WORKING PAPERS

N° 1533

May 2024

“Mechanism Design with Costly Inspection”

Amirreza Ahmadzadeh and Stephan Waizmann

Toulouse School of Economics

Mechanism Design with Costly Inspection

Amirreza Ahmadzadeh* and Stephan Waizmann † ‡

May 10, 2024

* Toulouse School of Economics, amirreza.ahmadzadeh@tse-fr.eu

† Yale University, stephan.waizmann@yale.edu.

‡ We are indebted to our advisors (in alphabetical order) Johannes Hörner, Thomas Mariotti, Larry Samuelson, and Jean Tirole for their support and guidance, and thank Deniz Kattwinkel, Teemu Pekkarinen, Bernardo Ribeiro, François Salanié, Anna Sanktjohanser and especially Marina Halac as well as audiences at the Econometric Society European Winter Meeting '23, the HEC Economics Ph.D. conference '23, the Midwest Economic Theory conference '23, the TSE Theory workshop, the TSE Ph.D. workshop, and the Yale Micro lunch for their comments and feedback. Part of this research was conducted while Stephan Waizmann was visiting the Toulouse School of Economics whose hospitality is gratefully acknowledged. All errors are our own.

This paper studies how to combine screening menus and inspection in mechanism design.
A Principal procures a good from an Agent whose cost is his private information. The Principal has three instruments: screening menus — i.e., quantities and transfers — and (ex-ante) inspection. Inspection is costly but reveals the Agent's cost.
The combination of inspection and screening menus mitigates inefficiencies: the optimal mechanism procures an efficient quantity from all Agents whose cost of production is sufficiently low, regardless of whether inspection has taken place. However, quantity distortions still necessarily occur in optimal regulation; the quantity procured from Agents with higher production costs is inefficiently low. Both results are true regardless of the magnitude of inspection costs.
In contrast to settings without inspection, incentive compatibility constraints do not bind locally. This paper provides a method to address this challenge, characterizing which constraints bind.

Keywords: Mechanism Design, Verification, Principal-Agent, Inspection, Procurement.
JEL: D82, D86, L51.

1. Introduction

Contemporary approaches to regulation operate under the assumption that regulators have less information about important data than the entity targeted by the regulation. For instance, a government lacks knowledge about a defense manufacturer's production costs, or a municipality is uninformed about the cost of constructing new infrastructure. The literatures on regulation and procurement show how to design transfer schemes in order to alleviate the problems caused by such asymmetric information.

In practice, however, regulators have the capabilities to acquire direct knowledge of the unknown variable. For example, the Defense Contract Audit Agency (DCAA), an agency under the United States Department of Defense, conducts inspections that "are generally completed before contract award where DCAA evaluates [...] how much it will cost the contractor to provide goods or services to the government."[1] The U.S. General Service Administration has its own Office of Audits whose responsibilities include conducting audits in procurement cases and construction projects. Its "[a]udits of [construction projects] take place before a contract is awarded" and "include the evaluation of submitted cost or pricing data[...]."[2]

In this paper, we study how inspection and transfers are optimally combined. We take a mechanism design approach to regulation and allow the regulator to use transfers, inspection and quantity menus. We examine the trade-offs between using either instrument and characterize their optimal use in regulation.

In our model, an Agent produces a good that the Principal values. The Agent has private information about his cost of production, a continuous variable. The Principal aims to procure a cost-dependent quantity from the Agent. To elicit the Agent's cost, the Principal can design a transfer and quantity scheme, and she can also learn the Agent's cost through costly inspection.

The Principal offers a mechanism to the Agent. For each reported cost, a (direct) mechanism specifies a probability that the Agent is inspected, as well as contingent quantities and transfers. More precisely, the mechanism specifies a quantity the Agent produces when not inspected and a transfer he receives as a function of the reported cost. When inspected, the mechanism specifies a quantity and a transfer conditional on the reported and the true cost, which has been observed through inspection. Crucially, the Agent is free to reject the mechanism ex–post and collect his outside option instead. This means the Principal cannot force the Agent to produce without reimbursing his cost of production. We place no further restrictions on the mechanism, including the magnitude of the transfers, and allow

[1]See DCAA (2023, p. 4).

[2]U.S. General Services Administration (2012, p. 12).

for stochastic inspection.

An optimal mechanism has the following features. First, incentives to the Agent are provided only through inspection and bonus payments when the Agent has reported his type truthfully. In case the Agent is not inspected, the Principal reimburses him his reported cost of production and sets his payoff at the outside option. Payments after inspection are strictly higher than the Agent's cost when he has reported his true type.

Second, the Agent produces the first–best quantity even when not inspected if his cost is low enough. While reminiscent of the familiar "no distortion at the top"-result in screening problems, this is a stronger property: there is an interval of types that produce the first-best quantity regardless of whether they are inspected or not. Consequently, there is no downward distortion of quantities for an interval of low-cost types. The quantity procured from types with higher costs is distorted downwards from the first–best benchmark.

Third, no type is inspected with probability 1. Inspecting a type with probability 1 is too costly for the Principal. In particular, this shows that restricting mechanisms to have deterministic inspection is never optimal. Moreover, low-cost types are inspected with a constant probability.

Lastly, reporting higher costs need not trigger inspection with a higher probability. We show there exist parameter values such that the inspection probability in an optimal mechanism is not monotone in the Agent's reported cost. We also provide sufficient conditions for the inspection probability to be monotonically increasing in the reported cost.

Our results highlight the efficiency implications of combining screening menus with costly inspection. When inspection takes place before production, and quantities and payments can be made contingent on its outcome, inefficiencies are mitigated: the optimal mechanism procures an efficient quantity from all agents whose cost of production is sufficiently low. However, quantity distortions still necessarily occur in optimal regulation; the quantity procured from agents with higher production costs is inefficiently low. Both results are true regardless of the magnitude of inspection costs.

From a methodological point of view, we combine the literatures on costly state verification and monopolistic screening. The literature on costly state verification has focused on the trade-off between costly information rents and costly inspection. The literature on monopolistic screening has emphasized the role of transfers and quantity distortions in providing incentives. Our model combines both aspects. It allows us to study the trade-off between quantity distortions and inspection costs in providing incentives to the Agent.

A major challenge is that incentive constraints do not bind locally in any optimal mechanism. Consequently, we cannot employ standard techniques based on the envelope theorem or the first-order approach. We provide a way to overcome this challenge. We are able to analytically characterize *which* incentive constraints bind in an optimal mechanism. Knowing which incentive constraints bind is needed to determine the quantities, transfers and inspection probabilities in an optimal mechanism. We characterize the binding incentives constraint as the unique solution to a system of differential equations.

The rest of the paper is organized as follows. The next section introduces the model. We provide our results in Section 3. The main proofs are relegated to the Appendix. In Section 4 we discuss our contribution to the literature. Section 5 concludes. The Online Appendix contains auxiliary results and proofs of technical lemmas.

2. Model

There is a Principal ("she"), and an Agent ("he"). The Agent produces a good that the Principal aims to procure. The Agent's per unit cost of production $\theta \in [\underline{\theta}, \overline{\theta}], 0 < \underline{\theta} < \overline{\theta}$, is his private information. As is common, we call the Agent's cost his type. The Principal believes the Agent's type is distributed according to the prior F. We assume the distribution F admits a density f that is continuously differentiable and bounded away from 0. The Principal aims to induce a type-contingent quantity allocation $q(\cdot) \in \mathbb{R}_+$.

The Principal has the ability to inspect the Agent's type. When the Principal inspects the Agent, she learns his type perfectly.[3] Inspection costs $\kappa > 0$ to the Principal. Moreover, the Principal can pay a transfer to the Agent.

The utility of the Agent from an allocation q and a transfer t given his type θ is $U(q, \theta) + t$. We follow Mussa and Rosen (1978) and assume that the Agent's utility from an allocation q is linear in his type, i.e., $U(q, \theta) = -\theta q$.

The utility of the Principal is $V(q) - t - \kappa \mathbb{1}_{\text{inspection}}$. We make the following standard assumption about the Principal's preferences.

Assumption 1 *$V : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, $V' > 0$, $V'' < 0$, and satisfies the Inada conditions $V'(q) \to_{q \searrow 0} \infty$ and $V'(q) \to_{q \to \infty} 0$.*

The Principal offers a mechanism $\mathbb{M}$ to the Agent. It is without loss to focus on direct mechanisms; see Lemma 8. A direct mechanism is a tuple $\mathbb{M} = \big(x(\cdot), q^I(\cdot, \cdot), t^I(\cdot, \cdot), q^N(\cdot), t^N(\cdot)\big)$. Here, $x(\hat{\theta})$ denotes the probability that the Agent is inspected when he reports type $\hat{\theta}$. When the Agent is inspected, the Principal learns his type θ, and the Agent produces $q^I(\hat{\theta}, \theta)$ and receives the transfer $t^I(\hat{\theta}, \theta)$. Note that quantity and transfer after inspection depend both on

[3]This assumption can be relaxed. See the literature review for a discussion.

the Agent's report $\hat{\theta}$ and his true type θ, which the Principal has learned through inspection. When the Agent is not inspected, he produces the quantity $q^N(\hat{\theta})$ and gets paid the transfer $t^N(\hat{\theta})$, conditional only on his report.

We assume that the Agent can reject the mechanism ex–post:[4] after the Agent observes if he has been inspected and observes the quantity he needs to produce as well as the transfer he receives, the Agent can walk away and secure a payoff of 0. When the Agent walks away from the mechanism, no production takes place, and the Principal does not pay the transfer. Standard arguments show that it is without loss of generality to focus on mechanisms that induce the Agent to not walk away.[5] This is also without loss of optimality, as no mechanism that induces the Agent to reject on path is optimal for the Principal. Throughout, we assume that the Agent does not reject the mechanism when indifferent. A direct mechanism that the Agent does not reject ex-post must satisfy, for every type θ and report $\hat{\theta}$, the following *obedience constraints*,

$$\begin{aligned} -q^I(\hat{\theta},\theta)\theta + t^I(\hat{\theta},\theta) &\geq 0; \\ -q^N(\theta)\theta + t^N(\theta) &\geq 0. \end{aligned} \tag{obedience constraints}$$

The Principal's problem is to choose a mechanism $\mathbb{M}$ that induces truth-telling and satisfies the obedience constraint. We make the weak technical assumption that the Principal has to inspect the Agent with probability at least $\underline{x}$ after any report, for a small but positive $\underline{x} > 0$. An optimal mechanism does not exist without this restriction.[6] Importantly, the optimal mechanism does not depend on the minimal inspection probability $\underline{x}$ in a meaningful way.[7]

[4]Assuming the Agent can reject the mechanism ex-post puts a lower bound on the payoff the Agent receives. Without a lower bound on the Agent's payoff after inspection, the Principal is able to implement the first-best quantity at a cost arbitrarily close to 0 by inspecting the Agent with a vanishingly small probability and driving the Agent's payoff to $-\infty$ if the Agent has not reported his true type. The assumption that the Agent can reject the mechanism ex-post is stronger than putting a lower bound on transfers. Indeed, the Principal could make the Agents payoff arbitrarily small even with bounded transfers by requiring an arbitrarily large quantity.

[5]See Lemma 9.

[6]See the discussion following Lemma 4 and Lemma 10 for proof.

[7]See Lemma 4 for a precise statement.

Formally, the Principal's problem is:

$$
\begin{aligned}
\sup_{\substack{x(\cdot),q^I(\cdot,\cdot),t^I(\cdot,\cdot),\\ q^N(\cdot),t^N(\cdot)}} \int_{\underline{\theta}}^{\overline{\theta}} & x(\theta)\Big(V(q^I(\theta,\theta)) - t^I(\theta,\theta) - \kappa\Big) \\
& + (1-x(\theta))\Big(V(q^N(\theta)) - t^N(\theta)\Big)\,\mathrm{d}F(\theta) \\
& \text{subject to, for all } \theta, \hat{\theta} \\
& x(\theta)\left(-q^I(\theta,\theta)\theta + t^I(\theta,\theta)\right) + (1-x(\theta))\left(-q^N(\theta)\theta + t^N(\theta)\right) \\
& \geq x(\hat{\theta})\left(-q^I(\hat{\theta},\theta)\theta + t^I(\hat{\theta},\theta)\right) + (1-x(\hat{\theta}))\left(-q^N(\hat{\theta})\theta + t^N(\hat{\theta})\right); \\
& -q^N(\theta)\theta + t^N(\theta) \geq 0; \\
& -q^I(\hat{\theta},\theta)\theta + t^I(\hat{\theta},\theta) \geq 0; \\
& \underline{x} \leq x(\hat{\theta}) \leq 1.
\end{aligned}
$$

Denote the problem by $\mathcal{P}_{\underline{x}}$. Throughout, we assume that $\mathcal{P}_{\underline{x}}$ admits a solution in the space of bounded and measurable functions. The first inequality constraint is the incentive compatibility constraint for type θ. It requires that type θ prefers reporting his true type θ to reporting any other type $\hat{\theta}$. The next two inequalities are the obedience constraints: the Agent prefers honoring the mechanism to walking away, both after inspection and without inspection. The last inequality ensures that the probability of inspection is between $\underline{x}$ and 1.

Under Assumption 1, for every type θ, the first-best quantity

$$
q^{FB}(\theta) = \arg\max_q \; V(q) - q\theta
$$

exists and is unique. We assume that there is no interval (θ', θ'') such that $q^{FB}(\theta) = 1/(c_1\theta - c_2)$ for $\theta \in (\theta', \theta'')$ for positive constants $c_1 > 0, c_2 \in [\underline{\theta}, \overline{\theta}]$.[8]

[8] The assumption guarantees that for no type θ, $q^{FB}(\theta')(\theta' - \theta)$ is constant on an interval of types θ'. The condition states that the payoff of type θ when producing $q^{FB}(\theta')$ and receiving a transfer of $\theta' q^{FB}(\theta')$ is not constant in θ'. The condition ensures that the set of binding incentive constraints in the optimal mechanism is "well-behaved", The assumption holds if there is no interval (q_1, q_2) such that $V(q) = c_1 \ln(q) + c_2 q + c_3$ for constants $c_1, c_2 > 0, c_3$ and all $q \in (q_1, q_2)$. We remark that this assumption is generically satisfied.

3. Analysis and Results

In this section, we derive properties of an optimal mechanism. All arguments in this section can be made precise; see Appendix A and the Online Appendix for formal proofs.

We proceed by identifying features each optimal mechanism must exhibit. We first show that incentives in an optimal mechanism are provided only through payments after truthful inspection. Next we establish that local incentive compatibility constraints do not bind in an optimal mechanism. We show how we overcome the technical difficulties caused by non–locally binding IC constraints. Third, we discuss the quantities and inspection probabilities in an optimal mechanism. Lastly, we provide sufficient conditions for the probability of inspection to be increasing in the reported cost. We finish with a characterization of the optimal mechanism.

3.1. Providing incentives

We start the analysis with a simple observation. Recall that the incentive constraint for type θ reads

$$\begin{aligned} & x(\theta)\left(-q^I(\theta,\theta)\theta + t^I(\theta,\theta)\right) + (1-x(\theta))\left(-q^N(\theta)\theta + t^N(\theta)\right) \\ \geq\ & x(\hat\theta)\left(-q^I(\hat\theta,\theta)\theta + t^I(\hat\theta,\theta)\right) + (1-x(\hat\theta))\left(-q^N(\hat\theta)\theta + t^N(\hat\theta)\right) \text{ for all } \hat\theta. \end{aligned} \tag{IC}$$

Observe that the quantity and transfer after inspection, $q^I(\hat\theta,\theta)$ and $t^I(\hat\theta,\theta)$, for $\hat\theta \neq \theta$ do not enter the incentive constraints for any type $\theta' \neq \theta$. Therefore, reducing $t^I(\hat\theta,\theta)$ and increasing $q^I(\hat\theta,\theta)$ relaxes the incentive constraints. Moreover, $t^I(\hat\theta,\theta)$ and $q^I(\hat\theta,\theta)$ do not affect the Principal's payoff. Therefore, due to the obedience constraints, it is without loss of optimality to restrict attention to mechanisms such that, for every θ,

$$-q^I(\hat\theta,\theta)\theta + t^I(\hat\theta,\theta) = 0\ \forall\hat\theta \neq \theta.$$

This is intuitive: it is optimal to punish the Agent for misreporting his type as harshly as possible. In the remainder of the paper, we abuse notation and write $t^I(\theta)$ and $q^I(\theta)$ instead of $t^I(\theta,\theta)$ and $q^I(\theta,\theta)$, respectively.

The second observation concerns the quantity after inspection. The optimal quantity after inspection is the first-best quantity, i.e., $q^I(\theta) = q^{FB}(\theta)$ for every type θ. To see this, note that $q^I(\theta)$ affects incentives only for type θ. Therefore, changing $q^I(\theta)$ to $q^{FB}(\theta)$ and making a compensatory change in $t^I(\theta)$ by $\theta(q^{FB}(\theta) - q^I(\theta))$ leaves the incentive constraints unaffected. Clearly, this change does not violate the obedience constraints either. However, the Principal's payoff increases.

This property of the optimal mechanism is also intuitive. The rent that has to be paid to the Agent to induce truth-telling is not affected by the quantity after inspection. Therefore, it cannot be optimal to distort the quantity from its first-best level.

Lemma 1 *In every optimal mechanism, the quantity after inspection and truth-telling is first-best,*

$$q^I(\theta, \theta) \equiv q^I(\theta) = q^{FB}(\theta).$$

Moreover, it is without loss of generality to consider mechanisms that punish lies as harshly as possible: for all $\theta \neq \hat{\theta}$,

$$-q^I(\hat{\theta}, \theta)\theta + t^I(\hat{\theta}, \theta) = 0.$$

Denote by $\pi(\theta)$ the rent of type θ, i.e., the highest payoff type θ can obtain by reporting any type $\hat{\theta}$. The previous Lemma implies that the Agent receives a payoff above 0 when he misreports his type only in case he is not inspected. When he is not inspected, his payoff is the transfer he receives minus his cost of production, $-q^N(\hat{\theta})\theta + t^N(\hat{\theta})$. Consequently, the information rent of type θ is

$$\pi(\theta) = \sup_{\hat{\theta}} (1 - x(\hat{\theta}))(-q^N(\hat{\theta})\theta + t^N(\hat{\theta})).$$

A mechanism is incentive compatible if and only if the Agent's payoff from reporting his true type is at least as high as his information rent. Moreover, the obedience constraints imply that the information rent is non-negative, $\pi \geq 0$.

Consider a type θ and recall that every report is inspected with positive probability. Because the Agent is risk–neutral with respect to transfers, his incentive to report his true type depends solely on the expected transfer (less the cost of production), $x(\theta)t^I(\theta) + (1 - x(\theta))t^N(\theta)$. Therefore, his incentives are preserved if a decrease in the transfer without inspection, $t^N(\theta)$, is compensated by an increase in the transfer after inspection, $t^I(\theta)$, such that the expected transfer remains constant. Because the Principal is risk-neutral with respect to transfers as well, this change leaves her payoff unaffected. Reducing the transfer without inspection that is paid to type θ, however, reduces the information rent that has to be paid to types $\theta' \neq \theta$. Consequently, for any incentive-compatible mechanism, there is a payoff-equivalent mechanism, implementing the same quantity allocation, such that the Agent gets paid his reported cost of production when not inspected. This shows the first part of the next Proposition.

Proposition 1 1. *For every incentive compatible mechanism that satisfies the obedience constraints there exists such a mechanism such that the transfer* *without inspection equals the cost of production, i.e.,*

$$t^N(\theta) = \theta q^N(\theta),$$

and both mechanisms have the same quantity allocation and inspection probability. Moreover, both mechanisms yield the same payoff to the Principal.

2. *In any optimal mechanism,*

$$q^N(\theta) = q^{FB}(\theta) \text{ or } t^N(\theta) = \theta q^N(\theta).$$

3. *For $\delta > 0$ let*

$$B_\delta = \{\hat{\theta} | t^N(\hat{\theta}) \geq q^N(\hat{\theta})\hat{\theta} + \delta\}$$

and

$$\hat{\theta}_\delta(\theta) = \{\hat{\theta} | (1 - x(\hat{\theta}))(-q^N(\hat{\theta})\theta + t^N(\hat{\theta})) \geq \pi(\theta) - \delta > 0\}.$$

If, for a positive measure of types θ,

$$\hat{\theta}_\delta(\theta) \subset B_\delta,$$

and B_δ has positive measure, then the mechanism $\mathbb{M}$ is not optimal.

Proof in Appendix A.1.

The first part of the Proposition 1 says that any allocation that can be implemented in a direct mechanism can be implemented by a mechanism such that $t^N(\theta) = \theta q^N(\theta)$. All incentives to the Agent can be provided through payments when the Agent is inspected and found to have reported his true type. When the Agent is not inspected, he is merely reimbursed his cost of production. That also implies a second result: restricting attention to mechanisms that satisfy $t^N(\theta) = \theta q^N(\theta)$ does not rule out any meaningful multiplicity of the optimal mechanism. The remainder of the Proposition illuminates to what extent this property is required in any optimal mechanism.

