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"Optimally Stubborn"

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Optimally Stubborn*

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Abstract

Models on reputational bargaining have introduced a perturbation with simple behavioral types as a way of refining payoff predictions for the rational type. I show that this outcome refinement is not robust to the specification of the behavioral type. More specifically, I consider a slight relaxation of the strategy restriction on behavioral types relative to the literature, allowing behavioral types to choose their initial demands. I show that with this relaxation any feasible payoff can be achieved in equilibrium for the rational type when the probability of facing a behavioral type is small. My results highlight the implications of different perturbations for economic applications.

1 Introduction

Models on reputational bargaining (Myerson 1991, Abreu and Gul 2000 [AG] and follow-up papers) have shown that the so-called Rubinstein-Stahl outcome is the unique outcome in a large class of bargaining protocols when the game is perturbed with simple behavioral types. This is a remarkable result given that non-cooperative bargaining models are known to provide profusion of equilibria. I re-examine the robustness of this result with respect to the nature of the perturbation and show that perturbing the bargaining game with behavioral types does not necessarily lead to outcome refinements. More specifically, I show that if we slightly relax the perturbation used, there is Folk-theorem-like payoff multiplicity. Any payoff, and also delay of any length, can arise in equilibrium.

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Consider two agents trying to agree on a division of some surplus. The literature on non-cooperative bargaining has struggled to come up with a clear prediction as to how and when agents agree. Following Myerson (1991), AG remodel this problem by introducing a simple perturbation. They assume each agent is behavioral with a small probability. A behavioral type always makes some fixed exogenous demand and never concedes to anything less. Behavioral types in AG have no choice over their actions. This perturbation selects the Rubinstein-Stahl outcome as the unique outcome for a large class of bargaining protocols.

Given this dramatic result, one naturally wonders about the role of the particular perturbation introduced. The perturbation I consider involves allowing the behavioral type to chooses his initial demand – in other words, he chooses which posture he wants to portray. I thereby relax the strategy restriction on behavioral types relative to AG. Otherwise, I essentially follow AG.[1] More specifically, I consider a bargaining game with two types of players: rational and stubborn. The game consists of two stages: a demand stage and a concession stage. Players simultaneously make demands. The game ends when one player concedes, i.e., agrees to his opponent's demand. Rational players can concede at each point in time. Stubborn players can choose their initial demand, but cannot concede to their opponent. With this seemingly small relaxation, I show that any payoff can be achieved in equilibrium when the probability of facing a behavioral type is small. This result is robust to refinements such as *D1*. In other words, perturbing the bargaining game with a slightly more flexible behavioral type radically changes the payoff predictions.

My proof is constructive. I give a full characterization of pooling equilibria.[2] In fact, the class of strategies that span this Folk-theorem-like payoff multiplicity have a specific structure which I discuss further below. What gives rise to the multiplicity is that when the behavioral types choose their initial demands, the probability that a demand is believed to come from a stubborn type can vary across demands. This means that players can be compensated for making otherwise "unattractive" demands by being more likely to be believed to be stubborn; similarly, players can be deterred from making otherwise "attractive" out-of-equilibrium demands by being believed to be rational. In this way, any payoff can be

[1] See Section 2 and 3 for details.

[2] However, the result holds regardless of whether types pool or separate in equilibrium.

achieved in equilibrium.

In the model with AG's behavioral types, provided that the "right" stubborn type is present, the rational type can guarantee himself the Rubinstein-Stahl payoff.[3,4] Similarly, Kambe (1999) shows that when players do not know their type at the time of choosing their initial demands, the unique equilibrium outcome is the Rubinstein-Stahl outcome.[5] In Wolitzky (2012), as in Kambe (1999), players do not know their type when announcing postures, but non-stationary types are allowed. In this setting, Wolitzky (2012) derives a non-zero lower bound on the rational type's payoff.[6]

My result highlights the importance of asking what these behavioral types are supposed to represent and hence, which perturbations to the bargaining game are reasonable. Clearly, how we should model these behavioral types depends on the economic application we have in mind. My model is suited to analyzing situations where the agent himself is well aware whether making demands or threats will restrict his flexibility, but these constraints are not common knowledge. For instance, a political leader might make demands in an international negotiation and privately know that backing down from this initial demand later is impossible or politically very costly. He may face a so-called "audience cost:" a domestic political cost because the foreign policy leadership is seen as unsuccessful.[7] A political leader knows whether or not it will be costly to back down from a threat or demand once it has been made (i.e., whether or not there is an audience cost), but other international actors may not be able to observe this about a political leader they are negotiating with. Instead adversaries update

[3]Note that this prediction does not rely on initial demands being sequential. In particular, the same result holds in a simultaneous-move version of AG.

[4]More generally, for any payoff up to one half, one can find a distribution that gives rise to that payoff in equilibrium.

[5]If players do not know their type at the time of choosing their demand, demands cannot convey information about a player's type. As a result, players strictly prefer intermediate demands over extreme demands, leading to faster agreement.

[6]More tangentially related are Abreu and Pearce (2007) and Abreu, Pearce and Stacchetti (2015) who show a similar equilibrium refinement when perturbing their game with slightly more sophisticated behavioral types.

[7]See Fearon (1994) and follow-up papers. Most closely related are Ozyurt (2014) and Ozyurt (2015 a and b).

their beliefs about whether a leader will act as if he has high audience cost as a function of the demands or threats made. Anticipating whether an announcement of a demand or threat imposes constraints on negotiations down the line and how making such a demand or threat will influence the adversaries' belief about a leader's future constraints, a political leader can adjust the initial demand or threat accordingly. In this way, my paper speaks to the topic of audience costs in international negotiations and disputes.

In AG, players know their type before choosing initial demands and in contrast to my model, the behavioral type has no choice to make. These assumptions may be suited to analyzing situations where an agent has experienced an exogenous shock to their flexibility prior to starting the negotiation. For example, a manager may delegate the communication of a demand to an employee but leaving the employee with no room to actually negotiate with the bargaining partner. When the employee has no further means of relaying any information prior to agreement back to the manager, the employee in this situation then plays the role of the behavioral type as modeled in AG. In Kambe (1999) and Wolitzky (2012), players do not know their type ex ante, but rather may become committed once bargaining postures have been announced. Assuming that players become committed with some exogenous probability may be suited to analyzing situations where an agent's flexibility depends on some exogenous shocks ex post – for example, a poor outcome in a fundraising round may lead a company to impose ex post constraints on a manager's flexibility in making decisions.

