

JANUARY 2024

“Acting in the Darkness: Towards some
Foundations for the *Precautionary Principle*”

Louise Guillouet and David Martimort

ACTING IN THE DARKNESS: TOWARDS SOME FOUNDATIONS FOR THE
PRECAUTIONARY PRINCIPLE

January 5, 2024

LOUISE GUILLOUET^a AND DAVID MARTIMORT^b

ABSTRACT. Invoked to guide actions under irreversibility, uncertainty and limited information, the *Precautionary Principle* states that decision-makers should act cautiously unless the consequences of acts are known. We consider a setting where the stock of past actions, passed a tipping point which remains unknown, increases the probability of a catastrophe. When past acts are observable, decision-makers can reconstruct the whole evolution of stock and beliefs and follow an optimal trajectory. Otherwise, and in accordance with the Precautionary Principle, they act cautiously, remaining too optimistic on their ability to delay the tipping point. This suboptimal behaviour has minor consequences on welfare.

KEYWORDS. *Precautionary Principle*, Environmental Risk, Tipping Point, Uncertainty and Irreversibility.

JEL CODES. D83, Q55.

1. INTRODUCTION

ON THE PRECAUTIONARY PRINCIPLE. The major environmental and health issues that pertain to our modern *risk society* are most often due to our own production and consumption.¹ When dealing with such risks, decision-making is complicated by two features that make the standard tools of cost-benefit analysis of limited value. The first specificity is that consumption and production choices might entail much irreversibility. The most salient example is given by global warming. Pollutants have been accumulating in the atmosphere from the beginning of the industrial era, leading to a steady increase in temperature. All current or planned efforts against global warming consist in controlling the growth rate of temperature, with little hope of reducing it. Another example is given by Genetically Modified Organisms (*GMOs*) crops whose production may profoundly

*We thank seminar participants to the *Environmental Economics Seminar* and to the *Frontier in Environmental Economics 2017 Workshop* at Paris School of Economics, *ETH Zurich Workshop on Environmental Economics 2017*, *World Congress of Environmental and Resource Economics 2018*, and Montpellier, *CREST-ENSAE-Polytechnique*, Columbia University, the *3rd IO Workshop* in Bergamo 2019, Toronto Rotman, Duke University, University British Columbia Vancouver, University Washington Seattle, University of Liverpool, *CAMS-EHESS Paris*, University of Glasgow, University Saint-Andrews, Toulouse School of Economics and *22nd Louis-André Gérard-Varet Conference on Public Economics* in Marseille, and Florida State University for useful comments on earlier versions of this paper. This work has benefited from helpful discussions with Sarah Auster, Bruno Jullien, R. Vijay Krishna, Perrin Lefévre, Paul-Henri Moisson, Jérôme Pouyet, Bernard Salanié, Eric Seré, Jean Tirole, Nicolas Treich. We especially thank Antoine Bommier, Ivar Ekeland, Bard Harstad and Ali Lazrak for detailed comments and Javier Gonzales-Morin for excellent research assistance. This paper has also been funded by the Agence Nationale de la Recherche under grant ANR-17-EURE-0010 (Investissements d'Avenir program) and by the *ERC (MARKLIN)*. The usual disclaimer applies.

^aColumbia University, New York, louise.guillouet@columbia.edu

^bToulouse School of Economics, Toulouse, david.martimort@tse-fr.eu

¹See Beck (1992).

modify the surrounding biotope without any possibility of going backwards because of irreversible mutations.²

The second feature of those problems is that the costs and benefits of any decision have to be assessed under significant uncertainty and limited information. Although the consequences of acting might be detrimental to the environment, the extent to which it is so and the probability of harmful events remain to a large extent unknown to decision-makers when acting.

The policy guidelines that have been adopted to structure decision-making and regulation in those contexts greatly vary from one country to the other. To illustrate, while *GMOs* are authorized for human consumption in the United States without labelling, it is compulsory to label them in 64 other countries throughout the world and they are actually forbidden in most of the European Union. To further guide decision-making, the so called *Precautionary Principle* has been repeatedly invoked. The original idea is due to philosopher Hans Jonas' *Vorsorgeprinzip*, or *Principle of Foresight*.³ This *Principle* states that decision-makers should recognize the long-term irreversible consequences of their current actions, and refrain from undertaking any such action if there is no proof that it would not negatively affect future generations' well-being.⁴

At least since its inception, there has always been a lively debate, mainly led by philosophers and political scientists, on whether the *Precautionary Principle* offers a convenient guide for decision-making under uncertainty. On the one hand, the fact that it serves as a background for some regulatory policies suggests that it should be judged on normative grounds. On the other hand, that doubts always exist on the fact that its adoption might actually do more harm, by hindering innovation and growth, than good, by protecting human health or the environment instead emphasizes that this notion has a more positive nature and that the *Precautionary Principle* just describes and justifies behavior which might remain suboptimal in practice.⁵

These conflicting views on the value of a *Precautionary Principle* raise a number of questions that certainly deserve further theoretical inquiries. What does it mean for decision-makers to take actions without having evidence on the future consequences of those acts in a context that entails irreversibility and uncertainty? How do current decision-makers form beliefs on the underlying risks when learning (or not) from past experience and how do those beliefs impact on their current actions? What differences, if any, are there

²Other examples include hydraulic fracturing to exploit shale gas (which implies irreversible pollution of underground water reserves), authorizing the use of bisphenol A or glyphosate (which are both potential sources of cancers), relying excessively on antibiotic use and *de facto* creating antimicrobial resistance, relying exclusively on nuclear energy (with potential severe environmental destruction and health issues in case of an catastrophe).

³Sometimes translated and referred to as the *Principle of Responsibility*.

⁴The *Precautionary Principle* was acknowledged by the United Nations in 1992, during the *Rio Earth Summit*, and perhaps expressed less restrictively as: "Where there are threats of serious and irreversible damage, lack of full scientific certainty shall not be used as a reason for postponing cost-effective measures to prevent environmental degradation." A similar principle was invoked in the French 2004 *Charter on Environment* (Loi constitutionnelle n 2005-205 du 1 mars 2005 relative à la Charte de l'environnement) that is now part of the French Constitution. Any risk, health or environmental regulation must thus comply with the legal framework that the *Precautionary Principle* contributes to build.

⁵See Sunstein (2005), Gardiner (2006), Giddens (2011), O'Riordan (2013) for informal discussions and Immordino (2003) for a survey of the relevant literature for economics.

between action plans that such decision-makers might follow with and without information on the consequences of past acts? In other words, what sort of externalities across decision-makers are induced by the lack of information? Does this lack of information tilt decision-making towards taking more precaution? And finally, does it really matter to abide to such *Precautionary Principle* from a welfare viewpoint?

MODEL. To address those issues, we consider a model of dynamic decision-making under irreversibility, uncertainty and limited information. This model stands as a rough metaphor for how society takes decisions under the threat of global warming. Hereafter, an action (consumption/production) taken at any point in time yields a flow payoff. Past actions have an irreversible impact. More precisely, the stock of past actions affects the arrival rate of an environmental catastrophe. A catastrophe is a major disruptive event. All opportunities for consumption/production disappear afterwards.⁶ Passed a tipping point, the probability of a catastrophe jumps up.⁷ Only the distribution of possible tipping points is known. Whether the tipping point has been passed or not remains ignored. In this context, an optimal trajectory should *a priori* follow a feedback rule that stipulates actions in terms of the level of stock and beliefs on where the tipping point lies.

COMMON KNOWLEDGE OF THE TIPPING POINT. Suppose first that the tipping point is known. All past actions contribute to approaching this tipping point; an *Irreversibility Effect*. In this benchmark, the sole state variable needed to describe the state of the system is the stock of past actions. Because of discounting and because all past actions play the same role in approaching the tipping point, the optimal feedback rule, which only depends on stock, requires lower actions as the stock increases during an early phase. Distortions below the myopic optimum are driven by the concern for irreversibility. Once the tipping point has been passed, actions no longer impact the arrival rate. The flow payoff is then maximized when jumping to a higher myopic optimum.

UNCERTAINTY ON THE TIPPING POINT. Suppose now that only the distribution of possible values for the tipping point is known.⁸ The decision-maker now acts in the darkness, taking into account not only the irreversibility of his earlier actions but also his beliefs on whether the tipping point has been passed or not. When acting, the decision-maker only knows that there has been no catastrophe up to that date. Acting today changes how likely it is that the tipping point will be passed in the near future and thus affects posterior beliefs in case no catastrophe takes place. The state of the system is now best described by appending to the stock of past actions another state variable (called the *regime survival ratio* in what follows) that reflects beliefs on whether the tipping point has been passed or not. The optimal feedback rule now determines how the current action depends on both the existing stock of past actions and current beliefs. As the decision-

⁶See Cropper (1976), Gjerde, Grepperud, and Kverndokk (1999) and Clarke and Reed (1994) for a similar assumption.

⁷Tipping points models are frequently used in ecology and in climatology (Lenton et al., 2008). To illustrate, a recent report by the World Bank argues that “As global warming approaches and exceeds 2-degrees Celsius, there is a risk of triggering nonlinear tipping elements. Examples include the disintegration of the West Antarctic ice sheet leading to more rapid sea-level rise. The melting of the Arctic permafrost ice also induces the release of carbon dioxide, methane and other greenhouse gases which would considerably accelerate global warming.” See <http://whrc.org/project/arctic-permafrost>.

⁸Kriegler et al. (2009) offers a view of what experts might think of those distributions of tipping points. Roe and Baker (2007) argues that whether past actions have already triggered a change of regimes might remain unknown for a while.

maker becomes more pessimistic and believes that it is more likely that the tipping point has been passed, jumping towards the myopic optimum becomes more attractive.

STOCK-MARKOV EQUILIBRIA. This optimal path helps to understand how trajectories are modified under more realistic assumptions. First, instead of having a single decision-maker choosing actions, the trajectory is viewed as the outcome of a game with different selves acting at different points in time. Those selves choose actions that prevail only for an infinitesimal period of time; a so called *impulse deviation*. Second, we consider that, when acting, those selves might have only limited information on the consequences of past acts. At a *Stock-Markov Equilibrium* (thereafter *SME*), those selves adopt a feedback rule based only on stock. In practice, while the stock of pollutants in the atmosphere can be easily verified, this stock might not be a sufficient statistics to form correct beliefs on whether the tipping point is likely to have been passed or not.

Those modeling assumptions are meant to echo the framework in which the *Precautionary Principle* is invoked. First, the concern that current actions may negatively impact future generations is captured by having different decision-makers, each endowed with the discounted flow of future payoffs, acting at different points in time. Second, the fact that current selves have only limited information on the consequences of past acts is a necessary ingredient to assess whether equilibrium actions are more cautious as suggested by the *Precautionary Principle*. When taken in tandem, those assumptions allow us to assess whether and under which circumstances an optimal trajectory can be decentralized as a non-cooperative equilibrium among selves. It also allows us to understand whether informational constraints on the consequences of past acts call for more precaution.

OBSERVABLE IMPULSE DEVIATIONS. Suppose first that impulse deviations are observable by future selves. In any such *SME*, actions remain below the myopic optimum because of the *Irreversibility Effect*. Since impulse deviations are observable, future selves will certainly believe that the tipping point is more likely to have been passed following a deviation that has increased the stock they inherited; a *Pessimistic Stigma*. Thinking that the tipping point is more likely to have been passed, yet no catastrophe has occurred, future selves no longer adopt a safe stance and actions jump towards the myopic optimum.

An optimal trajectory can always be implemented as a *SME* when impulse deviations are observable.⁹ The intuition is simple. At the optimum, a complete feedback rule defines actions in terms of the two state variables which are stock and beliefs. Yet, since stock and beliefs evolve on a one-dimensional manifold along the optimal trajectory, the optimal complete feedback rule naturally induces a *Stock-Markov* feedback rule on path. By construction, actions being the same with those two rules, beliefs evolve similarly. Off path, future selves can always reconstruct the evolution of beliefs from the observed impulse deviation by their predecessors and the conjecture that, beyond such deviation, all selves abide to the equilibrium feedback rule. This construction aligns the choices of various selves acting at different points in time with what a long-lived planner would do even if those selves have limited commitment power. Information that subsequent selves may have on possible deviations keeps predecessors in check.

⁹In passing, this argument also shows that such an equilibrium always exists since we demonstrate that the optimization problem has always a solution.

NON-OBSERVABLE IMPULSE DEVIATIONS. In contrast, consider the scenario where impulse deviations cannot be detected by future selves. In practice, the consequences of an act at a given point in time may only be revealed after a lag. It might be due to the fact that scientific knowledge is not advanced enough to assess those consequences right away; a context often referred to as being that of “*full scientific certainty*” in the parlance of the *Precautionary Principle*. An alternative justification is that those selves might have bounded rationality and a limited ability to process information.

Because impulse deviations are not observable, an informational externality now arises across decision-makers. Future selves can no longer infer that the tipping point is more likely to have been passed if they have not been able to observe past impulse deviations that had increased stock levels. The equilibrium feedback rule now entails a more prudent behavior. Actions are always too low in comparison with what the optimal trajectory would request. Along such a low-action trajectory, the tipping point is thought to be unlikely to have been passed yet; which in turn justifies adopting a more prudent behavior. This scenario gives foundations for the *Precautionary Principle* invoked by real-life decision-makers. Numerical simulations nevertheless suggest that the lack of information on past behavior does not entail a large welfare cost compared to the observable deviation scenario (less than 5%); softening concerns about the use of the *Precautionary Principle*.

ORGANIZATION. Section 2 reviews the literature. Section 3 presents the model. Section 4 analyzes the case where the tipping point is known. Section 5 deals with the scenario where only the distribution of the tipping point is known. Section 6 analyzes the properties of *SME* and shows that one such equilibrium implements the optimal trajectory when impulse deviations are observable. Section 7 characterizes equilibrium feedback rules with non-observable impulse deviations. Section 8 provides numerical simulations that show that, under a broad range of parameter values, the two scenarios with and without observable deviations provide close-by solutions. Section 9 briefly recaps our results and discusses possible extensions. Proofs are relegated into Appendices.

2. LITERATURE REVIEW

IRREVERSIBILITY, UNCERTAINTY AND INFORMATION. Arrow and Fisher (1974), Henry (1974) and Freixas and Laffont (1984) were the first to show how a decision-maker should take more preventive stances when the consequences of irreversible choices are uncertain. Epstein (1980) has discussed general conditions under which this *Irreversibility Effect* prevails. In those models, information is exogenous whereas in many contexts in environmental economics, actions also determine information structures. Hereafter, the probability of having passed the tipping point depends on the stock of past actions. Models with endogenous information structures are scarce. Freixas and Laffont (1984) have studied a scenario in which more flexible actions increase the quality of future information, thus confirming the existence of the *Irreversibility Effect*. Miller and Lad (1984) have challenged this view in a model of conservation in which irreversible actions might also be more informative. Salmi, Laiho and Murto (2019) study the trade-off faced by a decision-maker who must choose between acting now, which means taking a less informed decision but generating information that is useful in the sequel, and acting later, when being more informed. Greater actions accelerate the convergence of beliefs towards the true state.¹⁰

¹⁰Some authors have argued that the irreversibility literature suggests that current abatements of greenhouse gas emissions should be greater when more information will be available in the future

ECONOMICS AND THE PRECAUTIONARY PRINCIPLE. Gollier, Jullien and Treich (2000) have built on the insights of the irreversibility literature to give some economic content to the *Precautionary Principle*. These authors interpret the *Precautionary Principle* as the incentives of a decision-maker to reduce actions below the level that would otherwise be optimal without uncertainty, when actions are taken before learning information. Much in the spirit of Kolstad (1996), Gollier, Jullien and Treich (2000) have proposed a two-period model of pollution accumulation with exogenous information and drawn conclusions on specific forms of utility functions that induce more precaution. Asano (2010) has focused on the comparison of optimal environmental policies without and with ambiguity, showing that lack of confidence forces decision-makers to hasten policy adoption. In those models, decisions are always optimal although constrained by informational requirements.¹¹ In Gollier, Jullien and Treich (2000), scientific information is an exogenous process and the authors are interested in comparing actions with and without such information. In contrast, we stress that beliefs on the state of the system are by and large endogenous, determined by the history of past actions and the information structure.

ON TIPPING POINTS AND CATASTROPHES. Catastrophic outcomes due to stock pollutants have been analyzed by Cropper (1976), Heal (1984) and Clarke and Reed (1994) among others. In those models, the probability of a catastrophe (be it irreversible or temporary) is increasing in the stock. Tsur and Zemel (1995) have investigated a problem of optimal resource extraction when extraction affects the probability that the resource becomes obsolete passed a certain threshold. When this threshold is unknown, the initial state affects the optimal path and there is less resource exploitation than under certainty. Sims and Finoff (2016) have studied how irreversibility in environmental damage and irreversibility in sunk cost investment interact in a model with tipping point uncertainty. Focusing on the optimal control of atmospheric pollution, Tsur and Zemel (1996) have shown how uncertainty on a tipping point introduces a multiplicity of possible equilibria. Tsur and Zemel (2021) have studied trajectories with state-dependent catastrophe thresholds. Contrary to us, these authors have focused on the case where the mere fact that the stock of pollutants has passed the tipping point is immediately learned by the decision-maker.¹² We instead assume that this event remains ignored. To capture this ignorance, another state variable reflecting the decision-maker's beliefs is introduced. This addition bears some resemblance to Crépin and Nævdal (2020)'s analysis. For the sake of realism, these authors have also added to state-dependent catastrophe models based on pollutants (or temperature) another state variable, the stress of the system, that triggers changes of regime only when it itself passes a threshold. Van der Ploeg (2014) has analyzed how uncertainty on tipping points may modify the design of an optimal dynamic path for carbon taxes. Lemoine and Traeger (2014) have investigated optimal policy in a context where decision-makers learn over the location of the tipping point over time from observing how the system responds. In the context of policies against global warming, they demonstrate that the possibility of regime switching significantly increases the optimal carbon tax. A similar empirical assessment has been obtained in Cai and Lontzek (2019). Finally, Liski and Salanié (2020) have also studied a model with unknown tipping points

(Chichilnisky and Heal, 1993; Beltratti, Chichilnisky and Heal, 1995; Kolstad, 1996; Gollier, Jullien and Treich, 2000; among others). Others like Ulph and Ulph (2012) have pointed out that the sufficient conditions given by Epstein (1980) for the *Irreversibility Effect* to hold may fail even in simple models of global warming.

¹¹This feature is shared by other models in the field like Immordino (2000) and Gonzales (2008).

¹²On this, see also Nævdal (2006).

and uncertainty applied to climate change and pandemic crisis. A decision-maker chooses a control variable whose stock may trigger a catastrophe which occurs with stochastic delay, passed the tipping point. These authors are particularly concerned with conditions ensuring whether actions are monotonic over time.

3. THE MODEL

TECHNOLOGY. Actions taken by a decision-maker, say DM , over time put the environment at risk. Time is continuous. Let $r > 0$ be the discount rate. Let $\mathbf{x} = (x(\tau))_{\tau \geq 0}$ (resp. $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$) denote an action plan (resp. the continuation of a plan from date t on).

A catastrophe may arise; an event that follows a Poisson process with a (non-homogeneous) rate $\theta(t)$. That rate depends on the stock $X(t) = \int_0^t x(\tau) d\tau$ of past actions that have already been taken before date t . More precisely, we postulate

$$(3.1) \quad \theta(t) = \theta_0 + \Delta \mathbb{1}_{\{X(t) > \bar{X}\}}$$

where \bar{X} is a *tipping point*; which for the time being is supposed to be common knowledge. Although it remains quite close to a homogeneous Poisson process, and indeed it is so before and after the tipping point, this specification features dependence on past actions. Indeed, when the stock of past actions $X(t)$ passes \bar{X} , the rate jumps from θ_0 to $\theta_1 > \theta_0$. Let $\Delta = \theta_1 - \theta_0 > 0$ measure this jump.

PREFERENCES. Action $x(t)$ yields a flow payoff (net of the action cost) at date t worth $u(x(t))$. Although, we most often keep a general formulation, some of our results (optimal feedback rules and Hamilton-Bellman-Jacobi equations for value functions) are expressed in a crisper way by taking a quadratic specification, namely

$$u(x(t)) \equiv \zeta x(t) - \frac{x^2(t)}{2}.$$

where $\zeta > 0$ is the marginal benefits of action (the consumption side) and $\frac{x^2(t)}{2}$ its cost (the production side). The set of feasible actions is $\mathcal{X} = [0, 2\zeta]$ so that flow payoff remains non-negative under all circumstances below.¹³

To capture its detrimental and irreversible impact, we assume that, if a catastrophe arises at date t , the flow payoff is no longer realized from that date on. A justification for this extreme assumption is that production may no longer be possible afterwards.¹⁴

NOTATIONS. We start with the simplest scenario where DM has no control over the arrival rate of a catastrophe, i.e., the case of a homogeneous Poisson process and we assume that the tipping point is at $\bar{X} = 0$. DM 's expected payoff can thus be written as:

$$\int_0^{+\infty} e^{-\lambda_1 t} u(x(t)) dt$$

¹³For simplicity, we assume that there is no flow damage $D(X(t))$ due to the stock of past pollutant but this possibility could be added to the model, although at the cost of unnecessary complications.

¹⁴This assumption is made for simplicity. A more general model would allow for an arbitrary number of catastrophes with possibly changes in the production/consumption structure following each of those events. This additional complexity would not add anything in terms of insights.

where $\lambda_1 = r + \theta_1$ stands for the effective discount rate that applies once the possibility of a catastrophe is taken into account. Since he cannot influence the arrival rate of the catastrophe, DM maximizes current payoff at any point in time by choosing the *myopic action*

$$x^m(t) = \zeta \quad \forall t \geq 0.$$

For future reference, the myopic payoff once the tipping point has been passed writes as

$$\mathcal{V}_\infty = \frac{u(\zeta)}{\lambda_1}.$$

4. WHEN THE TIPPING POINT IS KNOWN

Let \bar{T} be the earliest date at which the tipping point is reached.¹⁵ With that notation at hands, we may rewrite DM 's expected payoff as:

$$\int_0^{\bar{T}} e^{-\lambda_0 t} u(x(t)) dt + e^{-\lambda_0 \bar{T}} \int_{\bar{T}}^{+\infty} e^{-\lambda_1 (t-\bar{T})} u(x(t)) dt.$$

where $\lambda_0 = r + \theta_0$ stands for the effective discount rate before the tipping point. The first integral thus stems for welfare before the tipping point. The second integral stands for welfare after the tipping point, weighted by the probability of survival up to the date \bar{T} at which the tipping point is reached, namely $e^{-\lambda_0 \bar{T}}$. The arrival rate of a catastrophe from that date on has now jumped up and payoffs beyond date \bar{T} are more heavily discounted. Passed the tipping point, the probability of a catastrophe jumps up. It is important to stress that, afterwards, current actions have no longer any impact on the probability of a catastrophe.¹⁶

DYNAMIC PROGRAMMING. Consider an action plan $\mathbf{x}_0 = \{x(\tau)\}_{\tau \geq 0}$ from date 0 onwards. If the stock at date 0 were X , the stock process $\hat{X}(\tau; X)$ would evolve as

$$(4.1) \quad \hat{X}(\tau; X) = X + \int_0^\tau x(s) ds.$$

After having passed the tipping point at date \bar{T} , DM always chooses the myopic optimal action ζ and gets, from that date on, a discounted continuation payoff worth \mathcal{V}_∞ . Let accordingly define the current value function $\mathcal{V}^k(X; \bar{X})$ as

$$(4.2) \quad \mathcal{V}^k(X; \bar{X}) \equiv \sup_{\mathcal{A}_0^k} \int_0^{\bar{T}} e^{-\lambda_0 \tau} u(x(\tau)) d\tau + e^{-\lambda_0 \bar{T}} \mathcal{V}_\infty^{17}$$

where the set of feasible trajectories is

$$\mathcal{A}_0^k = \left\{ \mathbf{x}_0, \hat{X}(\cdot) \text{ s.t. (4.1) and } \hat{X}(\bar{T}; X) = \bar{X} \text{ for some } \bar{T} \in [0, +\infty) \right\}.$$

¹⁵In all scenarios below, we allow for the possibility that $\bar{T} = +\infty$, i.e., the decision-maker might choose to never reach the tipping point. Our notations are consistent with this possibility. As Proposition 1 below demonstrates, the tipping point (when known) or the finite upper bound on its distribution (when unknown) is always reached in finite time.

¹⁶Of course, in an extension with multiple tipping points, this feature of our model could easily be relaxed although at the cost of some notational burden.

¹⁷It should be clear that the current value function $\mathcal{V}^k(X; \bar{X})$ and optimal decision-rule $\sigma^k(X; \bar{X})$ so obtained only depend on the distance $Y = \bar{X} - X$ to the tipping point. In other words, there exist two functions $\bar{\mathcal{V}}$ and $\bar{\sigma}$ such that $\mathcal{V}^k(X; \bar{X}) \equiv \bar{\mathcal{V}}(\bar{X} - X)$ and $\sigma^k(X; \bar{X}) \equiv \bar{\sigma}(\bar{X} - X)$. For the sake of comparing value functions and feedback rules under different scenarios, we nevertheless express the optimality conditions found below in terms of $\mathcal{V}^k(X; \bar{X})$ and $\sigma^k(X; \bar{X})$.

Equipped with these notations, we are now ready to further characterize the value function and the optimal feedback rule.

PROPOSITION 1 *The value function $\mathcal{V}^k(X; \bar{X})$ is continuously differentiable on $[0, \bar{X})$ and satisfies the following Hamilton-Bellman-Jacobi equation*

$$(4.3) \quad \dot{\mathcal{V}}^k(X; \bar{X}) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^k(X; \bar{X})}, \quad \forall X < \bar{X}.^{18}$$

$\mathcal{V}^k(X; \bar{X})$ is decreasing and strictly concave for $X \in [0, \bar{X})$ with the boundary condition

$$(4.4) \quad \mathcal{V}^k(X; \bar{X}) = \mathcal{V}_\infty \quad \forall X \geq \bar{X}.$$

The optimal feedback rule is such that

$$(4.5) \quad \sigma^k(X; \bar{X}) = \begin{cases} \zeta + \dot{\mathcal{V}}^k(X; \bar{X}) & \text{for } X \in [0, \bar{X}), \\ \zeta & \text{for } X \geq \bar{X}. \end{cases}$$

Moreover, $\sigma^k(X; \bar{X})$ is decreasing in X for $X \in [0, \bar{X})$.

