_n)$, we have, for each $x \in \mathcal{I}$,

$$J^i(x,\tau,\Gamma^j_n) \leq J^i(x,\Gamma^i_n,\Gamma^j_n).$$

Taking the limit on both sides, it follows that, for each $\tau \in \mathcal{T}$ such that $\tau \neq \tau_{S^j}$ a.s. and for each $x \in \mathcal{I}$,

$$J^i(x,\tau,\Gamma^j) \leq J^i(x,\Gamma^i,\Gamma^j). \tag{5.7}$$

To conclude, notice that a stopping time τ such that $\tau = \tau_{S^j}$ and $R^i(X_\tau) < G^i(X_\tau)$ with positive probability cannot be optimal, as player 1 would prefer, conditionally on this event, to wait indefinitely so as to let player j stop first. Hence, (5.7) holds without restriction for all $\tau \in \mathcal{T}$. We conclude that $(\mu^i, S^i) \in PBR^i(\mu^j, S^j)$ and therefore that the graph of Φ^i is closed. Hence the result. $\square$

6 Contractibility and the AR Property

The aim of this section is twofold. First, we establish in Proposition 6.6 that the space $(\mathcal{M}(\mathcal{I}), \vartheta)$ is contractible, which, together with Proposition 4.1, completes the proof of Theorem 2.9. Second, we establish in Proposition 6.7 that the correspondence Φ defined in (3.12) has contractible values, which together with the results of the preceding sections completes the proof of Theorem 2.10. Both proofs rely on an explicit construction of a contraction of the space $\mathcal{M}(\mathcal{I})$ using convolutions.

Let us first introduce more tools from general topology of metric spaces.

Definition 6.1. *A metric space (E, d) is an absolute neighborhood retract (ANR) if, for any continuous map $f : E \to E'$ into a metric space (E', d') such that f is an homeomorphism between E and $f(E)$ and $f(E)$ is closed in E', there exists an open set U such that $f(E) \subset U$ and a continuous map $g : U \to f(E)$ such that for all $x \in f(E)$, $g(x) = x$ (i.e., $f(E)$ is a retract of some neighborhood U).*

From Definition 2.12, it is clear that an AR is an ANR, and we have the following characterization of ARs.

Proposition 6.2 ([35, Theorem 8.2])**.** *(E, d) is an AR if and only if it is a contractible ANR.*

This equivalence is useful as there are sufficient conditions for a metric space to be an ANR. The first one can be stated as follows.

Proposition 6.3 ([35, Proposition 8.3])**.** *A metrizable convex subset of a locally convex Hausdorff topological vector space is an ANR.*

The second sufficient condition we will use states that the closure (in the strong sense of homopotopy-denseness defined below) of an ANR is still an ANR.

Definition 6.4. *Let (E, d) be a metric space and $A \subset E$. A is said to be homotopy-dense in E if there exists a continuous map $H : E \times [0, 1] \to E$ such that $H(\cdot, 0) = \mathrm{Id}_E$ and $H(E \times (0, 1]) \subset A$.*

Proposition 6.5 ([41, Corollary 6.6.7])**.** *Let (E, d) be a metric space and A a homotopy-dense subset of E. Then E is an ANR if and only if A is an ANR.*

The first main result of this section can be stated as follows.

Proposition 6.6. *There exists a continuous map $H : \mathcal{M}(\mathcal{I}) \times [0,1] \to \mathcal{M}(\mathcal{I})$ such that*

1. *for each $m \in \mathcal{M}(\mathcal{I})$, $H(m,0) = m$ and $H(m,1) = 0$;*

2. *for all $\varepsilon \in (0,1]$ and $m \in \mathcal{M}(\mathcal{I})$, $H(m,\varepsilon) \in \mathcal{M}_{\text{loc}}(\mathcal{I})$.*

Before proving Proposition 6.6, let us first show how this result enables us to complete the proof of Theorem 2.9.

Proof of Theorem 2.9. By Proposition 4.1, $\mathcal{M}(\mathcal{I})$ is compact and metrizable. Proposition 6.6 implies that $\mathcal{M}(\mathcal{I})$ is contractible and that $\mathcal{M}_{\text{loc}}(\mathcal{I})$ is homotopy-dense in $\mathcal{M}(\mathcal{I})$. The topology induced by ϑ on $\mathcal{M}_{\text{loc}}(\mathcal{I})$ coincides with the topology of vague convergence by Proposition 4.1(3), and thus $\mathcal{M}_{\text{loc}}(\mathcal{I})$ can be identified with a convex subset of the vector space of linear functionals on $\mathcal{C}_c(\mathcal{I})$ endowed with the vague topology. Therefore, $\mathcal{M}_{\text{loc}}(\mathcal{I})$ is a convex subset of a locally convex Hausdorff topological vector space and is metrizable. We deduce from Proposition 6.3 that $\mathcal{M}_{\text{loc}}(\mathcal{I})$ is an ANR. Because $\mathcal{M}_{\text{loc}}(\mathcal{I})$ is homotopy-dense in $\mathcal{M}(\mathcal{I})$ and an ANR, we conclude by Proposition 6.5 that $\mathcal{M}(\mathcal{I})$ is an ANR. As $\mathcal{M}(\mathcal{I})$ is also contractible, we conclude from Proposition 6.2 that it is an AR. Hence the result. $\square$

Let us come back to the proof of Proposition 6.6.

Proof of Proposition 6.6. The proof consists of two steps.

Step 1 We first show that it is sufficient to prove the result for $\mathcal{I} = \mathbb{R}$. Let $\psi : \mathcal{I} \to \mathbb{R}$ denote a $\mathcal{C}^1$-diffeomorphism and assume that a function H satisfying properties 1 and 2 exists with $\mathcal{I} = \mathbb{R}$. Define

$$\widehat{H}(m,\varepsilon) := H(m \circ \psi^{-1}, \varepsilon) \circ \psi, \quad (m,\varepsilon) \in \mathcal{M}(\mathcal{I}) \times [0,1],$$

where $m \circ \psi^{-1}$ denotes the image of the measure m by ψ, defined for each $B \in \mathcal{B}(\mathbb{R})$ by $m \circ \psi^{-1}(B) := m(\psi^{-1}(B))$, and $\nu \circ \psi$ the image of the measure ν by ψ^{-1}, so that

$$\forall (m,\varepsilon,A) \in \mathcal{M}(\mathcal{I}) \times [0,1] \times \mathcal{B}(\mathcal{I}),\ \widehat{H}(m,t)(A) = H(m \circ \psi^{-1}, \varepsilon)(\psi(A)).$$

We just need to check that $\widehat{H}$ is continuous and satisfies Properties 1 and 2 in Proposition 6.6. Property 1 is immediate and Property 2 follows from the fact that ψ preserves compact sets. To prove continuity, it is sufficient to prove that the mappings

$$\mathcal{M}(\mathcal{I}) \to \mathcal{M}(\mathbb{R}) : m \mapsto m \circ \psi^{-1} \text{ and } \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathcal{I}) : \nu \mapsto \nu \circ \psi$$

are continuous. As the arguments for the two mappings are similar, we only consider the first one. Thus suppose that $m_n \to m$ in $\mathcal{M}(\mathcal{I})$. If O is open in $\mathbb{R}$, then $\psi^{-1}(O)$ is open in

$\mathcal{I}$ and

$$\liminf_{n\to\infty} m_n \circ \psi^{-1}(O) = \liminf_{n\to\infty} m_n(\psi^{-1}(O)) \geq m(\psi^{-1}(O)) = m \circ \psi^{-1}(O).$$

Similarly, if K is compact in $\mathbb{R}$, then $\psi^{-1}(F)$ is compact in $\mathcal{I}$ and

$$\limsup_{n\to\infty} m_n \circ \psi^{-1}(K) = \limsup_{n\to\infty} m_n(\psi^{-1}(K)) \leq m(\psi^{-1}(K)) = m \circ \psi^{-1}(K).$$

We conclude that $m_n \circ \psi^{-1} \to m \circ \psi^{-1}$ in $\mathcal{M}(\mathbb{R})$, as desired. Notice for later use that, by choosing ψ as a $\mathcal{C}^1$-diffeomorphism, we ensure that, if, for some $\varepsilon \in [0,1]$, $H(m \circ \psi^{-1}, \varepsilon)$ is absolutely continuous with respect to Lebesgue measure, then so is $\widehat{H}(m, \varepsilon)$.

Step 2 We now prove the result for $\mathcal{I} = \mathbb{R}$. Define, for all $(m, \varepsilon, x) \in \mathcal{M}(\mathbb{R}) \times (0,1] \times \mathbb{R}$,

$$h(m, \varepsilon, x) := \min\left\{\int_{\mathbb{R}} \rho_\varepsilon(x-y)\, m(\mathrm{d}y), \frac{1}{\varepsilon^2}\right\}, \tag{6.1}$$

where ρ_ε is a continuous function with compact support $[-\varepsilon, \varepsilon]$ defined by

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon} \max\left\{1 - \frac{|x|}{\varepsilon}, 0\right\}, \quad x \in \mathbb{R}.$$

One can easily verify the following properties for all $\varepsilon \in (0,1]$ and $c \in (0,1)$:

$$0 \leq \rho_\varepsilon \leq \frac{1}{\varepsilon} \mathbb{1}_{(-\varepsilon,\varepsilon)}, \quad \int_{\mathbb{R}} \rho_\varepsilon(x)\,\mathrm{d}x = 1, \quad \lim_{\delta\to\varepsilon} \|\rho_\delta - \rho_\varepsilon\|_\infty = 0, \quad \rho_\varepsilon \geq \frac{(1-c)}{\varepsilon} \mathbb{1}_{(-c\varepsilon, c\varepsilon)}. \tag{6.2}$$

Define then, for all $(m, \varepsilon) \in \mathcal{M}(\mathbb{R}) \times (0,1]$,

$$H(m, \varepsilon) := (1-\varepsilon) h(m, \varepsilon, .) \cdot \lambda,$$

where λ denotes Lebesgue measure, and $H(m, 0) := m$. Notice that, for all $(m, \varepsilon) \in \mathcal{M}(\mathbb{R}) \times (0,1]$, the measure $H(m, \varepsilon)$ is absolutely continuous with respect to Lebesgue measure and has a bounded density, so that $H(m, \varepsilon) \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R})$.

Because $H(\cdot, 0) = \mathrm{Id}_{\mathcal{M}(\mathbb{R})}$ and $H(\cdot, 1) = 0$, we only need to check that H is jointly continuous on $\mathcal{M}(\mathbb{R}) \times [0,1]$. Thus consider a sequence $(\varepsilon_n, m_n) \to (\varepsilon, m)$ in $\mathcal{M}(\mathbb{R}) \times [0,1]$.

Case 1: $\varepsilon > 0$ With no loss of generality, we can assume that $\varepsilon_n > 0$ for all $n \geq 0$. Denoting $d(., C)$ the usual distance to a set C in $\mathbb{R}$, define the sets

$$E^+(m, \varepsilon) := \{x \in \mathbb{R} : d(x, e(m)) > \varepsilon\},$$

$$E^-(m, \varepsilon) := \{x \in \mathbb{R} : d(x, e(m)) < \varepsilon\},$$

$$E^0(m, \varepsilon) := \{x \in \mathbb{R} : d(x, e(m)) = \varepsilon\}.$$

We examine these three sets separately in cases numbered 1.1, 1.2, 1.3, respectively.

Case 1.1 For each $x \in E^+(m, \varepsilon)$, we have $d(x, e(m)) > \varepsilon$, so that $\rho_\varepsilon(x - \cdot) \in \mathcal{C}_c^+(\mathbb{R} \setminus e(m))$.