Despite the payoff multiplicity, I am able to make predictions concerning behavior. In fact, I fully characterize behavior in all symmetric equilibria as the probability that a player is behavioral vanishes. Types may pool or separate in equilibrium.[8] In the limit as the fraction of stubborn types vanishes, if types pool, they pool on either one or two demands. If types fully separate, the rational type does not randomize over multiple demands and the stubborn type randomizes over at most two demands. In the limit, if types semi-separate,

[8]A pooling demand is a demand that is made by both types with positive probability – such a demand does not perfectly reveal a player's type. A separating demand is a demand that is exclusively made by one type with positive probability – such a demand perfectly reveals a player's type. If all demands are pooling (separating) demands, I refer to the equilibrium as a pooling (separating) equilibrium. If an equilibrium is neither pooling nor separating, I refer to it as semi-separating.

the rational type may randomize over arbitrarily many separating demands and the stubborn type only makes pooling demands.[9]

In the setting of AG, reputational forces lead to a unique equilibrium behavior on path, in which types pool.[10] Compte and Jehiel (2002) show that if players have access to sufficiently attractive outside options, the equilibrium outcome remains unique but is pinned down by the outside options rather than the reputational forces in AG. In other words, types in Compte and Jehiel (2002) separate in equilibrium. Atakan and Ekmekci (2014) and Ozyurt (2015 a and b) show that if outside options are endogenously determined in a market setup, reputational forces determine the equilibrium behavior of the rational type. In this vein, this paper shows that when the behavioral type is given a choice over its initial demand, both pooling and separating equilibria exist, and so reputational forces may or may not determine the equilibrium outcome.

Given the difference in the preferences of the two types, it is perhaps not surprising that types can separate in equilibrium when the stubborn type is given choice over his initial demand. When facing a compatible demand, the payoff to the two types is the same. Instead when facing an incompatible demand, the rational type can concede and the stubborn type cannot. This means that when the stubborn type faces another stubborn type, his payoff is zero. As a result, players may have incentives to separate by type in equilibrium. When they do, the stubborn type makes lower demands than the rational type given any positive ex ante probability of stubbornness.

What is perhaps more surprising is that types can pool over multiple demands in equilibrium. The intuition for this lies in how the two types of players differ in their preferences over demands. First, players (regardless of their type) face a tradeoff between the amount

[9]For the most part, I focus on behavior in the limit, i.e., as the ex ante probability of a player being behavioral vanishes. This allows for the cleanest expressions and helps with tractability for obvious reasons. However, I am able to characterize all possible demand configurations regardless of the ex ante probability of stubbornness. Moreover, existence results for all separating equilibria as well as pooling and semi-separating equilibria with at most three pooling demands are established more generally. The reader is referred to Section 5 for details.

[10]In AG, this result relies on an assumption imposed on the distribution over behavioral types that ensures types pool in equilibrium. If this assumption is dropped, separating equilibria exist in their model as well.

received if the opponent concedes and the speed with which the opponent concedes. Fixing the concession behavior of the opponent, a player receives a higher payoff the more he demands. But everything else being equal, the higher a player's demand, the slower the opponent concedes. This tradeoff makes intermediate demands particularly attractive (as in AG). Second, this trade-off is not the same for the two types. When demands are compatible, the two types receive the same payoff. When demands are incompatible, the cost of being stubborn (relative to being rational) is smaller the higher the demands. Higher demands imply a longer war of attrition and hence, the stubborn type's cost of not being able to concede is paid "far in the future."[11] As a result, arbitrarily fixing the opponent's distribution over demands, the difference in the two types' payoffs as a function of one's own demand is never monotone. This failure of single-crossing implies that types can be made indifferent the same set of demands, and hence, that types can pool over multiple demands in equilibrium.[12]

My paper is closely related to two strands of literature. The first is the literature on reputational bargaining discussed above (beyond those papers already mentioned, see Abreu and Sethi 2003; Fanning 2016, 2018 and 2021; Kim 2009 among others). By extension, the paper is also related to the literatures on bargaining (Nash 1953; Stahl 1972; Rubinstein 1982) and on reputation (Fudenberg and Levine 1989 and 1992) more generally.

Moreover, this paper also contributes to the literature on crisis bargaining and audience costs. The closest paper is Ozyurt (2014) who builds on Fearon's seminal paper (Fearon 1994). In Fearon (1994), two players fight over a prize. At each point in time, each player chooses to wait, concede or attack. Conceding means giving the prize to the opponent and paying an "audience cost" (for backing down). Waiting increases the audience cost in the future. Attacking is costly for both the attacker and the attacked - but the cost to each player is private information. The key difference in Ozyurt (2014) relative to Fearon (1994) is that there is a small probability that a player is committed to waiting. Hence, unlike in my model and much like in AG, the stubborn type in Ozyurt (2014) has no choice to make.

[11]A rational player is willing to wait to concede as long as he is uncertain about the opponent's type. However, once the player is certain he is facing a stubborn opponent, he strictly prefers to concede. Hence, the length of the war of attrition determines the payoff difference between the two types.

[12]For further details, see Section 4.

Ozyurt (2014) shows that like in AG, there exists a finite time after which no concession takes place. This can be either because a player is known to be stubborn, or because at this point a player strictly prefers to attack over conceding. Ozyurt's main result is that having higher audience costs is advantageous to a player if and only if he is the stronger player in the sense of AG.[13]

The structure of this paper is as follows. I first describe the model in Section 2. Section 3 analyzes the benchmark case with the strategy restriction on the behavioral type as modeled in AG. Section 4 presents the main result. In Section 5 characterizes all equilibria. In Section 6, I discuss the robustness of the results. Section 7 concludes the paper. Unless stated otherwise, all proofs are in Appendix.

2 Model

The model and the notation (mostly) follow AG. Time is continuous, and the horizon is infinite. Two players decide on how to split a unit surplus. At time $t = -1$, players i and j simultaneously announce demands, α_i and α_j, with $\alpha_i, \alpha_j \in [0, 1]$. If $\alpha_i + \alpha_j \leq 1$, the demands are said to be *compatible*. In this case, the game ends. If $\alpha_i + \alpha_j > 1$, the demands are *incompatible*. In this case, a concession game starts at $t = 0$. The game ends when one player concedes. Concession means agreeing to the opponent's demand.

Each player i is rational with probability $1 - z$ and stubborn with probability z, where $z \in (0, 1)$. Before the game starts, each player privately learns whether he is stubborn or rational. A rational player $i = 1, 2$ can make any demand $\alpha_i \in [0, 1]$ at time 0 and concede to his opponent at any point in time. Stubborn player i can choose his initial demand $\alpha_i \in [0, 1]$ but cannot concede to his opponent. Note that this is unlike in AG, where a stubborn player cannot choose his initial demand.[14]

A strategy for a stubborn player, i, σ_i^S, is defined by a probability distribution s_i on $[0, 1]$ (his demand α_i). A strategy for a rational player i, σ_i^R, is defined by a probability

[13] This implies that in a symmetric model, having higher audience costs is not an advantage.

[14] In AG, there are $K + 1$ types of players: one rational type and K stubborn types, where K is an arbitrary finite number. A stubborn player of type α_i in AG always demands α_i, accepts any demand of at least α_i, and rejects all smaller demands. They assume an exogenously given finite set of stubborn types: $C = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ with $\alpha_1 + \alpha_K > 1$ and $\alpha_K < 1$.

distribution r_i on $[0,1]$ (his demand α_i) and a collection of cumulative distributions $F^{r,i}_{\alpha_i,\alpha_j}$ on $\mathcal{R}_+ \cup \{\infty\}$ for all $\alpha_i + \alpha_j > 1$. $F^{r,i}_{\alpha_i,\alpha_j}(t)$ is the probability of rational player i conceding to player j by time t (inclusive), given α_i, α_j. The probability of player i conceding by time t is given by:

$$F^{i}_{\alpha_i,\alpha_j}(t) = (1 - \pi_i(\alpha_i))\, F^{r,i}_{\alpha_i,\alpha_j}(t),$$

where

$$\pi_i(\alpha_i) = \frac{z s_i(\alpha_i)}{z s_i(\alpha_i) + (1-z) r_i(\alpha_i)} \tag{1}$$

is the posterior probability that player i is stubborn immediately after it is observed that i demands α_i at time zero given σ_i^R and σ_i^S. Therefore,

$$\lim_{t\to\infty} F^{i}_{\alpha_i,\alpha_j}(t) \leq 1 - \pi_i(\alpha_i).$$

Note that $F^{i}_{\alpha_i,\alpha_j}(0)$ may be positive. It is the probability that i immediately concedes to j. Throughout, all probability distributions over demands are assumed to have finite support (unless stated otherwise). Hence, I summarize the distribution over demands by the vector of weights assigned to the demands in its support when the support is understood and write the corresponding indicator function otherwise.