The second part of the Proposition states that either the transfer without inspection equals the cost of production, or the quantity without inspection is the first-best quantity. To see why this is true, suppose a type θ is paid more than his cost of production, $q^N(\theta)\theta$, when not inspected and produces strictly less than his first-best quantity. Then one can raise the quantity without inspection, keeping the transfer $t^N(\theta)$ constant, and adjust $t^I(\theta)$ to offset the increase in the production cost. This change does not affect the incentive constraint for any type θ' but raises the Principal's payoff. Consequently, the original mechanism was not optimal.

The third part of the Proposition is more subtle. It is best explained using finitely many types. Suppose a type θ^1 receives a transfer without inspection strictly higher than his cost of production, i.e., $-q^N(\theta^1)\theta^1 + t^N(\theta^1) > 0$, and suppose there exists another type θ^2 that is indifferent between reporting his true type and reporting θ^1, and strictly prefers reporting θ^1 to reporting any other type $\hat{\theta} \neq \theta^1, \theta^2$. Then the information rent that has to be paid to θ^2 to induce truth-telling can be reduced by reducing $t^N(\theta^1)$ and increasing $t^I(\theta^1)$. Thus, in an optimal mechanism, no type gets a transfer strictly higher than his cost of production when not inspected unless no other type finds it optimal to mimic this type. This is the intuitive meaning of the third part of Proposition 1. Its statement needs to account for the fact that a single type has 0 measure and thus does not affect the Principal's payoff.

To summarize, Proposition 1 shows that all incentives to the Agent *can* be provided only through payments above the outside option when inspected. But the Lemma also shows that, in any optimal mechanism, all incentives (almost) *must* be provided through payments when inspected.

We henceforth restrict attention to mechanisms such that $-q^N(\theta)\theta + t^N(\theta) = 0$.

Using that $-q^N(\theta)\theta + t^N(\theta) = 0$, it is straightforward to pin down the transfer after inspection. Because slack incentive constraints for some types θ cannot be optimal, the transfer after inspection must satisfy

$$x(\theta)\left(-q^{FB}(\theta)\theta + t^I(\theta)\right) = \sup_{\hat{\theta}} (1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta).$$

Here, we used $-q^N(\theta)\theta + t^N(\theta) = 0$ on the left–hand side of IC for type θ as well as on the right-hand side for type $\hat{\theta}$.

Observe that the possibility of inspection implies that every quantity allocation $(q^I(\cdot), q^N(\cdot))$ can be implemented in a direct mechanism. Consequently, we face no further restrictions on the quantities and inspection probability when solving for an optimal mechanism.

This reduces the Principal's problem to

$$\max_{q^N(\cdot), 1 \geq x(\cdot) \geq \underline{x}} \int x(\theta)\left(V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa\right) + (1 - x(\theta))\left(V(q^N(\theta)) - q^N(\theta)\theta\right) - \sup_{\hat{\theta}}(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta)\,\mathrm{d}F(\theta) \qquad \text{(unconstrained problem)}$$

The term in the first line is the social welfare from an allocation (x, q^N, q^I). The term on the second line is the information rent.

We effectively have reduced the problem to an unconstrained maximization problem. In the next section, we argue that the problem is not amenable to standard techniques. This is due to the sup-term in the objective function and the arg max , if it exists, not being known a priori. The next section shows how we overcome this problem.

3.2. Global incentive compatibility

In this subsection, we deal with the main technical challenge we face when deriving the optimal mechanism: global incentive compatibility. We will see that incentive constraints do not bind locally, but only globally. One major advantage of locally binding incentive compatibility constraints is not the fact that the constraints bind locally but that we know *which* constraints bind. Nonetheless, we manage to characterize which incentive constraints bind. Readers who are not interested in the methodological details may wish to skim this section.

Recall that the information rent of type θ is $\pi(\theta) = \sup_{\hat{\theta}}(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta)$. Since incentive constraints bind, type θ's payoff equals his information rent. Moreover, the information rent for a type θ is zero if and only if for all types $\theta^{'} > \theta$, $(1 - x(\theta^{'}))q^N(\theta^{'}) = 0$.

The expression of the information rent has a remarkable implication: incentive constraints do not bind locally for any type that receives a positive payoff $\pi(\theta) > 0$. The quantity without inspection $q^N(\theta)$ is bounded from above by the first–best quantity. Thus, for every type θ there exists a $\delta > 0$ such that $(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta) < \pi(\theta)$ for all $\hat{\theta}$ that are within δ of θ. Hence, reporting a type $\hat{\theta}$ within δ of θ gives type θ a payoff strictly less than his information rent.

A second implication is that incentive constraints only bind upwards: reporting a type $\hat{\theta} < \theta$ yields a negative payoff. Hence, downward incentive constraints are slack for each type that receives a positive payoff.

Define the correspondence[9]

$$\hat{\theta}(\theta) = \arg\max_{\hat{\theta}} \ (1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta).$$

The correspondence gives the binding incentive constraints for type θ; that is, for any $\hat{\theta} \in \hat{\theta}(\theta)$, the Agent is indifferent between reporting his true type θ and type $\hat{\theta}$ because, by our choice of $t^I(\theta)$,

$$x(\theta)(-q^{FB}(\theta)\theta + t^I(\theta)) = (1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta).$$

The correspondence $\hat{\theta}(\cdot)$ depends on the entire functions $q^N(\cdot)$ and $x(\cdot)$. Moreover,

[9] When local incentive constraints bind, $\hat{\theta}(\theta) = \{\theta\}$. In particular, when local incentive constraints are sufficient and necessary for global incentive compatibility, $\hat{\theta}(\cdot)$ does not depend on the quantity. When the allocation is monotone in the type, transfers exist such that the incentive constraints hold, even without inspection. Consequently, standard optimal control techniques can be applied. This is not possible in our case: not only are the incentive constraints not locally binding, but the set of binding constraints depends on the entire functions $x(\cdot)$ and $q^N(\cdot)$.

it is a priori not clear that the correspondence is well-behaved in an optimal mechanism. In particular, $\hat{\theta}(\cdot)$ may be empty– or multi-valued. Nonetheless, we can show the following technical Lemma.

Lemma 2 *There exists a solution such that*

1. $(1-x(\cdot))q^N(\cdot)$ *is a differentiable function that is strictly decreasing when positive;*

2. $\hat{\theta}(\cdot)$ *is single-valued and, viewed as a function, increasing.*

Proof in Appendix B.4

The intuition behind the continuity and monotonicity of $(1-x(\cdot))q^N(\cdot) \equiv Q^N(\cdot)$ is the following. Suppose that $Q^N(\cdot)$ is weakly increasing on the interval $(\theta', \theta^\dagger)$. Then $Q^N(\theta^\dagger)(\theta^\dagger - \theta) > Q^N(\hat{\theta})(\hat{\theta} - \theta)$ for all $\hat{\theta} \in (\theta', \theta^\dagger)$ for every type $\theta \leq \theta'$. Consequently, no type wants to mimic any such $\hat{\theta}$ and the Principal can receive a higher payoff by increasing $q^N(\cdot)$ on the interval $(\theta', \theta^\dagger)$.

Continuity of $Q^N(\cdot)$ holds for a similar reason. Suppose $Q^N(\cdot)$ has a downward jump at θ'. Since $Q^N(\cdot)$ is decreasing, no type wants to mimic any $\hat{\theta} \in (\theta', \theta' + \delta)$ for some small but positive δ. Consequently, the Principal can increase her payoff by increasing $q^N(\cdot)$ on $(\theta', \theta' + \delta)$.

Continuity of $Q^N(\theta)$ implies that $\hat{\theta}(\cdot)$ is not empty–valued. The other statements in the Lemma are tedious to prove. See the proof for details.

Recall that, in a model where inspection is not possible, monotonicity of $q^N(\cdot)$ is necessary and sufficient for the existence of a transfer scheme $t^N(\cdot)$ that implements $q^N(\cdot)$ in an incentive compatible manner. However, when inspection is feasible, monotonicity of $q^N(\cdot)$ or $Q^N(\cdot)$ is not required for incentive compatibility; any function $q^N(\cdot)$ can be implemented in an incentive compatible mechanism, however weirdly behaved. The monotonicity result in Lemma 2 is due not to the feasibility, but the optimality of implementing an allocation $Q^N(\cdot)$. Allocations for which $Q^N(\cdot)$ is not strictly decreasing are not optimal because they entail types no other type wants to mimic, and consequently, for which quantity distortions cannot be optimal.

Lemma 2 shows that the set of binding incentive constraints is well-behaved. Unfortunately, this is not enough to solve for the optimal mechanism: we also need to know which constraints bind. Yet, we are able to characterize the set of binding incentive constraints.

Lemma 3 *For all $\theta \in \left(\hat{\theta}(\underline{\theta}), \overline{\theta}\right)$ such that $\pi(\theta) > 0$, the function $\hat{\theta}(\cdot)$ is strictly increasing, differentiable and obeys the differential equation*

$$(\hat{\theta}(\theta) - \theta) f(\theta) = \hat{\theta}'(\theta) \left(V'(q^N(\hat{\theta}(\theta))) - \hat{\theta}(\theta) \right) f(\hat{\theta}(\theta)) \qquad (\text{ode } \hat{\theta})$$

with an appropriate boundary condition.
Proof in Appendix A.2.

The intuition behind this result is closely related to results in mechanism design without inspection. Recall that, under regularity conditions, the second-best quantity in a setting without inspection, call it $y(\theta)$, solves[10]

$$V'(y(\theta)) = \theta + \frac{F(\theta)}{f(\theta)}.$$

Multiplying both sides by $y(\theta)$, rearranging and integrating over an arbitrary interval $I \subset [\underline{\theta}, \overline{\theta}]$ yields

$$\int_I y(\theta)[V'(y(\theta)) - \theta] \, \mathrm{d}F(\theta) = \int_I y(\theta) \frac{F(\theta)}{f(\theta)} \, \mathrm{d}F(\theta).$$

The left-hand side is a (weighted measure) of the distortion of the second-best quantity $y(\theta)$ away from the first-best quantity. Recalling that the information rent π^{SB} in the problem without inspection solves $\dot{\pi}^{SB}(\theta) = -y(\theta)$ and hence

$$\int_{[\underline{\theta}, \overline{\theta}]} \pi^{SB}(\theta) \, \mathrm{d}F(\theta) = \int_{[\underline{\theta}, \overline{\theta}]} y(\theta) \frac{F(\theta)}{f(\theta)} \, \mathrm{d}F(\theta),$$

so that the equation above states that the distortion needs to equal the change in the information rent. Similarly, we can show that, in our setting with inspection,

$$\int_I (1 - x(\theta)) q^N(\theta) [V'(q^N(\theta)) - \theta] \, \mathrm{d}F(\theta) = \int_{\hat{\theta}(\theta) \in I} \pi(\theta) \, \mathrm{d}F(\theta).$$

The only difference with the case without inspection is that the integration of the information rent is over the set $\hat{\theta}(\theta) \in I$ to account for the fact that local incentive constraints do not bind. The differential equation for $\hat{\theta}(\cdot)$ in Lemma 3 follows from the equation in the last display.

Lemmas 2 and 3 have another implication. For every type θ there is exactly one type $\hat{\theta} > \theta$ such that type θ is indifferent between reporting his true type and type

[10] See, e.g., Laffont and Martimort (2002, pp. 134).

$\hat{\theta}$. Conversely, for every type $\hat{\theta}$ above a threshold θ_1 there exists exactly one type θ that is indifferent between reporting his true type and the type $\hat{\theta}$.

3.3. Optimal quantity and inspection probability

In this section, we describe the quantities and the inspection probability in an optimal mechanism. We first show that the quantity does not depend on the minimal inspection probability, $\underline{x}$. We then characterize quantities explicitly, making use of the results we obtained in the last section for the binding incentive constraints.

We first start with an observation about the minimal inspection probability $\underline{x}$. The minimal inspection probability affects the optimal mechanism only through the inspection probability. In particular, $\underline{x}$ does not affect quantities in an optimal mechanism.

Lemma 4 *The quantities in an optimal mechanism do not depend on the minimal inspection probability. Formally, let* $\underline{x}, \underline{x}' \in (0,1)$*. Then there is a solution* $\mathbb{M}_{\underline{x}} = \left(x_{\underline{x}}(\cdot), q^I_{\underline{x}}(\cdot,\cdot), t^I_{\underline{x}}(\cdot,\cdot), q^N_{\underline{x}}(\cdot), t^N_{\underline{x}}(\cdot)\right)$ *to* $\mathcal{P}_{\underline{x}}$ *and a solution to* $\mathcal{P}_{\underline{x}'}$, $\mathbb{M}_{\underline{x}'} = \left(x_{\underline{x}'}(\cdot), q^I_{\underline{x}'}(\cdot,\cdot), t^I_{\underline{x}'}(\cdot,\cdot), q^N_{\underline{x}'}(\cdot), t^N_{\underline{x}'}(\cdot)\right)$*, such that*

$$\left(q^I_{\underline{x}'}(\cdot,\cdot), q^N_{\underline{x}'}(\cdot), t^N_{\underline{x}'}(\cdot)\right) = \left(q^I_{\underline{x}}(\cdot,\cdot), q^N_{\underline{x}}(\cdot), t^N_{\underline{x}}(\cdot)\right).$$

Moreover, $x_{\underline{x}}$ *and* $x_{\underline{x}'}$ *are related by*

$$\frac{1 - x_{\underline{x}}(\theta)}{1 - \underline{x}} = \frac{1 - x_{\underline{x}'}(\theta)}{1 - \underline{x}'}$$

and

$$t^I_{\underline{x}}(\theta) - \theta q^{FB}(\theta) = \frac{1}{x_{\underline{x}}(\theta)} \left(\frac{1 - \underline{x}}{1 - \underline{x}'} - (1 - x_{\underline{x}}(\theta))\right)(t^I_{\underline{x}'} - \theta q^{FB}(\theta)).$$

Proof in Appendix B.6.

Lemma 4 states that the minimal inspection probability does not affect the quantities in an optimal mechanism. The result relies on two factors. First, the optimal quantity after inspection is independent of the inspection probability and the quantity without inspection. Second, inspection probabilities affect information rents in the same way as they affect the social welfare trade-off between inspection and no-inspection. In particular, scaling the probability of not inspecting, $1 - x(\theta)$, by the same factor for all types θ does not alter the trade-off between higher costs of inspection, lower quantities without inspection compared to inspection, and lower information rents. Consequently, the minimal probability of inspection affects the mechanism only up to scale of inspection.

Lemma 4 justifies our interpretation of the lower bound on the inspection probability as a purely technical assumption. Quantities in an optimal mechanism are not affected by the minimal inspection probability. With the exception of the transfer after inspection, $t^I(\cdot)$, all limits exist as the lower bound vanishes. In particular, for types θ that are inspected with the minimal probability, $x_{\underline{x}}(\theta) = \underline{x}$ ($\iff x_{\underline{x}'}(\theta) = \underline{x}'$), the transfer after inspection, $t^I(\cdot)$, grows unboundedly as the minimal inspection probability vanishes:

$$t^I_{\underline{x}}(\theta) - \theta q^{FB}(\theta) = \frac{1-\underline{x}}{\underline{x}} \left(\frac{1}{1-\underline{x}'} - 1 \right) (t^I_{\underline{x}'} - \theta q^{FB}(\theta)) \to \infty$$

as $\underline{x} \to 0$. This is the reason for the non-existence of an optimal mechanism without a lower bound on the inspection probability. However, the expected transfer after inspection, $x(\theta)t^I(\theta)$, has a well-defined limit for every type θ. Moreover, for the quantity and transfer without inspection, the limit coincides with the quantity and transfer for each positive but fixed lower bound $\underline{x}$. The Principal's payoff converges as well.

Lemma 5 *For any $\underline{x} \in (0, 1)$, the quantity and inspection probabilities are almost everywhere unique in every optimal mechanism.*
Proof in Appendix B.7.

The uniqueness of quantities and the inspection probability implies that the optimal mechanism is unique up to some multiplicity in transfers. Multiplicity in transfers is limited by the results of Proposition 1.

Proposition 2 *The following holds in the optimal mechanism.*

1. *Low-cost types produce their first-best quantity and are inspected with the minimal probability: there exists a $\theta_1 > \underline{\theta}$ such that*

$$\textit{for all} \;\; \theta < \theta_1, \;\; x(\theta) = \underline{x} \;\; \textit{and} \;\; q^N(\theta) = q^{FB}(\theta).$$

2. *Intermediate cost types are inspected with the minimal probability $\underline{x}$ and produce a quantity strictly less than first-best: there exists a $\theta_2, \theta_1 < \theta_2 \leq \overline{\theta}$, such that $x(\theta) = \underline{x}$ and $q^N(\theta) < q^{FB}(\theta)$ for all types $\theta \in [\theta_1, \theta_2)$.*

3. *for all types θ such that $\underline{x} < x(\theta) < 1$ the quantity without inspection is strictly less than first-best, strictly decreasing in θ and independent of $x(\theta)$. It is given as the unique solution $q = q^N(\theta)$ to*

$$V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa = V(q) - qV'(q). \quad \text{(quantity-interior-inspection)}$$

Proof in Appendix A.3.

The intuition behind the first part of Proposition 2 is the following. Type $\underline{\theta}$ receives the highest information rent. It is not optimal for type $\underline{\theta}$ to mimic any type $\hat{\theta}$ with $(1-\underline{x})q^{FB}(\hat{\theta})(\hat{\theta}-\underline{\theta}) < \pi(\underline{\theta})$. In particular, type $\underline{\theta}$ does not want to mimic types $\hat{\theta}$ that are close to $\underline{\theta}$. Because the information rent is continuous and decreasing in the type, a similar argument holds for types close enough to $\underline{\theta}$. Consequently, there exists a type θ_1 such that no other type finds it optimal to mimic a type $\hat{\theta} < \theta_1$.

However, if no type finds it optimal to mimic $\hat{\theta}$, distorting the quantity $q^N(\hat{\theta})$ downwards from the first-best does not lower information rents. Hence, it is not optimal to distort the quantity for type $\hat{\theta} < \theta_1$. Consequently, $q^N(\hat{\theta}) = q^{FB}(\hat{\theta})$ for such types. Moreover, since no type finds it optimal to mimic such types, inspecting them with probability strictly greater than $\underline{x}$ does not reduce the total information rents. It is therefore optimal to inspect those types with the lowest feasible probability $\underline{x}$.

The result regarding the quantities is reminiscent of the familiar "no-distortion-at-the-top" property. However, it is much stronger. It states that there is a positive mass of types $[\underline{\theta}, \theta_1)$ which produce the first-best quantity when not inspected. For such types, it is not optimal to distort quantities downward from first-best.

The property that quantities are not distorted for a positive mass of types follows from two properties. First, incentive constraints only bind upwards. Second, incentive constraints do not bind locally. Hence, there must be an interval of types which no other type wants to mimic. Distorting quantities is not optimal, however, when doing so does not decrease information rents. Consequently, there is an interval of low cost types that produce the first-best quantity even when not inspected.

The second part of Proposition 2 states that intermediate-cost types are inspected with the minimal inspection probability $\underline{x}$. However, these types produce a quantity less than the first-best quantity when not inspected. For types in some range $[\theta_1, \theta_2)$ there exists a type θ that is indifferent between reporting his true type and mimicking some type in $[\theta_1, \theta_2)$. Consequently, distorting quantities downwards in this range reduces the information rent of some types. Hence, types in that range produce less than the first-best quantity. We explain why it is optimal to distort the quantity downwards instead of increasing the probability of inspection for such types after discussing the last part of Proposition 2.

For the last part of Proposition 2, recall that the information rent is $\pi(\theta) = \sup_{\hat{\theta}}(1-x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta}-\theta)$. Distorting quantities without inspection downwards and increasing the probability of inspection are substitutes for reducing information rents. In particular, the information rent is unaffected if the quantity is increased by a positive factor and the probability of not being inspected is decreased antiproportionally. Increasing $q^N(\theta)$ raises the Principal's payoff by

$$V'(q^N(\theta))\theta - q^N(\theta)\theta.$$

The anti-proportional change in the inspection probability raises the Principal's payoff by

$$V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa - \left(V(q^N(\theta)) - \theta q^N(\theta)\right).$$

In an optimal mechanism, this change cannot increase the Principal's payoff. Such a change is feasible whenever $\underline{x} < x(\theta) < 1$. Consequently, when the inspection probability is interior in an optimal mechanism, both changes must offset each other: the quantity without inspection must satisfy quantity-interior-inspection.

The Proposition implies that the quantity without inspection does not depend on the distribution of types, at least when $x(\theta) > \underline{x}$.[11] Moreover, the quantity without inspection $q^N(\theta)$ given by equation quantity-interior-inspection is strictly less than the type's first-best quantity, $q^{FB}(\theta)$, for any $\kappa > 0$.[12]

Using the third part of Proposition 1, we can now return to the second part to explain why the probability of inspection for intermediate types is kept at the minimal probability $\underline{x}$. Recall that Lemma 2 states that $(1 - x(\cdot)q^N(\cdot)$ is continuous. Moreover, the quantity without inspection is bounded from above by the first-best quantity. As just noted, when $\underline{x} < x(\theta) < 1$, the quantity without inspection is strictly less than first-best, $q^N(\theta) < q^{FB}(\theta)$. In particular, $(1 - x(\theta))q(\theta) < (1 - \underline{x})q^{FB}(\theta)$ when q satisfies quantity-interior-inspection. Hence, there must be an interval of types, $[\theta_1, \theta_2)$, that are inspected with the minimal probability $\underline{x}$; for otherwise continuity of $(1 - x(\cdot))q^N(\cdot)$ would be violated.