It then follows from Proposition 4.1(3) that

$$\int_{\mathbb{R}} \rho_\varepsilon(x-y)\, m_n(\mathrm{d}y) \to \int_{\mathbb{R}} \rho_\varepsilon(x-y)\, m(\mathrm{d}y) < \infty.$$

Moreover, $\rho_{\varepsilon_n}(x-\cdot) \to \rho_\varepsilon(x-\cdot)$ uniformly, and, for n sufficiently large, all the supports of these functions are contained in a compact subset K of $\mathbb{R} \setminus e(m)$. As $\limsup_{n\to\infty} m_n(K) \leq m(K) < \infty$ and $\varepsilon > 0$, we have, by (6.2),

$$\int_{\mathbb{R}} |\rho_{\varepsilon_n}(x-y) - \rho_\varepsilon(x-y)|\, m_n(\mathrm{d}y) \leq \|\rho_{\varepsilon_n} - \rho_\varepsilon\|_\infty\, m_n(K) \to 0.$$

This implies

$$\int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y) \to \int_{\mathbb{R}} \rho_\varepsilon(x-y)\, m(\mathrm{d}y)$$

and therefore, by (6.1),

$$h(m_n, \varepsilon_n, x) \to h(m, \varepsilon, x),$$

as desired.

Case 1.2 The set $E^-(m,\varepsilon) = e(m) + (-\varepsilon, \varepsilon)$ is open. Let $x \in E^-(m,\varepsilon)$ and $z \in e(m)$ such that $|x-z| < \varepsilon$. Letting $c \in (\frac{|x-z|}{\varepsilon}, 1)$, we have, by (6.2),

$$\rho_\varepsilon(x-\cdot) \geq \frac{(1-c)}{\varepsilon}\, \mathbb{1}_{(x-c\varepsilon, x+c\varepsilon)}.$$

For n sufficiently large, $\|\rho_\varepsilon - \rho_{\varepsilon_n}\|_\infty \leq \frac{c}{\varepsilon}$, and thus

$$\rho_{\varepsilon_n}(x-\cdot) \geq \frac{(1-2c)}{\varepsilon}\, \mathbb{1}_{(x-c\varepsilon, x+c\varepsilon)}.$$

Using that $z \in (x-c\varepsilon, x+c\varepsilon) \cap e(m)$, we have $\lim_{n\to\infty} m_n((x-c\varepsilon, x+c\varepsilon)) = \infty$ and thus

$$\int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y) \geq \frac{1-2c}{\varepsilon}\, m_n((x-c\varepsilon, x+c\varepsilon)) \to \infty.$$

It follows that, for n sufficiently large, $h(m, \varepsilon_n, x) = \frac{1}{\varepsilon_n^2}$ by (6.1). On the other hand, $\int_{\mathbb{R}} \rho_\varepsilon(x-y)\, m(\mathrm{d}y) = \infty$, so that $h(m,\varepsilon,x) = \frac{1}{\varepsilon^2}$ by (6.1) again, and we conclude that

$$h(m_n, \varepsilon_n, x) \to h(m, \varepsilon, x),$$

as desired.

Case 1.3 Observe first that the set $E^0(m,\varepsilon)$ is countable and thus has Lebesgue measure 0; indeed, $e(m)^c$ is open and thus is a countable union of disjoint open intervals, each of which contains at most two points in $E^0(m,\varepsilon)$. Given $\phi \in \mathcal{C}_c^+(\mathcal{I})$, thanks to the analysis of Cases 1.1 and 1.2, we can apply the bounded convergence theorem to deduce that

$$L_\phi(H(m_n,\varepsilon_n)) = \int_{\mathbb{R}} \phi(x) h(m_n,\varepsilon_n,x)\,\mathrm{d}x \to \int_{\mathbb{R}} \phi(x) h(m,\varepsilon,x)\,\mathrm{d}x = L_\phi(H(m,\varepsilon)).$$

Because $e(H(m,\varepsilon)) = \emptyset$, we conclude by Proposition 4.1(3) that

$$H(m_n,\varepsilon_n) \to H(m,\varepsilon),$$

as desired.

Case 2: $\varepsilon = 0$ For each n such that $\varepsilon_n = 0$, we have $H(m_n,0) = m_n$, so we may assume with no loss of generality that $\varepsilon_n > 0$ for all n. To prove that $H(m_n,\varepsilon_n) \to H(m,0) = m$, we use Proposition 4.1(3). We check each property in turn.

Property 1 Given an open subset O of $\mathbb{R}$ such that $O \cap e(m) \neq \emptyset$, we have to prove that $\lim_{n\to\infty} H(m_n,\varepsilon_n)(O) = \infty$. Let us fix some $c \in (0,1)$, and let $z \in O \cap e(m)$ and $\delta > 0$ be such that $(z-\delta, z+\delta) \subset O$. There exists some n_0 such that, for each $n \geq n_0$, $(y - c\varepsilon_n, y + c\varepsilon_n) \subset O$ for all $y \in (z-\delta, z+\delta)$. By (6.1)–(6.2), we have

$$\begin{aligned} H(m_n,\varepsilon_n)(O) &= (1-\varepsilon_n)\int_{\mathbb{R}} \mathbb{1}_O(x) h(m_n,\varepsilon_n,x)\,\mathrm{d}x \\ &= (1-\varepsilon_n)\int \mathbb{1}_O(x) \min\left\{\int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y) dm_n(y), \frac{1}{\varepsilon_n^2}\right\}\mathrm{d}x \\ &\geq (1-\varepsilon_n)\int_{\mathbb{R}} \mathbb{1}_O(x) \min\left\{\frac{1-c}{\varepsilon_n} m_n((x-c\varepsilon_n, x+c\varepsilon_n)), \frac{1}{\varepsilon_n^2}\right\}\mathrm{d}x. \end{aligned} \tag{6.3}$$

In turn, letting $A_n := \{x \in O : (1-c)m_n((x-c\varepsilon_n, x+c\varepsilon_n)) < \frac{1}{\varepsilon_n}\}$, we have

$$\begin{aligned} &(1-\varepsilon_n)\int_{\mathbb{R}} \mathbb{1}_O(x) \min\left\{\frac{1-c}{\varepsilon_n} m_n((x-c\varepsilon_n, x+c\varepsilon_n)), \frac{1}{\varepsilon_n^2}\right\}\mathrm{d}x \\ &\quad= \frac{1-\varepsilon_n}{\varepsilon_n}\int_{\mathbb{R}} \mathbb{1}_O(x) \min\left\{(1-c) m_n((x-c\varepsilon_n, x+c\varepsilon_n)), \frac{1}{\varepsilon_n}\right\}\mathrm{d}x \\ &\quad= \frac{1-\varepsilon_n}{\varepsilon_n}\int_{\mathbb{R}} \left[\mathbb{1}_{A_n}(x)(1-c) m_n((x-c\varepsilon_n, x+c\varepsilon_n)) + \mathbb{1}_{O\setminus A_n}(x)\frac{1}{\varepsilon_n}\right]\mathrm{d}x \\ &\quad= \frac{1-\varepsilon_n}{\varepsilon_n}\left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{A_n}(x)(1-c)\mathbb{1}_{(x-c\varepsilon_n, x+c\varepsilon_n)}(y)\, m_n(\mathrm{d}y)\,\mathrm{d}x + \frac{1}{\varepsilon_n}\lambda(O\setminus A_n)\right] \\ &\quad= \frac{1-\varepsilon_n}{\varepsilon_n}\left[\int_{\mathbb{R}} (1-c)\lambda(A_n \cap (y-c\varepsilon_n, y+c\varepsilon_n))\, m_n(\mathrm{d}y) + \frac{1}{\varepsilon_n}\lambda(O\setminus A_n)\right], \end{aligned} \tag{6.4}$$

where the last equality follows from Fubini's theorem. Suppose, by way of contradiction, that, along some subsequence, $H(m_n,\varepsilon_n)(O)$ is bounded above by a constant M. Then, from (6.3)–(6.4), it must be that $\frac{1-\varepsilon_n}{\varepsilon_n^2}\lambda(O\setminus A_n) \leq M$. Because $(y-c\varepsilon_n, y+c\varepsilon_n) \subset O$, we deduce that, for each $n \geq n_0$ in the subsequence and each $y \in (z-\delta, z+\delta)$,

$$\begin{aligned}\lambda(A_n \cap (y - c\varepsilon_n, y + c\varepsilon_n)) &= \lambda((y - c\varepsilon_n, y + c\varepsilon_n)) - \lambda((O \setminus A_n) \cap (y - c\varepsilon_n, y + c\varepsilon_n)) \\ &\geq 2c\varepsilon_n - M \frac{\varepsilon_n^2}{1-\varepsilon_n} \\ &\geq c\varepsilon_n\end{aligned}$$

for n sufficiently large. It follows from (6.3)–(6.4) that, along this subsequence,

$$H(m_n, \varepsilon_n)(O) \geq (1-\varepsilon_n)c(1-c)m_n((z-\delta, z+\delta)) \to \infty,$$

a contradiction. We conclude that $H(m_n, \varepsilon_n)(O) \to \infty$, as desired.

Property 2 Given $\phi \in \mathcal{C}_c^+(\mathbb{R} \setminus e(m))$, we have to prove that $L_\phi(H(m_n, \varepsilon_n)) \to L_\phi(m)$. Let $K := \operatorname{supp} \phi$ and, for each $x \in \mathbb{R}$, $K_x := K + [-x, x]$. Because K is a compact subset of the open set $\mathbb{R} \setminus e(m)$, there exists $\delta > 0$ such that $K_\delta \cap e(m) = \emptyset$. Because $\lim_{n\to\infty} m_n = m$, we have $\limsup_{n\to\infty} m_n(K_\delta) < \infty$, and hence $\limsup_{n\to\infty} m_n(K_{\varepsilon_n}) < \infty$. Moreover,

$$\begin{aligned}H(m_n, \varepsilon_n)(K) &\leq \int_K \int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \int_K \rho_{\varepsilon_n}(x-y)\, \mathrm{d}x\, m_n(\mathrm{d}y) \\ &\leq \int_{\mathbb{R}} \mathbb{1}_{K_{\varepsilon_n}}(y) \int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, \mathrm{d}x\, m_n(\mathrm{d}y) \\ &= m_n(K_{\varepsilon_n}),\end{aligned}$$

where the first inequality follows from (6.1), the first equality follows from Fubini's theorem, the second inequality follows from the definition of ρ_ε, and the last equality follows from (6.2). We deduce from this that $\limsup_{n\to\infty} H(m_n, \varepsilon_n)(K) < \infty$. Now, recall that

$$L_\phi(H(m_n, \varepsilon_n)) = \int_{\mathbb{R}} \phi(x) h(m_n, \varepsilon_n, x)\, \mathrm{d}x = \int_{\mathbb{R}} \phi(x) \min\left\{\int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y), \frac{1}{\varepsilon_n^2}\right\} \mathrm{d}x.$$

By (6.2), for n sufficiently large,

$$\int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y) \leq \frac{1}{\varepsilon_n} m_n((x-\varepsilon_n, x+\varepsilon_n)) \leq \frac{1}{\varepsilon_n} m_n(K_\delta) \leq \frac{1}{\varepsilon_n^2}$$

for all $x \in K$, where the last inequality follows from $\limsup_{n\to\infty} m_n(K_\delta) < \infty$. Therefore, for n sufficiently large, we have, by Fubini's theorem,

$$\begin{aligned}L_\phi(H(m_n, \varepsilon_n)) &= \int_{\mathbb{R}} \phi(x) \int_{\mathbb{R}} \rho_{\varepsilon_n}(x-y)\, m_n(\mathrm{d}y)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(x)\rho_{\varepsilon_n}(x-y)\, \mathrm{d}x\, m_n(\mathrm{d}y)\end{aligned}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \phi(u+y)\rho_{\varepsilon_n}(u)\,\mathrm{d}u\, m_n(\mathrm{d}y).$$