ACTIONS PROFILE. The optimal action goes through two distinct phases. Before reaching the tipping point, actions remain below the myopic optimum. Those actions have a long-lasting impact since they may contribute to passing the tipping point earlier on. Reducing those actions decreases the probability that a catastrophe arises earlier. The quantity $-\dot{\mathcal{V}}^k(X; \bar{X})$ found on the r.h.s. of (4.5) is in fact the Lagrange multiplier for the irreversibility constraint

$$(4.6) \quad \int_0^{\bar{T}} x(\tau) d\tau = \bar{X} - X.$$

As X increases without having yet reached \bar{X} , this irreversibility constraint becomes more demanding, and the value function decreases. Actions are below the myopic optimum to account for this *Irreversibility Effect*.

The optimal action decreases over time before the tipping point. All actions taken during this first phase contribute the same to the overall stock. Because of discounting, DM prefers to choose higher actions earlier on and lower ones when approaching the tipping point. Expressed in terms of the value function, this monotonicity means that $\mathcal{V}^k(X; \bar{X})$ is strictly concave over this first phase while it becomes flat once the tipping point has been passed. By then, DM knows that his actions will no longer have any impact on the arrival rate of a catastrophe and thus chooses the myopic optimum.

TIPPING POINT. Because actions are now lower than the myopic optimum over the first phase, the tipping point is reached at a date¹⁹

$$(4.7) \quad \bar{T}^k = \bar{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}}\right) \frac{1 - e^{-\lambda_0 \bar{T}^k}}{\lambda_0} > \bar{T}^m = \frac{\bar{X}}{\zeta}$$

¹⁸At $X = \bar{X}$, this derivative is in fact a left-derivative but we use the same notation for simplicity.

¹⁹See the Appendix for details.

where \bar{T}^m is the time necessary to reach the tipping point when acting myopically. The intuition for this result is as follows. By pushing a bit further in the future the date \bar{T}^k at which the tipping point is reached by a small amount $d\bar{T}$, DM incurs a welfare loss since, over the first phase, the action is below the myopic optimum. DM is therefore getting less than the optimal payoff over a longer period of time. Pushing a bit further in the future the date \bar{T}^k also hardens the feasibility constraint. Finally, increasing \bar{T}^k maintains the arrival rate of a catastrophe at its low level θ_0 over that extended period. By doing so, DM is less likely to lose the myopic payoff $u(\zeta)$ in case a catastrophe occurs.

BOUNDS. Next proposition provides bounds on payoffs and actions. As we will see below, those bounds will also prevail when the location of the tipping point remains uncertain.

PROPOSITION 2 $\mathcal{V}^k(X; \bar{X})$ and $\sigma^k(X; \bar{X})$ admit the following bounds

$$(4.8) \quad \mathcal{V}_\infty \leq \mathcal{V}^k(X; \bar{X}) < \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty \quad \forall X,$$

$$(4.9) \quad \zeta \sqrt{\frac{\lambda_0}{\lambda_1}} \leq \sigma^k(X; \bar{X}) \leq \zeta \quad \forall X.$$

Because the rate of arrival of a catastrophe remains low till the tipping point is passed, the value function remains above its long-term limit \mathcal{V}_∞ reached beyond that point. The upper bound on the value function is the payoff corresponding to choosing always the myopic action but in the scenario where the tipping point would never be passed and the effective discount rate remains λ_0 . The upper bound on actions is simply the myopic optimum $x^m = \zeta$. The lower bound $\zeta \sqrt{\frac{\lambda_0}{\lambda_1}}$ is the action that ends the phase where the *Irreversibility Effect* is at play.

5. UNCERTAINTY: VALUE FUNCTION AND FEEDBACK RULE

Suppose now that DM does not know where the tipping point lies. Switching to the myopic optimum once the tipping point has been passed is no longer possible since DM remains ignorant on whether this event occurred or not. Accordingly, let denote by F the distribution of possible values for the tipping point and by f its (positive) density function. This distribution has a finite support $[0, \bar{X}]$ (i.e., $\bar{X} < +\infty$) and, for simplicity, no mass point.²⁰

5.1. Preliminaries

BELIEFS. Consider a history of past actions \mathbf{x}^t with no catastrophe up to date t and a stock reached at that date given by $\hat{X}(t; 0) = \int_0^t x(s) ds$. To evaluate DM 's continuation payoff, we need to compute his posterior beliefs $f(\tilde{X}|t, \mathbf{x}^t) d\tilde{X}$ that the tipping point lies within the interval $[\tilde{X}, \tilde{X} + d\tilde{X}]$ given that past history \mathbf{x}^t at date t . This posterior density $f(\tilde{X}|t, \mathbf{x}^t)$ should take into account that, if the tipping point lies at $\tilde{X} \leq \hat{X}(t; 0)$, the arrival rate has already jumped from θ_0 to θ_1 at an earlier date $T(\tilde{X}; 0) \leq t$. If instead

²⁰In contrast, the **RUNNING EXAMPLE** below entails mass points but adapting the analysis is straightforward.

the tipping point is at $\tilde{X} > \hat{X}(t; 0)$, the arrival rate remains θ_0 . A key variable to describe how the posterior density evolves is thus the probability of survival up to date t when the path of past actions is \mathbf{x}^t , namely

$$(5.1) \quad H(t, \mathbf{x}^t) = \int_0^{\hat{X}(t;0)} f(\tilde{X}) e^{-\theta_0 T(\tilde{X};0)} e^{-\theta_1(t-T(\tilde{X};0))} d\tilde{X} + \int_{\hat{X}(t;0)}^{+\infty} f(\tilde{X}) e^{-\theta_0 t} d\tilde{X}.$$

After manipulations, we obtain:

$$(5.2) \quad H(t, \mathbf{x}^t) = e^{-\theta_0 t} \left(1 - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau; 0)) e^{\Delta \tau} d\tau \right).^{21}$$

When $\hat{X}(\tau; 0)$ is close to 0, the likelihood of having passed the tipping point is also close to 0. The survival probability is then nearly that obtained when the arrival rate of a catastrophe is known to be θ_0 for sure. As $\hat{X}(\tau; 0)$ increases towards \bar{X} , it becomes more likely that the tipping point has been passed and the survival probability accordingly decreases. Of course, the shape of the distribution function F matters to evaluate this probability. As F puts more mass around the origin, it is more likely that the tipping point has been passed early on and the survival probability diminishes.

For future reference, let us define the *regime survival ratio* $\hat{Z}(t, \mathbf{x}^t)$ as

$$(5.3) \quad \hat{Z}(t, \mathbf{x}^t) = H(t, \mathbf{x}^t) e^{\theta_0 t} \quad \forall t \geq 0.$$

It is the ratio between the survival probability $H(t, \mathbf{x}^t)$ at date t following a history \mathbf{x}^t and the survival probability $e^{-\theta_0 t}$ that would prevail had the tipping point never been passed.²² This ratio actually reflects DM 's beliefs on whether the tipping point has been passed or not. The faster the trajectory moves towards \bar{X} , the faster $\hat{Z}(t, \mathbf{x}^t) = 1 - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau; 0)) e^{\Delta \tau} d\tau$ decreases. If the trajectory stays close to $X = 0$, $\hat{Z}(t, \mathbf{x}^t)$ decreases very slowly. In other words, a higher value of $\hat{Z}(t, \mathbf{x}^t)$ can be viewed as reflecting greater optimism for DM . DM still thinks that the tipping point is ahead.

RUNNING EXAMPLE. Suppose that F has Dirac masses q at 0 and $1 - q$ at \bar{X} . In other words, DM is uncertain whether the tipping point is passed right away or whether it will be later found at \bar{X} . For any $t > 0$ and history \mathbf{x}^t that has not yet reached \bar{X} , the probability of survival is a convex combination of exponential discounting:

$$H(t, \mathbf{x}^t) = q e^{-\theta_1 t} + (1 - q) e^{-\theta_0 t}.$$

From this, it follows that the *regime survival ratio* before reaching \bar{X} becomes

$$\hat{Z}(t, \mathbf{x}^t) = 1 - q + q e^{-\Delta t}.$$

Note that $\hat{Z}(t, \mathbf{x}^t)$ is decreasing in t ; capturing the fact that DM becomes more pessimistic as approaching the highest possible value of the tipping point \bar{X} . ■

²¹See the Proof of Lemma B.1 in the Appendix.

²²Since the survival probability is bounded below by $e^{-\theta_1 t}$, the regime survival ratio itself lies within $(e^{-\Delta t}, 1]$.

VALUE FUNCTION. The value function $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ is, by definition, DM 's continuation payoff starting from date t onwards given the past history \mathbf{x}^t . This function is computed with the posterior density function $f(\tilde{X}|t, \mathbf{x}^t)$ that the tipping point lies ahead of the current stock $X = \hat{X}(t; 0)$ reached at date t (i.e., for $\tilde{X} \geq \hat{X}(t; 0)$) given that, following past history, no catastrophe has yet occurred. For $\tau \geq t$, the stock (denoted with a slight abuse of notations by $\hat{X}(\tau; X, t)$) will evolve according to the stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$. Lemma 1 provides a compact representation for this value function.

LEMMA 1 *The value function $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ satisfies*

$$(5.4) \quad \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, \hat{X}(\tau; X, t) = X + \int_t^\tau x(s) ds} \int_0^{+\infty} e^{-\int_0^\tau \left(\lambda_0 - \frac{\frac{d\hat{Z}}{ds}(t+s, \mathbf{x}^{t+s})}{\hat{Z}(t+s, \mathbf{x}^{t+s})} \right) ds} u(x(t+\tau)) d\tau.$$

The representation (5.4) of the value function suggests that the state of the system is best described by adding to the stock X a second state variable, the *regime survival ratio* Z that reflects how beliefs evolve. Two trajectories that have reached the same stock X with the same beliefs Z at a given date should have the same continuation. Instead, two trajectories that have reached the same stock but with different beliefs might be pursued differently. If the regime switch is thought as having been likely (Z small), DM will pursue with higher actions since he has less incentives to take a precautionary stance.

5.2. Representation of the Value Function

To complete the state of the system, we must thus add to the law of motion for the stock, namely

$$(5.5) \quad \dot{X}(\tau) = x(\tau),$$

the law of motion for the regime survival ratio.²³ Differentiating (5.3) and using (5.2) yields

$$(5.6) \quad \dot{Z}(\tau) = \Delta(1 - F(X(\tau)) - Z(\tau)).$$

Integrating (5.6) with the initial condition $Z(0) = Z$, we get the following expression for the regime survival ratio $Z(\tau)$:

$$(5.7) \quad Z(\tau) = 1 - \underbrace{\Delta e^{-\Delta\tau} \int_0^\tau F(X(s)) e^{\Delta s} ds}_{\text{Memoryless Evolution}} - \underbrace{(1 - Z)e^{-\Delta\tau}}_{\text{Pessimistic Stigma}}.$$

This expression highlights how the evolution of beliefs actually superposes two effects. Suppose that DM keeps no memory of what happened in the past. He is naively believing to start with $Z = 1$, only knowing about the current level of stock X and considering, from that point on, the ensuing trajectory $X(t)$ given by (5.5). The first term on the r.h.s. of (5.7) captures how such a naive DM would evaluate the consequences of pursuing this

²³Kamien and Schwartz (1971), Reed (1989) and Tsur and Zemel (1995) have developed dynamic optimization models which all have in common to use the survival probability as a state variable. The difference in our setting comes from the fact that this survival probability depends on where the trajectory lies in the distribution of possible tipping points.

trajectory on future beliefs. Instead, whenever DM starts with some grain of pessimism inherited from past history, i.e., starting with a level of Z less than 1, this (negative) *Pessimistic Stigma* is carried on in the future (although at a decreasing rate) and all the more so as Z is lower; an effect that is captured by the second term on the r.h.s. of (5.7).

Finally, (5.6) also implies that, once a trajectory $X(\tau)$ has reached the upper bound \bar{X} at a date \bar{T} , the regime survival ratio evolves from then on as²⁴

$$(5.8) \quad Z(\tau) = Z(\bar{T})e^{-\Delta(\tau-\bar{T})} \quad \forall \tau \geq \bar{T}.$$

REMARK. For future reference, it is worth noticing that (5.6) together with the initial condition $Z(0) = Z$ imply that necessarily

$$(5.9) \quad Z(\tau) > 1 - F(X(\tau)) \quad \forall \tau \geq 0$$

and thus

$$(5.10) \quad \dot{Z}(\tau) < 0.$$

The first of those inequalities can be readily interpreted. Indeed, $1 - F(X(\tau))$ is the probability that the tipping point lies above $X(\tau)$. Consider an alternative scenario where the fact of having passed the tipping point would be always immediately known (which also means that when not having crossed the tipping point yet, the rate of arrival of a catastrophe is known to be θ_0). The probability of survival conditional on not having crossed the tipping point yet at date τ along a path $X(\tau)$ would thus be $(1 - F(X(\tau)))e^{-\theta_0\tau}$. The regime survival ratio in that scenario would be $1 - F(X(\tau))$. Henceforth, (5.9) can be interpreted as saying that not knowing whether the tipping point has been passed, decision-makers somehow remain more optimistic. The second inequality (5.10) simply means that those decision-makers nevertheless become more pessimistic over time. ■

Using (5.4) and (5.8), we can now get a representation of the value function in terms of the bi-dimensional state variable (X, Z) . Let accordingly define the value function $\mathcal{V}^e(X, Z)$ for $X \geq 0$ and any $Z \in (0, 1]$ as

$$(5.11) \quad \mathcal{V}^e(X, Z) = \sup_{\mathcal{A}} \int_0^{\bar{T}} e^{-\int_0^\tau (\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}) ds} u(x(\tau)) d\tau + e^{-\int_0^{\bar{T}} (\lambda_0 - \frac{\dot{Z}(s)}{Z(s)}) ds} \mathcal{V}_\infty.$$

where the set of admissible trajectories is

$$\mathcal{A} = \{\mathbf{x}, X(\cdot), Z(\cdot), \bar{T} \text{ s.t. } (5.5), (5.6), X(0) = X, X(\bar{T}) = \bar{X}, Z(0) = Z\}.$$

Starting from any pair (X, Z) , DM looks for an optimal arc that reaches \bar{X} at date \bar{T} . From that date on, DM knows for sure that the tipping point has been passed and chooses the myopic optimum. In fact, the tipping point might have already been passed a long time ago but DM could not know it for sure before reaching \bar{X} .

The expression (5.11) showcases that, under uncertainty, the effective discount rate

$$\lambda^e(\tau) \equiv \lambda_0 - \frac{\dot{Z}(\tau)}{Z(\tau)}$$

²⁴Once the stock level is beyond the support of F , the probability to be in the low-risk regime is 0.

is time-dependent. Using the regime survival ratio as a state variable keeps track of this time-dependency. The choice of an action at any given date has no direct impact on how this implicit discount rate evolves since the law of motion (5.6) for beliefs does not depend on current action. Yet, because stock and beliefs evolve over time, this implicit discount rate keeps on changing and DM must take this into account to assess how his future payoffs should be discounted. Specifically, DM is using $\lambda^e(\tau) \approx \lambda_0$ to discount future payoffs earlier on but, eventually, will switch to $\lambda^e(\tau) \approx \lambda_1$. The hazard rate $-\dot{Z}(\tau)/Z(\tau)$ measures how information contained in the fact that no catastrophe has yet happened is incorporated into this implicit discounting.

PROPOSITION 3 *The value function $\mathcal{V}^e(X, Z)$ satisfies:*

$$(5.12) \quad \frac{\partial \mathcal{V}^e}{\partial X}(X, Z) = -\zeta + \sqrt{2\lambda^e(X, Z)\mathcal{V}^e(X, Z) - 2\Delta(1 - F(X) - Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z)} \quad a.e.$$

where

$$(5.13) \quad \lambda^e(X, Z) = \lambda_0 - \frac{\Delta(1 - F(X) - Z)}{Z}$$

together with the boundary conditions

$$(5.14) \quad \mathcal{V}^e(X, Z) = \mathcal{V}_\infty \quad \forall X \geq \bar{X}, \forall Z \in (0, 1].$$

The optimal feedback rule is

$$(5.15) \quad \sigma^e(X, Z) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X, Z).$$

The comparison of the Hamilton-Bellman-Jacobi equations with and without uncertainty is instructive. The first difference between (4.3) and (5.12) is related to how future payoffs are discounted. As discussed earlier, under uncertainty, the effective discount rate is now time-dependent. As a thought experiment, suppose that the evolution of the hazard rate $-\dot{Z}(\tau)/Z(\tau)$ were exogenously given. The implicit discount rate being low earlier on and higher later, the optimal solution would call for taking larger actions earlier on. In our model, this dynamic evolution is endogenous. Current actions modify stock and beliefs and somewhat control the evolution of the hazard rate $-\dot{Z}(\tau)/Z(\tau)$.

The second difference comes from a new term, not present under complete information, $-2\Delta(1 - F(X) - Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z)$ on the r.h.s. of (5.12). Less optimistic stances, i.e., lower values of Z are associated with lower continuation values (i.e., $\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z) < 0$). Along the optimal trajectory, this new term is negative.²⁵ Being less optimistic and thinking that the tipping point has already been passed, DM certainly chooses to increase actions.

Finally, the comparison of the feedback rule (5.15) with its complete information counterpart (4.5) shows that the term $\frac{\partial \mathcal{V}^e}{\partial X}(X, Z)$ can again be interpreted as an opportunity cost of irreversibility. This cost now depends on beliefs. The consequences of such beliefs on actions can be further illustrated in the framework of our example.

RUNNING EXAMPLE (CONTINUED). When $q = 0$, we have $F(X) = 0$ for all $X \in [0, \bar{X})$ and it is straightforward to check that the solution to (5.12) and (5.14) is $\mathcal{V}^e(X, Z) \equiv$

²⁵Indeed, we have $-\dot{Z}(\tau)/Z(\tau) = -\frac{\Delta(1-F(X(\tau))-Z(\tau))}{Z(\tau)} > 0$.

$\mathcal{V}^k(X; \bar{X})$. When $q = 1$, we instead have $F(X) = 1$ for all $X \in (0, \bar{X}]$ and the solution to (5.12) and (5.14), for $Z = 1$ is then $\mathcal{V}^e(X, 1) \equiv \mathcal{V}_\infty$.

Although $\mathcal{V}^e(X, Z)$ cannot be expressed in closed form for $q > 0$, both the profile of optimal actions \mathbf{x}^e along the trajectory starting from $X = 0$ and $Z = 1$, and the delay \bar{T}^e till reaching the tipping point, can be solved explicitly.

PROPOSITION 4 *Suppose that F has Dirac masses q at 0 and $1 - q$ at \bar{X} . The optimal trajectory starting from $X = 0$ and $Z = 1$ has the following features.*

- The date \bar{T}^e at which \bar{X} is reached solves

$$(5.16) \quad \bar{T}^e = \bar{T}^m + \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) \int_0^{\bar{T}^e} \frac{Z(\bar{T}^e)}{Z(\tau)} e^{-\lambda_0(\bar{T}^e - \tau)} d\tau > \bar{T}^m$$

where the regime survival ratio is

$$(5.17) \quad Z(\tau) = 1 - q + qe^{-\Delta\tau} \quad \forall \tau \in [0, \bar{T}^e].$$

- The optimal action is decreasing over $\tau \in [0, \bar{T}^e)$ and equal to the myopic optimum thereafter:

$$(5.18) \quad x^e(\tau) = \begin{cases} \zeta \left(1 - e^{-\lambda_0(\bar{T}^e - \tau)} \frac{Z(\bar{T}^e)}{Z(\tau)} \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) \right) < \zeta & \text{for } \tau \in [0, \bar{T}^e), \\ \zeta & \text{for } \tau \geq \bar{T}^e. \end{cases}$$

The *Irreversibility Effect* is again at play as long as the highest possible values of the tipping point has not been passed. Actions remain below the myopic optimum over that first phase.

Yet, actions are higher than when the tipping point is known to lie at \bar{X} for sure. This result is illustrated by observing that the last actions before passing \bar{X} has now been raised towards the myopic solution in comparison with the scenario where the tipping point is known to be at \bar{X} for sure:

$$(5.19) \quad x^e(\bar{T}^{e-}) = \zeta \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} > \zeta \sqrt{\frac{\lambda_0}{\lambda_1}}.$$

Under uncertainty, the date at which the stock equals the maximum possible tipping point \bar{X} is reached earlier on:

$$\bar{T}^e < \bar{T}^k.$$

Intuitively, there is now a chance that the tipping point has already been passed before, so that optimal actions are closer to the myopic optimum. It can also be readily checked that as q goes to 0 (resp. 1), \bar{T}^e converges towards \bar{T}^k (resp. \bar{T}^m).

REMARK. The *feedback rule* $\sigma^e(X, Z)$ defines the trajectory both in terms of the overall stock $X^e(\tau; 0, 1)$ but also of the beliefs $Z^e(\tau; 0, 1)$ starting from the initial conditions

($X = 0, Z = 1$). Provided that actions remain positive at all points in time,²⁶ $X^e(\tau; 0, 1)$ is itself invertible. There is thus a one-to-one relationship between the current stock and beliefs. Even though the value function is computed for a broader set of values of those states variables, stock and beliefs evolve altogether along a one-dimensional manifold at the optimum. This remark plays a key role in what follows. ■

6. IMPLEMENTATION: *STOCK-MARKOV* EQUILIBRIA WITH OBSERVABLE DEVIATIONS

The value function $\mathcal{V}^e(X, Z)$ is a mere technical device to use dynamic programming techniques and compute a feedback rule $\sigma^e(X, Z)$ that guides behavior along the optimal trajectory. There are two ways of thinking about this device. First, this feedback rule may be viewed as a machine that determines actions that a planner would take at each point in time in response to the evolution of stock and beliefs along the trajectory. In the spirit of mechanism design, this machine is a social contract. This social contract is complete in that it specifies actions in terms of observable and verifiable variables. Second, and it is a direct consequence of the *Principle of Dynamic Programming*, such feedback rule can alternatively be viewed as a Perfect-Markov equilibrium strategy among various selves of this decision-maker. In this seemingly non-cooperative scenario, those selves acting at different points in time have only a limited ability to commit to an action over an infinitesimal period of time. They adopt Markov-strategies based on the state variable (X, Z) . Because those selves are endowed with the same objectives and the same information than what a long-lived planner would have, their choice of the best impulse deviation obviously replicates that of this planner.

Hereafter, we instead ask whether a more parsimonious decentralization of an optimal trajectory is also reached as a Perfect-Markov equilibrium if those selves were to adopt less complete *Stock-Markov* feedback rules that only depend on the stock X . Our motivation for looking at such a restriction on equilibrium strategies is that, in practice, only the stock of pollutants in the atmosphere can be easily verified and this stock might not be a sufficient statistics to form correct beliefs on whether the tipping point has likely been passed or not. Even though selves are still endowed with the same objectives, this restriction on feasible strategies may bite and affect the implemented action plan. We will show below that the extent to which it is so depends on whether impulse deviations are observable or not (Section 7 below).

GAME FORM. We now consider a game in continuous time among selves of the decision-maker who act at different points in time. At any point in time τ , the current self DM_τ has limited commitment ability. He can only choose an action $x(\tau)$ over an interval of infinitesimal length of time $[\tau, \tau + \varepsilon]$. DM_τ 's objective is to maximize intertemporal welfare from that date on given whatever information is available to him at date τ . In this section, we suppose that DM_τ can observe whatever actions may have been undertaken by all his predecessors $DM_{\tau'}$ for $\tau' < \tau$ both on and off equilibrium path. Section 7 below will entertain the opposite scenario where those actions remain unknown.

Stock-Markov equilibria for that game form are supported by *Stock-Markov* feedback rules. At any such equilibrium, the self DM_τ in charge at date τ sticks to the strategy

²⁶Proposition 7 shows that actions along the optimal trajectory are positive.

$\sigma^o(X)$ when the stock has reached level X because he expects future selves to abide to that rule as well following the subsequent evolution of the system.²⁷

Along any such *Stock-Markov* trajectory, the stock $X^o(\tau; X)$ evolves as

$$(6.1) \quad \frac{\partial X^o}{\partial \tau}(\tau; X) = \sigma^o(X^o(\tau; X)) \text{ with } X^o(0; X) = X.$$

The various selves should also be able to reconstruct the regime survival ratio that applies, along the equilibrium path, for each possible level of the stock and, by that means, correctly infer how to discount future payoffs. Let denote by $Z^o(X)$ such function. From (5.6), the regime survival ratio $Z(\tau; X)$, that starts from value $Z^o(X)$ at date 0 and that is consistent with the *Stock-Markov* feedback rule $\sigma^o(X)$ from that date on evolves as

$$(6.2) \quad \frac{\partial Z}{\partial \tau}(\tau; X) = \Delta(1 - F(X^o(\tau; X)) - Z(\tau; X)) \text{ with } Z(0; X) = Z^o(X).$$

Since conjectures on how the regime survival ratio evolves along the trajectory are correct at equilibrium, we must also have

$$(6.3) \quad Z(\tau; X) = Z^o(X^o(\tau; X)) \quad \forall \tau \geq 0, X \geq 0.$$

Taken together, those conditions dictate how the regime survival ratio evolves with the current stock along the trajectory. By differentiating (6.3) with respect to τ , we get

$$(6.4) \quad \sigma^o(X) \dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0$$

with the initial condition

$$(6.5) \quad Z^o(0) = 1.$$

We may now define a *Stock-Markov value function* $\mathcal{V}^o(X)$ as the intertemporal payoff along such a *Stock-Markov* trajectory as

$$(6.6) \quad \mathcal{V}^o(X) = \int_0^{+\infty} e^{-\int_0^\tau (\lambda_0 - \sigma^o(X^o(s; X)) \frac{\dot{Z}^o(X^o(s; X))}{Z^o(X^o(s; X))}) ds} u(\sigma^o(X^o(\tau; X))) d\tau.$$

This definition showcases how future payoffs are discounted at a rate

$$\lambda_0 - \sigma^o(X^o(s; X)) \frac{\dot{Z}^o(X^o(s; X))}{Z^o(X^o(s; X))}$$

that depends on the regime survival ratio along the *Stock-Markov* trajectory.