Proposition 2 is mute on the existence of a region of types that are inspected with probability above the minimal inspection probability, $x(\theta) > \underline{x}$. In fact, there are parameters such that no type is inspected with probability greater than $\underline{x}$. This happens, for example, when the cost of inspection, κ, is large.[13] However, the existence of a region in which types are inspected with probability $\underline{x}$ but produce

---

[11] The set of types for which $x(\theta) > \underline{x}$ depends on the distribution of types.

[12] Denote by $\tilde{q}(\theta)$ the optimal quantity in a model where inspection is not possible. There are parameters such that $\tilde{q}(\theta) > q^N(\theta)$, that is, for some types the quantity without inspection is distorted more than in the setting where inspection is not possible. For example, let $V(q) = \ln(q)$ so that $q^{FB}(\theta) = 1/\theta$, and $q^N(\theta) = e^{-\kappa}/\theta$ for $1 > x(\theta) > \underline{x}$. Assuming $[\underline{\theta}, \overline{\theta}] = [1, 2]$ and a uniform distribution over types, in the model where inspection is not feasible, the optimal quantity is $\tilde{q}(\theta) = \frac{1}{\theta(1+\ln(\theta))}$. If the cost of inspection, κ, is large enough, $e^\kappa > 1 + \ln(\theta)$ so that $\tilde{q}(\theta) > q^N(\theta)$.

[13] More precisely, there exists a $\underline{\kappa} > 0$ such that for all $\kappa \geq \underline{\kappa}$, all types θ are inspected with the minimal probability $x \equiv \underline{x}$ in any optimal mechanism. To see this, note that the solution to equation quantity-interior-inspection converges to 0 as $\kappa \to \infty$. By our assumption that $V'(q) \to_{q \to 0} \infty$, it cannot happen that a positive mass of types is inspected with probability strictly above $\underline{x}$ when κ is large enough.

strictly less than their first-best quantity is guaranteed by Proposition 2. This implies that the optimal contract does not implement the first-best quantity with probability 1.

The following example illustrates the results of Proposition 2 and Lemma 3.

Example 1 *Assume $V(q) = \ln(q)$.*[14] *Then $q^{FB}(\theta) = 1/\theta$, and the quantity without inspection given in quantity-interior-inspection satisfies $q^N(\theta) = e^{-\kappa}/\theta$.*

Let $f(\theta) = \alpha/\theta$ for some $\alpha > 0$, and $\underline{\theta} = 1$. Then the differential equation characterizing the binding incentive constraints, ode $\hat{\theta}$, for $x(\theta) > \underline{x}$ reads

$$(\hat{\theta}(\theta) - \theta)f(\theta) = \hat{\theta}'(\theta)\left(V'(q^N(\hat{\theta}(\theta))) - \hat{\theta}(\theta)\right)f(\hat{\theta}(\theta)) = \hat{\theta}'(\theta)\hat{\theta}(\theta)\left(e^{\kappa} - 1\right)f(\hat{\theta}(\theta)),$$

or equivalently,

$$\frac{\hat{\theta}(\theta) - \theta}{\theta} = \hat{\theta}'(\theta)\left(e^{\kappa} - 1\right),$$

with end point condition $\hat{\theta}(\overline{\theta}) = \overline{\theta}$.[15] *The unique solution is*

$$\hat{\theta}(\theta) = \frac{\overline{\theta}^{1-A}}{1-A}\theta^A - \frac{A\theta}{1-A},$$

where $A = 1/(e^{\kappa} - 1)$. If, for example, $\overline{\theta} = 2$ and $\kappa = \ln(3/2)$ then $A = 2$ and

$$\hat{\theta}(\theta) = 2\theta - \frac{\theta^2}{2}.$$

The next Lemma states how to derive the optimal quantity without inspection for types $\theta \geq \theta_1$ that are inspected with the minimal probability $\underline{x}$.

Lemma 6 *Suppose $x(\hat{\theta}) = \underline{x}$ for $\hat{\theta} \in [\theta', \theta^{\dagger}]$ for some $\theta' \geq \theta_1$. Then the quantity without inspection $q(\hat{\theta}) \equiv q^N(\hat{\theta})$ for types $\hat{\theta} \in [\theta', \theta^{\dagger}]$ solves*

$$-q'(\hat{\theta})(\hat{\theta} - \theta) = q(\hat{\theta}),$$
$$\hat{\theta} = \hat{\theta}(\theta) \ \text{solves} \ \ (ode\,\hat{\theta}),$$

for $\theta \in [\hat{\theta}^{-1}(\theta'), \hat{\theta}^{-1}(\theta^{\dagger})]$ with an appropriate boundary condition for $\hat{\theta}(\cdot)$ and $q(\cdot)$.
Proof in Appendix B.5.

[14] $V(q) = \ln(q)$ does not satisfy $(1/q^{FB}(\theta))'' \neq 0$. Nevertheless, our proofs remain correct in this case.

[15] This will follow from Lemma 7.

To see why the differential equation must hold, recall that $(1-x(\cdot))q^N(\cdot)$ is strictly decreasing by Lemma 2. Consequently, $\hat{\theta}$ must satisfy the first-order condition for maximizing $(1-x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta}-\theta)$. Since the inspection probability is constant for $\hat{\theta} \in [\theta_1, \theta_2)$, the first-order conditions yield the differential equation in the Lemma. The optimal quantity and the binding incentive constraints $\hat{\theta}$ can and must be determined simultaneously.

Moreover, the differential equation for $\hat{\theta}$ implies that the quantity $q(\theta)$ in Lemma 6 is strictly less than the first-best quantity: the right-hand side in the differential equation ode $\hat{\theta}$ vanishes if $q^N(\hat{\theta})$ is the first-best quantity. By the second part of Lemma 2, $\hat{\theta}(\cdot)$ is strictly increasing, which requires that $q^N(\hat{\theta}) < q^{FB}(\hat{\theta})$.

Knowing the optimal quantities, $q^N(\cdot)$, and the binding incentive constraints, $\hat{\theta}(\cdot)$, we can recover the inspection probability, $x(\cdot)$, in an optimal mechanism. Recall that $Q^N(\cdot) = (1-x(\cdot))q^N(\cdot)$ is strictly decreasing and differentiable. Moreover, the set of binding incentive constraints, $\hat{\theta}(\cdot)$, is a strictly increasing function. Consequently, for every θ, $\hat{\theta}(\theta)$ satisfies the first–order condition

$$-x'(\hat{\theta})q^N(\hat{\theta})(\hat{\theta}-\theta) + (1-x(\hat{\theta}))\frac{\partial q^N}{\partial \hat{\theta}}(\hat{\theta})(\hat{\theta}-\theta) + (1-x(\hat{\theta}))q^N(\hat{\theta}) = 0. \quad \text{(ode-}x(\cdot))$$

Recall that $\hat{\theta}(\cdot)$ is determined solely by the quantity without inspection, $q^N(\cdot)$: the inspection probability $x(\cdot)$ does not show up in equation (ode $\hat{\theta}$). Hence, we can determine the optimal inspection probability as the solution to a differential equation, with an appropriate boundary condition.[16] Details on deriving $x(\cdot)$ are provided in Appendix B.9.

Example 1 (continued) *Define $y(\theta) = 1 - x(\theta)$. By ode-$x(\cdot)$, we know*

$$-\frac{\mathrm{d}ln(y(\hat{\theta}))}{\mathrm{d}\hat{\theta}} = \frac{\partial\, ln(\frac{\hat{\theta}-\theta}{\hat{\theta}})}{\partial\, \hat{\theta}}.$$

Therefore

$$-\ln(y(\theta))\Big|_t^{\overline{\theta}} = \int_t^{\overline{\theta}} \frac{\partial\, \ln(\frac{\hat{\theta}-\theta}{\hat{\theta}})}{\partial\, \hat{\theta}}\, \mathrm{d}\hat{\theta} = \int_t^{\overline{\theta}} \left(\frac{1}{\hat{\theta}-\theta} - \frac{1}{\hat{\theta}}\right) \mathrm{d}\hat{\theta}$$

$$= \int_{\hat{\theta}^{-1}(t)}^{\hat{\theta}^{-1}(\overline{\theta})} \left(\frac{1}{\hat{\theta}(\theta)-\theta}\right) \hat{\theta}'(\theta)\, \mathrm{d}\theta - \ln(\theta)\Big|_t^{\overline{\theta}}$$

[16] See Theorem 1.

From the differential equation for $\hat{\theta}(\theta)$, we know $\frac{1}{\theta(e^{\kappa}-1)} = \frac{\hat{\theta}'(\theta)}{\hat{\theta}(\theta)-\theta}$. *Therefore*

$$-\ln(y(\theta))\Big|_{t}^{\overline{\theta}} = \frac{1}{e^{\kappa}-1}\ln(\theta)\Big|_{\hat{\theta}^{-1}(t)}^{\hat{\theta}^{-1}(\overline{\theta})} - \ln(\theta)\Big|_{t}^{\overline{\theta}}.$$

Equivalently,

$$\frac{y(\theta)}{y(\overline{\theta})} = \frac{\overline{\theta}^{A-1}\theta}{(\hat{\theta}^{-1}(\theta))^{A}} = \frac{\theta}{(1-A)\theta + A\hat{\theta}^{-1}(\theta)}.$$

Assuming again that $\overline{\theta} = 2$ and $A = 2$, we have

$$x(\theta) = 1 - \left(1 - x(\overline{\theta})\right)\frac{2\theta}{(2-\sqrt{4-2\theta})^2}.$$

The last result in this section, Lemma 7, states that it is not optimal to inspect a type with probability 1.

Lemma 7 *In an optimal mechanism, no type is inspected with probability* 1.
Proof in Appendix B.8.

The intuition behind this result is the following. Suppose there is a set of types $(\theta', \theta^{\dagger})$ with $\underline{\theta} < \theta' < \theta^{\dagger} < \overline{\theta}$ that are inspected with probability 1 and for some type $\hat{\theta}, \theta^{\dagger} < \hat{\theta} < \overline{\theta}$, the probability of inspection is less than 1, $x(\hat{\theta}) < 1$. Then for all types $\theta < \theta'$, the information rent is bounded away from 0 since $\pi(\theta) \geq (1 - x(\hat{\theta}))q^{N}(\hat{\theta})(\hat{\theta} - \theta) > 0$. Then there exists some $\overline{x} < 1$ such that $(1 - \overline{x})q^{FB}(\tilde{\theta})(\tilde{\theta} - \theta) < \pi(\theta)/2$ for all $\theta < \theta', \tilde{\theta} \in (\theta', \theta^{\dagger})$. Consequently, one can alter the mechanism such that the probability of inspection equals $\overline{x}$ on $(\theta', \theta^{\dagger})$ without affecting the information rent of other types. This change reduces the cost of inspection and thus increases the Principal's payoff. The reason why there is no threshold above which all types are inspected with probability 1 is more subtle; see the proof of Lemma 7.

3.4. Optimal mechanism

The preceding sections discuss properties any optimal mechanism must possess. In this section, we combine the results. We exhibit a complete characterization of the optimal mechanism. Theorem 1 states the properties of an optimal mechanism.

For a fixed type $\theta_1 \in (\underline{\theta}, \overline{\theta})$ let $(q_1, \hat{\theta}_1)$ be the solution to

$$-q'(\hat{\theta})(\hat{\theta} - \theta) = q(\hat{\theta}),$$
$$\hat{\theta} = \hat{\theta}(\theta) \text{ solves } (\text{ode}\,\hat{\theta}),$$

with the boundary conditions $q_1(\theta_1) = q^{FB}(\theta_1), \hat{\theta}_1(\underline{\theta}) = \theta_1$. With $q_1(\theta)$ as defined above, denote

$$\theta_2 = \min\{\overline{\theta}, \inf\{\theta | q_1(\theta) < q_2(\theta)\}\}, \tag{θ_2}$$

where $q_2(\theta)$ is the solution to equation quantity-interior-inspection.

Theorem 1 *An optimal mechanism in which the inspection probability is weakly increasing is uniquely characterized by the threshold $\theta_1 = \hat{\theta}(\underline{\theta})$. The threshold θ_1 defines a second threshold θ_2 given by equation (θ_2). These thresholds divide the type space into three regions: low cost types $[\underline{\theta}, \theta_1]$, intermediate cost types $(\theta_1, \theta_2]$, and high cost types $(\theta_2, \overline{\theta}]$. The quantity without inspection, q^N, for low and high cost types is given as in Proposition 2, part 1 and part 3, respectively. The quantity without inspection for intermediate cost types is given by $q_1(\cdot)$. The inspection probability is equal to the minimal inspection probability for low and intermediate cost types, and is given by the solution to ode-$x(\cdot)$ for high cost types.*

Proof in Appendix A.4.

Theorem 1 does not state that threshold θ_2 is strictly below $\overline{\theta}$. When the cost of inspection, κ, is sufficiently large, the threshold equals the upper bound of the type space $\theta_2 = \overline{\theta}$. In that case only the regimes for low-cost and intermediate-cost types are part of the optimal mechanism. Conversely, when the cost of inspection is sufficiently small, $\theta_2 < \overline{\theta}$, and all three regimes exist.

A comment on the hypothesis of increasing inspection probability in Theorem 1 is in order. The hypothesis is stronger than needed. It is sufficient that the inspection probability is strictly greater than the minimal probability, i.e., $x(\theta) > \underline{x}$, for types above the threshold θ_2. Even if that is not the case, we can characterize the optimal mechanism: the regimes for quantity and inspection probability between θ_1 and θ_2, and between θ_2 and $\overline{\theta}$ alternate. More precisely, there is a sequence of thresholds $(\theta_i)_{i=1}^n$ such that the quantity without inspection is given as in Lemma 6 and $x(\theta) = \underline{x}$ for $\theta \in (\theta_{2k-1}, \theta_{2k})$, whereas the quantity without inspection is given by equation quantity-interior-inspection for $\theta \in [\theta_{2k}, \theta_{2k+1}]$.

Example 1 (continued) *We continue with the Example 1. Suppose that $\underline{x} = 0.1$ and recall that the lower bound affects the inspection probability only up to scale. For* [17] *$\theta_1 = 1.38$, $\theta_2 = 1.64$. This yields $x(\overline{\theta}) = 0.636$. Consequently, the inspection*

[17]This is the unique (numerically derived) threshold θ_1 such that all necessary conditions in Theorem 1 are satisfied.

probability is:

$$x(\theta) = \begin{cases} 0.1, & \text{for } \theta \in [1, 1.64], \\ 1 - 0.364\frac{2\theta}{(2-\sqrt{4-2\theta})^2}, & \text{for } \theta \in (1.64, 2]. \end{cases}$$

Now we derive $\hat{\theta}(\cdot)$ and $q^N(\cdot)$ for $\theta \le \theta_2$. For ease of computation, we solve for $h(\cdot) = \hat{\theta}^{-1}(\cdot)$ instead of $\hat{\theta}(\cdot)$. For $q(\cdot) = q^N(\cdot)$, the system of differential equations for $\theta \le \theta_2$ is given as

$$h'(\theta) = \frac{(1/q(\theta) - \theta)h(\theta)}{(\theta - h(\theta))\theta},$$
$$q'(\theta) = -\frac{q(\theta)}{\theta - h(\theta)}.$$

The initial and end point conditions are:

$$h(\theta_2) = 2 - \sqrt{4 - 2\theta_2}, \quad q(\theta_2) = \frac{e^{-\kappa}}{\theta_2}, \quad h(\theta_1) = \underline{\theta}.$$

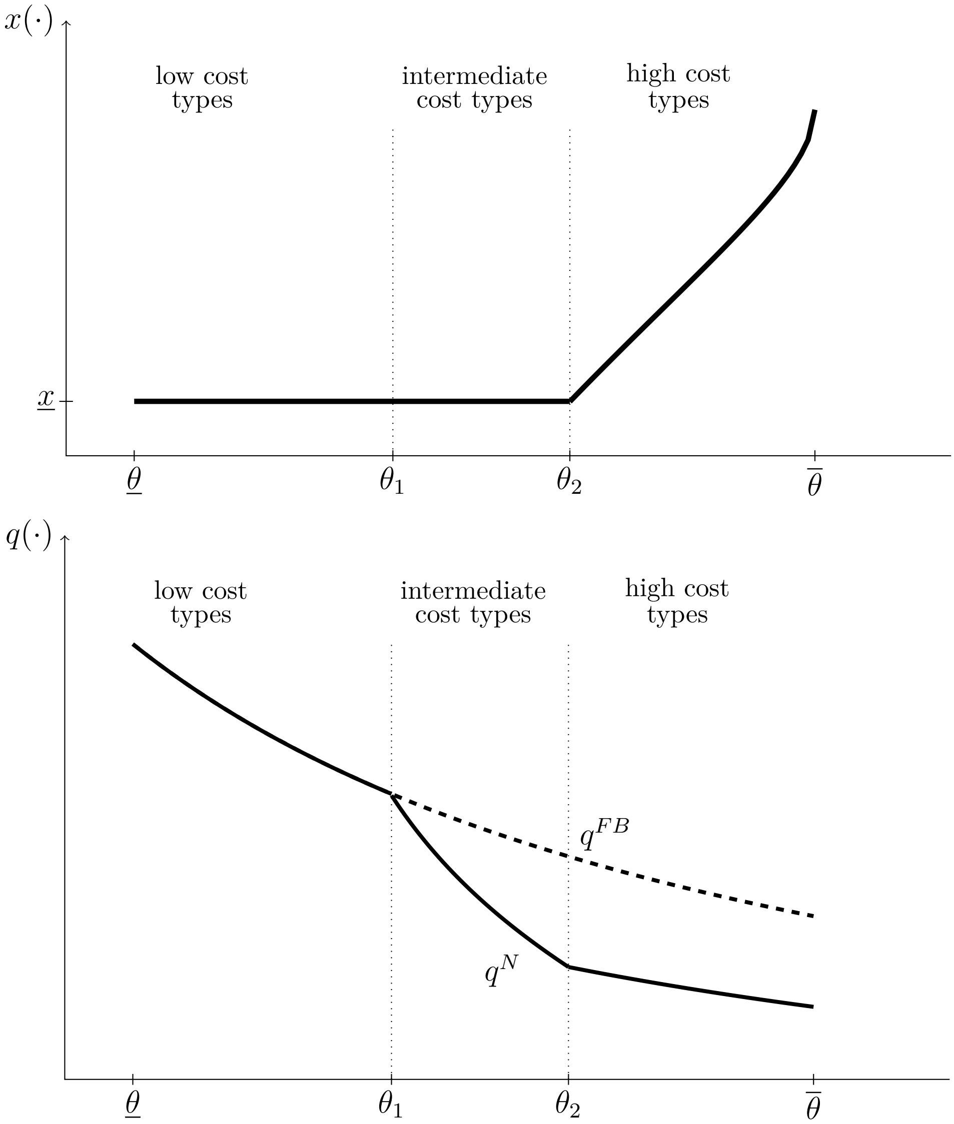
Solving the above differential equation (numerically) we get $h(\cdot)$, and $q(\cdot)$. The below figure express these functions.

Finally, note that for $\theta \le \theta_1 = 1.38$, $q^N(\theta) = q^{FB}(\theta) = 1/\theta$. Figure 1 shows the the optimal mechanism. Low cost types produce the first best quantity. Intermediate cost types produce the quantity derived using the differential equation above. High cost types produce the quantity given by equation quantity-interior-inspection. The inspection probability equals the minimal inspection probability for low and intermediate cost types. It is strictly higher than the minimal inspection probability for high cost types.

Figure 2 shows the binding incentive constraints, i.e., the function $\hat{\theta}(\cdot)$. θ_1 is the lowest type any other type wants to mimic. Higher types want to mimic higher types.

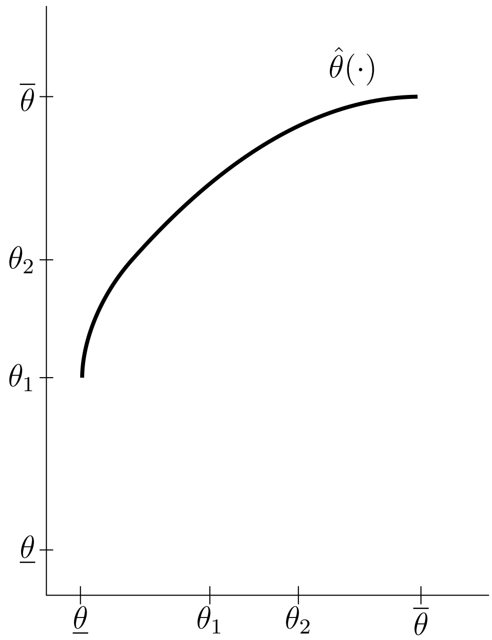
3.5. Monotonicity of inspection

So far, we have not shown that the probability of inspection is monotone in the Agent's type. In fact, this need not be the case; example 2 shows that the inspection probability can be strictly decreasing in the reported cost for some types. However, we provide sufficient conditions on the primitives of the model that guarantee that the inspection probability is increasing in the Agent's type. Proposition 3 states the sufficient conditions.