Using that $\int_{\mathbb{R}} \rho_{\varepsilon_n}(u)\,\mathrm{d}u = 1$, we obtain

$$\begin{aligned}
|L_\phi(H(m_n,\varepsilon_n)) - L_\phi(m_n)| &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [\phi(u+y) - \phi(y)]\rho_{\varepsilon_n}(u)\,\mathrm{d}u\, m_n(\mathrm{d}y) \right| \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\phi(u+y) - \phi(y)|\rho_{\varepsilon_n}(u)\,\mathrm{d}u\, m_n(\mathrm{d}y) \\
&= \int_{K_{\varepsilon_n}} \int_{\mathbb{R}} |\phi(u+y) - \phi(y)|\rho_{\varepsilon_n}(u)\,\mathrm{d}u\, m_n(\mathrm{d}y) \\
&\leq \omega_\phi(\varepsilon_n) m_n(K_{\varepsilon_n}) \\
&\to 0, \qquad (6.5)
\end{aligned}$$

where, recalling that $\phi \in \mathcal{C}_c^+(\mathcal{I} \setminus e(m))$ and thus is uniformly continuous, ω_ϕ denotes the modulus of continuity of ϕ. As $m_n \to m$ and $\operatorname{supp}\phi \cap e(m) = \emptyset$, we have $L_\phi(m_n) \to L_\phi(m)$ by Proposition 4.1(3). From this and (6.5), we conclude that

$$L_\phi(H(m_n,\varepsilon_n)) \to L_\phi(m),$$

as desired. Hence the result. □

Because the correspondence Φ defined by (3.13) and characterized by (3.14) has nonempty values by Proposition 3.1 and a closed graph by Proposition 5.2, and because $\mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I})$ is a compact AR as the product of two compact ARs [35, Exercise 8.4], the following result enables us to apply Theorem 2.14 to Φ and thereby to complete the proof of Theorem 2.10.

Proposition 6.7. *The correspondence Φ has contractible values.*

Proof. Recall that Φ take values in $\mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I})$ and that, denoting $m^i \in \mathcal{M}(\mathcal{I})$ the measure associated to the pair (μ^i, S^i), we have

$$\forall (m^1, m^2) \in \mathcal{M}(\mathcal{I}) \times \mathcal{M}(\mathcal{I}),\ \Phi(m^1, m^2) = \Phi^1(m^2) \times \Phi^2(m^1).$$

Because the product of two contractible spaces is contractible, it is therefore sufficient to prove that, for all $i = 1, 2$ and $m^j \in \mathcal{M}(\mathcal{I})$, $\Phi^i(m^j)$ is contractible in $\mathcal{M}(\mathcal{I})$. The measure m^j being fixed, we have, by (3.12),

$$\Phi^i(m^j) = \{m \in \mathcal{M}(\mathcal{I}) : \underline{S}^i \subset e(m) \subset \overline{S}^i,\ m(\mathcal{I} \setminus \overline{S}^i) = 0\},$$

where $\underline{S}^i$ and $\overline{S}^i$ are the closed subsets of $\mathcal{I}$ defined in Section 3. The open set $O = \mathcal{I} \setminus \underline{S}^i$ can be written as a countable union $O = \bigcup_{k \geq 0} O_k$ of disjoint open intervals $O_k \subset \mathcal{I}$. For each $k \geq 0$, let $F_k := \overline{S}^i \cap O_k$, which is closed in O_k. Every measure $m \in \Phi^i(m^j)$ can in turn be written as

$$m = \overline{m} + \sum_{k \geq 0} m_k, \tag{6.6}$$

where $\overline{m}$ is defined by

$$\overline{m}(A) = \begin{cases} \infty & \text{if } A \cap \underline{S}^i \neq \emptyset \\ 0 & \text{if } A \cap \underline{S}^i = \emptyset \end{cases}, \quad A \in \mathcal{B}(\mathcal{I}),$$

and, for each $k \geq 0$, m_k is the restriction of m to O_k that we identify with a (not necessarily regular) measure on $\mathcal{I}$ through the formula

$$m_k(A) = m(A \cap O_k), \quad A \in \mathcal{B}(\mathcal{I}).$$

Reciprocally, given any sequence $(m_k)_{k \geq 0}$ of regular measures on O_k, the formula (6.6) defines a regular measure on $\mathcal{I}$.

The proof consists of two steps. In Step 1, we prove that the contraction H constructed in Proposition 6.6 can be modified to obtain, for each $k \geq 0$, a contraction of the set of measures in $\mathcal{M}(O_k)$ concentrated on F_k. To do so, we simply compose, up to a diffeomorphism, H with the projection on F_k. In Step 2, we paste together a family of such contractions using (6.6) to obtain a contraction of $\Phi^i(m^j)$.

Step 1 We first prove that, for each $k \geq 0$, the set

$$C_k := \{m \in \mathcal{M}(O_k) : m(O_k \setminus F_k) = 0\}$$

is contractible for the topology induced by ϑ on $\mathcal{M}(O_k)$. Notice that C_k is closed in $\mathcal{M}(O_k)$ as the mapping $m \mapsto m(O_k \setminus F_k)$ is lsc and nonnegative. Fix some $k \geq 0$ and let $\psi_k : O_k \to \mathbb{R}$ be a $\mathcal{C}^1$-diffeomorphism. Then the map $\widehat{\psi}_k : \mathcal{M}(O_k) \to \mathcal{M}(\mathbb{R})$ defined by

$$\widehat{\psi}_k(m) = m \circ \psi_k^{-1}, \quad m \in \mathcal{M}(O_k),$$

is a homeomorphism (see the proof of Proposition 6.6). Letting $\widehat{F}_k := \psi_k(F_k)$ and $\widehat{C}_k := \widehat{\psi}_k(C_k)$, consider then the map $p_k : \mathbb{R} \to \widehat{F}_k$ defined by

$$p_k(x) = \max\left\{y \in \widehat{F}_k : |x - y| = \inf_{z \in \widehat{F}_k} |x - z|\right\}, \quad x \in \mathbb{R},$$

which is a right-continuous version of the orthogonal projection on $\widehat{F}_k$. The map p_k is nondecreasing and continuous outside of an at most countable set D_k. It is easy to verify that for any interval $U \subset \mathbb{R}$, $p_k^{-1}(U)$ is an interval and that

$$p_k^{-1}(U) = p_k^{-1}(U \cap \widehat{F}_k) \text{ and } p_k^{-1}(U) \cap \widehat{F}_k = U \cap \widehat{F}_k.$$

Define a map $\widehat{H}_k : \mathcal{M}(\mathbb{R}) \times [0,1] \to \mathcal{M}(\mathbb{R})$ by

$$\widehat{H}_k(m,\varepsilon) = H(m,\varepsilon) \circ p_k^{-1}, \quad (m,\varepsilon) \in \mathcal{M}(\mathbb{R}) \times [0,1],$$

where $H : \mathcal{M}(\mathbb{R}) \times [0,1] \to \mathcal{M}(\mathbb{R})$ denotes the contraction of $\mathcal{M}(\mathbb{R})$ explicitly constructed in Step 2 of the proof of Proposition 6.6. By construction, $\widehat{H}_k(m,0) = m \circ p_k^{-1}$ and $\widehat{H}_k(m,1) = 0$ for all $m \in \mathcal{M}(\mathbb{R})$. In particular, $\widehat{H}_k(m,0) = m$ for all $m \in \widehat{C}_k$. Therefore, to prove that $\widehat{H}_k$ is a contraction of $\widehat{C}_k$, we only need to check that it is jointly continuous on $\widehat{C}_k \times [0,1]$ and takes values in $\widehat{C}_k$. Thus consider a sequence $(m_n, \varepsilon_n) \to (m,\varepsilon)$ in $\widehat{C}_k \times [0,1]$. We distinguish two cases.

Case 1: $\varepsilon > 0$ With no loss of generality, we can assume that $\varepsilon_n > 0$ for all $n \geq 0$. First, let $U \subset \mathbb{R}$ be an open interval. Then $p_k^{-1}(U)$ is an interval, whose interior we denote by U'. For each $n \geq 0$, we have

$$\widehat{H}_k(m_n,\varepsilon_n)(U) = H(m_n,\varepsilon_n)(p_k^{-1}(U)) = H(m_n,\varepsilon_n)(U'),$$

where the second equality follows from the fact that $H(m_n,\varepsilon_n)$ is absolutely continuous and that $p_k^{-1}(U) \setminus U'$ consists of at most one point, which, whenever it exists, is the left-endpoint of $p_k^{-1}(U)$. Using that H is continuous, we conclude by the same argument that

$$\liminf_{n\to\infty} \widehat{H}_k(m_n,\varepsilon_n)(U) \geq H(m,\varepsilon)(U') = \widehat{H}_k(m,\varepsilon)(U). \tag{6.7}$$

Next, let $K \subset \mathbb{R}$ be a compact interval. Then $p_k^{-1}(K)$ is an interval, but it is not necessarily bounded. However, because the measures m_n belong to $\widehat{C}_k$, using the definition of H and letting G_k be the convex hull of $F_k + [-1,1]$, we have $H(m_n,\varepsilon_n)(\mathbb{R} \setminus G_k) = 0$ and therefore

$$H(m_n,\varepsilon_n)(p_k^{-1}(K)) = H(m_n,\varepsilon_n)(p_k^{-1}(K) \cap G_k)$$

for all $n \geq 0$. We claim that $p_k^{-1}(K) \cap G_k$ is a (possibly empty) bounded interval. That it is an interval follows from the fact that it is the intersection of two intervals. Now, suppose, by way of contradiction, that it is not bounded. Then there exists an unbounded monotone sequence $(x_m)_{m\geq 0}$ in $p_k^{-1}(K) \cap G_k$. With no loss of generality, assume that this sequence is increasing. Because $p_k(x_m) \in K$ for all $m \geq 0$, we have, for any sufficiently large m,

$$p_k(x_m) = x^* := \max K \cap \widehat{F}_k \text{ and } (x^*, x_m) \cap \widehat{F}_k = \emptyset.$$

Letting $m \to \infty$, this implies $(x^*, \infty) \cap \widehat{F}_k = \emptyset$ and in turn that $G_k \subset (-\infty, x^* + 1]$, a contradiction as $x_m \in G_k$ for all $m \geq 0$. The claim follows. Define K' as the closure of $p_k^{-1}(K) \cap G_k$, we have

$$\widehat{H}_k(m_n,\varepsilon_n)(K) = H(m_n,\varepsilon_n)(p_k^{-1}(K) \cap G_k) = H(m_n,\varepsilon_n)(K'),$$

where the second equality follows from the fact that $H(m_n, \varepsilon_n)$ is absolutely continuous and that $K' \setminus (p_k^{-1}(K) \cap G_k)$ is finite. Using that H is continuous, we conclude by the same argument that

$$\limsup_{n\to\infty} \widehat{H}_k(m_n, \varepsilon_n)(K) \leq H(m, \varepsilon)(K') = \widehat{H}_k(m, \varepsilon)(K). \tag{6.8}$$

It follows from (6.7)–(6.8) that $\widehat{H}_k$ is continuous at (m, ε).