Player i's discount rate is $\rho > 0$, for $i = 1, 2$. The continuous-time bargaining problem is denoted $B = \{z, \rho\}$. If $\alpha_i + \alpha_j \leq 1$ at $t = 0$, player i receives α_i and $1 - \alpha_j$ with probability 1/2. Suppose that $\bar{\alpha} = (\alpha_i, \alpha_j)$ is observed at time 0, with $\alpha_i + \alpha_j > 1$. Then, (rational) player i's expected payoff from conceding at time t, given strategy profile $\bar{\sigma} = (\sigma_i, \sigma_j)$, where $\sigma_i = \left(\sigma_i^R, \sigma_i^S\right)$, is:

$$\begin{aligned} U_i(t, \sigma^j \mid \bar{\alpha}) = \alpha_i \int_{y<t} e^{-\rho y} dF^j_{\bar{\alpha}}(y) + \frac{\alpha_i + 1 - \alpha_j}{2} \left(F^j_{\bar{\alpha}}(t) - F^j_{\bar{\alpha}}(t^-)\right) e^{-\rho t} \\ + (1-\alpha_j)\left(1 - F^j_{\bar{\alpha}}(t)\right) e^{-\rho t}, \end{aligned} \tag{2}$$

where $F^j_{\bar{\alpha}}(t^-) = \lim_{y \uparrow t} F^j_{\bar{\alpha}}(y)$.[15] Hence, player i receives the discounted value of his demand α_i if player j concedes to i before i concedes to j. If the players concede simultaneously,

[15] In the following, whenever I refer to a player's payoff from conceding to his opponent, I refer to a rational player's payoff given that the stubborn type cannot concede.

then player i receives his own demand and the complement of player j's demand with equal probability. Player i receives the discounted value of the complement of player j's demand, $1-\alpha_j$, if player i concedes first. Player i's expected payoff from never conceding is:

$$U_i(\infty, \sigma^j \mid \bar{\alpha}) = \alpha_i \int_{y\in[0,\infty)} e^{-\rho y} dF^j_{\bar{\alpha}}(y). \tag{3}$$

This is a stubborn player's payoff from facing a demand that is incompatible with his own demand. Since $F^i_{\alpha_i,\alpha_j}$ describes the concession behavior of a player, unconditional on his type, a rational player i's concession behavior is described by:

$$\frac{1}{1-\pi_i(\alpha_i)} F^i_{\alpha_i,\alpha_j}.$$

Therefore, a rational player i's expected utility from a mixed action F^i conditional on $\bar{\alpha} = (\alpha_i, \alpha_j)$ being observed at time 0 is:

$$U_i(\bar{\sigma} \mid \bar{\alpha}) = \frac{1}{1-\pi_i(\alpha_i)} \int_{y\in[0,\infty)} U_i(y, \sigma_j \mid \bar{\alpha}) dF^i_{\bar{\alpha}}(y). \tag{4}$$

A rational player i's expected utility from the strategy profile $\bar{\sigma}$ is:

$$\begin{aligned} U_i(\bar{\sigma}) = & \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j \leq 1-\alpha_i} \frac{\alpha_i + 1 - \alpha_j}{2} \left((1-z) r_j(\alpha_j) + z s_j(\alpha_j)\right) \right) \\ & + \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j > 1-\alpha_i} U_i(\bar{\sigma} \mid \alpha_i, \alpha_j) \left((1-z) r_j(\alpha_j) + z s_j(\alpha_j)\right) \right). \end{aligned} \tag{5}$$

The first term is the payoff a rational player receives from demanding α_i when $\alpha_i + \alpha_j \leq 1$. The second term is the payoff from demanding α_i when facing an incompatible demand. At this stage, it is useful for me to introduce two pieces of notation. I denote the probability of player j facing demand α_i by q_i, i.e.,

$$q_i = (1-z) r_i(\alpha_i) + z s_i(\alpha_i).$$

I follow the literature in modeling the bargaining as a war of attrition (rather than allowing players to revise their demands). This is inspired by Myerson's (1991) insight that revising one's demand reveals rationality, so that it is equivalent to conceding (in the context of his model, which is closely related, but not identical to mine).[16]

For the analysis in $B = \{z, \rho\}$, I use the solution concept of (weak) Perfect Bayesian equilibrium (PBE). A PBE is a profile of strategies $\sigma^* = (\sigma_1^*, \sigma_2^*)$ and a system of initial beliefs mapping demands into probabilities that a player is stubborn,

$$\pi_i : [0,1] \to [0,1] \quad \text{for} \;\; i = 1, 2,$$

such that

1. the strategy maximizes a player's expected utility (given beliefs), and

2. if an information set is reached with positive probability given the strategy profile, beliefs are formed according to Bayes' rule; and if an information set is not reached with positive probability given the strategy profile, beliefs are arbitrary probabilities that a player is stubborn.

Henceforth, equilibrium refers to weak PBE (see Fudenberg and Tirole, 1991 for a definition).[17] I focus on symmetric equilibria with finite support. By symmetric equilibria I mean equilibria where $r_i = r_j$ and $s_i = s_j$. In other words, I focus on equilibria where the identity of a player does not matter. Only his type does. This already suffices to establish payoff multiplicity.[18] By finite support, I mean that types randomize over finitely many demands in the initial stage.

3 Benchmark

The insight of AG is that perturbing a bargaining game with simple behavioral types leads to a unique payoff prediction for a large class of bargaining protocols. More precisely, AG show

[16]AG show that any convergent sequence of equilibrium outcomes within a broad family of discrete-time games must converge to the unique continuous-time equilibrium outcome as the maximum time between consecutive opportunities to revise demands goes to 0. Of course, the modeling of AG differs from mine in some respects as outlined and discussed in Section 3.

[17]To the extent that the concession behavior is a direct consequences of the demands made, I refer to an equilibrium by its support and the probabilities associated with that support.

[18]Note that given the assumption of symmetry, the multiplicity here refers to the delay rather than the division of surplus.

that the outcome for a large class of bargaining protocols converges to the unique continuous-time limit when the time between consecutive demands goes to 0. In this section, I recall the unique outcome of AG's continuous-time game. To this end, consider the continuous-time game described before but with a different strategy restriction on the behavioral type. In particular, there is an exogenously given set of stubborn types $C = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$, where $\alpha_k < \alpha_{k+1}$ and $\alpha_K < 1$.[19] A stubborn player of type α_i always demands α_i, accepts any offer of at least α_i, and rejects all smaller offers. AG assume throughout that $\alpha_1 + \alpha_K > 1$, i.e., the highest type is incompatible with the lowest type.[20] Moreover, assume that players make initial demands sequentially (as in AG) rather than simultaneously (as in my model). For each of the two propositions in this section, I will be precise if and how it extends to the model with simultaneous moves.