This figure shows the the optimal mechanism for Example 1. The first panel depicts the inspection probabilities. The horizontal axis depicts the type space and the vertical axis the inspection probability. For low and intermediate cost types, the inspection probability equals the minimal inspection probability. For high cost types, the inspection probability is strictly above the minimum $\underline{x}$.
The second panel shows the quantities in the optimal mechanism. The horizontal axis depicts the type space and the vertical axis quantities. The dashed line shows the first-best quantity. The solid line shows the quantity without inspection in the optimal mechanism. The quantity without inspection is strictly decreasing. Low cost types produce the first-best quantity when not inspected. The quantity without inspection is strictly less than the first-best quantity for intermediate and high cost types.

Figure 1: The optimal mechanism for Example 1.



This figure shows the binding incentive constraints, i.e., the function $\hat{\theta}(\cdot)$, for Example 1. The horizontal and vertical axis depict the type space. For a type θ on the horizontal axis the graph shows the type $\hat{\theta}$ on the vertical axis such that the Agent of type θ is indifferent between reporting his true type and reporting $\hat{\theta}$. No type wants to mimic a type lower than θ_1. Higher types want to mimic higher types. Moreover, for every type $\hat{\theta} \geq \theta_1$ on the vertical axis there is exactly one type θ on the horizontal axis indifferent between mimicking $\hat{\theta}$ and reporting truthfully.

Figure 2: Binding incentive constraints in the optimal mechanism of Example 1.

Proposition 3 *Let $q(\cdot)$ be given as in equation quantity-interior-inspection. Then the probability of inspection is weakly increasing if one of the following three conditions hold:*

1. *For all $\theta \in [\underline{\theta}, \overline{\theta}]$, $\theta \mapsto \theta q(\theta)$ is a weakly increasing function.*
2. *The third derivative of the Principal's valuation for quantity is negative, $V''' \leq 0$, and the following inequality is true:*

$$\frac{1}{\overline{\theta} - \underline{\theta}} \geq -\frac{q'(\underline{\theta})}{q(\underline{\theta})} = -\frac{q^{FB}(\underline{\theta})}{q^2(\underline{\theta}) V''(q(\underline{\theta}))}.$$

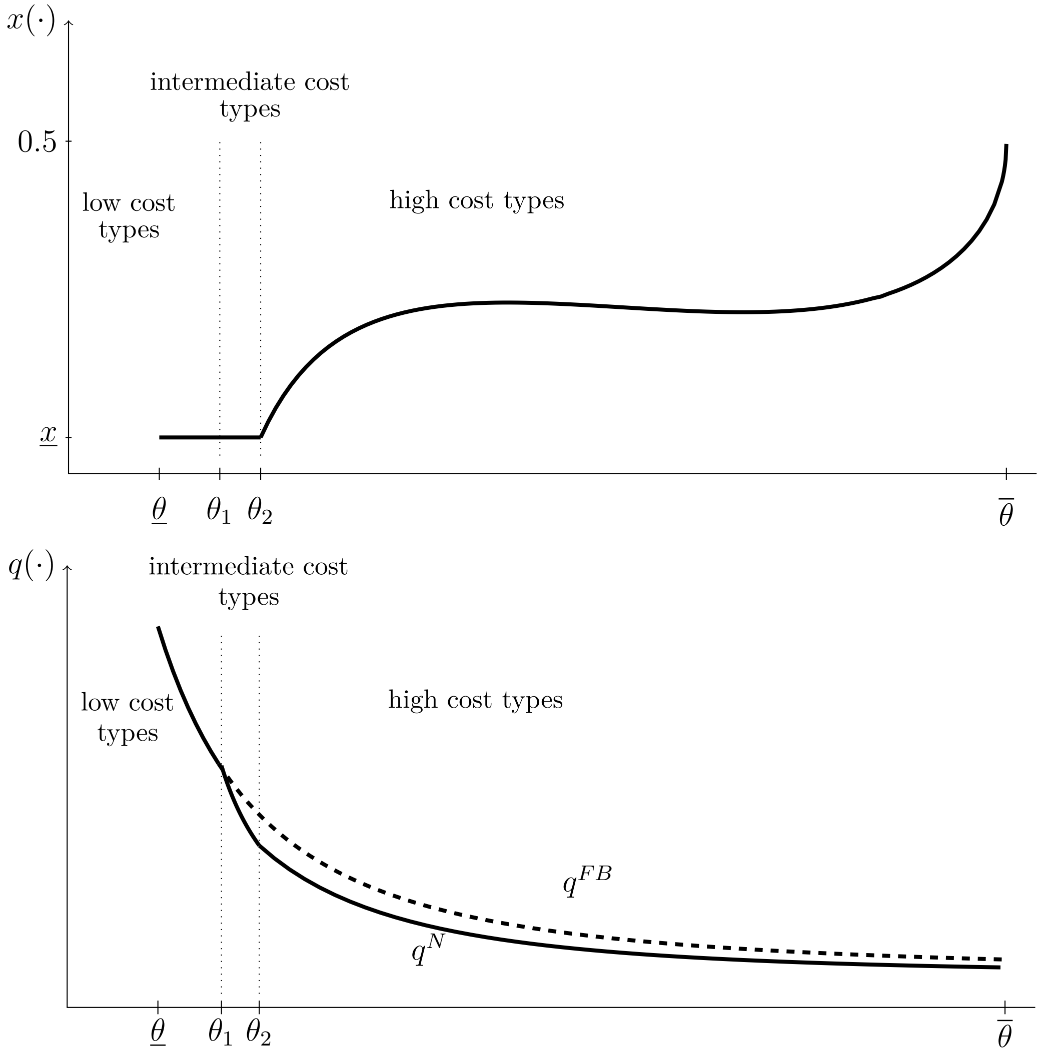
3. *The Principal's preferences over quantities are of the CARA type, $V(q) = q^{1-\alpha}/(1-\alpha)$, for $\alpha \geq 1$.*

Proof in Appendix A.5.

The conditions in Proposition 3 are straightforward to verify. The idea behind the proof of Proposition 3 is to solve for the slope of the inspection probability using ode-$x(\cdot)$. One then provides a lower bound on the slope of the inspection probability. The conditions in Proposition 3 guarantee that the lower bound is positive.

The lower bounds used in deriving the conditions for Proposition 3 do not rely on solving for $\hat{\theta}(\cdot)$ explicitly, but hold when $\theta \leq \hat{\theta}(\theta) \leq \overline{\theta}$ for all types θ. This is the reason why the conditions in Proposition 3 do not depend on the distribution of types F.

Example 2 *This example shows that the optimal probability of inspection is not increasing in the reported cost. Assume $V(q) = 2\sqrt{q}$, $\kappa = 0.0495$, $[\underline{\theta}, \overline{\theta}] = [1.4, 7]$, $\underline{x} = 0.01$, and $F(\cdot)$ is the uniform distribution. The first-best quantity is $q^{FB}(\theta) = \theta^{-2}$. Because $\overline{\theta} < 1/\kappa$, the solution to equation quantity-interior-inspection is given by $q(\theta) = (1/\theta - \kappa)^2$. We numerically solve ode-$x(\cdot)$ to obtain the inspection probability in the optimal mechanism. The first panel in Figure 3 plots the probability of inspection in the optimal mechanism. The inspection probability is constant and equal to the minimal inspection probability for low and intermediate cost types. For high cost types, the probability of inspection is strictly above $\underline{x}$. However, the inspection probability is strictly decreasing on an interval of high cost types. The second panel in Figure 3 plots the quantities in the optimal mechanism for this example.*



This figure shows the inspection probability in the optimal mechanism for Example 2. Types are depicted on the horizontal axis. The vertical axis shows the inspection probability $x(\cdot)$. The inspection probability equals the minimal probability of inspection, $\underline{x}$, for low and intermediate cost types $\theta \in [\underline{\theta}, \theta_2]$. The inspection probability increases strictly at the cut-off θ_2, but decreases strictly for higher types. In particular, the probability of inspection is not monotonically increasing in the type.
This figure shows the quantity allocation in the optimal mechanism for Example 2. Types are depicted on the horizontal axis. The vertical axis shows quantities. The dashed line is the first-best quantity. The solid line shows the quantity without inspection in the optimal mechanism. Low cost types between $\underline{\theta}$ and θ_1 produce the first-best quantity when not inspected. Intermediate and high cost types produce strictly less than the first-best quantity when not inspected.

Figure 3: Quantity in the optimal mechanism for Example 2.

4. Literature Review

The early literature on costly state verification — Townsend (1979), Diamond (1984), and Gale and Hellwig (1985) — and our paper share the assumption that inspection perfectly reveals the Agent's private information. Townsend (1979) was the first to study mechanisms with costly state verification. He observed that deterministic verification need not be optimal, but did not provide a characterization of optimal stochastic inspection. In contrast to us, Diamond (1984) assumes the Principal cannot condition her inspection decision on the Agent's report. In Gale and Hellwig (1985) the Agent's private information is binary as opposed to the compact interval in our model. This assumption simplifies the analysis significantly.

The classic paper on monopolistic screening, Mussa and Rosen (1978), does not allow inspection of the Agent's type. Incentives therefore need to be provided through transfers and quantity distortions. In particular, incentive constraints bind locally in an optimal mechanism when inspection is not feasible.

The paper closest to ours is Palonen and Pekkarinen (2022). They assume that the allocation in case of inspection is exogenous; i.e., the functions $q^I(\hat{\theta}, \theta)$ and $t^I(\hat{\theta}, \theta)$ are a primitive of their model. In our paper, the Principal chooses $q^I(\hat{\theta}, \theta)$ and $t^I(\hat{\theta}, \theta)$. The functions Palonen and Pekkarinen (2022) assume are, in general, not optimal for the Principal in our setting.

Dana, Larsen, and Moshary (2024) study a mechanism design problem in which the Agent has a uniform cost of misreporting his true type. This leads to incentive constraints that do not bind locally, as in our model. In our model, however, incentive constraints do not bind locally for a different reason: transfers equal the cost of production when the Agent is not inspected. Optimal mechanisms in Dana, Larsen and Moshary (2024) have the "no-distortion-at-the-top"-property of Proposition 2. However, the methods we develop are different from the ones Dana, Moshary and Larsen (2024) use. Their method does not allow them to pin down the set of binding incentive constraints in the optimal mechanism.

Alaei et al. (2022) study an auction model in which the auctioneer can inspect the bidder's valuation ex–post.[18] There are three differences to our paper: first, Alaei et al. (2022) assume inspection is costless to the Principal; second, in their paper, the allocation of the good depends on the reported type, but not on the true type; third, their auctioneer has no valuation for the good.

Mookherjee and Png (1989), Border and Sobel (1987), and Chander and Wilde (1998) study wealth extraction with audits. In their models, there is no allocation or production of a good other than monetary transfers. Melumad and Mookherjee (1989) study taxation with audits and the provision of a public good. In their

[18] See also the literature on auctions with contingent payments as surveyed in Skrzypacz (2013).

model, all agents consume the same quantity of the public good whereas in our model the allocation is type-dependent. Moreover, they rule out transfers to the Agent's after truthful reports.

One application of our model is monopoly regulation. Baron and Besanko (1984) add costly inspection to the seminal work of Baron and Myerson (1982). In contrast to our paper, Baron and Besanko (1984) assume that the Principal cannot pay the Agent above her outside option: using our notation, they assume $-q^I(\theta,\theta)\theta + t^I(\theta,\theta) = 0$. We do not make this assumption, and in fact show, that this cannot hold in an optimal mechanism; see Proposition 1. [19] [20] Moreover, verification in their model is imperfect whereas it is perfect in our paper.

We assume that inspection is perfect: when inspecting, the Principal observes the true type. This is a stronger assumption than needed. Our results apply when inspection is imperfect, but satisfies two properties. First, when inspecting, the Principal knows whether inspection was successful and whether she observed the true type or whether inspection was not successful. Put differently, if the Principal inspects and observes the reported type she can distinguish between the report having been truthful and inspection being unsuccessful. Second, the probability of successful inspection does not depend on the type or report. The last feature distinguishes our approach from papers with probabilistic verification, e.g., Ball and Kattwinkel (2022). The optimal mechanisms in models of probabilistic verification depend on the details of the verification technology, and require tools different from ours to analyze.

There is a literature on mechanism design with costly state verification and without transfers, e.g., Ben-Porath et al. (2014), Erlanson and Kleiner (2019), Halac and Yared (2020), Kattwinkel and Knoepfle (2023), and Ahmadzadeh (2024). Models with and without transfers differ in their predictions as well as in the techniques needed to analyze them.

5. Conclusion

This paper examines how the ability to learn the private information of a contracting party affects the optimal mechanism. In a procurement problem, a Principal can use transfers and costly inspection to induce a cost-specific allocation. We derive the optimal mechanism in this framework.

Our contributions are twofold. First, combining inspection and transfers yields new insights. The most important one is that all incentives are provided through

[19] Baron and Besanko (1984) state in their footnote 18 that their assumption is with loss of optimality, but do not discuss its effect on the allocation.

[20] Khalil (1997) studies a similar problem, but assumes the Principal cannot commit to inspect the Agent.

bonus payments after truthful inspection. This implies that there is no quantity distortion for types with sufficiently low costs. However, the quantity for high cost types is necessarily distorted from the efficient outcome.

Our second contribution is methodological. A major challenge when solving for the optimal mechanism is that incentive constraints do not bind locally. We have provided a way to explicitly characterize which incentive constraints bind. We believe similar proof strategies can be employed in other models where local incentive constraints do not bind.

We presented our results in the context of procurement and regulation. However, the main insights carry over if we had presented them in a setting of a monopolist selling a good, as well as other applications. Suppose a seller (the Principal) sells a good in different quality levels to a buyer (the Agent) and has the opportunity to learn the buyer's valuation at a cost, e.g., via a third-party data broker. In that case, the seller extracts all surplus from the buyer unless she decides to learn the buyer's valuation. After learning the buyer's valuation, the seller sells him a higher quality of the good at a price strictly below his valuation. Moreover, the discount after learning the buyer's valuation can be so large that the buyer receives a net payment from the seller.

A. Appendix

A.1. Proof of Proposition 1

Proof.

1. Define $q^N(\cdot) = \widetilde{q^N}(\cdot)$, $q^I(\cdot,\cdot) = \widetilde{q^I}(\cdot,\cdot)$, and $x(\cdot) = \widetilde{x}(\cdot)$. By Lemma 1, set $t^I(\hat{\theta},\theta) = q^I(\hat{\theta},\theta)\theta$ for all $\hat{\theta} \neq \theta$. For θ such that $\widetilde{x}(\theta) = x(\theta) = 0$, set $t^N(\theta) = \widetilde{t^N}(\theta)$ and $t^I(\theta,\theta) = \widetilde{t^I}(\theta,\theta)$. For θ such that $\widetilde{x}(\theta) = x(\theta) > 0$, set $t^I(\theta,\theta) = \widetilde{t^I}(\theta,\theta)\theta + \frac{1-x(\theta)}{x(\theta)}(-q^N(\theta) + \widetilde{t^N}(\theta))$. Since $\widetilde{M}$ satisfies obedience, $q^N(\theta)\theta + \widetilde{t^N}(\theta) \geq 0$. It is easy to see that the payoff to the Principal is equal under $\mathbb{M}$ and $\widetilde{\mathbb{M}}$. Moreover, $\mathbb{M}$ is incentive compatible and satisfies the obedience constraints.

2. Fix θ with $0 < x(\theta) < 1$ and suppose that $q^{FB}(\theta) > q^N(\theta)$ and $t^N(\theta) > \theta q^N(\theta)$. Increase $q^N(\theta)$ by $\delta > 0$ small enough so that $q^N(\theta)+\delta \leq q^{FB}(\theta)$ and $t^N(\theta) \geq \theta(q^N(\theta)+\delta)$ and increase $t^I(\theta)$ by $\delta\theta(1-x(\theta))/x(\theta)$. These changes leave the incentive compatibility constraints and the obedience constraints satisfied but increase the payoff to the Principal.

3. For each type $\theta \in B_\delta$ change the mechanism so that

$$t^N(\theta) \rightsquigarrow q^N(\theta)\theta \text{ and } t^I(\theta) \rightsquigarrow t^I(\theta) + \frac{x(\theta)}{1-x(\theta)}(t^N(\theta) - \theta q^N(\theta)).$$

The IC and obedience constraints for such types continue to hold. Moreover, the IC constraints for types $\hat{\theta} \in \hat{\theta}_\delta(\theta)$ are slack. Therefore, decreasing $t^I(\hat{\theta})$ or $t^N(\hat{\theta})$ infinitesimally for such types preserves incentives. Since the set of such types has positive measure, these changes raise the Principal's objective, in contradiction to the optimality of $\mathbb{M}$.

■

A.2. Proof of Lemma 3

Proof. We use three claims to prove the Lemma. Define

$$\tilde{\theta} = \min\{\theta | \pi(\theta) = 0\}.$$

Let $(x, q \equiv q^N)$ attain be the functions that attain the maximum in the unconstrained problem. Fix an interval $[I^-, I^+] \subset (\hat{\theta}(\underline{\theta}), \tilde{\theta})$. Let $\eta(q)(\theta) = q(\theta)\mathbb{1}_{\theta \in [I^-, I^+]}$, and $G(x, q)$ be the integrand in the objective of the unconstrained problem , i.e.,

$$\begin{aligned} G(x,q) =& x(\theta)\left(V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa\right) + (1-x(\theta))\left(V(q^N(\theta)) - q^N(\theta)\theta\right) \\ &- \sup_{\hat{\theta}}(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta). \end{aligned}$$

Define $g(\beta)$ for $\beta \in (-1, 1) \setminus \{0\}$ as

$$g(\beta) = \frac{G(x, q + \beta\eta) - G(x, q)}{\beta}.$$

Claim 1 *In an optimal mechanism,*

$$\begin{aligned} &\lim_{\beta \to 0^-} g(\beta) \\ &= \int_{t \in [I^-, I^+]} (1 - x(t))q(t)\left(V'(q(t)) - t\right) \mathrm{d}F(t) - \int_{\hat{\theta}(t) \in (I^-, I^+)} \pi(t) \, \mathrm{d}F(t) \geq 0, \\ &\lim_{\beta \to 0^+} g(\beta) = \lim_{\beta \to 0^-} g(\beta) - \int_{\hat{\theta}(t) \in \{I^-, I^+\}} \pi(t) \, \mathrm{d}F(t) \leq 0. \end{aligned}$$

Proof. We will compute $\lim_{\beta\to 0^-} g(\beta)$, and $\lim_{\beta\to 0^+} g(\beta)$. If limits exist, then

$$\lim_{\beta\to 0^-} g(\beta) \geq 0, \quad \text{and} \quad \lim_{\beta\to 0^+} g(\beta) \leq 0.$$

First let $\beta < 0$. Define

$$\chi(\theta) = \max_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]\setminus(I^-,I^+)} (1 - x(\hat{\theta}))q(\hat{\theta})(\hat{\theta} - \theta).$$

Note that $\chi(\theta)$ is well defined since $[\underline{\theta},\overline{\theta}]\setminus(I^-,I^+)$ is compact and $Q(\cdot)$ is continuous. Define set $I(\beta)$

$$I(\beta) = \{\hat{\theta}(\theta)|\chi(\theta) \leq (1+\beta)\pi(\theta)\}.$$

A directional derivative for $\beta < 0$ gives us

$$\begin{aligned}\lim_{\beta\to 0^-} g(\beta) = \int_{t\in[I^-,I^+]} &(1 - x(t))\left(V'(q(t))q(t) - q(t)t\right) \mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in I(\beta)} \pi(t)\,\mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in(I^-,I^+)\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in[\underline{\theta},\overline{\theta}]\setminus(I^-,I^+)} \frac{\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t)\end{aligned}$$

We show that the above limit for each integral exists, and we compute it. For the first integral, the limit can go inside the integral since inside is uniformly bounded above. Note that $\pi(\theta)$ of θ for $\hat{\theta}(t) \in I(\beta)$ changes to $(1+\beta)\pi(\theta)$. For types θ such that $\hat{\theta}(\theta) \in (I^-,I^+)\setminus I(\beta)$. For types θ such that $\hat{\theta}(\theta) \in [\underline{\theta},\overline{\theta}]\setminus(I^-,I^+)$ do not change (the last integral).[21] Therefore we can rewrite

$$\begin{aligned}\lim_{\beta\to 0^-} g(\beta) = \int_{t\in[I^-,I^+]} &(1 - x(t))\left(V'(q(t))q(t) - q(t)t\right) \mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in I(\beta)} \pi(t)\,\mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in(I^-,I^+)\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t).\end{aligned}$$

[21]Note that $\{I^-,I^+\} \subset [\underline{\theta},\overline{\theta}]\setminus(I^-,I^+)$.