Case 2: $\varepsilon = 0$ For each n such that $\varepsilon_n = 0$, we have $H_k(m_n, 0) = m_n$, so we may assume with no loss of generality that $\varepsilon_n > 0$ for all n. First, let $U \subset \mathbb{R}$ be an open interval. Then $p_k^{-1}(U)$ is an interval, whose interior we denote by U'. As in Case 1, we have

$$\liminf_{n\to\infty} \widehat{H}_k(m_n, \varepsilon_n)(U) \geq H(m, 0)(U') = m(U').$$

Notice that $p_k^{-1}(U) \setminus U'$ consists of at most one point, which, whenever it exists, is the left-endpoint of $p_k^{-1}(U)$ and does not belong to $\widehat{F}_k$. Thus $m(U') = m(p_k^{-1}(U))$ as m is concentrated on $\hat{F}_k$. Using the properties of p_k and the fact that $m \in \widehat{C}_k$, it follows that

$$m(U') = m(p_k^{-1}(U)) = m(U \cap \widehat{F}_k) = m(U).$$

We conclude that

$$\liminf_{n\to\infty} \widehat{H}_k(m_n, \varepsilon_n)(U) \geq m(U') = m(U) = \widehat{H}_k(m, 0)(U). \tag{6.9}$$

Next, let $K \subset \mathbb{R}$ be a compact interval, and let K' be the closure of $p_k^{-1}(K) \cap G_k$. As in Case 1, we have

$$\limsup_{n\to\infty} \widehat{H}_k(m_n, \varepsilon_n)(K) \leq H(m, 0)(K') = m(K').$$

Using the properties of p_k and the fact that $m \in \widehat{C}_k$, we have

$$m(K) = m(K \cap \widehat{F}_k) = m(p_k^{-1}(K)) = m(p_k^{-1}(K) \cap G_k).$$

Because G_k is a closed interval, any point in $K' \setminus (p_k^{-1}(K) \cap G_k)$ must be an endpoint of $p_k^{-1}(K)$ which does not belong to $p_k^{-1}(K)$ and thus cannot belong to $\widehat{F}_k$. It follows that

$$m(K) = m(p_k^{-1}(K) \cap G_k) = m(K').$$

We conclude that

$$\limsup_{n\to\infty} \widehat{H}_k(m_n, \varepsilon_n)(K) \leq m(K') = m(K) = \widehat{H}_k(m, 0)(K). \tag{6.10}$$

It follows from (6.7)–(6.8) that $\widehat{H}_k$ is continuous at $(m,0)$.

The analyses of Cases 1–2 above imply that $\widehat{H}_k$ is a contraction of $\widehat{C}_k$. Define then

$$H_k^*(m,\varepsilon) := \widehat{H}_k(m\circ\psi_k^{-1},\varepsilon)\circ\psi_k, \quad (m,\varepsilon)\in C_k\times[0,1].$$

By composition, H_k^* is continuous and, from the properties of $\widehat{H}_k$, we have

$$\forall m\in C_k,\ H_k^*(m,0)=m \text{ and } H_k^*(m,0)=0.$$

Hence H_k^* is a contraction of C_k, as desired.

Step 2 We now prove that $\Phi^i(m^j)$ is contractible. Define the map $H^*:\Phi^i(m^j)\times[0,1]\to\Phi^i(m^j)$ by

$$H^*(m,\varepsilon) := \overline{m}+\sum_{k\geq 0}H_k^*(m_k,\varepsilon), \quad (m,\varepsilon)\in\Phi^i(m^j)\times[0,1], \tag{6.11}$$

where $\overline{m}$ and the measures $(m_k)_{k\geq 0}$ are defined in (6.6) and, for each $k\geq 0$, H_k^* is the map constructed in Step 1. We only need to prove that H^* is continuous; indeed, that

$$\forall m\in\Phi^i(m^j),\ H^*(m,0)=m \text{ and } H^*(m,1)=\overline{m}$$

follows directly from (6.6), (6.11), and the properties of the maps H_k^*. Thus consider a sequence $(m_n,\varepsilon_n)\to(m,\varepsilon)$ in $\Phi^i(m^j)\times[0,1]$. Notice first that, for each $k\geq 0$, the sequence $(m_{k,n})_{n\geq 0}$ converges to m_k in $\mathcal{M}(O_k)$, so that

$$H_k^*(m_{k,n},\varepsilon_n)\to H_k^*(m_k,\varepsilon).$$

First, let U be an open subset of $\mathcal{I}$. If $U\cap\underline{S}^i\neq\emptyset$, then

$$H^*(m_n,\varepsilon_n)(U)=\infty\to\infty=H^*(m,\varepsilon)(U).$$

If $U\cap\underline{S}^i=\emptyset$, then U is the disjoint union of the open sets $U\cap O_k$, and we have

$$\begin{aligned}\liminf_{n\to\infty}H^*(m_n,\varepsilon_n)(U) &= \liminf_{n\to\infty}\sum_{k\geq 0}H_k^*(m_n,\varepsilon_n)(U\cap O_k)\\ &\geq \sum_{k\geq 0}\liminf_{n\to\infty}H_k^*(m_{k,n},\varepsilon_n)(U\cap O_k)\\ &\geq \sum_k H_k^*(m_k,\varepsilon)(U\cap O_k)\\ &= H^*(m,\varepsilon)(U).\end{aligned}$$

Next, let K be a compact subset of $\mathcal{I}$. If $K \cap \underline{S}^i \neq \emptyset$, then

$$H^*(m_n, \varepsilon_n)(K) = \infty \to \infty = H^*(m, \varepsilon)(K).$$

If $K \cap \underline{S}^i = \emptyset$, then K is the disjoint union of the compact sets $K \cap O_k$, and we have

$$\begin{aligned}
\limsup_{n\to\infty} H^*(m_n, \varepsilon_n)(K) &= \limsup_{n\to\infty} \sum_{k\geq 0} H^*_k(m_n, \varepsilon_n)(K \cap O_k) \\
&\leq \sum_{k\geq 0} \limsup_{n} H^*_k(m_{k,n}, \varepsilon_n)(K \cap O_k) \\
&\leq \sum_{k\geq 0} H^*_k(m_k, \varepsilon)(K \cap O_k) \\
&= H^*(m, \varepsilon)(K).
\end{aligned}$$

It follows that H^* is continuous at (m, ε). Hence the result. □

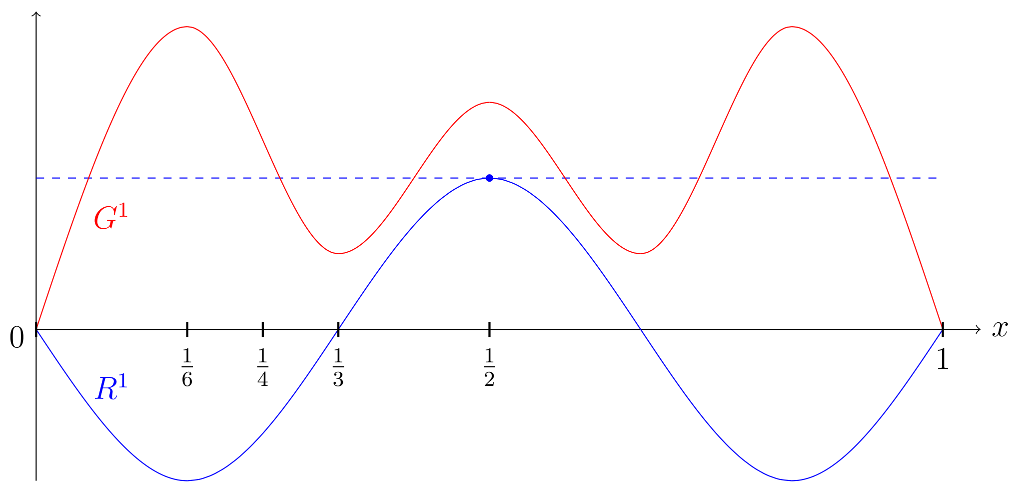
7 An Example

We consider in this section the diffusion X with state space $\mathcal{I} = (0, 1)$ solution of the SDE

$$dX_t = X_t(1 - X_t)\, \mathrm{d}W_t. \tag{7.1}$$

This process satisfies the assumptions of section 2 and is a martingale appearing in filtering equations (see, e.g., [33]) that satisfies $X_\infty := \lim_{t\to\infty} X_t \in \{0, 1\}$ a.s. We let the discount rate r be equal to 0. In order to satisfy Assumption A2, the payoff functions R^i and G^i, $i = 1, 2$, must converge to 0 at both boundaries 0 and 1.

The payoff functions, which we consider as functions on $[0, 1]$ equal to 0 at 0 and 1, are represented in Figure 1. Assumption A1 is satisfied as all these functions are bounded.



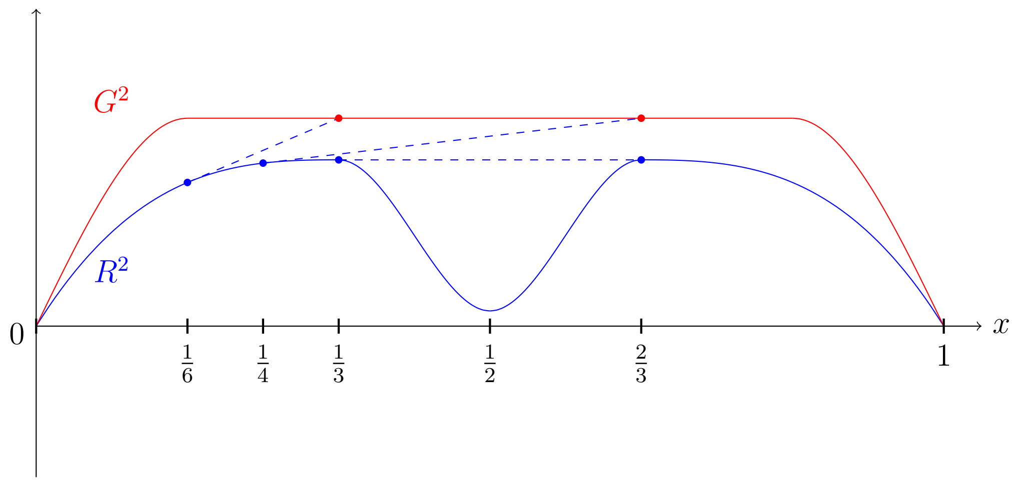


Figure 1: The players' payoff functions.

We shall not give explicit formulas for these functions, as this does not help for the proof. The properties of these functions that will be useful are the following:

1. For R^1, R^2, G^1, and G^2 are symmetric around $\frac{1}{2}$;
2. $G^1 > R^1$ on $(0,1)$, $R^1 < 0$ on $(0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$, and $R^1 > 0$ on $(\frac{1}{3}, \frac{2}{3})$;
3. G^1 is decreasing on $[\frac{1}{6}, \frac{1}{3}]$, and $G^1(\frac{1}{4}) > R^1(\frac{1}{2}) > G^1(\frac{1}{3})$;
4. $G^2 > R^2$ on $(0,1)$, and G^2 is concave on $(0,1)$ and constant on $[\frac{1}{6}, \frac{5}{6}]$;
5. R^2 is strictly concave and $\mathcal{C}^2$ on $(0, \frac{1}{3}]$, $(R^2)'(\frac{1}{3}) = 0$, $R^2 < R^2(\frac{1}{3})$ on $(\frac{1}{3}, \frac{2}{3})$, and
$$R^2(\tfrac{1}{6}) + (R^2)'(\tfrac{1}{6})(\tfrac{1}{3} - \tfrac{1}{6}) = G^2(\tfrac{1}{3}) \text{ and } R^2(\tfrac{1}{4}) + (R^2)'(\tfrac{1}{4})(\tfrac{2}{3} - \tfrac{1}{4}) = G^2(\tfrac{2}{3}). \tag{7.2}$$

The central result of this section can then be stated as follows.

