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The War of Attrition under Uncertainty: Theory and Robust Testable Implications[*]

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Abstract

We study a generic model of the war of attrition with symmetric information and stochastic payoffs that depend on a homogeneous linear diffusion. We first show that a player's mixed Markov strategy can be represented by an intensity measure over the state space together with a subset of the state space over which the player concedes with probability 1. We then show that, if players are asymmetric, then, in all mixed-strategy Markov-perfect equilibria, these intensity measures must be discrete, and characterize any such equilibrium through a variational system for the players' value functions. We illustrate these findings by revisiting the standard model of exit in a duopoly under uncertainty and construct a mixed-strategy Markov-perfect equilibrium in which attrition takes place on path despite firms having different liquidation values. We show that firms' stock prices comove negatively over the attrition zone and exhibit resistance and support patterns documented by technical analysis.

Keywords: War of Attrition, Mixed-Strategy Equilibrium, Uncertainty.
JEL Classification: C61, D25, D83.

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1 Introduction

The war of attrition is a workhorse to model situations in which, at any point of time, each player has to decide whether to hold fast or to concede and forfeit a prize to its opponent. Examples include animal conflict (Maynard Smith (1974)), public good provision (Bliss and Nalebuff (1984)), exit from a declining industry (Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986)), labor strikes (Kennan and Wilson (1989)), delays in agreement to stabilization policies (Alesina and Drazen (1991)), competition in technological standards (Bulow and Klemperer (1999)), bargaining (Abreu and Gul (2000)), investment decisions under learning externalities (Décamps and Mariotti (2004)), and boycotts (Egorov and Hardstad (2017)). A growing literature attempts to test the predictions of these models and to estimate the welfare cost of delayed concessions (Hendricks and Porter (1996), Ghemawat (1997), Geraghty and Wiseman (2008), Wang (2009), Takahashi (2015)).

However, theoretical and empirical applications of war-of-attrition models face several challenges. The first is the multiplicity of equilibria, both in pure and mixed strategies, that characterize these models (Riley (1980), Hendricks, Weiss, and Wilson (1988)). It is thus important to identify testable implications of these models that are robust, in the sense that they hold in a large class of equilibria. The second is to account for observable asymmetries in players' characteristics, which many applied models disregard for simplicity. The third is to allow for stochastic payoffs, so as to capture uncertainty about the future evolution of, say, market conditions. The present paper is an attempt at addressing these issues in a unified framework. In so doing, it identifies a new class of mixed-strategy equilibria that have novel and robust empirical implications.

To this end, we study a generic model of the war of attrition with symmetric information, stochastic payoffs, and potentially asymmetric players, which embeds the earlier models of Lambrecht (2001), Murto (2004), Steg (2015), and Georgiadis, Kim, and Kwon (2022). Two players initially present on a market face uncertainty about future market conditions—for instance, the future price of a relevant commodity, or the future state of market demand. Market conditions evolve according to an homogenous linear diffusion. Each player has the option to exit the market, which he may exert at any point in time. Specifically, both players continuously observe the evolution of market conditions; based on this information, each player then decides whether to remain in the market or to irreversibly exit, which terminates the game. In a Markovian way, the players' continuation payoffs when a player decides to exit the market only depend on current market conditions. Besides, there is a second-mover advantage in the sense that, if and when a player exits first, his continuation payoff is lower than the continuation payoff he would have obtained if the other player had exited first given the same market conditions. All payoff-relevant variables—the law of evolution of market conditions and the players' payoff functions—are assumed to be common knowledge. Our running example, inspired by Georgiadis, Kim, and Kwon (2022), features two firms that may exit a market by liquidating their assets—say, because market demand deteriorates too much—but would meanwhile individually fare better as a monopolist than as a duopolist. Firms may be asymmetric in that one firm may have a lower liquidation value than its opponent, and so is less willing to exit the market.

Given the payoff structure we postulate, it is natural to focus on Markov-perfect equilibria in which players' exit decisions at any point in time only depend on current market conditions (Maskin and Tirole (2001)). Our first contribution is to provide a precise definition of mixed Markov strategies that allows for rich possibilities of randomization for the players. Specifically, our first main result, Theorem 1, shows that a randomized stopping time for any player i—as defined by Touzi and Vieille (2002) by introducing an auxiliary randomizing device à la Aumann (1964)—is Markovian if and only if it can be represented by a pair (μ^i, S^i), where μ^i is a measure over the state space of the diffusion representing player i's stopping intensity, and S^i is a subset of the state space over which player i stops with probability 1. The interpretation is that player i exits the market with positive but finite intensity over the support of μ^i, and with infinite intensity over the set S^i.

Well-known examples of this characterization include pure strategies—that is, standard stopping times—as in Lambrecht (2001) and Murto (2004), in which the intensity measure μ^i is degenerate, and mixed strategies in which μ^i is absolutely continuous with respect to Lebesgue measure, as in Steg (2015) and Georgiadis, Kim, and Kwon (2022). These authors characterize pure-strategy Markov-perfect equilibria, and, in the case of symmetric players, a regular mixed-strategy Markov-perfect equilibrium in which players exit the market according to the same absolutely continuous intensity measure. In this regular equilibrium, attrition is maximal in the sense that each player obtains the payoff he would obtain when facing a stubborn opponent threatening never to exit the market. For instance, when two identical firms compete for a monopoly position as in our running example, both firms have the same equilibrium market value, which is equal, over an interval of market conditions, to the common liquidation value of their assets—a stark prediction that is unlikely to match the data. Pure-strategy equilibria do not lend themselves to interesting testable implications either, as no attrition takes place on the equilibrium path.

These examples, however, do not exhaust the range of possibilities made available by our general representation of mixed Markov strategies. In particular, one can conceive of such strategies in which the measure μ^i is singular with respect to Lebesgue measure. Such strategies need not be artificial nor exotic. For instance, μ^i may be a Dirac measure at a given point x^i of the state space, weighted by some positive coefficient a^i. The interpretation is that, each time market conditions reach x^i, player i exits the market with finite intensity a^i, a strategy that can be obtained as the limit of mixed Markov strategies defined on discretized state spaces with increasingly finer mesh, or as the limit of Markov strategies with absolutely continuous intensity measures with supports degenerating to $\{x^i\}$.

Our second main result, Theorem 2, precisely shows that, if players are asymmetric—for instance, if firms in our running example have different liquidation values—then mixed-strategy Markov-perfect equilibrium are singular, involving strategies with discrete intensity measures. At each point in the support of these measures, the corresponding player is indifferent between exiting and remaining in the market. This implies that the state space is partitioned into intervals in which players alternate between being in a dominated position (with a continuation payoff close to the value he could secure if facing a stubborn opponent) or in a dominant position (with a continuation payoff significantly above that value). Our third main result, Theorem 3, characterizes these singular mixed-strategy Markov-perfect equilibria through a variational system satisfied by the two players' continuation value functions. Solving for these equilibria then becomes a relatively simple numerical task. Importantly, this characterization also applies when players are symmetric.

We illustrate these findings in our running example by providing sufficient conditions ensuring that this variational system has a solution. Under these conditions, which allow for firms having different liquidation values, there exists a mixed-strategy Markov-perfect equilibrium in which one player uses a strategy with a Dirac intensity measure, while the other player uses a pure strategy. Specifically, the firm with the lowest liquidation value randomizes between remaining in the market and exiting at the exit threshold for market conditions that would be optimal if its opponent were stubborn. By contrast, the firm with the highest liquidation value exits with probability 1 if market conditions fall below a lower threshold, the value of which is determined precisely so as to meet its opponent's indifference condition. The intensity with which the firm with the lowest liquidation value exits the market is in turn chosen so as to make its opponent just willing to exit at this lower threshold. The conditions we provide ensure that this equilibrium exists when firms have the same liquidation values, and that it is robust to some asymmetry in the firms' liquidation values, as long as it is not too large. This contrasts with the regular mixed-strategy Markov-perfect equilibrium of the symmetric game, which has no counterpart when there is the slightest asymmetry between the firms (Georgiadis, Kim, and Kwon (2022)).

A robust property of the novel class of mixed-strategy equilibria we identify in this paper is that, at any point of the state space at which a player randomizes between exiting or remaining in the market, the equilibrium value function of its opponent has a kink, reflecting that exit by the randomizing player is unpredictable given current market conditions. In our running example, the randomizing firm's total value goes down to its liquidation value at any such point, while the total value of its opponent reaches a peak. Novel asset-pricing implications ensue when these firms are publicly traded.

First, along any path of the diffusion process modeling the evolution of market conditions, the firms' stock prices and their volatilities fluctuate randomly over the attrition region, moving in opposite directions as long as none of them exits the market. These negative comovements of firms' stock prices and their volatilities stand in sharp contrast with the predictions of the regular mixed-strategy Markov-perfect equilibrium that arises when firms have identical liquidation values, in which firms' stock prices are the same and stay constant and equal to their liquidation value over the attrition region.

Second, when the stock price of the nonrandomizing firm reaches its peak, two events may occur. Either the randomizing firm does not exit the market, causing the nonrandomizing firm's stock price to bounce downward. Or the randomizing firm exits the market, causing the nonrandomizing firm's stock price to jump upwards to its value as a monopolist. Because exit by the randomizing firm is unpredictable, these downward bounces exactly compensate for this upward jump. As a result, rational investors have no means to arbitrage away the profits associated to these downward bounces by short-selling the nonrandomizing firm's stock at its peak without incurring the risk of a sudden upward jump in its price. We argue that this pattern is consistent with what technical analysis describes as a resistance level in stock prices, for which our analysis provides an explanation in a setting in which stock prices are only driven by fundamentals.

Finally, it may be objected that our construction does not contribute to solving the multiplicity problem that plagues standard models of the war of attrition: if anything, we exhibit additional equilibria that have been disregarded in the literature, both in the case of symmetric players (Steg (2015)) and in the case of asymmetric players (Georgiadis, Kim, and Kwon (2022)). However, the above discussion points out that equilibria that are robust to even slight asymmetries between players share a common structure, and lead to similar testable implications. In that sense, our results offer a robust characterization of equilibrium outcomes in the war of attrition under uncertainty.

Related Literature

This paper belongs to the large literature on the continuous-time war of attrition, starting with the seminal contribution of Maynard Smith (1974) on animal conflict. Ghemawat and Nalebuff (1985) study a war of attrition between duopolists who must decide when to exit from a declining industry. Hendricks, Weiss, and Wilson (1988) offer an exhaustive characterization of pure- and mixed-strategy equilibria in the war of attrition with symmetric information when players have potentially asymmetric payoffs that are deterministic functions of time. Riley (1980), Bliss and Nalebuff (1984), and Fudenberg and Tirole (1986) extend the analysis to asymmetric-information setups where, for instance, a firm is uncertain about its opponent's cost. In the same vein, Décamps and Mariotti (2004) study an investment game that has the structure of a war of attrition because a firm's investment generates additional information for its opponent about the return of a common-value project.

With the exception of the last paper—which, however, considers a very special Poisson information structure—these papers confine their analysis to situations in which players' payoffs are deterministic. By contrast, a small literature, starting with Lambrecht (2001) and Murto (2004), examines the case where players in a war of attrition have symmetric information, but are uncertain about their future payoffs, which are driven by a diffusion process. Lambrecht (2001) analyzes the order in which firms go bankrupt in an industry, and how this order is influenced by aggregate factors and firm-specific factors such as their financial structure. Murto (2004) studies a stochastic version of Ghemawat and Nalebuff's (1985) exit model, and shows that a firm with a lower liquidation value may actually end up exiting the market first in equilibrium, despite being a priori more enduring than its opponent. These papers allow for asymmetries between players, but restrict attention to pure-strategy Markov-perfect equilibria.[1] By contrast, Steg (2015) characterizes the regular mixed-strategy Markov-perfect equilibrium of the symmetric game.

Closest to the present paper in this literature is Georgiadis, Kim, and Kwon (2022). In a setting that extends Murto (2004), they show that, as soon as firms have different liquidation values, there exists no mixed-strategy Markov-perfect equilibrium in which firms exit the market according to absolutely continuous intensity measures. This shows that the regular mixed-strategy Markov-perfect equilibrium characterized by Steg (2015) is not robust to even small asymmetries between firms. They conclude that, when firms are asymmetric, only pure-strategy Markov-perfect equilibria exist, and, therefore, that no attrition can actually take place on the equilibrium path. Our analysis shows that this conclusion is unwarranted once the possibility for firms to exit the market according to Markovian randomized stopping times with singular intensity measures is accounted for.

We have borrowed from Touzi and Vieille (2002) our concept of a randomized stopping time, which they introduced to show that continuous-time zero-sum Dynkin games admit a value. A technical contribution of the present paper is to provide a characterization of Markovian randomized stopping times in terms of an intensity measure and a stopping region. This characterization may prove useful for the study of general stochastic timing games in which the state variable is driven by a Brownian motion, without postulating a monotone

[1] This is also the case in Fine and Li's (1989) discrete-time duopolistic model of exit from an industry in which demand stochastically declines over time.

reward structure as in Huang and Li (1990).

The paper is organized as follows. Section 2 describes the model. Section 3 provides rigorous definitions of our strategy and equilibrium concepts, as well as preliminary properties of Markov-perfect equilibria. Section 4 heuristically shows how to construct a mixed-strategy Markov-perfect equilibrium involving a singular intensity measure for one of the players. Section 5 states our main characterization results. The main Appendix provides the proofs of Theorems 1–3. The Online Supplement collects detailed proofs of technical lemmas and claims used in the derivation of these theorems.

2 The Model

2.1 A General Model of War of Attrition under Uncertainty

We study a war of attrition with symmetric information between two players, 1 and 2, facing uncertainty about future market conditions. In what follows, i (he) refers to an arbitrary player and j (she) to his opponent. Time is continuous and indexed by $t \geq 0$. Both players observe the evolution of market conditions; based on this information, each player decides whether to *hold fast*, that is, to remain in the market, or to *concede*, that is, to exit the market, an irreversible decision that effectively terminates the game.

The evolution of market conditions is modeled as a one-dimensional time-homogeneous diffusion process $X \equiv (X_t)_{t\geq 0}$ defined over the canonical space $(\Omega, \mathcal{F}, \mathbf{P}_x)$ of continuous trajectories with $X_0 = x$ under $\mathbf{P}_x$, that is solution in law to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad t \geq 0, \tag{1}$$

driven by some Brownian motion $W \equiv (W_t)_{t\geq 0}$. The state space for X is an interval $\mathcal{I} \equiv (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$, and b and σ are continuous functions, with $\sigma > 0$ over $\mathcal{I}$. We assume that α and β are inaccessible (natural) endpoints for the diffusion. Therefore, X is regular over $\mathcal{I}$ and the SDE (1) admits a weak solution that is unique in law.

Player 1 chooses a (random) time τ^1 and player 2 chooses a (random) time τ^2. Both players discount future payoffs at a constant rate $r > 0$. For each $i = 1, 2$, the expected payoff of player i is[2]

$$J^i(x, \tau^1, \tau^2) = \mathbf{E}_x\Big[1_{\{\tau^i \leq \tau^j\}}\,\mathrm{e}^{-r\tau^i} R^i(X_{\tau^i}) + 1_{\{\tau^i > \tau^j\}}\,\mathrm{e}^{-r\tau^j} G^i(X_{\tau^j})\Big]. \tag{2}$$

The payoff functions R^i and G^i in (2) are continuous over their domain $\mathcal{I}$ and satisfy $G^i \geq R^i$, with $G^i(x) > R^i(x)$ for x above some threshold $\alpha^i < \beta$.[3] Therefore, if player i concedes at

[2] By convention, we let $f(X_\tau) \equiv 0$ over $\{\tau = \infty\}$ for any Borel function f and any random time τ.

[3] Notice that one may have $\alpha^i \leq \alpha$. If $\alpha^i > \alpha$, then $G^i = R^i$ over $(\alpha, \alpha^i]$; this reflects that, for low values of x, it may be optimal for player i to exit the market even as a monopolist.

time $\tau^i \leq \tau^j$, then he obtains a payoff $R^i(X_{\tau^i})$, whereas, if player j concedes at time $\tau^j < \tau^i$ and $X_{\tau^j} > \alpha^i$, then player i obtains a strictly higher payoff $G^i(\tau^j)$ than the payoff $R^i(\tau^j)$ he would have obtained by conceding at time τ^j. The payoff functions R^i and G^i, $i = 1, 2$, are assumed to be common knowledge among the players; hence the only primitive source of uncertainty in the model is the diffusion process (1), whose realizations are observed by both players. We study the resulting war of attrition with symmetric information and uncertain payoffs under technical assumptions that we now present.

2.2 Technical Assumptions

We first recall useful properties of the solution X to the SDE (1). We next detail the assumptions on the payoff functions R^i and G^i and emphasize useful properties of the optimal stopping problem

$$V_{R^i}(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x[\mathrm{e}^{-r\tau} R^i(X_\tau)] \tag{3}$$

faced by player i when player j is *stubborn*, that is, plays $\tau^j = \infty$; here $\mathcal{T}$ is the set of all stopping times of the usual augmentation $(\mathcal{F}_t)_{t \geq 0}$ of the natural filtration generated by X over the canonical space, whose definition is recalled in Online Supplement S.1. We refer to (3) as player i's *stand-alone exit problem*, in which he cannot benefit from player j conceding.

Discount Factors The infinitesimal generator of X is defined for functions $u \in \mathcal{C}^2(\mathcal{I})$ by

$$\mathcal{L}u(x) \equiv b(x)u'(x) + \frac{1}{2}\sigma^2(x)u''(x), \quad x \in \mathcal{I}. \tag{4}$$

That $\sigma > 0$ over $\mathcal{I}$ ensures that the ordinary differential equation (ODE) $\mathcal{L}u - ru = 0$ admits a two-dimensional space of solutions in $\mathcal{C}^2(\mathcal{I})$, which is spanned by two positive fundamental solutions ψ and ϕ, respectively strictly increasing and strictly decreasing, that are uniquely defined up to a linear transformation. Because the boundaries α and β of $\mathcal{I}$ are natural, we know in particular that

$$\lim_{x \to \alpha+} \psi(x) = 0, \quad \lim_{x \to \beta-} \psi(x) = \infty, \quad \lim_{x \to \alpha+} \phi(x) = \infty, \quad \lim_{x \to \beta-} \phi(x) = 0. \tag{5}$$

Letting $\tau_y \equiv \inf\{t \geq 0 : X_t = y\}$ be the hitting time by X of $y \in \mathcal{I}$ from $X_0 = x$, we then obtain the following formula for the expected discount factor associated to x and τ_y:

$$\mathbf{E}_x[\mathrm{e}^{-r\tau_y}] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{if} \quad x \leq y \\ \frac{\phi(x)}{\phi(y)} & \text{if} \quad x > y \end{cases}. \tag{6}$$

Assumptions on the Payoff Functions Our assumptions on R^i and G^i are in line with Décamps, Gensbittel, and Mariotti's (2021) model of real options under technological breakthroughs. For each $i = 1, 2$, we assume that $R^i \in \mathcal{C}^2(\mathcal{I})$, and that it satisfies

A1 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}|R^i(X_t)|] < \infty$.

A2 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt}R^i(X_t) = 0$ $\mathbf{P}_x$-almost surely.

A3 There exists $x_0^i \in \mathcal{I}$ such that $\mathcal{L}R^i - rR^i < 0$ over (α, x_0^i) and $\mathcal{L}R^i - rR^i > 0$ over (x_0^i, β).

A1 guarantees that the family $(\mathrm{e}^{-r\tau}R^i(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable. A1–A2 imply the useful growth property

$$\lim_{x\to\alpha+} \frac{R^i(x)}{\phi(x)} = \lim_{x\to\beta-} \frac{R^i(x)}{\psi(x)} = 0. \tag{7}$$

A3 intuitively captures the idea that, as long as the market conditions remain in the portion (α, x_0^i) of the state space, the gains from staying in the market decline if no player has conceded yet. This guarantees that the optimal stopping region $\{x \in \mathcal{I} : V_{R^i}(x) = R^i(x)\}$ for problem (3) is of the form $(\alpha, x_{R^i}]$ for some threshold $x_{R^i} < x_0^i$, so that

$$V_{R^i}(x) = \begin{cases} R^i(x) & \text{if } \ x \leq x_{R^i} \\ \frac{\phi(x)}{\phi(x_{R^i})} R^i(x_{R^i}) & \text{if } \ x > x_{R^i} \end{cases}. \tag{8}$$

The smooth-fit property applies at x_{R^i}, that is, $R^{i\prime}(x_{R^i}) = \frac{\phi'(x_{R^i})}{\phi(x_{R^i})} R^i(x_{R^i})$ (Peskir and Shiryaev (2006), Dayanik and Karatzas (2003, Corollary 7.1)). It follows from standard optimal stopping theory that $(\mathrm{e}^{-rt}V_{R^i}(X_t))_{t\geq 0}$ is a supermartingale and that $\mathcal{L}V_{R^i} - rV_{R^i} \leq 0$ over $\mathcal{I} \setminus \{x_{R^i}\}$. The following lemma holds.

Lemma 1 *$V_{R^i} > 0$ over $\mathcal{I}$ and $R^i > 0$ over $(\alpha, x_{R^i}]$.*

We assume that $G^i \in \mathcal{C}^1(\mathcal{I})$, that G^i is piecewise $\mathcal{C}^2$ over $\mathcal{I}$, and that it satisfies

A4 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}G^i(X_t)] < \infty$.

A5 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt}G^i(X_t) = 0$ $\mathbf{P}_x$-almost surely.

A6 $G^i \geq V_{R^i}$ over $\mathcal{I}$ and $G^i(x) > V_{R^i}(x)$ if and only if $x > \alpha^i$ for some $\alpha^i < x_{R^i}$.

A7 $\mathcal{L}G^i - rG^i \leq 0$ everywhere $G^{i\prime\prime}$ is defined.

The interpretation of A7 is that player i would rather obtain the payoff $G^i(X_t)$ sooner than later. This is the case, for instance, when G^i is the value function of an ulterior optimal stopping problem faced by the winner of the war of attrition. From (3) and A6–A7, we have $G^i > R^i \vee 0$ over $\mathcal{I}$, so that, by Lemma 1, $G^i > 0$ over $\mathcal{I}$; hence A4 guarantees that the family $(\mathrm{e}^{-r\tau}G^i(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable. A4–A5 imply the useful growth property

$$\lim_{x\to\alpha+} \frac{G^i(x)}{\phi(x)} = \lim_{x\to\beta-} \frac{G^i(x)}{\psi(x)} = 0. \tag{9}$$

2.3 A Running Example: Exit in Duopoly

Consider the following model of exit in duopoly, in the spirit of Murto (2004) or Giorgiadis, Kim, and Kwon (2022). Two firms are initially present on the market. As long as both firms remain in the market, each earns a flow duopoly profit X_t, where X follows a geometric Brownian motion with drift $b < r$ and volatility σ,

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad t \geq 0,$$

over the state space $\mathcal{I} \equiv (0, \infty)$. If firm i concedes at time τ^i, then its assets are liquidated for a value $l^i > 0$. If firm j concedes at time τ^j, then firm i thereafter enjoys a flow monopoly profit mX_t for some $m > 1$, until it in turn exits the market to receive its liquidation value l^i. Thus the expected discounted profit or total value of every firm i given exit times τ^i and τ^j is given by

$$F^i(x, \tau^1, \tau^2) \equiv \mathbf{E}_x\left[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt} X_t\,\mathrm{d}t + 1_{\{\tau^i \leq \tau^j\}}\,\mathrm{e}^{-r\tau^i} l^i + 1_{\{\tau^i > \tau^j\}}\,\mathrm{e}^{-r\tau_j} V_m^i(X_{\tau^j})\right],$$

where V_m^i is firm i's value function as a monopolist,

$$V_m^i(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\left[\int_0^{\tau} \mathrm{e}^{-rt} m X_t\,\mathrm{d}t + \mathrm{e}^{-r\tau} l^i\right].$$

Letting $E(x) \equiv \mathbf{E}_x\left[\int_0^\infty \mathrm{e}^{-rt} X_t\,\mathrm{d}t\right] = \frac{x}{r-b}$, $R^i \equiv l^i - E$, and $G^i \equiv V_m^i - E$, we obtain the expression (2) for $J^i(\cdot, \tau^1, \tau^2) \equiv F^i(\cdot, \tau^1, \tau^2) - E$. Standard computations (see, for instance, Dixit and Pindyck (1994)) yield

$$x_{R^i} = \frac{\rho^-}{\rho^- - 1}\,(r - b) l^i \quad \text{and} \quad \alpha^i = \frac{x_{R^i}}{m},$$

where

$$\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Notice that $G^i(x) = R^i(x) = l^i - E(x)$ for all $x \in (\alpha, \alpha^i]$. It is easy to check that this specification satisfies A1–A7. We will use it in Section 4 to illustrate our results.

3 Mixed Strategies and Equilibrium Concept

Our key methodological contribution is to allow players to play randomized stopping times. We first recall the definition and basic properties of randomized stopping times. Imposing a Markov restriction leads to our first main result, which is a characterization theorem for Markovian randomized stopping times. We then define the concept of Markov-perfect equilibrium and give some important properties of best replies.

3.1 Randomized Stopping Times

One classical definition of a randomized stopping time consists, following Aumann (1964), in enlarging the probability space; this compensates for the absence of a natural measurable structure over the space of stopping times. For every player $i = 1, 2$, the corresponding enlarged probability space is $(\Omega^i, \mathcal{F}^i) \equiv (\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]))$, endowed with the product probability $\mathbf{P}^i_x \equiv \mathbf{P}_x \otimes Leb$, where $\mathcal{B}([0,1])$ is the Borel σ-field over $[0,1]$ and Leb is Lebesgue measure. We borrow the following definition from Touzi and Vieille (2002).

Definition 1 *A* randomized stopping time *for player $i = 1, 2$ is a $\mathcal{F} \otimes \mathcal{B}([0,1])$-measurable function $\gamma^i : \Omega^i \to \mathbb{R}_+$ such that, for Leb-almost every $u^i \in [0,1]$, $\gamma^i(\cdot, u^i) \in \mathcal{T}$. The process $\Gamma^i \equiv (\Gamma^i_t)_{t \geq 0}$ defined by*

$$\Gamma^i_t(\omega) \equiv \int_{[0,1]} 1_{\{\gamma^i(\omega,u^i) \leq t\}} \, \mathrm{d}u^i, \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{10}$$

is the conditional cumulative distribution function *(ccdf) of the randomized stopping time γ^i. The process $\Lambda^i \equiv (\Lambda^i_t)_{t \geq 0}$ defined by*

$$\Lambda^i_t(\omega) \equiv 1 - \Gamma^i_t(\omega), \quad (\omega, t) \in \Omega \times \mathbb{R}_+, \tag{11}$$

is the conditional survival function *(csf) of the randomized stopping time γ^i.*

It is immediate that the ccdf process Γ^i defined by (10) takes values in $[0,1]$ and has nondecreasing and right-continuous trajectories. The following lemma shows that the process Γ^i is adapted and provides a useful representation.

Lemma 2 *The ccdf process Γ^i is $(\mathcal{F}_t)_{t \geq 0}$-adapted and, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Gamma^i_t(\omega) = \mathbf{P}^i_x[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega) \tag{12}$$

for $\mathbf{P}_x$-almost every $\omega \in \Omega$.

By convention, we let $\Gamma^i_{0-} \equiv 0$. This allows us in what follows to interpret integrals of the form $\int_{[0,\tau)} \cdot \, \mathrm{d}\Gamma^i_t$ in the Stieltjes sense for any ccdf Γ^i.

If the players use randomized stopping times γ^1 and γ^2, then their expected payoffs are defined over the product probability space $\Omega \times [0,1] \times [0,1]$ with canonical element (ω, u^1, u^2), endowed with the product probability $\overline{\mathbf{P}}_x \equiv \mathbf{P}_x \otimes Leb \otimes Leb$. Specifically, we have

$$J^i(x, \gamma^1, \gamma^2) \equiv \overline{\mathbf{E}}_x \Big[1_{\{\gamma^i \leq \gamma^j\}} \, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i}) + 1_{\{\gamma^i > \gamma^j\}} \, \mathrm{e}^{-r\gamma^j} G^i(X_{\gamma^j}) \Big], \tag{13}$$

where $\gamma^1 \equiv \gamma^1(\omega, u^1)$ and $\gamma^2 \equiv \gamma^2(\omega, u^2)$, reflecting that players 1 and 2 use the independent randomization devices u^1 and u^2, respectively.

The following lemma, which is somewhat standard in the literature (Touzi and Vieille (2002), Riedel and Steg (2017)), shows that we may equivalently work with the family of ccdf processes Γ^i.

Lemma 3 *If the players use randomized stopping times with ccdfs* Γ^1 *and* Γ^2*, then their expected payoffs write as*

$$J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\right]. \tag{14}$$

Moreover, any nondecreasing, right-continuous, $\mathcal{F}_t$*-adapted,* $[0,1]$*-valued process* Γ^i *is the ccdf of the randomized stopping time* $\hat{\gamma}^i$ *defined by*

$$\hat{\gamma}^i(u^i) \equiv \inf\{t \geq 0 : \Gamma^i_t > u^i\}. \tag{15}$$

3.2 Markovian Randomized Stopping Times

Our goal in this paper is to characterize equilibria in which players concede according to mixed Markov strategies that only depend on current market conditions. Notice that such strategies have to be defined for any initial market conditions $x \in \mathcal{I}$. We will need the following standard definition (Revuz and Yor (1999, Chapter I, §3)).

Definition 2 *Let* $Y \equiv (Y_t)_{t\geq 0}$ *be the coordinate process over the canonical space* Ω*, defined by* $Y_t(\omega) \equiv \omega_t$ *for all* $\omega \in \Omega$ *and* $t \geq 0$*. Then, for each* $t \geq 0$*, the shift operator* $\theta_t : \Omega \to \Omega$ *is defined by* $Y_s \circ \theta_t \equiv Y_{s+t}$ *for all* $s \geq 0$*.*

In words, the effect of θ_t on a trajectory ω is to forget the part of the trajectory prior to time t and to shift back the remaining part by t units of time. We are now ready to define our notion of a Markovian randomized stopping time.