We show the last integral is zero. The reason is

$$0 = \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \frac{\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \le \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t)$$
$$\le \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \frac{(1+\beta)\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) = \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \pi(t)\,\mathrm{d}F(t).$$

Since $\hat\theta(\theta)$ is a function, then $\lim_{\beta\to 0^-} I(\beta) = \cup_{\beta<0} I(\beta) = (I^-, I^+)$, by Squeeze Theorem we conclude

$$0 \le \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \le \lim_{\beta\to 0^-} \int_{\hat\theta(t)\in(I^-,I^+)\setminus I(\beta)} \pi(t)\,\mathrm{d}F(t) = 0.$$

Therefore

$$\lim_{\beta\to 0^-} g(\beta) = \int_{t\in[I^-,I^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) - \int_{\hat\theta(t)\in(I^-,I^+)} \pi(t)\,\mathrm{d}F(t).$$

For $\beta>0$, define set $I(\beta)$

$$I(\beta) = \{\hat\theta(\theta) | (1+\beta)\chi(\theta) \ge \pi(\theta)\},$$

where

$$\chi(\theta) = \max_{\tilde\theta\in[I^-,I^+]} (1-x(\tilde\theta))q(\tilde\theta)(\tilde\theta-\theta).$$

A directional derivative for $\beta>0$ gives us

$$\begin{aligned}
\lim_{\beta\to 0^+} g(\beta) = & \int_{t\in[I^-,I^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) \\
& - \lim_{\beta\to 0^+} \int_{\hat\theta(t)\in[I^-,I^+]} \frac{(1+\beta)\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \\
& - \lim_{\beta\to 0^+} \int_{\hat\theta(t)\in[\underline{\theta},\overline{\theta}]\setminus I(\beta)} \frac{\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \\
& - \lim_{\beta\to 0^+} \int_{\hat\theta(t)\in I(\beta)\setminus[I^-,I^+]} \frac{(1+\beta)\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t).
\end{aligned}$$

We show that the above limit for each integral exists, and we compute it. For the first integral, the limit can go inside the integral since inside is uniformly bounded above. We show the last integral is zero

$$0 = \int_{\hat{\theta}(t)\in I(\beta)\setminus[I^-,I^+]} \frac{\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \le \int_{\hat{\theta}(t)\in I(\beta)\setminus[I^-,I^+]} \frac{(1+\beta)\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t)$$
$$\le \int_{\hat{\theta}(t)\in I(\beta)\setminus[I^-,I^+]} \frac{(1+\beta)\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) = \int_{\hat{\theta}(t)\in I(\beta)\setminus[I^-,I^+]} \pi(t)\,\mathrm{d}F(t).$$

Since $\hat{\theta}(\theta)$ is a function, then $\lim_{\beta\to 0^+} I(\beta) = \cap_{\beta>0} I(\beta) = [I^-, I^+]$, by Squeeze Theorem we conclude

$$-\lim_{\beta\to 0^+} \int_{\hat{\theta}(t)\in I(\beta)\setminus[I^-,I^+]} \frac{(1+\beta)\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) = 0.$$

Therefore

$$\lim_{\beta\to 0^+} g(\beta) = \int_{t\in[I^-,I^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) - \int_{\hat{\theta}(t)\in[I^-,I^+]} \pi(t)\,\mathrm{d}F(t).$$

Finally, we have

$$\begin{aligned}
&\lim_{\beta\to 0^-} g(\beta)\\
&= \int_{t\in[I^-,I^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) - \int_{\hat{\theta}(t)\in(I^-,I^+)} \pi(t)\,\mathrm{d}F(t) \ge 0,\\
&\lim_{\beta\to 0^+} g(\beta) = \lim_{\beta\to 0^-} g(\beta) - \int_{\hat{\theta}(t)\in\{I^-,I^+\}} \pi(t)\,\mathrm{d}F(t) \le 0.
\end{aligned}$$

□

Claim 2 *For all θ with $\pi(\theta) > 0$, $\hat{\theta}(\cdot)$ is a strictly increasing function.*

Proof. We will consider two steps. In step 1, we show the correspondence $\hat{\theta}^{-1}(\cdot)$ is a function (hence strictly increasing) in $(\hat{\theta}(\underline{\theta}), \tilde{\theta})$. In step 2, we show $\hat{\theta}^{-1}(\hat{\theta})$ for $\hat{\theta} = \hat{\theta}(\underline{\theta})$ is single valued.

Step 1) By contradiction assume for $\hat{\theta} \in (\hat{\theta}(\underline{\theta}), \tilde{\theta})$, correspondence $\hat{\theta}^{-1}(\hat{\theta})$ is not single valued. There exists a sequence $\hat{\theta}-\delta_n^- < \hat{\theta}$ converging from left to $\hat{\theta}$, and a sequence $\hat{\theta}+\delta_n^+ > \hat{\theta}$ converging from right to $\hat{\theta}$, such that $\hat{\theta}^{-1}(\hat{\theta}+\delta_n^+)$, and $\hat{\theta}^{-1}(\hat{\theta}-\delta_n^-)$ are single valued for all $n \in \mathbb{N}$.[22] Let $I_n^+ = \hat{\theta} + \delta_n^+$, and $I_n^- = \hat{\theta} - \delta_n^-$. Therefore

[22]In fact there at most countable points $\hat{\theta}$ that $\hat{\theta}^{-1}(\hat{\theta})$ is not single valued. Therefore the sequence exists.

$\int_{\hat{\theta}(t)\in\{I_n^-,I_n^+\}} \pi(t)\,\mathrm{d}F(t) = 0$. Hence $\lim_{\beta\to 0^+} g(\beta,n) = \lim_{\beta\to 0^-} g(\beta,n) = 0$, using Claim 1 implies (by abusing of notation $g(\beta,n)$ is defined similar to $g(\beta)$ in Claim 1, but for interval $[I_n^-, I_n^+]$)

$$\int_{\hat{\theta}(t)\in[I_n^-,I_n^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) = \int_{\hat{\theta}(t)\in[I_n^-,I_n^+]} \pi(t)\,\mathrm{d}F(t).$$

We know $\cap_{n\in\mathbb{N}}[I_n^-, I_n^+] = \hat{\theta}$. When n to infinity the left side goes to zero. Therefore the right side should go to zero as well, but the right side will be

$$\int_{\hat{\theta}(t)\in\hat{\theta}} \pi(t)\,\mathrm{d}F(t).$$

Since $\pi(t) > 0$, then $\hat{\theta}^{-1}(\hat{\theta})$ is single valued. Otherwise, the above integral will be strictly positive.

Step 2) The proof will be the same with some adaptations. Note that we assume $\kappa > 0$ (otherwise $\pi(\theta) = 0$ for all θ). Then $\hat{\theta}(\underline{\theta}) \neq \tilde{\theta}$. Fix an interval $I = (\hat{\theta}(\underline{\theta}) - \delta, \hat{\theta}(\underline{\theta}) + \delta)$, and define $g(\beta)$ similar to the previous step. First let $\beta < 0$. Define

$$\chi(\theta) = \max_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]\setminus I} (1-x(\hat{\theta}))q(\hat{\theta})(\hat{\theta}-\theta).$$

Define set $I(\beta) \subset I$

$$I(\beta) = \{\hat{\theta}(\theta) | \chi(\theta) \leq (1+\beta)\pi(\theta)\}$$

The directional derivative for $\beta < 0$ gives us

$$\begin{aligned}\lim_{\beta\to 0^-} g(\beta) = \int_{t\in I} &(1-x(t))\left(V'(q(t)) - q(t)t\right)\,\mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in I(\beta)} \pi(t)\,\mathrm{d}F(t)\\ &- \lim_{\beta\to 0^-}\int_{\hat{\theta}(t)\in I\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t).\end{aligned}$$

We show that the above limit for each integral exists, and we compute it. For the first integral, the limit can go inside the integral since inside is uniformly bounded above. The last integral is zero since

$$
\begin{aligned}
0 = \int_{\hat\theta(t)\in I\setminus I(\beta)} \frac{\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) &\le \int_{\hat\theta(t)\in I\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \\
\le \int_{\hat\theta(t)\in I\setminus I(\beta)} \frac{(1+\beta)\pi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) &= \int_{\hat\theta(t)\in I\setminus I(\beta)} \pi(t)\,\mathrm{d}F(t).
\end{aligned}
$$

$\hat\theta(\theta)$ is a function, then $\lim_{\beta\to 0^-} I(\beta) = \cup_{\beta<0} I(\beta) = [\hat\theta(\hat\theta), \hat\theta(\hat\theta)+\delta)$, by Squeeze Theorem we conclude

$$
0 \le \int_{\hat\theta(t)\in I\setminus I(\beta)} \frac{\chi(t)-\pi(t)}{\beta}\,\mathrm{d}F(t) \le \lim_{\beta\to 0} \int_{\hat\theta(t)\in I\setminus I(\beta)} \pi(t)\,\mathrm{d}F(t) = 0.
$$

Therefore

$$
\lim_{\beta\to 0^-} g(\beta) = \int_{t\in I} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) - \int_{\hat\theta(t)\in[\hat\theta(\hat\theta),\hat\theta(\hat\theta)+\delta)} \pi(t)\,\mathrm{d}F(t).
$$

For $\beta > 0$, define set $I(\beta)$

$$
I(\beta) = \{\hat\theta(\theta) | (1+\beta)\chi(\theta) \ge \pi(\theta)\},
$$

where

$$
\chi(\theta) = \max_{\hat\theta\in I} (1-x(\hat\theta))q(\hat\theta)(\hat\theta-\theta).
$$

The same argument leads us

$$
\lim_{\beta\to 0^+} g(\beta) = \int_{t\in I} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) - \int_{\hat\theta(t)\in[\hat\theta(\hat\theta),\hat\theta(\hat\theta)+\delta]} \pi(t)\,\mathrm{d}F(t).
$$

Sending $\delta > 0$ to zero, the left side goes to zero, therefore the right side should go to zero. Which implies that $\hat\theta^{-1}(\underline{\theta})$ is single valued. □

Claim 3 *For all $(I^-, I^+) \in \mathbb{R}_+^2$ such that $[I^-, I^+] \subset [\hat\theta(\underline{\theta}), \tilde\theta]$ we have*

$$
\int_{t\in[I^-,I^+]} (1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t) = \int_{\hat\theta(t)\in[I^-,I^+]} \pi(t)\,\mathrm{d}F(t),
$$

Proof. Using Claims 1 and 2

$$\begin{aligned}&\lim_{\beta\to 0^+} g(\beta)\\&=\int_{t\in[I^-,I^+]}(1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t)-\int_{\hat\theta(t)\in(I^-,I^+)}\pi(t)\,\mathrm{d}F(t)\geq 0,\\&\lim_{\beta\to 0^+} g(\beta)=\lim_{\beta\to 0^-} g(\beta)-\int_{\hat\theta(t)\in\{I^-,I^+\}}\pi(t)\,\mathrm{d}F(t)\leq 0.\end{aligned}$$

We can conclude $\lim_{\beta\to 0^+} g(\beta)=\lim_{\beta\to 0^-} g(\beta)=0$, and

$$\int_{t\in[I^-,I^+]}(1-x(t))q(t)\left(V'(q(t))-t\right)\,\mathrm{d}F(t)=\int_{\hat\theta(t)\in[I^-,I^+]}\pi(t)\,\mathrm{d}F(t).$$

Note that I^+ can be $\tilde\theta$ since the above equality holds for all I^+ close to $\tilde\theta$, and $\int_{\hat\theta(t)\in\tilde\theta}\pi(t)\,\mathrm{d}F(t)=0$. In addition, I^- can be $\hat\theta^{-1}(\underline{\theta})$ since $\hat\theta^{-1}(\underline{\theta})$ is single valued. Hence the above equality holds even when $I^-=\hat\theta(\underline{\theta})$.□

Now we prove the Lemma. By Claim 2, $\hat\theta(\cdot)$ is strictly increasing, so both $\hat\theta(\cdot)$, and $\hat\theta^{-1}(\cdot)$ are differentiable almost everywhere. By Claim 3, for all $(I^-,I^+)\in\mathbb{R}^2_+$ such that $[I^-,I^+]\subset[\hat\theta(\underline{\theta}),\tilde\theta]$

$$\begin{aligned}&\int_{t\in[I^-,I^+]}(1-x(t))\left(V'(q(t))q(t)-q(t)t\right)\,\mathrm{d}F(t)\\&=\int_{t\in[I^-,I^+]}(1-x(t))q(t)(t-\hat\theta^{-1}(t))\,\mathrm{d}F(\hat\theta^{-1}(t)),\end{aligned}$$

and

$$\begin{aligned}&\int_{t\in[I^-,I^+]}q(t)(1-x(t))\left(\left(V'(q(t))-t\right)f(t)-(t-\hat\theta^{-1}(t))\frac{d\,\hat\theta^{-1}(t)}{d\,t}f(\hat\theta^{-1}(t))\right)\,\mathrm{d}t\\&=0.\end{aligned}$$

The above equality holds for all $[I^-,I^+]\subset[\hat\theta(\underline{\theta}),\overline{\theta})$. Using the Fundamental Theorem of Calculus for all t such that $\pi(t)>0$, and $t\in[\hat\theta(\underline{\theta}),\overline{\theta})$, almost everywhere (at all differentiable points of $\hat\theta^{-1}(\cdot)$) we have

$$\left(V'(q(t))-t\right)f(t)=(t-\hat\theta^{-1}(t))\frac{d\,\hat\theta^{-1}(t)}{d\,t}f(\hat\theta^{-1}(t)).$$

Fix $t^*\in(\hat\theta(\underline{\theta}),\overline{\theta})$. We want to show $\hat\theta^{-1}(\cdot)$ is differentiable at t^*. We know $\hat\theta^{-1}(\cdot)$ is almost everywhere differentiable, so there are two sequences t_L, and t_R converging to t^* from left and right such that $\hat{\theta}^{-1}(\cdot)$ is differentiable at each point of them. So, the above equation holds at each point t_L, and t_R. Finally, the left side of the below equation is continuous in t, we conclude

$$\frac{(V'(q(t))-t)\,f(t)}{(t-\hat{\theta}^{-1}(t))f(\hat{\theta}^{-1}(t))} = \left.\frac{d\,\hat{\theta}^{-1}(t)}{d\,t}\right|_{t\in\{t^{*+},t^{*-}\}}.$$

This means that $\hat{\theta}^{-1}(t)$ is differentiable at t^*. ■

A.3. Proof of Proposition 2

Proof.

1. For $\delta>0$ denote $B_\delta(\theta)=\{\hat{\theta}|(1-x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta}-\theta)\geq\pi(\theta)-\delta\}$. Define

$$\theta_1=\inf_{\theta}\bigcap_{\delta>0}\bigcup_{\theta\in[\underline{\theta},\overline{\theta}]}B_\delta(\theta).$$

For all θ and $\hat{\theta}$, define

$$\chi(\hat{\theta},\theta)=(1-x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta}-\theta).$$

We show that $\theta_1>\underline{\theta}$. In any optimal mechanism, $q^N(\theta)\leq q^{FB}(\theta)$ and $\pi(\underline{\theta})>0$. Hence, for a $\delta>0$ small enough there exists $\theta^\dagger>\underline{\theta}$ such that $\pi(\underline{\theta})-\chi(\hat{\theta},\underline{\theta})\leq 2\delta$ for all $\hat{\theta}<\theta^\dagger$. Since $\pi(\cdot)$ and $\chi(\hat{\theta},\cdot)$ are continuous in θ, $\pi(\theta)-\chi(\hat{\theta},\theta)\leq\delta$ for all $\hat{\theta}<\theta^\dagger$ and $\theta\leq\theta^{\dagger\dagger}$ or a $\theta^{\dagger\dagger}>\underline{\theta}$. Since $\chi(\hat{\theta},\theta)<0$ for all $\theta>\hat{\theta}$, we conclude that $\theta_1>\underline{\theta}$.

Therefore given $\hat{\theta}$, there exists $\delta>0$ such that $\pi(\theta)-\chi(\hat{\theta},\theta)>\delta$ for all θ, and $\hat{\theta}<\theta_1$. Hence there exists β° such that for all $\beta\in(\beta^\circ,-\beta^\circ)$, and for all θ

$$\pi(\theta)-(1+\beta)\chi(\hat{\theta},\theta)>\frac{\delta}{2}.$$

Now maximize point-wise: an admissible variation for $q^N(\theta)$ is $(1+\beta)q^N(\theta)$ for $\beta\in(\beta^\circ,-\beta^\circ)$. This variation does not change $\pi(\theta)$, so it changes only $V(q^N(\theta))-\theta q^N(\theta)$ in the objective of the unconstrained problem. Since $V(q)-\theta q$ has a unique maximizer for all θ, we conclude $q^N(\theta)=q^{FB}(\theta)$ for all $\theta<\theta_1$. Therefore for all $\theta<\theta_1$

$$V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa<V(q^N(\theta))-q^N(\theta)\theta.$$

If $x(\theta)>\underline{x}$, by a similar argument using an admissible variation of $x(\theta)$ to

$(1+\beta)x(\theta)$ for $\beta < 0$, we can increase the objective, without changing $\pi(\theta)$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$.

2. By Lemma 2, $Q^N(\cdot) = (1 - x(\cdot))q^N(\cdot)$ is continuous, so that $\lim_{\theta \searrow \theta_1}(1 - x(\theta))q^N(\theta) = (1 - \underline{x})q^{FB}(\theta_1)$ from Proposition 2, part 1. Moreover, $q^N(\cdot) \leq q^{FB}(\cdot)$ and $x(\cdot) \geq \underline{x}$ imply that $q^N(\theta_1) = q^{FB}(\theta_1)$.
Suppose toward a contradiction that for every n large enough there exists $\tilde{\theta}_n \in [\theta_1, \theta_1 + 1/n)$ with $x(\tilde{\theta}_n) > \underline{x}$. Then $q^N(\tilde{\theta}_n) = q^*(\tilde{\theta}_n)$ where $q^*(\cdot)$ is given as in Proposition 2, part 3. By definition of $q^*(\cdot)$ and $q^{FB}(\cdot)$,

$$\inf_{\theta \in [\underline{\theta}, \overline{\theta}]} q^{FB}(\theta) - q^*(\theta) > 0.$$

Consequently,

$$\limsup_n (1 - x(\tilde{\theta}_n))q^N(\tilde{\theta}_n) < (1 - \underline{x})q^{FB}(\theta_1),$$

in contradiction to continuity of $(1 - x(\cdot))q^N(\cdot)$. Thus, there exist $\theta_2 > \theta_1$ such that $x(\theta) = \underline{x}$ for all $\theta_1 \leq \theta \leq \theta_2$.

3. An admissible variation is to change $q^N(\theta)$ to $(1+\beta)q^N(\theta)$, and $1 - x(\theta)$ to $\frac{1-x(\theta)}{1+\beta}$ for some $\beta > 0$. Rewriting the integrand of the objective for type θ:

$$\begin{aligned}
&V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa \\
&+ \left(\frac{1 - x(\theta)}{1+\beta}\right)\big(V((1+\beta)q^N(\theta)) - (1+\beta)q^N(\theta)\theta - (V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa)\big) \\
&- \sup_{\hat{\theta}} \left(\frac{1 - x(\hat{\theta})}{1+\beta}\right)\Big((1+\beta)q^N(\hat{\theta})(\hat{\theta} - \theta)\Big).
\end{aligned}$$

Note that $\pi(\cdot)$ does not change. A derivative with respect to β gives us:

$$\begin{aligned}
&-\left(\frac{1 - x(\theta)}{(1+\beta)^2}\right)\big(V((1+\beta)q^N(\theta)) - (1+\beta)q(\theta)\theta - (V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa)\big) \\
&+ \left(\frac{1 - x(\theta)}{1+\beta}\right)\left(\frac{\partial V((1+\beta)q^N(\theta))}{\partial q^N(\theta)} q^N(\theta) - q^N(\theta)\theta\right).
\end{aligned}$$

The derivative at $\beta = 0$ must be zero, and $x(\theta) < 1$.

$$-V(q(\theta)) + q^N(\theta)\theta + (V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa) + \frac{\partial V(q^N(\theta))}{\partial q^N(\theta)} q^N(\theta) - q^N(\theta)\theta = 0.$$

Therefore,

$$V(q^{FB}(\theta)) - q^{FB}(\theta)\theta - \kappa = V(q^N(\theta)) - \frac{\partial V(q^N(\theta))}{\partial q^N(\theta)} q^N(\theta).$$

■

A.4. Proof of Theorem 1

Proof. The existence of the threshold $\theta_1 < \overline{\theta}$ follows from Proposition 2, part 1. Uniqueness of the threshold θ_1 follows from Lemma 5 and the characterization in part 2 of Proposition 2.

First, we show that $\theta_1 = \hat{\theta}(\underline{\theta})$. Suppose that $\theta_1 > \hat{\theta}(\underline{\theta})$. By the first part of Proposition 2, $q^N(\theta') = q^{FB}(\theta')$ for $\theta' \in (\hat{\theta}(\underline{\theta}), \theta_1)$. Fix such a θ' and let θ satisfy $\hat{\theta}(\theta) = \theta'$. By Lemma 3, $\hat{\theta}(\theta) = \theta = \theta'$. This implies that $\pi(\theta) = 0$ and, since $\pi(\cdot)$ is decreasing, $\pi(\theta^\dagger) = 0$ for all $\theta^\dagger \in [\theta', \overline{\theta}]$. The later requires $(1 - x(\theta^\dagger))q^N(\theta^\dagger) = 0$, in contradiction to Lemma 7.
Suppose that $\theta_1 < \hat{\theta}(\underline{\theta})$ and let $\theta \in (\theta_1, \hat{\theta}(\underline{\theta}))$. By Proposition 2, $q^N(\theta) < q^{FB}(\theta)$. Since $\pi(\theta) > (1 - x(\theta))(\theta - \theta')$ for all θ', an infinitesimal increase in $q^N(\theta)$ does not affect information rents. However, this change raises the value of the unconstrained problem, a contradiction to the optimality of the mechanism. Consequently, $\hat{\theta}(\underline{\theta}) = \theta_1$.