I denote the probability that stubborn player i is of type α_k by $s_i(\alpha_k)$. Hence, s_i is a probability distribution on C. The continuous-time bargaining problem is denoted $B^{AG} = \{(C, z, s_i, \rho)_{i=1}^2\}$. Proposition 1 (AG) establishes the existence and uniqueness of the equilibrium outcomes with a given distribution of stubborn types. In the class of symmetric equilibria, this result extends to the case when initial moves are simultaneous.[21]

Proposition 1 (AG, Proposition 2)**.** *For any bargaining game B^{AG}, a PBE exists. Furthermore, all equilibria yield the same distribution over outcomes.*

The unique equilibrium outcome in this game can be characterized by the two choices that a rational player makes: (1) when to concede and (2) whom to mimic. In the equilibrium, after the initial choice of demands, (i) at most one player immediately concedes with positive probability; (ii) players concede at a constant rate that makes the opponent indifferent between waiting and conceding; and (iii) there is a finite time, call it T_0, by which

[19] Given the focus on symmetric equilibria in this paper, in the following, I focus on the case where types are symmetric, i.e., the stubborn types of player 1 are identical to the stubborn types of player 2.

[20] This assumption is necessary to derive unique payoff predictions, regardless of whether moves are sequential or simultaneous. When moves are sequential it is also sufficient; when moves are simultaneous, it is sufficient in the class of symmetric equilibria.

[21] However, fixing a bargaining game B^{AG}, not all equilibria yield the same distribution over outcomes when dropping the assumption of symmetry. This is due to the fact that the rational type may separate in asymmetric equilibria.

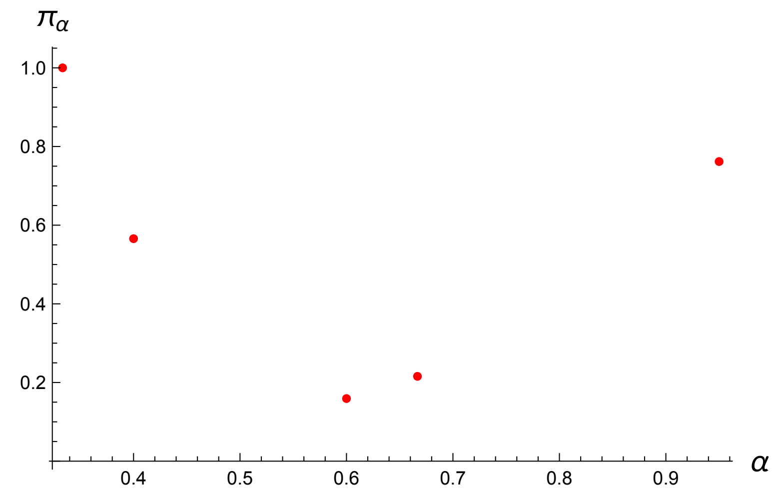

Figure 1: Probability of stubbornness conditional on the demand α, $\pi(\alpha)$, in a PBE with exogenous stubborn types (see body for parameters).

the posterior probability of stubbornness reaches 1 simultaneously for both players and concessions by the rational type stop. Moreover, any demand above some threshold is mimicked with positive probability.

I illustrate the mimicking behavior of the rational type in Figure 1. More precisely, in this figure I consider a simultaneous-move version of AG. The figure shows the posterior probability of stubbornness in an equilibrium.[22] We can see that the lowest demand is not mimicked by the rational type, i.e., $\pi\left(\alpha|\alpha=\frac{1}{3}\right)=1$. On the other hand, any demand of $\frac{2}{5}$ or higher is mimicked with positive probability, i.e., $\pi\left(\alpha|\alpha\geq\frac{2}{5}\right)<1$. The U-shaped structure of the posterior probability above the threshold is driven by the concept of strength, as defined and discussed below.

Let me be more precise regarding the rate of concession and the stopping time of the rational type (which applies regardless of the bargaining protocol at the initial stage). Player i is indifferent between waiting and conceding if the net cost of waiting is equal to the net benefit of waiting:

$$\rho(1-\alpha_j)=(\alpha_i-(1-\alpha_j))\frac{f^j_{\alpha_j,\alpha_i}(t)}{1-F^j_{\alpha_j,\alpha_i}(t)},$$

where $f^j_{\alpha_j,\alpha_i}(t)=dF^j_{\alpha_j,\alpha_i}(t)/dt$. By waiting, a player loses the value of concession over the

[22]In particular, I choose a PBE with five stubborn types $C=\{\frac{1}{3},\frac{2}{5},\frac{3}{5},\frac{2}{3},\frac{19}{20}\}$, and $z=\frac{1}{3}$.

next instant, which, given a player's impatience, is given by the LHS. The first term on the RHS captures the benefit from being conceded to relative to conceding. The second term on the RHS is the probability with which the opponent concedes in the next instant conditional on not yet having conceded. Therefore, after time 0, player j demanding α_j concedes to player i demanding α_i at a rate

$$\lambda_j^{\alpha_j,\alpha_i} = \frac{\rho(1-\alpha_j)}{\alpha_i+\alpha_j-1}.$$

Note that the rate at which player j concedes is decreasing in player i's demand: the more a player demands, the more he receives conditional on his opponent conceding. Therefore, the rate with which the opponent has to concede to make a player indifferent is lower the higher the player's demand is. Note also that this rate of concession is time-independent. However, only the rational type concedes, which implies that the probability of facing a rational opponent is decreasing over time. Hence, the rational type's rate of concession is increasing over time.

Requirement (iii) pins down the identity of the player who concedes at time 0 and the probability with which this happens.[23] Let $T_i^{\alpha_i,\alpha_j}$ denote the time at which player i is stubborn with probability 1 conditional on not conceding with positive probability at time 0. Then, the time T_0 is given by:

$$T_0 = \min\{T_1^{\alpha_1,\alpha_2}, T_2^{\alpha_2,\alpha_1}\},$$

where

$$T_i^{\alpha_i,\alpha_j} = -\frac{1}{\lambda_i^{\alpha_i,\alpha_j}} \log \pi_i(\alpha_i)$$

for $i = 1, 2$. Player i is *stronger* than player j if and only if $T_i^{\alpha_i,\alpha_j} < T_j^{\alpha_j,\alpha_i}$. In other words, a player is stronger the sooner is the time at which he is known to be stubborn. Note that

$$T_i^{\alpha_i,\alpha_j} < T_j^{\alpha_j,\alpha_i} \iff \pi_i(\alpha_i)^{\frac{1}{1-\alpha_i}} > \pi_j(\alpha_j)^{\frac{1}{1-\alpha_j}}.$$

For the rest of the paper, I will denote a player's strength by $\mu_i(\alpha_i)$, where

$$\mu_i(\alpha_i) = \pi_i(\alpha_i)^{\frac{1}{1-\alpha_i}}.$$

[23]For intuition for (iii), see AG page 10.