Definition 3 *A randomized stopping time for player* $i = 1, 2$ *with csf* $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0, 1]$ *is* Markovian *if, for all* $x \in \mathcal{I}$*,* $\tau \in \mathcal{T}$*, and* $s \geq 0$*,*

$$\tau(\omega) < \infty \quad \textit{implies} \quad \Lambda^i_{\tau(\omega)+s}(\omega) = \Lambda^i_{\tau(\omega)}(\omega)\Lambda^i_s(\theta_{\tau(\omega)}(\omega)) \tag{16}$$

for $\mathbf{P}_x$*-almost every* $\omega \in \Omega$*, or, more compactly,* $\Lambda^i_{\tau+s} = \Lambda^i_\tau(\Lambda^i_s \circ \theta_\tau)$ *over the event* $\{\tau < \infty\}$*.*

Definition 3 can be intuitively understood as follows. According to Definition 1 and Lemma 2, $\Lambda^i_{\tau+s}$ is the probability that player i concedes after time $\tau + s$ conditionally on $\mathcal{F}_{\tau+s}$. The Markov restriction then states that, at time τ, and conditionally on the fact that player i did not concede by then, the probability that he holds fast for at least s additional units of time should not depend on the trajectory of X prior to time τ. This probability is thus given by $\Lambda^i_s \circ \theta_\tau$, that is, the probability induced by the randomized strategy applied to the shifted trajectory. Formula (16) then follows from the standard formula for conditional probabilities.

Processes satisfying (16) are known as multiplicative functionals of the Markov process X and are studied in the literature on general Markov processes (Blumenthal and Getoor

(1968)). Combining a result by Sharpe (1971) with the classical representation result of additive functionals of regular diffusions (Borodin and Salminen (2002, Part I, Chapter II, Section 4, §23)), we can deduce the following characterization result for Markovian randomized stopping times.

Theorem 1 *For each $i = 1, 2$, $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0, 1]$ is the csf of a Markovian randomized stopping time for player i if and only if there exists a closed set $S^i \subset \mathcal{I}$ and a Radon measure*[4] *μ^i over $\mathcal{I} \setminus S^i$ such that, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Lambda_t^i(\omega) = 1_{\{t<\tau_{S^i}(\omega)\}} \, \mathrm{e}^{-\int_{\mathcal{I}\setminus S^i} L_t^y(\omega)\, \mu^i(\mathrm{d}y)} \tag{17}$$

for $\mathbf{P}_x$-almost every $\omega \in \Omega$, where

$$L_t^y \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(y-\varepsilon, y+\varepsilon)}(X_s) \sigma^2(X_s) \, \mathrm{d}s \tag{18}$$

is the local time of X at (y, t), and

$$\tau_{S^i} \equiv \inf \{t \geq 0 : X_t \in S^i\}$$

is the hitting time by X of S^i. In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is continuous over $[0, \tau_{S^i}(\omega))$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$.

The interpretation of (17) is that player i concedes with probability 1 over S^i, and with positive but finite intensity over $\operatorname{supp} \mu^i$. The relation (17) allows us in what follows to indifferently refer to a Markov strategy for player i as a ccdf Γ^i, a csf Λ^i, or a pair (μ^i, S^i); we shall use these notations interchangeably in the definition of players' payoffs. Three special cases of the representation (17) are worth mentioning.

The Pure Stopping Case If $\mu^i \equiv 0$, then the Markov strategy $(0, S^i)$ is just the pure stopping time τ_{S^i}. This is the class of Markov strategies considered by Murto (2004).

The Absolutely Continuous Case If $\mu^i \equiv g^i \cdot Leb$ is absolutely continuous with density g^i with respect to Lebesgue measure, then, using the occupation time formula (Revuz and Yor (1999, Chapter VI, §1, Corollary 1.6)), the corresponding csf writes as

$$\Lambda_t^i = 1_{\{t<\tau_{S^i}\}} \, \mathrm{e}^{-\int_{\mathcal{I}} L_t^y g^i(y) \, \mathrm{d}y} = 1_{\{t<\tau_{S^i}\}} \, \mathrm{e}^{-\int_0^t g^i(X_s) \sigma^2(X_s) \, \mathrm{d}s}. \tag{19}$$

Outside S^i, this strategy consists for player i in conceding according to a Poisson process with stochastic intensity $\lambda^i(X_t) \equiv g^i(X_t)\sigma^2(X_t)$; that is, during a short time interval $[t, t + \mathrm{d}t)$, he concedes with probability 1 if $X_t \in S^i$ and with probability $\lambda^i(X_t)\, \mathrm{d}t$ otherwise. This is the class of Markov strategies considered by Steg (2015) and Giorgiadis, Kim, and Kwon (2022).

[4] Recall that a Radon measure over an open set $U \subset \mathbb{R}$ is a nonnegative Borel measure that is locally finite in the sense that every point of U has a neighborhood having finite μ-measure.

The Singular Case If $\mu^i \equiv a^i \delta_{x^i}$, where $a^i > 0$ and δ_{x^i} is the Dirac mass at $x^i \in \mathcal{I} \setminus S$, then the corresponding csf writes as

$$\Lambda_t^i = 1_{\{t<\tau_{S^i}\}} \,\mathrm{e}^{-a^i L_t^{x^i}}. \tag{20}$$

In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is singular over $[0, \tau_{S^i}(\omega))$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$ such that the trajectory of X crosses x^i; that is, its derivative is zero for *Leb*-almost every $t \in [0, \tau_{S^i}(\omega))$, though $\Lambda_t^i(\omega)$ is not constant as it decreases each time X crosses x^i. To the best of our knowledge, Markov strategies with singular csfs have not been considered in the literature. Yet there is no reason to discard such strategies, as they naturally emerge as limits of more familiar ones. Here are two illustrations:

(i) First, discretize the state space (and possibly the time space) and consider Markov strategies for player i prescribing him to concede with positive intensity when the current state is x^i. Then, with appropriate normalizations, the natural limit of such strategies when the mesh of the discretization goes to 0 corresponds to a distribution with hazard rate proportional to the local time of the diffusion at x^i.

(ii) Second, consider the Markov strategy that, outside S^i, consists for player i in conceding according to a Poisson process with stochastic intensity $\lambda_\varepsilon^i(X_t) \equiv \frac{a^i}{2\varepsilon}\,\sigma^2(X_t)\,1_{(x^i-\varepsilon, x^i+\varepsilon)}$ for $a^i > 0$ and some small $\varepsilon > 0$. By (19), the corresponding csf writes as

$$\Lambda_{\varepsilon,t}^i = 1_{\{t<\tau_{S^i}\}} \,\mathrm{e}^{-\int_0^t \lambda_\varepsilon^i(X_s)\,\mathrm{d}s}.$$

From the definition (18) of the local time $L_t^{x^i}$ of X at (x^i, t), we deduce that, for each $t \geq 0$, $\Lambda_{\varepsilon,t}^i$ converges $\mathbf{P}_x$-almost surely to Λ_t^i in (20) as ε goes to 0.

Let us finally mention an important property of a Markov strategy, such as (20), associated to a singular intensity measure with an atom at x^i. Using the properties of the local time, one can check[5] that the total probability of conceding before time $\mathrm{d}t$ starting from x^i is of order $\sqrt{\mathrm{d}t}$, whereas the same quantity is of order $\mathrm{d}t$ for a Markov strategy, such as (19), associated to an absolutely continuous intensity measure. As we will see in Sections 4–5, this particular singular behavior will create points of nondifferentiability in the players' equilibrium value functions.

3.3 Markov-Perfect Equilibrium and Properties of Best Replies

We are now ready to define our equilibrium concept and to provide some basic properties of best replies. Our first result, which we will repeatedly use in what follows, illustrates the standard fact that a player, given the behavior of his opponent, cannot improve his payoff merely by randomizing over pure strategies.

[5] This may be done for example by adapting the method used in Peskir (2019, Lemma 15).

Lemma 4 *For each $x \in \mathcal{I}$ and for any pair of randomized stopping times with ccdfs (Γ^1, Γ^2),*

$$J^1(x, \Gamma^1, \Gamma^2) \leq \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, \Gamma^2),$$
$$J^2(x, \Gamma^1, \Gamma^2) \leq \sup_{\tau^2 \in \mathcal{T}} J^2(x, \Gamma^1, \tau^2).$$

This motivates the following definition.

Definition 4 *A* Markov-Perfect Equilibrium *(MPE) is a profile of Markov strategies $((\mu^1, S^1), (\mu^2, S^2))$ such that, for each $x \in \mathcal{I}$,*

$$J^1(x, (\mu^1, S^1), (\mu^2, S^2)) = \bar{J}^1(x, (\mu^2, S^2)) \equiv \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, (\mu^2, S^2)),$$
$$J^2(x, (\mu^1, S^1), (\mu^2, S^2)) = \bar{J}^2(x, (\mu^1, S^1)) \equiv \sup_{\tau^2 \in \mathcal{T}} J^2(x, (\mu^1, S^1), \tau^2).$$

That is, for each $i = 1, 2$, (μ^i, S^i) is a perfect best reply *(pbr) for player i to (μ^j, S^j), and $\bar{J}^i(\cdot, (\mu^j, S^j))$ is player i's* best-reply value function *(brvf) to (μ^j, S^j).*

When no confusion can arise as to the strategy of player j, we write $\bar{J}^i$ instead of $\bar{J}^i(\cdot, (\mu^j, S^j))$. The next proposition provides useful general properties of pbr and brvf, and is key to establish our main results.

Proposition 1 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf $\bar{J}^i$, then $V_{R^i} \leq \bar{J}^i \leq G^i$. Furthermore,*

(i) $S^1 \cap S^2 \cap (\alpha^i, \beta) = \emptyset$;

(ii) $S^i \subset C^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = R^i(x)\}$;

(iii) $\operatorname{supp} \mu^i \setminus S^j \subset C^i$ *and* $\operatorname{supp} \mu^i \cap S^j \subset D^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = G^i(x)\}$;

(iv) $S^i \cup (\operatorname{supp} \mu^i \setminus S^j) \subset (\alpha, x_{R^i}]$;

(v) $(0, S^i)$ *is also a pbr to* (μ^j, S^j); *more generally,* $(\tilde{\mu}^i, S^i)$ *is a pbr to* (μ^j, S^j) *for any* $\tilde{\mu}^i$ *such that* $\operatorname{supp} \tilde{\mu}^i \subset C^i \cup S^j$.

Property (i) intuitively states that player i should never concede when market conditions x are such that player j concedes with probability 1 and player i's payoff from conceding is strictly less than the payoff from letting player j concede, that is, $x \in S^j$ and $G^i(x) > V_{R^i}(x)$. Property (ii) simply expresses the fact that player i's brvf coincides with R^i over the portion S^i of the state space over which he concedes with probability 1. Property (iii) states that player i's payoff is R^i when he concedes with positive intensity outside of player j's stopping region S^j. Property (iv) reflects that player i should never concede when market conditions are above the optimal threshold x_{R^i} for his stand-alone exit problem; intuitively, this is because waiting for X to drop down to x_{R^i} before conceding is player i's optimal strategy even in the worst-case scenario in which player j is stubborn, that is, $(\mu^j, S^j) = (0, \emptyset)$. Finally, property (v) states that, when conceding with positive intensity outside of S^i, player i should be indifferent between holding fast and conceding.

Remark Some authors (see, for instance, Murto (2004)) include, as a refinement in the definition of an MPE, the requirement that $(\alpha, \alpha^i] \subset S^i$ for all i. The rationale for this assumption is that, because $G^i = V_{R^i} = R^i$ over $(\alpha, \alpha^i]$, holding fast further below α^i would be weakly dominated for player i by conceding with probability 1 over this interval. For instance, being stubborn is a best reply for player i over (α, α^i) only if player j concedes with probability 1 over this interval, except perhaps over a set of Lebesgue measure 0. This behavior is not per se inconsistent with an MPE, but it is not consistent with trembling-hand perfection in the spirit of Selten (1975), see Ghemawat and Nalebuff (1985) for a discussion of a similar point in a deterministic model. Hereafter, we do not systematically impose this refinement, especially in Section 5 where this allows to simplify notation; however, we will indicate which MPEs can be modified so as to satisfy it.

We close this section with an important global regularity result.

Proposition 2 *If $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE, then, for each $i = 1, 2$, player i's brvf $\bar{J}^i$ is continuous over $\mathcal{I}$.*

4 Heuristic Derivation and Testable Implications of a Singular Mixed-Strategy MPE

We first recall within our general framework two standard MPEs, respectively in pure and mixed strategies, that have been emphasized in the literature. Based on these examples and on our characterization theorem for Markovian randomized stopping times, we next describe a novel type of MPE involving a singular strategy for one of the two players. Our heuristic derivation leads to a variational system that turns out to fully characterize this candidate MPE. We provide sufficient conditions for the existence of a solution to this variational system in the context of the running example described in Section 2.3. We finally compare the resulting singular mixed-strategy MPE with the two above standard MPEs and discuss its asset-pricing implications.

4.1 A Pure-Strategy MPE

We say that player 1 is *as least as enduring* as player 2 if $\alpha^1 \leq \alpha^2$ and $x_{R^1} \leq x_{R^2}$; intuitively, player 1 is at least as willing to hold fast as player 2. Suppose then that player 1 threatens to hold fast maximally and concede only at $\tau^1 = \inf\{t \geq 0 : X_t \leq \alpha^1\}$. Then, because $\alpha^1 \leq \alpha^2$, we have $G^2(X_{\tau^1}) = R^2(X_{\tau^1})$ by definition of α^2. In light of (2)–(3), this implies that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$,

$$J^2(x, \tau^1, \tau^2) = \mathbf{E}_x\left[\mathrm{e}^{-r\tau^1 \wedge \tau^2} R^2(X_{\tau^1 \wedge \tau^2})\right] \leq V_{R^2}(x).$$

Thus a pbr for player 2 to τ^1 is to concede at $\tau^2 = \inf\{t \geq 0 : X_t \leq x_{R^2}\}$. As for player 1, if player 2 concedes at τ^2, then, for each $x \in \mathcal{I}$,

$$\mathbf{E}_x\left[\mathrm{e}^{-r\tau^2} G^1(X_{\tau^2})\right] \geq R^1(x).$$

For $x \leq x_{R^2}$, this follows from the fact that $G^1(x) \geq R^1(x)$ by A6, with a strict inequality if $x > \alpha^1$. For $x > x_{R^2}$, this follows from A6 again along with the fact that the process $(\mathrm{e}^{-rt} V_{R^1}(X_t))_{t \geq 0}$ is a martingale up to $\tau_{x_{R^1}}$, the hitting time by X of x_{R^1}, which is no less than τ^2 because $x_{R^1} \leq x_{R^2}$ by assumption. Thus a pbr for player 1 to τ^2 is to concede at τ^1. This implies the following result, which has many counterparts in the literature (see, for instance, Ghemawat and Nalebuff (1985), Décamps and Mariotti (2004), Murto (2004), Giorgiadis, Kim, and Kwon (2022)).

Proposition 3 *If player 1 is at least as enduring as player 2, then* $((0, (\alpha, \alpha^1]), (0, (\alpha, x_{R^2}]))$ *is a pure-strategy MPE.*

In the case of a small asymmetry between the players, $((0, (\alpha, x_{R^1}]), (0, \emptyset))$ is also an MPE in which the more enduring player 1 follows his stand-alone optimal strategy because the less enduring player 2 is stubborn (Giorgiadis, Kim, and Kwon (2022)). However, this MPE does not satisfy Murto's (2004) trembling-hand-perfection refinement, because, for $x \in (\alpha^1, \alpha^2)$, player 2's strategy is no longer a best response when player 1 does not concede with probability 1 in any small enough neighborhood of x. Nevertheless, Murto (2004) shows that, when we allow player 1's stopping set S^1 to exhibit a gap, there may exist an MPE satisfying this refinement in which, when $x > x_{R^1}$, player 1 exits first when X reaches x_{R^1}.

4.2 A Regular Mixed-Strategy MPE in the Symmetric Case

Suppose now that players are symmetric, in the weak sense that they are as enduring as each other, $\alpha^1 = \alpha^2 \equiv \alpha^*$ and $x_{R^1} = x_{R^2} \equiv x^*$. This is of course the case when the players have identical payoff functions, $R^1 = R^2$ and $G^1 = G^2$. The following result, which restates in our framework earlier results in the literature (Steg (2015), Georgiadis, Kim, and Kwon (2022))[6],

[6] A related construction also appears in Kwon and Palczewski (2022). There, a symmetric Bayesian equilibrium is constructed in a model with asymmetric information and a continuum of types. The pure strategies, seen as randomized strategies assimilating the types as randomization devices, use absolutely continuous intensities depending on X and on an auxiliary belief process.

characterizes a regular mixed-strategy MPE in which the players concede with absolutely continuous intensities over the interval $(\alpha^*, x^*]$.

Proposition 4 *If the players are as enduring as each other, then the strategy profile*

$$((\lambda^1(x)\sigma^{-2}(x)\,\mathrm{d}x, (\alpha, \alpha^*]), (\lambda^2(x)\sigma^{-2}(x)\,\mathrm{d}x, (\alpha, \alpha^*]))$$

defined, for each $i = 1, 2$, *by*

$$\lambda^i(x) \equiv \frac{rR^j(x) - \mathcal{L}R^j(x)}{G^j(x) - R^j(x)}\, 1_{\{\alpha^* < x \leq x^*\}}, \tag{21}$$

is a mixed-strategy MPE.

Following Steg (2015, Theorem 5.1), the MPE constructed in Proposition 4 is such that each player exits the market with an intensity function λ^i with support $(\alpha^*, x^*]$. This intensity is constructed so that, at each point of this interval, each player is indifferent between holding fast and conceding; for instance, in our running example, each firm over this interval obtains its liquidation value and hence a flat payoff. In equilibrium, the value function of each player coincides with the value function of his stand-alone exit problem (3). Thus, in expectation, the war of attrition yields no benefit to either player.

4.3 A Singular Mixed-Strategy MPE

When there is no uncertainty about future payoffs, the war of attrition with symmetric or asymmetric players admits mixed-strategy equilibria in which players' strategies are described, over an interval of exit times, by absolutely continuous distributions (see, for instance, Hendricks, Weiss, and Wilson (1988)); this is notably the case in the limiting case of our model where $\mu = \sigma \equiv 0$, so that market conditions are constant. However, this result has no counterpart under Brownian uncertainty when players are asymmetric. Specifically, Georgiadis, Kim, and Kwon (2022) have shown that, when players are not as enduring as each other, there exists no mixed-strategy MPE in which the players concede with absolutely continuous intensities. For all that, it would be incorrect to conclude that only pure-strategy MPEs exist, and thus that attrition cannot take place when players are asymmetric. This section argues for this claim by describing an MPE involving a singular strategy for one of the two players. For the sake of simplicity, the analysis below remains at a heuristical level. A full justification of our arguments is provided in Section 5.

From now on, assume as in Section 4.1 that player 1 is at least as enduring as player 2, that is, $\alpha^1 \leq \alpha^2$ and $x_{R^1} \leq x_{R^2}$. This covers the case of asymmetric players, as well as the limiting case of symmetric players. Consider then the following equation in x:

$$R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)}\, G^1(x). \tag{22}$$

Lemma S.5 in the Online Supplement shows that (22) admits a unique solution $\underline{x}^2 \in (\alpha^1, x_{R^1})$. In words, the threshold $\underline{x}^2$ is such that, if player 2 threatens to concede only at $\tau^2 = \inf\{t \geq 0 : X_t \leq \underline{x}^2\}$, then, at x_{R^1}, player 1 is indifferent between conceding and obtaining $R^1(x_{R^1})$ immediately and waiting for player 2 to concede at τ^2 and obtaining $G^1(\underline{x}^2)$ only then. Our goal is to construct an MPE in which player 1 randomizes between holding fast and conceding at x_{R^1} and player 2 concedes only at τ^2. Using the characterization of randomized stopping times provided in Theorem 1, this amounts to finding a constant $a^1 > 0$ such that the strategy profile $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1]), (0, (\alpha, \underline{x}^2]))$ is an MPE in which player 1 concedes, with positive but finite intensity, only at x_{R^1}.

4.3.1 Necessary Conditions

To this end, we first assume that such an MPE exists, and we derive necessary conditions for the brvf $\bar{J}^1$ and $\bar{J}^2$. An obvious preliminary observation is that $\bar{J}^1 \geq R^1$ and $\bar{J}^2 \geq R^2$, because, given current market conditions x, every player i can guarantee himself the payoff $R^i(x)$ by exiting the market immediately.

Player 1 Player 1, whose strategy involves randomization at x_{R^1}, should be indifferent at x_{R_1} between conceding and holding fast until τ^2. This implies that his brfv $\bar{J}^1$ must be $\mathcal{C}^2$ over $(\underline{x}^2, \beta)$, with $\bar{J}^1(x_{R^1}) = R^1(x_{R^1})$ (value matching). Because $\bar{J}^1 \geq R^1$, it follows in turn that $\bar{J}^{1\prime}(x_{R^1}) = R^{1\prime}(x_{R^1})$ as well (smooth pasting). Moreover, by standard dynamic-programming arguments, $\bar{J}^1$ must satisfy the ODE $\mathcal{L}\bar{J}_1 - r\bar{J}_1 = 0$ over $(\underline{x}^2, \beta)$ (see, for instance, Dixit and Pindyck (1994)). This leads to

$$\bar{J}^1(x) = \frac{\phi(x)}{\phi(x_{R^1})}\, R^1(x_{R^1}), \quad x \in (\underline{x}^2, \beta). \tag{23}$$

In particular, $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$: player 1 does not benefit from the war of attrition over $[x_{R^1}, \beta)$. By contrast, $\bar{J}^1 > V_{R^1}$ over $[\underline{x}^2, x_{R^1})$, reflecting that player 1 can hope that player 2 may concede at $\underline{x}^2$ before he himself concedes at x_{R^1}.

Player 2 Player 2 plays a pure strategy and hopes to benefit from player 1 conceding at x_{R^1}. We guess that $\bar{J}^2$ is $\mathcal{C}^2$ over $(\underline{x}^2, \beta) \setminus \{x_{R^1}\}$, with $\bar{J}^2(\underline{x}^2) = R^2(\underline{x}^2)$ (value-matching) and $\bar{J}^{2\prime}(\underline{x}^2) = R^{2\prime}(\underline{x}^2)$ (smooth pasting), and that it satisfies the ODE $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over that region. There remains to characterize the behavior of $\bar{J}^2$ at x_{R^1}. Because player 1 randomizes at x_{R^1} between holding fast and conceding, we expect that $G^2(x_{R^1}) > \bar{J}^2(x_{R^1}) > R^2(x_{R^1})$. This, along with the properties of the local time highlighted in Section 3.2, implies that $\bar{J}^2$ is not differentiable at x_{R^1}. Indeed, starting from x_{R^1}, player 1 concedes in a small time interval of length $\mathrm{d}t$ with probability $\mathbf{E}_{x_{R^1}}[\Gamma_{\mathrm{d}t}] = a^1 c\sqrt{\mathrm{d}t} + o(\sqrt{\mathrm{d}t})$, where $\Gamma_{\mathrm{d}t} = 1 - \mathrm{e}^{-a^1 L^{x_{R^1}}_{\mathrm{d}t}}$ and c is a positive constant. If player 1 concedes, then player 2 benefits from the follower payoff $G^2(x_{R^1})$, while if player 1 holds fast, then player 2 achieves the value $\bar{J}^2(X_{\mathrm{dt}})$. Thus we have

$$\bar{J}^2(x_{R^1}) = a^1 c\sqrt{\mathrm{dt}}\, G^2(x_{R^1}) + (1 - a^1 c\sqrt{\mathrm{dt}})\, \mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}} \bar{J}^2(X_{\mathrm{dt}})] + o(\sqrt{\mathrm{dt}}). \tag{24}$$

Now, suppose, by way of contradiction, that J^2 is $\mathcal{C}^2$ in a neighborhood of x_{R^1}. Then, from Itô's formula,

$$\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}} \bar{J}^2(X_{\mathrm{dt}})] = \bar{J}^2(x_{R^1}) + (\mathcal{L}\bar{J}^2 - r\bar{J}^2)(x_{R^1})\, \mathrm{dt} + o(\mathrm{dt}). \tag{25}$$

Plugging (25) into (24) yields $a^1 c[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})]\sqrt{dt} + o(\sqrt{\mathrm{dt}}) = 0$, a contradiction as $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$ and a^1 and c are positive constants. This is an indication that $\bar{J}^2$ is not differentiable at x_{R^1}; let us denote by $\Delta\bar{J}^{2\prime}(x_{R^1}) \equiv \bar{J}^{2\prime+}(x_{R^1}) - \bar{J}^{2\prime-}(x_{R^1})$ the corresponding derivative jump. From the Itô–Tanaka–Meyer formula, which generalizes Itô's formula to functions, such as $\bar{J}^2$, that can be written as the difference of two convex functions (Karatzas and Shreve (1991, Theorem 3.7.1 and Problem 3.6.24)), we have

$$\begin{aligned}
\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}} \bar{J}^2(X_{\mathrm{dt}})] &= \bar{J}^2(x_{R^1}) + \mathbf{E}_{x_{R^1}}\bigg[\int_0^{\mathrm{dt}} \mathrm{e}^{-rs}(\mathcal{L}\bar{J}^2 - r\bar{J}^2)(X_s)\, \mathrm{d}s \\
&\qquad + \int_0^{\mathrm{dt}} \mathrm{e}^{-rs}\bar{J}^{2\prime-}(X_s)\sigma(X_s)\, \mathrm{d}W_s + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1}) L_{\mathrm{dt}}^{x_{R^1}}\bigg] \\
&= \bar{J}^2(x_{R^1}) + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1}) c\sqrt{dt} + o(\sqrt{\mathrm{dt}}),
\end{aligned} \tag{26}$$

where the second equality follows from the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over $(\underline{x}^2, \beta) \setminus \{x_{R^1}\}$ and from the properties of local time. Plugging (26) into (24) yields

$$a^1[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})] + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1}) = 0. \tag{27}$$

Notice from $G^2(x_{R^1}) > J^2(x_{R^1})$ and (27) that $\Delta\bar{J}^{2\prime}(x_{R^1}) < 0$. Intuitively, player 2 gets more and more optimistic as X approaches x_{R^1}, but is disappointed if X crosses x_{R^1} yet player 1 does not concede at x_{R^1}.

The Variational System Our discussion so far leads to the following variational system: find a constant $a^1 > 0$, and two functions $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2\})$ and $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2, x_{R^1}\})$ such that

$$w^1 \geq R^1 \text{ over } \mathcal{I}, \tag{28}$$

$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (\underline{x}^2, \beta), \tag{29}$$

$$w^1 = G^1 \text{ over } (\alpha, \underline{x}^2], \tag{30}$$

$$w^1(x_{R^1}) = R^1(x_{R^1}), \tag{31}$$

$$w^1(\beta-) = 0, \tag{32}$$

$$w^2 \geq R^2 \text{ over } \mathcal{I}, \tag{33}$$

$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (\underline{x}^2, \beta) \setminus \{x_{R^1}\}, \tag{34}$$

$$w^2 = R^2 \text{ over } (\alpha, \underline{x}^2], \tag{35}$$

$$w^{2\prime}(\underline{x}^2) = R^{2\prime}(\underline{x}^2), \tag{36}$$

$$a^1[G^2(x_{R^1}) - w^2(x_{R^1})] + \frac{1}{2}\,\Delta w^{2\prime}(x_{R^1}) = 0, \tag{37}$$

$$w^2(\beta-) = 0. \tag{38}$$

4.3.2 Sufficient Conditions

It is an implication of our main characterization result, Theorem 3, that, if $(a^1, \bar{J}^1, \bar{J}^2)$ is a solution to the variational system (28)–(38), then $\bar{J}^1$ is the brfv to $(0, (\alpha, \underline{x}^2])$ and w^2 is the brfv to $(a^1\delta_{x_{R^1}}, (\alpha, \alpha^1])$, so that, according to the construction in Section 4.3.1, $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1]), (0, (\alpha, \underline{x}^2])$ is an MPE. As for $\bar{J}^1$, we have already seen that (28)–(29) and (31) pin down a unique solution, given by (23), which satisfies $\bar{J}^1(\underline{x}^2) = G^1(\underline{x}^2)$ by definition of $\underline{x}^2$. As for $\bar{J}^2$, the analysis is a bit more delicate due to the presence of the derivative jump $\Delta\bar{J}^{2\prime}(x_{R^1})$ at x_{R^1}, which, by (37), is pinned down by the intensity a^1 with which player 1 exits at x_{R^1}. In our running example, it can be shown that, as long as the asymmetry between the players is small or nonexistent, and provided that $b > 0$ and that m is sufficiently large, one can indeed find a positive value for a^1 such that (33)–(38) holds. The following result then holds.[7]

Proposition 5 *In the running example, if the firms' liquidation values $l^1 \leq l^2$ are close enough to each other, and if m is sufficiently large and $b > 0$, then there exists a mixed-strategy MPE $((a^1\delta_{x_{R^1}}, (\alpha, \alpha^1]), (0, (\alpha, \underline{x}^2])$ in which the more enduring firm 1 randomizes between holding fast and conceding at x_{R^1} while the less enduring firm 2 exits with probability 1 as soon as market conditions fall below $\underline{x}^2 < x_{R^1}$.*

4.3.3 Comparisons with Other MPEs

We now compare this MPE with the other types of MPEs that have been emphasized in the literature, using our running example as an illustration.

The MPE constructed in Proposition 5 differs from the pure-strategy MPE of Proposition 3 in that, for $x \geq x_{R^1}$, it is the more enduring firm 1, with the lowest liquidation value, that does not benefit from the war of attrition; indeed, we have $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \beta)$, while $\bar{J}^2 > V_{R^2}$ over this portion of the state space. The reason is that firm 2 adopts a tougher

[7] Numerical simulations suggest that, when firms' liquidation values $l^1 \leq l^2$ are close enough to each other, the variational system (28)–(38) admits a solution whatever the parameter values of the model if $b > 0$, and, if $b < 0$, as long as $m \in [1, C]$ for some constant C that increases with σ.