Proposition 7.1. *Consider the BWoA with underlying diffusion process solution to (7.1) and payoff functions illustrated in Figure 1 and satisfying Properties 1–5. Then,*

(i) there exists no pure-strategy MPE;

(ii) the randomized stopping times $(\mu^1, S^1) := (\alpha\delta_{\frac{1}{2}}, \emptyset)$ and $(\mu^2, S^2) := (0, (0, x^*] \cup [1-x^*, 1))$ form an MPE, where x^* is the unique solution in $(\frac{1}{4}, \frac{1}{3})$ of $G^1(x^*) = R^1(\frac{1}{2})$ and
$$\alpha = \frac{(R^2)'(x^*)}{G^2(\frac{1}{2}) - R^2(x^*) - (R^2)'(x^*)(\frac{1}{2} - x^*)} > 0.$$

To prove the Proposition 7.1(i), we will use a semi-harmonic characterization of best replies that can be found in [2], proven in a more general framework. To deduce the statement below from [2], we use that the fine topology associated to X coincides with the usual topology in $(0,1)$, that all points of $(0,1)$ are regular for X, and that super-harmonic functions are just concave functions because X is a martingale and $r=0$.

Theorem 7.2 ([2, Theorem 5.3])**.** *Let $\bar{J}^i$ denote the pbr value function to some pure strategy $(0,S^j)$. Then $\bar{J}^i$ is continuous and is the pointwise minimum of the family of continuous functions $u:(0,1)\to\mathbb{R}$ satisfying $R^i \le u \le \operatorname{cav} G^i$, $u=G^i$ on S^j, and u is concave on each connected component of $(0,1)\setminus S^j$, where $\operatorname{cav} G^i$ is the smallest concave function bounded below by G^i.*

Notice that, for one-dimensional continuous diffusions, the characterization given in Theorem 7.2 has a local character, in the sense that the restriction of $\bar{J}^i$ to any connected component (a,b) of $(0,1)\setminus S^j$ is the smallest concave function bounded below by R^i that is equal to G^i at a and b (where, if $a=0$, this equality means that the limit at $0+$ is 0, as implied by the inequalities $R^i \le u \le \operatorname{cav} G^i$ together with the fact that $\operatorname{cav} G^i(0)=0$, and similarly if $b=1$).

Proof of Proposition 7.1. (i) Suppose, by way of contradiction, that a pure-strategy MPE $((0,S^1),(0,S^2))$ exists. We will use Theorem 7.2 several times during the proof, as well as the fact that, if $(0,S^i)$ is a pbr to $(0,S^j)$, then $S^i \subset \overline{S}^i = \{\bar{J}^i = R^i\}$ (Proposition 2.8), where $\bar{J}^1, \bar{J}^2$ are the players' equilibrium brvfs.

We first claim that

$$S^1 \subset [\tfrac{1}{3}, \tfrac{2}{3}]. \tag{7.3}$$

Let $x \in (0,\frac{1}{3})$. If $x \in S^2$, then $x \notin S^1$ as $\bar{J}^1(x) = G^1(x) > R^1(x)$ on $(0,1)$. If $x \notin S^2$, let (a,b) denote the connected component of $(0,1)\setminus S^2$ containing x. By Theorem 7.2, $\bar{J}^1$ is concave on (a,b), bounded below by R^1, and equal to G^1 at a and b. As $G^1 \ge 0$, if follows that $\bar{J}^1(x) \ge 0 > R^1(x)$ and thus $x \notin S^1$. A symmetric result holds for $(\frac{2}{3},1)$. Hence (7.3). The claim follows.

We next claim that

$$S^2 \subset (0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1). \tag{7.4}$$

The proof is similar to that of (7.3) and is thus omitted. The claim follows.

Now, we claim that, if $S^1 = S^1 \cap [\frac{1}{3},\frac{2}{3}] \neq \emptyset$, then it must be that

$$\exists (x_0,x_1) \in [\tfrac{1}{6},\tfrac{1}{4}] \times [\tfrac{3}{4},\tfrac{5}{6}],\ S^2 = (0,x_0] \cup [x_1,1). \tag{7.5}$$

Let $z_0 := \min S^1 \in [\frac{1}{3}, \frac{2}{3}]$. Using (7.2) along with the fact that R^2 is strictly concave and increasing on $[\frac{1}{6}, \frac{1}{4}]$, we obtain that the mapping $x \mapsto R^2(x) + (R^2)'(x)(z_0 - x)$ is decreasing on $[\frac{1}{6}, \frac{1}{4}]$, larger or equal to $G^2(z_0) = G^2(\frac{1}{3})$ at $\frac{1}{6}$ and smaller or equal to $G^2(z_0) = G^2(\frac{2}{3})$ at $\frac{1}{4}$. It follows that there exists a unique point $x_0 \in [\frac{1}{6}, \frac{1}{4}]$ such that

$$R^2(x_0) + (R^2)'(x_0)(z_0 - x_0) = G^2(z_0).$$

By Theorem 7.2, the restriction of $\bar{J}^2$ to $(0, z_0]$ is the smallest concave function bounded below by R^2 and bounded above by G^2 which is equal to G^2 at 0 and at z_0. It follows that $\bar{J}^2 = R^2$ on $[0, x_0]$ and that $\bar{J}^2(x) = R^2(x_0) + (R^2)'(x_0)(x - x_0)$ for all $x \in [x_0, z_0]$. A symmetric argument on the interval $[z_1, 1)$ with $z_1 := \max S^1$ shows that there exists $x_1 \in [\frac{3}{4}, \frac{5}{6}]$ such that $\bar{J}^2 = R^2$ on $[x_1, 1]$ and $\bar{J}^2(x) = R^2(x_1) + (R^2)'(x_1)(x - x_1)$ for all $x \in [z_1, x_1]$. Finally, it must be that $\bar{J}^2 = G^2$ on $[z_0, z_1]$: first, on $[z_0, z_1]$, $\bar{J}^2 \le G^2$, with equality on S^1, and G^2 is constant; second, $\bar{J}^2$ is concave on any connected component (a, b) of $(z_0, z_1) \setminus S^1$ and equal to G^2 at a and b. Therefore $\bar{J}^2$ is constant on any such interval, and thus $\bar{J}^2 = G^2$ on $[z_0, z_1]$. Hence (7.5). The claim follows.

Now, if (7.5) holds, then it must be that $S^1 = \emptyset$. Indeed, any continuous function that is equal to G^1 on S^2 and is concave on (x_0, x_1) is strictly larger than R^1, so that $\bar{J}^1 > R^1$ using again Theorem 7.2. Because $S_1 \neq \emptyset$ implies (7.5), we deduce that it must be that $S^1 = \emptyset$. We deduce from this that $S^2 = (0, \frac{1}{3}] \cup [\frac{2}{3}, 1)$, and hence that $\bar{J}^1(\frac{1}{2}) = G^1(\frac{1}{3}) = G^1(\frac{2}{3})$ because $r = 0$ and X is a martingale such that $X_\infty \in \{0, 1\}$ a.s., a contradiction as $G^1(\frac{1}{3}) = G^1(\frac{2}{3}) < R^1(\frac{1}{2})$ and $\bar{J}^1 \ge R^1$. We conclude that no pure-strategy MPE exists.

(ii) Let $\bar{J}^i$ denote the brvf to (μ^j, S^j) for the randomized stopping times defined in the statement of the proposition. First, we easily see that $\bar{J}^1$ is equal to G^1 on S^2 and is constant and equal to $R^1(\frac{1}{2})$ on $[x^*, 1 - x^*]$. It follows that $\overline{S}^1 = \{\frac{1}{2}\}$. To show that $(0, S^1) = (0, \emptyset)$ is a best reply to $(0, S^2)$, just note that the expected payoff from not stopping starting from any point in $(x^*, 1 - x^*)$ is equal to $G^1(x^*) = G^1(1 - x^*) = R^1(\frac{1}{2})$ because $r = 0$ and X is a martingale such that $X_\infty \in \{0, 1\}$ a.s. From Proposition 2.8(iv), we conclude that $(0, \alpha' \delta_{\frac{1}{2}})$ is a pbr to $(0, S^2)$ for any nonnegative α'.

Notice then that $\alpha > 0$. Indeed, R^2 is strictly concave and increasing on $(\frac{1}{4}, \frac{1}{3}]$, so that $(R^2)'(x^*) > 0$ and

$$R^2(x^*) + (R^2)'(x^*)(\tfrac{1}{2} - x^*) < R^2(\tfrac{1}{4}) + (R^2)'(\tfrac{1}{4})(\tfrac{1}{2} - \tfrac{1}{4}) < R^2(\tfrac{1}{4}) + (R^2)'(\tfrac{1}{4})(\tfrac{2}{3} - \tfrac{1}{4}) = G^2(\tfrac{1}{2})$$

by (7.2) as $G^2(\frac{1}{2}) = G^2(\frac{2}{3})$. Let us prove that $(0, S^2)$ is a pbr to $(\alpha \delta_{\frac{1}{2}}, \emptyset)$. By Proposition 3.3, it is sufficient to prove that $\bar{J}^2$ is equal to R^2 on S^2 and is strictly larger than R^2 on

$(x^*, 1-x^*)$. Let $w^2 : (0,1) \to \mathbb{R}$ be equal to R^2 on S^2 and such that

$$w^2(x) = R^2(x^*) + (R^2)'(x^*)(x - x^*), \quad x \in [x^*, \tfrac{1}{2}],$$
$$w^2(x) = R^2(1-x^*) + (R^2)'(1-x^*)[x - (1-x^*)], \quad x \in [\tfrac{1}{2}, 1-x^*].$$

Notice that $w^2 > R^2$ on $(x^*, 1-x^*)$, that w^2 is $\mathcal{C}^1$ over $(0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and piecewise $\mathcal{C}^2$, and that w^2 is solution to the variational system

$$\begin{aligned} w^2(0+) = w^2(1-) &= 0, \\ w^2 &= R^2 \text{ on } (0, x^*] \cup [1-x^*, 1), \\ (w^2)'(x^*) &= (R^2)'(x^*), \\ (w^2)'(1-x^*) &= (R^2)'(1-x^*), \\ (w^2)'' &= 0 \text{ on } (x^*, \tfrac{1}{2}) \cup (\tfrac{1}{2}, 1-x^*), \\ (w^2)'' &< 0 \text{ on } (0, x^*) \cup (1-x^*, 1), \\ \alpha[G^2(\tfrac{1}{2}) - w^2(\tfrac{1}{2})] + \tfrac{1}{2}\Delta(w^2)'(\tfrac{1}{2}) &= 0. \end{aligned}$$

Proceeding along the same lines as in [12, Lemma A.4], the proof that $\bar{J}^2 = w^2$ now follows from a standard verification argument based on the Itô–Tanaka–Meyer formula. First, let us observe that for $\tau \in \mathcal{T}$ and denoting by L the local time of X at $\frac{1}{2}$, we have

$$J^2(x, (\mu^1, S^1), \tau) = \mathbb{E}_x\left[R^2(X_\tau)\Lambda^1_\tau + \int_{[0,\tau)} G^2(X_s)\Lambda^1_s \,\mathrm{d}L_s\right]. \tag{7.6}$$