The weaker player j has to concede with sufficient probability at time 0 that conditional on not conceding, and given the concession rates, his probability of stubbornness reaches 1 at the same time as player i. In particular, the probability of immediate concession by player j is given by:

$$F^j_{\alpha_j,\alpha_i}(0) = \max\left\{1 - \left(\frac{\mu_j(\alpha_j)}{\mu_i(\alpha_i)}\right)^{1-\alpha_j}, 0\right\}. \tag{6}$$

The derivation follows AG. The strength of player j relative to player i depends on (i) how likely j is thought to be stubborn conditional on his demand and (ii) how high j's demand is. Clearly, the more likely a player is thought to be stubborn, the more willing the opponent is to concede. The higher a player's demand, the more willing his opponent is to wait. This is because conditional on giving up, a player obtains less the higher his opponent's demand. Hence, the lower the demand a player makes, the stronger he is because it makes his opponent more willing to concede. Everything else being equal, a player's payoff is increasing in his strength. Consider an incompatible pair of demands. In equilibrium, a weak player is not conceded to with positive probability at time 0. He is indifferent between waiting and conceding and hence must receive what he would receive by conceding immediately. A strong player is conceded to with positive probability at time 0, in which case the player obtains what he demanded, which he strictly prefers over conceding himself. If the strong player is not conceded to at time 0, he also simply receives what he would have received by conceding immediately. The probability with which the opponent concedes to the strong player is strictly increasing in the player's strength. This yields a tradeoff: The more a player demands, the more a player receives conditional on being conceded to immediately. However, the more a player demands, the lower the probability with which the opponent concedes at time 0. This makes intermediate demands particularly attractive for the rational type.

Let me return to the U-shaped posterior probability of stubbornness in Figure 1. Suppose that player j demands α_j with probability 1 and is thought to be stubborn with probability $\pi(\alpha_j)$. Then, fixing the probability of player i being stubborn, the preferences of a rational player i are single-peaked in his own demand α_i: he trades off the probability with which his opponent concedes at time 0, with how high his payoff is conditional on his opponent conceding. This implies that in equilibrium, the conditional probability of stubbornness must be single-bottomed in α_i, as Figure 1 shows.

Proposition 2 (AG, Corollary in Section 5)**.** *Let $B_n^{AG} = \{(C, z_n, s_i, \rho)_{i=1}^2\}$ be a sequence of continuous-time bargaining games such that $\lim_{n\to\infty} z_n = 0$. Let ϵ be the mesh of $C \cup \{0,1\}$.*[24] *Then, for n sufficiently large, the equilibrium payoff of (rational) player i is at least $\frac{1}{2} - \epsilon$, and hence, the inefficiency due to delay is at most 2ϵ.*

Proposition 2 states that as the probability of a player being stubborn goes to 0, delay and inefficiency disappear provided that the "right" behavioral type is present. This result also holds when moves are simultaneous. By the right type, I mean the type whose presence alone, i.e., regardless of which other behavioral types are present, allows for payoff predictions. The right type in this sense is the type making a demand proportional to a player's patience. In the symmetric discounting case, the right type is then a type demanding 1/2, and a rational type can guarantee himself a payoff of 1/2 in the limit. The loose argument for why the right type is the type demanding 1/2 is as follows. Since there cannot be any delay, it must mean that with almost probability 1, a player's demand is immediately accepted. Since this is true for both players, it must mean that a player can mimic type 1/2.[25,26]

4 Main Result

Returning to the model of this paper, where behavioral types choose their initial demands, my main result is that the slight relaxation of the strategy restriction on behavioral types leads to Folk-theorem-like payoff multiplicity for the rational type. The proof is constructive and so this section starts with an existence result for pooling equilibria with two demands. By pooling equilibria, I mean equilibria where the set of demands over which a player randomizes is identical for the stubborn and the rational type. Given the symmetry assumption, I simplify notation in the remainder. In particular, $r_i(\alpha_k) = r_k$, $s_i(\alpha_k) = s_k$, $\mu_i(\alpha_k) = \mu_k$ etc.

[24] I.e., $\epsilon = \max_k (\alpha_{k+1} - \alpha_k)$, ordering the demands from smallest to largest.

[25] More generally, a rational player in AG obtains a payoff proportional to his patience.

[26] Note that if the right behavioral type is not present, delay and hence inefficiency persist in AG. For instance, if $C = \{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$, then the limit payoff of the rational type is $\frac{2}{5}$.

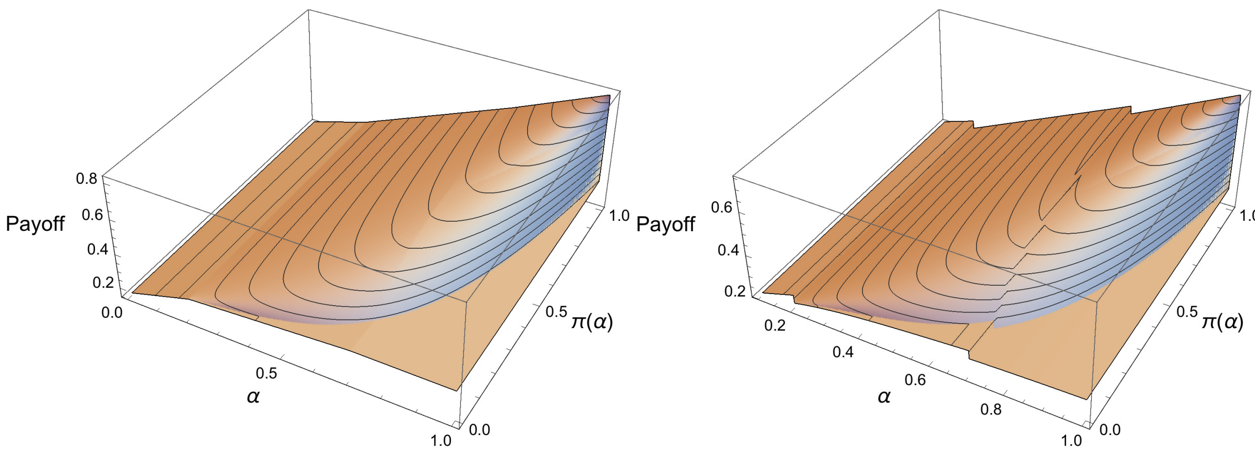

Figure 2: 3D-Payoff profile for a rational type (left) and a stubborn type (right) for a fixed set of demands and posterior probabilities of the opponent.

Proposition 3. *Fix a sequence $z^n \to 0$ and fix $a_1 \in (0, 1/2]$ and $a_2 \in (1-a_1, 1]$. Then there exists N such that for any $n > N$, a corresponding convergent sequence of pooling equilibria $(\alpha_1^n, \alpha_2^n, r^n, s^n)$ satisfying*

$$\lim_{n\to\infty} \alpha_1^n = a_1, \quad \lim_{n\to\infty} \alpha_2^n = a_2, \tag{7}$$

and

$$\lim_{n\to\infty} \begin{pmatrix} r_1^n \\ r_2^n \end{pmatrix} = \begin{cases} \frac{2(a_1+a_2-1)}{2a_2-1}, \\ \frac{1-2a_1}{2a_2-1}, \end{cases} \quad \text{and} \quad \lim_{n\to\infty} \begin{pmatrix} s_1^n \\ s_2^n \end{pmatrix} = \begin{cases} \frac{1-a_2}{2-a_1-a_2}, \\ \frac{1-a_1}{2-a_1-a_2} \end{cases} \tag{8}$$

exists.