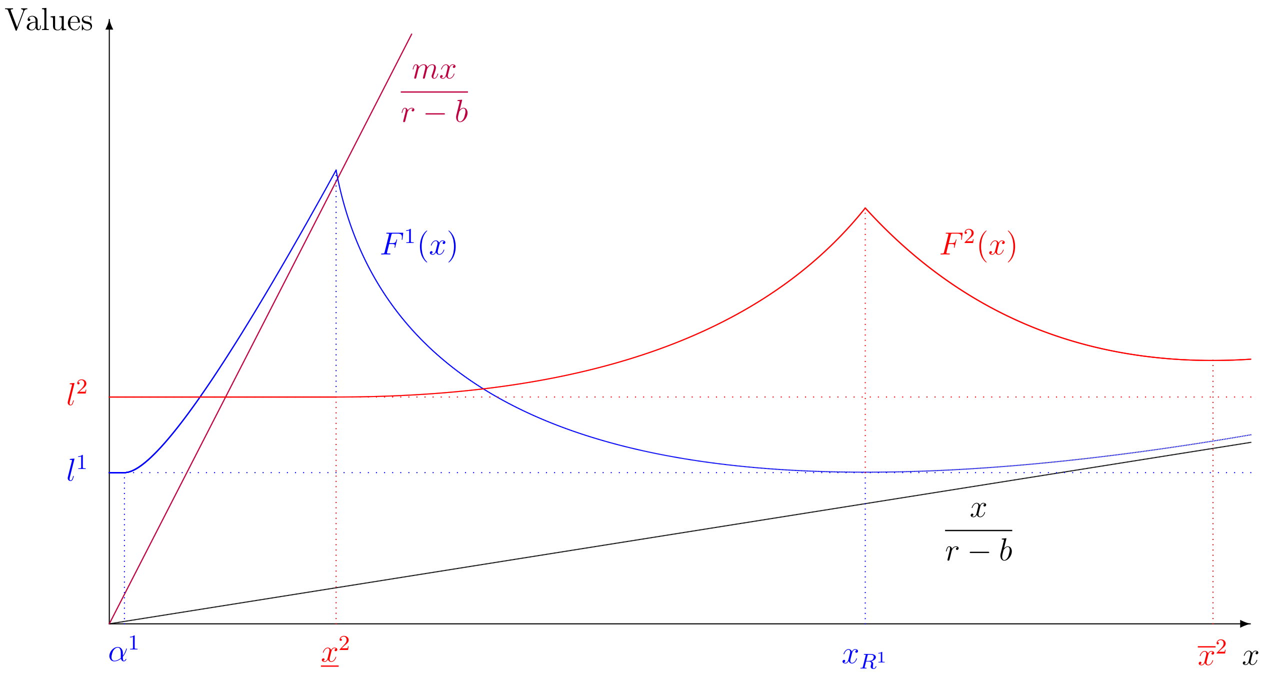

Figure 1: The value of never exiting the market in a duopoly (in black), the value of never exiting the market in a monopoly (in purple), the more enduring firm's value (in blue) and the less ensuring firm's value (in red) in the singular mixed-strategy MPE of Proposition 5.

stance by threatening to exit the market only at $\underline{x}^2 < x_{R^1} < x_{R^2}$, which makes firm 1 indifferent between holding fast and conceding at x_{R^1}. By construction, this MPE satisfies the requirement that $(\alpha, \alpha^i] \subset S^i$ for every firm i, as in Murto (2004).

It should also be noted that, in this singular mixed-strategy MPE, we have $\max S^1 \vee \max S^2 = \underline{x}^2 < x_{R^1} \leq x_{R^2}$. This contrasts with pure-strategy MPEs, in which one always have $\max S^1 \vee \max S^2 \in \{x_{R^1}, x_{R^2}\}$. Thus mixing by firm 1 delays the time at which a firm must necessarily exit the market. In particular, the difference with the pure-strategy MPE characterized by Murto (2004), in which the stopping set S^1 of firm 1 exhibits a gap below $\max S^1 = x_{R^1}$ and firm 1 is the first to exit the market at x_{R^1} when $x > x_{R^1}$, is that firm 1 does not exit with probability 1 at x_{R^1}. This leads to a richer dynamics, whereby, on the equilibrium path, every firm i can alternate between being in a dominated position (with a value close to V_{R^i}) or in a dominant position (with a value significantly above V_{R^i}); specifically, firm 1 is in a dominant position when X is close to $\underline{x}^2$, while firm 2 is in a dominant position when X is close to x_{R^1}. As we show in Theorem 2 and Corollary 1, this alternation phenomenon is a robust feature of any singular mixed-strategy MPE.

Figure 1 illustrates this point by plotting the firms' total value functions $F^i \equiv \bar{J}^i + E$ in the singular mixed-strategy MPE constructed in Proposition 5. Notice that, over $(\alpha, \underline{x}_2]$, firm 1's total value F^1 coincides with its value V^1_m as a monopolist, because firm 2 exits the market with probability 1 at any point of this interval. It can also be checked that $F^{2\prime-}(x_{R^1}) > 0 > F^{2\prime+}(x_{R^1})$, reflecting that firm 2's total value F^2 reaches a local maximum when $X_t = x_{R^1}$.

In the limiting case of symmetric firms, in which $l^1 = l^2 \equiv l$, $\alpha^1 = \alpha^2 \equiv \alpha^*$, and $x_{R^1} = x_{R^2} \equiv x^*$, it is interesting to contrast the predictions of this singular mixed-strategy MPE with those of the regular mixed-strategy MPE characterized in Proposition 4. In the latter case, there exists a whole interval $(\alpha^*, x^*]$ of the state space over which the probability of any firm exiting the market over a small time interval of length $\mathrm{d}t$ is itself of order $\mathrm{d}t$. Moreover, over $(\alpha^*, x^*]$, the two firms' total values are constant and equal to their common liquidation value l as long as no firm exits the market; thus attrition leads to a complete dissipation of rents. By contrast, in the singular mixed-strategy MPE, the probability that firm 1, starting at $x_{R^1} = x^*$, exits the market over a small time interval of length $\mathrm{d}t$ is now of higher order $\sqrt{\mathrm{d}t}$. It is also apparent from Figure 1 that the singular mixed-strategy MPE generates less dissipation of rents for any initial market condition $x > \alpha^*$ and thus Pareto-dominates the regular mixed-strategy MPE, and that the firms' total values are not monotonic in market conditions, unlike in the regular mixed-strategy MPE.

When firms are asymmetric, with $l^1 < l^2$, a regular mixed-strategy MPE does not exist (Georgiadis, Kim, and Kwon (2020)) and the appropriate benchmark is the pure-strategy MPE characterized by Murto (2004). As already noted, because firm 1 does not exit with probability 1 at x_{R^1}, firm 2's total value at x_{R^1}, $F^2(x_{R^1})$, must be less than its value as a monopolist, $V^2_m(x_{R^1})$. Because firm 1's total value satisfies $F^1 = V_{R^1} + E$ over $[x_{R^1}, \infty)$ and thus coincides with his stand-alone total value, this implies that, for any initial market condition $x > x_{R^1}$, the singular mixed-strategy MPE constructed in Proposition 5 is ex-ante Pareto dominated by any pure-strategy MPE in which player 1 concedes at x_{R^1}. Thus, even when firms have asymmetric liquidation values, wasteful attrition takes place with positive probability on the equilibrium path, in contrast with the conclusion drawn by Georgiadis, Kim, and Kwon (2020).

Finally, it should be noted that, whereas the regular mixed-strategy MPE exhibited in Proposition 4 in the limiting case of symmetric firms has no counterpart when there is even the slightest degree of asymmetry in the firms' liquidation values, the singular mixed-strategy MPE constructed in Proposition 5 also exists in the symmetric case and is robust to asymmetry.

4.3.4 Asset-Pricing Implications

We now draw the asset-pricing implications of the MPE constructed in Proposition 5.

Assets and Investors Suppose that the two firms in the running example are all-equity firms whose stocks are traded on a frictionless financial market. At any time $t \geq 0$, every firm i's stock distributes its profit to its shareholders in the form of an instantaneous payout

$X_t\,\mathrm{d}t$ as long as neither firm has conceded, and a 0 or $mX_t\,\mathrm{d}t$ payout otherwise, according to whether or not firm i is the first firm to have conceded. For simplicity, shareholders are assumed to be risk-neutral and their information set at any time t is

$$\hat{\mathcal{F}}_t \equiv \mathcal{F}_t \vee \sigma(1_{\{\gamma^1 \leq s\}}, 0 \leq s \leq t). \tag{39}$$

Thus shareholders continuously observe the evolution of market conditions and are informed of when any of the firms concedes.[8] Our goal is to characterize the dynamics of firms' stock prices up to the first time $\tau^c \equiv \gamma^1 \wedge \tau_{\underline{x}^2}$ at which one of them concedes.

Stock Prices Because shareholders are risk-neutral, every firm i's stock-price process $(V_t^{i,\tau^c})_{t\geq 0}$ stopped at τ^c is given, for each $t \geq 0$, by

$$\begin{aligned} V_t^{1,\tau^c} &\equiv F^1(X_{t\wedge\tau^c}), \\ V_t^{2,\tau^c} &\equiv F^2(X_{t\wedge\tau^c}) + [V_m^2(x_{R^1}) - F^2(x_{R^1})]1_{\{t\wedge\tau_{\underline{x}^2} \geq \gamma^1\}}, \end{aligned}$$

where the second term in the definition of V_t^{2,τ^c} reflects that, over $\{\tau_{\underline{x}^2} > \gamma^1\}$, firm 1 is the first to concede, so that firm 2's market value jumps upwards to its value as a monopolist at γ^1. Notice that there is no analogous term in the definition of V_t^{1,τ^c} for $\{\gamma^1 > \tau_{\underline{x}^2}\}$, because $F^1(\underline{x}^2) = V_m^1(\underline{x}^2)$ as $\tau_{\underline{x}^2}$, unlike γ^1, is predictable given the shareholders' information. Applying Itô's formula to F^1 and the Itô–Tanaka–Meyer formula to F^2 yields

$$V_t^{1,\tau^c} = F_1(x) + \int_0^{t\wedge\tau^c} [rF^1(X_s) - X_s]\,\mathrm{d}s + \int_0^{t\wedge\tau^c} \sigma X_s F^{1\prime}(X_s)\,\mathrm{d}W_s, \tag{40}$$

$$\begin{aligned} V_t^{2,\tau^c} &= F_2(x) + \int_0^{t\wedge\tau^c} [rF^2(X_s) - X_s]\,\mathrm{d}s + \int_0^{t\wedge\tau^c} \sigma X_s F^{2\prime -}(X_s)\,\mathrm{d}W_s \\ &\quad + [V_m^2(x_{R^1}) - F^2(x_{R^1})](1_{\{t\wedge\tau_{\underline{x}^2} \geq \gamma^1\}} - a^1 L_{t\wedge\tau^c}^{x_{R^1}}) \end{aligned} \tag{41}$$

for all $t \geq 0$. We now discuss the implications of (40)–(41).

The Martingale Property The stock-price processes $(V_t^{i,\tau^c})_{t\geq 0}$, $i = 1, 2$, share common features as each corresponds to the market value of an all-equity firm that delivers to its risk-neutral shareholders an instantaneous payout X_t per unit of time as long as neither firm concedes. In particular, for each $i = 1, 2$, the discounted cum-dividend stock-price process $(\mathrm{e}^{-rt\wedge\tau^c} V_t^{i,\tau^c} + \int_0^{t\wedge\tau^c} \mathrm{e}^{-rs} X_s\,\mathrm{d}s)_{t\geq 0}$ is a martingale with respect to the shareholders' filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$—as must be the case in the absence of arbitrage opportunities.

For firm 1, the martingale property readily follows from (40); the function F^1 is $\mathcal{C}^2$ on $(\underline{x}^2, \infty)$ and the analysis of firm 1's stock price is the same as in the corporate-finance models of Merton (1974), Leland (1994), and Goldstein, Ju, and Leland (2001), except that it is not

[8] Notice that, for each $t \geq 0$, the information that firm 2 has conceded by time t is already included in $\hat{\mathcal{F}}_t$ because $\sigma(1_{\{\tau_{\underline{x}^2} \leq s\}}, 0 \leq s \leq t) \subset \mathcal{F}_t \subset \hat{\mathcal{F}}_t$ as $\tau_{\underline{x}^2}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted.

stopped with probability 1 at $\tau_{x_{R^1}}$, resulting in a nonmonotonicity of firm 1's stock price as a function of the current payout level.

For firm 2, the martingale property is more subtle and deserves some comment. At first sight, it may seem that the presence of the nonstandard local-time term $L^{x_{R^1}}_{t\wedge\tau^c}$ in (41) creates an arbitrage opportunity. Indeed, a candidate arbitrage strategy consists in selling firm 2's stock each time $X_t = x_{R^1}$ at price $F^2(x_{R^1})$ and then repurchasing firm 2's stock at price $F^2(X_{t+\mathrm{d}t}) < F^2(x_{R^1})$ at time $t+\mathrm{d}t$; to a naive investor, this strategy seems to yield a gain of order $\mathrm{d}L^{x_{R^1}}_t$ each time $X_t = x_{R^1}$.[9] However, this does not account for the possibility that firm 1 may exit when $X_t = x_{R^1}$, causing firm 2's stock price to jump upwards to $V^2_m(x_{R^1})$. Once this risk is taken into account, the expected gain of this strategy is exactly zero, reflecting that the term $1_{\{t\wedge\tau_{\underline{x}^2}\geq\gamma^1\}} - a^1 L^{x_{R^1}}_{t\wedge\tau^c}$ in (41) is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingale.[10]

Comovements of Stock Prices and their Volatilities A testable implication of the singular mixed-strategy MPE constructed in Proposition 5 that follows immediately from Figure 1 is that, along any path of X, the firms' stock prices move in opposite directions as long as no firm exits the market and market conditions do not wander too much above x_{R^1}—that is, above the level $\overline{x}^2$ indicated on Figure 1, beyond which both F^1 and F^2 are strictly increasing. The general results of Section 5 imply that these negative comovements of firms' stock prices over the attrition zone—here, the interval $(\underline{x}^2, \overline{x}^2)$—are a robust feature of any singular mixed-strategy MPE. Figure 2 illustrates sample paths of the firms' stock prices before any firm exits the market. Negative comovements of stock prices occur in bad times, when market conditions are in the interval $(\underline{x}^2, \overline{x}^2)$ and thus current cash-flows are relatively low, while positive comovements of stock prices occur in good times, when market conditions are in the interval $(\overline{x}_2, \infty)$ and thus current cash-flows are relatively high.[11]

As predicted by (41), each time $X_t = x_{R^1}$ without firm 1 exiting the market, firm 2's stock price is continuously reflected downward by an amount $[V^2_m(x_{R^1}) - F^2(x_{R^1})]a^1\mathrm{d}L^{x_{R^1}}_{t\wedge\tau^c}$. Moreover, because $F^{2\prime-}(x_{R^1}) > 0 > F^{2\prime+}(x_{R^1})$ and F^2 is convex over $(\underline{x}^2, x_{R^1})$ and (x_{R^1}, ∞), (41) predicts that the volatility of firm 2's stock price peaks when X approaches x_{R^1}—so that firm 2's stock price approaches the reflecting boundary $F^2(x_{R^1})$—while it drops to zero when X approaches $\underline{x}^2$—so that firm 2's stock price approaches its liquidation value l^2. Notice

[9] This strategy is in the spirit of Karatzas and Shreve (1998, Appendix B) and Jarrow and Protter (2005, Theorem 4.3), who show that the presence of a singular term—such as the local time of a diffusion at a given level—in the dynamics of a cum-dividend stock prices leads to arbitrage opportunities. Of course, this is not the case for an ex-dividend stock-price process, as in dynamic security-design models (DeMarzo and Sannikov (2006), Biais, Plantin, Mariotti, and Rochet (2007)) or cash-management models (Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011)).

[10] This last property is established in Online Supplement S.4.

[11] It should be noted that this occurs despite our assumption in the running example that firms' cash-flows are perfectly correlated. Notice also that, in good times, firms' stock prices are not perfectly correlated as $F^1 \neq F^2$, though their correlation goes to 1 as market conditions go to ∞.

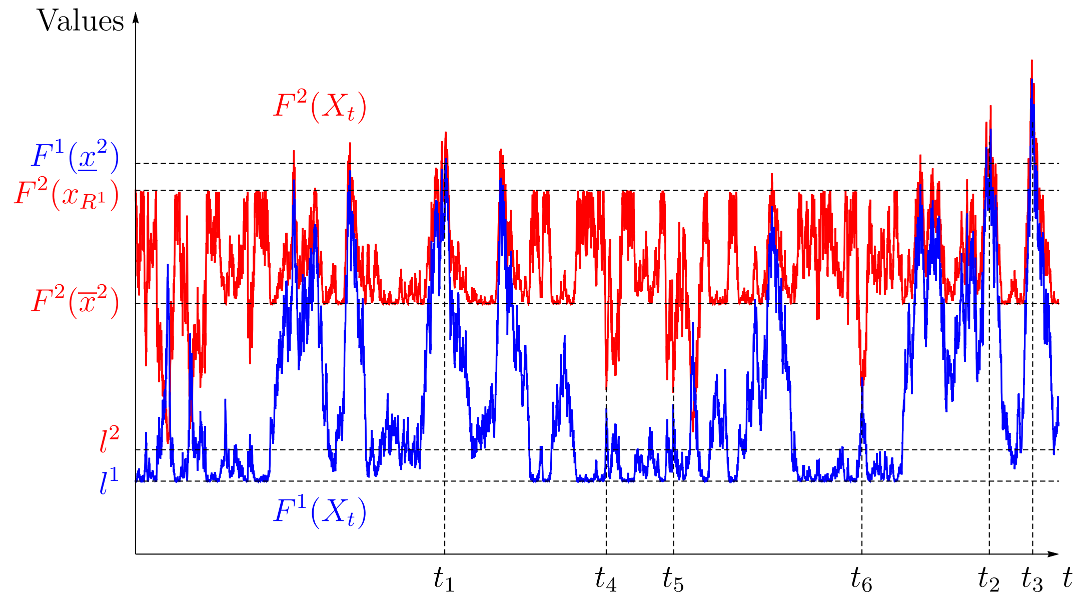

Figure 2: Sample paths of firm 1's stock price (in blue) and firm 2's stock price (in red) before any firm exits the market. The selected parameter values are $l^1 = 0.97$, $l^2 = 1$, $m = 2$, $\mu = 0.15$, $r = 0.8$, and $\sigma = 0.3$.

that, if X does wander above $\overline{x}^2$, the volatility of firm 2's stock price also drops to zero each time $X_t = \overline{x}^2$ and firm 2's stock price is locally bounded below by $F^2(\overline{x}^2)$.

Similarly, because $F^{1\prime +}(\underline{x}^2) < 0$ and F^1 is convex over $(\underline{x}^2, \infty)$, (40) predicts that the volatility of firm 1's stock price peaks when X approaches $\underline{x}^2$—so that firm 2's stock price approaches its value $V_m^1(\underline{x}^2)$ as a monopolist—and drops to zero when X approaches x_{R^1}—so that firm 1's stock price approaches its liquidation value l^1. Therefore, a testable implication of the singular mixed-strategy MPE constructed in Proposition 5 is that the volatilities of the firms' stock prices move in opposite directions as long as no firm exits the market and market conditions remain in the attrition zone $(\underline{x}^2, \overline{x}^2)$. The general results of Section 5 imply that these negative comovements of firms' stock-price volatilities over the attrition zone $(\underline{x}^2, \overline{x}^2)$ are a robust feature of any singular mixed-strategy MPE.

A Rationale for Resistance and Support Levels Technical analysts claim that they can predict financial price movements using limited information sets, including past prices (Edwards, Magee, and Bassetti (2013)). Faced with a chart such as Figure 2, a technical analyst unaware of the fundamental relationship between market conditions and stock prices would interpret $F^2(x_{R^1})$ as a predictable *resistance level* for firm 2's stock price, at which upward trends tend to be reversed.[12] Similarly, he may interpret l^1 and $F^2(\overline{x}^2)$ as predictable *support levels* for firm 1's and firm 2's stock prices.[13] Our analysis provides a rationale for

[12]Thus resistance of firm 2's stock price requires that market conditions hit x_{R^1}, see Figure 1.

[13]Thus support of firm 1's stock price requires that market conditions hit x_{R^1}, and support of firm 2's stock price requires that market conditions hit $\overline{x}^2$, see Figure 1.

these well-documented stylized facts while maintaining the assumption that stock prices are only driven by fundamentals.[14]

A *breakup* of the resistance level $F^2(x_{R^1})$ for firm 2 can in turn occur in two types of circumstances. First, the market may leave the attrition zone $(\underline{x}^2, \overline{x}^2)$, in which case a large improvement in market conditions may lead firm 2's stock price to break, in a continuous way, the resistance level $F^2(x_{R^1})$. This is for instance what happens at times t_1, t_2, and t_3 in Figure 2. Notice that these times are preceded and followed by episodes of positive comovements of stock prices, reflecting that market conditions have left the attrition zone $(\underline{x}_2, \overline{x}_2)$ following an improvement in cash-flows. Second, firm 1 may concede at x_{R^1}, causing an upward jump in firm 2's stock price from $F^2(x_{R^1})$ to its value $V^2_m(x_{R^1})$ as a monopolist. This second case is in line with the observation often made in technical analysis that, when prices rise above their resistance levels, they tend to do so decisively. In contrast with continuous breakups, such a discontinuous breakup can only occur at a relatively low level of cash-flows, and is preceded by an episode of negative comovements of stock prices. A *breakdown* of the support level $F^2(\overline{x}^2)$ can, in turn, only happen in a continuous way, and only after firm 2's stock price has reached its resistance level $F^2(x_{R^1})$. This is for instance what happens at times t_4, t_5, and t_6 in Figure 2. Of course, a breakdown of the support level l^1 for firm 1 is impossible.

Although technical analysis does not provide a consistent theoretical explanation for decisive breakups, a reason sometimes adduced is that breakups of resistance levels are triggered by large changes in the fundamentals that are above investors' expectations, such as unpredictable changes in earnings, management, or, as in our model, competition. This is exactly what happens in our model. Where we differ from technical analysis is that the downward bounces in firm 2's stock price at the resistance level $F^2(x_{R^1})$ are no more predictable from past prices than the upward jump in firm 2's stock price that occurs when firm 1 exits the market at x_{R^1}; formally, this is because these downward bounces exactly compensate for this upward jump. Thus, as pointed above, rational investors have no means to arbitrage away the profits associated to these downward bounces by short-selling firm 2's stock without incurring the risk of a sudden upward jump in firm 2's stock price.

5 Main Results

This section presents our main results, which generalize the analysis in Section 4.3. We first provide a necessary condition for mixed-strategy MPEs, establishing that any such MPE

[14]The interpretation of l^1 and $F^2(\overline{x}^2)$ as support levels for firms 1 and 2, respectively, is a little less clear-cut than that of $F^2(x_{R^1})$ as a resistance level for firm 2. Indeed, the volatilities of firm 1's and firm 2's stock prices vanish at x_{R^1} and $\overline{x}^2$, respectively, making it less likely to detect a trend reversal at l^1 and $F^2(\overline{x}^2)$ than at $F^2(x_{R^1})$, where the volatility of firm 2's stock price reaches a peak, see Figure 2.

is either singular and exhibits an alternating threshold structure, or—and only whenever $x_{R^1} = x_{R^2}$ as in Section 4.2—is regular, involving absolutely continuous intensity measures. We then characterize singular MPEs by a variational system satisfied by the two players' value functions, which provides the required ground for Proposition 5.

5.1 The Alternating Structure of Singular Mixed-Strategy MPEs

The proofs of our main results make use of an additional regularity assumption, which we maintain in the remainder of the paper.

A8 The functions b, σ, and $R^{i\prime\prime}$ are locally Lipschitz.

By convention, we let $\max \emptyset \equiv \alpha$ and, for any MPE $((\mu^1, S^1), (\mu^2, S^2))$, we let $s^i \equiv \max S^i$. The following result then holds.

Theorem 2 *For any mixed-strategy MPE* $((\mu^1, S^1), (\mu^2, S^2))$,

(i) if $x_{R^1} \neq x_{R^2}$, *then the restrictions of the intensity measures* μ^1 *and* μ^2 *to* $(s^1 \vee s^2, \beta)$ *are purely atomic*;

(ii) if $x_{R^1} = x_{R^2}$, *either the restrictions of the intensity measures* μ^1 *and* μ^2 *to* $(s^1 \vee s^2, \beta)$ *are purely atomic, or they are absolutely continuous, with densities characterized by (21) with* α^* *replaced by* $\alpha^1 \vee \alpha^2$.

Theorem 2 first confirms the basic insight of Georgiadis, Kim, and Kwon (2022), according to which there exists no mixed-strategy MPE with absolutely continuous intensity measures when $x_{R^1} \neq x_{R^2}$. Thus, if a mixed-strategy MPE exists at all in this case, it must feature intensity measures that are singular with respect to Lebesgue measure. The key information provided by Theorem 2 is that these measures must be discrete, which, for instance, rules out intensity measures with Cantor-set types of supports.

The proof can be sketched as follows.

Let us consider a mixed-strategy MPE $((\mu^1, S^1), (\mu^2, S^2))$, supposing one exists. First, Proposition 1(iv) implies $\max \operatorname{supp} \mu^i \cap (s^1 \vee s^2, \beta) \leq x_{R^i}$ for every player i; we show that this must in fact be an equality for the largest maximum of the supports. Next, Proposition 1(v) and dynamic-programming arguments imply that the brvf $\bar{J}^i$ of every player i satisfies the ODE $\mathcal{L}u - ru = 0$ over any interval (q, q') where player j does not concede; it also follows from Proposition 1 that $\bar{J}^i \geq V_{R^i}$ and that $\bar{J}^i(q^i) = R^i(q^i)$ for all $q^i \in \operatorname{supp} \mu^i$. Fixing such an interval (q, q'), and assuming that $q, q' \in \operatorname{supp} \mu^j$, we deduce from this that there must exist a single point $q^i \in (q, q') \cap \operatorname{supp} \mu^i$ at which player i is indifferent between conceding or holding fast and randomizes accordingly. The reason why such a point q^i must exist is that, otherwise, player j would expect, starting from any initial market condition $x \in (q, q')$, to obtain either $R^j(q)$ or $R^j(q')$ when leaving this interval. However, because $\mathcal{L}R^j - rR^j < 0$ over (q, q') as $q' \le x_{R^j} < x_0^j$, player j would be strictly better off conceding and obtaining $R^j(x)$ at x, a contradiction. It follows that $\bar{J}^i$ coincides with the solution to the ODE $\mathcal{L}u - ru = 0$ that is tangent to R^i at q^i. This, together with the property $\mathcal{L}R^i - rR^i < 0$, implies that q^i is unique. As a consequence, the set of accumulation points of the supports of μ^1 and μ^2 in $(s^1 \vee s^2, \beta)$ must coincide.

Consider first the asymmetric case $x_{R^1} \neq x_{R^2}$, and, to fix ideas, assume that $q_1^1 = x_{R^1}$ and $q_1^1 \geq q_1^2$ for $q_1^i \equiv \max \operatorname{supp} \mu^i$. We verify that it is not optimal for player 2 to concede at q_1^1. Therefore, q_1^1 must be an isolated point of $\operatorname{supp} \mu^1$ and $q_1^1 > q_1^2$. Iterating this argument and using the preceding remarks, we show that, for each $i = 1, 2$, and for any two consecutive points $q_n^i > q_{n+1}^i > s^1 \vee s^2$ in the support of μ^i, there must exist a single point $q_n^j \in (q_{n+1}^i, q_n^i)$ in the support of μ^j at which player j is indifferent between conceding or holding fast and randomizes accordingly. We thus obtain two decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$, with either $N^1 = N^2 = \infty$ or $0 \le N^1 - N^2 \le 1$, which are intertwined in the sense that $q_1^1 > q_1^2 > q_2^1 > q_2^2 > \ldots$ as long as these thresholds are defined. We also show that if $N^1 = N^2 = \infty$, any such intertwined sequences can only converge to α. (This is where A8 is needed.) These two sequences characterize the restrictions of μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$. As a result, when $x_{R^1} \neq x_{R^2}$, any mixed-strategy MPE must fall into one of three categories, which are delineated in Corollary 1 below.

In the symmetric case $x_{R^1} = x_{R^2}$, analogous arguments show that the common set of accumulation points of the supports of μ^1 and μ^2 is either empty or equal to $(s^1 \vee s^2, x_{R^1}]$. In the latter case, analytic arguments imply that the measures μ^i are absolutely continuous, with densities characterized by (21) for $\alpha^* \equiv s^1 \vee s^2$.

Corollary 1 *Let $((\mu^1, S^1), (\mu^2, S^2))$ be a singular mixed-strategy MPE. Then, for every player i, $\operatorname{supp} \mu^i \cap (s^1 \vee s^2, \beta) = \{q_n^i : n = 1, \ldots, N^i\}$ for intertwined decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$ satisfying, with no loss of generality, $q_1^1 > q_1^2$. Moreover, $q_1^1 = x_{R^1}$ and one of the following three conditions holds:*

1. *$N^1 = N^2 \equiv N \in \mathbb{N} \setminus \{0\}$ and $q_N^1 > q_N^2 > s^1 > s^2$;*

2. *$N^1 = N^2 + 1 \equiv N \in \mathbb{N} \setminus \{0\}$ and $q_{N-1}^2 > q_N^1 > s^2 > s^1$, with $q_0^2 \equiv \beta$ by convention;*

3. *$N^1 = N^2 = \infty$ and $\lim_{n \to \infty} q_n^1 = \lim_{n \to \infty} q_n^2 = s^1 = s^2 = \alpha$, so that $S^1 = S^2 = \emptyset$.*

In an MPE of type 1, player 1 exits the market with probability 1 at s^1, and player 2 has the lowest randomization threshold. In an MPE of type 2, player 1 has the lowest randomization threshold, and player 2 exits the market with probability 1 at s^2—the example of Section 4.3 is a case in point, with $N^1 = 1$ and $N^2 = 0$. In an MPE of type 3, neither player exits the market with probability 1 at any point of the state space, and players keep randomizing all the way down to α. It should be noted that an MPE of type 3 can exist only if $\alpha^1 = \alpha^2 = \alpha$; indeed, every player i such that $\alpha^i > \alpha$ would not be willing to delay exiting the market over (α, α^i) if his opponent were to do the same.