For future reference, we define the intertemporal payoff once the tipping point has been passed for sure but, being ignorant of that event, all future selves still rely on the feedback rule $\sigma^o(X)$ to choose actions, as

$$(6.7) \quad \varphi^o(X) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X))) d\tau.$$

²⁷Of course, a *Stock-Markov* feedback rule should specify that $\sigma^o(X) = \zeta$ for $X \geq \bar{X}$ but, in order to save on notations, this expression of the continuation will be kept implicit in what follows.

IMPULSE DEVIATIONS. To express the equilibrium requirement that sticking to the feedback rule $\sigma^o(X)$ is optimal at any point along the trajectory, we follow an approach that was developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015), Ekeland and Lazrak (2006, 2008, 2010) and more recently used in Auster, Che and Mierendorff (2023). These authors have analyzed various macroeconomic, growth or dynamic decision-making models with time-inconsistency problems. To model non-cooperative action choices by various decision-makers (or selves of the same decision-maker), these authors have imported the notion of perfect-Markov equilibrium, familiar in discrete-time models, to a continuous time setting. The idea is to look at the benefits of deviating from the feedback rule for periods of commitment which are of arbitrarily small length; deriving from there conditions for the sub-optimality of such deviations and thus properties of the equilibrium feedback rule.

To this end, consider a possible deviation that would consist for the current self in committing to an action x for a period of length ε , reaching a stock level $X + x\varepsilon$, before jumping back to the above feedback rule σ^o . For such a deviation, actions are thus

$$(6.8) \quad y(x, \varepsilon, \tau; X) = \begin{cases} x & \text{if } \tau \in [0, \varepsilon], \\ \sigma^o(\hat{X}(x, \varepsilon, \tau; X)) & \text{if } \tau > \varepsilon \end{cases}$$

while the whole stock trajectory is modified as

$$(6.9) \quad \hat{X}(x, \varepsilon, \tau; X) = \begin{cases} X + x\tau & \text{if } \tau \in [0, \varepsilon], \\ X + x\varepsilon + \int_{\varepsilon}^{\tau} \sigma^o(\hat{X}(x, \varepsilon, s; X)) ds & \text{if } \tau \geq \varepsilon. \end{cases}$$

By adopting such impulse deviation, the regime survival ratio would also change as

$$(6.10) \quad \hat{Z}(x, \varepsilon, \tau; X) = 1 - \Delta e^{-\Delta\tau} \int_0^{\tau} F(\hat{X}(x, \varepsilon, s; X)) e^{\Delta s} ds - (1 - Z^o(X)) e^{-\Delta\tau}.$$

From this, we may define DM 's deviation payoff $\hat{\mathcal{V}}(x, \varepsilon; X)$ as

$$(6.11) \quad \hat{\mathcal{V}}(x, \varepsilon; X) = \int_0^{+\infty} e^{-\int_0^{\tau} \left(\lambda_0 - \frac{\partial \hat{Z}}{\partial s}(x, \varepsilon, s; X) \right) ds} u(y(x, \varepsilon, \tau; X)) d\tau.$$

When ε is made arbitrarily small, we will refer to such deviations as *impulse deviations*.

That all future selves are able to observe any impulse deviation that the current decision-maker may entertain allows those selves to reconstruct the evolution of beliefs as expressed in (6.10). When considering the consequences of any impulse deviation, the current decision-maker should thus assess those consequences on his intertemporal payoff by applying the implicit discounting that follows from the evolution of beliefs so induced. This inference is clear in the expression of the continuation payoff on the right-hand side of (6.11).

STOCK-MARKOV EQUILIBRIUM WITH OBSERVABLE IMPULSE DEVIATIONS. We start with a definition.

DEFINITION 1 *A triplet $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$ is a SME with observable impulse deviations if the following conditions hold.*

1. $\mathcal{V}^o(X)$ as defined by (6.6) cannot be improved upon by any impulse deviation of the form (6.8)-(6.9) for ε made arbitrarily small:

$$(6.12) \quad \mathcal{V}^o(X) = \max_{x \in \mathcal{X}} \lim_{\varepsilon \rightarrow 0^+} \hat{\mathcal{V}}(x, \varepsilon; X).$$

2. $\sigma^o(X)$ is optimal for ε made arbitrarily small:

$$(6.13) \quad \sigma^o(X) \in \arg \max_{x \in \mathcal{X}} \lim_{\varepsilon \rightarrow 0^+} \hat{\mathcal{V}}(x, \varepsilon; X).$$

3. $Z^o(X)$ is consistent with the feedback rule $\sigma^o(X)$ and satisfies (6.4)-(6.5).

Item 1. requires to approximate the deviation payoff $\hat{\mathcal{V}}(x, \varepsilon; X)$ to the first order in ε and look for the optimal action that maximizes such approximation; an optimality condition that is expressed in Item 2. Those two steps are familiar from applying the *Principle of Dynamic Programming* in contexts with time-consistent plans. Item 3. follows from the consistency condition (6.3) which states that the optimal evolution of beliefs is dictated by the *Stock-Markov* feedback rule. This step is more novel. Of course, the evolution of the survival ratio should be consistent with this feedback rule.

PROPERTIES OF $(\mathcal{V}^o(X), \sigma^o(X))$. Developing the equilibrium conditions in Definition 1 yields important properties.

PROPOSITION 5 *At any (continuously differentiable) SME, with observable impulse deviations, the Stock-Markov value function $\mathcal{V}^o(X)$ satisfies the following functional equation*

$$(6.14) \quad \dot{\mathcal{V}}^o(X) = -\zeta - \frac{\dot{Z}^o(X)}{Z^o(X)} \mathcal{V}^o(X) + \sqrt{2\lambda_0 \mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2} \quad \forall X \in [0, \bar{X}]$$

together with the boundary condition

$$(6.15) \quad \mathcal{V}^o(X) = \mathcal{V}_\infty \quad \forall X \geq \bar{X}.$$

The corresponding Stock-Markov feedback rule writes as

$$(6.16) \quad \sigma^o(X) = \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} (\mathcal{V}^o(X) - \varphi^o(X)).$$

The formula for the feedback rule in (6.16) bears some resemblance with its counterpart (4.5) that was found under complete information. To understand the changes, it is useful to come back on the expression of the *Stock-Markov* value function (6.6). Starting from a current stock X with current beliefs $Z^o(X)$ on the equilibrium path, consider an impulse deviation consisting in increasing by a marginal amount dx the current action $\sigma^o(X)$ over an interval of infinitesimal length ε . Since the current stock increases by $dX = \varepsilon dx$, such impulse deviation reduces the *Stock-Markov* value function by

$$(6.17) \quad -\dot{\mathcal{V}}^o(X) \varepsilon dx.$$

This impact can be decomposed into three different components. First, this impulse deviation yields a marginal benefit on current payoff over the infinitesimal interval worth

$$(6.18) \quad (\zeta - \sigma^o(X))\varepsilon dx.$$

Second, this impulse deviation also increases the implicit discount rate that applies to future payoffs by

$$\frac{\dot{Z}^o(X)}{Z^o(X)}\varepsilon dx < 0.$$

The corresponding impact on continuation payoff is thus a reduction in continuation payoff worth

$$(6.19) \quad \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X)\varepsilon dx < 0.$$

This effect decreases current action. Importantly, it is entirely due to the induced change in stock. It takes as given the evolution of beliefs and would be also present if the rate $\frac{\dot{Z}^o(X)}{Z^o(X)}$ at which the survival ratio evolves was taken as given. This will be the case in Section 7 below here we investigate the scenario of non-observable impulse deviations.

Because here it is observable by future selves, an impulse deviation has nevertheless also a long-lasting effect on beliefs as highlighted by formula (6.10). A marginal increase in the stock worth εdx makes it more likely that the tipping point has been passed within the infinitesimal interval where this impulse deviation applies. It brings an extra grain of pessimism over the whole future trajectory. From (6.10), this deviation indeed impacts the *Pessimistic Stigma* by a term which, at a date τ beyond the impulse deviation, is

$$\dot{Z}^o(X)e^{-\Delta\tau}\varepsilon dx < 0.$$

Passed the tipping point, payoffs would be discounted at rate λ_1 if this event was observed leading to an intertemporal gain worth $\varphi^o(X)$. The benefit of believing that the tipping point is more likely to have been passed following this impulse deviation is thus

$$(6.20) \quad -\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\varepsilon dx > 0.$$

Since a more pessimistic decision-maker chooses higher actions, this last effect increases current action.

Gathering (6.17), (6.18), (6.19) and (6.20) above finally yields Condition (6.16) which characterizes the optimal feed-back rule.

Reciprocally, a triplet $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$ that satisfies (6.14), (6.15), (6.16) and the consistency requirements (6.4)-(6.5) forms a *SME*. This point is exploited in Proposition 6 below to show that an optimal arc can be implemented as a *SME*.

REMARK. Consider the alternative scenario where *DM* remains ignorant on where the tipping point lies but, thanks to hard scientific evidence, immediately learns it upon

passing it.²⁸ DM thus knows that his payoffs should be discounted at rate λ_0 as long as he has not yet learned having passed the tipping point. The dynamics of the system is thus fully summarized by the stock X that can be used as the sole state variable. Observe also that the probability of not having yet switched regime is then $1 - F(X)$ and that, once the tipping point has been passed, the myopic action is chosen which yields a continuation payoff \mathcal{V}_∞ . Denoting by $\mathcal{V}^u(X)$ the value function conditionally on not having yet learned that the tipping point has been passed, we may adapt our previous analysis to express this value function as

$$(6.21) \quad \mathcal{V}^u(X) = \int_0^{+\infty} e^{-\int_0^\tau (\lambda_0 + \sigma^u(X^u(s; X)) \frac{f(X^u(s; X))}{1-F(X^u(s; X))}) ds} u(\sigma^u(X^u(\tau; X))) d\tau$$

and get the optimal feedback rule $\sigma^u(X)$ as

$$(6.22) \quad \sigma^u(X) = \zeta + \dot{\mathcal{V}}^u(X) - \frac{f(X)}{1-F(X)} (\mathcal{V}^u(X) - \mathcal{V}_\infty).$$

This formula bears some obvious resemblance with (6.16). Upon learning that he has passed the tipping point, an event whose hazard rate is $\frac{f(X)}{1-F(X)}$, DM knows for sure that the continuation payoff drops from $\mathcal{V}^u(X)$ to \mathcal{V}_∞ . In order to postpone this drop, DM reduces current actions. ■

IMPLEMENTING THE OPTIMAL TRAJECTORY. The evolution of beliefs along a *SME* is completely fixed by the feedback rule on path. If DM expects future selves to stick to a *Stock-Markov* rule that implements the optimal action profile, he also expects beliefs to be modified as expected at the optimum. Hence, when considering the possible benefits of an observable impulse deviation, there is nothing that distinguishes the current self when he is playing the *SME* defined in Proposition 6 from a planner who would be considering the impact of a marginal change of action on the future stream of payoffs at the same point in time. Because impulse deviations are observable, future selves will modify beliefs as a long-lived planner would also do and will accordingly choose the same actions thereafter.

PROPOSITION 6 *Suppose that impulse deviations are observable, an optimal path can be implemented as a SME,²⁹ $(\mathcal{V}^o(X), \sigma^o(X), Z^o(X))$, such that*

$$(6.23) \quad \mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X)) \text{ and } \sigma^o(X) = \sigma^e(X, Z^o(X)) \quad \forall X$$

with $Z^o(X)$ being consistent with the feedback rule $\sigma^o(X)$ and satisfying (6.4)-(6.5).

²⁸This scenario is analyzed in Tsur and Zemel (1996, 2021) among others and is isomorphic to Loury (1979)'s analysis of how to exploit a resource with unknown reserve. In that model as well, when DM has reached the limits of the resource stock, he immediately knows it and stops consuming from that date on.

²⁹The difficulty in directly proving existence of a *SME* comes from the fact that the differential equation (6.14) for $\mathcal{V}^o(X)$ depends on DM 's payoff $\varphi^o(X)$ in case the tipping point has been passed which itself depends on the *Stock-Markov* feedback rule computed over the whole future trajectory. Local existence results are of little help given that non-local property. Proposition 6 overcomes this difficulty, in proving the existence of a *SME* indirectly from the existence of an optimal path.

BOUNDS. This implementation of the optimum is useful to get bounds on payoffs and actions at the optimum. Proposition 7 below provides tight bounds on the *Stock-Markov* value function and the feedback rule for any *SME*, and in particular the one, described in Section 6, that implements the optimal trajectory.

PROPOSITION 7 $\mathcal{V}^o(X)$, $\varphi^o(X)$ and $\sigma^o(X)$ admit the following bounds:

$$(6.24) \quad \varphi^o(X) \leq \mathcal{V}_\infty \leq \mathcal{V}^o(X) \leq \mathcal{V}_\infty \left(1 + \frac{\Delta}{\lambda_0} (1 - F(X)) \right) \quad \forall X \in [0, \bar{X}],$$

$$(6.25) \quad \zeta \sqrt{\frac{\lambda_0}{\lambda_1}} \leq \sigma^o(X) \leq \zeta \quad \forall X \in [0, \bar{X}].$$

These bounds are similar to those in the no-uncertainty scenario of Section 4. The dynamics with and without uncertainty are in fact similar. To illustrate, the lower bound on $\mathcal{V}^o(X)$ is readily obtained by following a non-equilibrium strategy consisting in adopting the myopic action under all circumstances. For X below but close enough to \bar{X} , the stock has already gone through most possible values of the tipping point. From (6.24), the *Stock-Markov* value function converges towards \mathcal{V}_∞ from above and is continuous at this point.³⁰ There, the current action has almost no longer any influence on the arrival rate of a catastrophe which is almost surely θ_1 . On the other hand, the lower bound on possible actions is independent on the distribution F and again found for scenario where the tipping point is located at \bar{X} for sure.

RUNNING EXAMPLE (CONTINUED). Consider the trajectory starting from $X = 0$ and $Z = 1$. From the expression of the optimal action (5.18), the stock evolves as

$$(6.26) \quad X^e(\tau) = \begin{cases} \zeta \left(\tau - \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) \int_0^\tau \frac{Z(\bar{T}^e)}{Z(s)} e^{-\lambda_0(\bar{T}^e - s)} ds \right) & \text{for } \tau \in [0, \bar{T}^e], \\ \bar{X} + \zeta(\tau - \bar{T}^e) & \text{for } \tau \geq \bar{T}^e. \end{cases}$$

Together with (5.17), this expression allows us to recover an almost closed form for $X^o(Z)$ (the inverse function of $Z^o(X)$) for $Z \in [1 - q + qe^{-\Delta\bar{T}^e}, 1]$ as

$$(6.27) \quad X^o(Z) = \zeta \left(-\frac{1}{\Delta} \ln \left(1 + \frac{Z-1}{q} \right) - \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) \int_0^{-\frac{1}{\Delta} \ln \left(1 + \frac{Z-1}{q} \right)} \frac{Z(\bar{T}^e)}{Z(s)} e^{-\lambda_0(\bar{T}^e - s)} ds \right).$$

It can be readily verified that

$$\dot{X}^o(Z(\bar{T}^e)) = \frac{\zeta e^{\Delta\bar{T}^e}}{q\Delta} \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}}.$$

³⁰The *Stock-Markov* value function is not necessarily differentiable at \bar{X} though it admits a right- and a left-derivative. This is so because the optimal action may have an upwards jump at that point; a case that arises when the distribution of tipping point has a mass point at \bar{X} as in our RUNNING EXAMPLE. Continuity of the feedback rule at \bar{X} holds when F has no mass point. See the Appendix for more details.

We thus get $\lim_{q \rightarrow 1} \dot{X}^o(Z(\bar{T}^e)) = 0$ or, equivalently, $\lim_{q \rightarrow 1} \dot{Z}^o(\bar{X}^-) = -\infty$. Intuitively, when q is close to one, the function $Z^o(X)$ also remains close to one for most values of X , only decreasing very quickly towards $1 - q + qe^{-\Delta \bar{T}^m}$ when X comes close to \bar{X} . Finally, the optimal action at \bar{X}^- , namely $\sigma^o(\bar{X}^-) = x^e(\bar{T}^e)$ (which is expressed in (5.19)) indeed converges towards the lowest bound $\zeta \sqrt{\frac{\lambda_0}{\lambda_1}}$ as q goes to zero. ■

7. STOCK-MARKOV EQUILIBRIA WITH NON-OBSERVABLE DEVIATIONS

We now consider a scenario where the self DM_τ in charge over a period of infinitesimal length around date τ does not observe any impulse deviations that his predecessors $DM_{\tau'}$ for $\tau' < \tau$ may have entertained. Only the current level of the stock $X = X(\tau)$ remains observable for DM_τ . In practice, the consequences of an action at a given point in time may only be detected after a lag. Hereafter, we will take the polar view that the lag for detecting any impulse deviation is infinite. One possible justification is that scientific knowledge might not be sufficiently advanced to assess those consequences right away. An alternative explanation is that the selves might have bounded rationality and limited ability to process information. Accordingly, we need to slightly modify the notion of *SME* to account for the non-observability of impulse deviations.

SME WITH NON-OBSERVABLE IMPULSE DEVIATIONS. In any such *SME*, all selves conjecture that the feedback rule $\sigma^{no}(X)$ is adopted. Accordingly, they all believe that the regime survival ratio evolves according to

$$(7.1) \quad \sigma^{no}(X) \dot{Z}^{no}(X) = \Delta(1 - F(X) - Z^{no}(X)) \quad \forall X \geq 0^{31}$$

with the initial condition

$$(7.2) \quad Z^{no}(0) = 1.$$

Because necessarily $\sigma^{no}(X) = \zeta$ for $X > \bar{X}$, (7.1) immediately imply

$$(7.3) \quad Z^{no}(X) = Z^{no}(\bar{X}) e^{-\frac{\Delta}{\zeta}(X - \bar{X})} \quad \forall X > \bar{X}.$$

For any stock $X \leq \bar{X}$, we may now define the *Stock-Markov value function with non-observable deviations* $\mathcal{V}^{no}(X)$ along such *SME* as:

$$(7.4) \quad \mathcal{V}^{no}(X) = \int_0^{+\infty} e^{-\int_0^\tau (\lambda_0 - \sigma^{no}(X^{no}(s; X)) \frac{\dot{Z}^{no}(X^{no}(s; X))}{Z^{no}(X^{no}(s; X))}) ds} u(\sigma^{no}(X^{no}(\tau; X))) d\tau.$$

IMPULSE DEVIATIONS. An impulse deviation again entails a modification of the action profile as specified in (6.8) and an ensuing evolution of the stock as in (6.9). Because impulse deviations are now not observable, a deviation by DM_τ has no impact on the degree of pessimism that his followers $DM_{\tau'}$, for $\tau' > \tau$ adopt. They still believe that the regime survival ratio evolves on path as specified in (7.1) and (7.2). Of course, an impulse deviation made earlier on modifies the current stock and affects where the regime survival ratio lies along this trajectory. This point is made clear in the following expression of the payoff for such a deviation:

$$(7.5) \quad \hat{\mathcal{V}}^{no}(x, \varepsilon; X) = \int_0^{+\infty} e^{-\int_0^\tau (\lambda_0 - \frac{\partial \hat{X}}{\partial s}(x, \varepsilon, s; X) \frac{\dot{Z}^{no}(\hat{X}(x, \varepsilon, s; X))}{Z^{no}(\hat{X}(x, \varepsilon, s; X))}) ds} u(y(x, \varepsilon, \tau; X)) d\tau.$$

From there, we deduce the following definition.

DEFINITION 2 *A triplet $(\mathcal{V}^{no}(X), \sigma^{no}(X), Z^{no}(X))$ is a SME with non-observable deviations if the following conditions hold.*

1. $\mathcal{V}^{no}(X)$ as defined by (7.4) cannot be improved upon by any impulse deviation of the form (6.8)-(6.9) for ε made arbitrarily small:

$$(7.6) \quad \mathcal{V}^{no}(X) = \max_{x \in \mathcal{X}} \lim_{\varepsilon \rightarrow 0^+} \hat{\mathcal{V}}^{no}(x, \varepsilon; X).$$

2. $\sigma^{no}(X)$ is optimal for ε made arbitrarily small:

$$(7.7) \quad \sigma^o(X) \in \arg \max_{x \in \mathcal{X}} \lim_{\varepsilon \rightarrow 0^+} \hat{\mathcal{V}}^{no}(x, \varepsilon; X).$$

3. $Z^{no}(X)$ is consistent with the feedback rule $\sigma^{no}(X)$ and satisfies (7.1)-(7.2).

This definition looks very much alike Definition 1. Both definitions require first, that impulse deviations should not improve payoffs locally (Item 1.) and second, that the evolution of the regime survival ratio should be consistent with the feedback rule (Item 3.). The key difference between Definitions 1 and 2 comes from the fact that deviations payoffs are written differently. With observable deviations, continuation payoffs following an impulse deviation are modified to account for how the regime survival ratio carries over changes in the *Pessimistic Stigma*. With non-observable deviations, subsequent decision-makers are more naive. The sole impact of an impulse deviation on continuation payoff is to change the level of stock and thus the implicit discount rate that applies to how they compute future payoffs. Decision-makers take the evolution of beliefs as fixed when considering a deviation.

REMARKS. Two implicit assumptions in our analysis should be justified. First, each self only knows the current level of stock when acting. Suppose instead, that he would have known for how long the project has been run, or at which point in time he is acting. Conjecturing that previous selves have abided to the *Stock-Markov* feedback rule that prevails at equilibrium and comparing with the current stock he is observing would allow this self to detect that (at least) one deviation has taken place earlier on, even if he might not be able to infer at which date it was. Assuming that only the current stock is observed avoids such inference and accordingly simplifies the analysis. This assumption is akin to suppose that selves are naive and have limited memory. They can just keep track of the level of stock but cannot figure out the precise actions path that induces such stock. Alternatively, it could be that the initial level of stock remains unknown so that correct inferences on whether a deviation took place are not feasible either.

Second, because impulse deviations are non-observable, all selves believe that the regime survival ratio still evolves as on path, i.e., as in (7.1). Instead, when deviating at date τ , DM_τ knows that the correct evolution of beliefs is given by (6.10). This difference *a priori* implies that, beyond the commitment period whose length is infinitesimal, the discounted intertemporal streams of utilities evaluated with DM_τ 's beliefs and that of his future selves $DM_{\tau'}$ for $\tau' > \tau$ differ. To fix this issue, focus on the main consequences of non-observability in the simpler scenario and again simplify the analysis, we assume that DM_τ cares about the intertemporal payoff of his subsequent selves; thus considering their own beliefs when evaluating his future payoffs. From this, we may thus define DM 's deviation payoff $\hat{\mathcal{V}}(x, \varepsilon; X)$ as in (7.5). ■

EQUILIBRIUM PROPERTIES. Next proposition echoes our findings in Proposition 5 but now considering a scenario with non-observable deviations.

PROPOSITION 8 *At any (continuously differentiable) SME with non-observable impulse deviations, the Stock-Markov value function $\mathcal{V}^{no}(X)$ satisfies the following Hamilton-Bellman-Jacobi differential equation*

$$(7.8) \quad \dot{\mathcal{V}}^{no}(X) = -\zeta - \frac{\dot{Z}^{no}(X)}{Z^{no}(X)} \mathcal{V}^{no}(X) + \sqrt{2\lambda_0 \mathcal{V}^{no}(X)} \quad \forall X \in [0, \bar{X}]$$

together with the boundary condition

$$(7.9) \quad \mathcal{V}^{no}(X) = \mathcal{V}_\infty \quad \forall X \geq \bar{X}.$$

The corresponding Stock-Markov feedback rule writes as

$$(7.10) \quad \sigma^{no}(X) = \zeta + \dot{\mathcal{V}}^{no}(X) + \frac{\dot{Z}^{no}(X)}{Z^{no}(X)} \mathcal{V}^{no}(X) \quad \forall X \in [0, \bar{X}].$$

The feedback rule with non-observable deviations (7.10) is much like its counterpart (6.16) found when those deviations are observable. Yet, the term (6.20) is missing. To explain this omission, consider again increasing by a small amount dx the current action $\sigma^{no}(X)$ over an interval of infinitesimal length ε , starting from a current stock X with current beliefs $Z^{no}(X)$. If this impulse deviation is non-observable, future selves, when choosing their own actions, only consider its impact on the observable stock which has increased by εdx . The comparison with observable deviations is thus straightforward.

First, this non-observable impulse deviation still impacts current payoff because the feedback rule $\sigma^{no}(X)$ requires a change in action at this new level of stock. This term is again given by (6.17). Second, this impulse deviation also increases the implicit discount rate; a term which is still captured by (6.18). Yet, with a non-observable deviation, the regime survival ratio $Z^{no}(X)$ is taken as given over the whole trajectory. Had such a deviation been observable, DM_τ instead would have known that increasing current action also means that future beliefs will carry on some *Pessimistic Stigma* and this pessimism makes it more attractive for future selves, $DM_{\tau'}$ for $\tau' > \tau$ who think that the tipping point may have been passed, to further increase actions later on. With a non-observable deviation, this motive for raising actions disappears and actions remain low.

At equilibrium, the feedback rule now calls for excessively low actions in comparison with the optimal trajectory. Indeed, in any SME with observable deviations, we have

$$\sigma^o(X) > \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} \mathcal{V}^o(X).$$

With low actions early on, the conjectured evolution of beliefs remains quite optimistic. Each self thinks that the tipping point remains unlikely to have been already passed when he acts and, in response, adopts a prudent behavior. This prudent behavior is of course excessive in comparison with the optimal trajectory. Yet, it is self-fulfilling.

RUNNING EXAMPLE (CONTINUED). The trajectory under a SME with non-observable impulse deviations can again be computed in (almost) closed form.

PROPOSITION 9 Suppose that F has Dirac masses q at 0 and $1 - q$ at \bar{X} . The trajectory under a SME with non-observable impulse deviations starting from $X = 0$ and $Z = 1$ has the following features.