Applying the Itô–Tanaka–Meyer formula to the process $(\Lambda^1_t w^2(X_t))_{t\geq 0}$ yields

$$\begin{aligned} w^2(x) = \Lambda^1_\tau w^2(X_\tau) &- \int_{[0,\tau)} w^2(X_s)\,\mathrm{d}\Lambda^1_s - \int_{[0,\tau)} \Lambda^1_s (w^2)'(X_s)\,\mathrm{d}X_s \\ &- \frac{1}{2}\int_{[0,\tau)} \Lambda^1_s (w^2)''(X_s) X_s^2 (1-X_s)^2\,\mathrm{d}s - \frac{1}{2}\Delta(w^2)'(\tfrac{1}{2})\int_{[0,\tau)} \Lambda^1_s\,\mathrm{d}L_s. \end{aligned} \tag{7.7}$$

Because $(w^2)'' \leq 0$ on $(0,1) \setminus \{x^*, \frac{1}{2}, 1-x^*\}$, with equality on $(x^*, 1-x^*) \setminus \{\frac{1}{2}\}$, we have

$$-\int_{[0,\tau)} \Lambda^1_s (w^2)''(X_s) X_s^2 (1-X_s)^2\,\mathrm{d}s \geq 0. \tag{7.8}$$

From the last line of the variational system for w^2 and the properties of L, we have

$$\begin{aligned} -\frac{1}{2}\Delta(w^2)'(\tfrac{1}{2})\int_{[0,\tau)} \Lambda^1_s\,\mathrm{d}L_s &= \alpha\big[G^2(\tfrac{1}{2}) - w^2(\tfrac{1}{2})\big]\int_{[0,\tau)} \Lambda^1_s\,\mathrm{d}L_s \\ &= \int_{[0,\tau)} \alpha G^2(X_s)\Lambda^1_s\,\mathrm{d}L_s - \int_{[0,\tau)} \alpha \Lambda^1_s w^2(X_s)\,\mathrm{d}L_s \\ &= \int_{[0,\tau)} G^2(X_s)\,\mathrm{d}\Gamma^1_s + \int_{[0,\tau)} w^2(X_s)\,\mathrm{d}\Lambda^1_s. \end{aligned} \tag{7.9}$$

We deduce that

$$
\begin{aligned}
w^2(x) &\geq \mathbb{E}_x\left[\Lambda^1_\tau w^2(X_\tau) + \int_{[0,\tau)} G^2(X_s)\,\mathrm{d}\Gamma^1_s\right] \\
&\geq \mathbb{E}_x\left[\Lambda^1_\tau R^2(X_\tau) + \int_{[0,\tau)} G^2(X_s)\,\mathrm{d}\Gamma^1_s\right] \\
&= J^2(x,(\mu^1,S^1),\tau), \qquad (7.10)
\end{aligned}
$$

where the first inequality follows from (7.7)–(7.9) along with the fact that the stochastic integral in (7.7) is a centered integrable variable as X is a bounded martingale, the second inequality follows from the fact that $w^2 \geq R^2$ on $(0,1)$, and the equality follows from (7.6). Taking the supremum over $\tau \in \mathcal{T}$ in (7.10) yields $w^2 \geq \bar{J}^2$. It is easy to check that the above inequalities turn into equalities when $\tau = \tau_{S^2}$, which concludes the proof that $w^2 = \bar{J}^2$. Hence the result. □

Let us conclude this section by explaining why the pure Nash equilibria constructed using the method of Hamadene and Zhang [23] need not be Markovian. In our example, the algorithm in [23] actually stops after two iterations and leads to the following equilibrium: Assume first that player 1 never stops. Then, as shown in the proof of Proposition 7.1, a pure best reply of player 2 is to use the hitting time τ_{S^2} with $S^2 := (0,\frac{1}{3}] \cup [\frac{2}{3},1)$. In turn, facing the strategy $(0,S^2)$, a pure best reply of player 1 is to use the hitting time τ_{S^1}, where, letting $\bar{J}^1$ denote the brvf of player 1 against $(0,S^2)$, $S^1 := \{\bar{J}^1 = R^1\}$ is a nonempty subset of $(\frac{1}{3},\frac{2}{3})$, see again the proof of Proposition 7.1. Define then the stopping time

$$
\tau^1 = \mathbb{1}_{\tau_{S^1}<\tau_{S^2}}\,\tau_{S^1} + \mathbb{1}_{\tau_{S^2}<\tau_{S^1}}\,\infty.
$$

This strategy consists for player 1 in stopping in S^1 if X did not visit S^2 before, and to never stop if X visits S^2 before S^1 (one could say that player 1 threatens to play ∞ if player 2 does not stop in S^2). On the one hand, τ^1 is a best reply to τ_{S^2} as it gives the same payoff to player 1 as τ_{S^1} against τ_{S^2}. On the other hand, whereas, as shown in the proof of Proposition 7.1, τ_{S^2} is not a best reply to τ_{S^1}, it turns out that τ_{S^2} is a best reply to τ^1 and that (τ^1,τ_{S^2}) is a Nash equilibrium. Indeed, when facing the strategy τ^1, player 2 will not stop if S^1 is reached before S^2 as $G^2 > R^2$, and player 2 will not stop before X reaches S^1 or S^2 as this would give him a strictly smaller payoff than playing τ_{S^2}. However, if X reaches S^2 before S^1, player 2 believes that player 1 will never stop in the future, and thus the best player 2 can do is to play a best reply against the stopping time ∞, that is, to stop in S^2. Notice that we may reverse the roles of the players in this construction and obtain another Nash equilibrium in which player 2 plays a non-Markovian strategy.

Appendix

A.1 Proof of Proposition 2.8

Point (a) follows from the fact that stopping immediately is suboptimal in problem (3.3). Point (b) follows from the fact that, for $x \in S^j$, the payoff of player i is $G^i(x)$ if he does not stop immediately and $R^i(x) \leq G^i(x)$ otherwise. Let us now prove point (c). Under the stated condition, by continuity, there exist $C \in \mathbb{R}$ and $\varepsilon, \delta > 0$ such that

$$\forall y \in [x-\delta, x+\delta],\ G^i(y) \geq C \geq R^i(y) + \varepsilon. \tag{A.1}$$

Using that $\tau^i = \infty$ is suboptimal in problem (3.3), we have for all $y \in [x-\delta, x+\delta]$, letting τ_x and τ_δ denote respectively the hitting time of x and the exit time of $[x-\delta, x+\delta]$:

$$\begin{aligned}
\bar{J}^i(y) &\geq \mathbb{E}_y\left[\int_{[0,\infty)} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right] \\
&= \mathbb{E}_y\left[\int_{[0,\tau_x]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right] \\
&= \mathbb{E}_y\left[\mathbb{1}_{\tau_x<\tau_\delta} \int_{[0,\tau_x]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right] + \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[0,\tau_x]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right] \\
&\geq C\mathbb{E}_y[\mathrm{e}^{-r\tau_x} \mathbb{1}_{\tau_x<\tau_\delta}] + \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[0,\tau_x]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right],
\end{aligned} \tag{A.2}$$

where the second inequality follows from the fact that $\int_{[0,\tau_x]} \mathrm{d}\Gamma^j_s = 1$ when $\tau_x < \infty$ as $x \in S^j$. Consider the last term on the right hand side of (A.2). We have

$$\begin{aligned}
&\mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[\tau_\delta,\tau_x]} \mathrm{e}^{-rs} |G^i(X_s)|\, \mathrm{d}\Gamma^j_s\right] \\
&\quad= \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta}\, \mathrm{e}^{-r\tau_\delta} \left[|G^i(X_{\tau_\delta})|(\Gamma^j_{\tau_\delta} - \Gamma^j_{\tau_\delta-}) + \Lambda^j_{\tau_\delta} \int_{(\tau_\delta,\tau_x]} \mathrm{e}^{-r(s-\tau_\delta)} |G^i(X_s)|\, \mathrm{d}(\Gamma^j_s \circ \theta_{\tau_\delta})\right]\right] \\
&\quad= \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta}\, \mathrm{e}^{-r\tau_\delta} \Lambda^j_{\tau_\delta-} \int_{[\tau_\delta,\tau_x]} \mathrm{e}^{-r(s-\tau_\delta)} |G^i(X_s)|\, \mathrm{d}(\Gamma^j_s \circ \theta_{\tau_\delta})\right],
\end{aligned}$$

where the first equality follows from (2.10), and the second equality follows from the facts that $\Gamma^j_{\tau_\delta} - \Gamma^j_{\tau_\delta-} = \Lambda^j_{\tau_\delta-} - \Lambda^j_{\tau_\delta}$ and that Λ^j is continuous except at τ_{S^j} where it jumps to 0. Using this result, we have, for some constant $C' > 0$,

$$\begin{aligned}
&\left|\mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[0,\tau_x]} \mathrm{e}^{-rs} G^i(X_s)\, \mathrm{d}\Gamma^j_s\right]\right| \\
&\leq \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[0,\tau_\delta)} \mathrm{e}^{-rs} |G^i(X_s)|\, \mathrm{d}\Gamma^j_s\right] + \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \int_{[\tau_\delta,\tau_x]} \mathrm{e}^{-rs} |G^i(X_s)|\, \mathrm{d}\Gamma^j_s\right] \\
&\leq \sup_{[x-\delta,x+\delta]} |G^i|\, \mathbb{E}_y[\mathbb{1}_{\tau_x>\tau_\delta}] + \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta}\, \mathrm{e}^{-r\tau_\delta} \Lambda^j_{\tau_\delta-} \int_{[\tau_\delta,\tau_x]} \mathrm{e}^{-r(s-\tau_\delta)} |G^i(X_s)|\, \mathrm{d}(\Gamma^j_s \circ \theta_{\tau_\delta})\right]
\end{aligned}$$

$$\leq \sup_{[x-\delta,x+\delta]} |G^i| \, \mathbb{E}_y[\mathbb{1}_{\tau_x>\tau_\delta}] + \mathbb{E}_y\left[\mathbb{1}_{\tau_x>\tau_\delta} \, \mathbb{E}_{X_{\tau_\delta}}\left[\sup_{t\geq 0} \mathrm{e}^{-rt}|G^i(X_t)|\right]\right]$$
$$\leq C'\mathbb{P}_y[\tau_x > \tau_\delta], \tag{A.3}$$

where the third inequality follows from the Markov property, and the fourth inequality follows from assumption A1 along with the fact that $X_{\tau_\delta} \in \{x-\delta, x+\delta\}$ $\mathbb{P}_y$-almost surely. From (A.2)–(A.3), we deduce that

$$\bar{J}^i(y) \geq C\mathbb{E}_y[\mathrm{e}^{-r\tau_x}\mathbb{1}_{\tau_x<\tau_\delta}] - C'\mathbb{P}_y[\tau_x > \tau_\delta].$$

The above lower bound is a continuous function of y that is equal to C at x and to $-C'$ at $x-\delta$ and $x+\delta$. Therefore, by (A.1), there exists $\delta' \in (0,\delta)$ such that $\bar{J}^i(y) > R^i(y)$ for all $y \in [x-\delta', x+\delta']$. This proves (c).

Finally, points (i)–(iv) can be proven exactly as in [12, Proposition 1]. Hence the result.