The upshot from Theorem 2 and Corollary 1 is that, when players have different stand-alone optimal exit thresholds, alternation is a robust feature of any singular mixed-strategy MPE, which generalizes the insights from Section 4.3. In the attrition region, players randomize between conceding and holding fast at isolated thresholds. Thus, as in the MPE constructed in Proposition 5, players may alternate on the equilibrium path between being in a dominated position or in a dominant position; the difference is that both players may now randomize on the equilibrium path, leading to a richer set of equilibrium outcomes. In an MPE of type 1 and type 2, this process may persist until one player eventually reaches his stopping region and exits the market with probability 1. By contrast, in an MPE of type 3, exit must take place at a randomization threshold.

Corollary 1 fully characterizes equilibrium outcomes for an MPE of type 3, because any market conditions in $\mathcal{I}$ can be reached with positive probability from any initial market conditions $x \in \mathcal{I}$. The same holds true for MPEs of types 1 and 2, provided $x > x_{R^1}$, with $q_1^1 > q_1^2$ by convention. Indeed, for any such MPE $((\mu^1, S^1), (\mu^2, S^2))$ and for each $x > x_{R^1}$, there exists an outcome-equivalent MPE $((\tilde{\mu}^1, \tilde{S}^1), (\tilde{\mu}^2, \tilde{S}^2))$ such that $\operatorname{supp} \tilde{\mu}^i = \operatorname{supp} \mu^i \cap (s^1 \vee s^2, \beta)$ for every player i and $\tilde{S}^1 = (\alpha, s^1)$ and $\tilde{S}^2 = \emptyset$ (for an MPE of type 1), or $\tilde{S}^1 = \emptyset$ and $\tilde{S}^2 = (\alpha, s^2)$ (for an MPE of type 2).

By contrast, Corollary 1 does not determine equilibrium outcomes of MPEs of types 1 and 2 for lower initial market conditions. First, as in Murto (2004), it may be possible to construct MPEs in which the stopping regions S^1 and S^2 exhibit gaps. Second, these gaps may themselves include randomization thresholds at which players exit the market with positive but finite intensity.

5.2 The Characterization Result

Our final theorem provides a necessary and sufficient condition for the existence of an MPE of type 2. Analogous results hold for MPEs of types 1 and 3; their statements and proofs proceed along similar lines, and are omitted for the sake of brevity.

Theorem 3 *Let $N \in \mathbb{N} \setminus \{0\}$ and let be given*

- *two finite sequences $(q_n^1)_{n=1}^N$ and $(q_n^2)_{n=0}^{N-1}$ of numbers in $\mathcal{I}$, with $q_0^2 \equiv \beta$ by convention, and a number $s^2 \in \mathcal{I}$ such that $q_1^1 = x_{R^1} > q_1^2 > q_2^1 > \ldots > q_{N-1}^1 > q_{N-1}^2 > q_N^1 > s^2$;*

- *two finite sequences* $(a_n)_{n=1}^N$ *and* $(b_n)_{n=0}^{N-1}$ *of positive real numbers.*

Then the strategy profile $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2]))$, *with* $\sum_{n=1}^0 \equiv 0$ *by convention, is an MPE of type 2 if and only if* $s^2 > \alpha^2$ *and there exists two functions* $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q_n^2 : 1 \leq n \leq N-1\} \cup \{s^2\}))$ *and* $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q_n^1 : 1 \leq n \leq N\} \cup \{s_2\}))$ *that satisfy the variational system*

$$w^1 \geq R^1 \text{ over } \mathcal{I}, \tag{42}$$

$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (s^2, \beta) \setminus \{q_n^2 : 1 \leq n \leq N-1\}, \tag{43}$$

$$w^1 = G^1 \text{ over } (\alpha, s^2], \tag{44}$$

$$w^1(q_n^1) = R^1(q_n^1), \; 1 \leq n \leq N, \tag{45}$$

$$b_n[G^1(q_n^2) - w^1(q_n^2)] + \frac{1}{2}\,\Delta w^{1\prime}(q_n^2) = 0, \; 1 \leq n \leq N-1, \tag{46}$$

$$w^1(\beta-) = 0, \tag{47}$$

$$w^2 \geq R^2 \text{ over } \mathcal{I}, \tag{48}$$

$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (s^2, \beta) \setminus \{q_n^1 : 1 \leq n \leq N\}, \tag{49}$$

$$w^2 = R^2 \text{ over } (\alpha, s^2], \tag{50}$$

$$w^2(q_n^2) = R^2(q_n^2), \; 1 \leq n \leq N-1, \tag{51}$$

$$w^{2\prime}(s^2) = R^{2\prime}(s^2), \tag{52}$$

$$a_n[G^2(q_n^1) - w^2(q_n^1)] + \frac{1}{2}\,\Delta w^{2\prime}(q_n^1) = 0, \; i = 1 \leq n \leq N, \tag{53}$$

$$w^2(\beta-) = 0. \tag{54}$$

Moreover, whenever $\alpha^1 \leq \alpha^2$, $((\sum_{n=1}^N a_n \delta_{q_n^1}, (\alpha, \alpha^1], (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2]))$ *is an outcome-equivalent MPE that satisfies Murto's (2004) refinement.*

The proof of Theorem 3 is based on the properties obtained in the proof of Theorem 2, together with classical methods employed in verification theorems for optimal-stopping and stopping-game theory. In particular, conditions (46) and (53) are obtained by applying the Itô–Tanaka–Meyer formula.

The ultimate justification for Proposition 5 follows from applying Theorem 3 for $N = 1$, which yields the variational system (28)–(38). MPEs of type 2 in our running example for $N > 1$ have implications that are similar to those of the MPE of type 2 constructed in Proposition 5. In particular, firms' stock prices in the attrition region comove negatively and have volatilities that also comove negatively. The difference is that firms' stock prices now exhibit several resistance levels—specifically, $F^1(q_n^2)$, $1 \leq n \leq N-1$, for firm 1, and $F^2(q_n^1)$, $1 \leq n \leq N$, for firm 2—resulting in a richer price dynamics. MPEs of types 1 and 3 lead to similar robust predictions.

Importantly, the variational characterization in Theorem 3 and the analogous results for MPEs of types 1 and 3 hold for both symmetric and asymmetric players. Thus our results provide a characterization of mixed-strategy MPE outcomes in the war of attrition under uncertainty that are robust to even the slightest degree of heterogeneity between players.

6 Concluding Remarks

This paper has offered a detailed study of mixed-strategy MPE outcomes in the symmetric-information war of attrition when future payoffs are driven by a homogenous linear diffusion. Our contribution is threefold.

First, we have provided a characterization result for Markov strategies in terms of an intensity measure over the state space together with a subset of the state space over which the player concedes with probability 1. This covers the usual cases of pure strategies and of mixed strategies in which intensity measures are absolutely continuous over the state space. In addition, this representation allows for mixed Markov strategies with singular intensity measures, a possibility that has been disregarded in the literature.

Second, we have argued that, far from being artificial or exotic, such singular strategies are key to the identification of robust mixed-strategy MPE outcomes, both in the cases of symmetric and asymmetric players. We have provided a variational characterization of singular mixed-strategy MPEs and we have shown that they are characterized by intertwined sequences of randomization thresholds for the players. As a result, players on the equilibrium path typically alternate between being in a dominated position or in a dominant position, a novel prediction in the literature.

Third, we have seen that, in the standard model of exit in a duopoly, this characterization leads to new testable asset-pricing implications when firms are publicly traded. Namely, the firms' stock prices, as well as their volatilities, comove negatively over the attrition zone and exhibit resistance and support patterns documented by technical analysis. This contrasts with the predictions of the standard regular mixed-strategy MPE that only exists when firms are symmetric, in which firms' stock prices are perfectly aligned and are constant and equal to the firms' common liquidation value over the attrition zone.

Taken together, our results show that mixed-strategy MPEs that are robust to even slight asymmetries between players' payoffs share a common structure, and lead to qualitatively similar empirical implications. This yields rich and robust predictions for the war of attrition under uncertainty—something that is precluded by focusing on pure-strategy MPEs, or regular mixed-strategy MPEs of symmetric games, whose implications are too stark to fruitfully lend themselves to applied analysis. Our hope is that these insights may pave new avenues for empirical work.

Appendix

Notation To ease the exposition, we shall say that a property of the trajectories $\omega \in \Omega$ is satisfied *almost surely* (a.s.) if, for each $x \in \mathcal{I}$, it is satisfied for $\mathbf{P}_x$-almost every $\omega \in \Omega$.

Proof of Theorem 1. (Necessity) We hereafter omit the index i for the sake of clarity. If Λ is the csf of a Markovian randomized stopping time, then, for all $t, s \geq 0$,

$$\Lambda_{t+s} = \Lambda_t(\Lambda_s \circ \theta_t) \text{ a.s.} \tag{A.1}$$

In particular, applying this property at $t = s = 0$ yields $\Lambda_0 = (\Lambda_0)^2$ and, hence, $\Lambda_0 \in \{0, 1\}$ a.s. In the terminology of Blumenthal and Getoor (1968, Definition III.1.1), Λ is a right-continuous multiplicative functional of X adapted to $(\mathcal{F}_t)_{t\geq 0}$. The set $E_\Lambda \equiv \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 1] = 1\}$ is called the set of permanent points for Λ. Using Blumenthal's 0–1 law (Blumenthal and Getoor (1968, Proposition I.5.17)) and the fact that $\Lambda_0 \in \{0, 1\}$ a.s., we have $\mathcal{I} \setminus E_\Lambda = \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 0] = 1\}$. The stopping time $\tau \equiv \inf\{t > 0 : \Lambda_t = 0\} \in \mathcal{T}$ is called the lifetime of Λ. The proof consists of three steps.

Step 1 In order to apply the main result of Sharpe (1971), we need to check that Λ is an exact multiplicative functional in the sense of Blumenthal and Getoor (1968, Definition III.4.13). According to Blumenthal and Getoor (1968, Proposition III.5.9) it is sufficient to prove that, for all $x \in \mathcal{I} \setminus E_\Lambda$ and $t > 0$,

$$\lim_{u \downarrow 0} \mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] = 0. \tag{A.2}$$

To this end, notice that, for any such x and t, and for each $u \in (0, t)$, we have

$$1_{\{t-u \geq \tau_x \circ \theta_u\}}(\Lambda_{t-u} \circ \theta_u) = 0 \tag{A.3}$$

$\mathbf{P}_x$-almost surely. Indeed, if $t - u \geq \tau_x \circ \theta_u(\omega)$ for some $\omega \in \Omega$, then the trajectory $\theta_u(\omega)$ crosses x over the interval $[0, t-u]$. Because, by (16), $\Lambda_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega)) = \Lambda_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega))$ $\Lambda_0(\theta_{\tau_x \circ \theta_u(\omega)}(\theta_u(\omega)) = 0$ $\mathbf{P}_x$-almost surely as $x \in \mathcal{I} \setminus E$, it follows that $\Lambda_{t-u}(\theta_u(\omega)) = 0$ as the mapping $s \mapsto \Lambda_s(\theta_u(\omega))$ is nonincreasing and nonnegative; hence (A.3). This implies in particular that, for $u < \frac{t}{2}$,

$$\mathbf{E}_x[\Lambda_{t-u} \circ \theta_u] \leq \mathbf{P}_x[t - u < \tau_x \circ \theta_u] = \mathbf{E}_x[\mathbf{P}_{X_u}[t - u < \tau_x]] \leq \mathbf{E}_x\left[\mathbf{P}_{X_u}\left[\frac{t}{2} < \tau_x\right]\right]. \tag{A.4}$$

The mapping $y \mapsto \mathbf{P}_y[\frac{t}{2} < \tau_x]$ is bounded and $\lim_{y \to x} \mathbf{P}_y[\frac{t}{2} < \tau_x] = 0$ as X is a regular diffusion. Hence (A.2) follows from (A.4) by bounded convergence along with the fact that $\lim_{u \downarrow 0} X_u = x$ $\mathbf{P}_x$-almost surely. Exactness of Λ implies that E_Λ is open and thus that $\mathcal{I} \setminus E_\Lambda$ is closed, see Blumenthal and Getoor (1968, page 126, last paragraph) together with the fact that the fine topology over $\mathcal{I}$ associated to X coincides with the usual topology, see Blumenthal and Getoor (1968, Definition II.4.1 and Exercise II.4.16).

Step 2 We are now in a position to apply Sharpe (1971, Theorem 7.1, Formula (7.1)), which expresses Λ_t as the product of three factors.

1. The first factor is equal to 1 because X has continuous trajectories, so that the terms $F(X_{s-}, X_s)$ vanish as $F = 0$ over the diagonal of $\mathcal{I}$, see Sharpe (1971, Theorem 5.1 and proof of Theorem 7.1).

2. The second factor can be written as $1_{\{t<\tau_B\}}$, where τ_B is the hitting time by X of a Borel subset B of $\mathcal{I}$; this is because the lifetime of X is infinite and X has continuous trajectories. In turn, because X is a diffusion process and $\sigma > 0$ over $\mathcal{I}$, this term is a.s. equal to $1_{\{t<\tau_S\}}$, where S is the closure of B.

3. The third factor is of the form $\mathrm{e}^{-\int_0^t f(X_s)\,\mathrm{d}A_s}$, where $f : \mathcal{I} \to \mathbb{R}_+$ is Borel-measurable and A is a continuous additive functional of X (Revuz and Yor (1999, Chapter X, §1, Definition 1.1)) such that the mapping $x \mapsto \mathbf{E}_x[\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}A_t]$ is bounded.

Thus, for each $t \geq 0$, we have the representation

$$\Lambda_t = 1_{\{t<\tau_S\}}\,\mathrm{e}^{-\int_0^t f(X_s)\,\mathrm{d}A_s} \text{ a.s.} \tag{A.5}$$

Moreover, the integral $\int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-almost surely finite for all $t < \tau_S$ except maybe for x in an M-polar set, where M is the multiplicative functional defined by $M_t \equiv 1_{\{t<\tau_S\}}$ for all $t \geq 0$ (Blumenthal and Getoor (1968, II.2.18 and III.1.4)). According to Sharpe (1971, Definition, page 29), $B \subset \mathcal{I}$ is an M-polar set if there exists a nearly Borel subset (Blumenthal and Getoor (1968, Definition I.10.21)) $C \supset B$ of $\mathcal{I}$ such that the hitting time by X of C is a.s. greater or equal to the lifetime of M, that is, τ_S. Hence, because the trajectories of X are continuous and S is closed, an M-polar set must be a subset of S, and it follows that $\int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-almost surely finite for all $t < \tau_S$ and $x \in \mathcal{I} \setminus S$. Finally, observe that we can with no loss of generality assume that $f = 0$ over S, as replacing f by $f1_{\mathcal{I}\setminus S}$ does not alter the right-hand side of (A.5).

Step 3 Using the classical representation result for additive functionals of X (Borodin and Salminen (2002, Part I, Chapter I, Section 4, §23)), there exists a Radon measure ν over $\mathcal{I} \setminus S$ such that $A_t = \int_{\mathcal{I}\setminus S} L_t^y\,\nu(\mathrm{d}y)$ a.s. Therefore, for each $t < \tau_S$,

$$\tilde{A}_t \equiv \int_0^t f(X_s)\,\mathrm{d}A_s = \int_0^t \int_{\mathcal{I}\setminus S} f(X_s)\,\mathrm{d}L_s^y\,\nu(\mathrm{d}y) = \int_{\mathcal{I}\setminus S} L_t^y f(y)\,\nu(\mathrm{d}y) \text{ a.s.}$$

We claim that $\mu \equiv f \cdot \nu$ is a Radon measure, which concludes the first part of the proof. To this end, we only need to prove that μ is locally finite. Indeed, if it were not so, then there would exist $x \in \mathcal{I} \setminus S$ such that $\int_{[x-\varepsilon,x+\varepsilon]} f(y)\,\nu(\mathrm{d}y) = \infty$ for all $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon] \subset \mathcal{I} \setminus S$. For each $t > 0$, $L_t^x(\omega) > 0$ for all ω in a set of $\mathbf{P}_x$-probability 1. Therefore, as the local time of X is a.s. jointly continuous (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), we have that, for any such ω, there exists $\varepsilon(\omega) > 0$ such that

$[x-\varepsilon(\omega), x+\varepsilon(\omega)] \subset \mathcal{I}\setminus S$ and $L^y_t(\omega)>0$ for all $y\in[x-\varepsilon(\omega), x+\varepsilon(\omega)]$. This implies that, if $0<t<\tau_S(\omega)$, then

$$\tilde{A}_t(\omega)=\int_{\mathcal{I}\setminus S} L^y_t(\omega) f(y)\,\mu(\mathrm{d}y) \geq \min_{y\in[x-\varepsilon(\omega),x+\varepsilon(\omega)]} L^y_t(\omega)\int_{[x-\varepsilon(\omega),x+\varepsilon(\omega)]} f(y)\,\mu(\mathrm{d}x)=\infty.$$

Because $\mathbf{P}_x[\tau_S>0]=1$ as $x\in\mathcal{I}\setminus S$, this contradicts the fact that, for each $t<\tau_S$, $\tilde{A}_t=\int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-almost surely finite. The claim follows.

(Sufficiency) Reciprocally, if S is a closed subset of $\mathcal{I}$ and μ is a Radon measure over $\mathcal{I}\setminus S$, then the process defined by

$$\Lambda_t=1_{\{t<\tau_S\}}\,\mathrm{e}^{-\int_{\mathcal{I}\setminus S} L^y_t\,\mu(\mathrm{d}y)}$$

is well-defined and, as the local time of X is a strong additive functional of X (Revuz and Yor (1999, Chapter X, §1, Proposition 1.2)), is a right-continuous multiplicative functional that satisfies (16). In particular, $\Gamma\equiv 1-\Lambda$ satisfies the assumptions of Lemma 3 and thus is the ccdf of a randomized stopping time. Hence the result. ■

Proof of Theorem 2. Let $s\equiv s^1\vee s^2$ and $E^i\equiv \operatorname{supp}\mu^i\cap(s,\beta)$ for $i=1,2$. E^i is a relatively closed subset of (s,β) that can be written as a disjoint union $E^i=A^i\cup K^i$, where A^i is the set of accumulation points of E^i in (s,β), which is relatively closed in (s,β), and K^i is the (countable) set of isolated points of E^i. Observe that $E^i\subset(s,x_{R^i}]$ by Proposition 1(iv) as $E^i\cap S^j=\emptyset$. If $E^1=E^2=\emptyset$, there is nothing to prove and the MPE under consideration is outcome-equivalent to a pure-strategy MPE. Let us otherwise denote by $\bar{J}^i$ player i's equilibrium value function. The proof then consists of four steps and repeatedly uses assertions (i)–(iii) of Lemma A.1 below. (We shall later use assertion (iv) in the proof of Theorem 3.)

Lemma A.1 *Let u be a $\mathcal{C}^2$ function defined over an open interval $(a,b)\subset\mathcal{I}$ and such that $\mathcal{L}u-ru=0$ over (a,b). Then, the following holds:*

(i) if $b=\beta$, $u(\beta-)=0$, $u(a+)=R^i(a)$, and $u\geq V_{R^i}$ over (a,β), then $a=x_{R^i}$;

(ii) if $u\geq V_{R^i}$ over (a,b), then $\{x\in(a,b):u(x)=R^i(x)\}$ contains at most one point;

(iii) if $b\leq x_{R^i}$, $u(b-)=R^i(b)$, and either $a>\alpha$ and $u(a+)=R^i(a)$ or $a=\alpha$ and $u(a+)=0$, then $u<R^i$ over (a,b);

(iv) if $\alpha<a\leq x_{R^i}$, $u\geq R^i$ over (a,b), $u(a)=R^i(a)$, and $u'(a+)>R^{i\prime}(a)$, then, for every sufficiently small $\varepsilon>0$, the function f_ε solution to $\mathcal{L}f-rf=0$ over $(a-\varepsilon,a+\varepsilon)$ with $f_\varepsilon(a-\varepsilon)=R^i(a-\varepsilon)$ and $f_\varepsilon(a+\varepsilon)=u(a+\varepsilon)$ satisfies $f_\varepsilon(a)>u(a)$.

Step 1 We first claim that every connected component (a,b) of $(s,\beta)\setminus E^i$ such that (a) $a>s$ or $a=s=s^i$ or $a=s=\alpha$, and (b) $b\leq x_{R^i}$, contains exactly one point of E^j.

Suppose first, by way of contradiction, that $E^j\cap(a,b)=\emptyset$. By Proposition 1(v), the strategy $(0,S^i)$ is a pbr to the strategy (μ^j,S^j). Therefore, τ_{S^i} is a solution to the optimal-stopping problem $\bar{J}^i(x)=\sup_{\tau^i\in\mathcal{T}}J^i(x,\tau^i,(\mu^j,S^j))$. Letting τ be the first exit time of X from (a,b), we have $\tau_{S^i}\geq\tau$ $\mathbf{P}_x$-almost surely for all $x\in(a,b)$. We deduce from this that the brvf $\bar{J}^i$ satisfies, for each $x\in(a,b)$,

$$\bar{J}^i(x)=J^i(x,(0,S^i),(\mu^j,S^j))=\mathbf{E}_x[\mathrm{e}^{-r\tau}\bar{J}^i(X_\tau)], \tag{A.6}$$

where the last inequality follows from the strong Markov property (S.7). As $E^j\cap(a,b)=\emptyset$, it then follows from standard arguments that

$$\bar{J}^i \text{ is } \mathcal{C}^2 \text{ and } \mathcal{L}\bar{J}^i-r\bar{J}^i=0 \text{ over } (a,b). \tag{A.7}$$

Now, consider the conditions in the claim. First, if $a>s$, then $a\in\operatorname{supp}\mu^i$ by definition of a connected component of $(s,\beta)\setminus E^i$, and thus $\bar{J}^i(a)=R^i(a)$ by Proposition 1(iii); the same reasoning shows that $\bar{J}^i(b)=R^i(b)$. Next, if $a=s=s^i$, then $\bar{J}^i(a)=R^i(a)$ by Proposition 1(iii). Finally, if $a=s=\alpha$, then τ coincides with the hitting time of b, and thus (A.6) and (5)–(6) together imply, letting x go to $\alpha+$, that $\bar{J}^i(a+)=0$. Thanks to (A.7) and $b\leq x_{R^i}$, we are thus in a position to apply Lemma A.1(iii); we obtain $\bar{J}^i<R^i$ over (a,b), a contradiction as $\bar{J}^i\geq V_{R^i}$ over $\mathcal{I}$. Therefore, $E^j\cap(a,b)\neq\emptyset$. Finally, using the same arguments as for the derivation of (A.7), it must be that $\bar{J}^j$ satisfies $\mathcal{L}\bar{J}^j-r\bar{J}^j=0$ over (a,b). Because $\bar{J}^j\geq V_{R^j}$, Lemma A.1(ii) implies that $E^j\cap(a,b)$ contains exactly one point. The claim follows. It should be noted that the same arguments show that every interval $(a,b)\subset(s,\beta)$ such that (a) $a>s$ or $a=s=s^i$ or $a=s=\alpha$, (b) $b\leq x_{R^i}$, and (c) $\bar{J}^i(a+)=R^i(a+)$ and $\bar{J}^i(b)=R^i(b)$ contains at least one point of E^j.

Step 2 We next claim that $A^1=A^2$.

Let $x\in A^i$. Suppose first, by way of contradiction, that $x\notin E^j$. Then there exists $\varepsilon>0$ such that $(x-\varepsilon,x+\varepsilon)\cap E^j=\emptyset$, where ε can be chosen sufficiently small so that $x-\varepsilon>s$. As x is an accumulation point of E^i and E^i is relatively closed in (s,β), one of the two following conditions must hold:

(i) $(x-\varepsilon,x+\varepsilon)$ includes a connected component (a,b) of $(s,\beta)\setminus E^i$ such that $a>s$ and $b\leq x_{R^i}$;

(ii) E^i includes a nondegenerate interval $\mathcal{I}_0\subset(x-\varepsilon,x+\varepsilon)$ that contains x.

In case (i), the connected component (a,b) must contain one point of E^j by Step 1, a contradiction. In case (ii), notice that $\mathcal{I}_0\cap S^j=\emptyset$ by definition of s, E^i, and E^j. Thus, by Proposition 1(ii), it must be that $\bar{J}^i = R^i$ over $\mathcal{I}_0$. On the other hand, because $(0, S^i)$ is also a pbr to (μ^j, S^j) and $E^j \cap \mathcal{I}_0 = \emptyset$, we obtain as in Step 1 that $\bar{J}^i$ must be $\mathcal{C}^2$ and satisfy $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over the interior of $\mathcal{I}_0$. But then $\mathcal{L}R^i - rR^i = 0$ over a nondegenerate interval, a contradiction by A3. We conclude that $x \in E^j$ and in turn that $A^i \subset E^j$. Let us now prove that $A^i \subset A^j$. If $x \in A^i$ belongs to the relative closure of $A^i \setminus \{x\}$ in (s, β), then $x \in A^j$ as $A^i \setminus \{x\} \subset E^j$. If not, then x must be the limit of a sequence of points $(x_n)_{n \geq 1}$ in K^i, which we can assume to be strictly monotone. By Step 1, for every sufficiently large n, the interval formed by two consecutive elements x_n and x_{n+1} of this sequence contains exactly one point y_n of E^j, and thus $x = \lim_{n \to \infty} y_n \in A^j$ as it is an accumulation point of E^j. We conclude that $A^i \subset A^j$ and in turn that $A^i = A^j$ by exchanging the role of the players. The claim follows.

Step 3 We then claim that, if $A^1 = A^2 = \emptyset$ and $K^1 \cup K^2 \neq \emptyset$, then the measures μ^1 and μ^2 are discrete or degenerate, with at least a nondegenerate one, and their supports are described by one of the cases in Corollary 1.

By assumption, μ^1 and μ^2 are discrete measures and their supports have no accumulation points in $(s, x_{R^i}]$. Therefore, either their supports are finite, or they are infinite, with s as a unique accumulation point. In both cases, for each $i = 1, 2$, $E^i = K^i \equiv \{q^i_n : 1 \leq n \leq N^i\}$ for some decreasing sequence $(q^i_n)_{n=1}^{N^i}$ in $(s, x_{R^i}]$, with N^i finite or infinite, and possibly equal to 0 for some i, in which case μ^i is degenerate. We now establish three key properties of the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, which together imply the claim.

First, it must be that $q^i_1 = x_{R^i}$ for some i. Indeed, suppose that $K^i \neq \emptyset$ and $\max E^j \leq q^i_1$, where $\max \emptyset = -\infty$. We first have $\bar{J}^i(q^i_1) = R^i(q^i_1)$ by Proposition 1(ii)–(iii) as $q^i_1 > s \geq s^j$. Next, because $E^j \cap (q^i_1, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that $\bar{J}^i$ is $\mathcal{C}^2$ and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (q^i_1, β). As a result, $\bar{J}^i = A\phi + B\psi$ over this interval for some constants A and B. From this, it follows in turn that $\bar{J}^i(\beta^-) = 0$. Indeed, by Lemma 1 and Proposition 1, we have $0 \leq \bar{J}^i \leq G^i$, which, together with (9), implies $B = 0$. That $\bar{J}^i(\beta-) = 0$ follows then from (5). Finally, $J^i \geq V_{R^i}$ by Proposition 1. Thus $\bar{J}^i$ satisfies all the conditions of Lemma A.1(i), from which we conclude that $q^i_1 = x_{R^i}$.

Next, it must be that the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, are intertwined. Indeed, Step 1 implies that, if at least one of these sequence has at least two elements, then, between two consecutive elements of each sequence, there must be exactly one element of the other sequence. Similarly, if $1 \leq N^i < \infty$ and $s = s^i$ or $s = \alpha$, then $s < q^i_{N^i}$ and there must be one element of K^j in $(s, q^i_{N^i})$. These properties have two main implications. (a) First, the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, have no common element. Indeed, suppose, by way of contradiction, that $q^1 = q^2 = q$ for two components of these two sequences. We distinguish two cases. If at least one of the sets K^1 and K^2 is not a singleton, then, because K^1 and

K^2 have s as their only possible accumulation point, there exists some $i = 1, 2$ for which the distance $\inf_{q' \in K^i \setminus \{q\}} |q' - q| > 0$ is minimized, with $\inf_{q' \in \emptyset} |q' - q| \equiv \infty$ for all $q \in \mathcal{I}$ by convention. Let this minimal distance be reached at q'. But then, as argued above, there must exist $q'' \in K^j$ in between q and q', so that $|q'' - q| < |q' - q|$, in contradiction with the definition of q'. If both K^1 and K^2 are singletons, then it must be that $K^1 = K^2 = \{x_{R^1}\} = \{x_{R^2}\}$ by the first property above. Applying Step 1 to the connected component (s^i, x_{R^i}) of $(s, \beta) \setminus E^i$ for a player i such that $s = s^i$, we obtain that (s^i, x_{R^i}) contains exactly one point of E^j, a contradiction. (b) Second, and as a result, if $\max S^j \cup E^j < q_1^i$, then either N^i is finite and $N^j \in \{N^i - 1, N^i\}$, or $N^i = N^j = \infty$.

Finally, if $N^1 = N^2 = \infty$, the sequences $(q_n^i)_{n \geq 1}$, $i = 1, 2$, must converge to α, so that $s = \alpha$ and $S^1 = S^2 = \emptyset$. This is a consequence of the following general lemma.

Lemma A.2 *Let $((\mu^1, S^1), (\mu^2, S^2))$ be a mixed-strategy MPE for which there exists two intertwined decreasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ of isolated points in $\operatorname{supp} \mu^1 \cap (s, \beta)$ and $\operatorname{supp} \mu^2 \cap (s, \beta)$, respectively, such that, for $i = 1, 2$, $\operatorname{supp} \mu^i \cap (\inf_{n \geq 1} \chi_n^i, \chi_1^i] = \{\chi_n^i : n \geq 1\}$. Then these two sequences converge to α. Similarly, there are no intertwined increasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ of isolated points in $\operatorname{supp} \mu^1 \cap (s, \beta)$ and $\operatorname{supp} \mu^2 \cap (s, \beta)$, respectively, such that, for $i = 1, 2$, $\operatorname{supp} \mu^i \cap [\chi_0^i, \sup_{n \geq 1} \chi_n^i) = \{\chi_n^i : n \geq 1\}$.*

The claim follows.