Define $\tilde{\theta}_2 = \min\{\overline{\theta}, \inf\{\theta | x(\theta) > \underline{x}\}\}$. Denote θ_2 the unique threshold defined by (θ_2). We show that $\theta_2 = \tilde{\theta}_2$. Observe that $x(\theta) = \underline{x}$ and the quantity without inspection equals $q^N(\theta) = q_1(\theta)$ for $\theta \in (\theta_1, \tilde{\theta}_2]$, according to Lemma 6.

Suppose $\tilde{\theta}_2 > \theta_2$. Because $q_1(\cdot), q_2(\cdot)$ are continuous, there is a positive mass of types $\theta \in (\theta_2, \tilde{\theta}_2)$ with $q^N(\theta) = q_1(\theta) < q_2(\theta)$ and $x(\theta) = \underline{x}$. A variational argument similar to the one in A.3, part 3, shows that $q^N(\theta) < q_2(\theta)$ requires $x(\theta) = 1$ in any optimal mechanism, a contradiction.

Suppose $\theta_2 > \tilde{\theta}_2$. Then $x(\theta) > \underline{x}$ for $\theta > \theta_2$ by the definition of $\tilde{\theta}_2$ and the hypothesis that the inspection probability is weakly increasing. Moreover, $x(\theta) < 1$ by Lemma 7. Part 3 of Proposition 2 implies that $q^N(\theta) = q_2(\theta)$ for $\theta > \tilde{\theta}_2$.

Integration of ode $\hat{\theta}$ yields

$$\int_{\theta\in[\theta_2,\overline{\theta}]}(1-x(\theta))q_2(\theta)\left(V'(q_2(\theta))-\theta\right)\,\mathrm{d}F(\theta)=\int_{\hat{\theta}(\theta)\in[\theta_2,\overline{\theta}]}\pi(\theta)\,\mathrm{d}F(\theta).$$

Therefore the Principal's payoff in the unconstrained problem equals

$$\int_{\underline{\theta}}^{\tilde{\theta}_2}(1-\underline{x})\left(V(q_1(\theta))-q_1(\theta)V'(q_1(\theta))-\left(V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right)\right)\,\mathrm{d}F(\theta)$$
$$+E_\theta\left[V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right]\equiv P_{\tilde{\theta}_2}.$$

Now consider an alternative inspection policy such that $x(\theta)=\underline{x}$ for all $\theta\leq\theta_2$. The payoff in the unconstrained problem under this inspection policy is

$$P_{\theta_2}\equiv\int_{\underline{\theta}}^{\theta_2}(1-\underline{x})\left(V(q_1(\theta))-\theta V'(q_1(\theta))-\left(V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right)\right)\,\mathrm{d}F(\theta)$$
$$+E_\theta\left[V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right].$$

Since $\theta_2>\tilde{\theta}_2$ and

$$V(q_1(\theta))-\theta V'(q_1(\theta))-\left(V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right)\geq 0,$$

$P_{\theta_2}\geq P_{\tilde{\theta}_2}$. Since the original inspection probability was optimal, we conclude $P_{\tilde{\theta}_2}=P_{\theta_2}$, which implies $V(q_1(\theta))-\theta V'(q_1(\theta))-\left(V(q^{FB}(\theta))-q^{FB}(\theta)\theta-\kappa\right)=0$ almost everywhere on $(\tilde{\theta}_2,\theta_2)$. This furthermore implies that the inspection policy $x(\theta)$ remains constant at $\underline{x}$ on the interval $(\tilde{\theta}_2,\theta_2)$ in the initial inspection policy, in contradiction to the definition of $\tilde{\theta}_2$.

By definition of $\tilde{\theta}_2$ and the hypothesis that the inspection probability is increasing, Proposition 2, part 3, implies that $q^N(\theta)$ is given by quantity-interior-inspection on $(\theta_2,\overline{\theta}]$. Moreover, $x(\theta)=\underline{x}$ for $\theta\in(\theta_1,\theta_2]$, and the quantity without inspection is given as in Lemma 6 with the boundary conditions $\hat{\theta}(\underline{\theta})=\theta_1$ and, by continuity, $q^N(\theta_1)=q^{FB}(\theta_1)$; that is, $q^N(\theta)=q_1(\theta)$ on $(\theta_1,\theta_2]$. The statements regarding the probability of inspection follow from Proposition 2 and the discussion at the end of Section 3.3. ■

A.5. Proof of Proposition 3

Proof. From Lemma 2, $(1-x(\cdot))q^N(\cdot)$, is strictly decreasing and differentiable. Moreover, $\hat{\theta}(\cdot)$ is single–valued. Given θ, $\hat{\theta}(\theta)=\hat{\theta}$ is solves the FOC to $\max_{\hat{\theta}}(1-$

$x(\hat{\theta}))q(\hat{\theta})(\hat{\theta}-\theta)$. Therefore,

$$\frac{x^{'}(\hat{\theta})}{1-x(\hat{\theta})}=q^{'}(\hat{\theta})(\hat{\theta}-\theta)+q(\hat{\theta})\geq 0$$
$$\iff \frac{1}{\hat{\theta}-\theta}\geq -\frac{q^{'}(\hat{\theta})}{q(\hat{\theta})}.$$

1. Since $q^{'}<0$, $q(\hat{\theta})+q^{'}(\hat{\theta})(\hat{\theta}-\theta)\geq q(\hat{\theta})+q^{'}(\hat{\theta})\hat{\theta}$. The RHS of the last inequality is the derivative of $\theta\mapsto\theta q(\theta)$.

2. When $V^{'''}\leq 0$, $-q^{'}(\hat{\theta})/q(\hat{\theta})$ is decreasing. Moreover, $\hat{\theta}-\theta\leq\overline{\theta}-\underline{\theta}$. The claim follows.

3. $q(\theta)=\left(\frac{\alpha}{\alpha\theta^{1-\frac{1}{\alpha}}-\theta^{1-\frac{1}{\alpha}}+\left(\theta^{-1/\alpha}\right)^{1-\alpha}+\alpha\kappa-\kappa}\right)^{\frac{1}{\alpha-1}}$. Since $1/(\hat{\theta}-\theta)>1/\hat{\theta}$, the inequality in the last display is satisfied if $1/\hat{\theta}\geq -q'(\hat{\theta})/q(\hat{\theta})$. Simple algebra shows that this inequality is true for $\alpha\geq 1$.

■

References

Ahmadzadeh, Amirreza (2024). "Costly State Verification with Ex-post Participation Constraint", mimeo.

Alaei, Saeed et al. (2022). "Optimal Auction Design with Deferred Inspection and Reward", mimeo.

Ball, Ian and Kattwinkel, Deniz (2022). "Probabilistic Verification in Mechanism Design", mimeo.

Baron, David and Besanko, David (1984). "Regulation, Asymmetric Information, and Auditing", *The RAND Journal of Economics*, 15(4), pp. 447–470.

Baron, David and Myerson, Roger (1982). "Regulating a Monopolist with Unknown Costs", *Econometrica*, 50(4), pp. 911–930.

Ben-Porath, Elchanan, Dekel, Eddie and Lipman, Barton (2014). "Optimal Allocation with Costly Verification", *The American Economic Review*, 104(12), pp. 3779–3813.

Border, Kim and Sobel, Joel (1987). "Samurai Accountant: A Theory of Auditing and Plunder", *Review of Economic Studies*, 54, pp. 525–540.

Chander, Parkash and Wilde, Louis L. (1998). "A General Characterization of Optimal Income Tax Enforcement", *The Review of Economic Studies*, 65, pp. 165–183.

Dana, James, Larsen, Brad and Moshary, Sarah (2022). "Fake IDs and Arbitrage. Price Discrimination with Misreporting Costs", mimeo.

Defense Contract Audit Agency (DCAA) (2023). "Report to Congress on FY 2022 Activities", available here.

Diamond, Douglas (1984). "Financial Intermediation and Delegated Monitoring", *Review of Economic Studies*, 51, pp. 393–414.

Erlanson, Albin and Kleiner, Andreas (2020). "Costly verification in collective decisions", *Theoretical Economics*, 15, pp. 923–954.

Gale, Douglas and Hellwig, Martin (1985). "Incentive-Compatible Debt Mechanisms: The One-Period Problem", *Review of Economic Studies*, 52, pp. 647–663.

General Services Administration Office of Inspector General (2012). *Procurement Fraud Handbook*, available here.

Halac, Marina and Yared, Pierre (2020). "Commitment versus Flexibility with Costly Verification", *Journal of Political Economy*, 128(2), pp. 4523–4573.

Kattwinkel, Deniz and Knoepfle, Jan (2023). "Costless Information and Costly Verification: A Case for Transparency", *Journal of Political Economy*, 131.2, pp. 504–548.

Khalil, Fahad (1997). "Auditing without Commitment", *The RAND Journal of Economics*, 28(4), pp. 629–640.

Laffont, Jean-Jacques and Martimort, David (2002). *The Theory of Incentives: The Principal-Agent Model*, Princeton, NJ, Oxford, UK: Princeton University Press.

Laffont, Jean-Jacques and Tirole, Jean (1986). "Using Cost Observation to Regulate Firms", *Journal of Political Economy*, 94(3), pp. 614–641.

Melumad, Nahum D. and Mookherjee, Dilip (1989). "Delegation as commitment: the case of income tax audits", *The RAND Journal of Economics*, 20.2, pp. 139-163.

Mookherjee, Dilip and Png, Ivan (1989). “Optimal Auditing, Insurance, and Redistribution”, *The Quarterly Journal of Economics*, 104.2, pp. 399-415.

Mussa, Michael and Rosen, Sherwin (1978). “Monopoly and Product Quality”, *Journal of Economic Theory*, 18, pp. 301–317.

Palonen, Petteri and Pekkarinen, Teemu (2022) “Optimal Regulation with Costly Verification”, mimeo.

Skrzypacz, Andrzej (2013). “Auctions with contingent payments — An overview”, *International Journal of Industrial Organization*, 31, pp. 666–675.

Townsend, Robert (1979). “Optimal Mechanisms and Competitive Markets with Costly State Verification”, *Journal of Economic Theory*, 21, pp. 265–293.

B. Online Appendix

B.1. Revelation Principle

Let $\mathcal{M}$ be an arbitrary set, the message space. A mechanism on the message space $\mathcal{M}$ is a tuple $\tilde{\mathbb{M}} = \left(\tilde{x}(\cdot), \tilde{q}^N(\cdot), \tilde{t}^N(\cdot), \tilde{q}^I(\cdot,\cdot), \tilde{t}^I(\cdot,\cdot)\right)$ where $\tilde{q}^N : \mathcal{M} \mapsto \mathbb{R}_+, \tilde{q}^I : \mathcal{M} \times [\underline{\theta}, \overline{\theta}] \mapsto \mathbb{R}_+, \tilde{t}^N : \mathcal{M} \mapsto \mathbb{R}, \tilde{t}^I : \mathcal{M} \times [\underline{\theta}, \overline{\theta}] \mapsto \mathbb{R}, \tilde{x} : \mathcal{M} \mapsto [0,1]$.

Let A be an allocation rule, $A(\theta, \theta') = \left(x(\theta), q^N(\theta), t^N(\theta), q^I(\theta, \theta'), t^I(\theta, \theta')\right)$. Say that the mechanism $\mathbb{M}$ on the message space $\mathcal{M}$ implements the allocation rule A and satisfies obedience if for all types θ there exists a message $m^*(\theta) \in \mathcal{M}$ such that
(i)

$$\begin{aligned}
&\tilde{x}(m^*(\theta)) \left(-\tilde{q}^I(m^*(\theta), \theta)\theta + \tilde{t}^I(m^*(\theta), \theta)\right) \\
&+ (1 - \tilde{x}(m^*(\theta))) \left(-\tilde{q}^N(m^*(\theta))\theta + \tilde{t}^N(m^*(\theta))\right) \\
&\geq \tilde{x}(m) \left(-\tilde{q}^I(m, \theta)\theta + \tilde{t}^I(m, \theta)\right) + (1 - \tilde{x}(m)) \left(-\tilde{q}^N(m)\theta + \tilde{t}^N(m)\right);
\end{aligned}$$

for all $m \in \mathcal{M}$;
(ii)

$$\begin{aligned}
-\tilde{q}^N(m^*(\theta))\theta + \tilde{t}^N(m^*(\theta)) &\geq 0 \;\forall \theta; \\
-\tilde{q}^I(m, \theta)\theta + \tilde{t}^I(m, \theta) &\geq 0 \;\forall \theta, m;
\end{aligned}$$

(iii)

$$A(\theta, \theta') = \left(\tilde{x}(m^*(\theta)), \tilde{q}^N(m^*(\theta)), \tilde{t}^N(m^*(\theta)), \tilde{q}^I(m^*(\theta), \theta'), \tilde{t}^I(m^*(\theta), \theta')\right).$$

Lemma 8 *Suppose the mechanism $\tilde{\mathbb{M}}$ on the message space $\mathcal{M}$ implements the allocation rule A and satisfies obedience. Then the allocation rule can be implemented with the direct mechanism $\mathcal{M} = [\underline{\theta}, \overline{\theta}]$, $\mathbb{M} = A$ which satisfies obedience.*

Proof. We can construct a direct revelation mechanism $q(\cdot) \equiv \tilde{q}^N(m^*(\cdot))$, $t^N(\cdot) \equiv \tilde{t}^N(m^*(\cdot))$, $q^I(\cdot,\cdot) \equiv \tilde{q}^I(m^*(\cdot),\cdot)$, $t^I(\cdot,\cdot) \equiv \tilde{t}^I(m^*(\cdot),\cdot)$, and $x(\cdot) \equiv \tilde{x}(m^*(\cdot))$. Since the mechanism $\tilde{\mathbb{M}}$ satisfies the IC constraints and limited commitment constraints, the direct revelation mechanism satisfies IC constraints and the obedience constraint.
■

B.2. Obedience constraints

Lemma 9 *1. Suppose* $\tilde{\mathbb{M}} = \left(\tilde{x}, \tilde{q^I}, \tilde{t^I}, \tilde{q^N}, \tilde{t^N}\right)$ *is an incentive-compatible direct mechanism with* $\tilde{x}(\cdot) \geq \underline{x}$*, and the Agent's strategy is sequentially rational. Then there exists an incentive compatible direct mechanism* $\mathbb{M} = \left(x, q^I, t^I, q^N, t^N\right)$ *such that*

$$-q^I(\hat{\theta}, \theta)\theta + t^I(\hat{\theta}, \theta) \geq 0 \ \forall \hat{\theta}, \theta,$$
$$-q^N(\theta)\theta + t^N(\theta) \geq 0 \ \forall \theta,$$

and the Principal's payoff is at least as high under $\mathbb{M}$ *than under* $\tilde{\mathbb{M}}$*.*

2. *Suppose the direct mechanism* $\mathbb{M} = \left(x, q^I, t^I, q^N, t^N\right)$ *is incentive compatible and the Agent behaves sequentially rational. If, for a positive measure of types,* $(1 - x(\theta))(-q^N(\theta)\theta + t^N(\theta)) < 0$ *or* $-q^I(\theta, \theta)\theta + t^I(\theta, \theta) < 0$*, then* $\mathbb{M}$ *is not optimal for the Principal.*

Proof.

1. Since $\tilde{\mathbb{M}}$ is incentive compatible, for all θ and $\hat{\theta}$,

$$\begin{aligned} &\tilde{x}(\theta) \max\{0, \tilde{q^I}(\theta, \theta)\theta + \tilde{t^I}(\theta, \theta)\} + (1 - \tilde{x}(\theta)) \max\{0, -\tilde{q}^N(\theta)\theta + \tilde{t^N}(\theta)\} \\ \geq\ &\tilde{x}(\hat{\theta}) \max\{0, \tilde{q^I}(\hat{\theta}, \theta)\theta + \tilde{t^I}(\hat{\theta}, \theta)\} + (1 - \tilde{x}(\hat{\theta})) \max\{0, -\tilde{q}^N(\hat{\theta})\theta + \tilde{t^N}(\hat{\theta})\}. \end{aligned}$$

Take θ such that $-\tilde{q^I}(\hat{\theta}, \theta)\theta + \tilde{t^I}(\hat{\theta}, \theta) < 0$ for some $\hat{\theta}$. For that θ and $\hat{\theta}$ set $q^I(\hat{\theta}, \theta) = \min\{q^{FB}(\theta), \tilde{q^I}(\hat{\theta}, \theta)\}$ and $t^I(\hat{\theta}, \theta) = q^I(\hat{\theta}, \theta)\theta$. Then the new mechanism is still incentive-compatible and yields a weakly higher payoff to the Principal.
Now consider a type θ such that $-\tilde{q^N}(\theta)\theta + \tilde{t^N}(\theta) < 0$. Change the mechanism such that $q^N(\theta) = 0, t^N(\theta) = \theta q^N(\theta) = 0$ for some. Set $x(\theta) = \tilde{x}(\theta)$. Then the Principal's payoff from type θ remains unaffected since (ignoring the terms in case of inspection)

$$\begin{aligned} &(1 - x(\theta))(V(q^N(\theta)) - t^N(\theta)) = (1 - x(\theta))V(0) \\ &= (1 - \tilde{x}(\theta)) \left[\mathbb{1}_{-\tilde{q^N}(\theta)\theta + \tilde{t^N}(\theta) \geq 0}(V(\tilde{q^N}(\theta) - \tilde{t^N}(\theta) - V(0)) + V(0)\right]. \end{aligned}$$

Moreover, the payoff type θ' receives when mimicking θ is 0 under $\mathbb{M}$ and

$$(1 - \tilde{x}(\theta)) \max\{0, -\tilde{q^N}(\theta)\theta' + \tilde{t^N}(\theta)\} \geq 0$$

under $\tilde{\mathbb{M}}$. Consequently, $\mathbb{M}$ is incentive-compatible.

2. Fix θ such that $(1-x(\theta))(-q^N(\theta)\theta+t^N(\theta)) < 0$. Then the Agent walks away on path. Change the mechanism such that $q^N(\theta) = \delta$ and $t^N(\theta) = \theta\delta$ for a $\delta > 0$ to be determined. For each other type θ', change the transfer after inspection to $t^I(\theta', \theta') = \tilde{t}^I(\theta', \theta') + \frac{1-x(\theta)}{x(\theta')}\delta \max\{0, \theta - \theta'\}$. The incentive constraints continue to hold after the change. Since $x(\theta) < 1$, the Principal's payoff is point-wise larger for δ sufficiently small because $\lim_{q\searrow 0} V'(q) = \infty$ by Assumption 1. The Principal's payoff increases strictly if the measure of types for which the obedience constraints do not hold on path is positive. The argument for the case $-q^I(\theta, \theta)\theta + t^I(\theta, \theta) < 0$ is similar and therefore omitted.

■

B.3. Non-existence

Lemma 10 *Problem $\mathcal{P}_0$ does not have a solution.*

Proof. Assume toward a contradiction that problem $\mathcal{P}_0$ has a maximizer $\mathbb{M}_0 = \big(x(\cdot), q^N(\cdot), t^N(\cdot), q^I(\cdot,\cdot), t^I(\cdot,\cdot)\big)$.

- Step 1: There exists an optimal mechanism such that $x(\theta) > 0 \implies t^N(\theta) = \theta q^N(\theta)$ and $q^I(\theta', \theta)\theta = t^I(\theta', \theta) = 0$ for all $\theta' \neq \theta$. Restrict attention to such mechanisms.

- Step 2: Let $\Theta_0 = \{\theta \in [\underline{\theta}, \overline{\theta}] | x(\theta) = 0\}$. Denote $U(\theta) = x(\theta)(-\theta q^I(\theta) + t^I(\theta)) + (1-x(\theta))(-\theta q^N(\theta) + t^N(\theta))$. Suppose $\Theta_0 \cap \{\theta | U(\theta) > 0\}$ has positive measure. Then

$$q^N(\theta) = q^{FB}(\theta)\ \forall \theta \in \Theta_0 \cap \{\theta | U(\theta) > 0\}.$$

- Step 3: For almost all $\theta \notin [\underline{\theta}, \overline{\theta}] \cap \Theta_0$, $q^N(\theta) = q$ solves quantity-interior-inspection.

- Step 4: Let $\pi(\theta) = \sup_{\hat{\theta}}(1 - x(\hat{\theta}))(t^N(\hat{\theta}) - \theta q^N(\hat{\theta}))$. Incentive compatibility requires $U(\theta) \geq \pi(\theta)$. For $\delta > 0$ let

$$\hat{\theta}_\delta(\theta) = \{\hat{\theta} | (1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta) \geq \pi(\theta) - \delta > 0\}.$$

If, for some δ>0, $\hat{\theta}_\delta(\theta) \subset \Theta_0$ for a positive measure of types θ, then $\mathbb{M}_0$ does not achieve the maximum.

- Step 5: Suppose there exists a $\delta > 0$ such that

$$B_\delta = [\underline{\theta}, \overline{\theta}] \setminus \cup_\theta \hat{\theta}_\delta(\theta)$$

has positive mass. Then $x(\theta) = 0$ on B_δ.

- Step 7: Let $\xi(\theta) = \min\{\xi | q^{FB}(\theta + \xi) - q(\theta) \leq 0, \theta + \xi \leq \overline{\theta}\}$ and $0 < \xi = \min\{\overline{\theta} - \underline{\theta}, \inf \xi(\theta)\}$ where $q(\theta)$ solves quantity-interior-inspection. Then

$$\{\theta | x(\theta) > 0, \exists \theta' \in (\theta, \theta + \xi) : x(\theta') = 0\}$$

has measure 0.