- The date $\bar{T}^{no} > \bar{T}^k$ at which \bar{X} is reached solves

$$(7.11) \quad \bar{T}^m = \sqrt{\frac{\lambda_0}{\lambda_1}} \left(\int_0^{\bar{T}^{no}} \sqrt{\frac{Z(T^{no})}{Z(\tau)}} e^{-\lambda_0(\bar{T}^{no}-\tau)} d\tau \right) + \lambda_0 \int_0^{\bar{T}^{no}} \left(\int_\tau^{\bar{T}^{no}} \sqrt{\frac{Z(s)}{Z(\tau)}} e^{\lambda_0(\tau-s)} ds \right) d\tau$$

where $Z(\tau)$ is given by

$$(7.12) \quad Z(\tau) = 1 - q + qe^{-\Delta\tau} \quad \forall \tau \in [0, \bar{T}^{no}].$$

- The action $x^{no}(\tau)$ satisfies

$$(7.13) \quad x^{no}(\tau) = \begin{cases} \zeta \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}} \left(\sqrt{Z(T^{no})} e^{-\lambda_0\bar{T}^{no}} \sqrt{\frac{\lambda_0}{\lambda_1}} + \lambda_0 \int_\tau^{\bar{T}^{no}} \sqrt{Z(s)} e^{-\lambda_0s} ds \right) < \zeta & \text{for } \tau \in [0, \bar{T}^{no}), \\ \zeta & \text{for } \tau \geq \bar{T}^{no}. \end{cases}$$

To illustrate the tendency for choosing low actions when impulse deviations are non-observable, observe that the last action before jumping to the myopic optimum is always lower than with observable deviations:

$$x^{no}(\bar{T}^{no}) = \sqrt{\frac{\lambda_0}{\lambda_1}} < x^o(\bar{T}^e) = x^e(\bar{T}^e) = \sqrt{\frac{\lambda_0}{\lambda_1} + \frac{q\Delta e^{-\Delta\bar{T}^e}}{1 - q + qe^{-\Delta\bar{T}^e}}}.$$

■

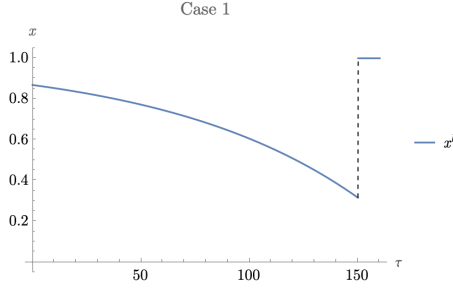
8. NUMERICAL SIMULATIONS

The debate on the relevance of the *Precautionary Principle* would really matter if the trajectories with and without observability of deviations were significantly different in terms of welfare levels. In this respect, the numerical simulations we are now presenting suggest that the lack of information on past behavior might not entail a significant welfare cost. This result softens concerns about the use of the *Precautionary Principle* in practice. The two trajectories with and without observability mainly differ at early dates but are very close afterwards; and this result holds under a broad range of scenarios.

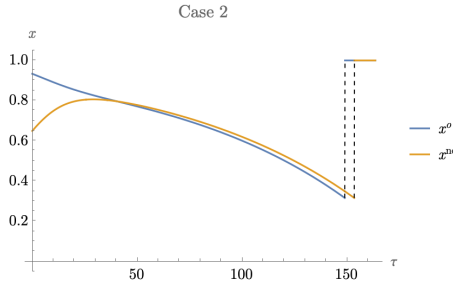
Again, our RUNNING EXAMPLE is still useful to quantify on how the various actions and beliefs profiles compare. To fix ideas, we suppose that the highest possible value of the tipping point is located at $\bar{X} = 100$ while the myopic action is $x^m = \zeta = 1$. We consider several scenarios with relevant values of the various parameters.

SCENARIO 1: KNOWN TIPPING POINT AT \bar{X} . We first choose $r = 0.01$ as the value of the interest rate. We will also assume that before the tipping point the rate of arrival of a

catastrophe is very small, namely $\theta_0 = 0.001$, while it jumps to $\theta_1 = 0.1$ afterwards.³² The tipping point is reached at date $\bar{T}^k = 150.257$, which is significantly higher than in the myopic scenario that, thanks to our normalizations, corresponds to $\bar{T}^m = 100$. Next figure represents the action profile $x^k(\tau)$ when the tipping point is known. The intuition is that decreasing the action pushes back the switch to the higher risk scenario, but it comes at a utility cost. Because of discounting, the decision-maker decreases the action over time before reaching the tipping point. Similar patterns will be found under uncertainty on the value of the tipping point.



SCENARIO 2: UNCERTAIN TIPPING POINT. We now keep all previous parameters the same but suppose that there is an equal probability to pass the tipping point at zero and at \bar{X} , i.e., $q = \frac{1}{2}$. Under uncertainty, we expect to find different dates at which the upper bound \bar{X} is now reached depending on whether deviations are observable or not. In fact, we compute $\bar{T}^0 = 149.026$ and, as expected, a higher value $\bar{T}^{no} = 153.535$. Yet, the difference is less than 3 %. This minor difference comes from the fact that the two action profiles $x^o(\tau)$ and $x^{no}(\tau)$ are themselves close to each other. Interestingly, the action path is non-monotonous for the non-observable case. The intuition is that at the beginning, decision-makers are rather pessimistic and decide to enjoy flow payoffs by increasing actions. Conditional on no catastrophe having yet happened, after a while it becomes more likely that the tipping was not in fact at 0, and so that actions are again reduced to push back the switch.

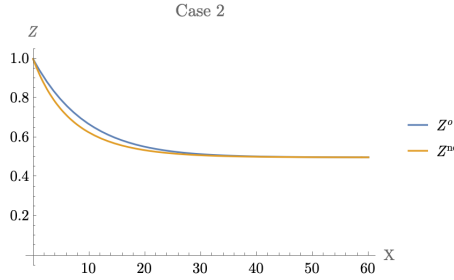


Although quite similar after a while, actions in the two scenarios mostly differ at the beginning of the trajectory. In the non-observable scenario, decision-makers start with a very low action and then increase actions over a first phase as they become more pessimistic and believe that the tipping point is more likely to have been passed. In a second phase, decision-makers adopt actions which are close to those in the observable scenario. The regime survival ratios in both scenarios become very flat after a while and the existing pessimistic stigma that pertains to the observable-deviation scenario has not enough magnitude to significantly distinguish trajectories in the two scenarios. In other

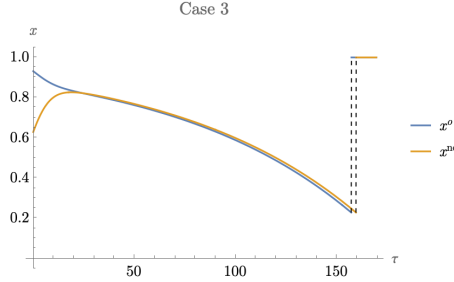
³²This latter value is actually consistent with those chosen by Besley and Dixit (2019) in a related context, although those authors posit that the arrival rate is a smooth and nonlinear function while we adopt a step function.

words, for most of the trajectory, there is little impact of observing past deviations on actual choices.

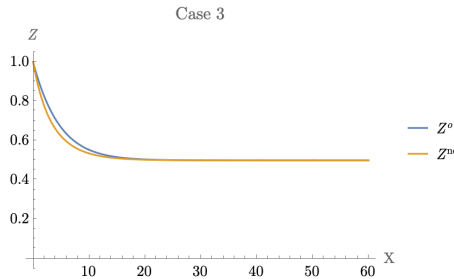
Turning now to the regime survival ratios, we first observe that, since actions are higher when deviations are observable, the stock with observable deviations $X^o(\tau)$ accumulates over time faster than the stock $X^{no}(\tau)$ with non-observable deviations. Let denote by $\tau^{-1,o}(X)$ and $\tau^{-1,no}(X)$ the corresponding inverse functions. Using (5.17) and (7.12) allows us to recover the expressions of the regime survival ratios in terms of X and to check that $Z^{no}(X) = Z(\tau^{-1,no}(X)) \leq Z(\tau^{-1,o}(X)) = Z^o(X)$ as confirmed on next figure. Notice that in this scenario and the following ones, the asymptote is at 0.5 because after enough time DMs are sure that the tipping point was not at 0, but at \bar{X} .



SCENARIO 3: INCREASE IN THE RATE OF ARRIVAL OF A CATASTROPHE. Keeping all other parameters as in SCENARIO 1, consider increasing the rate of arrival of a catastrophe up to $\theta_1 = 0.2$. This change increases delays before reaching the maximal value of the tipping point but does not change the fact that trajectories with observable and non-observable deviations are very close. The switching times now differ by less than 1.5%, at $\bar{T}^0 = 157.147$ and $\bar{T}^{no} = 159.494$ and the action profiles $x^o(\tau)$ and $x^{no}(\tau)$ are described as follows.

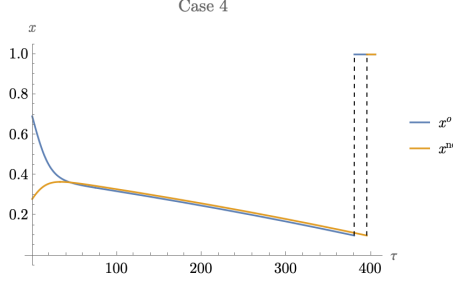


The main effect of increasing the rate of arrival of a catastrophe is to make regime survival ratios decrease faster as shown below.

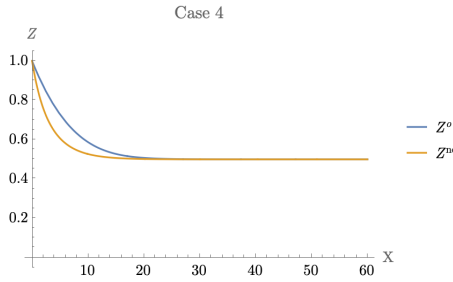


SCENARIO 4: ZERO DISCOUNTING. Consider now the case of zero discounting (i.e., $r = 0$) as advocated by Stern (2007) and suppose again that $\theta_1 = 0.1$. In this scenario, the sole source of discounting comes from the probability of a catastrophe that suppresses future payoffs. Because this event is unlikely before having crossed the tipping point, low

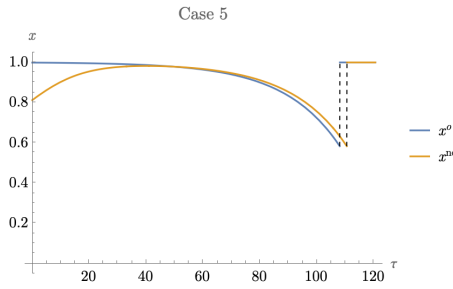
actions are now chosen in a first phase that lasts longer. Indeed, we find $\bar{T}^0 = 380.429$ and $\bar{T}^{no} = 395.302$. Yet, the difference between the scenarios with and without observable deviations is mild; those delays now differing by less than 4%. With almost no-discounting, the future matters a lot and the *Irreversibility Effect* is quite significant. As a result, both $x^o(\tau)$ and $x^{no}(\tau)$ remain low for a very long time while still being very close.



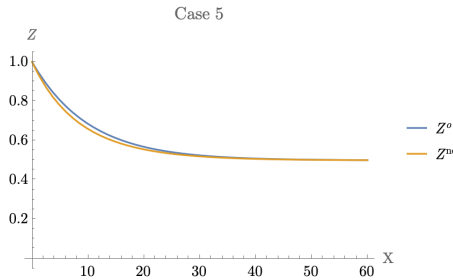
With almost no-discounting, the regime survival ratios decrease slowly as shown below.



SCENARIO 5: HIGH DISCOUNTING. In his critique of the *Stern Review*, Weitzman (2007) advocated a much higher discount rate, namely $r = 0.05$. The first consequence of high discounting is to shorten the delays before reaching \bar{X} . We find that the switching times, namely $\bar{T}^0 = 108.114$ and $\bar{T}^{no} = 110.596$, differ by less than 2.3%. The second consequence is that both actions $x^o(\tau)$ and $x^{no}(\tau)$ although still nearby are now closer to the myopic solution for a long time. Indeed, a high discount rate makes behaving myopically more attractive; leaving distortions needed to satisfy the irreversibility constraint only for the very last periods before reaching \bar{X} .



With high discounting, the regime survival ratios in both scenarios are now almost the same.



9. CONCLUDING REMARKS

We have considered a dynamic decision-making problem with irreversibility, uncertainty and possibly limited information on the consequences of past actions. Increasing current actions makes it more likely to pass a tipping point and thus increases the likelihood of an environmental catastrophe but the location of such tipping point remains unknown through the process. The optimal trajectory follows a feedback rule that depends not only on the stock of past actions but also on beliefs on whether the tipping point has been passed or not. This trajectory can be implemented as a decentralized equilibrium where decision-makers, acting at different points in time and sharing the same objectives, have limited commitment power and adopt a *Stock-Markov* feedback rule that only depends on stock. This implementation requires that impulse deviations are observable by followers. Indeed, upon observing such deviations, future decision-makers are able to reconstruct the evolution of beliefs and act as what a planner would do at the optimal trajectory.

Instead, when impulse deviations are non-observable, the equilibrium feedback rule entails more prudent actions. When actions have been kept low in the past, decision-makers remain quite optimistic on the fact that the tipping point has not been passed yet. In response, they also refrain from taking large actions to avoid any irreversible move.

This framework has allowed us to discuss the relevance of the *Precautionary Principle* that states that one should not act when the consequences of those acts remain unknown. Numerical simulations nevertheless suggest that a trajectory so constrained remains close to the optimum under broad circumstances. The lack of information across decision-makers might thus not be so damageable to society, softening concerns regarding the use of the *Precautionary Principle*.

Our model could be modified along several interesting dimensions. First, signals on the location of the tipping point could be exogenously learned by the decision-maker as the trajectory evolves; maybe thanks to scientific progresses and verifiable expertise. As the trajectory comes closer to the tipping point, decision-makers may accumulate enough evidence on the decreasing distance to the regime switch. The history of past actions then would not suffice to determine beliefs. Because the mapping between stock and beliefs would no longer be one-to-one, a *Stock-Markov* feedback rule would certainly fail to replicate the optimal trajectory even when deviations are observable. Yet, we conjecture that, with observable deviations, a *Stock-Markov* feedback rule might still require higher actions as hard signals on the fact that the tipping point has been passed are learnt while, with non-observable deviations, more prudent actions would be implemented.

Second, while we assumed a strong form of irreversibility in our analysis, many applications in environmental economics feature some sort of limited reversibility. The stock of some pollutants in the atmosphere might have some depreciation over time. The analysis of this scenario could be readily mapped into our framework, at the cost of a more complex dynamics. To illustrate, consider the scenario where the tipping point is known. When the depreciation rate is small enough, an optimal trajectory may require higher actions earlier on since the stock does not evolve so quickly towards the tipping point. Later, once the tipping point has been just passed, moderate actions might be preferred with the hope that the stock will reverse below that threshold. In other words, the myopic optimum might only be attractive once the tipping point has been passed a while ago. We might thus conjecture that depreciation attenuates variations in actions. How much

of this tempering effect remains when there is uncertainty on where the tipping point lies remains to be explored. We conjecture that the scenario with non-observable deviations might lead to less fluctuations in actions.

Third, we have assumed that decision-makers were concerned by the payoff of an infinitely-lived agent. In the case of overlapping generations, decision-makers at different points in time have instead different objectives. Earlier selves might be consuming too much and the *Precautionary Principle*, viewed as a ban on actions, could improve welfare of future generations. Relatedly, future selves may not be fully rational and put an excessive weight on the most recent information they might learn. In this case, any impulse deviation that increases current action will drive future beliefs towards over-pessimism and a fast move towards the myopic optimum. Under those conditions, the scenario with non-observable deviations might lead to more moderate actions and be more attractive from a welfare point of view.

Other political considerations could be at play. First, experts who could detect the consequences of previous acts may have their own objectives and manipulate information on those consequences. This incentive problem of expertise is less pronounced when trajectories are less sensitive to beliefs, a scenario which would bring the analysis closer to the case of non-observable deviations. Second, consider the possibility that rotating decision-makers with different preferences are democratically elected for periods of finite length. If a first decision-maker knows he is about to step down from power and be replaced with another decision-maker who will prefer higher actions, he might as well enact laws that stipulate limits on future actions. The *Precautionary Principle* is now akin to a political constraint on future decision-makers.

REFERENCES

- Arrow, K. and A. Fisher (1974). "Environmental Preservation, Uncertainty, and Irreversibility," *The Quarterly Journal of Economics*, 88: 312-319.
- Asano, T. (2010). "*Precautionary Principle* and the Optimal Timing of Environmental Policy under Ambiguity," *Environmental and Resource Economics*, 47: 173-196.
- Asheim, G and I. Ekeland (2015). "Resource Conservation across Generations in a Ramsey-Chichilnisky Game," *Economic Theory*, 61: 611-639.
- Auster, S., Y.-K. Che and K. Mierendorff (2023). "Prolonged Learning and Hasty Stopping: the Wald Problem with Ambiguity," forthcoming *The American Economic Review*.
- Beck, U. (1992). *Risk Society: Towards a New Modernity*. Sage.
- Beltratti, A., G. Chichilnisky and G. Heal (1995). "Sustainable Growth and the Green Golden Rule," in *The Economics of Sustainable Development*, A. Halpern ed. Cambridge University Press.
- Benveniste, L. and J. Scheinkman (1979). "On the Differentiability of the Value Function in Dynamic Models of Economics." *Econometrica*, 47: 727-732.
- Besley, T. and A. Dixit (2019). "Comparing Alternative Policies against Environmental Catastrophes," *Proceedings of the National Academy of Sciences* 116.12: 5270-5276.
- Cai, Y. and T. Lontzek (2019). "The Social Cost of Carbon with Economic and Climate Risks," *Journal of Political Economy*, 127: 2684-2734.

- Chichilnisky, G. and G. Heal (1993). "Global Environmental Risks," *The Journal of Economic Perspectives*, 7: 65-86.
- Clarke, H. and W. Reed (1994). "Consumption/Pollution Tradeoffs in an Environment Vulnerable to Pollution-related Catastrophic Collapse," *Journal of Economic Dynamics and Control*, 18: 991-1010.
- Crépin, A. and E. Nævdal (2020). "Inertia Risk: Improving Economic Models of Catastrophes," *Scandinavian Journal of Economics*, 122:1259-85.
- Cropper, M. (1976). "Regulating Activities with Catastrophic Environmental Effects," *Journal of Environmental Economics and Management*, 3: 1-15.
- Ekeland, I. , L. Karp and R. Sumaila (2015). "Equilibrium Resource Management with Altruistic Overlapping Generations," *Journal of Environmental Economics and Management*, 70: 1-16.
- Ekeland, I. and A. Lazrak (2006). "Being Serious about Non-Commitment: Subgame-Perfect Equilibrium in Continuous Time," arXiv:math/0604264.
- Ekeland, I. and A. Lazrak (2008). "Equilibrium Policies when Preferences Are Time Inconsistent," arXiv:0808.3790.
- Ekeland, I. and A. Lazrak (2010). "The Golden Rule When Preferences Are Time Inconsistent," *Mathematical Financial Economics*, 4: 29-55.
- Ekeland, I. and T. Turnbull (1983). *Infinite-Dimensional Optimization and Convexity*, The University of Chicago Press.
- Epstein, L. (1980). "Decision-Making and the Temporal Resolution of Uncertainty," *International Economic Review*, 21: 269-283.
- Freixas, X. and J.J. Laffont (1984). "On the Irreversibility Effect," in *Bayesian Models in Economic Theory*, M. Boyer and R. Kihlstrom eds., 105-114, North-Holland.
- Gardiner, S. (2006). "A Core Precautionary Principle," *Journal of Political Philosophy*, 14: 33-60.
- Gjerde, J., S. Grepperud and S. Kverndokk. (1999). "Optimal Climate Policy under the Possibility of a Catastrophe," *Resource and Energy Economics*, 21: 289-317.
- Giddens, A. (2011). *The Politics of Climate Change*, Polity.
- Gollier, C., B. Jullien and N. Treich (2000). "Scientific Progress and Irreversibility: An Economic Interpretation of the Precautionary Principle," *Journal of Public Economics*, 75: 229-253.
- Gonzales, F. (2008). "Precautionary Principle and Robustness for a Stock Pollutant with Multiplicative Risk," *Environmental and Resource Economics*, 41: 25-46.
- Henry, C. (1974). "Investment Decisions Under Uncertainty: The Irreversibility Effect," *The American Economic Review*, 64: 1006-1012.
- Immordino, G. (2000). "Self-Protection, Information and the Precautionary Principle," *The Geneva Papers on Risk and Insurance Theory*, 25: 179-187.
- Immordino, G. (2003). "Looking for a Guide to Protect the Environment: The Development of the Precautionary Principle", *Journal of Economic Surveys*, 17: 629-644.
- Kamien, N. and N. Schwartz (1971). "Optimal Maintenance and Sale Age for a Machine subject to Failure," *Management Science*, 17: 495-504.
- Karp, L. (2005). "Global Warming and Hyperbolic Discounting," *Journal of*

- Public Economics*, 89: 261-282.
- Karp, L. (2007). "Non-Constant Discounting in Continuous Time," *Journal of Economic Theory*, 132: 557-568.
- Karp, L. and I. Lee. (2003). "Time-Consistent Policies," *Journal of Economic Theory*, 112: 353-364.
- Kolstad, C. (1996). "Fundamental Irreversibilities in Stock Externalities," *Journal of Public Economics*, 60: 221-233.
- Kriegler, E., J. Hall, H. Held, R. Dawson and H. Schellnhuber (2009). "Imprecise Probability Assessment of Tipping Points in the Climate System," *Proceedings of the national Academy of Sciences*, 106: 5041-5046.
- Lemoine, D. and C. Traeger (2014). "Watch Your Step: Optimal Policy in a Tipping Climate," *The American Economic Journal: Economic Policy*, 6: 137-166.
- Lenton, T. , H. Held, E. Kriegler, J. Hall, W. Lucht, S. Rahmstorf and H. Schellnhuber (2008). "Tipping Elements in the Earth's Climate System," *Proceedings of the National Academy of Sciences*, 106: 1786-1793.
- Liski, M. and F. Salanié (2020). "Tipping Points, Delays, and the Control of Catastrophes," <https://www.tse-fr.eu/fr/publications/catastrophes-delays-and-learning>.
- Loury, G. (1978). "The Optimal Exploitation of an Unknown Reserve," *The Review of Economic Studies*, 45: 621-636.
- Miller, J. and F. Lad (1984). "Flexibility, Learning, and Irreversibility in Decisions : A Bayesian Approach," *Journal of Environmental Economics and Management*, 11: 161-172.
- Nævdal E. (2006). "Dynamic Optimization in the Presence of Threshold Effects when the Location of the Threshold is Uncertain with an Application to a Possible Disintegration of the Western Antarctic Ice Sheet," *Journal of Economics Dynamics and Control*, 30: 1131-58.
- Nemytskii, V. and V. Stepanov (1989). *Qualitative Theory of Differential Equations*, Dover.
- O'Riordan, T. (2013). *Interpreting the Precautionary Principle*, Routledge.
- Reed, W. (1989). "Optimal Investment in the Protection of a Vulnerable Biological Resource," *Natural Resource Modelling*, 3: 463-480.
- Rio Declaration on Environment and Development. (1992), United Nations.
- Roe, G. and M. Baker (2007). "Why Is Climate Sensitivity So Unpredictable?" *Science*, 318: 629-632.
- Salmi, J., Laiho, T., and Murto, P. (2019). "Endogenous Learning from Incremental Actions," mimeo.
- Seierstad, A. and K. Sydsaeter (1987). *Optimal Control Theory with Economic Applications*, North-Holland.
- Sims, C. and D. Finoff (2016). "Opposing Irreversibilities and Tipping Point Uncertainty," *Journal of the Association of Environmental and Resource Economists*, 3: 985-1022.
- Stern, N. (2007). *The Economics of Climate Change: The Stern Review*, Cambridge University Press.
- Sunstein, C. (2005). *Laws of Fear*, Cambridge University Press.
- Tsur, Y. and A. Zemel (1995). "Uncertainty and Irreversibility in Groundwater Resource Management," *Journal of Environmental Economics and*

Management, 29: 149-161.

Tsur, Y. and A. Zemel (1996). "Accounting for Global Warming Risks: Resource Management under Event Uncertainty," *Journal of Economic Dynamics and Control*, 20: 1289-1305.

Tsur, Y. and A. Zemel (2021). "Resource Management under Catastrophic Threats," *Annual Review of Resource Economics*, 13: 403-425.

Ulph, A. and D. Ulph (1997). "Global Warming, Irreversibility and Learning," *The Economic Journal*, 107: 636-650.

Van der Ploeg, F. (2014). "Abrupt Positive Feedback and the Social Cost of Carbon," *European Economic Review*, 67: 28-41.

Weitzman, M. (2007). "A Review of the 'Stern Review on the Economics of Climate Change'," *Journal of Economic Literature*, 45: 703-724.

APPENDIX A: KNOWN TIPPING POINT

PROOFS OF PROPOSITION 1 AND PROPOSITION 2: Consider an action plan $\mathbf{x}_t = \{x(\tau)\}_{\tau \geq t}$ from date t onwards. If the stock at date t is X , the stock process $\hat{X}(\tau; X, t)$ from that date on evolves as:

$$(A.1) \quad \hat{X}(\tau; X, t) = X + \int_t^\tau x(s)ds.$$

In the text, we slightly abuse notations and, for simplicity, write $\hat{X}(\tau; X) \equiv \hat{X}(\tau; X, 0)$, in which case the stock trajectory evolves as (4.1). Let define the value function $\tilde{\mathcal{V}}^k(X, t; \bar{X})$, conditionally on having not yet faced a catastrophe, with a survival probability being $e^{-\theta_0 t}$ in this scenario where the value of the tipping point is known being at \bar{X} , as

$$\tilde{\mathcal{V}}^k(X, t; \bar{X}) \equiv \sup_{\bar{T}, \mathbf{x}_t, X(\cdot) \text{ s.t. (A.1) and } \hat{X}(\bar{T}; X, t) = \bar{X}} \int_t^{\bar{T}} e^{-\lambda_0(\tau-t)} u(x(\tau)) d\tau + e^{-\lambda_0(\bar{T}-t)} \mathcal{V}_\infty. \quad {}^{33}$$

First, observe that we can write $\tilde{\mathcal{V}}^k(X, t; \bar{X}) = \mathcal{V}^k(X; \bar{X})$ for all $t \geq 0$, where the current value function $\mathcal{V}^k(X; \bar{X})$ is defined in (4.2).