Step 4 We finally claim that, if $A \equiv A^1 = A^2 \neq \emptyset$, then $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, $s = \alpha^1 \vee \alpha^2$, and, for each $i = 1, 2$, the restriction of μ^i to $(s, x_R]$ is absolutely continuous with density $\sigma^{-2} \lambda^i$, where λ^i is given by (21) with s instead of α^*.

We first show that $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, and $s = \alpha^1 \vee \alpha^2$. The argument is fourfold.

We first claim that $A \subset (s, x_{R^1} \wedge x_{R^2}]$ is an interval. Indeed, suppose, by way of contradiction, that this is not so. Then there exists an interval $(a, b) \subset (s, \beta) \setminus A$ such that $a > s$ and $a, b \in A$. Because (a, b) cannot be a connected component of both $(s, \beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (a, b) \neq \emptyset$ for some i. Fix some $\chi_1^i \in K^i \cap (a, b)$. Then $\bar{J}^i(a) = R^i(a)$ and $\bar{J}^i(\chi_1^i) = R^i(\chi_1^i)$ by Proposition 1(ii)–(iii) as $a > s \geq s^j$, so that $K^j \cap (a, \chi_1^i) \neq \emptyset$ by the final remark of Step 1. Because $\chi_1^i \in (a, b)$ is not an accumulation point of E^j, we have $\chi_1^i > \chi_1^j \equiv \sup K^j \cap (a, \chi_1^i) \in K^j$. Applying this argument recursively, we obtain two infinite intertwined decreasing sequences $(\chi_n^1)_{n \geq 1}$ and $(\chi_n^2)_{n \geq 1}$ in $K^1 \cap (a, b)$ and $K^2 \cap (a, b)$, respectively. Because these sequences are bounded below by $a > s$ and (a, b) is a connected component of $(s, \beta) \setminus A$, they both converge to a. Moreover, arguing as in Step 3, it is easy to check that $\operatorname{supp} \mu^1 \cap (\inf_{n \geq 1} \chi_n^1, \chi_1^1] = \{\chi_n^1 : n \geq 1\}$, and similarly for player 2. Thus, by Lemma A.2, it must be the case that $a = \alpha$, a contradiction as $a > s$.

The claim follows. As A is relatively closed in (s,β), $\sup A = \max A \in A$.

We next claim that $\max A = x_{R^1} = x_{R^2}$. Indeed, suppose first, by way of contradiction, that $\max A < x_{R^1} \wedge x_{R^2}$. Arguing as in Step 3, we obtain that $x_{R^i} \in K^i$ for some $i = 1, 2$. Hence $\bar{J}^i(\max A) = R^i(\max A)$ and $\bar{J}^i(x_{R^i}) = R^i(x_{R^i})$ by Proposition 1(ii)–(iii) as $\max A > s \geq s^j$, so that $K^j \cap (\max A, x_{R^1} \wedge x_{R^2}] \neq \emptyset$ by the final remark of Step 1 along with the fact that $K^j \subset (s, x_{R^j}]$. We can then repeat the above argument, leading again to a contradiction. We conclude that $\max A = x_{R^1} \wedge x_{R^2} = x_{R^j}$ for some $j = 1, 2$, so that $\max E^i \geq x_{R^j}$. Now, $\bar{J}^i(x_{R^j}) = R^i(x_{R^j})$ by Proposition 1(ii)–(iii) as $x_{R^j} > s \geq s^j$. Because $E^j \cap (x_{R^j}, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that $\bar{J}^i$ is $\mathcal{C}^2$ and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (x_{R^j}, β). Finally, we can use similar arguments as in Step 3 to show that $\bar{J}^i(\beta-) = 0$. As $\bar{J}^i \geq V_{R^i}$ by Proposition 1, $\bar{J}^i$ satisfies all the conditions of Lemma A.1(i), from which we conclude that $x_{R^j} = x_{R^i} \equiv x_R$. The claim follows.

We then claim that $\inf A = s$. Indeed, suppose, by way of contradiction, that $\inf A > s$. Because $(s, \inf A)$ cannot be a connected component of both $(s,\beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (a, b) \neq \emptyset$ for some i. Fixing some $\chi^i_1 \in K^i \cap (\max A, x^i_R)$, we can then mirror the above argument to obtain two infinite intertwined *increasing* sequences $(\chi^1_n)_{n\geq 1}$ and $(\chi^2_n)_{n\geq 1}$ in K^1 and K^2, respectively, converging to $\inf A$, and such that for $i = 1, 2$, $\operatorname{supp} \mu^i \cap [\chi^i_1, \sup_{n\geq 1} \chi^i_n) = \{\chi^i_n : n \geq 1\}$, a contradiction by Lemma A.2. We conclude that $\inf A = s$ and thus that $A = (s, x_R]$. The claim follows.

We finally claim that $s = \alpha^1 \vee \alpha^2$. Notice first that $s \geq \alpha^1 \vee \alpha^2$ by Lemma S.4(ii) in the Online Supplement. Now, suppose, by way of contradiction that $s > \alpha^1 \vee \alpha^2$ and $s \in S^i$. Then, by Proposition 1(i), $s \notin S^j$, so that $\bar{J}^j(s) = G^j(s)$. But $\bar{J}^j(s+) = R^j(s) < G^j(s)$ as $(s, x_{R^j}] \subset \operatorname{supp} \mu^j$ and $s > \alpha^j$, a contradiction as $\bar{J}^j$ is continuous by Proposition 2. The claim follows.

We have thus shown that, if $A \neq \emptyset$, then $A = (s, x_R]$, with $s = \alpha^1 \vee \alpha^2$ and $x_R = x_{R^1} = x_{R^2}$. By Proposition 1(iii), it follows that, for each $i = 1, 2$, $\bar{J}^i = R^i$ over $(s, x_R]$. Therefore, by Lemma 3 and Proposition 1(v),

$$\bar{J}^i(x) = f^i(x, \mu^j) \equiv \mathbf{E}_x\left[\int_{[0,\tau_s)} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}\Gamma^j_t + \mathrm{e}^{-r\tau_s} R^i(s)(1 - \Gamma^j_{\tau_s})\right] = R^i(x) \tag{A.8}$$

for all $x \in (s, x_R]$, where $\Gamma^j_t \equiv 1 - \mathrm{e}^{-\int_{(s,x_R]} L^y_t\,\mu^j(\mathrm{d}y)}$. Notice that the right-hand side of (A.8) does not depend on μ^j, so that neither does $f^i(x, \mu^j)$ in equilibrium for all $x \in (s, x_R]$. Consider then the measure $\bar{\mu}^j \equiv \sigma^{-2}\lambda^j \cdot Leb$ over (α^i, β), where

$$\lambda^j(x) \equiv \frac{rR^i(x) - \mathcal{L}R^i(x)}{G^i(x) - R^i(x)}\, 1_{\{\alpha^i < x \leq x_R\}}.$$

Adapting the arguments in Steg (2015) and Georgiadis, Kim, and Kwon (2022), it can be verified that, as in Proposition 4, the pair $(((\alpha, \alpha^2], \bar{\mu}^1), ((\alpha, \alpha^1], \bar{\mu}^2))$ is an MPE with equilibrium value functions (V_{R^1}, V_{R^2}). In particular, Proposition 1 implies that player j's strategy makes player i indifferent between holding fast and conceding over $(\alpha^i, x_{R^i}]$, which, together with the Markov property, implies that $\bar{\mu}^j$ is solution to (A.8).

To conclude, we show that $\mu^j = \bar{\mu}^j_{|\mathcal{B}((s,\beta))}$, that is, (A.8) has a unique solution over the Borel σ-field $\mathcal{B}((s,\beta))$. The strong Markov property implies that, for each $x \in (s, x_R]$ and for every stopping time $\tau < \tau_s$,

$$f^i(x, \mu^j) = \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}\Gamma^j_t + \mathrm{e}^{-r\tau} f^i(X_\tau, \mu^j)\right]$$
$$= \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}\Gamma^j_t + \mathrm{e}^{-r\tau} R^i(X_\tau)\right].$$

Because $f^i(x, \bar{\mu}^j) = f^i(x, \mu^j) = R^i(x)$ and similar equalities hold for $f^i(x, \bar{\mu}^j)$, it follows that

$$\mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}(\Gamma^j_t - \bar{\Gamma}^j_t)\right] = 0,$$

where $\bar{\Gamma}^j_t \equiv 1 - \mathrm{e}^{-\int_{(s,x_R]} L^y_t\,\bar{\mu}^j(\mathrm{d}y)}$. Because this equality holds for any stopping time $\tau < \tau_s$, the process $u \mapsto M_u \equiv \int_{[0,u]} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}(\Gamma^j_t - \bar{\Gamma}^j_t)$ is a martingale over $[0, \tau_s)$ (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Therefore, being a continuous process of bounded variation, it is indistinguishable from 0 over $[0, \tau_s)$. As $G^i > V_{R^i} > 0$ by Lemma 1, it follows that the process $u \mapsto \Gamma^j_u - \bar{\Gamma}^j_u = \int_{[0,u]} \frac{\mathrm{e}^{rt}}{G^i(X_t)}\,\mathrm{d}M_t$ is indistinguishable from 0 over $[0, \tau_s)$, so that the processes $u \mapsto \int_{(s,x_{R^i}]} L^y_u\,\mu^j(\mathrm{d}y)$ and $u \mapsto \int_{(s,x_{R^i}]} L^y_u\,\bar{\mu}^j(\mathrm{d}y)$ are indistinguishable from each other over $[0, \tau_s)$. In turn, these two processes can be seen as additive functionals of the diffusion X over (s, β), where s is modified into a killing boundary. This implies that $\mu^j = \bar{\mu}^j$, because both the measure associated to an additive functional of a diffusion and the killing measure of a diffusion are unique (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §4, and Section 4, §23)). Hence the result. ∎

Proof of Theorem 3. (Necessity) Let $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q^1_n}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q^2_n}, (\alpha, s^2])))$ be an MPE of type 2, and consider the brvf $\bar{J}^2$ to (μ^1, S^1). Our goal is to show that $\bar{J}^2$ satisfies the variational system (48)–(54).

We start with some simple observations. First, $\bar{J}^2 \in \mathcal{C}^0(\mathcal{I})$ by Proposition 2, as requested. Second, we know from Proposition 1 that $\bar{J}^2 \geq V_{R^2}$ over $\mathcal{I}$ and from (3) that $V_{R^2} \geq R^2$ over $\mathcal{I}$. Hence $\bar{J}^2$ satisfies (48). Third, $\bar{J}^2 = R^2$ over $S^2 = (\alpha, s^2]$ by Proposition 1(ii). Hence $\bar{J}^2$ satisfies (50). Fourth, $\operatorname{supp} \mu^2 = \{q^2_n : 1 \leq n \leq N-1\} \subset \{x \in \mathcal{I} : \bar{J}^2(x) = R^2(x)\}$ by Proposition 1(iii). Hence $\bar{J}^2$ satisfies (51). Fifth, as in Steps 1 and 3 of the proof of Theorem 2, it can be verified that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q^1_1, β) and (s^2, q^1_N), and that $\bar{J}^2 = T^2_{q^2_n}$ over (q^1_{n+1}, q^1_n) for $1 \leq n \leq N-1$, where T^2_q is the solution to $\mathcal{L}u - ru = 0$ that is tangent to R^2 at q. Hence $\bar{J}^2$ satisfies (49). Sixth, as in Step 3 of the proof of Theorem 2, the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q^1_1, β) implies that $\bar{J}^2 = A\phi + B\psi$ over this interval for some constants A, B, and the fact that $0 \leq \bar{J}^2 \leq G^2$ together with (9) implies that $B = 0$ and thus $\bar{J}^2(\beta-) = 0$. Hence $\bar{J}^2$ satisfies (54). Seventh, and as a result, $\bar{J}^2 \in \mathcal{C}^2(\mathcal{I} \setminus (\{q^1_n : 1 \leq n \leq N\} \cup \{s_2\}))$ and $|\bar{J}^{2\prime-}(x)| \vee |\bar{J}^{2\prime+}(x)| < \infty$ for all $x \in \{q^1_n : 1 \leq n \leq N\} \cup \{s_2\}$, as requested.

Let us now check that $\bar{J}^2$ satisfies (52). Because $\bar{J}^2 \geq R^2$, with equality at s^2, it must be that $\bar{J}^{2\prime+}(s^2) \geq R^{2\prime}(s^2)$. Suppose, by way of contradiction, that this inequality is strict. Consider the stopping time $\tau_\varepsilon \equiv \inf\{t \geq 0 : X_t \notin (s^2 - \varepsilon, s^2 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $\alpha < s^2 - \varepsilon < s^2 + \varepsilon < q^1_N$. Define $f_\varepsilon(x) \equiv \mathbf{E}_x[\mathrm{e}^{-r\tau_\varepsilon} \bar{J}^2(X_{\tau_\varepsilon})]$ for $x \in (s^2 - \varepsilon, s^2 + \varepsilon)$. Recalling that τ_{S^2} is a best reply to (μ^1, S^1) by Proposition 1(v) and invoking the strong Markov property, we obtain that $f_\varepsilon(x)$ is the payoff of player 2 against (μ^1, S^1) when using the non-Markovian stopping time $\tau_\varepsilon + \tau_{S^2} \circ \theta_{\tau_\varepsilon}$ that consists in holding fast up to τ_ε and then conceding the first time X hits S^2 in the continuation game. By construction, $\mathcal{L} f_\varepsilon - r f_\varepsilon = 0$ over $(s^2 - \varepsilon, s^2 + \varepsilon)$. Applying Lemma A.1(iv) with $i = 2$, $a = s^2$, $b = q^1_N$, and $u = \bar{J}^2$, we deduce that $f_\varepsilon(s^2) > \bar{J}^2(s^2)$ for ε sufficiently small, a contradiction as (μ^2, S^2) is a pbr to (μ^1, S^1). Hence $\bar{J}^2$ satisfies (52).

Let us finally check that $\bar{J}^2$ satisfies (53). We shall use the following lemma, which provides two alternative expressions for $\bar{J}^2$ that result from the Markov property and the Itô–Tanaka–Meyer formula, respectively.

Lemma A.3 *Let $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q^1_n}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q^2_n}, (\alpha, s^2]))$ be an MPE of type 2. Then, for all $x \in \mathcal{I}$ and $\tau \in \mathcal{T}$,*

$$\bar{J}^2(x) = \mathbf{E}_x \Bigg[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} \mathrm{e}^{-rt} G^2(q^1_n) \Lambda^1_t a_n \, \mathrm{d}L^{q^1_n}_t + 1_{\{\tau_{S^2} < \tau\}} \mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda^1_{\tau_{S^2}} + 1_{\{\tau_{S^2} \geq \tau\}} \mathrm{e}^{-r\tau} \bar{J}^2(X_\tau) \Lambda^1_\tau \Bigg], \tag{A.9}$$

and

$$\bar{J}^2(x) = \mathbf{E}_x \Bigg[\sum_{n=1}^N \int_{[0, \tau \wedge \tau_{S^2})} \mathrm{e}^{-rt} \left[\bar{J}^2(q^1_n) a_n - \frac{1}{2} \Delta \bar{J}^{2\prime}(q^1_n) \right] \Lambda^1_t \, \mathrm{d}L^{q^1_n}_t + 1_{\{\tau_{S^2} < \tau\}} \mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}}) \Lambda^1_{\tau_{S^2}} + 1_{\{\tau_{S^2} \geq \tau\}} \mathrm{e}^{-r\tau} \bar{J}^2(X_\tau) \Lambda^1_\tau \Bigg]. \tag{A.10}$$

An immediate implication of (A.9)–(A.10) is that, for each $\tau \in \mathcal{T}$,

$$\mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau)} 1_{\{\tau_{S^2} > t\}} \mathrm{e}^{-rt} G^2(q^1_n) \Lambda^1_t a_n \, \mathrm{d}L^{q^1_n}_t \right] = \mathbf{E}_x \left[\sum_{n=1}^N \int_{[0, \tau)} 1_{\{\tau_{S^2} > t\}} \mathrm{e}^{-rt} \left[\bar{J}^2(q^1_n) a_n - \frac{1}{2} \Delta \bar{J}^{2\prime}(q^1_n) \right] \Lambda^1_t \, \mathrm{d}L^{q^1_n}_t \right].$$

Equivalently, for each $\tau \in \mathcal{T}$, $\mathbf{E}_x[M_\tau] = \mathbf{E}_x[M_0] = 0$, where

$$M_t \equiv \sum_{n=1}^{N} \int_{[0,t)} 1_{\{\tau_{S^2}>s\}} \, \mathrm{e}^{-rs} \left\{ a_n[G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2} \, \Delta \bar{J}^{2\prime}(q_n^1) \right\} \Lambda_s^1 \, \mathrm{d}L_s^{q_n^1} \tag{A.11}$$

for all $t \geq 0$. It follows that the process $(M_t)_{t \geq 0}$ is a martingale (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Because it is a continuous process of bounded variation, it must then be that, for each $\tau \in \mathcal{T}$,

$$M_\tau = M_0 = 0 \tag{A.12}$$

$\mathbf{P}_x$-almost surely. Now, suppose, by way of contradiction, that

$$a_n[G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2} \, \Delta \bar{J}^{2\prime}(q_n^1) \neq 0 \tag{A.13}$$

for some n such that $1 \leq n \leq N$. Let $x \equiv q_n^1$ and $\tau_\varepsilon \equiv \inf\{t \geq 0 : X_t \notin (q_n^1 - \varepsilon, q_n^1 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $q_{n+1}^1 < q_n^1 - \varepsilon < q_n^1 + \varepsilon < q_{n-1}^1$, with $q_0^1 \equiv \beta$ and $q_{N+1}^1 \equiv s^2$ by convention. From the properties of local time, we have that, for each $t > 0$, $L_t^{q_n^1} > 0$ $\mathbf{P}_{q_n^1}$-almost surely (see, for instance, Revuz and Yor (1999, Chapter VI, §2, Proof of Proposition 2.5)). It then follows from (A.11) and (A.13) that $M_{\tau_\varepsilon} \neq 0$ $\mathbf{P}_{q_n^1}$-almost surely, a contradiction by (A.12). Hence $\bar{J}^2$ satisfies (53). This completes the proof that $\bar{J}^2$ satisfies the variational system (48)–(54). The proof that $\bar{J}^1$ satisfies the variational system (42)–(47) is similar, and is omitted for the sake of brevity.

(Sufficiency) That the variational system (42)–(54) fully characterizes the players' value functions in MPEs of type 2 is an immediate consequence of the following verification lemma.

Lemma A.4 *Let w^1 and w^2 be solutions to the systems (42)–(47) and (48)–(54), respectively, for a given $N \in \mathbb{N} \setminus \{0\}$ and four sequences $(q_n^1)_{n=1}^N$, $(q_n^2)_{n=0}^{N-1}$, $(a_n)_{n=1}^N$, $(b_n)_{n=0}^{N-1}$ and a number s^2 as in the statement of Theorem 3. Then, for each $i = 1, 2$,*

$$w^i(x) \geq \sup_{\tau \in \mathcal{T}} J^i(x, \tau, (\mu^j, S^j)), \tag{A.14}$$

$$w^i(x) = J^i(x, (\mu^1, S^1), (\mu^2, S^2)), \tag{A.15}$$

where $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} b_n \delta_{q_n^2}, (\alpha, s^2]))$. In particular, $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE.

(Refinement) Let us finally prove the last assertion of Theorem 3. On the one hand, we have $\alpha^1 \leq \alpha^2 \leq s^2$, where the second inequality follows from Lemma S.4(ii) in the Online Supplement, and thus it can be easily checked that, for each $x \in \mathcal{I}$,

$$J^1(x, (\mu^1, (\alpha, \alpha^1]), (\mu^2, S^2)) = J^1(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^1(x),$$

which implies that $(\mu^1, (\alpha, \alpha^1])$ is a pbr to (μ^2, S^2). On the other hand, using (13) along with the fact that $G^2 = R^2$ over $(\alpha, \alpha^1]$ as $\alpha^1 \leq \alpha^2$, it is easily checked that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$,

$$J^2(x, (\mu^1, (\alpha, \alpha^1]), \tau^2) = J^2(x, (\mu^1, \emptyset), \tau^2 \wedge \tau_{(\alpha,\alpha^1]}) \leq \bar{J}^2(x)$$

and

$$J^2(x, (\mu^1, (\alpha, \alpha^1]), (\mu^2, S^2)) = J^2(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^2(x),$$

which implies that (μ^2, S^2) is a pbr to $(\mu^1, (\alpha, \alpha^1])$. Hence the result. ■

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Supplement to "The War of Attrition under Uncertainty: Theory and Robust Testable Implications": Additional Proofs

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Abstract

Section S.1 provides useful preliminary results. Section S.2 gathers proofs of lemmas that appear elsewhere in the literature or follow directly from existing results. Section S.3 provides the proofs of Propositions 1 and 2. Section S.4 gathers proofs of results in Section 4. Section S.5 proves two key lemmas for Theorem 2. Section S.6 proves two key lemmas for Theorem 3.

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S.1 Preliminaries

S.1.1 The Fundamental Filtration

We start with some definitions (Revuz and Yor (1999, Chapter I, §4)). The process X is defined over the canonical space $(\Omega, \mathcal{F})$ of continuous trajectories, and $\mathbf{P}_\mu$ denotes the law of the process X given an initial distribution $\mu \in \Delta(\mathcal{I})$, where $\Delta(\mathcal{I})$ is the space of Borel probability measures over $\mathcal{I}$. We denote by $(\mathcal{F}_t^0)_{t\geq 0}$ the natural filtration $(\sigma(X_s; s \leq t))_{t\geq 0}$ generated by X, and we let $\mathcal{F}_\infty^0 \equiv \sigma(\bigcup_{t\geq 0} \mathcal{F}_t^0)$. For each $\mu \in \Delta(\mathcal{I})$, we denote by $\mathcal{F}_\infty^\mu$ the completion of $\mathcal{F}_\infty^0$ with respect to $\mathbf{P}_\mu$, and, for each $t \geq 0$, we let $\mathcal{F}_t^\mu$ be the augmentation of $\mathcal{F}_t^0$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}_\infty^\mu$-measurable sets. The usual augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ is then defined by $\mathcal{F}_t \equiv \bigcap_{\mu\in\Delta(\mathcal{I})} \mathcal{F}_t^\mu$ for all $t \geq 0$. Because the process X is a Feller process in the sense of Revuz and Yor (1999, Chapter III, §2, Definition 2.5) and a standard process in the sense of Blumenthal and Getoor (1968, Chapter I, Definition 9.2), the filtration $(\mathcal{F}_t)_{t\geq 0}$ is actually right-continuous. As usual in this literature, we say that a property of the trajectories $\omega \in \Omega$ is satisfied *almost surely* if it is satisfied $\mathbf{P}_\mu$-almost surely for all $\mu \in \mathcal{I}$ or, equivalently, $\mathbf{P}_x$-almost surely for all $x \in \mathcal{I}$.

S.1.2 A Useful Change of Variables

Dayanik and Karatzas (2003) introduced an elegant change of variables that we use in several proofs. Specifically, for each $x \in \mathcal{I}$, define $\zeta(x) \equiv \frac{\phi(x)}{\psi(x)}$, which is strictly decreasing in x and maps $\mathcal{I}$ onto $(0, \infty)$. Then, for any function $g : \mathcal{I} \to \mathbb{R}$, define the function $\hat{g}$ by

$$\hat{g}(y) \equiv \frac{g}{\psi} \circ \zeta^{-1}(y), \quad y \in (0, \infty). \tag{S.1}$$

Observe that $\hat{\phi}(y) = y$ and $\hat{\psi}(y) = 1$ for all $y \in (0, \infty)$. A direct computation shows that, if $g \in \mathcal{C}^2(\mathcal{I})$, then

$$\hat{g}''(\zeta(x)) = \frac{2\phi(x)^3}{[\varrho\sigma(x)p'(x)]^2} (\mathcal{L}g - rg)(x), \quad x \in \mathcal{I}, \tag{S.2}$$

where p is the scale function of the diffusion X, which is uniquely defined up to an affine transformation by

$$p(x) \equiv \int_c^x \exp\left(-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}\,\mathrm{d}z\right) \mathrm{d}y, \quad x \in \mathcal{I}, \tag{S.3}$$

for some fixed $c \in \mathcal{I}$ (Karatzas and Shreve (1991, Chapter 5, Section 5, §B)), and

$$\varrho \equiv \frac{\psi'(x)\phi(x) - \psi(x)\phi'(x)}{p'(x)} > 0, \tag{S.4}$$

the ratio of the Wronskian of ψ and ϕ and of the derivative of the scale function, is a constant independent of x by Abel's theorem. From A3 and (S.2), we deduce that $\hat{R}^{i\prime\prime}(\zeta(x)) < 0$ for all $x \in (\alpha, x_0^i)$ or, equivalently, that $\hat{R}^{i\prime\prime}(y) < 0$ for all $y \in (\zeta(x_0^i), \infty)$ and thus, in particular, for all $y \in (\zeta(x_{R^i}), \infty)$ as $x_{R^i} < x_0^i$. From A7 and (S.2), we deduce that $\hat{G}^{i\prime\prime} \leq 0$ everywhere $\hat{G}^{i\prime\prime}$ is defined. Another useful remark is that, from Lemma 1 and A6, we have $G^i > 0$ over $\mathcal{I}$. Thus, $\hat{G}^i > 0$ over $(0, \infty)$, and (9) implies

$$\lim_{y \to 0} \hat{G}^i(y) = \lim_{y \to \infty} \frac{\hat{G}^i(y)}{y} = 0. \tag{S.5}$$

S.2 Basic Lemmas

Proof of Lemma 1. The proof proceeds along the same lines as that of Lemma 1 in Décamps, Gensbittel, and Mariotti (2021). The result follows. ■

Proof of Lemma 2. For each $\mu \in \Delta(\mathcal{I})$, ω and u^i are independent under $\mathbf{P}^i_\mu \equiv \mathbf{P}_\mu \otimes Leb$, and hence, for each $t \geq 0$,

$$\Gamma^i_t(\omega) = \mathbf{P}^i_\mu[\gamma^i \leq t \,|\, \mathcal{F}](\omega)$$

for $\mathbf{P}_\mu$-almost every $\omega \in \Omega$. We may assume that $\gamma(\cdot, u^i) \in \mathcal{T}$ for all u^i, as we can replace γ^i by the constant stopping time 0 for all u^i in a Borel set of Lebesgue measure zero without modifying the process Γ^i. Therefore, for all $u^i \in [0, 1]$ and $t \geq 0$, we have $\{\omega \in \Omega : \gamma^i(\omega, u^i) \leq t\} \in \mathcal{F}_t$ as $\gamma(\cdot, u^i) \in \mathcal{T}$. Using Corollary 2 in Solan, Tsirelson, and Vieille (2012), this implies that Γ^i_t is measurable with respect to the augmentation of $\mathcal{F}_t$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}^\mu_\infty$-measurable sets, which coincides with $\mathcal{F}^\mu_t$. As this is true for all $\mu \in \Delta(\mathcal{I})$, we deduce that Γ^i is adapted with respect to $\mathcal{F}_t$. In particular, letting $\mu \equiv \delta_x$ yields

$$\Gamma^i_t(\omega) = \mathbf{P}^i_x[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega)$$

for $\mathbf{P}_x$-almost every $\omega \in \Omega$ by the law of iterated expectations. The result follows. ■

Proof of Lemma 3. Suppose that, for each $i = 1, 2$, γ^i is a randomized stopping time with ccdf Γ^i. We have

$$\begin{aligned}
\overline{\mathbf{E}}_x\Big[1_{\{\gamma^i \leq \gamma^j\}}\, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i})\Big] &= \int_0^1 \int_0^1 \mathbf{E}_x\Big[1_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}}\, \mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)})\Big] \mathrm{d}u^j\, \mathrm{d}u^i \\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \int_0^1 1_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}}\, \mathrm{d}u^j\right] \mathrm{d}u^i \\
&= \int_0^1 \mathbf{E}_x\Big[\mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda^j_{\gamma^i(u^i)-}\Big] \mathrm{d}u^i \\
&= \mathbf{E}_x\left[\int_0^1 \mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda^j_{\gamma^i(u^i)-}\, \mathrm{d}u^i\right] \\
&= \mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-}\, \mathrm{d}\Gamma^i_t\right],
\end{aligned}$$

where the second and fourth equalities follow from Fubini's theorem, and the third equality follows from the definition of Λ^j. The last equality follows from observing that, for $\mathbf{P}_x$-almost every $\omega \in \Omega$, $t \mapsto \Gamma^i_t(\omega)$ is the cdf of the random variable $\gamma^i(\omega, \cdot)$ defined on the probability space $([0,1], \mathcal{B}([0,1]), Leb)$ and taking values in $[0, \infty]$, where $\Gamma^i_\infty(\omega) \equiv 1$ by convention; Fubini's theorem then implies that the random variable $u^i \mapsto \mathrm{e}^{-r\gamma^i(\omega,u^i)} R^i(X_{\gamma^i(\omega,u^i)}) \Lambda^j_{\gamma^i(\omega,u^i)-}$ is Lebesgue integrable over $[0,1]$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$,[1] and we can thus apply the usual formula for the expectation. The proof for the second term appearing in (13) and (14) is similar and thus omitted.