- Step 8: Either $x > 0$ almost everywhere or $\exists \theta' : x(\theta) = 0 \iff \theta \leq \theta'$. $x > 0$ implies $[\underline{\theta}, \underline{\theta} + \delta) \in B_\delta$ for some $\delta > 0$ small enough, a contradiction. The second condition implies $(\theta', \theta' + \delta) \in B_\delta$ for some $\delta > 0$ small enough or $\theta' = \overline{\theta} \iff x \equiv 0$; both lead to a contradiction.

■

B.4. Proof of Lemma 2

Proof. Throughout the proof, denote $q(\cdot) \equiv q^N(\cdot)$, $Q(\cdot) \equiv (1 - x(\cdot))q^N(\cdot)$ and $y(\theta) = 1 - x(\theta)$.
Recall the objective

$$\max_{q(\cdot) \geq 0,\, 1 \geq x(\cdot) \geq \underline{x}} \int x(\theta) \left(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \right) + (1 - x(\theta)) \left(V(q(\theta)) - q(\theta)\theta \right) \\ - \sup_{\hat{\theta}} (1 - x(\hat{\theta})) q(\hat{\theta})(\hat{\theta} - \theta) \, \mathrm{d}F(\theta),$$

and $\pi(\theta) = \sup_{\hat{\theta}} (1 - x(\hat{\theta})) q(\hat{\theta})(\hat{\theta} - \theta)$. For all θ and $\hat{\theta}$ define $\chi(\hat{\theta}, \theta) = (1 - x(\hat{\theta})) q(\hat{\theta})(\hat{\theta} - \theta)$.

We divide the proof of the Lemma into several claims.

Claim 4 *Suppose $(y(\cdot), q(\cdot))$ solve the optimization problem. Then $Q(\cdot)$ is (almost everywhere) equal to a decreasing continuous function on $\{\theta | Q(\theta) > 0\}$.*

Proof. The proof of the claim is divided into four steps.

1. **Suppose the set** $\{\theta | \exists \theta' > \theta \wedge Q(\theta') \geq Q(\theta) > 0\}$ **has positive measure. Then** $(y(\cdot), q(\cdot))$ **are not optimal.**
   Fix θ, θ' such that $\theta' > \theta \wedge Q(\theta') > Q(\theta) > 0$. By the last inequality, $y(\theta) \neq 0$. If $q(\theta) < q^{FB}(\theta)$, change $q(\theta) \rightsquigarrow \min\{q^{FB}(\theta), Q(\theta')/y(\theta)\} > q(\theta)$. Then, by definition of θ, θ', this change leaves $\sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta} - \tilde{\theta})$ unchanged for all $\tilde{\theta}$ but raises the objective point-wise. If $Q(\theta) = Q(\theta')$ one can raise $q(\theta)$ to $\min\{q^{FB}(\theta), Q(\theta')/y(\theta) + 1/n\}$ for an $n \in \mathbb{N}$ large enough without affecting $\sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta} - \tilde{\theta})$. If $q(\theta) = q^{FB}(\theta)$, $G(\theta) > 0$; moreover, $Q(\theta') \leq (1 - \underline{x})q^{FB}(\theta')$ implies $y(\theta) < 1 - \underline{x}$. Hence, one can raise $y(\theta)$ without affecting $\sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta} - \tilde{\theta})$ for any $\tilde{\theta}$.
   If there is a positive measure of such point, the objective increases strictly.

2. Q is almost everywhere equal to a decreasing (strictly decreasing when positive) function $\tilde{Q}$ such that for all θ

$$\sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta} - \theta) = \sup_{\hat{\theta}} \tilde{Q}(\hat{\theta})(\hat{\theta} - \theta).$$

   Denote $A = \{\theta | \forall \theta' > \theta : Q(\theta') < Q(\theta)\}$. By Step 1, A has measure 1. Define the function $\tilde{Q}(\cdot)$ by

$$\tilde{Q}(\theta) = \begin{cases} Q(\theta) & \theta \in A, \\ \sup_{\theta' > \theta, \theta' \in A} Q(\theta'). \end{cases}$$

   Note that $\tilde{Q}$ is well-defined because A has measure 1. It is straightforward to see that $\tilde{Q}(\cdot)$ has the required properties.

3. Suppose $Q(\cdot)$ is decreasing. Then we may assume it is left-continuous.
   Recall that decreasing functions have at most countably many discontinuity points. For each $\theta \in [\underline{\theta}, \overline{\theta}]$ define a new function $\tilde{Q}(\theta)$ by

$$\tilde{Q}(\theta) = \limsup_{\theta' \to \theta} Q(\theta').$$

   Since $Q(\cdot)$ is decreasing, $\tilde{Q}(\cdot)$ is left-continuous. Moreover, the value of the objective under $Q(\cdot)$ and $\tilde{Q}(\cdot)$ is the same, and $Q(\theta) = \tilde{Q}(\theta)$ for almost all θ.
   Henceforth, assume $Q(\cdot)$ is decreasing and left-continuous.

4. Suppose $Q(\cdot)$ is left-continuous, strictly decreasing when positive and has a discontinuity. Then $Q(\cdot)$ is not optimal.
   Let θ^1 be a discontinuity point:

$$\liminf_{\theta' \to \theta^1} Q(\theta') = \lim_{\theta' \to \theta^1 +} Q(\theta') > 0.$$

Since there is a discontinuity at θ^1, Q is strictly decreasing and left-continuous there exists a $\varepsilon > 0$ such that for all $\delta > 0$,

$$Q(\theta^1) \geq Q(\theta') + \varepsilon$$

for all $\theta' \in (\theta^1, \theta^1 + \delta)$. Note that for any $\theta \in [\underline{\theta}, \overline{\theta}], \hat{\theta} \in (\theta^1, \theta^1 + \delta)$

$$Q(\hat{\theta})(\hat{\theta} - \theta) \leq (Q(\theta^1) - \varepsilon)(\delta + \theta^1 - \theta) \leq Q(\theta^1)(\theta^1 - \theta)$$

where the last inequality holds for all $\delta > 0$ small enough. Note that, since $\pi(\theta^1) > 0$ there exists a $\delta > 0$ such that for all $\theta \in (\theta^1 - \delta, \theta^1)$,

$$Q(\theta^1)(\theta^1 - \theta) < \pi(\theta^1) < \pi(\theta).$$

This implies that, raising $Q(\theta')$ by $\varepsilon/2$ for all $\theta' \in (\theta^1, \theta^1 + \delta)$ does not affect

$$\sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta} - \theta)$$

for any θ provided δ is small enough. This change, however, increases the objective, in contradiction to the optimality of Q. □

From now we consider optimal $x(\cdot), q(\cdot)$ such that $Q(\cdot)$ is continuous and strictly decreasing. Recall the definition of the correspondence

$$\hat{\theta}(\theta) = \arg\max_{\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]} (1 - x(\tilde{\theta}))q(\tilde{\theta})(\tilde{\theta} - \theta).$$

Since $Q(\cdot)$ is a continuous function on $[\underline{\theta}, \overline{\theta}]$, the correspondence is non-empty valued.

Claim 5 $\hat{\theta}(\cdot)$ *is upper hemicontinuous with nonempty and compact values.*

Proof. This follows from Berge's Maximum Theorem.□

Claim 6 *Let* $\theta' < \theta^\dagger$. *Then* $\sup \hat{\theta}(\theta') \leq \inf \hat{\theta}(\theta^\dagger)$.

Proof. Assume $\breve{\theta} \in \hat{\theta}(\theta')$, and $\breve{\breve{\theta}} \in \hat{\theta}(\theta^\dagger)$. Thus

$$Q(\breve{\theta})(\breve{\theta} - \theta') \geq Q(\breve{\breve{\theta}})(\breve{\breve{\theta}} - \theta'),$$
$$Q(\breve{\breve{\theta}})(\breve{\breve{\theta}} - \theta^\dagger) \geq Q(\breve{\theta})(\breve{\theta} - \theta^\dagger).$$

Therefore

$$(\theta^\dagger - \theta^{'})(Q(\check{\theta}) - Q(\check{\check{\theta}})) \geq 0.$$

Since $Q(\cdot)$ is strictly decreasing, $\check{\theta} \leq \check{\check{\theta}}$.□

Claim 7 *If $\theta^{'} < \theta^\dagger$, and $\theta^{'}, \theta^\dagger \in \hat{\theta}(\check{\theta})$ for a type $\check{\theta}$, then $q(\theta'') = q^{FB}(\theta'')$, and $x(\theta'') = \underline{x}$ for all $\theta'' \in (\theta^{'}, \theta^\dagger)$.*

Proof. By Lemma 6 we know $\hat{\theta}(\theta) = [\inf \hat{\theta}(\theta), \sup \hat{\theta}(\theta)]$. The below definitions will be useful throughout the proof. For $\tilde{\gamma} > 0$, and type θ

$$J(\tilde{\gamma}) = [\inf \hat{\theta}(\check{\theta}) + \tilde{\gamma}, \sup \hat{\theta}(\check{\theta}) - \tilde{\gamma}];$$
$$\chi(\theta, \tilde{\gamma}) = \sup_{\theta'' \in J(\tilde{\gamma})} \chi(\theta'', \theta);$$
$$I(\beta, \tilde{\gamma}) = \{\theta | \pi(\theta) - (1+\beta)\chi(\theta, \tilde{\gamma}) \leq 0\}.$$

An admissible variation of $q(\theta'')$ for $\theta'' \in J(\tilde{\gamma})$ is $(1+\beta)q(\theta'')$ for small enough $\beta > 0$. A directional derivative of the objective for $\beta > 0$ gives us

$$\lim_{\beta \to 0^+} \int_{t \in J(\tilde{\gamma})} (1 - x(t)) \left(\frac{\partial V((1+\beta)q(t))}{\partial q(t)} q(t) - q(t)t \right) \mathrm{d}F(t)$$
$$- \lim_{\beta \to 0^+} \int_{t \in I(\beta, \tilde{\gamma})} \frac{(1+\beta)\chi(t, \tilde{\gamma}) - \pi(t)}{\beta} \mathrm{d}F(t) \leq 0.$$

The above inequality is correct only if the above limits exist for each integral. We will compute the above limit, hence it exists. We claim the last integral is zero in the above inequality because

$$0 = \int_{t \in I(\beta, \tilde{\gamma})} \frac{\pi(t) - \pi(t)}{\beta} \mathrm{d}F(t) \leq \int_{t \in I(\beta, \tilde{\gamma})} \frac{(1+\beta)\chi(t, \tilde{\gamma}) - \pi(t)}{\beta} \mathrm{d}F(t)$$
$$\leq \int_{t \in I(\beta, \tilde{\gamma})} \frac{(1+\beta)\pi(t) - \pi(t)}{\beta} \mathrm{d}F(t) = \int_{t \in I(\beta, \tilde{\gamma})} \pi(t) \mathrm{d}F(t).$$

If we show $\lim_{\beta \to 0^+} \int_{t \in I(\beta, \tilde{\gamma})} \pi(t) \mathrm{d}F(t) = 0$, then by the Squeeze Theorem we conclude that $\lim_{\beta \to 0^+} \int_{t \in I(\beta, \tilde{\gamma})} \frac{(1+\beta)\chi(t, \tilde{\gamma}) - \pi(t)}{\beta} \mathrm{d}F(t) = 0$. For this purpose we show $\cap_{\beta > 0} I(\beta, \tilde{\gamma}) = \lim_{\beta \to 0^+} I(\beta, \tilde{\gamma}) = \check{\theta}$. First we know $\check{\theta} \in \lim_{\beta \to 0^+} I(\beta, \tilde{\gamma})$, since $\pi(\check{\theta}) = Q(\theta'')(\theta'' - \check{\theta})$ for all $\theta'' \in [\inf \hat{\theta}(\theta), \sup \hat{\theta}(\theta)]$. Assume there exists $\tilde{\theta} \neq \check{\theta}$ such that $\tilde{\theta} \in \lim_{\beta \to 0^+} I(\beta, \tilde{\gamma})$. Fix $\beta > 0$. This means that for all $i \in \mathbb{N}$, there exists θ_i such that $\pi(\tilde{\theta}) \leq Q(\theta_i)(\theta_i - \tilde{\theta})(1 + \frac{\beta}{i})$, and $\theta_i \in J(\tilde{\gamma})$. The sequence $\{\theta_i\}_{i=1}^\infty$ has a subsequence with a convergence point, call it $\mathring{\theta}$. Since $\frac{\beta}{i}$ converges to zero and $Q(\cdot)$

is a continuous function, we will have $\pi(\tilde{\theta}) \leq Q(\mathring{\theta})(\mathring{\theta} - \tilde{\theta})$. This is a contradiction since $\mathring{\theta} \in J(\tilde{\gamma})$, and by Claim 6, $J(\tilde{\gamma}) \cap \hat{\theta}(\tilde{\theta}) = \emptyset$. Therefore $\lim_{\beta \to 0^+} I(\beta, \tilde{\gamma}) = \check{\theta}$ for all $\tilde{\gamma} > 0$. Since $\pi(\cdot)$ is bounded we can conclude $\lim_{\beta \to 0^+} \int_{t \in I(\beta, \tilde{\gamma})} \pi(t) \, \mathrm{d}F(t) = 0$. Finally the directional derivative becomes

$$\lim_{\beta \to 0^+} \int_{t \in J(\tilde{\gamma})} (1 - x(t)) \left(\frac{\partial V((1+\beta)q(t))}{\partial q(t)} q(t) - q(t)t \right) \mathrm{d}F(t) \leq 0,$$

$$\text{or} \int_{t \in J(\tilde{\gamma})} (1 - x(t))q(t) \left(\frac{\partial V(q(t))}{\partial q(t)} - t \right) \mathrm{d}F(t) \leq 0.$$

The second inequality follows from Dominated Convergence since the integrand is uniformly bounded from above. The same analysis applies to all intervals that are strictly inside $\hat{\theta}(\check{\theta})$, since $\tilde{\gamma} > 0$ was arbitrary. So for all intervals $I \subset \hat{\theta}(\check{\theta})$.

$$\int_{t \in I} (1 - x(t))q(t) \left(\frac{\partial V(q(t))}{\partial q(t)} - t \right) \mathrm{d}F(t) \leq 0.$$

We know $V'(q(t)) > t$ for all t such that $q(t) < q^{FB}(t)$. Therefore except measure zero points of $\hat{\theta}(\check{\theta})$, we have $q(t) = q^{FB}(t)$. By an argument similar to the one in the proof of part 3, Proposition 2 (which does not rely on Lemma 2), we know if $q(t) = q^{FB}(t)$, then $x(t) = \underline{x}$. Therefore for almost all $t \in \hat{\theta}(\check{\theta})$, we have $q(t) = q^{FB}(t)$, and $x(t) = \underline{x}$. Since $Q(\cdot)$ is a continuous function, for all $t \in \hat{\theta}(\check{\theta})$, we have $q(t) = q^{FB}(t)$, and $x(t) = \underline{x}$.□

Claim 8 *$\hat{\theta}(\cdot)$ is single-valued on $\{\theta | \pi(\theta) > 0\}$.*

Proof. If $\theta', \theta^\dagger \in \hat{\theta}(\check{\theta})$, then by Claim 7, we have $\pi(\check{\theta}) = (1 - \underline{x})q^{FB}(\theta'')(\theta'' - \check{\theta})$, for all $\theta'' \in (\theta', \theta^\dagger)$. This means that $q^{FB}(\theta'') = \frac{\pi(\check{\theta})}{(1-\underline{x})(\theta'' - \check{\theta})}$ which implies that $\frac{1}{q^{FB}(\theta'')} = C\theta'' + D$ where $C = \frac{1-\underline{x}}{\pi(\check{\theta})}$, and $D = \frac{-\check{\theta}(1-\underline{x})}{\pi(\check{\theta})}$ for all $\theta'' \in (\theta', \theta^\dagger)$, in contradiction to our assumption. □

Claim 9 *Let $\mathbb{M}$ be an optimal mechanism such that $Q(\cdot)$ is strictly decreasing and continuous. Then $Q(\cdot)$ is differentiable for all $\hat{\theta} \in (\hat{\theta}(\underline{\theta}), \theta^\dagger)$ where $\theta^\dagger = \min\{\theta' | Q(\theta') = 0\}$.*

Proof. Fix a point $\hat{\theta} \in (\hat{\theta}(\underline{\theta}), \theta^\dagger)$. Since $Q(\cdot)$ is strictly decreasing, it is almost everywhere differentiable. Hence, there exist sequences $(\theta_i^L)_i$, and $(\theta_i^R)_i$ such that $(\hat{\theta}(\theta_i^L))_i$, and $(\hat{\theta}(\theta_i^R))_i$ converge to $\hat{\theta}$ from the left and the right, respectively, and

$Q(\cdot)$ is differentiable at every point in the sequence. By definition of $\hat{\theta}(\cdot)$, necessary conditions for all for all $i \in \mathbb{N}$ are

$$Q'(\hat{\theta}(\theta_i^R))(\hat{\theta}(\theta_i^R) - \theta_i^R) + Q(\hat{\theta}(\theta_i^R)) = 0,$$
$$Q'(\hat{\theta}(\theta_i^L))(\hat{\theta}(\theta_i^L) - \theta_i^L) + Q(\hat{\theta}(\theta_i^L)) = 0.$$

Therefore

$$Q'(\hat{\theta}(\theta_i^L)) = \frac{-Q(\hat{\theta}(\theta_i^L))}{(\hat{\theta}(\theta_i^L) - \theta_i^L)}, \text{ and } Q'(\hat{\theta}(\theta_i^R)) = \frac{-Q(\hat{\theta}(\theta_i^R))}{(\hat{\theta}(\theta_i^R) - \theta_i^R)}.$$

Since $Q(\cdot)$ and $\hat{\theta}(\cdot)$ is a continuous function, the right-hand side of both expressions converges so that

$$Q'(\hat{\theta}^-) = \frac{-Q(\hat{\theta})}{\hat{\theta} - \hat{\theta}^{-1}(\hat{\theta})} = Q'(\hat{\theta}^+).$$

Thus $Q(\cdot)$ is differentiable at $\hat{\theta}$.□

This ends the proof of Lemma 2. ■

B.5. Proof of Lemma 6

Proof.
By definition

$$\hat{\theta}(\theta) = \underset{\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]}{\arg\max}\, (1 - x(\tilde{\theta}))q(\tilde{\theta})(\tilde{\theta} - \theta).$$

By Lemma 2, $\hat{\theta}(\cdot)$ is strictly increasing. Consequently, the solution of the above optimization problem cannot be a corner solution. Writing the first order condition for θ such that $\hat{\theta}(\theta) \leq \theta_2$, and using the fact that $x(\cdot)$ for a neighborhood of $\hat{\theta}(\theta)$ is constant, give us for $q(\cdot) = q^N(\cdot)$

$$(q)'(\hat{\theta}(\theta)) \times (\hat{\theta}(\theta) - \theta) + q(\hat{\theta}(\theta)) = 0.$$

■

B.6. Proof of Lemma 4

Proof. Recall that we can restrict ourselves to solutions such that $t^N(\theta) = \theta q^N(\theta)$, $q^I(\theta) = q^{FB}(\theta)$ and

$$x(\theta)t^I(\theta) = x(\theta)q^I(\theta) + \sup_{\hat{\theta}}(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta).$$

Hence, we can write problem $\mathcal{P}_{\underline{x}}$ equivalently as

$$\begin{aligned} \max_{x(\cdot),q^N(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} & x(\theta)\Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big) \\ & + (1 - x(\theta))\Big(V(q^N(\theta)) - \theta q^N(\theta)\Big) - \sup_{\hat{\theta}}(1 - x(\hat{\theta}))q^N(\hat{\theta})(\hat{\theta} - \theta)\,\mathrm{d}F(\theta) \\ & \text{subject to} \\ & \underline{x} \leq x(\hat{\theta}) \leq 1. \end{aligned}$$

Using the notation $y(\theta) = 1 - x(\theta)$, we see the problem is equivalent to

$$\begin{aligned} \max_{y(\cdot),q^N(\cdot)} & \int_{\underline{\theta}}^{\overline{\theta}} \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\,\mathrm{d}F(\theta) \\ & + \int_{\underline{\theta}}^{\overline{\theta}} y(\theta)\Big(V(q^N(\theta)) - \theta q^N(\theta) - \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\Big) \\ & - \sup_{\hat{\theta}} y(\hat{\theta})q^N(\hat{\theta})(\hat{\theta} - \theta)\,\mathrm{d}F(\theta) \\ & 0 \leq y(\hat{\theta}) \leq 1 - \underline{x}. \end{aligned}$$

This problem is equivalent to

$$\begin{aligned} \max_{y(\cdot),q^N(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} & \frac{y(\theta)}{1 - \underline{x}}\Big(V(q^N(\theta)) - \theta q^N(\theta) - \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\Big) \\ & - \sup_{\hat{\theta}} \frac{y(\hat{\theta})}{1 - \underline{x}} q^N(\hat{\theta})(\hat{\theta} - \theta)\,\mathrm{d}F(\theta) \\ & \text{subject to} \\ & 0 \leq y(\hat{\theta}) \leq 1. \end{aligned}$$

The claim follows immediately. ■

B.7. Proof of Lemma 5

Proof. Throughout we dispose of a suitable set of measure 0. Recall that we can solve the equivalent problem

$$\begin{aligned}\max_{y(\cdot),Q(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} & y(\theta)\Big(V\left(\frac{Q(\theta)}{y(\theta)}\right) - \theta\frac{Q(\theta)}{y(\theta)} - \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\Big) \\ & - \sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta}-\theta)\,\mathrm{d}F(\theta) \\ & \text{subject to} \\ & 0 \leq y(\theta) \leq 1 - \underline{x}.\end{aligned}$$

where $y(\theta) = 1 - x(\theta)$ and $Q(\theta) = y(\theta)q^N(\theta)$.