Take now $X < \bar{X}$ and fix ε small enough so that $X + x\varepsilon < \bar{X}$. Denote $\mathcal{D}(\varepsilon) = \{x \text{ s.t. } X + x\varepsilon < \bar{X}\}$. By the *Principle of Dynamic Programming* when applied to (4.2), we must have

$$\mathcal{V}^k(X; \bar{X}) \equiv \sup_{x \in \mathcal{D}(\varepsilon)} \int_0^\varepsilon e^{-\lambda_0 \tau} u(x) d\tau + e^{-\lambda_0 \varepsilon} \mathcal{V}^k(X + x\varepsilon; \bar{X}).$$

Taking first-order Taylor approximations when $\mathcal{V}^k(X; \bar{X})$ is continuously differentiable in X , we may rewrite this problem as

$$\mathcal{V}^k(X; \bar{X}) = \sup_{x \in \mathcal{D}(\varepsilon)} \varepsilon u(x) + (1 - \lambda_0 \varepsilon)(\mathcal{V}^k(X; \bar{X}) + x\varepsilon \dot{\mathcal{V}}^k(X; \bar{X})).$$

The corresponding Hamilton-Bellman-Jacobi equation writes as

$$(A.2) \quad \lambda_0 \mathcal{V}^k(X; \bar{X}) = \max_x x \dot{\mathcal{V}}^k(X; \bar{X}) - \frac{1}{2}(x - \zeta)^2 + \lambda_1 \mathcal{V}_\infty$$

³³This expression of $\tilde{\mathcal{V}}^k(X, t; \bar{X})$ is valid both for $X < \bar{X}$, and for $X \geq \bar{X}$ provided that we use the convention $\bar{T} = t$ in that latter case.

together with the boundary condition (4.4).

The maximand of the r.h.s. of (A.2) is obtained for the optimal feedback rule (4.5). Inserting this feedback rule into (A.2) yields

$$(A.3) \quad \lambda_0 \mathcal{V}^k(X; \bar{X}) = \zeta \dot{\mathcal{V}}^k(X; \bar{X}) + \frac{(\dot{\mathcal{V}}^k(X; \bar{X}))^2}{2} + \lambda_1 \mathcal{V}_\infty.$$

Solving this second-degree polynomial for $\dot{\mathcal{V}}^k(X; \bar{X})$ and taking the root ensuring that $\sigma^k(X; \bar{X})$ as given by (4.5) remains positive yields (4.3).

COMPARATIVE STATICS. Define

$$(A.4) \quad \hat{\mathcal{V}}(X) = \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty.$$

From (4.3), we have $\dot{\mathcal{V}}^k(X; \bar{X}) \leq 0$ if and only if $\mathcal{V}^k(X) \leq \hat{\mathcal{V}}(X)$. Observe that $\mathcal{V}^k(\bar{X}; \bar{X}) < \hat{\mathcal{V}}(\bar{X})$ because of (4.4). Moreover, $\mathcal{V}^k(X; \bar{X})$ were to cross $\hat{\mathcal{V}}(X)$ at $X_1 < \bar{X}$, we would have $\dot{\mathcal{V}}^k(X_1; \bar{X}) = 0$. Observe that $\hat{\mathcal{V}}(X)$ is a constant solution to (4.3). Suppose that $\mathcal{V}^k(X; \bar{X})$ were to cross $\hat{\mathcal{V}}(X)$ at $X_1 < \bar{X}$. By Cauchy-Lipschitz Theorem, the only solution to (4.3) which is such $\mathcal{V}^k(X_1; \bar{X}) = \hat{\mathcal{V}}(X_1)$ is such that $\mathcal{V}^k(X; \bar{X}) = \hat{\mathcal{V}}(X)$ for all $X \in [0, \bar{X}]$. This would contradict the boundary condition (4.4). Hence, necessarily, $\mathcal{V}^k(X; \bar{X})$ remains always below $\hat{\mathcal{V}}(X)$ and the r.h.s. inequality of (4.8) holds. From (4.3), it then follows that $\dot{\mathcal{V}}^k(X; \bar{X}) < 0$ for $X < \bar{X}$. From (4.4), we thus have necessarily $\mathcal{V}^k(X; \bar{X}) > \mathcal{V}_\infty$ for $X < \bar{X}$ and the l.h.s. inequality of (4.8) also holds.

Turning now to the optimal action. The r.h.s. inequality of (4.9) follows from (4.5) and $\dot{\mathcal{V}}^k(X; \bar{X}) < 0$ for $X < \bar{X}$. The l.h.s. inequality follows from the l.h.s. inequality in (4.8), together with (4.3) and (4.5).

Differentiating (A.3) with respect to X yields

$$(A.5) \quad (\dot{\mathcal{V}}^k(X; \bar{X}) + \zeta) \ddot{\mathcal{V}}^k(X; \bar{X}) = \lambda_0 \dot{\mathcal{V}}^k(X; \bar{X})$$

or

$$(A.6) \quad \left(1 + \frac{\zeta}{\dot{\mathcal{V}}^k(X; \bar{X})}\right) \ddot{\mathcal{V}}^k(X; \bar{X}) = \lambda_0.$$

Because $\dot{\mathcal{V}}^k(X; \bar{X}) < 0$ for $X \in [0, \bar{X})$ and $\sigma^k(X; \bar{X}) = \dot{\mathcal{V}}^k(X; \bar{X}) + \zeta > 0$, we deduce that $\ddot{\mathcal{V}}^k(X; \bar{X}) < 0$ for $X \in [0, \bar{X})$ and thus $\sigma^k(X; \bar{X})$ is decreasing.

VERIFICATION THEOREM. It is routine and thus omitted.

Q.E.D.

APPENDIX B: UNCERTAINTY

Preliminaries

We start by presenting the evolution of the posterior density function $f(\tilde{X}|t, \mathbf{x}^t)$. For future reference, notice that, as times passes, a stock process $\tilde{X}(t; 0)$ of the form (4.1) goes through various possible values \tilde{X} of the tipping point. We may thus also describe process by the time $T(\tilde{X}; 0)$ at which this stock reaches a level \tilde{X} .³⁴

³⁴If $\tilde{X}(t; 0)$ is smooth, increasing and differentiable in t with no flat part, $T(\tilde{X}; 0)$ is itself increasing and smooth and differentiable with a finite derivative.

LEMMA B.1 *The posterior density function $f(\tilde{X}|t, \mathbf{x}^t)$ conditional on not having a catastrophe up to date t following history \mathbf{x}^t satisfies:*

$$(B.1) \quad f(\tilde{X}|t, \mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t, \mathbf{x}^t)} f(\tilde{X}) & \text{if } \hat{X}(t; 0) \leq \tilde{X} \\ \frac{e^{-\theta_0 t} e^{-\Delta(t-T(\tilde{X}; 0))}}{H(t, \mathbf{x}^t)} f(\tilde{X}) & \text{otherwise.} \end{cases}$$

PROOF OF LEMMA B.1: We first compute the probability of survival $H(t, \mathbf{x}^t)$, i.e., the probability that there has been no catastrophe till date t following history \mathbf{x}^t , as (5.1). The first term on the r.h.s. of (5.1) stems for the probability that the tipping point is below $\hat{X}(t; 0)$, and the rate of survival then jumps up to θ_1 at a date $T(\tilde{X}; 0)$ before date t . The second term is the probability that the tipping point is above $\hat{X}(t; 0)$ and the rate of arrival of a catastrophe is still θ_0 . Denote these terms respectively by P_{1t} and P_{2t} . We immediately compute

$$(B.2) \quad P_{2t} = (1 - F(\hat{X}(t; 0)))e^{-\theta_0 t}.$$

Changing variables and letting $\hat{X}(\tau; 0) = \tilde{X}$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau; 0)d\tau = d\tilde{X}$, we rewrite

$$P_{1t} = \int_0^{\hat{X}(t; 0)} f(\tilde{X}) e^{-\theta_0 T(\tilde{X}; 0)} e^{-\theta_1(t-T(\tilde{X}; 0))} d\tilde{X} = \int_0^t f(\hat{X}(\tau; 0)) \frac{\partial \hat{X}}{\partial \tau}(\tau; 0) e^{-\theta_0 \tau} e^{-\theta_1(t-\tau)} d\tau.$$

Integrating by parts yields

$$(B.3) \quad P_{1t} = e^{-\theta_0 t} \left(\left[F(\hat{X}(\tau; 0)) e^{\Delta(\tau-t)} \right]_0^t - \Delta \int_0^t F(\hat{X}(\tau; 0)) e^{\Delta(\tau-t)} d\tau \right).$$

Inserting (B.2) and (B.3) into (5.1) finally yields the expression of the probability of survival up to date t in (5.2). From this expression, we compute the conditional density

$$f(\tilde{X}|t, \mathbf{x}^t) = \begin{cases} \frac{e^{-\theta_0 t}}{H(t, \mathbf{x}^t)} f(\tilde{X}) & \text{if } \hat{X}(t; 0) \leq \tilde{X} \\ \frac{e^{-\theta_0 T(\tilde{X}; 0)} e^{-\theta_1(t-T(\tilde{X}; 0))}}{H(t, \mathbf{x}^t)} f(\tilde{X}) & \text{otherwise.} \end{cases}$$

Simplifying yields (B.1). Q.E.D.

PROOFS OF LEMMA 1: Following a history of past actions \mathbf{x}^t , the stock $\hat{X}(\tau; X, t)$ will evolve as requested by (A.1) with a stream of future actions $\mathbf{x}_t = (x(\tau))_{\tau \geq t}$. Let $T(\tilde{X}; X, t)$ accordingly denote the inverse function defined for $\tilde{X} \geq X$. The value function $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ can be written as

$$(B.4) \quad \begin{aligned} \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv & \sup_{\mathbf{x}_t, X(\cdot) \text{ s.t. (A.1)}} \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) f(\tilde{X}|t, \mathbf{x}^t) d\tilde{X} \\ & + \int_X^{+\infty} \left(\int_t^{T(\tilde{X}; X, t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right. \\ & \left. + e^{-\theta_0(T(\tilde{X}; X, t)-t)} \int_{T(\tilde{X}; X, t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-T(\tilde{X}; X, t))} u(x(\tau)) d\tau \right) f(\tilde{X}|t, \mathbf{x}^t) d\tilde{X}. \end{aligned}$$

Taking into account the expression of the conditional density given in (B.1), we rewrite the expression of $\hat{\mathcal{V}}(t, \mathbf{x}^t)$ in (B.4) as

$$(B.5)$$

$$\begin{aligned}
e^{\theta_0 t} H(t, \mathbf{x}^t) \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv & \sup_{\mathbf{x}_t, X(\cdot) \text{ s.t. (A.1)}} \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) e^{-\Delta(t-T(\tilde{X};0))} f(\tilde{X}) d\tilde{X} \\
& + \int_X^{+\infty} \left(\int_t^{T(\tilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right. \\
& \left. + e^{-\theta_0(T(\tilde{X};X,t)-t)} \int_{T(\tilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-T(\tilde{X};X,t))} u(x(\tau)) d\tau \right) f(\tilde{X}) d\tilde{X}.
\end{aligned}$$

Let

$$\mathcal{I}_1 = \int_0^X \left(\int_t^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-t)} u(x(\tau)) d\tau \right) e^{-\Delta(t-T(\tilde{X};0))} f(\tilde{X}) d\tilde{X}$$

which rewrites as

$$(B.6) \quad \mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(\int_0^X e^{-\Delta(t-T(\tilde{X};0))} f(\tilde{X}) d\tilde{X} \right).$$

Changing variables and letting $\hat{X}(\tau;0) = \tilde{X}$ for $\tau \leq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0) d\tau = d\tilde{X}$, we also rewrite

$$\int_0^X e^{-\Delta(t-T(\tilde{X};0))} f(\tilde{X}) d\tilde{X} = \int_0^t e^{-\Delta(t-\tau)} f(\hat{X}(\tau;0)) \frac{\partial \hat{X}}{\partial \tau}(\tau;0) d\tau.$$

Integrating by parts, yields

$$\begin{aligned}
\int_0^X e^{-\Delta(t-T(\tilde{X};0))} f(\tilde{X}) d\tilde{X} &= e^{-\Delta t} \left(\left[F(\hat{X}(\tau;0)) e^{\Delta \tau} \right]_0^t - \Delta \int_0^t F(\hat{X}(\tau;0)) e^{\Delta \tau} d\tau \right) \\
&= F(X) - \Delta e^{-\Delta t} \int_0^t F(\hat{X}(\tau;0)) e^{\Delta \tau} d\tau
\end{aligned}$$

where the last equality follows from $\hat{X}(t;0) = X$. Inserting into (B.6) yields

$$(B.7) \quad \mathcal{I}_1 = \left(\int_t^{+\infty} e^{-\lambda_1(\tau-t)} u(x(\tau)) d\tau \right) \left(F(X) - \Delta e^{-\Delta t} \int_0^t F(X(s;0)) e^{\Delta s} ds \right).$$

We now compute

$$\begin{aligned}
\mathcal{I}_2 &= \int_X^{+\infty} \left(\int_t^{T(\tilde{X};X,t)} e^{-r(\tau-t)} e^{-\theta_0(\tau-t)} u(x(\tau)) d\tau \right. \\
&\quad \left. + e^{-\theta_0(T(\tilde{X};X,t)-t)} \int_{T(\tilde{X};X,t)}^{+\infty} e^{-r(\tau-t)} e^{-\theta_1(\tau-T(\tilde{X};X,t))} u(x(\tau)) d\tau \right) f(\tilde{X}) d\tilde{X}.
\end{aligned}$$

Changing variables and letting $\hat{X}(\tau;X,t) = \tilde{X}$ for $\tau \geq t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;X,t) d\tau = d\tilde{X}$ and $\hat{X}(t;X,t) = X$, we also rewrite

$$\mathcal{I}_2 = \int_t^{+\infty} \left(\int_t^\tau e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) f(\hat{X}(\tau;X,t)) \frac{\partial \hat{X}}{\partial \tau}(\tau;X,t) d\tau.$$

Integrating by parts yields

$$(B.8) \quad \mathcal{I}_2 = \left[F(\hat{X}(\tau;X,t)) \left(\int_t^\tau e^{-\lambda_0(s-t)} u(x(s)) ds + e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty}$$

$$-\Delta \int_t^{+\infty} F(\hat{X}(\tau; X, t)) e^{\Delta(\tau-t)} \int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \Big) d\tau.$$

Using that $\lim_{\tau \rightarrow +\infty} F(\hat{X}(\tau; X, t)) = 1$ if $\lim_{\tau \rightarrow +\infty} \hat{X}(\tau; X, t) = +\infty$ (which holds when the minimal action is positive at any point of time as we will see below), we get

$$(B.9) \quad \mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \\ - \Delta \int_t^{+\infty} F(\hat{X}(\tau; X, t)) e^{\Delta(\tau-t)} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau.$$

Integrating by parts, we obtain

$$\int_t^{+\infty} F(\hat{X}(\tau; X, t)) e^{\Delta\tau} \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) d\tau \\ = \left[\left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) \left(\int_\tau^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \right) \right]_t^{+\infty} \\ + \int_\tau^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau \\ = \int_\tau^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Inserting into (B.9), we thus obtain

$$(B.10) \quad \mathcal{I}_2 = \int_t^{+\infty} e^{-\lambda_0(s-t)} u(x(s)) ds - F(X) \int_t^{+\infty} e^{-\lambda_1(s-t)} u(x(s)) ds \\ - \Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_t^\tau F(\hat{X}(s; X, t)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Summing up (B.7) and (B.10) and taking into account that $\hat{X}(s; X, t)$ for $s \geq t$ is the continuation of the trajectory $\hat{X}(s; 0)$, i.e., $\hat{X}(s; X, t) \equiv \hat{X}(s; 0, 0) = \hat{X}(s; 0)$ (where the last equality slightly abuses notation) for $s \geq t$, yields

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau-t)} u(x(\tau)) d\tau - \Delta e^{-\Delta t} \int_t^{+\infty} e^{-\lambda_1(\tau-t)} \left(\int_0^\tau F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau)) d\tau$$

and thus

$$\mathcal{I} = \int_t^{+\infty} e^{-\lambda_0(\tau-t)} \left(1 - \Delta e^{-\Delta\tau} \int_0^\tau F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau)) d\tau.$$

Changing variables and setting $\tau' = \tau - t$ yields

$$(B.11) \quad \mathcal{I} = \int_0^{+\infty} e^{-\lambda_0\tau'} \left(1 - \Delta e^{-\Delta(\tau'+t)} \int_0^{\tau'+t} F(\hat{X}(s; 0)) e^{\Delta s} ds \right) u(x(\tau' + t)) d\tau'.$$

Generalizing (5.2) to paths that go till date $t + \tau$, we observe that the probability of survival up to date $t + \tau$ can be expressed in terms of the action plan $\mathbf{x}^{t+\tau}$ followed up to that date (that

plan includes all past actions taken up to date t , namely \mathbf{x}^t , and the actions planned from date t on $\mathbf{x}_t^{t+\tau}$) as

$$(B.12) \quad H(t + \tau, \mathbf{x}^{t+\tau}) = e^{-\theta_0(t+\tau)} \left(1 - \Delta e^{-\Delta(t+\tau)} \int_0^{t+\tau} F(\hat{X}(s; 0)) e^{\Delta s} ds \right).$$

Inserting into (B.11) and changing the name of dummy variables yields

$$(B.13) \quad \mathcal{I} = e^{\theta_0 t} \int_0^{+\infty} e^{-r\tau} H(t + \tau, \mathbf{x}^{t+\tau}) u(x(\tau + t)) d\tau.$$

Inserting into (B.5) yields

$$e^{\theta_0 t} H(t, \mathbf{x}^t) \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, \hat{X}(\cdot)} \int_0^{+\infty} e^{-\lambda_0 \tau} e^{\theta_0(t+\tau)} H(t + \tau, \mathbf{x}^{t+\tau}) u(x(t + \tau)) d\tau$$

$$\text{s.t. } \hat{X}(t + \tau; 0) = X + \int_0^\tau x(t + s) ds \text{ and } X = \int_0^\tau \bar{x}(s) ds.$$

which can be written as

$$(B.14) \quad \hat{Z}(t, \mathbf{x}^t) \hat{\mathcal{V}}(t, \mathbf{x}^t) \equiv \sup_{\mathbf{x}_t, \hat{X}(\tau; X, t) = X + \int_t^\tau x(s) ds} \int_0^{+\infty} e^{-\lambda_0 \tau} \hat{Z}(t + \tau, \mathbf{x}^{t+\tau}) u(x(t + \tau)) d\tau.$$

and, finally, (5.4) with the definition of $\hat{Z}(t + \tau, \mathbf{x}^{t+\tau})$ in (5.3).

Q.E.D.

Value Function

Next proposition provides some properties of the value function $\mathcal{V}^e(X, Z)$.

PROPOSITION B.1 *There exists a solution to the optimization problem (5.11). $Z\mathcal{V}^e(X, Z)$ is non-increasing in X , convex in Z , Lipschitz-continuous in both arguments and thus a.e. differentiable.*

At a higher stock, $\mathcal{V}^e(X, Z)$ is necessarily lower since the irreversibility constraints become more stringent as X comes closer to \bar{X} . Convexity of $Z\mathcal{V}^e(X, Z)$ in Z somehow means that information is valuable for DM .³⁵

PROOF OF PROPOSITION B.1: We first define $\mathcal{W}^e(X, Z)$ as

$$\mathcal{W}^e(X, Z) = Z\mathcal{V}^e(X, Z).$$

EXISTENCE. Existence of a solution to the optimization problem (B.15) follows from applying Filipov-Cesari Theorem with free final time (see Seierstad and Sydsaeter, 1987, Theorem 12, p. 145). To check that all conditions for this theorem are satisfied, first observe that \mathcal{X} is closed and bounded, while X is bounded above by \bar{X} on the relevant interval and Z is also bounded since $Z \in [0, 1]$. Denote

$$N(X, Z, \mathcal{X}, \tau) = \{e^{-\lambda_0 \tau} Z u(x) + \gamma \leq 0, x, \Delta(1 - F(X) - Z); \gamma \leq 0, x \in \mathcal{X}\}.$$

³⁵ From a technical viewpoint, this property implies that a standard result like Benveniste and Scheinkman (1979) that ensures (under some conditions) that the value function is differentiable when it is concave is not available here. Fortunately, Lipschitz-continuity ensures that such differentiability holds almost everywhere.

Let us check that $N(X, Z, \mathcal{X}, \tau)$ is convex for each (X, Z, τ) . Take a pair $(x_1, x_2) \in N(X, Z, \mathcal{X}, \tau) \times N(X, Z, \mathcal{X}, \tau)$, i.e., there exist $\gamma_i \leq 0$ such that $e^{-\lambda_0 \tau} Z u(x_i) + \gamma_i \leq 0$. Consider now $\lambda x_1 + (1 - \lambda)x_2$ for $\lambda \in [0, 1]$ and observe that

$$e^{-\lambda_0 \tau} Z u(\lambda x_1 + (1 - \lambda)x_2) \leq e^{-\lambda_0 \tau} Z (u(\lambda x_1 + (1 - \lambda)x_2) - \lambda u(x_1) - (1 - \lambda)u(x_2)) - \lambda \gamma_1 - (1 - \lambda)\gamma_2.$$

Define $\gamma = \lambda \gamma_1 + (1 - \lambda)\gamma_2 + e^{-\lambda_0 \tau} Z (\lambda u(x_1) + (1 - \lambda)u(x_2) - u(\lambda x_1 + (1 - \lambda)x_2))$ and observe that $\gamma \leq 0$ since u is concave and $\gamma_i \leq 0$. Moreover, we have

$$e^{-\lambda_0 \tau} Z u(\lambda x_1 + (1 - \lambda)x_2) + \gamma \leq 0.$$

Hence, $N(X, Z, \mathcal{X}, \tau)$ is convex as requested. From Filipov-Cesari Theorem, an optimal arc thus exists. Let denote by $(X^e(\tau; X, Z), Z^e(\tau; X, Z), x^e(\tau; X, Z), \bar{T}^e(\tau; X, Z))$ such an arc.

PROPERTIES. Inserting (5.7) into the r.h.s. of (B.15), we thus rewrite

$$\begin{aligned} \text{(B.15)} \quad \mathcal{W}^e(X, Z) = & \max_{\mathbf{x}, X(\cdot), T \text{ s.t. (5.5)}, X(0) = X, X(T) = X} (Z - 1) \left(\int_0^T e^{-\lambda_0 \tau} e^{-\Delta \tau} u(x(\tau)) d\tau \right. \\ & + \lambda_1 \mathcal{V}_\infty \int_T^\infty e^{-\lambda_0 \tau} e^{-\Delta \tau} d\tau \Big) + \int_0^T e^{-\lambda_0 \tau} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F(X(s)) e^{\Delta s} ds \right) u(x(\tau)) d\tau \\ & + \int_T^{+\infty} e^{-\lambda_0 \tau} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F(X(s)) e^{\Delta s} ds \right) \lambda_1 \mathcal{V}_\infty d\tau. \end{aligned}$$

Fixing an action path \mathbf{x} and taking $X' \geq X$, the corresponding stocks satisfy $X(s; X) \leq X(s; X')$. The r.h.s. of (B.15) is thus lower at X' for any action path. Taking the max-operator proves that $\mathcal{W}^e(X, Z)$ is non-increasing in X .

From (B.15), it also follows that $\mathcal{W}^e(X, Z)$ is convex as a maximum of linear functions of Z .

Consider an alternative pair (X', Z') . Because an arc which is optimal for (X', Z') , say $(X^e(\tau; X', Z'), Z^e(\tau; X', Z'), x^e(\tau; X', Z'), \bar{T}^e(X', Z'))$, is weakly suboptimal for (X, Z) , the following inequality holds:

$$\begin{aligned} \mathcal{W}^e(X, Z) \geq & (Z - 1) \left(\int_0^{\bar{T}^e(X', Z')} e^{-\lambda_0 \tau} e^{-\Delta \tau} u(x^e(\tau; X', Z')) d\tau + \lambda_1 \mathcal{V}_\infty \int_{\bar{T}^e(X', Z')}^\infty e^{-\lambda_0 \tau} e^{-\Delta \tau} d\tau \right) \\ & + \int_0^{\bar{T}^e(X', Z')} e^{-\lambda_0 \tau} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F \left(X + \int_0^s x^e(s'; X', Z') ds' \right) e^{\Delta s} ds \right) u(x^e(\tau; X', Z')) d\tau \\ & + \int_{\bar{T}^e(X', Z')}^{+\infty} e^{-\lambda_0 \tau} \left(1 - \Delta e^{-\Delta \tau} \int_0^\tau F \left(X + \int_0^s x^e(s'; X', Z') ds' \right) e^{\Delta s} ds \right) \lambda_1 \mathcal{V}_\infty d\tau. \end{aligned}$$

We express the r.h.s. in terms of $\mathcal{W}^e(X', Z')$ to get:

$$\begin{aligned} \text{(B.16)} \quad \mathcal{W}^e(X, Z) - \mathcal{W}^e(X', Z') \geq & (Z - Z') \left(\int_0^{\bar{T}^e(X', Z')} e^{-\lambda_1 \tau} u(x^e(\tau; X', Z')) d\tau + \lambda_1 \mathcal{V}_\infty \int_{\bar{T}^e(X', Z')}^\infty e^{-\lambda_1 \tau} d\tau \right) + \\ & \Delta \left(\int_0^{\bar{T}^e(X', Z')} e^{-\lambda_0 \tau} \left(\int_0^\tau \left(F \left(X' + \int_0^s x^e(s'; X', Z') ds' \right) \right) \right) \right) \end{aligned}$$

$$\begin{aligned}
& -F\left(X + \int_0^s x^e(s'; X', Z') ds'\right) e^{\Delta s} ds \Big) u(x^e(\tau; X', Z')) d\tau \Big) \\
& + \Delta \left(\int_{\bar{T}^e(X', Z')}^\infty e^{-\lambda_0 \tau} \left(\int_0^\tau \left(F\left(X' + \int_0^s x^e(s'; X', Z') ds'\right) \right. \right. \right. \\
& \left. \left. - F\left(X + \int_0^s x^e(s'; X', Z') ds'\right) \right) e^{\Delta s} ds \right) \lambda_1 \mathcal{V}_\infty d\tau \Big).
\end{aligned}$$

Permuting the roles of (X, Z) and (X', Z') , we deduce a similar inequality. Putting together those conditions implies

$$|\mathcal{W}^e(X, Z) - \mathcal{W}^e(X', Z')| \leq \mathcal{V}_\infty (\|f\|_\infty |X' - X| + |Z' - Z|).$$

From which, we deduce that there exists $k = 2\mathcal{V}_\infty \max\{\|f\|_\infty, 1\}$ such that

$$|\mathcal{W}^e(X, Z) - \mathcal{W}^e(X', Z')| \leq k \| (X', Z') - (X, Z) \|$$

where $\|\cdot\|$ denotes the Euclidian norm. Thus, $\mathcal{W}^e(X, Z)$ is Lipschitz continuous and thus a.e. differentiable.