Let us then verify that (15) defines a randomized stopping time in the sense of Definition 1. That $\hat{\gamma}^i(u^i) \in \mathcal{T}$ for *Leb*-almost every $u^i \in [0,1]$ is standard (Jacod and Shiryaev (2003, Proposition I.1.28)). The random variable $(\omega, u^i) \mapsto \hat{\gamma}^i(u^i)(\omega)$ is $\mathcal{F}_\infty \otimes \mathcal{B}([0,1])$-measurable as it is nondecreasing and right-continuous with respect to u^i. That the ccdf associated to $\hat{\gamma}^i$ is Γ^i is proven in De Angelis, Ferrari, and Moriarty (2018, Lemma 4.1), who use this representation as the definition of a randomized stopping time. The result follows. ■

Proof of Lemma 4. We focus on player 1, the proof for player 2 being symmetrical. Observe from (14) that, for each $\tau^1 \in \mathcal{T}$, player 1's payoff from playing τ^1 against Γ^2 is

$$J^1(x, \tau^1, \Gamma^2) = \mathbf{E}_x \left[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) \Lambda^2_{\tau^1-} + \int_{[0,\tau^1)} \mathrm{e}^{-rt} G^1(X_t) \, \mathrm{d}\Gamma^2_t \right]. \tag{S.6}$$

Letting $\hat{\gamma}^1$ be the randomized stopping time associated to the ccdf Γ^1 by (15), we have

$$\begin{aligned} J^1(x, \Gamma^1, \Gamma^2) &= \int_0^1 \mathbf{E}_x \left[\mathrm{e}^{-r\hat{\gamma}^1(u^1)} R^1(X_{\hat{\gamma}^1(u^1)}) \Lambda^2_{\hat{\gamma}^1(u^1)-} + \int_{[0,\hat{\gamma}^1(u^1))} \mathrm{e}^{-rt} G^1(X_t) \, \mathrm{d}\Gamma^2_t \right] \mathrm{d}u^1 \\ &= \int_0^1 J^1(x, \hat{\gamma}^1(u^1), \Gamma^2) \, \mathrm{d}u^1 \\ &\leq \sup_{u^1 \in [0,1]} J^1(x, \hat{\gamma}^1(u^1), \Gamma^2) \\ &\leq \sup_{\tau^1 \in \mathcal{T}} J^1(x, \tau^1, \Gamma^2). \end{aligned}$$

where the first equality follows along the same steps as in the proof of Lemma 3, and the second equality follows from (S.6). The result follows. ■

The following consequence of the strong Markov property will be used several times throughout this Online Supplement.

Lemma S.1 *If the players use Markovian randomized stopping times with ccdfs* (Γ^1, Γ^2)*, then, for all* $x \in \mathcal{I}$ *and* $\tau \in \mathcal{T}$*, their expected payoffs write as*

$$\begin{aligned} J^i(x, \Gamma^1, \Gamma^2) = \mathbf{E}_x \bigg[&\int_{[0,\tau)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-} \, \mathrm{d}\Gamma^i_t + \int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t) \Lambda^i_t \, \mathrm{d}\Gamma^j_t \\ &+ \mathrm{e}^{-r\tau} J^i(X_\tau, \Gamma^1, \Gamma^2) \Lambda^j_{\tau-} \Lambda^i_{\tau-} \bigg]. \end{aligned} \tag{S.7}$$

[1]Recall that, by convention, this random variable is equal to 0 if $\gamma^i(\omega, u^i) = \infty$.

Proof. It follows from Lemma 3 that

$$J^i(x,\Gamma^1,\Gamma^2) = \mathbf{E}_x\bigg[\int_{[0,\tau)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t + \mathrm{e}^{-r\tau} R^i(X_\tau)\Lambda^j_{\tau-}(\Gamma^i_\tau - \Gamma^i_{\tau-}) + \mathrm{e}^{-r\tau} G^i(X_\tau)\Lambda^i_\tau(\Gamma^j_\tau - \Gamma^j_{\tau-}) + \int_{(\tau,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{(\tau,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\bigg]. \tag{S.8}$$

Notice from (17) that the only jump of Λ^i occurs at τ_{S^i}, at which time Λ^i jumps down to 0 and remains there forever after, and similarly for Λ_j. Hence

$$\begin{aligned}
&\mathrm{e}^{-r\tau} R^i(X_\tau)\Lambda^j_{\tau-}(\Gamma^i_\tau - \Gamma^i_{\tau-}) + \mathrm{e}^{-r\tau} G^i(X_\tau)\Lambda^i_\tau(\Gamma^j_\tau - \Gamma^j_{\tau-}) \\
&\quad = 1_{\{\tau_{S^j}\geq\tau=\tau_{S^i}\}}\,\mathrm{e}^{-r\tau} R^i(X_\tau)\Lambda^j_{\tau-}\Lambda^i_{\tau-} + 1_{\{\tau_{S^i}>\tau=\tau_{S^j}\}}\,\mathrm{e}^{-r\tau} G^i(X_\tau)\Lambda^i_{\tau-}\Lambda^j_{\tau-} \\
&\quad = 1_{\{\tau_{S^j}\geq\tau=\tau_{S^i}\}}\,\mathrm{e}^{-r\tau} J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-} + 1_{\{\tau_{S^i}>\tau=\tau_{S^j}\}}\,\mathrm{e}^{-r\tau} J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^i_{\tau-}\Lambda^j_{\tau-} \\
&\quad = 1_{\{\tau\geq\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau} J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-},
\end{aligned} \tag{S.9}$$

where the last equality follows from the fact that $\mathrm{e}^{-r\tau} J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^i_{\tau-}\Lambda^j_{\tau-}$ vanishes over $\{\tau > \tau_{S^i}\wedge\tau_{S^j}\}$. On the other hand, we have

$$\begin{aligned}
&\int_{(\tau,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{(\tau,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t \\
&\quad = 1_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\bigg[\int_{(0,\infty)} \mathrm{e}^{-rt} R^i(X_{\tau+t})\Lambda^j_{(\tau+t)-}\,\mathrm{d}\Gamma^i_{\tau+t} + \int_{(0,\infty)} \mathrm{e}^{-rt} G^i(X_{\tau+t})\Lambda^i_{\tau+t}\,\mathrm{d}\Gamma^j_{\tau+t}\bigg] \\
&\quad = 1_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\Lambda^j_\tau\Lambda^i_\tau\bigg[\int_{(0,\infty)} \mathrm{e}^{-rt} R^i(X_t\circ\theta_\tau)(\Lambda^j_{t-}\circ\theta_\tau)\,\mathrm{d}(\Gamma^i_t\circ\theta_\tau) \\
&\qquad\qquad + \int_{(0,\infty)} \mathrm{e}^{-rt} G^i(X_t\circ\theta_\tau)(\Lambda^i_t\circ\theta_\tau)\,\mathrm{d}(\Gamma^2_t\circ\theta_\tau)\bigg] \\
&\quad = 1_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\Lambda^j_{\tau-}\Lambda^i_{\tau-}\bigg[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t\circ\theta_\tau)(\Lambda^j_{t-}\circ\theta_\tau)\,\mathrm{d}(\Gamma^i_t\circ\theta_\tau) \\
&\qquad\qquad + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t\circ\theta_\tau)(\Lambda^i_t\circ\theta_\tau)\,\mathrm{d}(\Gamma^j_t\circ\theta_\tau)\bigg],
\end{aligned}$$

where the second equality follows from (16). Taking expectations and applying the strong Markov property at τ yields

$$\mathbf{E}_x\bigg[\int_{(\tau,\infty)} \mathrm{e}^{-rt} R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{(\tau,\infty)} \mathrm{e}^{-rt} G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\bigg] = \mathbf{E}_x\big[1_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau} J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-}\big]. \tag{S.10}$$

Inserting (S.9) and (S.10) into (S.8) yields (S.7). The result follows. ■

S.3 Proofs of Propositions 1 and 2

Proof of Proposition 1. Suppose, with no loss of generality, that $i = 1$. We first prove that $V_{R^1} \leq \bar{J}^1 \leq G^1$. For the first inequality, let $\tau^1 \equiv \tau_{(\alpha,x_{R^1}]}$, the hitting time by X of

$(\alpha, x_{R^1}]$, and let $\hat{\gamma}^2(u)$ be defined by (15). Using Lemma 3 and $G^1 \geq V_{R^1}$ by A6, we obtain

$$\begin{aligned}
\bar{J}^1(x) &\geq J^1(x, \tau^1, \Gamma^2) \\
&= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))\,\mathrm{d}u \\
&\geq \int_0^1 \mathbf{E}_x\Big[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u
\end{aligned}$$

for all $x \in \mathcal{I}$. For each $u \in [0,1]$, we have

$$\mathrm{e}^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) \,|\, \mathcal{F}_{\hat{\gamma}^2(u)}\Big]$$

$\mathbf{P}_x$-almost surely over $\{\tau^1 > \hat{\gamma}^2(u)\}$. Thus, by the tower property of conditional expectation,

$$\mathbf{E}_x\Big[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)})\Big] = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1})\Big] = V_{R^1}(x),$$

and we conclude that, for each $x \in \mathcal{I}$,

$$\bar{J}^1(x) \geq \int_0^1 V_{R^1}(x)\,\mathrm{d}u = V_{R^1}(x).$$

For the second inequality, we have $R^1 \leq V_{R^1} \leq G^1$ by A6. Hence, for each $\tau^1 \in \mathcal{T}$,

$$\begin{aligned}
J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))\,\mathrm{d}u \\
&\leq \int_0^1 \mathbf{E}_x\Big[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} G^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u \\
&= \int_0^1 \mathbf{E}_x\Big[\mathrm{e}^{-r(\tau^1 \wedge \hat{\gamma}^2(u))} G^1(X_{\tau^1 \wedge \hat{\gamma}^2(u)})\Big]\,\mathrm{d}u \\
&\leq \int_0^1 G^1(x)\,\mathrm{d}u \\
&= G^1(x)
\end{aligned}$$

for all $x \in \mathcal{I}$, where the second inequality follows from the fact that $(\mathrm{e}^{-rt} G^1(X_t))_{t \geq 0}$ is a supermartingale by A7. We now prove properties (i)–(v) in turn.

(i) It is not optimal for player 1 to concede at $x \in S^2$ if $R^1(x) < G^1(x)$, that is, if $x > \alpha^1$. Therefore, if (μ^1, S^1) is a pbr to (μ^2, S^2), then $S^1 \cap S^2 \cap (\alpha^1, \beta) = \emptyset$.

(ii) This directly follows from the definition (13) of players' payoffs.

(iii) By Lemma 3, for each $x \in \operatorname{supp} \mu^1$, we have

$$\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2)\,\mathrm{d}u,$$

where $\hat{\gamma}^1(u) = \inf\{t \geq 0 : \Gamma^1_t > u\}$. Thus the inequality $J^1(x, \hat{\gamma}^1(u), \Gamma^2) \leq \bar{J}^1(x)$, which holds for all $u \in [0,1]$, must be an equality for all u in a set U of Lebesgue measure 1. By definition of Γ^1, $\hat{\gamma}^1(u) = \inf\{t \geq 0 : 1 - \mathrm{e}^{-\int_{\mathcal{I} \setminus S^1} L^y_t\, \mu^1(\mathrm{d}y)} > u\} \wedge \tau_{S^1}$ for all $u \in [0,1)$. Notice that $\hat{\gamma}^1(u) > 0$ $\mathbf{P}_x$-almost surely for all $u \in (0,1)$ as the mapping $t \mapsto 1_{\{t<\tau_{S^1}\}}\,\mathrm{e}^{-\int_{\mathcal{I}\setminus S^1} L_t^y\,\mu^1(\mathrm{d}y)}$ is continuous over $[0, \tau_{S^1})$ by Theorem 1. We claim that, because $x \in \operatorname{supp}\mu^1$, we also have $\lim_{u\to 0}\hat{\gamma}^1(u) = 0$ $\mathbf{P}_x$-almost surely. Indeed, $\hat{\gamma}^1(u,\omega)$ is nondecreasing with respect to u for all ω and converges to $\hat{\gamma}^1(0,\omega) = \inf\{t \geq 0 : \int_{\mathcal{I}\setminus S^1} L_t^y(\omega)\,\mu^1(\mathrm{d}y) > 0\} \wedge \tau_{S^1}(\omega)$. Let us fix a continuous version $(t,y) \mapsto L_t^y$ of the local time of X (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), and observe that $L_t^x > 0$ $\mathbf{P}_x$-almost surely for all $t > 0$. Thus there exist a sequence $(t_n)_{n\geq 1}$ converging to 0 and, for each $n \geq 1$, a set $\Omega_{t_n} \in \mathcal{F}$ of $\mathbf{P}_x$-probability 1 such that $L_{t_n}^x(\omega) > 0$ and $y \mapsto L_{t_n}^y(\omega)$ is continuous at x for all $\omega \in \Omega_{t_n}$. Now, $x \in \operatorname{supp}\mu^1$ and $\operatorname{supp}\mu^1$ being closed jointly imply that any open interval of $\mathcal{I}$ containing x has positive μ^1-measure. From these observations, it follows that, for each $n \geq 1$, $\int_{\mathcal{I}\setminus S^1} L_{t_n}^y(\omega)\,\mu^1(\mathrm{d}y) > 0$ for all $\omega \in \Omega_{t_n}$, so that $\hat{\gamma}^1(0,\omega) = 0$ for all $\omega \in \bigcap_{n\geq 1}\Omega_{t_n}$ and thus $\mathbf{P}_x$-almost surely, as claimed. Finally, for each $u \in U$,

$$\bar{J}^1(x) = J^1(x, \hat{\gamma}^1(u), \Gamma^2) = \mathbf{E}_x\left[\int_{[0,\hat{\gamma}^1(u))} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\hat{\gamma}^1(u)} R^1(X_{\hat{\gamma}^1(u)})\Lambda^2_{\hat{\gamma}^1(u)-}\right]. \tag{S.11}$$

Using bounded convergence to take the limit as $u \in U$ goes to 0, two cases must be distinguished. If $x \notin S^2$, then Γ_t^2 is continuous at $t = 0$, from which it follows that $\bar{J}^1(x) = R^1(x)$. If $x \in S^2$, then $\Gamma_{0-}^2 = 0$, $\Gamma_0^2 = 1$, and $\Lambda^2_{\hat{\gamma}^1(u)-} = 0$ for all $u \in (0,1)$, from which it follows that $\bar{J}^1(x) = G^1(x)$.

(iv) We claim that, for each $x \in \mathcal{I}$,

$$\bar{J}^1(x) \geq J^1(x, (0, (\alpha, x_{R^1}]), (\mu^2, S^2)) \geq J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)). \tag{S.12}$$

The first inequality in (S.12) directly follows from the fact that (μ^1, S^1) is a pbr to (μ^2, S^2). For the second one, recall that, by A6,

$$G^1(x) \geq V_{R^1}(x) = \sup_{\tau\in\mathcal{T}} \mathbf{E}_x\big[\mathrm{e}^{-r\tau} R^1(X_\tau)\big] = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1})\Big],$$

where $\tau^1 \equiv \tau_{(\alpha, x_{R^1}]}$. We have

$$\begin{aligned} J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))\,\mathrm{d}u \\ &= \int_0^1 \mathbf{E}_x\Big[1_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) + 1_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u. \end{aligned}$$

Over $\{\tau^1 > \hat{\gamma}^2(u)\}$, we have

$$\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \geq \mathrm{e}^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) \,|\, \mathcal{F}_{\hat{\gamma}^2(u)}\Big].$$

$\mathbf{P}_x$-almost surely by A6. Therefore, using the tower property of conditional expectation,

$$J^1(x, \tau^1, \hat{\gamma}^2(u)) \geq \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1})\Big],$$

which implies the second inequality of (S.12) upon integrating with respect to u. The conclusion follows from noticing that $J^1(x,(0,(\alpha,x_{R^1}]),(0,\emptyset)) = V_{R^1}(x) > R^1(x)$ for all $x > x_{R^1}$ and applying (ii) and the first assertion in (iii).

(v) Arguing as in (iii) yields that $\bar{J}^1(x) = \int_0^1 J^1(x,\hat{\gamma}^1(u),\Gamma^2)\,\mathrm{d}u$ for all u in a set U of Lebesgue measure 1. Moreover, using the explicit expression for $\hat{\gamma}^1(u)$ given in (iii), it is easy to check that $\lim_{u\to 1}\hat{\gamma}^1(u) = \tau_{S^1}$. Therefore, taking the limit in (S.11) as $u\in U$ goes to 1, we deduce that

$$\bar{J}^1(x) = \mathbf{E}_x\left[\int_{[0,\tau_{S^1})} \mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_{S^1}}R^1(X_{\tau_{S^1}})\Lambda^2_{\tau_{S^1}-}\right] = J^1(x,\tau_{S^1},\Gamma^2)$$

by bounded convergence, from which the first assertion follows. For the second assertion, let $\tilde{\Gamma}^1$ be the ccdf associated to $(\tilde{\mu}^1,S^1)$ and $\tilde{\gamma}^1(u) = \inf\{t\geq 0 : \tilde{\Gamma}^1_t > u\}$. By assumption,

$$\bar{J}^1(X_{\tilde{\gamma}^1(u)}) = R^1(X_{\tilde{\gamma}^1(u)}). \tag{S.13}$$

for all $u\in[0,1]$. On the one hand,

$$J^1(x,\tilde{\Gamma}^1,\Gamma^2) = \int_0^1 \mathbf{E}_x\left[\int_{[0,\tilde{\gamma}^1(u))} \mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\gamma}^1(u)}R^1(X_{\tilde{\gamma}^1(u)})\Lambda^2_{\tilde{\gamma}^1(u)-}\right]\mathrm{d}u. \tag{S.14}$$

On the other hand, using that $\bar{J}^1 = J^1(\cdot,\tau_{S^1},\Gamma^2)$ and applying the strong Markov property at $\tilde{\gamma}^1(u)$ in (14) yields

$$\begin{aligned}\bar{J}^1(x) &= \mathbf{E}_x\left[\int_{[0,\tilde{\gamma}^1(u))} \mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\gamma}^1(u)}\bar{J}^1(X_{\tilde{\gamma}^1(u)})\Lambda^2_{\tilde{\gamma}^1(u)-}\right]\\ &= \mathbf{E}_x\left[\int_{[0,\tilde{\gamma}^1(u))} \mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\gamma}^1(u)}R^1(X_{\tilde{\gamma}^1(u)})\Lambda^2_{\tilde{\gamma}^1(u)-}\right],\end{aligned} \tag{S.15}$$

where the second equality follows from (S.13). Integrating (S.15) with respect to u yields (S.14), from which the second assertion follows. Hence the result. ■

Proof of Proposition 2. Our argument requires some technical results on processes $A\equiv(A_t)_{t\geq 0}$ of the form $A_t \equiv \int_{\mathcal{I}\setminus S} L^x_t\,\mu(\mathrm{d}x)$, where $S\subset\mathcal{I}$ is a closed set and μ is a Radon measure over $\mathcal{I}\setminus S$. Precisely, if τ is the first exit time of $(a,b)\subset\mathcal{I}\setminus S$, with $[a,b]\subset\mathcal{I}$, then

$$\mathbf{E}_x[A_\tau] = \int_{\mathcal{I}\setminus S}\mathbf{E}_x[L^y_\tau]\,\mu(\mathrm{d}y) = \int_{(a,b)}\mathbf{E}_x[L^y_\tau]\,\mu(\mathrm{d}y) = \int_{(a,b)} 2[p'(y)]^{-1}\Phi_{a,b}(x,y)\,\mu(\mathrm{d}y), \tag{S.16}$$

where p' is the derivative of the scale function (S.3) of the diffusion X, and

$$\Phi_{a,b}(x,y) \equiv \frac{[p(x\wedge y)-p(a)][p(b)-p(x\vee y)]}{p(b)-p(a)}$$

is the Green function of the diffusion X killed at the boundaries a and b (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §11, and Section 2, §13)). It is easy to check that $\mathbf{E}_x[A_\tau]$ is finite if and only if, for some $x\in(a,b)$,

$$\int_a^x [p(y)-p(a)]\,\mu(\mathrm{d}y) < \infty \quad\text{and}\quad \int_x^b [p(b)-p(y)]\,\mu(\mathrm{d}y) < \infty.$$

A more precise result can be stated as follows (Cetin (2018, Theorem 2.1)):

$$A_{\tau_a} 1_{\{\tau_a<\tau_b\}} = \infty \text{ a.s. if } \int_a^x [p(y)-p(a)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a,b), \tag{S.17}$$

$$A_{\tau_a} 1_{\{\tau_a<\tau_b\}} < \infty \text{ a.s. otherwise.} \tag{S.18}$$

A symmetric result holds for b. The following lemma is key to our continuity result.

Lemma S.2 *For each $t \geq 0$, let $A_t \equiv \int_{(a,b)} L_t^y\,\mu(\mathrm{d}y)$ for some Radon measure μ over $(a,b) \subset \mathcal{I}$. Then the function h defined, for nonnegative constants C_a and C_b, by*

$$h(x) = \mathbf{E}_x\Big[C_a 1_{\{\tau_a<\tau_b\}}\,\mathrm{e}^{-A_{\tau_a}} + C_b 1_{\{\tau_b<\tau_a\}}\mathrm{e}^{-A_{\tau_b}}\Big], \quad x \in (a,b),$$

is nonnegative, p-convex,[2] and continuous over (a,b). Moreover, the limits $h(a+)$ and $h(b-)$ exist and are given by

$$h(a+) = \begin{cases} 0 & \text{if } \int_a^x [p(y)-p(a)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a,b) \\ C_a & \text{otherwise} \end{cases}, \tag{S.19}$$

$$h(b-) = \begin{cases} 0 & \text{if } \int_x^b [p(b)-p(y)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a,b) \\ C_b & \text{otherwise} \end{cases}. \tag{S.20}$$

Proof. First, h is clearly nonnegative. Next, applying the strong Markov property to $h(\lambda x_1 + (1-\lambda)x_2)$ at $\tau_{x_1} \wedge \tau_{x_2}$ yields

$$h(\lambda x_1 + (1-\lambda)x_2) = \mathbf{E}_{\lambda x_1+(1-\lambda)x_2}\Big[h(x_1)1_{\{\tau_{x_1}<\tau_{x_2}\}}\,\mathrm{e}^{-A_{\tau_{x_1}}} + h(x_2)1_{\{\tau_{x_2}<\tau_{x_1}\}}\,\mathrm{e}^{-A_{\tau_{x_2}}}\Big].$$

Using that $\mathrm{e}^{-A_t} \leq 1$, we then obtain from standard computations (Karatzas and Shreve (1991, Chapter 5, Section 5, §C)) that h is p-convex. Finally, that h is continuous follows from its being p-convex (Revuz and Yor (1999, Appendix, §3)).

Consider now (S.19). If $\int_a^x [p(y)-p(a)]\,\mu(\mathrm{d}y) = \infty$ for some $x \in (a,b)$, then by (S.17) $h(x) = \mathbf{E}_x\big[C_b 1_{\{\tau_b<\tau_a\}}\,\mathrm{e}^{-A_{\tau_b}}\big]$ and thus $0 \leq h(x) \leq C_b \mathbf{P}_x[\tau_b < \tau_a]$, which goes to 0 as x goes to a. Hence $h(a+) = 0$. If $\int_a^x [p(y)-p(a)]\,\mu(\mathrm{d}y) < \infty$ for some $x \in (a,b)$, then by (S.18) $\mathrm{e}^{-A_{\tau_a}} > 0$ $\mathbf{P}_x$-almost surely. If $(a_n)_{n\geq 1}$ is a decreasing sequence converging to a and strictly bounded above by x, then, applying the strong Markov property to $h(x)$ at τ_{a_n}, we have

$$h(x) = \mathbf{E}_x\Big[h(a_n)1_{\{\tau_{a_n}<\tau_b\}}\,\mathrm{e}^{-A_{\tau_{a_n}}} + C_b 1_{\{\tau_b<\tau_{a_n}\}}\,\mathrm{e}^{-A_{\tau_b}}\Big].$$

Using bounded convergence to take the limit along any subsequence $(h(a_{n_k}))_{k\geq 1}$ converging to some $z < \infty$, we obtain that

$$h(x) = \mathbf{E}_x\Big[z 1_{\{\tau_a<\tau_b\}}\,\mathrm{e}^{-A_{\tau_a}} + C_b 1_{\{\tau_b<\tau_{a_n}\}}\,\mathrm{e}^{-A_{\tau_b}}\Big],$$

[2]That is,

$$h(\lambda x_1 + (1-\lambda)x_2) \leq h(x_1)\,\frac{p(x_2) - p(\lambda x_1 + (1-\lambda)x_2)}{p(x_2)-p(x_1)} + h(x_2)\,\frac{p(\lambda x_1 + (1-\lambda)x_2) - p(x_1)}{p(x_2)-p(x_1)}$$

for all $x_1, x_2 \in (a,b)$ and $\lambda \in [0,1]$.

and thus $z = C_a$ as $\mathbf{E}_x\big[1_{\{\tau_a<\tau_b\}}\,\mathrm{e}^{-A_{\tau_a}}\big] > 0$. It follows that $\lim_{n\to\infty} h(a_n) = C_a$. Because this is true for any decreasing sequence $(a_n)_{n\geq 0}$ converging to a, this implies that $h(a+)$ exists and is equal to C_a. This concludes the proof of (S.19). The argument for (S.20) proceeds along similar lines, using (S.18). The result follows. ■

The proof of Proposition 2 relies on two preliminary lemmas.

Lemma S.3 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf $\bar{J}^i$, then the restriction of $\bar{J}^i$ to $[a, b]$ is continuous for any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$.*

Proof. Suppose, with no loss of generality, that $i = 1$. Given $x \notin S^1 \cup S^2$, and for each integer $n \geq 1$, let $\tilde{\tau}_n \equiv \tau_{x-\eta} \wedge \tau_{x+\varepsilon_n}$, where $\eta > 0$, $(\varepsilon_n)_{n\geq 1}$ is a decreasing sequence converging to 0, and $[x-\eta, x+\varepsilon_n] \subset \mathcal{I} \setminus (S^1 \cup S^2)$. Applying Lemma S.1 with $\tau \equiv \tilde{\tau}_n$ yields

$$\begin{aligned}
&\bar{J}^1(x)\\
&= \mathbf{E}_x\left[\int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} R^1(X_t)\Lambda^2_{t-}\,\mathrm{d}\Gamma^1_t + \int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} G^1(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\tau}_n}\bar{J}^1(X_{\tilde{\tau}_n})\Lambda^2_{\tilde{\tau}_n-}\Lambda^1_{\tilde{\tau}_n-}\right]\\
&= \mathbf{E}_x\left[\int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} R^1(X_t)\Lambda^2_t\,\mathrm{d}\Gamma^1_t + \int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} G^1(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\tau}_n}\bar{J}^1(X_{\tilde{\tau}_n})\Lambda^2_{\tilde{\tau}_n}\Lambda^1_{\tilde{\tau}_n}\right], \quad \text{(S.21)}
\end{aligned}$$

where the second equality follows from the fact that $\Lambda^i_{t-} = \Lambda^i_t$ over $\{t \leq \tilde{\tau}_n\}$. Consider a subsequence $(\bar{J}^1(x+\varepsilon_{n_k}))_{k\geq 1}$ converging to some z. Because η is fixed, $\tilde{\tau}_{n_k}$ goes to 0 $\mathbf{P}_x$-almost surely as k goes to ∞, and $\mathbf{P}_x[\tilde{\tau}_{n_k} = \tau_{x+\varepsilon_{n_k}}]$ goes to 1. The equality $X_{\tilde{\tau}_{n_k}} = 1_{\{\tilde{\tau}_{n_k}=\tau_{x-\eta}\}}(x-\eta) + 1_{\{\tilde{\tau}_{n_k}=\tau_{x+\varepsilon_{n_k}}\}}(x+\varepsilon_{n_k})$ then implies that $\bar{J}^1(X_{\tilde{\tau}_{n_k}})$ goes to z $\mathbf{P}_x$-almost surely as k goes to infinity. Using bounded convergence to take the limit in (S.21), and taking advantage of the fact that both Γ^1_t and Γ^2_t are continuous at $t = 0$ as $x \notin S^1 \cup S^2$, we obtain that $\bar{J}^1(x) = z$, from which it follows as in the proof of Lemma S.2 that $\bar{J}^1$ is right-continuous at x. The proof that $\bar{J}^1$ is left-continuous at x is similar.

Now, let us consider an interval $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$. That $\bar{J}^1$ is continuous over (a, b) follows from the preceding argument; but we need to check that $\bar{J}^1$ is right-continuous at a and left-continuous at b. We focus on a, the arguments for b being symmetrical. Because $S^1 \cup S^2$ is closed, the only difficulty arises when $a \in S^1 \cup S^2$. We distinguish two cases.

Case 1 Suppose first that $a \in S^1$, so that $\bar{J}^1(a) = R^1(a)$ by Proposition 1(ii). By Proposition 1(v), $\bar{J}^1 = J^1(\cdot, (0, S^1), (\mu^2, S^2))$. Applying Lemma S.1 with $\tau \equiv \tau_a \wedge \tau_b$ yields

$$\bar{J}^1(x) = \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\right]$$

for all $x \in (a, b)$. Moreover,

$$0 \leq \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t\right] \leq C\,\mathbf{E}_x\big[1 - \mathrm{e}^{-A^2_\tau}\big],$$

where C is an upper bound for G^1 over $[a, b]$. Because $a \notin S^2$, μ^2 is locally finite at a. Applying Lemma S.2 with $C_a = C_b \equiv 1$ and $\mu \equiv \mu^2$ then yields that $\mathbf{E}_x\big[1 - \mathrm{e}^{-A^2_\tau}\big]$ goes to 0 as $x > a$ goes to a. Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.2 also yields that $\mathbf{E}_x\big[\mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\big]$ goes to $\bar{J}^1(a) = R^1(a)$ as $x > a$ goes to a. Thus $\bar{J}^1$ is right-continuous at a.