Proposition 3 states that for any two demands (satisfying the stated conditions) a sequence of pooling equilibria exists. In the appendix, I prove a slightly stronger statement - in particular, in addition to Proposition 3 as stated, I prove that any convergent sequence of pooling equilibria with two demands satisfies (7) and (8).[27] The proof starts with showing uniqueness (in the sense of convergence as just stated), and then uses the Implicit Function Theorem to prove existence. Note that even in the limit, as the probability of stubbornness goes to 0, both types assign strictly positive probability to both demands. This may be surprising to the reader – it implies that unlike in standard models of signaling, preferences

[27]Here and thereafter converges in the equilibrium distribution of demands refers to weak convergence.

do not satisfy the single-crossing property.[28]

To gain some intuition for why such pooling equilibria exist, it is useful to consider the tradeoffs involved in choosing a demand. First, players (regardless of their type) face a tradeoff between the amount received if the opponent concedes and the speed with which the opponent concedes. Fixing the concession behavior of the opponent, a player receives more, the more he demands. However, the higher a player's demand, the slower the opponent concedes. This tradeoff makes intermediate demands particularly attractive and implies that a (rational) player's payoff is single-peaked in his own demand (as in AG). Second, the tradeoffs involved in choosing a demand differ across the two types. When demands are compatible, the two types receive the same payoff. When demands are incompatible, the rational type's (expected) payoff is higher than the stubborn type's payoff. When facing an incompatible demand, the rational type is able to concede while the stubborn type cannot. A rational player is willing to wait to concede as long as he is uncertain about the opponent's type. However, once the player assigns probability 1 to facing a stubborn opponent, he strictly prefers to concede. Yet, a stubborn type cannot do so. This leads to the rational type's payoff being higher than the stubborn type's payoff when demands are incompatible. Moreover, this payoff difference when facing an incompatible demand is smaller the higher the demands. The higher the demands, the slower the rate at which players concede. This implies that the higher the demands the longer the war of attrition. As a result, the time at which the rational type has a strict preference for conceding is "far into the future." Discounting then implies that the stubborn type's cost of not being able to concede is low when demands are high. Hence, conditional on incompatible demands, the payoff difference is smaller the higher the demands. Together this then implies that fixing the opponent's distribution over demands, the difference in the two types' payoffs as a function of a player's own demand is not monotone.[29] This failure of single-crossing enables both types of players

[28] Suppose indifference correspondences are plotted as a function of a player's own demand and the conditional probability of stubbornness, fixing the opponent's strategy. If preferences satisfy single-crossing, indifference correspondences of the rational type and the stubborn type cross once. If indifference correspondences cross once, then there exists no pair of demands such that both types are indifferent between these two demands. In other words, if preferences satisfy single-crossing there can be at most one pooling demand.

[29] More specifically, fixing the opponent's distribution over demands, the stubborn type's payoff is discon-

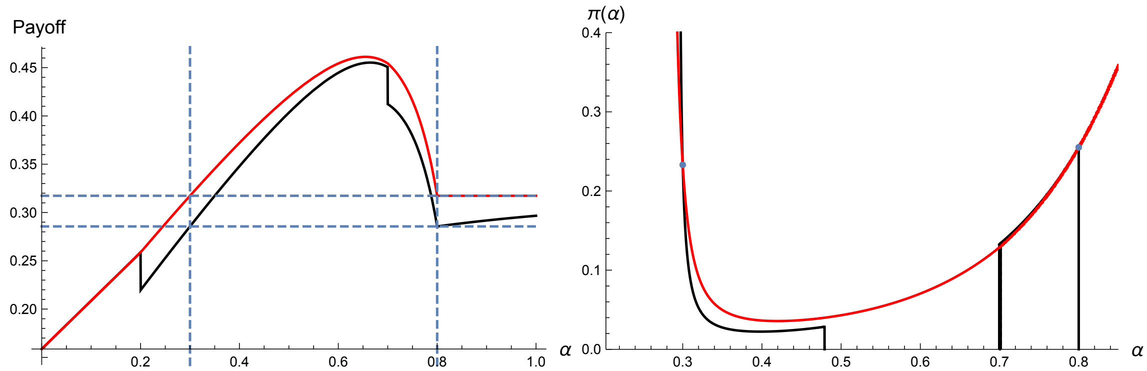

Figure 3: Cross-sections of the 3D-payoff profile (left) and indifference correspondences (right) for rational (red) and stubborn (black) type.

to be indifferent over the same set of demands.[30]

Figures 2 and 3 illustrate. Fix a pair of demands, say 3/10 and 8/10, over which the opponent j randomizes. Moreover, fix an associated probability of stubbornness for each of these demands.[31] Figure 2 shows the 3D-payoff profile of a rational and stubborn player i as a function of his own demand α and $\pi(\alpha)$. By payoff, I mean the lottery over equilibrium payoffs in the concession game when demands and the associated probabilities of stubbornness are drawn according to this distribution. Therefore, Figure 2 shows the equilibrium payoff of a rational (stubborn) player i when the opponent j mixes over 3/10 and 8/10, and I take α_i and $\pi(\alpha_i)$ as given (not necessarily optimal). Figure 3 shows a vertical and horizontal cross-section of Figure 2. Consider an equilibrium with $z = 1/100$, $\alpha_1 = 3/10$, and $\alpha_2 = 8/10$. The right panel of Figure 3 shows the indifference correspondences of a

tinuous at every point where the player's demand becomes incompatible with a demand that the opponent makes with positive probability. This is in short because if a player's own demand becomes incompatible with a demand that the opponent makes with positive probability, the probability of facing a stubborn type making an incompatible demand has increased. This then implies that the difference in payoff to the two types, as a function of a player's own demand is neither monotonous nor single-peaked.

[30]Given the solution concept of PBE, it is straightforward to deter deviations to other demands. One can simply assign probability 1 (or a "high" probability) to any deviation coming from the rational type. Revealing rationality when facing a potentially stubborn opponent leads to immediate concession. Hence, given such a belief, players have no incentive to deviate to other demands.

[31]In particular, for the illustration I use the equilibrium probabilities of stubbornness conditional on the demands 3/10 and 8/10 for $z = 1/100$.

rational and stubborn type in this equilibrium (rational type in red, stubborn type in black); in other words, this is a (horizontal) cross-section of Figure 2. The left panel of Figure 3 shows a (vertical) cross-section of the 3D-payoff profile of player i as a function of α_i and π_i. In particular, I take the cross-section through $(3/10, \pi(3/10))$ and $(8/10, \pi(8/10))$, where $\pi(3/10)$ and $\pi(8/10)$ are the equilibrium probabilities of stubbornness.