Q.E.D.

For future reference, we now define DM 's payoff along an optimal arc $(X^e(\tau; X, Z), Z^e(\tau; X, Z))$ for the stock and the regime survival ratio starting from arbitrary initial conditions (X, Z) in case the regime switch has already occurred as

$$(B.17) \quad \varphi^e(X, Z) = \int_0^{\bar{T}^e(X, Z)} e^{-\lambda_1 \tau} u(\sigma^e(X^e(\tau; X, Z), Z^e(\tau; X, Z))) d\tau + e^{-\lambda_1 \bar{T}^e(X, Z)} \mathcal{V}_\infty$$

where $\bar{T}^e(X, Z)$ is the date at which the highest possible value of the tipping point is reached, namely $X^e(\bar{T}^e(X, Z); X, Z) = \bar{X}$. Payoffs are discounted at a rate λ_1 once the tipping point has been passed. When $X \geq \bar{X}$, DM knows for sure that it has been the case. He adopts the myopic action with payoff \mathcal{V}_∞ and beliefs evolve according to (5.8). Because $\varphi^e(X, Z)$ is computed when discounting payoffs at rate λ_1 , while $\mathcal{V}^e(X)$ is computed by discounting at a lower rate λ_0 over a first phase, we necessarily have $\mathcal{V}^e(X, Z) \geq \varphi^e(X, Z)$. Although DM ignores having passed the tipping point, he knows that, if that happened, continuation payoffs are necessarily lower.

PROOF OF PROPOSITION 3: CHARACTERIZATION.

PROPOSITION B.2 *At any point of differentiability, $\mathcal{W}^e(X, Z)$ that solves (B.15) satisfies the following Hamilton-Bellman-Jacobi partial differential equation:*

$$(B.18) \quad \lambda_0 \mathcal{W}^e(X, Z) = \lambda_1 \mathcal{V}_\infty Z + \zeta \frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \frac{1}{2Z} \left(\frac{\partial \mathcal{W}^e}{\partial X}(X, Z) \right)^2 + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z).$$

The feedback rule is given by

$$(B.19) \quad \sigma^e(X, Z) = \zeta + \frac{1}{Z} \frac{\partial \mathcal{W}^e}{\partial X}(X, Z).$$

Moreover, we have

$$(B.20) \quad \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z) = \varphi^e(X, Z).$$

PROOF OF PROPOSITION B.2: For the sake of completeness and for future references, we remind below the well-known derivation of the Hamilton-Bellman-Jacobi equation satisfied by $\mathcal{W}^e(X, Z)$. Consider $Z \in [0, 1]$. Using the *Dynamic Programming Principle*, $\mathcal{W}^e(X, Z)$ satisfies

$$(B.21) \quad \mathcal{W}^e(X, Z) = \sup_{\mathcal{A}} \int_0^\varepsilon e^{-\lambda_0 t} Z(t) u(x(t)) dt + e^{-\lambda_0 \varepsilon} \mathcal{W}^e(X(\varepsilon; X, Z), Z(\varepsilon; X, Z)).$$

Consider now ε small enough and denote by x a fixed action over the interval $[0, \varepsilon]$. From (5.6) and (5.5), we get

$$X(\varepsilon; X, Z) = X + \varepsilon x + o(\varepsilon), Z(\varepsilon; X, Z) = Z + \varepsilon \Delta(1 - F(X) - Z) + o(\varepsilon)$$

where $\lim_{\varepsilon \rightarrow 0} o(\varepsilon)/\varepsilon = 0$.

When $\mathcal{W}^e(X, Z)$ is continuously differentiable, we can take a first-order Taylor expansion in ε of the maximand in (B.21) to write it as

$$\mathcal{W}^e(X, Z) + \varepsilon \left(Zu(x) + x \frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z) - \lambda_0 \mathcal{W}^e(X, Z) \right) + o(\varepsilon).$$

Inserting into (B.21) yields the following Hamilton-Bellman-Jacobi equation:

$$(B.22) \quad \lambda_0 \mathcal{W}^e(X, Z) = \sup_{x \in \mathcal{X}} \left\{ Zu(x) + x \frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z) \right\}.$$

FEEDBACK RULE. The maximand on the r.h.s. of (B.22) is strictly concave. It immediately follows that the feedback rule $\sigma^e(X, Z)$ is given by (B.19) when interior. Simplifying (B.22) by using the feedback rule (B.19) finally yields (B.18).

PARTIAL DIFFERENTIAL EQUATION. Rewriting the optimality conditions in terms of $\mathcal{V}^e(X, Z)$, (B.18) becomes

$$\lambda_0 \mathcal{V}^e(X, Z) = \lambda_1 \mathcal{V}_\infty + \zeta \frac{\partial \mathcal{V}^e}{\partial X}(X, Z) + \frac{1}{2} \left(\frac{\partial \mathcal{V}^e}{\partial X}(X, Z) \right)^2 + \frac{\Delta(1 - F(X) - Z)}{Z} \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z).$$

Solving this second-degree equation and keeping the solution that gives a positive feedback rule yields

$$(B.23) \quad \frac{\partial \mathcal{V}^e}{\partial X}(X, Z) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^e(X, Z) - 2 \frac{\Delta(1 - F(X) - Z)}{Z} \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z)}.$$

Denote the optimal solution to (B.15) by $(x^e(\tau; X, Z), X^e(\tau; X, Z), Z^e(\tau; X, Z), \bar{T}^e(X, Z))$. From (B.15), we can write

$$(B.24) \quad \mathcal{W}^e(X, Z) = \int_0^{\bar{T}^e(X, Z)} e^{-\lambda_0 \tau} Z^e(\tau; X, Z) u(x^e(\tau; X, Z)) d\tau + Z^e(\bar{T}^e(X, Z); X, Z) e^{-\lambda_0 \bar{T}^e(X, Z)} \mathcal{V}_\infty.$$

Integrating (5.6), we obtain

$$(B.25) \quad \tilde{Z}^e(\tau; X, Z) = (Z - 1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X^e(s; X, Z)) e^{\Delta s} ds \quad \forall \tau \geq 0, X, Z \geq 0$$

Applying the Envelope Theorem to (B.15) thus yields

$$(B.26) \quad \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z) = \varphi^e(X, Z)$$

or

$$Z \frac{\partial \mathcal{V}^e}{\partial Z}(X, Z) + \mathcal{V}^e(X, Z) = \varphi^e(X, Z)$$

where $\varphi^e(X, Z)$ is defined as in (B.17). Inserting into (B.23) and simplifying yields

$$\frac{\partial \mathcal{V}^e}{\partial X}(X, Z) = -\zeta + \sqrt{2\lambda_0 \mathcal{V}^e(X, Z) - 2 \frac{\Delta(1 - F(X) - Z)}{Z} \varphi^e(X, Z)}$$

which can be written as (5.12).

Q.E.D.

Q.E.D.

BOUNDS. For future references, it is useful to provide simple bounds on $\mathcal{V}^e(X, Z)$.

COROLLARY B.1

$$(B.27) \quad Z\mathcal{V}_\infty \leq Z\mathcal{V}^e(X, Z) \leq \left(F(X) + (1 - F(X)) \frac{\lambda_1}{\lambda_0} \right) \mathcal{V}_\infty \quad \forall X \geq 0, \forall Z \in (0, 1].$$

PROOF OF COROLLARY B.1: Observe that (5.6) and $F(X) \leq F(X^e(\tau; X, Z)) \leq 1$ imply

$$0 \leq \frac{d}{d\tau} (Z^e(\tau; X, Z) e^{\Delta\tau}) \leq \Delta(1 - F(X)) e^{\Delta\tau}.$$

Integrating between 0 and τ yields

$$0 \leq Z e^{-\Delta\tau} \leq Z^e(\tau; X, Z) \leq Z e^{-\Delta\tau} + (1 - F(X)) (1 - e^{-\Delta\tau}).$$

From this and the fact that $0 \leq Z \leq 1$, it follows that

$$(B.28) \quad 0 \leq Z e^{-\Delta\tau} \leq Z^e(\tau; X, Z) \leq F(X) e^{-\Delta\tau} + 1 - F(X) \leq 1.$$

Henceforth, the whole trajectory $Z^e(\tau; X, Z)$ always remains in the stable domain $[0, 1]$.

From the third inequality in (B.28), taking maximum on the r.h.s. of (B.15), the r.h.s. inequality of (B.27) follows. From the first inequality in (B.28), we immediately get the l.h.s. inequality of (B.27). *Q.E.D.*

A VERIFICATION THEOREM. Proposition B.3 below shows that the conditions given Proposition 3 to characterize the extended value function by means of an Hamilton-Bellman-Jacobi equation together with boundary conditions are in fact sufficient. We follow Ekeland and Turnbull (1983, Theorem 1, p. 6) to derive a *Verification Theorem*.

PROPOSITION B.3 *Assume first that there exists a continuously differentiable function $\mathcal{W}_0(X, Z)$ which satisfies:*

(B.29)

$$\lambda_0 \mathcal{W}_0(X, Z) \geq Z(t; X, Z) u(x) + x \frac{\partial \mathcal{W}_0}{\partial X}(X, Z) + \Delta(1 - F(X) - Z(t; X, Z)) \frac{\partial \mathcal{W}_0}{\partial Z}(X, Z) \quad \forall (x, X, Z);$$

and, second, that there exists an action profile X and a path $\bar{X}(t) = \int_0^t \bar{X}(\tau) d\tau$, $Z_0(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(\bar{X}(\tau)) e^{\Delta\tau} d\tau$ such that

$$(B.30) \quad \lambda_0 \mathcal{W}_0(\bar{X}(t), Z_0(t)) = Z_0(t) u(\bar{X}(t))$$

$$+ \bar{X}(t) \frac{\partial \mathcal{W}_0}{\partial X}(\bar{X}(t), Z_0(t)) + \Delta(1 - F(\bar{X}(t)) - Z_0(t)) \frac{\partial \mathcal{W}_0}{\partial Z}(\bar{X}(t), Z_0(t)) \quad \forall t \geq 0.$$

Then X is an optimal action profile with its associated path $(\bar{X}(t), Z_0(t))$.

PROOF OF PROPOSITION B.3: Suppose that a function $\mathcal{W}^e(X, Z)$ that satisfies conditions in Proposition B.2 is continuously differentiable. It is our candidate for the function $\mathcal{W}_0(X, Z)$ in the statement of Proposition B.3. By definition (B.22), we have

$$\lambda_0 \mathcal{W}^e(X, Z) = Zu(\sigma^e(X, Z)) + \sigma^e(X, Z) \frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z), \quad \forall (X, Z)$$

and thus

$$(B.31) \quad \lambda_0 \mathcal{W}^e(X, Z) \geq Zu(x) + x \frac{\partial \mathcal{W}^e}{\partial X}(X, Z) + \Delta(1 - F(X) - Z) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z), \quad \forall (x, X, Z)$$

where the inequality comes from the fact that $\sigma^e(X, Z)$ maximizes the r.h.s..

To get (B.30), we use again (B.22) but now applied to the path $(x^e(t), X^e(t), Z^e(t))$ where $X^e(t)$ is such that $\dot{X}^e(t) = x^e(t) = \sigma^e(X^e(t), Z^e(t))$ with $X^e(0) = 0$ and $Z^e(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(x^e(\tau)) e^{\Delta \tau} d\tau$.

Define now a value function $\widetilde{\mathcal{W}}^e(X, Z, t) = e^{-\lambda_0 t} \mathcal{W}^e(X, Z)$. By (B.31), we get

$$(B.32) \quad 0 \geq \frac{\partial \widetilde{\mathcal{W}}^e}{\partial t}(X, Z, t) + x \frac{\partial \widetilde{\mathcal{W}}^e}{\partial X}(X, Z, t) + \Delta(1 - F(X) - Z) \frac{\partial \widetilde{\mathcal{W}}^e}{\partial Z}(X, Z, t) + e^{-\lambda_0 t} Zu(x) \quad \forall (x, X, Z).$$

Using $X^e(t) = \sigma^e(X^e(t), Z^e(t))$, $Z^e(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(x^e(\tau)) e^{\Delta \tau} d\tau$ and (B.30), we get

$$(B.33) \quad 0 = \frac{\partial \widetilde{\mathcal{W}}^e}{\partial t}(X^e(t), Z^e(t), t) + x^e(t) \frac{\partial \widetilde{\mathcal{W}}^e}{\partial X}(X^e(t), Z^e(t), t) \\ + \Delta(1 - F(X^e(t)) - Z^e(t)) \frac{\partial \widetilde{\mathcal{W}}^e}{\partial Z}(X^e(t), Z^e(t), t) + e^{-\lambda_0 t} Z^e(t) u(X^e(t)) \quad \forall t \geq 0.$$

Take now an arbitrary action plan \mathbf{x} with the associated path $X(t) = \int_0^t x(\tau) d\tau$ and $Z(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(X(\tau)) e^{\Delta \tau} d\tau$. Eventually, this path crosses the upper bound \bar{X} at some \bar{T}^e . Let us fix an arbitrary $t > 0$. Integrating (B.32) along the path $(x(\tau), X(\tau), Z(\tau))$, we compute

$$0 \geq \int_0^t \left(\frac{\partial \widetilde{\mathcal{W}}^e}{\partial \tau}(X(\tau), Z(\tau), \tau) + x(\tau) \frac{\partial \widetilde{\mathcal{W}}^e}{\partial X}(X(\tau), Z(\tau), \tau) \right. \\ \left. + \Delta(1 - F(X(\tau)) - Z(\tau)) \frac{\partial \widetilde{\mathcal{W}}^e}{\partial Z}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) \right) d\tau$$

or

$$0 \geq \int_0^t \left(\frac{d\widetilde{\mathcal{W}}^e}{d\tau}(X(\tau), Z(\tau), \tau) + e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) \right) d\tau \quad \forall t \geq 0.$$

Integrating the first term on the r.h.s., we thus get

$$\widetilde{\mathcal{W}}^e(0, 0, 0) \geq \widetilde{\mathcal{W}}^e(X(t), Z(t), t) + \int_0^t e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau \quad \forall \tau \geq 0.$$

Because $\widetilde{\mathcal{W}}^e(X, Z, t) = e^{-\lambda_0 t} \mathcal{W}^e(X, Z) \geq 0$ for all (X, Z, t) , we obtain:

$$\mathcal{W}^e(0, 0) \geq e^{-\lambda_0 t} \mathcal{W}^e(X(t), Z(t)) + \int_0^t e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau \quad \forall \tau \geq 0.$$

Because of the boundary conditions (B.27), $e^{-\lambda_0 t} \mathcal{W}^e(X(t), Z(t))$ converges towards zero as $t \rightarrow +\infty$ for any feasible path. Moreover, for any such feasible path $\int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau$ exists. Henceforth, we get:

$$\mathcal{W}^e(0, 0) \geq \sup_{\mathbf{x}} \int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau$$

which shows that $(x^e(\tau), X^e(\tau), Z^e(\tau))$ is indeed an optimal path.

Q.E.D.

Optimal Path

The intertemporal date 0-payoff $\mathcal{V}^e(0, 1)$ is achieved by adopting the action profile $\sigma^e(X^e(\tau; 0, 1))$ for all $\tau \geq 0$ starting from the initial conditions $X = 0$ and $Z = 1$. Next Proposition provides necessary conditions for an optimal arc.

PROPOSITION B.4 *An optimal action path $x^e(t)$ satisfies the following necessary condition:*³⁶

$$(B.34) \quad x^e(\tau) = \zeta - \frac{\Delta e^{\lambda_0 \tau}}{Z^e(\tau)} \int_{\tau}^{\bar{T}^e} f(X^e(s)) e^{\Delta s} \left(\int_s^{\bar{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds$$

where, along the optimal trajectory, the probability of no-regime switch writes as

$$Z^e(t) = 1 - \Delta e^{-\Delta t} \int_0^t F(X^e(\tau)) e^{\Delta \tau} d\tau.$$

The upper bound on possible values of the tipping point \bar{X} is reached at a date $\bar{T}^e < \bar{T}^m$ such that

$$(B.35) \quad \bar{X} = \zeta \bar{T}^e - \int_0^{\bar{T}^e} \frac{\Delta e^{\lambda_0 \tau}}{Z^e(\tau)} \left(\int_{\tau}^{\bar{T}^e} f(X^e(s)) e^{\Delta s} \left(\int_s^{\bar{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds \right) d\tau.$$

PROOF OF PROPOSITION B.4 : From (5.4), DM 's intertemporal payoff writes as

$$(B.36) \quad \mathcal{V}^e(0, 1) \equiv \sup_{\mathcal{A}} \int_0^T e^{-\lambda_0 \tau} Z(\tau) u(x(\tau)) d\tau + \int_T^{+\infty} e^{-\lambda_0 \tau} Z(\tau) \lambda_1 \mathcal{V}_{\infty} d\tau.$$

EXISTENCE. It immediately follows that there exists a solution to problem (B.36) from the argument for existence in the Proof of Proposition 3.

MAXIMUM PRINCIPLE. Observe that, for $\tau \geq T$, (5.6) implies

$$(B.37) \quad Z(\tau) = Z(T) e^{-\Delta(\tau-T)}$$

and thus the scrap value on the r.h.s. of the maximand in (B.36) writes as

$$(B.38) \quad \int_T^{+\infty} e^{-\lambda_0 \tau} Z(\tau) \lambda_1 \mathcal{V}_{\infty} d\tau = Z(T) e^{-\lambda_0 T} \mathcal{V}_{\infty}.$$

We now define the Hamiltonian for this optimization problem as

$$(B.39) \quad \mathcal{H}^e(X, Z, x, \tau, \mu, \nu) = e^{-\lambda_0 \tau} Z u(x) + \mu x + \nu \Delta (1 - F(X) - Z)$$

³⁶We slightly abuse notations and omit the dependence on the initial conditions $(0, 1)$.

where μ and ν are respectively the costate variables for (4.1) and (5.6). The *Maximum Principle* with free final time and scrap value now gives us the following necessary conditions for optimality of an arc $(X^e(\tau), Z^e(\tau), x^e(\tau), \bar{T}^e)$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)

Costate variables. $\mu(\tau)$ and $\nu(\tau)$ are both continuously differentiable on \mathbb{R}_+ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial X}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

or

$$(B.40) \quad \dot{\mu}(\tau) = \Delta f(X^e(\tau))\nu(\tau) \quad \forall \tau \in [0, \bar{T}^e];$$

and

$$-\dot{\nu}(\tau) = \frac{\partial \mathcal{H}^e}{\partial Z}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau))$$

or

$$(B.41) \quad \dot{\nu}(\tau) = -e^{-\lambda_0 \tau} u(x^e(\tau)) + \Delta \nu(\tau) \quad \forall \tau \in [0, \bar{T}^e].$$

Transversality conditions. The boundary conditions $X^e(0) = 0$, $X^e(\bar{T}^e) = \bar{X}$ and $Z^e(0) = 1$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \bar{T}^e$ and on $\nu(\tau)$ at $\tau = 0$ only while

$$(B.42) \quad \nu(\bar{T}^e) = 0.$$

Free-end point conditions. The optimality condition with respect to \bar{T} writes as

$$(B.43) \quad \mathcal{H}^e(X^e(\bar{T}^e), Z^e(\bar{T}^e), x^e(\bar{T}^e), \bar{T}^e, \mu(\bar{T}^e), \nu(\bar{T}^e)) + \frac{d}{dT} \left(Z(T) e^{-\lambda_0 T} \right)_{T=\bar{T}^e} \mathcal{V}_\infty = 0.$$

Using (B.39), (B.42), (5.6) taken for \bar{T}^e (with the fact that F has no mass point at \bar{X}), namely

$$(B.44) \quad \dot{Z}(\bar{T}^e) = -\Delta Z(\bar{T}^e),$$

Condition (B.43) rewrites as

$$(B.45) \quad e^{-\lambda_0 \bar{T}^e} Z(\bar{T}^e) \left(u(x^e(\bar{T}^{e-})) - \lambda_1 \mathcal{V}_\infty \right) + \mu(\bar{T}^e) x^e(\bar{T}^{e-}) = 0$$

or

$$(B.46) \quad -\frac{1}{2} e^{-\lambda_0 \bar{T}^e} Z(\bar{T}^e) (x^e(\bar{T}^{e-}) - \zeta)^2 + \mu(\bar{T}^e) x^e(\bar{T}^{e-}) = 0$$

where $x^e(\bar{T}^{e-})$ denotes the l.-h. side limit of $x^e(\tau)$ as $\tau \rightarrow \bar{T}^{e-}$.

Control variable $x^e(\tau)$.

$$x^e(\tau) \in \arg \max_{x \geq 0} \mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \mu(\tau), \nu(\tau)).$$

Because $\mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \mu(\tau), \nu(\tau))$ is strictly concave in x , an interior solution satisfies

$$\frac{\partial \mathcal{H}^e}{\partial x}(X^e(\tau), Z^e(\tau), x^e(\tau), \mu(\tau), \nu(\tau)) = 0$$

or

$$(B.47) \quad x^e(\tau) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau)}{Z^e(\tau)}.$$

Characterization. Inserting (B.47) taken for \bar{T}^e into (B.46) yields

$$\frac{e^{\lambda_0 \bar{T}^e} \mu^2(\bar{T}^e)}{2Z^e(\bar{T}^e)} + \mu(\bar{T}^e) \zeta = 0.$$

The only solution consistent with a non-negative action at date \bar{T}^e is thus

$$(B.48) \quad \mu(\bar{T}^e) = 0.$$

From there, it follows that the optimal action is continuous at \bar{T}^e , namely

$$(B.49) \quad x^e(\bar{T}^{e-}) = x^e(\bar{T}^{e+}) = \zeta.$$

The solution for (B.41) that satisfies the transversality condition (B.42) is

$$(B.50) \quad \nu(\tau) = e^{\Delta \tau} \int_{\tau}^{\bar{T}^e} e^{-\lambda_1 s} u(x^e(s)) ds.$$

Inserting into (B.40) and integrating yields

$$\mu(\tau) = \mu(\bar{T}^e) - \int_{\tau}^{\bar{T}^e} \Delta f(X^e(s)) e^{\Delta s} \left(\int_s^{\bar{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds$$

or, using (B.48),

$$(B.51) \quad \mu(\tau) = - \int_{\tau}^{\bar{T}^e} \Delta f(X^e(s)) e^{\Delta s} \left(\int_s^{\bar{T}^e} e^{-\lambda_1 s'} u(x^e(s')) ds' \right) ds.$$

Inserting into (B.47), we obtain (B.34). Finally, the value of \bar{T}^e is obtained when $\int_0^{\bar{T}^e} x^e(\tau) d\tau = \bar{X}$ or (B.35). That $\bar{T}^e < \bar{T}^m$ is immediate.

Q.E.D.

APPENDIX C: *SME* WITH OBSERVABLE IMPULSE DEVIATIONS

For further reference, we now state the following Lemmatas.

LEMMA C.1

$$(C.1) \quad \frac{\partial X^o}{\partial X}(\tau; X) = \frac{\sigma^o(X^o(\tau; X))}{\sigma^o(X)} = \frac{\frac{\partial X^o}{\partial \tau}(\tau; X)}{\sigma^o(X)}.$$

PROOF OF LEMMA C.1: Starting with the definition of $X^o(\tau; X)$ we get:

$$\frac{\partial X^o}{\partial \tau}(\tau; X) = \sigma^o(X^o(\tau; X)).$$

Differentiating with respect to X and using Schwartz' Lemma (for $X^o(\tau; X)$ twice continuously differentiable) yields

$$\frac{\partial}{\partial \tau} \log \left(\frac{\partial X^o}{\partial X}(\tau; X) \right) = \dot{\sigma}^o(X^o(\tau; X)).$$

Integrating and taking into account that $X^o(0; X) = X$ yields

$$(C.2) \quad \frac{\partial X^o}{\partial X}(\tau; X) = \exp \left(\int_0^\tau \dot{\sigma}^o(X^o(s; X)) ds \right).$$

Using the stationarity of the feedback rule and differentiating with respect to t yields

$$(C.3) \quad \dot{\sigma}^o(X^o(\tau; X)) = \frac{\frac{\partial^2 X^o}{\partial \tau^2}(\tau; X)}{\frac{\partial X^o}{\partial \tau}(\tau; X)}.$$

Inserting into (C.2) and integrating yields

$$\frac{\partial X^o}{\partial X}(\tau; X) = \exp \left(\ln \left(\frac{\frac{\partial X^o}{\partial \tau}(\tau; X)}{\frac{\partial X^o}{\partial \tau}(0; X)} \right) \right)$$

and thus

$$\frac{\partial X^o}{\partial X}(\tau; X) = \frac{\sigma^o(X^o(\tau; X))}{\sigma^o(X^o(0; X))}.$$

Noticing that $X^o(0; X) = X$ yields (C.1).

Q.E.D.

LEMMA C.2

$$(C.4) \quad \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = \sigma^o(X^o(\tau; X)) \left(\frac{x}{\sigma^o(X)} - 1 \right).$$

PROOF OF LEMMA C.2: Take $\tau > \varepsilon$, we have

$$\hat{X}(x, \varepsilon, \tau; X) = X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(\hat{X}(x, \varepsilon, s; X)) ds$$

Now observe that, for $s \geq \varepsilon$, we have

$$\hat{X}(x, \varepsilon, s; X) = X^o(s - \varepsilon, X + x\varepsilon).$$

Hence, we rewrite

$$(C.5) \quad \hat{X}(x, \varepsilon, \tau; X) = X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(X^o(s - \varepsilon, X + x\varepsilon)) ds.$$

Differentiating with respect to ε yields

$$(C.6) \quad \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \int_0^\tau \dot{\sigma}^o(X^o(s; X)) \left(-\frac{\partial X^o}{\partial s}(s; X) + x \frac{\partial X^o}{\partial X}(s; X) \right) ds.$$

Inserting (C.1) into (C.6) yields

$$\frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1 \right) \int_0^\tau \dot{\sigma}^o(X^o(s; X)) \frac{\partial X^o}{\partial s}(s; X) ds.$$

Integrating the last term yields

$$(C.7) \quad \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} = x - \sigma^o(X) + \left(\frac{x}{\sigma^o(X)} - 1 \right) (\sigma^o(X^o(\tau, X)) - \sigma^o(X)).$$

Simplifying further yields (C.4).