Case 2 Suppose next that $a \in S^2$, so that $\bar{J}^1(a) = G^1(a)$ by Proposition 1(i) and (iii). Fix some $\varepsilon \in (0, b-a)$. As in Case 1 with $\tau \equiv \tau_a \wedge \tau_{a+\varepsilon}$, we have

$$\bar{J}^1(x) = \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\right]$$

for all $x \in (a, a+\varepsilon)$. If $\int_a^x [p(y) - p(a)]\,\mu^2(\mathrm{d}y) < \infty$, the proof proceeds along the same lines as in Case 1. Thus let us assume that $\int_a^x [p(y) - p(a)]\,\mu^2(\mathrm{d}y) = \infty$. Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.2 yields that $\mathbf{E}_x\big[\mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\big]$ goes to 0 as $x > a$ goes to a. Moreover,

$$\mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t\right] \geq \min_{y \in [a, a+\varepsilon]} G^1(y)\,\mathbf{E}_x\big[\mathrm{e}^{-r\tau} - \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\big].$$

By (6) and Lemma S.2, the last expectation goes to 1 as $x > 0$ goes to a. We deduce that $\liminf_{x \to a+} \bar{J}^1(x) \geq \min_{y \in [a, a+\varepsilon]} G^1(y)$ and thus that $\liminf_{x \to a+} \bar{J}^1(x) \geq G^1(a)$ by letting ε go to zero. Finally, we also have $\limsup_{x \to a+} \bar{J}^1(x) \leq G^1(a)$ as $\bar{J}^1 \leq G^1$ by Proposition 1, and this concludes the proof that $\bar{J}^1$ is right-continuous at a. The result follows. ■

Lemma S.4 *The following holds*:

(i) If (μ^i, S^i) is a pbr to (μ^j, S^j), then $(\alpha, \alpha^i] \subset S^1 \cup S^2$;

(ii) If $\alpha^1 < \alpha^2$ and $((\mu^1, S^1), (\mu^2, S^2))$ is an MPE, then S^1 and S^2 cannot both intersect $(\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2]$, so that either $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^1$ or $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^2$.

Proof. (i) Suppose, with no loss of generality, that $i = 1$, and recall that $\bar{J}^1 = R^1 = V_{R^1} = G^1$ over $(\alpha, \alpha^1]$. Suppose, by way of contradiction, that $x \in (\alpha, \alpha^1) \setminus (S^1 \cup S^2)$. Let $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$, with $b < \alpha^1$ and $x \in (a, b)$. Because $(0, S^1)$ is also a pbr to (μ^2, S^2) by Proposition 1(v), applying Lemma S.1 with $\tau \equiv \tau_{\mathcal{I} \setminus (a,b)}$ yields

$$\begin{aligned}
\bar{J}^1(x) &= \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau}\bar{J}^1(X_\tau)\Lambda^2_{\tau-}\right] \\
&= \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} R^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda^2_{\tau-}\right] \\
&= \int_0^1 \mathbf{E}_x\left[1_{\{\hat{\gamma}^2(u,\cdot) < \tau\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u,\cdot)} R^1(X_{\hat{\gamma}^2(u,\cdot)}) + 1_{\{\hat{\gamma}^2(u,\cdot) \geq \tau\}}\,\mathrm{e}^{-r\tau} R^1(X_\tau)\right] \mathrm{d}u \\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r(\hat{\gamma}^2(u,\cdot) \wedge \tau)} R^1(X_{\hat{\gamma}^2(u,\cdot) \wedge \tau})\right] \mathrm{d}u \\
&< R^1(x),
\end{aligned} \tag{S.22}$$

where the third equality follows along the same lines as in Lemma 3, and the inequality follows from A3 together with the fact that, for each $u > 0$, $\tau \wedge \hat{\gamma}^2(u, \cdot) > 0$ $\mathbf{P}_x$-almost surely as Γ^2 is continuous over $[0, \tau_{S^2})$ and $\tau_{S^2} > 0$ $\mathbf{P}_x$-almost surely. By (S.22), $J(x) < R^1(x)$, in contradiction with Proposition 1. Therefore, $(\alpha, \alpha^1) \subset S^1 \cup S^2$, from which (i) follows as $S^1 \cup S^2$ is closed.

(ii) Suppose, with no loss of generality, that $\alpha^1 < \alpha^2$. By Proposition 1(i), $S^1 \cap S^2 \cap (\alpha^1, \alpha^2] = \emptyset$, and, as shown in (i), $(\alpha^1, \alpha^2] \subset S^1 \cup S^2$. It follows that $S^1 \cap (\alpha^1, \alpha^2]$ and $S^2 \cap (\alpha^1, \alpha^2]$, which are both relatively closed sets in $(\alpha^1, \alpha^2]$, are complementary sets in $(\alpha^1, \alpha^2]$, and thus are both relatively open in $(\alpha^1, \alpha^2]$. As their union $(\alpha^1, \alpha^2]$ is a connected set, either one or the other must be empty. Thus either $(\alpha^1, \alpha^2] \subset S^1$ or $(\alpha^1, \alpha^2] \subset S^2$, from which (ii) follows as both S^1 and S^2 are closed sets. The result follows. ■

We are now ready to complete the proof of Proposition 2. We focus on the right-continuity of the functions $\bar{J}^i$, $i = 1, 2$, the arguments for their left-continuity being symmetrical. For any function $J : \mathcal{I} \to \mathbb{R}$ and for each $S \subset \mathcal{I}$, we denote by $J_{|S}$ the restriction of J to S. Suppose, with no loss of generality, that $\alpha^1 \leq \alpha^2$. For each $i = 1, 2$, $R^i = G^i$ over $(\alpha, \alpha^i]$ and $R^i \leq \bar{J}^i \leq G^i$ by Proposition 1. Thus $\bar{J}^i$ is continuous over $(\alpha, \alpha^i]$ and, in particular, over $(\alpha, \alpha^1]$. Moreover, by Lemma S.4(ii), J^1 coincides with R^1 or G^1 over $(\alpha^1, \alpha^2]$. We conclude that, for each $i = 1, 2$, $\bar{J}^i_{|(\alpha, \alpha^2]}$ is continuous. Notice that $\bar{J}^2$ is right-continuous at α^2 and that the same is true for $\bar{J}^1$ if $\alpha^1 = \alpha^2$. By Lemma S.3, for each $i = 1, 2$, $\bar{J}^i_{|[a,b]}$ is continuous for any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$; moreover, $\bar{J}^i_{|S^i} = R^i_{|S^i}$ and $\bar{J}^i_{|S^j} = G^i_{|S^j}$ are also continuous. Therefore, if $\bar{J}^1$ or $\bar{J}^2$ is not right-continuous at x, it must be that $x \geq \alpha^2$, that $x \in S^1 \cup S^2$, and that, for each $\varepsilon > 0$, $[x, x + \varepsilon)$ intersects both $S^1 \cup S^2$ and $\mathcal{I} \setminus (S^1 \cup S^2)$; we refer to this last property as Property P. We distinguish two cases.

Case 1 Let us first consider the case where $x \in S^2$ and $x > \alpha^2$, and suppose, by way of contradiction, that $\bar{J}^1$ or $\bar{J}^2$ is not right-continuous at x, so that Property P is satisfied. As $(\alpha^2, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^1$. Hence, because S^1 is closed, there exists $\varepsilon > 0$ such that $[x, x + \varepsilon) \cap S^1 = \emptyset$. If (a, b) is a connected component of the open set $[x, x + \varepsilon) \setminus S^2$, so that $a, b \in S^2$, then it must be that $\mu^1[(a, b)] > 0$. Indeed, suppose, by way of contradiction, that this is not the case. Then, for each $y \in (a, b)$, we have

$$\bar{J}^2(y) = J^2(y, (0, S^1), (\mu^2, S^2)) = \mathbf{E}_y\big[\mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})\big] < R^2(y)$$

by A3 as $b \leq x_{R^2}$ by Proposition 1(iv), in contradiction with Proposition 1. Thus $\mu^1[(a, b)] > 0$ and, by Proposition 1(iii), there exists some $y \in (a, b)$ such that $\bar{J}^1(y) = R^1(y)$. As this is true for every connected component of $[x, x + \varepsilon) \setminus S^2$, Property P implies that there exists a decreasing sequence $(y_n)_{n \geq 1}$ converging to x such that $\bar{J}^1(y_n) = R^1(y_n)$, as well as a sequence of connected components $((a_n, b_n))_{n \geq 1}$ of $[x, x + \varepsilon) \setminus S^2$ such that $y_n \in (a_n, b_n)$ for all $n \geq 1$

and whose length goes to zero as n goes to ∞. By Proposition 1, $\bar{J}^1(a_n) = G^1(a_n)$ and $\bar{J}^1(b_n) = G^1(b_n)$. Because $x > \alpha^1$, $G^1(x) > R^1(x)$. For each $n \geq 1$, because $(0, S^2)$ is a best reply to (μ^1, S^1) by Proposition 1(v), applying Lemma S.1 to $\tau_n \equiv \tau_{a_n} \wedge \tau_{b_n}$ yields

$$\begin{aligned}\bar{J}^1(y_n) &= \mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_n}\bar{J}^1(X_{\tau_n})\Lambda^2_{\tau_n-}\right] \\ &= \mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_n} G^1(X_{\tau_n})\Lambda^2_{\tau_n-}\right].\end{aligned}$$

G^1 and R^1 being locally Lipschitz, there exists $\varepsilon > 0$ such that, for any sufficiently large n,

$$G^1(y) > R^1(y_n) + \varepsilon, \quad y \in (a_n, b_n).$$

Hence, for any such n,

$$\bar{J}^1(y_n) \geq [R^1(y_n) + \varepsilon]\,\mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt}\,\mathrm{d}\Gamma^2_s + \mathrm{e}^{-r\tau_n}\Lambda^2_{\tau_n-}\right] \geq [R^1(y_n) + \varepsilon]\,\mathbf{E}_{y_n}[\mathrm{e}^{-r\tau_n}].$$

We have $\mathbf{E}_{y_n}[e^{-r\tau_n}] = A_n\phi(y_n) + B_n\psi(y_n)$, where the coefficients A_n and B_n are such that

$$A_n\phi(a_n) + B_n\psi(a_n) = A_n\phi(b_n) + B_n\psi(b_n) = 1.$$

If follows that these coefficients are bounded, and, therefore, as ϕ and ψ are locally Lipschitz, that $\mathbf{E}_{y_n}[\mathrm{e}^{-r\tau_n}]$ goes to 1 as n goes ∞. This, for n sufficiently large, contradicts the fact that $\bar{J}^1(y_n) = R^1(y_n)$. Thus $\bar{J}^1$ and $\bar{J}^2$ are right-continuous at x. The right-continuity of $\bar{J}^1$ and $\bar{J}^2$ at x in case $x \in S^1$ and $x > \alpha^2$ and the right-continuity of $\bar{J}^1$ at x in case $x \in S^2$ and $x = \alpha^2 > \alpha^1$ can be proven in a similar way.

Case 2 It remains only to prove that $\bar{J}^1$ is right-continuous at x in case $x \in S^1$ and $x = \alpha^2 > \alpha^1$. Suppose that Property P is satisfied so that $\bar{J}^1$ may not be right-continuous at x. As $(\alpha^1, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^2$. Hence, because S^2 is closed, there exists $\varepsilon > 0$ such that $[x, x+\varepsilon) \cap S^2 = \emptyset$. Notice that $\mu^2([x, x+\varepsilon)) < \infty$ as μ^2 is locally finite on $\mathcal{I} \setminus S^2$. If (a, b) is a connected component of the open set $(x, x+\varepsilon) \setminus S^1$, so that $a, b \in S^1$, then, for $y \in (a, b)$ and $\tau \equiv \tau_a \wedge \tau_b$, we have

$$\begin{aligned}\bar{J}^1(y) - R^1(y) &= J^1(y, (0, S^1), (\mu^2, S^2)) - R^1(y) \\ &= \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda^2_{\tau-}\right] - R^1(y) \\ &\geq 0, \end{aligned} \tag{S.23}$$

where the first equality follows from Proposition 1(v). We also have

$$\begin{aligned}&\bar{J}^1(y) - R^1(y) \\ &\quad = \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda^2_{\tau-}\right] - R^1(y)\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} R^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda^2_{\tau-} - R^1(y) + \int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma^2_t\right] \\
&= \int_0^1 \mathbf{E}_y\left[\mathrm{e}^{-r[\tau\wedge\hat{\gamma}^2(u,\cdot)]} R^1(X_{\tau\wedge\hat{\gamma}^2(u,\cdot)}) - R^1(y)\right]\mathrm{d}u + \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}(G^1(X_t) - R^1(X_t))\,\mathrm{d}\Gamma^2_t\right] \\
&= \int_0^1 \mathbf{E}_y\left[\int_0^{\tau\wedge\hat{\gamma}^2(u,\cdot)} (\mathcal{L}R^1 - rR^1)(X_t)\,\mathrm{d}t\right]\mathrm{d}u + \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma^2_t\right] \\
&\leq \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma^2_t\right], \qquad \text{(S.24)}
\end{aligned}
$$

where the third equality follows along the same lines as in the proof of Lemma 3, the fourth equality follows from Itô's formula, and the inequality follows from A3 and Proposition 1(iv). Letting $C > 0$ be an upper bound for $G^1 - R^1$ over $[x, x+\varepsilon)$, we then have

$$
\begin{aligned}
\mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma^2_t\right] &\leq C\,\mathbf{E}_y[\Gamma^2_\tau] \\
&= C\,\mathbf{E}_y[1 - \Lambda^2_\tau] \\
&= C\,\mathbf{E}_y[1 - \mathrm{e}^{-A^2_\tau}] \\
&\leq C\,\mathbf{E}_y[A^2_\tau].
\end{aligned}
$$

From (S.16), we have, for some positive constant C',

$$
\mathbf{E}_y[A^2_\tau] = \int_a^b 2[p'(z)]^{-1}\Phi_{a,b}(y,z)\,\mu^2(\mathrm{d}z) \leq C'\mu^2[(a,b)],
$$

as the mapping $z \mapsto 2[p'(z)]^{-1}\Phi_{a,b}(y,z)$ is uniformly bounded over $[x, x+\varepsilon)$. Property P implies that there exists a sequence $((a_n, b_n))_{n\in\mathbb{N}}$ of connected components of $[x, x+\varepsilon) \setminus S^1$ whose length goes to zero as n goes to ∞. Because μ^2 is locally bounded at x, it must be that $\mu^2[(a_n, b_n)]$ goes to 0 as n goes to ∞, and the inequalities $0 \leq \bar{J}^1(y) - R^1(y) \leq C\,C'\,\mu^2[(a_n, b_n)]$ along with the fact that the constants C and C' are independent of n imply that $\bar{J}^1$ is right-continuous at x. Hence the result. ■

S.4 Proofs for Section 4

Lemma S.5 *The equation*

$$
R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)}\,G^1(x) \qquad \text{(S.25)}
$$

has a unique solution $\underline{x}^2 \in (\alpha^1, x_{R^1})$ *and* $R_1(x_{R^1}) < \frac{\phi(x_{R^1})}{\phi(x)}\,G^1(x)$ *over* $(\underline{x}^2, \beta)$*.*

Proof. For each $x \in \mathcal{I}$, let $f(x) \equiv \frac{\phi(x)}{\phi(x_{R^1})}\,R^1(x_{R^1})$. Notice that $f = V_{R^1} \geq R^1$ over $[x_{R^1}, \beta)$ and that $f'(x_{R^1}) = R^{1\prime}(x_{R^1})$ by the smooth-fit property. Applying the change-of-variables formula (S.1) to f, a direct computation shows that $\underline{x}^2$ is a solution to (S.25) if and only if

$\zeta(\underline{x}^2)$ is a solution to

$$\hat{f}(y) = \hat{G}^1(y), \text{ that is, } \frac{R^1(x_{R^1})}{\phi(x_{R^1})}\, y = \hat{G}^1(y).$$

Because $f = V_{R^1}$ over $[x_{R^1}, \beta)$, it follows from A6 that $\hat{f} < \hat{G}^1$ over $(0, \zeta(x_{R^1})]$. Because $\hat{G}^1$ is positive, concave, and satisfies (S.5), it follows in turn that (S.25) admits a unique solution $\underline{x}^2 < x_{R^1}$, and that $\frac{\phi}{\phi(x_{R^1})}\, R^1(x_{R^1}) > G^1$ over $(\alpha, \underline{x}^2)$ and $\frac{\phi}{\phi(x_{R^1})}\, R^1(x_{R^1}) < G^1$ over $(\underline{x}^2, \beta)$. Finally, recall from A3 and A6 that $\alpha^1 < x_{R^1} < x_0^1$ and that $\hat{R}^1$ is strictly concave over $(\zeta(x_0^1), \infty)$. Therefore, $\hat{f} > \hat{R}^1$ over $(\zeta(x_{R^1}), \infty)$ as $\hat{f}$ is linear and tangent to $\hat{R}^1$ at $\zeta(x_{R^1})$. Hence, if $\alpha^1 > \alpha$, it must be that $\alpha^1 < \underline{x}^2$ as $G^1 = R^1$ over $(\alpha, \alpha^1]$. The result follows. ■

Proof of Proposition 5. The next lemma provides sufficient conditions for the variational system (28)–(38) to admit a solution.

Lemma S.6 *In the running example, if the firms' liquidation values $l^1 \leq l^2$ are close enough to each other, and if m is sufficiently large and $b > 0$, then there exists a constant $a^1 > 0$ and two functions $w^1 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2\})$ and $w^2 \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus \{\underline{x}^2, x_{R^1}\})$ solution to the variational system (28)–(38).*

Proof. We shall use the standard fact (see, for instance, Dixit and Pindyck (1994)) that, in the running example, $\phi(x) = x^{\rho^-}$ and $\psi(x) = x^{\rho^+}$ for all $x \in (0, \infty)$, where

$$\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \quad \text{and} \quad \rho^+ \equiv \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{S.26}$$

The proof then consists of two parts. We first characterize a candidate solution to (28)–(38) and provide sufficient conditions for its existence. We then show that these conditions are met under our parameter restrictions.

A Candidate Solution Using the notation of Section 2.3, we have

$$V_{R^i}(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x[\mathrm{e}^{-r\tau} R^i(X_\tau)] = \begin{cases} \frac{\phi(x)}{\phi(x_{R^i})}\,(l^i - \frac{1}{r-b}\, x_{R^i}) & \text{if } x > x_{R^i} \\ l^i - \frac{1}{r-b}\, x & \text{if } x \leq x_{R^i} \end{cases},$$

where $x_{R^i} = \frac{\rho^-}{\rho^- - 1}\,(r-b) l^i$. Similarly,

$$V_m^i(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\left[\int_0^\tau \mathrm{e}^{-rt} m X_t \,\mathrm{d}t + \mathrm{e}^{-r\tau} l^i\right] = \begin{cases} \frac{m}{r-b}\, x + \frac{\phi(x)}{\phi(\alpha^i)}\,(l^i - \frac{m}{r-b}\,\alpha^i) & \text{if } x > \alpha^i \\ l^i & \text{if } x \leq \alpha^i \end{cases},$$

where $\alpha^i = \frac{x_{R^i}}{m} < x_{R^i}$. Thus

$$G^i(x) = (V_m^i - E)(x) = \begin{cases} \frac{m-1}{r-b}\, x + \frac{\phi(x)}{(1-\rho^-)\phi(\alpha^i)}\, l^i & \text{if } x > \alpha^i \\ l^i - \frac{1}{r-b}\, x & \text{if } x \leq \alpha^i \end{cases}.$$

This allows us to rewrite (S.25) as

$$\frac{x^{\rho^-}}{(1-\rho^-)\left[\frac{\rho^-}{\rho^- -1}(r-b)l^1\right]^{\rho^-}}\,l^1 = \frac{m-1}{r-b}\,x + \frac{x^{\rho^-}m^{\rho^-}}{(1-\rho^-)\left[\frac{\rho^-}{\rho^- -1}(r-b)l^1\right]^{\rho^-}}\,l^1.$$

Solving this equation yields $\underline{x}^2 = \xi x_{R^1}$, where

$$\xi \equiv \left[\frac{1-m^{\rho^-}}{\rho^-(1-m)}\right]^{\frac{1}{1-\rho^-}} \in \left(\frac{1}{m},1\right). \tag{S.27}$$

It follows that the function w^1 defined by

$$w^1(x) \equiv \begin{cases} \frac{\phi(x)}{\phi(x_{R^1})}\left(l^1 - \frac{1}{r-b}\,x_{R^1}\right) & \text{if } x > \underline{x}^2 \\ G^1(x) & \text{if } x \leq \underline{x}^2 \end{cases}$$

is, by construction, solution to the variational system (28)–(32).

If a solution (w^2, a^1) to the variational system (33)–(38) exists, then, letting T^2_x denote the unique solution to $\mathcal{L}u - ru = 0$ that is tangent to R^2 at x, it must be that $w^2 = T^2_{\underline{x}^2}$ over $(\underline{x}^2, x_{R^1})$. Specifically, we have $T^2_{\underline{x}^2} = B\psi + C\phi$ with positive coefficients B and C given by[3]

$$B = \frac{-\phi'(\underline{x}^2)\left(l^2 - \frac{1}{r-b}\,\underline{x}^2\right) - \frac{1}{r-b}\,\phi(\underline{x}^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)} \quad \text{and} \quad C = \frac{\psi'(\underline{x}^2)\left(l^2 - \frac{1}{r-b}\,\underline{x}^2\right) + \frac{1}{r-b}\,\psi(\underline{x}^2)}{\psi'(\underline{x}^2)\phi(\underline{x}^2) - \psi(\underline{x}^2)\phi'(\underline{x}^2)}.$$

Similarly, we have $w^2 = A\phi$ over (x_{R^1}, ∞) for

$$A \equiv B\,\frac{\psi(x_{R^1})}{\phi(x_{R^1})} + C,$$

as required by the continuity of w^2 at x_{R^1}. It follows that

$$\Delta w^{2\prime}(x_{R^1}) = B\left[\frac{\psi(x_{R^1})}{\phi(x_{R^1})}\,\phi'(x_{R^1}) - \psi'(x_{R^1})\right] < 0$$

by (S.4). We deduce that, if

$$G^2(x_{R^1}) > T^2_{\underline{x}^2}(x_{R^1}) > T^2_{x_{R^2}}(x_{R^1}), \tag{S.28}$$

then

$$w^2 = 1_{(0,\underline{x}^2]}R^2 + 1_{(\underline{x}^2, x_{R^1}]}T^2_{\underline{x}^2} + 1_{(x_{R^1},\infty)}A\phi \quad \text{and} \quad a^1 = -\frac{\Delta w^{2\prime}(x_{R^1})}{G^2(x_{R^1}) - w^2(x_{R^1})}$$

is solution to the variational system (33)–(38). In (S.28), the first inequality ensures that $a^1 > 0$, while the second inequality ensures that $\underline{x}^2 < x_{R^2}$ and that $w^2 \geq R^2$ over (x_{R^1}, ∞). The convexity of $T^2_{\underline{x}^2}$ and the linearity of R^2 imply that $w^2 \geq R^2$ over $(\underline{x}^2, x_{R^1}]$.

[3] That B and C are positive can be seen as follows. First, the denominator of B and C is positive by (S.4). Second, because $\phi' < 0$, $\phi'' > 0$, $\underline{x}_2 < x_{R^1}$, and $l^2 \geq l^1$, the numerator of B is greater than or equal to $-\phi'(x_{R^1})\left(l^1 - \frac{1}{r-b}\,x_{R^1}\right) - \frac{1}{r-b}\,\phi(x_{R^1})$, which is equal to 0 by the smooth-pasting condition for (3) with $i = 1$. Third, because $\psi' > 0$, $\underline{x}_2 < x_{R^1}$, and $l^2 \geq l^1$, the numerator of C is positive.

Checking the Sufficient Conditions We now show that, if $l^1 \leq l^2$ are close enough to each other, and if m is sufficiently large and $b > 0$, then (S.28) holds. Letting $\Delta\rho \equiv \rho^+ - \rho^-$, direct computations lead to

$$B = \frac{\rho^-(\underline{x}^2)^{-\rho^+}}{\Delta\rho}(\xi l^1 - l^2) \text{ and } C = \frac{\rho^+(\underline{x}^2)^{-\rho^-}}{\Delta\rho}\left(l^2 - \frac{\rho^+ - 1}{\rho^+}\frac{\rho^-}{\rho^- - 1}\xi l^1\right).$$

Using that $\underline{x}^2 = \xi x_{R^1}$, we deduce from this that

$$T^2_{\underline{x}^2}(x_{R^1}) = Bx_{R^1}^{\rho^+} + Cx_{R^1}^{\rho^-} = \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho}(\xi l^1 - l^2) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho}\left(l^2 - \frac{\rho^+ - 1}{\rho^+}\frac{\rho^-}{\rho^- - 1}\xi l^1\right).$$

Now, we have $T^2_{x_{R^2}} = \frac{l^2}{(1-\rho^-)\phi(x_{R^2})}\,\phi$, so that

$$T^2_{x_{R^2}}(x_{R^1}) = \frac{l^2}{1-\rho^-}\left(\frac{l^1}{l^2}\right)^{\rho^-}.$$

If l^1 and l^2 are close enough to each other so that $l^1 \geq \frac{l^2}{m}$, then $x_{R^1} \geq \alpha^2$ and thus

$$G^2(x_{R^1}) = \frac{m-1}{r-\mu}x_{R^1} + \frac{\phi(x_{R^1})}{(1-\rho^-)\phi(\alpha^2)}l^2 = \frac{\rho^-}{\rho^- - 1}(m-1)l^1 + \frac{l^2}{1-\rho^-}\left(\frac{l^1}{l^2}\right)^{\rho^-} m^{\rho^-}.$$

Therefore, if $l^1 \geq \frac{l^2}{m}$, then (S.28) holds if and only if

$$\begin{aligned}
&\frac{\rho^-}{\rho^- - 1}(m-1)l^1 + \frac{l^2}{1-\rho^-}\left(\frac{l^1}{l^2}\right)^{\rho^-} m^{\rho^-} \\
&\qquad > \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho}(\xi l^1 - l^2) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho}\left(l^2 - \frac{\rho^+ - 1}{\rho^+}\frac{\rho^-}{\rho^- - 1}\xi l^1\right) \\
&\qquad > \frac{l^2}{1-\rho^-}\left(\frac{l^1}{l^2}\right)^{\rho^-}.
\end{aligned}$$

This is true for any close enough values of l^1 and l^2 if

$$\begin{aligned}
&\frac{\rho^-}{\rho^- - 1}(m-1) + \frac{1}{1-\rho^-}m^{\rho^-} \\
&\qquad > \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho}(\xi - 1) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho}\left(1 - \frac{\rho^+ - 1}{\rho^+}\frac{\rho^-}{\rho^- - 1}\xi\right) \\
&\qquad > \frac{1}{1-\rho^-}.
\end{aligned} \tag{S.29}$$

As for the second inequality in (S.29), notice from (S.27) that, as $\rho^- < 0$, ξ goes to 0 as m goes to ∞, and thus, as $\rho^+ > 0$ and $\rho^-(\xi - 1) > 0$, that the left-hand side goes to ∞ as m goes to ∞. Therefore, the second inequality in (S.29) is satisfied if m is sufficiently large. As for the first inequality in (S.29), notice from (S.27) that the right-hand side is of the order $m^{\frac{\rho^+}{1-\rho^-}}$ as m goes to ∞, while the left-hand side is of the order m. Therefore, the first inequality in (S.29) is satisfied if m is sufficiently large and $\rho^+ + \rho^- < 1$, that is, from (S.26), if $b > 0$. The result follows. ■

Given Lemma S.6, Proposition 5 is then a direct consequence of Theorem 3 as explained in the main text. Hence the result. ■

Lemma S.7 *Let γ^1 be a randomized stopping time of player 1 associated to $\Lambda^1 \equiv (\Lambda^1_t)_{t\geq 0} \equiv \big(e^{-a^1 L^{x_{R^1}}_t}\big)_{t\geq 0}$ of the form*

$$\gamma^1 \equiv \inf\{t \geq 0 : \Gamma^1_t > U^1\},$$

where U^1 is uniformly distributed over $[0,1]$ and independent of X. Then γ^1 is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-stopping time and its $(\hat{\mathcal{F}}_t)_{t\geq 1}$-predictable compensator is $(a^1 L^{x_{R^1}}_{t\wedge\gamma^1})_{t\geq 0}$, where $(\hat{\mathcal{F}}_t)_{t\geq 0}$ is the shareholders' filtration defined by (39). In particular, the processes $(1_{\{t\wedge\tau_{\underline{x}^2}\geq\gamma^1\}} - a^1 L^{x_{R^1}}_{t\wedge\tau^c})_{t\geq 0}$ and $(e^{-rt\wedge\tau^c} V^{2,\tau^c}_t + \int_0^{t\wedge\tau^c} e^{-rs} X_s\, ds)_{t\geq 0}$ are $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingales.