Given the characterization of pooling equilibria with two demands in Proposition 3, it is straightforward to see the payoff multiplicity for the rational type. In the limit, the equilibrium payoff for the rational type, v_r, is given by:

$$\lim_{z\to 0} v_r = \frac{2(a_1+a_2-1)}{2a_2-1}(1-a_1) + \frac{1-2a_1}{2a_2-1}(1-a_2) = \frac{1}{2} - \frac{\left(\frac{1}{2}-a_1\right)^2}{a_2-\frac{1}{2}}, \tag{9}$$

where recall that a_1 and a_2 are given by Proposition 3. The level of inefficiency is measured by the distance between 1/2 and the lower demand α_1 and between α_2 and 1, as (9) shows. It is clear that when α_1 is close to 0 (and hence, α_2 close to 1), a rational player's expected equilibrium payoff is close to 0. A demand α_2 close to 1 implies that a player almost certainly will face a demand of α_2 which induces a long war of attrition. If, on the other hand, α_1 is close to 1/2, a rational player's expected payoff is close to 1/2 (when players are equally patient). When demands are close to 1/2, the war of attrition is short and hence, the inefficiency is minimal. By adjusting α_1 and α_2, one can generate in this fashion any payoff between 0 and 1/2. Theorem 1 formalizes this insight. Note that when fixing α_1, a higher α_2 *increases* the limit equilibrium payoff. This may sound surprising at first, given that the symmetric equilibrium with the highest payoff is the one where both types demand 1/2 with probability 1. Conditional on facing a demand of α_2, a rational type receives $1-\alpha_2 < 1/2$ from demanding α_2. Hence, conditional on facing a demand of α_2, the rational payoff is higher the lower α_2 is. However, there is another effect that dominates: the probability that the rational type demands α_2 is decreasing in α_2 as can be seen from equation (9).

Theorem 1. *Fix any $v \in [0, 1/2]$. Then, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, a symmetric equilibrium exists such that the equilibrium payoff for a rational player is v.*

Proof. For any $v \in (0, 1/2]$, this follows immediately from Proposition 3. Fix any equilibrium characterized in Proposition 3. Denote the payoff of a rational player in this equilibrium by

v_r. Then, the limit of this payoff is given by equation 9. Fix any $\epsilon > 0$, and set $a_1 < \epsilon$. Then, $a_2 > 1 - \epsilon$. The result immediately follows. For $v = 0$, note that there exists a pooling equilibrium where both types demand $\alpha = 1$ with probability 1, providing a payoff of 0 to both players. □

In short, with a slightly relaxed strategy restriction on behavioral types relative to AG, a perturbation of the game with behavioral types has no refinement effect at all. This shows the importance of the particular perturbation introduced. I have argued that without an economic application in mind, there is no obvious reason to impose a particular strategy restriction. If an agent has experienced an exogenous shock to their flexibility prior to starting the bargaining process modeling behavioral types as in AG may be reasonable. In contrast, the strategy restriction imposed in this paper is more suited to analyzing situations where the agent is aware whether making a demand will restrict their flexibility (but this is not common knowledge). For instance, a political leader knows whether or not it will be costly to back down from a threat or demand once it has been made (i.e., whether there is an audience cost), but bargaining partners may not be able to observe this about a leader they are negotiating with.

Here, the payoff multiplicity is induced by the delay to agreement. For delay to disappear in the limit in AG, the "right" stubborn type is required to be present. In the symmetric discounting case, this is the type demanding 1/2. Theorem 1 shows that when the stubborn type is given choice over his initial demand, the right stubborn type may not be present. When he is not, delay (and, hence, inefficiency) does not disappear even when the probability of a player being stubborn vanishes. As explained in Section 6, this Folk-theorem-like payoff multiplicity is robust to (i) refinements such as *D1*, and to (ii) changes in the bargaining protocol: in particular, it holds when players move sequentially at the initial stage.

5 Characterization

Given the payoff multiplicity, it is quite surprising that predictions concerning behavior can be made nonetheless. Indeed, I characterize behavior in all symmetric equilibria as the probability that a player is behavioral vanishes. Before I do so, it is helpful to understand what the preferences of the two types imply for the demand configurations that can arise in equilibrium.

5.1 Preliminaries

This subsection establishes several general features of equilibria. Lemma 1 is concerned with the concession behavior at time 0. Lemmas 2 and 3 are concerned with the demand configurations that can arise.

Recall that players face a tradeoff between the amount received if their opponent concedes and the speed with which the opponent concedes. The following lemma is a straightforward consequence of this tradeoff for the rational type (and hence, already holds in AG). It implies that higher demands concede to lower demands at time 0. Moreover, it implies that the lower a player's demand (fixing the opponent's demand), the more likely he is being conceded to at time 0. Let $\underline{\alpha}$ denote the lowest demand in some finite set C.

Lemma 1. *Fix any set of demands C, where C is an arbitrary finite subset of $[0,1]$. In any symmetric equilibrium with support C, strength $\mu(\alpha)$ is decreasing in $\alpha \in C$, strictly so for pooling demands with $\alpha < 1 - \underline{\alpha}$.*

The key intuition for the proof is that the probability of immediate concession, $F^i_{\alpha_i,\alpha_j}(0)$, is increasing in $\mu_j(\alpha_j)$. If $\mu_j(\alpha_j)$ were increasing in α_j, a player would always benefit from increasing his demand α_j. This is inconsistent with a player being indifferent between demands.[32]

The difference between the rational and the stubborn type is the payoff when facing an incompatible demand coming from a potentially stubborn opponent. Suppose that a rational type faces an incompatible demand. In equilibrium, a rational type is willing to wait until he is sure to face a stubborn opponent. However, once he assigns probability 1 to facing a stubborn opponent, the rational type strictly prefers to concede. However, the stubborn type does not have this "option value of concession" – conditional on facing an incompatible demand from a stubborn opponent, a stubborn player receives a payoff of 0. Hence, when demands are incompatible, the expected payoff to a rational payoff is strictly higher than the payoff to a stubborn type. Therefore, unless every demand made with positive probability is compatible with every other demand, the equilibrium payoff for a rational player must

[32]In Appendix, I prove a slightly stronger statement - specifically, I do not restrict C to be finite.

be strictly higher than the payoff for a stubborn player. Suppose that every demand is compatible with every other demand made with positive probability, i.e., all demands have to lie below 1/2. This cannot be an equilibrium since a rational player would then strictly prefer to deviate to a demand above 1/2. Hence, the payoff a rational player receives is strictly higher than the payoff a stubborn player receives (unless C is a singleton set).

Lemma 2 imposes restrictions on the value of equilibrium demands, arising from the stubborn type's preferences over demands.

Lemma 2. *Fix any set of demands C, where C is an arbitrary finite subset of $[0, 1]$. In any symmetric equilibrium with support C, the following holds:*

1. *The lowest pooling demand in C is incompatible with the highest pooling demand in C.*
2. *Consider any two demands, α and α', that the stubborn type makes with positive probability: $\alpha, \alpha' \in \operatorname{supp} s$ with $\alpha < \alpha'$. Then there exists a demand $\alpha'' \in C$ that is compatible with α but not with α', i.e., $\alpha + \alpha'' \leq 1 < \alpha' + \alpha''$.*

While a proof can be found in Appendix, a heuristic argument for the first part of Lemma 2 is as follows. [33] Suppose that the lowest pooling demand were compatible with the highest pooling demand. Then, the payoff from making the lowest demand would be the same for a rational and a stubborn player. However, if it is the same for the lowest demand, it must be the same for every other demand with positive probability. By the above argument, this cannot be.