Q.E.D.

We first prove the following Lemma. on the properties of $Z(\tau; X)$ and $Z^o(X)$.

LEMMA C.3 *$Z(\tau; X)$ and $Z^o(X)$ satisfy the following conditions*

$$(C.8) \quad \sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) = \frac{\partial Z}{\partial \tau}(\tau; X) \quad \forall \tau \geq 0, X \geq 0,$$

$$(C.9) \quad \sigma^o(X) \dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \text{ with } Z^o(0) = 1.$$

$Z^o(X) \geq 1 - F(X)$ for all X with equality at $X = 0$ only, and thus $\dot{Z}^o(X) \leq 0$ when $\sigma^o(X) > 0$.

PROOF OF LEMMA C.3: Differentiating (6.3) with respect to τ yields

$$(C.10) \quad \frac{\partial Z}{\partial \tau}(\tau; X) = \dot{Z}^o(X^o(\tau; X)) \sigma^o(X^o(\tau; X)).$$

Differentiating (6.3) with respect to X and using (C.1) now yields

$$(C.11) \quad \frac{\partial Z}{\partial X}(\tau; X) = \dot{Z}^o(X^o(\tau; X)) \frac{\sigma^o(X^o(\tau; X))}{\sigma^o(X)}.$$

Gathering (C.10) and (C.11) yields (C.8). Using (C.8) and (6.3) and

$$(C.12) \quad Z(\tau; X) = (Z^o(X) - 1)e^{-\Delta\tau} + 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X^o(s; X)) e^{\Delta s} ds \quad \forall \tau \geq 0, X \geq 0,$$

finally yields (C.9).

Consider $Z_0(X) = 1 - F(X)$. Observe that $\dot{Z}_0(X) < 0$ when $f(X) > 0$. Observe also that $\dot{Z}^o(0) = 0 > \dot{Z}_0(0)$ when $\sigma^o(0) > 0$. Hence, $Z^o(X) > Z_0(X)$ in a starred-right neighborhood of 0. Suppose that $Z^o(X)$ crosses again $Z_0(X)$ for the first time at some $X_1 > 0$, the same reasoning as above shows that $\dot{Z}^o(X_1) = 0 > \dot{Z}_0(X_1)$ when $\sigma^o(X) > 0$ and thus $Z^o(X) < Z_0(X)$ in a starred-left neighborhood of X_1 ; a contradiction. Hence, $Z^o(X) \geq Z_0(X)$ for all X with equality at $X = 0$ only. From (C.8), $\dot{Z}^o(X) \leq 0$. *Q.E.D.*

Next Lemma provides a characterization of any continuously differentiable *SME* with *Stock-Markov* value function and feedback rule $(\mathcal{V}^o(X), \sigma^o(X))$.

LEMMA C.4 *If $\mathcal{V}^o(X)$ is continuously differentiable, the following necessary conditions hold:*

$$(C.13) \quad 0 = \max_{x \in \mathcal{X}} \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X),$$

$$(C.14) \quad \sigma^o(X) \in \arg \max_{x \in \mathcal{X}} \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X).$$

PROOF OF LEMMA C.4: If $\mathcal{V}^o(X)$ is continuously differentiable, $\hat{\mathcal{V}}(x, \varepsilon; X)$ is itself continuously differentiable in ε , and a first-order Taylor expansion in ε yields

$$(C.15) \quad \hat{\mathcal{V}}(x, \varepsilon; X) = \mathcal{V}^o(X) + \varepsilon \frac{\partial \hat{\mathcal{V}}}{\partial \varepsilon}(x, 0, X) + o(\varepsilon).$$

Hence, (6.12) amounts to (C.13). Conjectures being correct at equilibrium, (C.14) also holds. *Q.E.D.*

PROOF OF PROPOSITION 5: We define

$$(C.16) \quad \mathcal{W}^o(X) = Z^o(X) \mathcal{V}^o(X)$$

where

$$(C.17) \quad \mathcal{W}^o(X) = \int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau.$$

Next lemma turns to the properties of $\mathcal{V}^o(X)$ and $\varphi^o(X)$.

LEMMA C.5 *$\mathcal{V}^o(X)$ and $\varphi^o(X)$ satisfy the following system of first-order differential equations:*

$$(C.18) \quad \sigma^o(X) \left(\dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} \mathcal{V}^o(X) \right) = \lambda_0 \mathcal{V}^o(X) - u(\sigma^o(X)),$$

$$(C.19) \quad \sigma^o(X) \dot{\varphi}^o(X) = \lambda_1 \varphi^o(X) - u(\sigma^o(X)).$$

PROOF OF LEMMA C.5: Differentiating (C.17) with respect to X yields

$$\begin{aligned} \dot{\mathcal{W}}^o(X) &= \int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau; X) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial X}(\tau; X) d\tau \\ &\quad + \int_0^{+\infty} e^{-\lambda_0 \tau} \frac{\partial Z}{\partial X}(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau. \end{aligned}$$

Using (C.1), we rewrite this condition as

$$\begin{aligned} (C.20) \quad \sigma^o(X) \dot{\mathcal{W}}^o(X) &= \int_0^{+\infty} e^{-\lambda_0 \tau} Z(\tau; X) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \\ &\quad + \int_0^{+\infty} e^{-\lambda_0 \tau} \sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau. \end{aligned}$$

Integrating by parts the first integral above, we find

(C.21)

$$\begin{aligned} \sigma^o(X)\dot{\mathcal{W}}^o(X) &= \left[e^{-\lambda_0\tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X))) \right]_0^{+\infty} + \lambda_0 \int_0^{+\infty} e^{-\lambda_0\tau} Z(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau \\ &\quad + \int_0^{+\infty} e^{-\lambda_0\tau} \left(\sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) - \frac{\partial Z}{\partial \tau}(\tau; X) \right) u(\sigma^o(X^o(\tau; X))) d\tau. \end{aligned}$$

Using (C.8) and simplifying yields

$$(C.22) \quad \sigma^o(X)\dot{\mathcal{W}}^o(X) = \lambda_0 \mathcal{W}^o(X) - Z^o(X) u(\sigma^o(X)) \quad \forall X.$$

Using the definition of $\mathcal{W}^o(X)$ in (C.16) and simplifying yields (C.18).

Using (6.7) and differentiating with respect to X yields

$$\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau} u'(\sigma^o(X^o(\tau; X))) \frac{\partial X^o}{\partial X}(\tau; X) d\tau.$$

Using (C.1), we rewrite this condition as

$$(C.23) \quad \sigma^o(X)\dot{\varphi}^o(X) = \int_0^{+\infty} e^{-\lambda_1\tau} u'(\sigma^o(X^o(\tau; X))) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau.$$

Integrating by parts we obtain

$$\begin{aligned} \int_0^{+\infty} e^{-\lambda_1\tau} u'(\sigma^o(X^o(\tau; X))) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau &= \left[e^{-\lambda_1\tau} u(\sigma^o(X^o(\tau; X))) \right]_0^{+\infty} \\ &\quad + \lambda_1 \int_0^{+\infty} e^{-\lambda_1\tau} u(\sigma^o(X^o(\tau; X))) d\tau = -u(\sigma^o(X)) + \lambda_1 \varphi^o(X). \end{aligned}$$

Inserting into (C.23) ends the proof. Q.E.D.

By adopting the deviation (6.8)-(6.9), the probability of no-regime switch would also change as (6.10). We can thus write the benefit of a deviation as

$$(C.24) \quad \mathcal{W}(\varepsilon, x; X) = \mathcal{W}_1(\varepsilon, x; X) + \mathcal{W}_2(\varepsilon, x; X)$$

where

$$(C.25) \quad \mathcal{W}_1(\varepsilon, x; X) = (Z^o(X) - 1) \left(\int_0^\varepsilon e^{-\lambda_1\tau} u(x) d\tau + \int_\varepsilon^{+\infty} e^{-\lambda_1\tau} u(\sigma^o(\hat{X}(x, \varepsilon, \tau; X))) d\tau \right)$$

and

$$\begin{aligned} (C.26) \quad \mathcal{W}_2(\varepsilon, x; X) &= \int_0^\varepsilon e^{-\lambda_0\tau} \left(1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X + xs) e^{\Delta s} ds \right) u(x) d\tau \\ &\quad + \int_\varepsilon^{+\infty} e^{-\lambda_0\tau} \left(1 - \Delta e^{-\Delta\tau} \int_0^\tau F(\hat{X}(x, \varepsilon, \tau; X)) e^{\Delta s} ds \right) u(\sigma^o(\hat{X}(x, \varepsilon, \tau; X))) d\tau. \end{aligned}$$

From (C.25), we deduce

$$(C.27) \quad \frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1) \left(u(x) - u(\sigma^o(X)) \right)$$

$$+ \int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, s; X)|_{\varepsilon=0} d\tau \Bigg).$$

Using (C.4), this expression can be simplified as

$$(C.28) \quad \frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1) \left(u(x) - u(\sigma^o(X)) \right. \\ \left. + \left(\frac{x}{\sigma^o(X)} - 1 \right) \int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \right).$$

Integrating by parts, we also have

$$(C.29) \quad \int_0^{+\infty} e^{-\lambda_1 \tau} u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \\ = \left[e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X))) \right]_0^{+\infty} + \lambda_1 \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^o(X^o(\tau; X))) d\tau. \\ = -u(\sigma^o(X)) + \lambda_1 \varphi^o(X) = \sigma^o(X) \dot{\varphi}^o(X)$$

where the last equality follows from (C.19). Inserting into (C.28) yields

$$(C.30) \quad \frac{\partial \mathcal{W}_1}{\partial \varepsilon}(0, x, X) = (Z^o(X) - 1) \left(u(x) - u(\sigma^o(X)) + (x - \sigma^o(X)) \dot{\varphi}^o(X) \right).$$

From (C.26) and (6.10), we deduce

$$(C.31) \quad \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \\ + \int_0^{+\infty} e^{-\lambda_0 \tau} (Z(\tau; X) - (Z^o(X) - 1)e^{-\Delta \tau}) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, \tau; X)|_{\varepsilon=0} d\tau \\ + \int_0^{+\infty} e^{-\lambda_0 \tau} \left(-\Delta e^{-\Delta \tau} \int_0^\tau f(X^o(s; X)) \frac{\partial \hat{X}}{\partial \varepsilon}(x, \varepsilon, s; X)|_{\varepsilon=0} e^{\Delta s} ds \right) u(\sigma^o(X^o(\tau; X))) d\tau.$$

Using (C.4), this expression can be simplified as

$$(C.32) \quad \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \\ + \left(\frac{x}{\sigma^o(X)} - 1 \right) \left(\int_0^{+\infty} e^{-\lambda_0 \tau} (Z(\tau; X) - (Z^o(X) - 1)e^{-\Delta \tau}) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \right. \\ \left. + \int_0^{+\infty} e^{-\lambda_0 \tau} \left(-\Delta e^{-\Delta \tau} \int_0^\tau f(X^o(s; X)) \frac{\partial X^o}{\partial \tau}(s; X) e^{\Delta s} ds \right) u(\sigma^o(X^o(\tau; X))) d\tau \right).$$

Differentiating (C.12) with respect to X and using (C.1) yields

$$(C.33) \quad \sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) = \sigma^o(X) \dot{Z}^o(X) e^{-\Delta \tau} - \Delta e^{-\Delta \tau} \int_0^\tau f(X^o(s; X)) \frac{\partial X^o}{\partial s}(s; X) e^{\Delta s} ds.$$

Using (C.33), we now rewrite

$$(C.34) \quad \int_0^{+\infty} e^{-\lambda_0 \tau} \left(-\Delta e^{-\Delta \tau} \int_0^\tau f(X^o(s; X)) \frac{\partial X^o}{\partial \tau}(s; X) e^{\Delta s} ds \right) u(\sigma^o(X^o(\tau; X))) d\tau$$

$$= \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) - \sigma^o(X) \dot{Z}^o(X) e^{-\Delta \tau} \right) u(\sigma^o(X^o(\tau; X))) d\tau.$$

Integrating by parts, we also have

$$\begin{aligned} (C.35) \quad & \int_0^{+\infty} e^{-\lambda_0 \tau} (Z(\tau; X) - (Z^o(X) - 1)e^{-\Delta \tau}) u'(\sigma^o(X^o(\tau; X))) \dot{\sigma}^o(X^o(\tau; X)) \frac{\partial X^o}{\partial \tau}(\tau; X) d\tau \\ &= \left[e^{-\lambda_0 \tau} (Z(\tau; X) - (Z^o(X) - 1)e^{-\Delta \tau}) u(\sigma^o(X^o(\tau; X))) \right]_0^{+\infty} + \\ & \int_0^{+\infty} \left(\lambda_0 (Z(\tau; X) - (Z^o(X) - 1)e^{-\Delta \tau}) - \frac{\partial Z}{\partial \tau}(\tau; X) - \Delta (Z^o(X) - 1)e^{-\Delta \tau} \right) e^{-\lambda_0 \tau} u(\sigma^o(X^o(\tau; X))) d\tau. \\ &= \lambda_0 \mathcal{W}^o(X) - u(\sigma^o(X)) - \lambda_1 (Z^o(X) - 1) \varphi^o(X) - \int_0^{+\infty} e^{-\lambda_0 \tau} \frac{\partial Z}{\partial \tau}(\tau; X) u(\sigma^o(X^o(\tau; X))) d\tau. \end{aligned}$$

Using (C.34) and (C.35) and inserting into (C.32) yields

$$\begin{aligned} & \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \\ & + \left(\frac{x}{\sigma^o(X)} - 1 \right) \left(\lambda_0 \mathcal{W}^o(X) - u(\sigma^o(X)) - \lambda_1 (Z^o(X) - 1) \varphi^o(X) \right. \\ & \left. + \int_0^{+\infty} e^{-\lambda_0 \tau} \left(\sigma^o(X) \frac{\partial Z}{\partial X}(\tau; X) - \frac{\partial Z}{\partial \tau}(\tau; X) - \sigma^o(X) \dot{Z}^o(X) e^{-\Delta \tau} \right) u(\sigma^o(X^o(\tau; X))) d\tau \right). \end{aligned}$$

Using (C.8) and simplifying yields

$$\begin{aligned} (C.36) \quad & \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) \\ & + \left(\frac{x}{\sigma^o(X)} - 1 \right) \left(\lambda_0 \mathcal{W}^o(X) - Z^o(X) u(\sigma^o(X)) + (Z^o(X) - 1) u(\sigma^o(X)) - \sigma^o(X) \dot{Z}^o(X) \varphi^o(X) \right. \\ & \left. - \lambda_1 (Z^o(X) - 1) \varphi^o(X) \right). \end{aligned}$$

Using (C.22) and (C.19) and simplifying yields

$$(C.37) \quad \frac{\partial \mathcal{W}_2}{\partial \varepsilon}(0, x, X) = u(x) - u(\sigma^o(X)) + (x - \sigma^o(X)) \left(\dot{\mathcal{W}}^o(X) - (Z^o(X) - 1) \dot{\varphi}^o(X) - \dot{Z}^o(X) \varphi^o(X) \right).$$

Gathering (C.37) and (C.30) finally yields

$$\frac{\partial \mathcal{W}}{\partial \varepsilon}(0, x, X) = Z^o(X) \left(u(x) - u(\sigma^o(X)) \right) + (x - \sigma^o(X)) \left(\dot{\mathcal{W}}^o(X) - \dot{Z}^o(X) \varphi^o(X) \right).$$

Because $\frac{\partial \mathcal{W}}{\partial \varepsilon}(0, x, X)$ so obtained is strictly concave in x , the following first-order condition is necessary and sufficient for an interior optimum obtained from (C.13) and (C.14):

$$0 = \frac{\partial^2 \mathcal{W}}{\partial \varepsilon \partial x}(0, \sigma^o(X), X)$$

Developing, we find

$$(C.38) \quad \sigma^o(X) = \zeta + \frac{\dot{\mathcal{W}}^o(X)}{Z^o(X)} - \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X).$$

which writes as (6.16).

Inserting (6.16) into (C.18), we now obtain

$$\sigma^o(X) \left(\sigma^o(X) - \zeta + \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right) = \lambda_0 \mathcal{V}^o(X) - \lambda_1 \mathcal{V}_\infty + \frac{1}{2} (\sigma^o(X) - \zeta)^2.$$

Simplifying, we obtain

$$(C.39) \quad \left(\sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2 = 2\lambda_0 \mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2.$$

Taking then the highest root to (C.39), we obtain

$$(C.40) \quad \sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) = \sqrt{2\lambda_0 \mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)} \varphi^o(X) \right)^2}.$$

Inserting (6.16) into (C.40) and simplifying finally yields (6.14).

LIMITING BEHAVIOR. From (C.12) and the fact that $X^o(\tau; X) \geq \bar{X}$ for all $\tau \geq 0$ and $X \geq \bar{X}$, it follows that

$$(C.41) \quad Z(\tau; X) = Z^o(\bar{X}) e^{-\Delta \tau} \quad \forall \tau \geq 0, X \geq \bar{X}.$$

Inserting into (6.6) immediately yields (6.15). From there, it immediately follows that

$$(C.42) \quad \sigma^o(X) = \zeta \quad \forall X \geq \bar{X}.$$

Q.E.D.

PROOF OF PROPOSITION 6: Clearly (6.23) holds for $X \geq \bar{X}$. We turn to the more difficult case, $X \in [0, \bar{X})$. Consider the pair $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with a belief index $Z^o(X)$ now defined as

$$(C.43) \quad \sigma^e(X, Z^o(X)) \dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X))$$

with the boundary condition

$$(C.44) \quad Z^o(0) = 1.$$

Observe that, provided that $\sigma^e(X, Z)$ remains positive, such a $Z^o(X)$ is uniquely defined and satisfies the same properties as in Lemma C.3. In particular, $Z^o(X)$ is positive for all $X \in [0, \bar{X})$.

We shall prove that $\mathcal{V}^e(X, Z^o(X)) \equiv \mathcal{V}^o(X)$, $\sigma^e(X, Z^o(X)) \equiv \sigma^o(X)$ and $Z^o(X)$ as defined above altogether form a *SME*. To ease notations, define accordingly $\mathcal{W}^o(X)$ as in (C.16).

First, notice that, from (B.22), it immediately follows that, for $X \in [0, \bar{X})$,

$$(C.45) \quad \lambda_0 \mathcal{W}^e(X, Z^o(X)) = \sup_{x \in \mathcal{X}} \left\{ Z^o(X) u(x) + x \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \Delta(1 - F(X) - Z^o(X)) \frac{\partial \mathcal{W}^e}{\partial Z}(X, Z^o(X)) \right\}$$

where we remind that $\mathcal{W}^e(X, Z^o(X)) = Z^o(X) \mathcal{V}^e(X, Z^o(X))$.

Using (B.26) and (C.43), we rewrite (C.45) as

$$(C.46) \quad \lambda_0 \mathcal{W}^e(X, Z^o(X)) = \sup_{x \in \mathcal{X}} \left\{ Z^o(X)u(x) + x \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \sigma^e(X, Z^o(X)) \dot{Z}^o(X) \varphi^e(X, Z^o(X)) \right\}$$

where the maximand above is achieved for

$$(C.47) \quad \sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)} \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) \quad \forall X \in [0, \bar{X}).$$

Still using (B.26), we obtain the following expression of the total derivative of $\mathcal{W}^e(X, Z^o(X))$

$$(C.48) \quad \frac{d\mathcal{W}^e}{dX}(X, Z^o(X)) = \frac{\partial \mathcal{W}^e}{\partial X}(X, Z^o(X)) + \dot{Z}^o(X) \varphi^e(X, Z^o(X)) \quad \forall X \in [0, \bar{X}).$$

Inserting (C.48) into (C.47) yields

$$(C.49) \quad \sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)} \left(\frac{d}{dX} \mathcal{W}^e(X, Z^o(X)) - \dot{Z}^o(X) \varphi^e(X, Z^o(X)) \right) \quad \forall X \in [0, \bar{X}).$$

Also, (B.17) allows us to rewrite

$$(C.50) \quad \varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^e(\tilde{X}^e(\tau; X, Z^o(X)), \tilde{Z}^e(\tau; X, Z^o(X)))) d\tau.$$

At equilibrium, DM expects that the feedback rule $\sigma^o(X') = \sigma^e(X', Z^o(X'))$ prevails for all $X' > X$ and in particular for $X' = X^o(\tau; X)$ for $\tau > 0$. Observe that the future trajectory of stock and beliefs is thus such that $\tilde{X}^e(\tau; X, Z^o(X)) = X^o(\tau; X)$ and $\tilde{Z}^e(\tau; X, Z^o(X)) = Z^o(X^o(\tau; X))$ for all $\tau > 0$. Hence, we rewrite (C.50) as

$$\varphi^e(X, Z^o(X)) = \int_0^{+\infty} e^{-\lambda_1 \tau} u(\sigma^e(X^o(\tau; X), Z^o(X^o(\tau; X)))) d\tau$$

or

$$(C.51) \quad \varphi^o(X) = \varphi^e(X, Z^o(X)).$$

Inserting (C.51) into (C.49) yields

$$(C.52) \quad \sigma^e(X, Z^o(X)) = \zeta + \frac{1}{Z^o(X)} \left(Z^o(X) \frac{d}{dX} \mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X) (\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X)) \right) \quad \forall X \in [0, \bar{X}).$$

Rewriting (C.46), we obtain that $\mathcal{V}^e(X, Z^o(X))$ solves

$$(C.53) \quad \lambda_0 Z^o(X) \mathcal{V}^e(X, Z^o(X)) = \sup_{x \in \mathcal{X}} Z^o(X) u(x) + x \left(Z^o(X) \frac{d}{dX} \mathcal{V}^e(X, Z^o(X)) + \dot{Z}^o(X) (\mathcal{V}^e(X, Z^o(X)) - \varphi^o(X)) \right) + \sigma^e(X, Z^o(X)) \dot{Z}^o(X) \varphi^o(X)$$

where the maximum is achieved with $\sigma^e(X, Z^o(X))$ that satisfies (C.52).

From this, we now observe that $\mathcal{V}^o(X) \equiv \mathcal{V}^e(X, Z^o(X))$ and $\sigma^o(X) = \sigma^e(X, Z^o(X))$ altogether solve

$$(C.54) \quad \lambda_0 Z^o(X) \mathcal{V}^o(X) = \sup_{x \in \mathcal{X}} Z^o(X) u(x) + x \left(Z^o(X) \dot{\mathcal{V}}^o(X) + \dot{Z}^o(X) (\mathcal{V}^o(X) - \varphi^o(X)) \right)$$

$$+\sigma^o(X)\dot{Z}^o(X)\varphi^o(X)$$

where $\sigma^o(X)$, which achieves the maximum on the r.-h.s. above, satisfies

$$(C.55) \quad \sigma^o(X) = \zeta + \frac{1}{Z^o(X)} \left(Z^o(X)\dot{\mathcal{V}}^o(X) + \dot{Z}^o(X)(\mathcal{V}^o(X) - \varphi^o(X)) \right) \quad \forall X \in [0, \bar{X}).$$

Inserting (C.55) into (C.54), rearranging and simplifying yields that $\mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X))$ indeed satisfies (6.14) as requested with any (continuously differentiable) *SME*. Moreover, and from (5.14), the boundary condition (6.15) holds. Hence, $(\mathcal{V}^e(X, Z^o(X)), \sigma^e(X, Z^o(X)))$ together with the associated index $Z^o(X)$ that satisfies (C.43)-(C.44) form a *SME*. *Q.E.D.*

PROOF OF PROPOSITION 7: First, using (C.12) and noticing that $F(X) \leq F(X^o(\tau; X)) \leq 1$ for $\tau \geq 0$, we obtain the bounds

$$(C.56) \quad Z^o(X)e^{\Delta\tau} \leq Z(\tau; X) = Z^o(X^o(\tau; X)) \leq 1 - F(X) + F(X)e^{-\Delta\tau} \quad \forall \tau \geq 0, X \geq 0.$$

Inserting into the definition of $\mathcal{V}^o(X)$ given in (6.6) and integrating, we obtain

$$(C.57) \quad Z^o(X)\varphi^o(X) \leq Z^o(X)\mathcal{V}^o(X) \leq (1 - F(X))\frac{\lambda_1}{\lambda_0} + F(X)\varphi^o(X) \quad \forall X \geq 0.$$

Of course, we have

$$(C.58) \quad \varphi^o(X) \leq \mathcal{V}_\infty \quad \forall X \geq 0$$

which is the l.-h.s. inequality in (6.24). Inserting into (C.57) yields the r.-h.s. inequality in (6.24). The second inequality immediately follows from (6.23) and (B.27) taken for $Z = Z^o(X)$.

To obtain the r.-h.s. inequality in (6.25), first observe that (5.12), (5.15) and (6.23) imply

$$\sigma^o(X) \leq \sqrt{2\lambda_1\mathcal{V}_\infty} = \zeta$$

as requested. To obtain the l.-h.s. inequality in (6.25), observe that $\dot{Z}^o(X) \leq 0$ (from Lemma C.3) and $\varphi^o(X) \geq 0$ altogether imply

$$\sigma^o(X) \geq \sqrt{2\lambda_0\mathcal{V}^o(X)}.$$

Using the second left inequality in (6.24) yields the result. *Q.E.D.*

APPENDIX D: *SME* WITH NON-OBSERVABLE IMPULSE DEVIATIONS

PROOF OF PROPOSITION 8: Being given that each decision-maker takes as given the evolution of beliefs when looking for an optimal action, $\mathcal{V}^{no}(X)$ as defined by (7.4) and following Definition 2 solves

$$(D.1) \quad \mathcal{V}^{no}(X) = \sup_A \int_0^{+\infty} e^{-\int_0^\tau (\lambda_0 - \sigma^{no}(X^{no}(s; X)) \frac{\dot{Z}^{no}(X^{no}(s; X))}{Z^{no}(X^{no}(s; X))}) ds} u(\sigma^{no}(X^{no}(\tau; X))) d\tau.$$

where $Z^{no}(X)$ is consistent with the feedback rule $\sigma^{no}(X)$ that is optimal for problem (D.1) and satisfies (7.1)-(7.2).