A.2 Proof of Equation (3.10)

For the sake of completeness, we show how to deduce (3.10) from the arguments in [18]. Recall that Z^x is the Snell envelope on the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ of the process $\bar{Y}$ defined by

$$\bar{Y}_t := \int_{[0,t]} \mathrm{e}^{-rs}G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_t\,\mathrm{e}^{-rt}R^i(X_t), \quad t \geq 0,$$

and that $\hat{Z}$ is defined by

$$\widehat{Z}_t := \int_{[0,t]} \mathrm{e}^{-rs}G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_t\,\mathrm{e}^{-rt}\bar{J}^i(X_t), \quad t \geq 0.$$

First, it is clear that $\widehat{Z} \geq \bar{Y}$. Then, recall that (see [18, Lemma 3.4 and the references therein], noticing that we work on the smaller canonical space of continuous trajectories), for every stopping time τ of $(\mathcal{F}^0_t)_{t\geq 0}$ and every stopping time ρ of $(\mathcal{F}^0_{t+})_{t\geq 0}$ such that $\rho \geq \tau$, there exists an $\mathcal{F}^0_\tau \otimes \mathcal{F}^0_\infty$ measurable random variable $U : \Omega \times \Omega \to [0,\infty]$ such that

- $U(\omega,\tilde{\omega}) = 0$ if $\tau(\omega) = \infty$ or if $X_0(\tilde{\omega}) \neq X_\tau(\omega)$;
- for each $\omega \in \Omega$, $U(\omega,\cdot)$ is a stopping time of $(\mathcal{F}^0_{t+})_{t\geq 0}$;
- for all $\omega \in \Omega$ such that $\tau(\omega) < \infty$, $\rho(\omega) = \tau(\omega) + U(\omega, \theta_{\tau(\omega)}(\omega))$.

We deduce that, on the event $\{\tau < \infty\}$,

$$\mathbb{E}_x[\bar{Y}_\rho\,|\,\mathcal{F}_\tau] = \mathbb{E}_x\left[\int_{[0,\rho]} \mathrm{e}^{-rs}G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_\rho\,\mathrm{e}^{-r\rho}R^i(X_\rho)\,\middle|\,\mathcal{F}_\tau\right]$$

$$
\begin{aligned}
&= \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma_s^j + \int_{(\tau,\rho]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma_s^j + \Lambda_\rho^j\,\mathrm{e}^{-r\rho} R^i(X_\rho)\,\Big|\,\mathcal{F}_\tau\right] \\
&= \int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma_s^j + \Lambda_\tau^j\,\mathrm{e}^{-r\tau}\mathbb{E}_x\left[\int_{(0,U(\omega,\theta_\tau(\omega))]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}(\Gamma^j\circ\theta_\tau)_s\right. \\
&\qquad \left. + (\Lambda^j_{U(\omega,\theta_\tau(\omega))}\circ\theta_\tau)\,\mathrm{e}^{-rU(\omega,\theta_\tau(\omega))} R^i(X_{\tau+U(\omega,\theta_\tau(\omega))})\,\Big|\,\mathcal{F}_\tau\right] \\
&= \int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma_s^j + \Lambda_\tau^j\,\mathrm{e}^{-r\tau}\mathbb{E}_x\left[\int_{[0,U(\omega,\theta_\tau(\omega))]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}(\Gamma^j\circ\theta_\tau)_s\right. \\
&\qquad \left. + (\Lambda^j_{U(\omega,\theta_\tau(\omega))}\circ\theta_\tau)\,\mathrm{e}^{-rU(\omega,\theta_\tau(\omega))} R^i(X_{\tau+U(\omega,\theta_\tau(\omega))})\,\Big|\,\mathcal{F}_\tau\right] \\
&\le \int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma_s^j + \Lambda_\tau^j\,\mathrm{e}^{-r\tau}\bar{J}^i(X_\tau), \\
&= \widehat{Z}_\tau,
\end{aligned}
$$

where the third equality follows from (2.10) and the decomposition of stopping times, the fourth equality follows from the fact that $\Lambda_\tau^j = 0$ whenever $\Gamma^j\circ\theta_\tau$ has a jump at time 0 by (2.10), which allows us to replace the integral over $(0, U(\omega,\theta_\tau(\omega))]$ by an integral over $[0, U(\omega,\theta_\tau(\omega))]$, and the inequality follows from the Markov property. We deduce that $\mathbb{E}_x[\bar{Y}_\rho\,|\,\mathcal{F}_\tau] \le \widehat{Z}_\tau$ as it is an equality on $\{\tau=\infty\}$. Because for each $x\in\mathcal{I}$, every stopping time in $\mathcal{T}$ is $\mathbb{P}_x$-a.s. equal to a stopping time of $(\mathcal{F}^0_{t+})_{t\ge 0}$ [24, Lemma I.1.19]), we deduce that, for every stopping time τ of $(\mathcal{F}^0_t)_{t\ge0}$,

$$
Z^x_\tau = \operatorname*{ess\,sup}_{\rho\ge\tau,\,\rho\in\mathcal{T}} \mathbb{E}_x[\bar{Y}_\rho\,|\,\mathcal{F}_\tau] \le \widehat{Z}_\tau.
$$

To prove the reverse inequality, it is sufficient to prove that $\mathbb{E}_x[\widehat{Z}_\tau] \le \mathbb{E}_x[Z^x_\tau]$. By [18, Proposition 2.4],

$$
\forall \nu\in\Delta(\mathcal{I}),\ \int_{\mathcal{I}} \bar{J}^i(y)\,\nu(\mathrm{d}y) = \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_\nu[\bar{Y}_\rho],
$$

where $\mathcal{T}^0$ denotes the set of stopping times of the canonical filtration $(\mathcal{F}^0_t)_{t\ge0}$. Let $\hat{\nu}$ denote the finite measure on $\mathcal{I}$ defined by

$$
\hat{\nu}(A) := \mathbb{E}_x[\Lambda^j_\tau\,\mathrm{e}^{-r\tau}\mathbb{1}_A(X_\tau)], \quad A\in\mathcal{B}(\mathcal{I}).
$$

Whenever $\nu\ne 0$, define the probability $\nu := \frac{\hat{\nu}}{\hat{\nu}(\mathcal{I})}$. Then, denoting by $\tilde{\Omega}$ a copy of the canonical space endowed with the probabilities $\tilde{\mathbb{P}}_y := \mathbb{P}_y$ for $y\in\mathcal{I}$, we have

$$
\mathbb{E}_x[\Lambda^j_\tau\,\mathrm{e}^{-r\tau}\bar{J}^i(X_\tau)] = \int_{\mathcal{I}} \bar{J}^i(y)\,\hat{\nu}(\mathrm{d}y) = \hat{\nu}(\mathcal{I})\sup_{\rho\in\mathcal{T}^0}\mathbb{E}_\nu[\bar{Y}_\rho] = \sup_{\rho\in\mathcal{T}^0}\mathbb{E}_x[\Lambda^j_\tau\,\mathrm{e}^{-r\tau}\,\tilde{\mathbb{E}}_{X_\tau}[\bar{Y}_\rho]]. \tag{A.4}
$$

We deduce that

$$\begin{aligned}
\mathbb{E}_x[\widehat{Z}_\tau] &\le \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_\tau\,\mathrm{e}^{-r\tau}\,\tilde{\mathbb{E}}_{X_\tau}[\bar{Y}_\rho]\right] \\
&= \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_\tau\,\mathrm{e}^{-r\tau}\,\mathbb{E}_x\left[\int_{[0,\rho\circ\theta_\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}(\Gamma^j\circ\theta_\tau)_s\right.\right. \\
&\qquad\left.\left. + (\Lambda^j_{\rho\circ\theta_\tau}\circ\theta_\tau)\,\mathrm{e}^{-r(\rho\circ\theta_\tau)} R^i(X_{\tau+\rho\circ\theta_\tau}) \,\Big|\, \mathcal{F}_\tau\right]\right] \\
&= \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_x\left[\int_{[0,\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_\tau\,\mathrm{e}^{-r\tau}\,\mathbb{E}_x\left[\int_{(0,\rho\circ\theta_\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}(\Gamma^j\circ\theta_\tau)_s\right.\right. \\
&\qquad\left.\left. + (\Lambda^j_{\rho\circ\theta_\tau}\circ\theta_\tau)\,\mathrm{e}^{-r(\rho\circ\theta_\tau)} R^i(X_{\tau+\rho\circ\theta_\tau}) \,\Big|\, \mathcal{F}_\tau\right]\right] \\
&= \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_x\left[\int_{[0,\tau+\rho\circ\theta_\tau]} \mathrm{e}^{-rs} G^i(X_s)\,\mathrm{d}\Gamma^j_s + \Lambda^j_{\tau+\rho\circ\theta_\tau}\,\mathrm{e}^{-r(\tau+\rho\circ\theta_\tau)} R^i(X_{\tau+\rho\circ\theta_\tau})\right] \\
&= \sup_{\rho\in\mathcal{T}^0} \mathbb{E}_x[\bar{Y}_{\tau+\rho\circ\theta_\tau}] \\
&\le \mathbb{E}_x[Z^x_\tau],
\end{aligned}$$

where the first inequality follows from (A.4), the first equality follows from the strong Markov property, the second equality follows from the fact that $\Lambda^j_\tau = 0$ whenever $\Gamma^j\circ\theta_\tau$ has a jump at time 0, which allows us to replace the integral over $[0,\rho\circ\theta_\tau]$ by an integral over $(0,\rho\circ\theta_\tau]$, and the third equality follows from (2.10). This concludes the proof of (3.10).

A.3 Proof of Proposition 4.1

It is hereafter assumed without explicit mention that $\mathcal{M}(\mathcal{I})$ is endowed with the topology ϑ. The proof consists of three parts.

Metrizability We first prove that $\mathcal{M}(\mathcal{I})$ is metrizable. By Urysohn's metrization theorem (see, e.g., [19, Theorem 4.58]), it is sufficient to check that $\mathcal{M}(\mathcal{I})$ is Hausdorff, regular, and second countable.

First, we check that $\mathcal{M}(\mathcal{I})$ is second countable. By definition, the topology ϑ has a countable subbasis of neighborhoods defined by all the sets U, V of the form

$$U = U_{a,b,c} := \{m \in \mathcal{M}(\mathcal{I}) : m((a,b)) > c\} \text{ and } V = V_{a,b,d} := \{m \in \mathcal{M}(\mathcal{I}) : m([a,b]) < d\}$$

for all $a, b \in \mathcal{I}\cap\mathbb{Q}$, $c \in [0,\infty)\cap\mathbb{Q}$, and $d \in ((0,+\infty)\cap\mathbb{Q})\cup\{\infty\}$. Therefore, $\mathcal{M}(\mathcal{I})$ is second countable.

Next, we check that $\mathcal{M}(\mathcal{I})$ is regular. To this end, let B be a nonempty closed set in $\mathcal{M}(\mathcal{I})$ and $m \in \mathcal{M}(\mathcal{I})\setminus B$. We have to prove that B and m have disjoint neighborhoods.

The complement B^c of B is open and thus

$$B^c = \bigcup_{\alpha} \left(\bigcap_{k=1}^{n_\alpha} O_k^\alpha \right) \text{ and } B = \bigcap_{\alpha} \left(\bigcup_{k=1}^{n_\alpha} (O_k^\alpha)^c \right),$$

where α ranges through an arbitrary countable set, and each O_k^α is of the form U, V above. In particular, there exists α such that $m \in \bigcap_{k=1}^{n_\alpha} O_k^\alpha$ and $B \subset B^\alpha := \bigcup_{k=1}^{n_\alpha} (O_k^\alpha)^c$. Therefore, it is sufficient to prove the claim for B^α instead of B. Thus assume that $B = \bigcup_{k=1}^{n} (O_k)^c$. In turn, it is sufficient to prove the claim for each $(O_k)^c$ and then take the union of the neighborhoods of each set $(O_k)^c$, and the (finite) intersection of the neighborhoods of m. Thus assume that $B = O^c$ with O of the form U, V above. We accordingly distinguish two cases, depending on the form of O.