Proof. We only need to check that $Z \equiv (Z_t)_{t\geq 0} \equiv (1_{\{t\geq\gamma^1\}} - a^1 L^{x_{R^1}}_{t\wedge\gamma^1})_{t\geq 0}$ is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingale for all $\mathbf{P}_x$, $x \in \mathcal{I}$. Let $s \leq t$, and consider the random variable $U^1_s(\omega, u^1) \equiv u^1 1_{\{\Gamma^1_s(\omega)\geq u^1\}} + 1_{\{\Gamma^1_s(\omega)<u^1\}}$ over the probability space $\Omega^1 \equiv \Omega \times [0,1]$. It is easy to check that

$$\hat{\mathcal{F}}_s = \mathcal{F}_s \vee \sigma(U^1_s) \subset \mathcal{F}_\infty \vee \sigma(U^1_s).$$

From the definition of $\mathcal{F}_\infty$, we have

$$\mathbf{E}_x[Z_t - Z_s \,|\, \mathcal{F}_\infty \vee \sigma(U^1_s)] = \mathbf{E}_x[Z_t - Z_s \,|\, \omega, U^1_s].$$

A version of the conditional law of U^1 given $(\omega, U^1_s(\omega, U^1))$ is

$$1_{\{U^1_s<1\}}\, \delta_{U^1_s} + 1_{\{U^1_s=1\}}\, \mathcal{U}_{[\Gamma^1_s,1]},$$

where $\mathcal{U}_{[a,b]}$ denotes the uniform distribution over $[a,b]$. Hence

$$1_{\{U^1_s=1\}}\, \mathbf{P}_x[\gamma^1 \leq t \,|\, \omega, U^1_s] = 1_{\{U^1_s=1\}}\, \frac{\Gamma^1_t - \Gamma^1_s}{1 - \Gamma^1_s}.$$

We deduce that

$$\begin{aligned}
&\mathbf{E}_x[Z_t - Z_s \,|\, \omega, U^1_s] \\
&\quad = \mathbf{E}_x\Big[1_{\{s<\gamma^1\leq t\}} - a^1(L^{x_{R^1}}_{t\wedge\gamma^1} - L^{x_{R^1}}_s) \,|\, \omega, U^1_s\Big] \\
&\quad = \mathbf{E}_x\Big[1_{\{\Gamma^1_s<U^1\leq\Gamma^1_t\}} - a^1(L^{x_{R^1}}_{t\wedge\gamma^1} - L^{x_{R^1}}_s) \,|\, \omega, U^1_s\Big] \\
&\quad = \frac{1_{U^1_s=1}}{1-\Gamma^1_s}\Big[\Gamma^1_t - \Gamma^1_s - a^1 \int_s^t (L^{x_{R^1}}_u - L^{x_{R^1}}_s)\, d\Gamma^1_u - a^1(1-\Gamma^1_t)(L^{x_{R^1}}_t - L^{x_{R^1}}_s)\Big] \\
&\quad = 0,
\end{aligned}$$

where the fourth equality follows from the integration by parts formula and the fact that

$$\Gamma^1_t - \Gamma^1_s = \int_s^t a^1(1-\Gamma^1_u)\, dL^{x_{R^1}}_u.$$

We conclude that $\mathbf{E}_x[Z_t - Z_s \,|\, \hat{\mathcal{F}}_s] = 0$ by using the law of iterated conditional expectations. The result follows. ■

S.5 Proofs of Lemmas for Theorem 2

Proof of Lemma A.1. Recall that any solution $u \in \mathcal{C}^2((a,b))$ to the ODE $\mathcal{L}u - ru = 0$ is of the form $u = A\phi + B\psi$ for some constants A and B. Whenever needed, we use the change of variables (S.1) to reexpress the assumptions and the conclusions of (i)–(iv). For instance, $u(x) \geq V_{R^i}(x)$ for all $x \in (a,b)$ if and only if $\hat{u}(z) = Az + B \geq \hat{V}_{R^i}(z)$ for all $z \in (\zeta(b), \zeta(a))$. Recall also that $\hat{V}_{R^i} \in \mathcal{C}^1((0,\infty))$, that, for some $C^i > 0$, $\hat{V}_{R^i}(z) = C^i z > \hat{R}^i(z)$ for all $z \in (0, \zeta(x_{R^i}))$, and that $\hat{V}_{R^i} = \hat{R}^i$ is $\mathcal{C}^2$ and strictly concave over $[\zeta(x_{R^i}), \infty)$.

(i) The assumption $u(\beta-) = 0$ implies $B = 0$, and thus $\hat{u}(0+) = 0$. The assumption that $\hat{u} \geq \hat{V}_{R^i}$ over $(0, \zeta(a))$ implies $A \geq C^i$. If this inequality were strict, then we would have $Az > C^i z \geq \hat{R}^i(z)$ for all $z > 0$ as $\hat{V}_{R^i}$ is concave, in contradiction to the assumption $\hat{u}(\zeta(a)-) = A\zeta(a) = \hat{R}^i(\zeta(a))$. We conclude that $A = C^i$ and, from the properties of $\hat{V}_{R^i}$, that the unique solution to $Az = \hat{R}^i(z)$ is $\zeta(x_{R^i})$.

(ii) Notice that $V_{R^i} > R^i$ over (x_{R^i}, β), so that $\hat{V}_{R^i} > \hat{R}^i$ over $(0, \zeta(x_{R^i}))$. Hence, if there exists $z_0 \in (\zeta(b), \zeta(a))$ such that $\hat{u}(z_0) = \hat{R}^i(z_0)$, then it must be that $z_0 \geq \zeta(x_{R^i})$. In this case, $\hat{u}$ is tangent to the concave $\mathcal{C}^1$ function $\hat{V}_{R^i}$ at z_0. Over $[\zeta(x_{R^i}), \infty)$, $\hat{V}_{R^i} = \hat{R}^i$ is strictly concave. As a result, $\hat{u}(z) > \hat{R}^i(z)$ for all $z \neq z_0$ in $[\zeta(x_{R^i}), \infty) \cap (\zeta(b), \zeta(a))$, and thus for all $z \neq z_0$ in $(\zeta(b), \zeta(a))$ by the preceding remark.

(iii) If $a > \alpha$, then $\hat{u}$ is an affine function over $(\zeta(b), \zeta(a))$ that coincides with $\hat{R}^i$ at both boundaries of this interval. The fact that $\hat{R}^i$ is strictly concave over $[\zeta(x_{R^i}), \infty)$ together with $\zeta(b) \geq \zeta(x_{R^i})$ then implies that $\hat{u} < \hat{R}^i$ over $(\zeta(b), \zeta(a))$. If $a = \alpha$, then $u(a+) = 0$ implies that $u = B\psi$ for some constant B by (5), and thus that $\hat{R}^i(\zeta(b)) = \hat{u}(\zeta(b)) = B$. The function $\hat{R}^i$ is strictly concave and, by Lemma 1, positive over $[\zeta(x_{R^i}), \infty)$. It is thus increasing over this interval, which implies that $\hat{u} = B < \hat{R}^i$ over $(\zeta(b), \infty)$.

(iv) The function $\hat{u}$ satisfies $\hat{u}(z) = Az + B$ for all $z \in (\zeta(b), \zeta(a))$ for some constants A and B. A direct computation yields

$$A = \hat{u}'(\zeta(a)-) = \frac{\psi(a)u'(a+) - \psi'(a)u(a)}{\psi(a)^2\zeta'(a)} \quad \text{and} \quad \hat{R}^{i\prime}(\zeta(a)) = \frac{\psi(a)R^{i\prime}(a) - \psi'(a)R^i(a)}{\psi(a)^2\zeta'(a)},$$

so that, as $u(a) = R^i(a)$, $u'(a+) > R^{i\prime}(a)$, and $\zeta'(a) < 0$,

$$\hat{R}^{i\prime}(\zeta(a)) - A = \frac{R^{i\prime}(a) - u'(a+)}{\psi(a)\zeta'(a)} > 0.$$

Hence $\hat{R}^i(\zeta(a-\varepsilon)) > A\zeta(a-\varepsilon) + B$ for $\varepsilon > 0$ small enough. Similarly, the function $\hat{f}_\varepsilon$ satisfies $\hat{f}_\varepsilon(z) = A'z + B'$ for all $z \in (0,\infty)$ for some constants A' and B'. Moreover, $\hat{f}_\varepsilon(\zeta(a-\varepsilon)) = \hat{R}^i(\zeta(a-\varepsilon))$ and $\hat{f}_\varepsilon(\zeta(a+\varepsilon)) = \hat{u}(\zeta(a+\varepsilon))$. Hence

$$A'\zeta(a+\varepsilon) + B' = A\zeta(a+\varepsilon) + B \quad \text{and} \quad A'\zeta(a-\varepsilon) + B' > A\zeta(a-\varepsilon) + B,$$

so that $A'\zeta(a) + B' > A\zeta(a) + B$ as $\zeta(a) \in (\zeta(a+\varepsilon), \zeta(a-\varepsilon))$. The result follows. ■

Proof of Lemma A.2. As in the proof of Lemma S.6, let T_x^i denote, for each $x < x_{R^i}$, the unique solution to $\mathcal{L}u - ru = 0$ that is tangent to R^i at x. Then $T_x^i \geq R^i$ over (x_{R^i}, β) and $T_x^i \equiv A_x\phi + B_x\psi$ for some positive coefficients A_x and B_x.[4] For each $z \geq \zeta(x_{R^i})$, let $\hat{T}_z^i \equiv \widehat{T^i_{\zeta^{-1}(z)}}$ be the affine function tangent to $\hat{R}^i$ at z, which is given by

$$\hat{T}_z^i(y) = A_{\zeta^{-1}(z)}y + B_{\zeta^{-1}(z)} = \hat{R}^i(z) + \hat{R}^{i\prime}(z)(y - z), \quad y \in (0, \infty). \tag{S.30}$$

Now, suppose, by way of contradiction, that $\chi_\infty \equiv \lim_{n\to\infty} \chi_n^1 = \lim_{n\to\infty} \chi_n^2 > \alpha$. Also suppose, with no loss of generality, that $\chi_1^1 > \chi_1^2$, and let $y_{2n-1} \equiv \zeta(\chi_n^1)$ and $y_{2n} \equiv \zeta(\chi_n^2)$ for all $n \geq 1$. Because $(\chi_n^i)_{n\geq 1}$ is a sequence in $\operatorname{supp}\mu^i \cap (s, \beta)$ and, hence, in $(\alpha, x_{R^i}]$ by Proposition 1(iv), $(y_n)_{n\geq 1}$ is a sequence in $[\zeta(x_{R^i}), \infty)$. As in Step 3 of the proof of Theorem 2, that player 1 does not stop over the interval (χ_{n+1}^1, χ_n^1) and that $\chi_n^2 \in (\chi_{n+1}^1, \chi_n^1)$ belongs to the support of μ^2 implies that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (χ_{n+1}^1, χ_n^1) and that $\bar{J}^2 \geq V_{R^2}$ and $\bar{J}^2(\chi_n^2) = R^2(\chi_n^2)$. Moreover, as $\bar{J}^2$ is continuous, it coincides with $T^j_{\chi_n^2}$ on $[\chi_{n+1}^1, \chi_n^1]$. It follows that, for each $n \geq 1$, $\bar{J}^2(\chi_{n+1}^1) = T^j_{\chi_n^2}(\chi_{n+1}^1) = T^j_{\chi_{n+1}^2}(\chi_{n+1}^1)$, and a similar property holds for $\bar{J}^1$. Using (S.30) to rewrite these equalities yields, for each $n \geq 1$,

$$\hat{R}^1(y_{2n-1}) + \hat{R}^{1\prime}(y_{2n-1})(y_{2n} - y_{2n-1}) = \hat{R}^1(y_{2n+1}) + \hat{R}^{1\prime}(y_{2n+1})(y_{2n} - y_{2n+1}),$$
$$\hat{R}^2(y_{2n}) + \hat{R}^{2\prime}(y_{2n})(y_{2n+1} - y_{2n}) = \hat{R}^2(y_{2n+2}) + \hat{R}^{2\prime}(y_{2n+2})(y_{2n+1} - y_{2n+2}).$$

With $y < y' < y''$ three appropriate consecutive terms of the sequence $(y_n)_{n\geq 1}$, these equalities can be compactly rewritten for $i = 1, 2$ as

$$\hat{R}^i(y) + \hat{R}^{i\prime}(y)(y' - y) - \hat{R}^i(y') = \hat{R}^i(y'') + \hat{R}^{i\prime}(y'')(y' - y'') - \hat{R}^i(y'). \tag{S.31}$$

Using Taylor's theorem with integral remainder, (S.31) is equivalent to

$$-\int_y^{y'} (y' - z)\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z = -\int_{y'}^{y''} (z - y')\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z. \tag{S.32}$$

Because $\hat{R}^{i\prime\prime} < 0$ over $[y_1, \infty) \subset [\zeta(x_{R^i}), \infty)$, the right-hand side of (S.32) is increasing in y''. Therefore, given $y' > y \geq y_1$, if a solution $y'' > y'$ to (S.32) exists, it is unique. By assumption, $\lim_{n\to\infty} y_n = y_\infty \equiv \zeta(\chi_\infty) < \infty$. Moreover, because $\hat{R}^{i\prime\prime}$ is locally Lipschitz by A8, there exists $K > 0$ such that $|\hat{R}^{i\prime\prime}(z) - \hat{R}^{i\prime\prime}(y')| \leq K|z - y'|$ for all $z, y' \in [y_1, y_\infty]$. Thus

$$-\int_y^{y'} (y' - z)\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z \geq -R^{i\prime\prime}(y')\,\frac{(y' - y)^2}{2} - K\,\frac{(y' - y)^3}{3}, \tag{S.33}$$

$$-\int_{y'}^{y''} (z - y')\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z \leq -R^{i\prime\prime}(y')\,\frac{(y'' - y')^2}{2} + K\,\frac{(y'' - y')^3}{3}. \tag{S.34}$$

By (S.32), we have

$$(y'' - y')^2 + \frac{2K}{3|\hat{R}^{i\prime\prime}(y')|}\,(y'' - y')^3 \geq (y' - y)^2 - \frac{2K}{3|\hat{R}^{i\prime\prime}(y')|}\,(y' - y)^3. \tag{S.35}$$

[4] That A_x and B_x are positive follows from $x < x_{R^i}$ along the same lines as in Footnote 3.

Let C such that, for each $y' \in [y_1, y_\infty]$,

$$\frac{2K}{3|\hat{R}^{i\prime\prime}(y')|} \leq C.$$

Then, by (S.35), we have

$$(y'' - y')^2 + C(y'' - y')^3 \geq (y' - y)^2 - C(y' - y)^3.$$

Letting $u_n \equiv y_{n+1} - y_n$ for all $n \geq 1$, the upshot of the above analysis is that $h(u_{n+1}) \geq g(u_n)$, where $g(u) \equiv u^2 - Cu^3$ and $h(u) \equiv u^2 + Cu^3$. By assumption, $y_1 + \sum_{n\geq 1} u_n = y_\infty < \infty$, which implies that $\lim_{n\to\infty} u_n = 0$. Therefore, for n sufficiently large, $g(u_n) > 0$ and $u_{n+1} \geq h^{-1}(g(u_n))$, where h^{-1} denotes the inverse of h restricted to $[0, \infty)$. Because $h^{-1}(z) = \sqrt{z} - \frac{C}{2}\, z + o(z)$, we have $h^{-1}(g(u)) = u - Cu^2 + o(u^2)$. Hence

$$u_{n+1} \geq u_n - Cu_n^2 + o(u_n^2)$$

and, as a result,

$$\frac{1}{u_{n+1}} - \frac{1}{u_n} \leq \frac{1}{u_n}\left[\frac{1}{1 - Cu_n + o(u_n)} - 1\right] = C + o(1).$$

We obtain

$$\frac{1}{u_n} = \frac{1}{u_1} + \sum_{k=1}^{n-1}\left(\frac{1}{u_{k+1}} - \frac{1}{u_k}\right) \leq nC + o(n)$$

and thus

$$u_n \geq \frac{1}{nC} + o\left(\frac{1}{n}\right),$$

so that $\sum_{n\geq 1} u_n = \infty$, a contradiction. The case of increasing sequences, whose limit must be in $(\alpha, x_{R^i}]$, can be dealt in a similar way by replacing the inequalities (S.33) and (S.34) by an upper bound and a lower bound of the same type, respectively. The result follows. ■

S.6 Proofs of Lemmas for Theorem 3

Proof of Lemma A.3. From Proposition 1(v), if $((\mu^1, S^1), (\mu^2, S^2))$ is a MPE, then $(0, S^2)$ is a pbr to (μ^1, S^1). Applying the strong Markov property (S.7) to the value function of player 2 associated to the pair of Markov strategies $((\mu^1, S^1), (0, S^2))$ yields, for all $x \in \mathcal{I}$ and $\tau \in \mathcal{T}$,

$$\begin{aligned}\bar{J}^2(x) = \mathbf{E}_x\Bigg[&\sum_{n=1}^{N} \int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt} G^2(q_n^1)\Lambda_t^1 a_n \,\mathrm{d}L_t^{q_n^1} \\ &+ 1_{\{\tau_{S^2}<\tau\}}\, \mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}} + 1_{\{\tau_{S^2}\geq\tau\}}\, \mathrm{e}^{-r\tau} \bar{J}^2(X_\tau)\Lambda^1_\tau\Bigg],\end{aligned}$$

where we used that $\mathrm{d}\Gamma^1_t = \sum_{n=1}^N a_n \Lambda^1_t \,\mathrm{d}L^{q^1_n}_t$. This proves (A.9).

To prove (A.10), we apply the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r(\tau\wedge\tau_{S^2}\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k}$, where, for each $k\in\mathbb{N}$, $\tau_k \equiv \inf\{t\geq 0 : X_t \notin [\alpha_k,\beta_k]\}$ for some increasing sequence $([\alpha_k,\beta_k])_{k\in\mathbb{N}}$ of compacts intervals of $\mathcal{I}$ such that $\bigcup_{k\in\mathbb{N}}[\alpha_k,\beta_k] = \mathcal{I}$. Observe that $\mathbf{E}_x[\tau_k] < \infty$ (Karatzas and Shreve (1999, Chapter 5, Section 5, §C)) and that $X_t \in [\alpha_k,\beta_k]$ over $\{t\leq\tau_k\}$ $\mathbf{P}_x$-almost surely for all $x\in[\alpha_k,\beta_k]$. Moreover, because X does not explode in finite time, $\lim_{k\to\infty}\tau_k = \infty$ and, hence, $\lim_{k\to\infty}\tau\wedge\tau_k = \tau$ for all $\tau\in\mathcal{T}$. We obtain

$$\begin{aligned}
\bar{J}^2(x) = {}& \mathrm{e}^{-r(\tau\wedge\tau_{S^2}\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k} - \int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\bar{J}^2(X_t)\,\mathrm{d}\Lambda^1_t \\
&- \int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}\bar{J}^2(X_t) - r\bar{J}^2(X_t)] \prod_{n=1}^N 1_{\{X_t\neq q^1_n\}}\Lambda^1_t\,\mathrm{d}t \\
&- \int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\sigma(X_t)\bar{J}^{2\prime}(X_t) \prod_{n=1}^N 1_{\{X_t\neq q^1_n\}}\Lambda^1_t\,\mathrm{d}W_t \\
&- \frac{1}{2}\sum_{n=1}^N \Delta\bar{J}^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t.
\end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned}
\bar{J}^2(x) = \mathbf{E}_x\Bigg[& \mathrm{e}^{-r\tau\wedge\tau_{S^2}\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k} - \int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\bar{J}^2(X_t)\,\mathrm{d}\Lambda^1_t \\
&- \frac{1}{2}\sum_{n=1}^N \Delta\bar{J}^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t\Bigg],
\end{aligned}$$

where we have used the fact that $\bar{J}^2$ satisfies (49) and that

$$\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\sigma(X_t)\bar{J}^{2\prime}(X_t) \prod_{n=1}^N 1_{\{X_t\neq q^1_n\}}\Lambda^1_t\,\mathrm{d}W_t\right] = 0. \tag{S.36}$$

Indeed, notice that σ is continuous on I, and that $\bar{J}^2 \in \mathcal{C}^1(\mathcal{I}\setminus\{(q^1_n)_{1\leq n\leq N}\})$ with $|\bar{J}^{2\prime}(x+)| < \infty$ and $|\bar{J}^{2\prime}(x-)| < \infty$ for $x\in\{q^1_n : 1\leq n\leq N\}$. Thus there exists $C_k > 0$ such that $|\sigma(X_t)\bar{J}^{2\prime}(X_t)| \leq C_k$ over $\{t\leq\tau_{S^2}\wedge\tau_k\}$ $\mathbf{P}_x$-almost surely, which implies (S.36). Hence

$$\begin{aligned}
\bar{J}^2(x) = {}& \mathbf{E}_x\big[1_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}\big] + \mathbf{E}_x\Big[1_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\Big] \\
&+ \mathbf{E}_x\left[\sum_{n=1}^N\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\bar{J}^2(X_t)\Lambda^1_t a_n\,\mathrm{d}L^{q^1_n}_t\right] \\
&- \mathbf{E}_x\left[\frac{1}{2}\sum_{n=1}^N \Delta\bar{J}^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t\right].
\end{aligned}$$

Using that the measure $\mathrm{d}L^{q^1_n}_t$ only charges the set $\{t\geq 0 : X_t = q^1_n\}$, we obtain

$$\bar{J}^2(x) = \mathbf{E}_x\big[1_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}\big] + \mathbf{E}_x\Big[1_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\Big]$$

$$+\,\mathbf{E}_x\left[\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n-\frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]. \tag{S.37}$$

By the monotone convergence theorem,

$$\lim_{k\to\infty}\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)} \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n-\frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]$$
$$=\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n-\frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]$$

for all n, and

$$\lim_{k\to\infty}\mathbf{E}_x\left[1_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\,\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda_{\tau_{S^2}}^1\right]=\mathbf{E}_x\left[1_{\{\tau_{S^2}<\tau\}}\,\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda_{\tau_{S^2}}^1\right].$$

Because $0\leq\bar{J}^2\leq G^2$ by Proposition 1, it follows from A4 that the sequence $(1_{\{\tau\wedge\tau_k\leq\tau_{S^2}\}}\,\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k}))_{k\in\mathbb{N}}$ is uniformly integrable. Therefore, by Vitali's convergence theorem,

$$\lim_{k\to\infty}\mathbf{E}_x\left[1_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\,\mathrm{e}^{-r(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda_{\tau\wedge\tau_k}^1\right]=\mathbf{E}_x\left[1_{\{\tau_{S^2}\geq\tau\}}\,\mathrm{e}^{-r\tau}\bar{J}^2(X_\tau)\Lambda_\tau^1\right].$$

Finally, $1_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\,\mathrm{e}^{-r(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda_{\tau\wedge\tau_k}^1=1_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda_{\tau_k}^1$ over $\{\tau=\infty\}$. For k large enough, $x\in(\alpha_k,\beta_k)$. Hence

$$\begin{aligned}
&\mathbf{E}_x\left[1_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda_{\tau_k}^1\right]\\
&\quad\leq\mathbf{E}_x\left[1_{\{X_{\tau_k}=\alpha_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda_{\tau_k}^1\right]+\mathbf{E}_x\left[1_{\{X_{\tau_k}=\beta_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda_{\tau_k}^1\right]\\
&\quad\leq\mathbf{E}_x\left[\mathrm{e}^{-r\tau_k}\bar{J}^2(\alpha_k)\Lambda_{\tau_{\alpha_k}}^1\right]+\mathbf{E}_x\left[\mathrm{e}^{-r\tau_k}\bar{J}^2(\beta_k)\Lambda_{\tau_{\beta_k}}^1\right]\\
&\quad\leq\frac{\phi(x)}{\phi(\alpha_k)}\,G^2(\alpha_k)+\frac{\psi(x)}{\psi(\beta_k)}\,G^2(\beta_k).
\end{aligned}$$

Because $\bar{J}^2\geq0$, it then follows from the growth properties (9) that

$$\lim_{k\to\infty}\mathbf{E}_x\left[1_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda_{\tau_k}^1\right]=0.$$

Thus, letting k go to ∞ in (S.37) yields

$$\begin{aligned}
\bar{J}^2(x)=\mathbf{E}_x\Bigg[&\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n-\frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\\
&+1_{\{\tau_{S^2}<\tau\}}\,\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda_{\tau_{S^2}}^1+1_{\{\tau_{S^2}\geq\tau\}}\,\mathrm{e}^{-r\tau}\bar{J}^2(X_\tau)\Lambda_\tau^1\Bigg].
\end{aligned}$$

This shows (A.10). The result follows. ■

Proof of Lemma A.4. Suppose, with no loss of generality, that $i=2$ and $j=1$. First, let us observe that (S.7) leads to

$$J^2(x,(\mu^1,S^1),\tau)=\mathbf{E}_x\left[\mathrm{e}^{-r\tau}R^2(X_\tau)\Lambda_\tau^1+\sum_{n=1}^{N}\int_{[0,\tau)}\mathrm{e}^{-rt}G^2(X_t)\Lambda_s^1a_n\,\mathrm{d}L_t^{q_n^1}\right].$$

Let w^2 be a solution to (48)–(54). We apply the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r(\tau\wedge\tau_k)}w^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}$, where, for each $k\in\mathbb{N}$, τ_k is defined as in the proof of Lemma A.3. We obtain

$$\begin{aligned}
w^2(x) &= \mathrm{e}^{-r(\tau\wedge\tau_k)}w^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k} - \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\,\mathrm{d}\Lambda^1_t \\
&\quad - \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}w^2(X_t) - rw^2(X_t)]\prod_{n=1}^{N} 1_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\,\mathrm{d}t \\
&\quad - \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\sigma(X_t)w^{2\prime}(X_t)\prod_{n=1}^{N} 1_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\,\mathrm{d}W_t \\
&\quad - \frac{1}{2}\sum_{n=1}^{N}\Delta w^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t. \qquad \text{(S.38)}
\end{aligned}$$

From (50) and A3, we have $\mathcal{L}w^2 - rw^2 = \mathcal{L}R^2 - rR^2 \le 0$ over $(\alpha, s^2)\subset(\alpha, x_{R^2}]$. It then follows from (49) that

$$\mathbf{E}_x\left[-\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}w^2(X_t) - rw^2(X_t)]\prod_{n=1}^{N} 1_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\,\mathrm{d}t\right] \ge 0. \qquad \text{(S.39)}$$

Next, we have

$$\begin{aligned}
&\mathbf{E}_x\left[-\frac{1}{2}\sum_{n=1}^{N}\Delta w^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t\right] \\
&\quad = \mathbf{E}_x\left[\sum_{n=1}^{N} a_n[G^2(q^1_n) - w^2(q^1_n)]\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t\right] \\
&\quad = \mathbf{E}_x\left[\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\Lambda^1_t a_n\,\mathrm{d}L^{q^1_n}_t - \sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\Lambda^1_t a_n\,\mathrm{d}L^{q^1_n}_t\right] \\
&\quad = \mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\,\mathrm{d}\Gamma^1_t + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\,\mathrm{d}\Lambda^1_t\right], \qquad \text{(S.40)}
\end{aligned}$$

where the first equality follows from (53), the second equality follows from the fact that the measure $\mathrm{d}L^{q^1_n}_t$ only charges the set $\{t\ge 0 : X_t = q^1_n\}$, and the third equality follows from the representation (17). We obtain from (S.38)–(S.40) that

$$\begin{aligned}
w^2(x) &\ge \mathbf{E}_x\left[\mathrm{e}^{-r(\tau\wedge\tau_k)}w^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k} + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\,\mathrm{d}\Gamma^1_t\right] \\
&\ge \mathbf{E}_x\left[\mathrm{e}^{-r(\tau\wedge\tau_k)}R^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k} + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\,\mathrm{d}\Gamma^1_t\right],
\end{aligned}$$

where the first inequality follows from the fact that the stochastic integral in (S.38) is a centered square-integrable random variable as shown in the proof of Lemma A.3, and the second inequality follows from (48). Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x) \ge \mathbf{E}_x\left[\mathrm{e}^{-r\tau}R^2(X_\tau)\Lambda^1_\tau + \int_{[0,\tau)} \mathrm{e}^{-rt}G^2(X_t)\,\mathrm{d}\Gamma^1_t\right] = J^2(x,(\mu^1,S^1),\tau),$$

where the equality follows from (14). Taking the supremum over $\tau \in \mathcal{T}$ yields (A.14).

To establish (A.15), we apply the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r\tau_k} w^2(X_{\tau_k})\Lambda^1_{\tau_k}\Lambda^2_{\tau_k-}$. Taking expectations, we obtain

$$w^2(x) = \mathbf{E}_x\Bigg[\mathrm{e}^{-r\tau_k} w^2(X_{\tau_k})\Lambda^1_{\tau_k}\Lambda^2_{\tau_k-} - \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda^2_{t-}\,\mathrm{d}\Lambda^1_t$$
$$- \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda^1_t\,\mathrm{d}\Lambda^2_t - \frac{1}{2}\sum_{n=1}^N \Delta w^{2\prime}(q^1_n) \int_{[0,\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\Lambda^2_{t-}\,\mathrm{d}L^{q^1_n}_t\Bigg], \quad \text{(S.41)}$$

where, as in the proof of Lemma A.3, we have used that

$$\mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt}\sigma(X_t) w^{2\prime}(X_t)\prod_{n=1}^N 1_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\Lambda^2_{t-}\,\mathrm{d}W_s\right] = 0$$

and that

$$\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}\bar{w}^2(X_t) - rw^2(X_t)]\prod_{n=1}^N 1_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\Lambda^2_{t-}\,\mathrm{d}t\right] = 0,$$

which follows from (49) and from the fact that $\Lambda^2_{t-} = 1_{\{t\leq\tau_{S^2}\}}\,\mathrm{e}^{-\int_{\mathcal{I}} L^y_t\,\mu^2(\mathrm{d}y)}$ vanishes over $\{X_t < s^2\}$. Now, using that the measure $\mathrm{d}\Gamma^2_t$ only charges the set $\{t \geq 0 : w^2(X_t) = R^2(X_t)\}$, we have

$$\mathbf{E}_x\left[-\int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda^1_t\,\mathrm{d}\Lambda^2_t\right] = \mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t\right]$$
$$= \mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt} R^2(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t\right]. \quad \text{(S.42)}$$

Next, using (47), and following the same steps as for (S.40), we have

$$\mathbf{E}_x\left[-\frac{1}{2}\sum_{n=1}^N \Delta w^{2\prime}(q^1_n)\int_{[0,\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\Lambda^2_{t-}\,\mathrm{d}L^{q^1_n}_t\right]$$
$$= \mathbf{E}_x\left[\sum_{n=1}^N \int_{[0,\tau_k)} \mathrm{e}^{-rt} G^2(q^1_n)\Lambda^1_t\Lambda^2_{t-} a_n\,\mathrm{d}L^{q^1_n}_t - \sum_{n=1}^N \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(q^1_n)\Lambda^1_t\Lambda^2_{t-} a_n\,\mathrm{d}L^{q^1_n}_t\right]$$
$$= \mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\Lambda^2_{t-}\,\mathrm{d}\Gamma^1_t + \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda^2_{t-}\mathrm{d}\Lambda^1_t\right]. \quad \text{(S.43)}$$

We obtain from (S.41)–(S.43) that

$$w^2(x) = \mathbf{E}_x\Bigg[\mathrm{e}^{-r\tau_k} w^2(X_{\tau_k})\Lambda^1_{\tau_k}\Lambda^2_{\tau_k-} \quad \text{(S.44)}$$
$$+ \int_{[0,\tau_k)} \mathrm{e}^{-rt} R^2(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \int_{[0,\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\Lambda^2_{t-}\,\mathrm{d}\Gamma^1_t\Bigg]. \quad \text{(S.45)}$$

Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x) = \mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^2(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^2(X_t)\Lambda^2_{t-}\,\mathrm{d}\Gamma^1_t\right] = J^2(x, (\mu^1, S^1), (\mu^2, S^2)),$$

where the equality follows from (14). The result follows. ■

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