Let first define

$$(D.2) \quad \mathcal{W}^{no}(X) = Z^{no}(X)\mathcal{V}^{no}(X).$$

It is routine to show that, at any point of differentiability, $\mathcal{W}^{no}(X)$ satisfies the following Hamilton-Bellman-Jacobi equation for problem (7.4):

$$(D.3) \quad \lambda_0 \mathcal{W}^{no}(X) = \max_{x \in \mathcal{X}} Z^{no}(X)u(x) + x \dot{\mathcal{W}}^{no}(X).$$

The maximand is obtained for an interior solution

$$(D.4) \quad \sigma^{no}(X) = \zeta + \frac{\dot{\mathcal{W}}^{no}(X)}{Z^{no}(X)}.$$

Simplifying yields the *SME* feedback rule when impulse deviations are non-observable as in (7.10). Inserting (D.4) into (D.3) yields

$$\lambda_0 \mathcal{W}^{no}(X) = Z^{no}(X)\lambda_1 \mathcal{V}_\infty + \frac{(\dot{\mathcal{W}}^{no}(X))^2}{2Z^{no}(X)} + \zeta \dot{\mathcal{W}}^{no}(X).$$

Solving this second-degree equation in $\dot{\mathcal{W}}^{no}(X)$ yields

$$(D.5) \quad \dot{\mathcal{W}}^{no}(X) = Z^{no}(X) \left(-\zeta + \sqrt{2\lambda_0 \frac{\mathcal{W}^{no}(X)}{Z^{no}(X)}} \right).$$

Rewriting this condition in terms of $\mathcal{V}^{no}(X)$ yields (7.8).

The boundary condition (7.9) is immediate. For future reference, observe that it also writes in terms of $\mathcal{W}^{no}(X)$ as

$$(D.6) \quad \mathcal{W}^{no}(X) = \mathcal{Z}^{no}(X)\mathcal{V}_\infty \quad \forall X \geq \bar{X}.$$

Q.E.D.

EXISTENCE. Finally, our last result proves existence of a *SME* with non-observable impulse deviations. Its proof consists in studying the properties of the system of first-order differential equations satisfied by $(\mathcal{V}^{no}(X), Z^{no}(X))$ and showing that the boundary conditions at $X = 0$ and $X = \bar{X}$ for that system are satisfied.

PROPOSITION D.1 *A Stock-Markov value function with non-observable deviations $\mathcal{V}^{no}(X)$ and an associated feedback rule $\sigma^{no}(X)$ always exist.*

PROOF OF PROPOSITION D.1: We consider the flow of the differential system made of (7.1) and (D.5) with the initial condition for $\mathcal{Z}^{no}(X)$ given by (7.2) together with an arbitrary initial condition for $\mathcal{W}^{no}(X)$ given by

$$(D.7) \quad \mathcal{W}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty \right].$$

We look for such an initial value $\mathcal{W}^{no}(0)$ so that the terminal condition (D.6) is satisfied.

Observe that the system (7.1)-(D.5) is Lipschitz-continuous on the open domain

$$(D.8) \quad \mathcal{W}^{no}(X) > 0$$

We now define $\widetilde{\mathcal{W}}^{no}(Y) = \mathcal{W}^{no}(X)$, $Z^{no}(Y) = Z^{no}(X)$, $\tilde{\sigma}^{no}(Y) = \sigma^{no}(X)$ where $Y = 1 - F(X) \in [0, 1]$. Let also denote $R(Y) = f(F^{-1}(1 - Y))$ for all $Y \in [0, 1]$. First, notice that we also have $\dot{Z}^{no}(Y) = -\frac{\dot{Z}^{no}(X)}{R(Y)}$ and $\dot{\widetilde{\mathcal{W}}}^{no}(Y) = -\frac{\dot{\mathcal{W}}^{no}(X)}{R(Y)}$. Second, using (7.10) and (D.2), we rewrite

$$(D.9) \quad \tilde{\sigma}^{no}(Y) = \sqrt{2\lambda_0 \frac{\widetilde{\mathcal{W}}^{no}(Y)}{Z^{no}(Y)}}.$$

We now transform the system of first-order differential equations (7.1)-(D.5) as

$$(D.10) \quad \dot{\widetilde{\mathcal{W}}}^{no}(Y) = \frac{Z^{no}(Y)}{R(Y)}(\zeta - \tilde{\sigma}^{no}(Y)),$$

$$(D.11) \quad \dot{Z}^{no}(Y) = \frac{\Delta(Z^{no}(Y) - Y)}{R(Y)\tilde{\sigma}^{no}(Y)}.$$

together with the following boundary conditions

$$(D.12) \quad \widetilde{\mathcal{W}}^{no}(1) \in \left[0, \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty\right], \quad Z^{no}(1) = 1$$

and

$$(D.13) \quad \widetilde{\mathcal{W}}^{no}(0) = Z^{no}(0)\mathcal{V}_\infty.$$

Satisfying boundary conditions at the two end-points $Y = 0$ and $Y = 1$ requires a global analysis of the system. The first step consists in observing that the new system (D.10) can be transformed into an homogenous system expressed in terms of a variable $\tau \in \mathbb{R}_+$ such that (slightly abusing notations by not changing the names of variables although they now depend on τ)

$$(D.14) \quad \dot{\widetilde{\mathcal{W}}}^{no}(\tau) = Z^{no}(\tau)(-\zeta + \tilde{\sigma}^{no}(\tau)),$$

$$(D.15) \quad \dot{Z}^{no}(\tau) = \frac{\Delta(Y - Z^{no}(\tau))}{\tilde{\sigma}^{no}(Y)},$$

$$(D.16) \quad \dot{Y}(\tau) = -R(Y(\tau))$$

together with the following boundary conditions

$$(D.17) \quad \widetilde{\mathcal{W}}^{no}(0) \in \left[0, \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty\right], \quad Z^{no}(0) = 1, \quad Y(0) = 1$$

and

$$(D.18) \quad \lim_{\tau \rightarrow +\infty} \widetilde{\mathcal{W}}^{no}(\tau) - Z^{no}(\tau)\mathcal{V}_\infty = 0, \quad \lim_{\tau \rightarrow +\infty} Y(\tau) = 0.$$

Observe that $Y(\tau)$ is decreasing. Moreover, direct integration of (D.16) together with the third condition in (D.17) yields

$$(D.19) \quad \tau = \int_{Y(\tau)}^1 \frac{dY}{R(Y)}.$$

Consider now the hyperplans

$$\mathcal{D}_0 = \left\{ (\widetilde{W}, Z, Y) \in \mathbb{R}_+^3 \text{ s.t. } \widetilde{W} = \frac{\lambda_1 \mathcal{V}_\infty}{\lambda_0} Z \right\} \text{ and } \mathcal{D}_1 = \{(0, Z, Y) \in \mathbb{R}_+^3\}.$$

Observe that the segment for initial conditions

$$\mathcal{D}_3 = \left\{ (\widetilde{W}, Z, Y) \in \mathbb{R}_+^3 \text{ s.t. } \widetilde{W} \in \left[0, \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty\right], \quad Z = 1, \quad Y = 1 \right\}$$

lies in the cone of the positive orthant whose faces are the hyperplans \mathcal{D}_0 and \mathcal{D}_1 . Observe that the hyperplan

$$\mathcal{D}_4 = \left\{ (\widetilde{W}, Z, Y) \in \mathbb{R}_+^3 \text{ s.t. } \widetilde{W} = Z \mathcal{V}_\infty \right\}$$

belongs to that cone since $0 < \mathcal{V}_\infty < \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty$ and intersects \mathcal{D}_0 and \mathcal{D}_1 at the origin only.

Condition (D.19) shows that any trajectory is such that $Y(\tau)$ is decreasing and remains in the bandwith

$$\mathcal{D}_2 = \left\{ (\widetilde{W}, Z, Y) \in \mathbb{R}_+^3 \text{ s.t. } Y \in [0, 1] \right\}.$$

Moreover, Condition (D.19) also implies that a trajectory reaches $Y = 0$ in finite time if and only if $\int_0^1 \frac{dY}{R(Y)} < +\infty$. If instead $\int_0^1 \frac{dY}{R(Y)} = +\infty$, $Y = 0$ is only reached asymptotically.

Note that any solution to the system (D.14)-(D.15)-(D.16) with initial conditions (D.17) that would cross the hyperplan \mathcal{D}_0 at a time \bar{T} crosses it from below (from the fact that $\dot{\widetilde{W}}^{no}(\bar{T}) \leq 0$ and that direction is not in the hyperplan \mathcal{D}_0). Similarly, any solution to the system (D.14)-(D.15)-(D.16) with initial conditions (D.17) that would cross the hyperplan \mathcal{D}_1 at a time τ_1 reaches it from above (from the fact that $\dot{Z}^{no}(\tau_1) = +\infty$ and that direction is not in the hyperplan \mathcal{D}_1). Moreover, such trajectory stops there.

Because the system is continuous on the open positive cone defined by the faces \mathcal{D}_0 , \mathcal{D}_1 , and \mathcal{D}_2 , any trajectory starting from the segment \mathcal{D}_3 can be extended till it reaches the boundaries of this domain in finite time (Nemytskii and Stepanov, 1989, p. 307). Because the flow of the system is continuous, the image of \mathcal{D}_3 which is connected and compact consists of a continuous line \mathcal{L} that might lie on \mathcal{D}_0 , \mathcal{D}_1 , and \mathcal{D}_2 . Observe that, for the initial condition $\widetilde{W}(0) = \frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty$, the trajectory immediately crosses \mathcal{D}_0 and goes out of the cone. Similarly, for the initial condition $\widetilde{W}(0) = \frac{D}{\lambda_0}$, the trajectory immediately reaches \mathcal{D}_1 and stays there. By continuity of the flow of the differential system, trajectories with an initial condition $\widetilde{W}(0)$ in a neighborhood of $\frac{\lambda_1}{\lambda_0} \mathcal{V}_\infty$ goes through \mathcal{D}_0 while trajectories with an initial condition $\widetilde{W}(0)$ in a neighborhood of $\frac{D}{\lambda_0}$ reaches \mathcal{D}_1 . Two cases may *a priori* arise. First, \mathcal{L} may not go through the origin $(0, 0, 0)$. In this case, and by continuity, the part of \mathcal{L} that lies on \mathcal{D}_2 necessarily crosses \mathcal{D}_4 somewhere and the boundary problem has a solution such that $\lim_{\tau \rightarrow +\infty} \widetilde{W}(\tau) = \lim_{\tau \rightarrow +\infty} Z^{no}(\tau) \mathcal{V}_\infty > 0$ or, expressed in terms of original variables $\mathcal{W}^{no}(\bar{X}) = Z^{no}(\bar{X}) \mathcal{V}_\infty > 0$. Second, \mathcal{L} may go through the origin $(0, 0, 0)$. In this case, there is a trajectory that satisfies the boundary condition with $\lim_{\tau \rightarrow +\infty} \widetilde{W}(\tau) = \lim_{\tau \rightarrow +\infty} Z^{no}(\tau) \mathcal{V}_\infty = 0$ or expressed in terms of original variables $\mathcal{W}^{no}(\bar{X}) = Z^{no}(\bar{X}) \mathcal{V}_\infty = 0$.

Q.E.D.

APPENDIX E: RUNNING EXAMPLE

PROOF OF PROPOSITION 4: Observe that (5.7) rewrites now as

$$(E.1) \quad Z(\tau) = -(1 - Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau}.$$

It is straightforward to check that $Z(\tau) \geq 1 - q$ for all $\tau > 0$ when $Z \geq 1 - q$. Since the optimal trajectory starts from $Z = 1$, this condition always holds.

This expression of $Z(\tau)$ allows us to rewrite the definition (5.11) for $\mathcal{V}^e(X, Z)$ in a quasi-explicit form as

$$(E.2) \quad \begin{aligned} Z\mathcal{V}^e(X, Z) = \max_{\mathbf{x}, \bar{T}} \int_0^{\bar{T}} e^{-\lambda_0\tau} \left(-(1 - Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau} \right) u(x(\tau)) d\tau \\ + e^{-\lambda_0\bar{T}} \left(-(1 - Z)e^{-\Delta\bar{T}} + 1 - q + qe^{-\Delta\bar{T}} \right) \mathcal{V}_\infty \end{aligned}$$

subject to

$$(E.3) \quad \int_0^{\bar{T}} x(\tau) d\tau = \bar{X} - X.$$

Solving this problem is straightforward. Let denote by μ the multiplier for (E.3). We form the Lagrangean

$$\begin{aligned} \mathcal{L}(\mathbf{x}, \bar{T}) = \int_0^{\bar{T}} e^{-\lambda_0\tau} \left(-(1 - Z)e^{-\Delta\tau} + 1 - q + qe^{-\Delta\tau} \right) u(x(\tau)) d\tau \\ + e^{-\lambda_0\bar{T}} \left(-(1 - Z)e^{-\Delta\bar{T}} + 1 - q + qe^{-\Delta\bar{T}} \right) \mathcal{V}_\infty + \mu \left(\bar{X} - X - \int_0^{\bar{T}} x(\tau) d\tau \right). \end{aligned}$$

Pointwise optimization for this strictly concave objective yields the following expression of the optimal action at any point in time

$$(E.4) \quad \zeta - x^e(\tau) = \frac{\mu e^{\lambda_0\tau}}{Z(\tau)}$$

where, for simplicity, we omit the dependence on the state variables (X, Z) .

Integrating over $[0, \bar{T}^e]$ yields

$$(E.5) \quad \zeta \bar{T}^e - (\bar{X} - X) = \mu \int_0^{\bar{T}^e} \frac{e^{\lambda_0\tau}}{Z(\tau)} d\tau.$$

Optimizing now with respect to \bar{T} and assuming the quasi-concavity of the objective in \bar{T} yields the following necessary first-order condition

$$e^{-\lambda_0\bar{T}^e} Z(\bar{T}^e) u(x^e(\bar{T}^{e-})) + \mathcal{V}_\infty e^{-\lambda_0\bar{T}^e} \left(-\lambda_0 Z(\bar{T}^e) + \dot{Z}(\bar{T}^e) \right) = \mu x^e(\bar{T}^{e-})$$

where $x^e(\bar{T}^{e-})$ denotes the l.h.-s limit of $x^e(\tau)$ at \bar{T}^e . Simplifying, we get

$$\zeta x^e(\bar{T}^{e-}) - \frac{(x^e(\bar{T}^{e-}))^2}{2} + \mathcal{V}_\infty \left(-\lambda_0 + \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)} \right) = \mu \frac{e^{-\lambda_0\bar{T}^e}}{Z(\bar{T}^e)} x^e(\bar{T}^{e-})$$

Using (E.4) taken at $\tau = \bar{T}^e$, we rewrite the r.h.s. and get

$$\zeta x^e(\bar{T}^{e-}) - \frac{(x^e(\bar{T}^{e-}))^2}{2} + \mathcal{V}_\infty \left(-\lambda_0 + \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)} \right) = x^e(\bar{T}^{e-})(\zeta - x^e(\bar{T}^{e-}))$$

Simplifying further yields

$$x^e(\bar{T}^{e-}) = \zeta \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}}.$$

From (E.4) taken at $\tau = \bar{T}^e$, we then get

$$(E.6) \quad \mu \frac{e^{\lambda_0 \bar{T}^e}}{Z(\bar{T}^e)} = \zeta \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right).$$

Inserting (E.6) into (E.5) and (E.4) finally yields (E.7) and (E.8) respectively:

$$(E.7) \quad \bar{T}^e = \bar{T}^m + \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) e^{-\lambda_0 \bar{T}^e} Z(\bar{T}^e) \int_0^{\bar{T}^e} \frac{e^{\lambda_0 \tau}}{Z(\tau)} d\tau,$$

$$(E.8) \quad x^e(\tau) = \zeta \left(1 - e^{-\lambda_0(\bar{T}^e - \tau)} \frac{Z(\bar{T}^e)}{Z(\tau)} \left(1 - \sqrt{\frac{\lambda_0 - \frac{\dot{Z}(\bar{T}^e)}{Z(\bar{T}^e)}}{\lambda_1}} \right) \right) \quad \forall \tau \in [0, \bar{T}^e].$$

Specializing this solution to the case $X = 0$ and $Z = 1$ yields the optimal trajectory described in (5.18) and (5.16) with $Z(\tau)$ being given by (5.17). Because $\frac{e^{\lambda_0 \tau}}{Z(\tau)}$ is increasing, $x^e(\tau)$ is itself decreasing over $[0, \bar{T}^e]$.

Specializing further to the case $q = 0$ yields the optimal trajectory when the tipping point is known being at \bar{X} for sure. In this case, \bar{T}^k is given by (4.7) while the optimal action is now

$$(E.9) \quad x^k(\tau) = \begin{cases} \zeta \left(1 - e^{-\lambda_0(\bar{T}^k - \tau)} \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) \right) < \zeta & \text{for } \tau \in [0, \bar{T}^k), \\ \zeta & \text{for } \tau \geq \bar{T}^k. \end{cases}$$

Because $Z(\tau)$ is decreasing, one has

$$\bar{T}^k < \bar{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) e^{-\lambda_0 \bar{T}^k} \int_0^{\bar{T}^k} e^{\lambda_0 \tau} d\tau = \bar{T}^m + \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) \frac{1 - e^{-\lambda_0 \bar{T}^k}}{\lambda_0}.$$

Consider now the function $\delta(t) \equiv t - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) \frac{1 - e^{-\lambda_0 t}}{\lambda_0}$. We have $\delta(\bar{T}^k) = \bar{T}^m$, $\delta(0) = 0$ and $\delta'(t) = 1 - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) e^{-\lambda_0 t} > 0$. Hence, there is a unique positive root $0 < \bar{T}^k < \bar{T}^m$ for (5.16). *Q.E.D.*

PROOF OF PROPOSITION 9: The equilibrium trajectory starting from $X = 0$ solves

$$\max_{\mathbf{x}, X(\cdot), \bar{T}} \int_0^{\bar{T}} e^{-\lambda_0 \tau} Z^{no}(X(\tau)) u(x(\tau)) d\tau + e^{-\lambda_0 \bar{T}} Z^{no}(\bar{X}) \mathcal{V}_\infty$$

$$\text{subject to (5.5), } X(0) = X, \text{ and } X(T) = \bar{X},$$

where $Z^{no}(X)$ is given by (7.1) and (7.2).

Let denote by μ the costate variable for (5.5). The Hamiltonian for this control problem is

$$(E.10) \quad \mathcal{H}^{no}(X, x, \tau, \lambda) = e^{-\lambda_0 \tau} Z^{no}(X) u(x) + \mu x.$$

The *Maximum Principle* with free final time and scrap value gives us the following necessary conditions for an optimal arc $(X^{no}(\tau), x^{no}(\tau), \bar{T}^{no})$. (See Seierstad and Sydsaeter, 1987, Theorem 11, p. 143).)

Costate variable. $\mu(\tau)$ is continuously differentiable on \mathbb{R}_+ with

$$-\dot{\mu}(\tau) = \frac{\partial \mathcal{H}^{no}}{\partial X}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau))$$

or

$$(E.11) \quad -\dot{\mu}(\tau) = e^{-\lambda_0 \tau} \dot{Z}^{no}(X^{no}(\tau)) u(x^{no}(\tau)) \quad \forall \tau \in [0, \bar{T}^{no}].$$

Transversality conditions. The boundary conditions $X^{no}(0) = 0$ and $X^{no}(\bar{T}^{no}) = \bar{X}$ imply that there are no transversality conditions on $\mu(\tau)$ at both $\tau = 0$ and $\tau = \bar{T}^{no}$.

Control variable $x^{no}(\tau)$.

$$x^{no}(\tau) \in \arg \max_{x \geq 0} \mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau)).$$

Because $\mathcal{H}^{no}(X^{no}(\tau), x, \tau, \mu(\tau))$ is strictly concave in x , an interior solution satisfies

$$\frac{\partial \mathcal{H}^{no}}{\partial x}(X^{no}(\tau), x^{no}(\tau), \tau, \mu(\tau)) = 0$$

or

$$(E.12) \quad x^{no}(\tau) = \zeta + e^{\lambda_0 \tau} \frac{\mu(\tau)}{Z^{no}(X^{no}(\tau))}.$$

Free-end point conditions. The optimality condition with respect to \bar{T} writes as

$$(E.13) \quad \mathcal{H}^{no}(X^{no}(\bar{T}^{no}), x^{no}(\bar{T}^{no}), \bar{T}^{no}, \mu(\bar{T}^{no})) - \lambda_0 Z^{no}(\bar{X}) e^{-\lambda_0 \bar{T}^{no}} \mathcal{V}_\infty = 0.$$

From (E.12), we get

$$(E.14) \quad x^{no}(\bar{T}^{no}) = \zeta + e^{\lambda_0 \bar{T}^{no}} \frac{\mu(\bar{T}^{no})}{Z^{no}(\bar{X})}.$$

Using (E.10), (E.14), inserting into (E.13) and simplifying yields

$$\zeta x^{no}(\bar{T}^{no-}) - \frac{1}{2} \left(x^{no}(\bar{T}^{no-}) \right)^2 - \lambda_0 \mathcal{V}_\infty = x^{no}(\bar{T}^{no-}) (\zeta - x^{no}(\bar{T}^{no-}))$$

or

$$(E.15) \quad x^{no}(\bar{T}^{no-}) = \zeta \sqrt{\frac{\lambda_0}{\lambda_1}}.$$

where, to account for the discontinuity in action at \bar{T}^{no} , we denote by $x^{no}(\bar{T}^{no-})$ the l.-h. side limit of $x^{no}(\tau)$ as $\tau \rightarrow \bar{T}^{no-}$.

Characterization. Using (E.1) for the optimal arc starting from $Z = 1$, we get

$$(E.16) \quad Z(\tau) = 1 - q + qe^{-\Delta\tau}.$$

Along the trajectory, we must have

$$(E.17) \quad Z^{no}(X^{no}(\tau)) = Z(\tau) \quad \forall \tau \leq \bar{T}^{no}.$$

Differentiating, we get

$$(E.18) \quad \dot{Z}^{no}(X^{no}(\tau)) = \frac{\dot{Z}(\tau)}{x^{no}(\tau)} = -\frac{q\Delta e^{-\Delta\tau}}{x^{no}(\tau)}$$

Now, we rewrite (E.12) as

$$\mu(\tau) = Z^{no}(X^{no}(\tau))(x^{no}(\tau) - \zeta)e^{-\lambda_0\tau}.$$

Differentiating w.r.t. τ and using (E.18) yields the following ordinary differential equation for $x^{no}(\tau)$:

$$\dot{x}^{no}(\tau) - \left(\lambda_0 - \frac{\dot{Z}(\tau)}{2Z(\tau)} \right) x^{no}(\tau) = -\lambda_0\zeta.$$

It is routine to check that the solution of this ordinary differential equation is of the form

$$(E.19) \quad x^{no}(\tau) = \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}} \left(C_0 - \lambda_0\zeta \int_0^\tau e^{-\lambda_0s} \sqrt{Z(s)} ds \right)$$

for some constant C_0 . Using (E.15), this constant is determined as

$$\zeta \sqrt{\frac{\lambda_0}{\lambda_1}} = \frac{e^{\lambda_0\bar{T}^{no}}}{\sqrt{Z(\bar{T}^{no})}} \left(C_0 - \lambda_0\zeta \int_0^{\bar{T}^{no}} e^{-\lambda_0s} \sqrt{Z(s)} ds \right)$$

or

$$(E.20) \quad C_0 = \zeta \sqrt{\frac{\lambda_0}{\lambda_1}} e^{-\lambda_0\bar{T}^{no}} \sqrt{Z(\bar{T}^{no})} + \lambda_0\zeta \int_0^{\bar{T}^{no}} e^{-\lambda_0s} \sqrt{Z(s)} ds.$$

Integrating (E.19), the corresponding stock evolves according to

$$(E.21) \quad X^{no}(\tau) = C_0 \int_0^\tau \frac{e^{\lambda_0s}}{\sqrt{Z(s)}} ds - \lambda_0\zeta \int_0^\tau \frac{e^{\lambda_0s}}{\sqrt{Z(s)}} \left(\int_0^s e^{-\lambda_0s'} \sqrt{Z(s')} ds' \right) ds.$$

The value of \bar{T}^{no} is obtained from the terminal condition $X^{no}(\bar{T}^{no}) = \bar{X} = \zeta \bar{T}^m$. We get:

$$(E.22) \quad \zeta \bar{T}^m = C_0 \int_0^{\bar{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}} d\tau - \lambda_0\zeta \int_0^{\bar{T}^{no}} \frac{e^{\lambda_0\tau}}{\sqrt{Z(\tau)}} \left(\int_0^\tau e^{-\lambda_0s} \sqrt{Z(s)} ds \right) d\tau.$$

Simplifying and using (E.20) to express C_0 yields (7.11).

Inserting the expression of C_0 from (E.20) into (E.19), we obtain the expression of $x^{no}(\tau)$ for $\tau \leq \bar{T}^{no}$ given in (7.13). The expression $\tau \geq \bar{T}^{no}$ is straightforward.

Now, observing that $Z(\tau) \geq Z(\bar{T}^{no})$ for all $\tau \leq \bar{T}^{no}$, we obtain the following majoration of the r.-h. side of (7.11) as

$$\bar{T}^m < e^{-\lambda_0 \bar{T}^{no}} \left(\int_0^{\bar{T}^{no}} e^{\lambda_0 \tau} d\tau \right) \sqrt{\frac{\lambda_0}{\lambda_1}} + \lambda_0 \int_0^{\bar{T}^{no}} e^{-\lambda_0 \tau} \left(\int_0^\tau e^{-\lambda_0 s} ds \right) d\tau$$

or, after simplifying,

$$\bar{T}^m < \bar{T}^{no} - \left(1 - \sqrt{\frac{\lambda_0}{\lambda_1}} \right) \frac{1 - e^{-\lambda_0 \bar{T}^{no}}}{\lambda_0}.$$

From there and (4.7), it follows that $\bar{T}^{no} > \bar{T}^k$.

Q.E.D.