Case $B = U^c$ with $U = U_{a,b,c}$ Let $\delta > 0$ such that $m((a,b)) > c + 2\delta$. There exists $(a', b') \subset (a, b)$ such that $m((a', b')) > c + 2\delta$ by inner regularity, so that $U_{a',b',c+2\delta}$ is an open neighborhood of m. On the other hand $V_{a',b',c+\delta}$ is an open neighborhood of B as

$$\forall \nu \in B,\ \nu([a', b']) \leq \nu((a, b)) \leq c < c + \delta.$$

To conclude, notice that $V_{a',b',c+\delta}$ and $U_{a',b',c+2\delta}$ are disjoint.

Case $B = V^c$ with $V = V_{a,b,d}$ Notice that $m \notin B$ is equivalent to $m([a, b]) < d$, so that $m([a, b]) < \infty$. There exists $(a', b') \supset [a, b]$ such that $m([a', b']) < d$ by outer regularity. Thus let d', d'' such that $m([a', b']) < d' < d'' < d$, and observe that $B \subset U_{a',b',d''}$ whereas $m \in V_{a',b',d'}$. To conclude, notice that $V_{a',b',d'}$ and $U_{a',b',d''}$ are disjoint.

Therefore, $\mathcal{M}(\mathcal{I})$ is regular.

Finally, we check that $\mathcal{M}(\mathcal{I})$ is Hausdorff. As $\mathcal{M}(\mathcal{I})$ is regular, it is sufficient to prove that singletons are closed. Let $m_0 \in \mathcal{M}(\mathcal{I})$ and consider the closed set

$$\begin{aligned} &C(m_0) \\ &:= \bigcap_{a,b \in \mathcal{I} \cap \mathbb{Q}} \big(\{m \in \mathcal{M}(\mathcal{I}) : m((a,b)) \leq m_0((a,b))\} \cap \{m \in \mathcal{M}(\mathcal{I}) : m([a,b]) \geq m_0([a,b])\} \big). \end{aligned}$$

If $m \neq m_0$, then there exists an interval $(a, b) \subset \mathcal{I}$ such that $m((a, b)) \neq m_0((a, b))$. By inner regularity, we can assume that $a, b \in \mathbb{Q}$. If $m((a, b)) > m_0((a, b))$, then $m \notin C(m_0)$. If $m((a, b)) < m_0((a, b))$, then, by inner regularity, there exists an interval $[a', b'] \subset (a, b)$ such that $a', b' \in \mathbb{Q}$ and $m([a', b']) < m_0([a', b'])$, so that $m \notin C(m_0)$. We conclude that $C(m_0) = \{m_0\}$ and hence that singletons are closed. Therefore, $\mathcal{M}(\mathcal{I})$ is Hausdorff.

The proof that $\mathcal{M}(\mathcal{I})$ is metrizable is now complete.

Proofs of Assertions 1–3 We prove each assertion in turn.

(1) Any open set $O \subset \mathcal{I}$ can be written as $O = \bigcup_{n\geq 0} O_n$ for some nondecreasing sequence $(O_n)_{n\geq 0}$ such that each O_n is a finite disjoint union of open intervals with rational endpoints. The mapping $m \mapsto m(O_n)$ is lsc as a finite sum of lsc mappings, and the mapping $m \mapsto m(O)$ is lsc as the supremum of lsc mappings.

(2) Any compact set $K \subset \mathcal{I}$ can be written as $K = \bigcap_{n\geq 0} K_n$ for some nonincreasing sequence $(K_n)_{n\geq 0}$ such that each K_n is a finite disjoint union of compact intervals with rational endpoints. The mapping $m \mapsto m(K_n)$ is usc as a finite sum of usc mappings, and the mapping $m \mapsto m(F)$ is usc as the infimum of usc mappings.

(3) Suppose that $m_n \to m$ in $\mathcal{M}(\mathcal{I})$. If $O \cap e(m) \neq \emptyset$ with O open, then $m(O) = \infty$, and thus $m_n(O) \to \infty$ by point (1). Now, let $\phi \in \mathcal{C}_c^+(\mathcal{I} \setminus e(m))$, with support K. Because $m(K) < \infty$, by outer regularity, there exists a compact neighborhood K' of K such that $m(K') < \infty$. Then, by point (2), $\limsup_{n\to\infty} m_n(K') \leq m(K') < \infty$. The restrictions of the measures $(m_n)_{n\geq 0}$ to the open set $O' := \operatorname{int} K'$ are therefore locally finite for any sufficiently large n, and by [26, Lemma 4.1(iv)], converge vaguely to the restriction of m to O', which implies $L_\phi(m_n) \to L_\phi(m)$.

Conversely, suppose that the sequence $(m_n)_{n\geq 0}$ in $\mathcal{M}(\mathcal{I})$ and the measure m satisfy the properties that, for every open set O such that $O \cap e(m) \neq \emptyset$, $m_n(O) \to \infty$, and that, for each $\phi \in \mathcal{C}_c^+(\mathcal{I} \setminus e(m))$, $L_\phi(m_n) \to L_\phi(m)$. We want to prove that $m_n \to m$ in $\mathcal{M}(\mathcal{I})$. Let $a, b \in \mathcal{I} \cap \mathbb{Q}$. If $(a, b) \cap e(m) \neq \emptyset$, then, by the first property, $\liminf_{n\to\infty} m_n((a, b)) = \infty = m((a, b))$. If $(a, b) \cap e(m) = \emptyset$, let $(\phi_k)_{k\geq 0}$ be a nondecreasing sequence of continuous functions with compact support in (a, b) with pointwise limit $\mathbb{1}_{(a,b)}$. Then, by the second property, we have, for each k,

$$\liminf_{n\to\infty} m_n((a, b)) \geq \lim_{n\to\infty} L_{\phi_k}(m_n) = L_{\phi_k}(m),$$

and thus, by monotone convergence,

$$\liminf_{n\to\infty} m_n((a, b)) \geq m((a, b)). \tag{A.5}$$

If $[a, b] \cap e(m) \neq \emptyset$, then $\limsup m_{n\to\infty}([a, b]) \leq \infty = m([a, b])$. If $[a, b] \cap e(m) = \emptyset$, then $m([a, b]) < \infty$, and there exists a', b' such that $[a, b] \subset (a', b')$ and $m((a', b')) < \infty$ by outer regularity; in particular, $(a', b') \cap e(m) = \emptyset$. Let $(\phi_k)_{k\geq 0}$ be a nonincreasing sequence of continuous functions with compact support in (a', b') and pointwise limit $\mathbb{1}_{[a,b]}$. Then, by the second property, we have, for each k,

$$\limsup_{n\to\infty} m_n([a, b]) \leq \lim_{n\to\infty} L_{\phi_k}(m_n) = L_{\phi_k}(m),$$

and thus, by bounded convergence,

$$\limsup_{n\to\infty} m_n([a,b]) \le m([a,b]). \tag{A.6}$$

We conclude from (A.5)–(A.6) that $m_n \to m$ in $\mathcal{M}(\mathcal{I})$.

Compactness We finally prove that $\mathcal{M}(\mathcal{I})$ is compact. As $\mathcal{M}(\mathcal{I})$ is metrizable, it is sufficient to prove that it is sequentially compact, i.e., that any sequence $(m_n)_{n\ge0}$ has a convergent subsequence. The proof consists of three steps.

Step 1 Let $\mathcal{B} = \{O_1, O_2, ...\}$ denote a countable basis of open sets for $\mathcal{I}$. If $\limsup_{n\to\infty} m_n (O_1) = \infty$, then we extract a subsequence $(m_n^1)_{n\ge0}$ such that $\lim_{n\to\infty} m_n^1(O_1) = \infty$, otherwise we let $(m_n^1)_{n\ge0} := (m_n)_{n\ge0}$. Assuming that the subsequence $(m_n^k)_{n\ge0}$ for some $k \ge 1$ is constructed, if $\limsup_{n\to\infty} m_n^k(O_{k+1}) = \infty$, then we extract a subsequence $(m_n^{k+1})_{n\ge0}$ of $(m_n^k)_{n\ge0}$ such that $\lim_{n\to\infty} m_n^{k+1}(O_{k+1}) = \infty$, otherwise we let $(m_n^{k+1})_{n\ge0} := (m_n^k)_{n\ge0}$. By diagonal extraction, we obtain a subsequence $(m_n^*)_{n\ge0}$ of $(m_n)_{n\ge0}$ such that, for each $k \ge 1$, either $\lim_{n\to\infty} m_n^*(O_k) = \infty$ or $\limsup_{n\to\infty} m_n^*(O_k) < \infty$. Now, for all $x \in \mathcal{I}$, let $D_x := \{k \ge 1 : x \in O_k\}$, and notice that $\{x\} = \bigcap_{k\in D_x} O_k$. Define then

$$S := \big\{x \in \mathcal{I} : \forall k \in D_x,\ \lim_{n\to\infty} m_n^*(O_k) = \infty\big\}.$$

We claim that S is closed. Indeed, let $(x_p)_{p\ge0}$ be a sequence in S with limit $x \in \mathcal{I}$. For each $k \in D_x$, we have $x_p \in O_k$ for p sufficiently large and thus $k \in D_{x_p}$. Therefore, $\lim_{n\to\infty} m_n^*(O_k) = \infty$ for all $k \in D_x$, which proves that $x \in S$. The claim follows.

Step 2 Let $(K_p)_{p\ge0}$ be an increasing sequence of compact sets such that $\bigcup_{p=0}^{\infty} K_p = \mathcal{I} \setminus S$. We claim that $\limsup_{n\to\infty} m_n^*(K_p) < \infty$ for all $p \ge 0$. Indeed, each $x \in K_p$ is such that there exists $k \in D_x$ such that $\limsup_{n\to\infty} m_n^*(O_k) < \infty$. These open sets form an open covering of K_p, and we may therefore extract a finite open cover $(O_{k_1}, \dots, O_{k_r})$. We conclude that

$$\limsup_{n\to\infty} m_n^*(K_p) \le \limsup_{n\to\infty} \sum_{t=1}^{r} m_n^*(O_{k_t}) \le \sum_{t=1}^{r} \limsup_{n\to\infty} m_n^*(O_{k_t}) < \infty.$$

The claim follows. Because $\limsup_{n\to\infty} m_n(K_1) < \infty$, the restriction of m_n^* to K_1 is a finite measure for all sufficiently large n. By [26, Theorem 4.2], it admits a subsequence that converges weakly to some finite measure μ_1 on K_1.[6] Iterating the process and using diagonal extraction, we can extract a subsequence $(m_n^{**})_{n\ge0}$ of $(m_n^*)_{n\ge0}$ such that, for each $p \ge 0$, the sequence of the restrictions of the measures $(m_n^{**})_{n\ge0}$ to K_p converges weakly to some

[6] A sequence a finite measures $(\nu_n)_{n\ge0}$ on some metric space E converge weakly to ν if $\int_E \phi\, \mathrm{d}\nu_n \to \int_E \phi\, \mathrm{d}\nu$ for every bounded and continuous function $\phi : E \to \mathbb{R}$.

finite measure μ_p on K_p. By construction, the measures $(\mu_p)_{p\geq 0}$ are consistent in the sense that there exists a Radon measure μ on $\mathcal{I}\setminus S$ whose restriction to K_p is μ_p for all $p\geq 0$, and therefore the sequence of the restrictions of the measures $(m_n^{**})_{n\geq 0}$ to $\mathcal{I}\setminus S$ converges vaguely to μ.

Step 3 Define $m\in\mathcal{M}(\mathcal{I})$ such that $e(m):=S$ and $m|_{\mathcal{I}\setminus e(m)}:=\mu$. Then, by Assertion 3, the subsequence $(m_n^{**})_{n\geq 0}$ constructed in Step 2 converges in $\mathcal{M}(\mathcal{I})$ to m. Hence the result.

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