For two functions $y : [\underline{\theta}, \overline{\theta}] \to [0, 1 - \underline{x}], Q : [\underline{\theta}, \overline{\theta}] \to \mathbb{R}_+$, denote

$$\begin{aligned}G(y, Q) = \int_{\underline{\theta}}^{\overline{\theta}} & y(\theta)\Big(V\left(\frac{Q(\theta)}{y(\theta)}\right) - \theta\frac{Q(\theta)}{y(\theta)} - \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\Big) \\ & - \sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta}-\theta)\,\mathrm{d}F(\theta); \\ g(y, Q)(\theta) = {} & y(\theta)\Big(V\left(\frac{Q(\theta)}{y(\theta)}\right) - \theta\frac{Q(\theta)}{y(\theta)} - \Big(V(q^{FB}(\theta)) - \theta q^{FB}(\theta) - \kappa\Big)\Big) \\ & - \sup_{\hat{\theta}} Q(\hat{\theta})(\hat{\theta}-\theta)\end{aligned}$$

The Hessian of the function $(h, q) \mapsto hV(q/h)$ equals

$$\begin{pmatrix} \frac{q^2}{h^3}V''\left(\frac{q}{h}\right) & -\frac{q}{h^2}V''\left(\frac{q}{h}\right) \\ -\frac{q}{h^2}V''\left(\frac{q}{h}\right) & \frac{1}{h}V''\left(\frac{q}{h}\right) \end{pmatrix},$$

which is negative semi-definite for all $h, q > 0$. Hence, $(y, Q) \mapsto g(y, Q)(\theta)$ is concave for every θ and so is $(y, Q^N) \mapsto \int g(y, Q)(\theta)\,\mathrm{d}F(\theta)$.

Let $(y_1(\cdot), Q_1(\cdot)), (y_2(\cdot), Q_2(\cdot))$ be two solutions to the maximization problem. Let $\alpha \in (0, 1), y_\alpha = \alpha y_1 + (1-\alpha)y_2, Q_\alpha = \alpha Q_1 + (1-\alpha)Q_2$.

Suppose $y_1(\theta) = y_2(\theta)$, but $Q_1(\theta) \neq Q_2(\theta)$ for a positive mass of points θ. By strict concavity of $V(\cdot)$, $G(y_\alpha, Q_\alpha) > G(y_1, Q_1)$, a contradiction.

Suppose $y_1(\theta) \neq y_2(\theta)$, but $Q_1(\theta) = Q_2(\theta)$. By strict concavity of $V(\cdot)$ the map

$h \mapsto hV(1/h)$ is strictly concave. Hence, $G(y_\alpha, Q_\alpha) > G(y_1, Q_1)$, a contradiction.

Suppose $y_1(\theta) < y_2(\theta), Q_1(\theta) \neq Q_2(\theta)$ but $Q_1(\theta)/y_1(\theta) \neq Q_2(\theta)/y_2(\theta)$. By Proposition 2, part 3, $Q_1(\theta)/y_1(\theta)$ satisfies equation quantity-interior-inspection. Moreover, for each $\alpha \in (0,1)$, $Q_\alpha(\theta)/y_\alpha(\theta)$ needs to satisfy equation quantity-interior-inspection, in contradiction to $Q_1(\theta)/y_1(\theta) \neq Q_2(\theta)/y_2(\theta)$.

Hence, we conclude that $q_1^N(\theta) = Q_1(\theta)/y_1(\theta) = Q_2(\theta)/y_2(\theta) = q_2^N(\theta)$ for almost all θ.

Suppose $q^N(\theta) \neq q(\theta)$ where $q(\theta)$ solves equation quantity-interior-inspection. Then, by the same argument as in the proof of Proposition 2, part 3, $y_1(\theta) = y_2(\theta) = 1 - \underline{x}$ or $y_1(\theta) = y_2(\theta) = 0$. Hence, on $\{y_1 \neq y_2\}$, $q^N(\theta)$ solves equation quantity-interior-inspection.

By Claim 4, $Q_1(\cdot)$ and $Q_2(\cdot)$ are continuous and strictly decreasing when positive, and, without loss, left-continuous if they have a discontinuity at $\inf\{\theta | Q_i(\theta) = 0\}$. Assume toward a contradiction that $Q_1 \neq Q_2$.

Note that, for almost all $\theta' \in \{Q_1 \neq Q_2\}$ there exist a θ^i such that $\theta' \in \arg\max_{\hat{\theta}} Q_i(\hat{\theta})(\hat{\theta} - \theta^i)$; otherwise, an argument similar to the proof of Claim 4 shows that Q_i was not optimal. Denote $\check{\theta} = \inf\{\theta | Q_1(\theta) \neq Q_2(\theta)\}$. Since $Q_i(\cdot)$ is decreasing and continuous, there exists $\delta > 0$ such that for $\theta \in (\check{\theta}, \check{\theta} + \delta)$, $Q_1(\theta) > Q_2(\theta)$ and $Q_1'(\theta) > Q_2'(\theta)$ (almost everywhere).

Let θ be such that $(\check{\theta}, \check{\theta} + \delta) \supset \arg\max_{\hat{\theta}} Q_i(\hat{\theta})(\hat{\theta} - \theta)$ for $i = 1, 2$. For such a type θ,

$$\sup_{\hat{\theta}} (\alpha Q_1(\hat{\theta}) + (1-\alpha) Q_2(\hat{\theta}))(\hat{\theta} - \theta) < \alpha \sup_{\hat{\theta}} Q_1(\hat{\theta})(\hat{\theta} - \theta) + (1-\alpha) \sup_{\hat{\theta}} Q_2(\hat{\theta})(\hat{\theta} - \theta).$$

If $\{y_1 \neq y_2\}$ has positive measure, there is a positive mass of such types; a contradiction to the optimality of y_1 and y_2 .

■

B.8. Proof of Lemma 7

Proof. Let

$$\tilde{\theta} = \min\{\theta | \pi(\theta) = 0\}.$$

Step 1: if $\kappa > 0$, then full inspection, $x(\theta) = 1$ for all θ, is not optimal. Recall the objective

$$\max_{q(\cdot) \geq 0,\ 1 \geq x(\cdot) \geq \underline{x}} \int x(\theta) \left(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \right) + (1 - x(\theta)) \left(V(q(\theta)) - q(\theta)\theta \right) \\ - \sup_{\hat{\theta}} (1 - x(\hat{\theta})) q(\hat{\theta})(\hat{\theta} - \theta) \, \mathrm{d}F(\theta).$$

Set $x(\theta) = \underline{x}$ for $\theta \leq \check{\theta}$, and $x(\theta) = 1$ for $\theta > \check{\theta}$. Set $q(\theta) = q^{FB}(\theta)$ for all θ. We show there exists $\check{\theta} > \underline{\theta}$ such that, the value of the policy that we defined above is greater than full inspection. For this purpose, we should show

$$\int_{\underline{\theta}}^{\overline{\theta}} \big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta\big) \,\mathrm{d}F(\theta) + \int_{\underline{\theta}}^{\check{\theta}} -\underline{x}\kappa \,\mathrm{d}F(\theta) - \int_{\check{\theta}}^{\overline{\theta}} \kappa \,\mathrm{d}F(\theta) \\ - \int_{\underline{\theta}}^{\check{\theta}} \sup_{\hat{\theta} \in [\underline{\theta}, \check{\theta}]} (1 - \underline{x}) q^{FB}(\hat{\theta})(\hat{\theta} - \theta) \,\mathrm{d}F(\theta) \\ > \int_{\underline{\theta}}^{\overline{\theta}} \big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta\big) \,\mathrm{d}F(\theta) - \kappa.$$

Or

$$\int_{\underline{\theta}}^{\check{\theta}} \kappa - \underline{x}\kappa \,\mathrm{d}F(\theta) - \int_{\underline{\theta}}^{\check{\theta}} \sup_{\hat{\theta} \in [\underline{\theta}, \check{\theta}]} (1 - \underline{x}) q^{FB}(\hat{\theta})(\hat{\theta} - \theta) \,\mathrm{d}F(\theta) > 0.$$

Or

$$(1 - \underline{x}) \int_{\underline{\theta}}^{\check{\theta}} \left(\kappa - \sup_{\hat{\theta} \in [\underline{\theta}, \check{\theta}]} q^{FB}(\hat{\theta})(\hat{\theta} - \theta)\right) \mathrm{d}F(\theta) > 0.$$

By sending $\check{\theta}$ to $\underline{\theta}$, the inside of the integral becomes positive for a $\check{\theta} > \underline{\theta}$. Therefore there exists $\check{\theta}$ such that the above inequality holds.

Step 2: if $\kappa > 0$, then $\tilde{\theta} = \overline{\theta}$. Define $x_{[\theta', \overline{\theta}]}$, and $q_{[\theta', \overline{\theta}]}$, the solution to the problem $P_{[\theta', \overline{\theta}]}$, and define the value of this problem $W_{[\theta', \overline{\theta}]}$, where $P_{[\theta', \overline{\theta}]}$ is

$$\max_{x(\cdot) \in [\underline{x}, 1], q(\cdot)} \int_{\theta'}^{\overline{\theta}} x(\theta)\Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\Big) + (1 - x(\theta))\Big(V(q(\theta)) - q(\theta)\theta\Big) \\ - \sup_{\hat{\theta}} (1 - x(\hat{\theta})) q(\hat{\theta})(\hat{\theta} - \theta) \,\mathrm{d}F(\theta),$$

and $\theta' \in [\underline{\theta}, \overline{\theta}]$. We show if $\tilde{\theta} < \overline{\theta}$, then

$$W_{[\tilde{\theta}, \overline{\theta}]} = \int_{\tilde{\theta}}^{\overline{\theta}} \Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\Big) \,\mathrm{d}F(\theta).$$

The above equality means that the solution of the problem $P_{[\tilde{\theta}, \overline{\theta}]}$ is full inspection. This is a contradiction to the first step.

Toward a contradiction assume $\tilde{\theta} < \overline{\theta}$. The solution of the problem $P_{[\underline{\theta}, \overline{\theta}]}$ is

$x_{[\underline{\theta},\bar{\theta}]}$, and $q_{[\underline{\theta},\bar{\theta}]}$ which by abuse of notation we say x, and q. Since the inspection probability is equal to 1 for types above $\tilde{\theta}$, we can say the solution of $P_{[\underline{\theta},\bar{\theta}]}$ for $\theta \geq \tilde{\theta}$ is $x(\theta) = 1$, and $\hat{q}(\theta)$, where $\hat{q}(\theta)$ can be any function (since it is irrelevant). So we can say the solution of the problem $P_{[\underline{\theta},\bar{\theta}]}$ is $x(\theta)$, $q(\theta)$ for $\theta < \tilde{\theta}$, and for $\theta \geq \tilde{\theta}$, is $1 - \beta(1 - x_{[\tilde{\theta},\bar{\theta}]}(\theta))$, $q_{[\tilde{\theta},\bar{\theta}]}(\theta)$ when $\beta = 0$. Define

$$\check{x}(\theta,\beta) = \begin{cases} x(\theta) & \theta < \tilde{\theta}, \\ 1 - \beta(1 - x_{[\tilde{\theta},\bar{\theta}]}(\theta)) & \theta \geq \tilde{\theta}, \end{cases} \qquad \check{q}^N(\theta) = \begin{cases} q(\theta) & \theta < \tilde{\theta}, \\ q_{[\tilde{\theta},\bar{\theta}]}(\theta) & \theta \geq \tilde{\theta}. \end{cases}$$

Define $W_{[\underline{\theta},\bar{\theta}]}(\beta)$ to be

$$\int_{\underline{\theta}}^{\bar{\theta}} \check{x}(\theta,\beta)\Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\Big) + (1 - \check{x}(\theta,\beta))\Big(V(\check{q}^N(\theta)) - \check{q}^N(\theta)\theta\Big) \\ - \sup_{\hat{\theta}}(1 - \check{x}(\hat{\theta},\beta))\check{q}^N(\hat{\theta})(\hat{\theta} - \theta)\,\mathrm{d}F(\theta).$$

Therefore $W_{[\underline{\theta},\bar{\theta}]}(\beta = 0) = W_{[\underline{\theta},\bar{\theta}]}$. We will show

$$\lim_{\beta \to 0^+} \frac{W_{[\underline{\theta},\bar{\theta}]}(\beta) - W_{[\underline{\theta},\bar{\theta}]}(0)}{\beta}$$

exists and we will compute it. Note that if the limit exists, by optimality of x and q we know

$$\lim_{\beta \to 0^+} \frac{W_{[\underline{\theta},\bar{\theta}]}(\beta) - W_{[\underline{\theta},\bar{\theta}]}(0)}{\beta} \leq 0.$$

Define $\chi(\theta) = \max_{\hat{\theta} \in [\tilde{\theta},\bar{\theta}]} (1 - x_{[\tilde{\theta},\bar{\theta}]}(\hat{\theta}))q^N_{[\tilde{\theta},\bar{\theta}]}(\hat{\theta})(\hat{\theta} - \theta)$. Define the set $I(\beta) = \{\theta \in [\underline{\theta},\tilde{\theta}] | \pi(\theta) \leq \beta\chi(\theta)\}$. Compute $W_{[\underline{\theta},\bar{\theta}]}(\beta)$

$$W_{[\underline{\theta},\bar{\theta}]}(\beta) = \int_{\underline{\theta}}^{\bar{\theta}} \Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\Big)\,\mathrm{d}F(\theta) \\ + \int_{\underline{\theta}}^{\tilde{\theta}} (1 - x(t))\left(V(q(\theta)) - \theta q(\theta) - \left(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\right)\right)\,\mathrm{d}F(\theta) \\ + \int_{\tilde{\theta}}^{\bar{\theta}} \beta(1 - x_{[\tilde{\theta},\bar{\theta}]}(t))\left(V(q^N_{[\tilde{\theta},\bar{\theta}]}(\theta)) - \theta q^N_{[\tilde{\theta},\bar{\theta}]}(\theta) - \left(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa\right)\right)\,\mathrm{d}F(\theta) \\ - \int_{\tilde{\theta}}^{\bar{\theta}} \beta\chi(\theta)\,\mathrm{d}F(\theta) - \int_{\underline{\theta}}^{\tilde{\theta}} \max\left(\beta\chi(\theta), \pi(\theta)\right)\,\mathrm{d}F(\theta).$$

Therefore $W_{[\underline{\theta},\overline{\theta}]}(\beta) - W_{[\underline{\theta},\overline{\theta}]}(0)$ equals

$$= \int_{\tilde{\theta}}^{\overline{\theta}} \beta(1 - x_{[\tilde{\theta},\overline{\theta}]}(t)) \Big(V(q^N_{[\tilde{\theta},\overline{\theta}]}(\theta)) - \theta q^N_{[\tilde{\theta},\overline{\theta}]}(\theta) - \big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \big) \Big) \, \mathrm{d}F(\theta)$$
$$- \int_{\tilde{\theta}}^{\overline{\theta}} \beta\chi(\theta) \, \mathrm{d}F(\theta) - \int_{\underline{\theta}}^{\tilde{\theta}} \max(0, \beta\chi(\theta) - \pi(\theta)) \, \mathrm{d}F(\theta).$$

First we show

$$\lim_{\beta \to 0^+} \int_{\underline{\theta}}^{\tilde{\theta}} \frac{\max(0, \beta\chi(\theta) - \pi(\theta))}{\beta} \, \mathrm{d}F(\theta) = 0.$$

The reason is

$$0 \le \int_{\underline{\theta}}^{\tilde{\theta}} \frac{\max(0, \beta\chi(\theta) - \pi(\theta))}{\beta} \, \mathrm{d}F(\theta) = \int_{I(\beta)} \frac{\beta\chi(\theta) - \pi(\theta)}{\beta} \, \mathrm{d}F(\theta)$$
$$\le \int_{I(\beta)} \chi(\theta) \, \mathrm{d}F(\theta).$$

Since $\cap_{\beta > 0} I(\beta) = \lim_{\beta \to 0^+} I(\beta) = \tilde{\theta}$, and $\chi(\theta)$ is abounded above, then $\lim_{\beta \to 0^+} \int_{I(\beta)} \chi(\theta) \, \mathrm{d}F(\theta) = 0$. Therefore $\lim_{\beta \to 0^+} \int_{\underline{\theta}}^{\tilde{\theta}} \frac{\max(0, \beta\chi(\theta) - \pi(\theta))}{\beta} \, \mathrm{d}F(\theta) = 0$. Thus $\lim_{\beta \to 0^+} \frac{W_{[\underline{\theta},\overline{\theta}]}(\beta) - W_{[\underline{\theta},\overline{\theta}]}(0)}{\beta}$ is equal to

$$= \int_{\tilde{\theta}}^{\overline{\theta}} (1 - x_{[\tilde{\theta},\overline{\theta}]}(t)) \Big(V(q^N_{[\tilde{\theta},\overline{\theta}]}(\theta)) - \theta q^N_{[\tilde{\theta},\overline{\theta}]}(\theta) - \big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \big) \Big) \, \mathrm{d}F(\theta)$$
$$- \int_{\tilde{\theta}}^{\overline{\theta}} \chi(\theta) \, \mathrm{d}F(\theta) = W_{[\tilde{\theta},\overline{\theta}]} - \int_{\tilde{\theta}}^{\overline{\theta}} \Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \Big) \, \mathrm{d}F(\theta) \le 0$$

Note that by the Dominated convergence theorem, we can transfer the limit inside of the above integrals since the inside is bounded above. We know $W_{[\tilde{\theta},\overline{\theta}]} - \int_{\tilde{\theta}}^{\overline{\theta}} \Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \Big) \, \mathrm{d}F(\theta) \ge 0$ since full inspection is always feasible. Therefore

$$W_{[\tilde{\theta},\overline{\theta}]} = \int_{\tilde{\theta}}^{\overline{\theta}} \Big(V(q^{FB}(\theta) - q^{FB}(\theta)\theta - \kappa \Big) \, \mathrm{d}F(\theta).$$

A contradiction. ■

B.9. Computing the inspection probability

Let $q^*(\cdot)$ be the solution to equation quantity-interior-inspection. Define $y(\theta) = 1 - x(\theta)$. By the first order condition, we know

$$-\frac{\mathrm{d}\ln(y(\hat{\theta}))}{\mathrm{d}\hat{\theta}} = \frac{\partial\ \ln\left(q^*(\hat{\theta})(\hat{\theta}-\theta)\right)}{\partial\,\hat{\theta}}.$$

Therefore

$$-\ln(y(\theta))\Big|_t^{\overline{\theta}} = \int_t^{\overline{\theta}} \frac{\partial\ \ln\left(q^*(\hat{\theta})(\hat{\theta}-\theta)\right)}{\partial\,\hat{\theta}}\,\mathrm{d}\hat{\theta} = \int_t^{\overline{\theta}} \left(\frac{\partial\ \ln\left(q^*(\hat{\theta})\right)}{\partial\,\hat{\theta}} + \frac{\partial\ \ln(\hat{\theta}-\theta)}{\partial\,\hat{\theta}}\right)\mathrm{d}\hat{\theta}$$

$$= \ln(q^*(\hat{\theta}))\Big|_t^{\overline{\theta}} + \int_{\hat{\theta}^{-1}(t)}^{\hat{\theta}^{-1}(\overline{\theta})} \left(\frac{1}{\hat{\theta}(\theta)-\theta}\right)\hat{\theta}'(\theta)\,\mathrm{d}\theta$$

Using the differential equation for $\hat{\theta}(\theta)$, we know

$$\frac{f(\theta)}{\left(V'(q^N(\hat{\theta}(\theta))) - \hat{\theta}(\theta)\right) f(\hat{\theta}(\theta))} = \frac{\hat{\theta}'(\theta)}{\hat{\theta}(\theta)-\theta}.$$

Therefore

$$-\ln(y(\theta))\Big|_t^{\overline{\theta}} = \ln(q^*(\hat{\theta}))\Big|_t^{\overline{\theta}} + \int_{\hat{\theta}^{-1}(t)}^{\hat{\theta}^{-1}(\overline{\theta})} \frac{f(\theta)}{\left(V'(q^N(\hat{\theta}(\theta))) - \hat{\theta}(\theta)\right) f(\hat{\theta}(\theta))}\,\mathrm{d}\theta.$$

Rewriting, for $t \leq \overline{\theta}$ we have

$$y(\overline{\theta}) = exp\left(-\int_t^{\overline{\theta}} \frac{\partial\ ln\left(q^*(\hat{\theta})(\hat{\theta}-\theta)\right)}{\partial\,\hat{\theta}}\Bigg|_{\hat{\theta}(\theta)=\theta}\mathrm{d}\hat{\theta}\right) y(t).$$