The key heuristic argument for the second part of Lemma 2 is as follows. Fix a set of demands C. Suppose that player j makes all demands in C with positive probability, and suppose further that the rational type of player i is indifferent over all demands in C.[34] If the stubborn type of player i is indifferent, then the difference in expected payoff between a stubborn and a rational type must be identical for each demand i makes.

Conditional on facing a compatible demand from player j, the stubborn type does not pay a cost for being stubborn. Conditional on facing a certain incompatible demand, call it

[33] In Appendix, I prove a slightly stronger statement - specifically, I do not restrict C to be finite.

[34] The argument easily extends to the case where C contains separating demands.

α_j, the stubborn type's cost for being stubborn is higher the lower his own demand is. To see this, suppose that the opponent, say player j, demands α_K. Let me compare player i demanding α_1 versus demanding α_2 (although this generalizes to any other two consecutive demands when facing any other higher demand).

In equilibrium, player i is willing to wait until time $\bar{T}_i$ to concede (i.e., the time at which player j is known to be stubborn), where of course $\bar{T}_1 < \bar{T}_2$. Recall that (1) the probability of immediate concession by the opponent is decreasing in the demand (i.e., α_K is more likely to concede immediately to α_1 than to α_2); (2) the rate of concession by α_K is higher for α_1 than for α_2; and (3) conditional on j not conceding at $t = 0$, the expected payoff for the rational type of player i at any point before $\bar{T}_i$ is $1 - \alpha_K$ regardless of whether i demands α_1 or α_2. Now suppose that player i does wait until $\bar{T}_i$ to concede (despite j expecting him to concede at the appropriate rate). Then, the probability with which player j concedes (ever) is $1 - \mu_K^{1-\alpha_K}$ – i.e., regardless of the demand i makes. Hence, conditional on j not conceding immediately, the probability with which j concedes (ever) is higher when player i demanded α_2 than when player i demanded α_1. The stubborn type only pays a cost for being stubborn conditional on his opponent not conceding immediately. Hence, conditional on facing a certain demand, the stubborn type's cost for being stubborn is lower the higher the demand. Therefore, for the expected cost to the stubborn type to be identical for α_1 and α_2, the opponent must make a demand with positive probability, which imposes a cost of being stubborn on the player when demanding α_2 but not when demanding α_1. In other words, there must exist a demand that is compatible with α_1 but not with α_2.

Therefore, for the stubborn type to be indifferent over any two demands in C, the set (in the sense of set inclusion) of compatible equilibrium demands is strictly decreasing in the demand the stubborn type makes with positive probability.

The following lemma imposes structure on the ordering of pooling and separating demands (in terms of their values). It is driven by the difference in preferences over demands by the two types.

Lemma 3. *For any $z > 0$, in any equilibrium,*

1. *any separating demand by the stubborn type (if it exists) is smaller than the lowest demand assigned positive probability by the rational type,*

2. *the highest demand made with positive probability by the stubborn type is smaller than the lowest separating demand by the rational type (if it exists),*

3. *there exists at most one separating demand by the stubborn type.*

The key intuition for the first two parts of Lemma 3 are that the stubborn type is more concerned with facing an incompatible demand than the rational type (who can always concede). As a result, the stubborn type's separating demands are the lowest demands made and the rational type's separating demands are the highest demands made in equilibrium. The key intuition for the last part of the Lemma is that if the stubborn type separates, he is conceded to immediately by any rational opponent (regardless of the value of the stubborn type's demand). As a result, the stubborn type cannot be indifferent over multiple separating demands.

5.2 Limiting Behavior and Existence

The following proposition characterizes the limiting behavior of the rational type in any equilibrium. It states that there are two possible demand configurations for the rational type in equilibrium.

Proposition 4. *Let (z^n, r^n, s^n) be a convergent sequence of equilibrium triples with $\lim_{n\to\infty} z^n = 0$. Then* **either** *there exists some $a \le \frac{1}{2}$ such that*

$$\lim_{n\to\infty} r^n = \mathbb{1}_{\{1-a\}} \tag{10}$$

or *there exist $a_1 < \frac{1}{2}$ and $a_i > 1 - a_1$ for $i = 2, \ldots K$ such that*

$$\lim_{n\to\infty} r^n = \mathbb{1}_{\{a_1\}} 2\left(a_1 - \sum_{i\neq 1} r_i(1-\alpha_i)\right) + \sum_{i\neq 1} \mathbb{1}_{\{a_i\}} r_i. \tag{11}$$

Proof. See Online Appendix. ∎

Proposition 5 establishes existence. It states that both demand configurations for the rational type specified in Proposition 4 exist and further specifies whether such sequences of equilibria are pooling, separating or semi-separating.

Proposition 5. *(a) Fix a sequence* $z^n \to 0$ *and fix* $a \leq \frac{1}{2}$*. Then there exists* N *such that for* $n > N$*, a corresponding convergent sequence of pooling equilibria satisfying* $\lim_{n\to\infty} r^n = \mathbb{1}_{\{1-a\}}$ *exists. The same statement holds when pooling is replaced by separating or semi-separating.*

(b) Fix a sequence $z^n \to 0$*,* $K = 2$*,* $a_1 < \frac{1}{2}$ *and* $a_2 > 1 - a_1$*. Then there exists* N *such that for* $n > N$*, a corresponding convergent sequence of pooling equilibria satisfying*

$$\lim_{n\to\infty} r^n = \mathbb{1}_{\{a_1\}} \frac{2(a_1 + a_2 - 1)}{2a_2 - 1} + \mathbb{1}_{\{a_2\}} \frac{1 - 2a_1}{2a_2 - 1}$$

exists. The same statement holds when pooling is replaced by semi-separating.

(c) Fix a sequence $z^n \to 0$*,* $K > 2$*,* $a_1 < \frac{1}{2}$ *and* $a_i > 1 - a_1$ *for* $i = 2, \ldots, K$*. Then there exists* N *such that for* $n > N$*, a corresponding convergent sequence of semi-separating equilibria with*

$$\lim_{n\to\infty} s^n = \mathbb{1}_{\{a_1\}}$$

and

$$\lim_{n\to\infty} r^n = \mathbb{1}_{\{a_1\}} 2 \left(a_1 - \sum_{i\neq 1} r_i (1 - \alpha_i) \right) + \sum_{i \neq 1} \mathbb{1}_{\{a_i\}} r_i$$

exists.

Proof. See Online Appendix. □

The proofs of Proposition 4 and 5 build on Lemmas 1 to 3. More specifically, Lemma 1 helps establish convergence. The proofs of existence build on Lemmas 2 and 3.

It follows from Propositions 4 and 5, that types can pool, separate or semi-separate. If types pool in the limit, they pool on either one or two demands. If types separate in the limit, the rational type makes one demand and it can be shown that the stubborn type makes either one or two demands. If only the rational type separates in the limit, then there is one pooling demand and the rational type may separate over arbitrarily many demands. [35]

For an intuition for why types can pool over multiple demands, the reader is referred back to Section 4. The intuition for why types can separate in equilibrium is straightforward. When facing a compatible demand, the payoff to the two types is the same. Instead when facing an incompatible demand, the rational type can concede and the stubborn type cannot. This means that when the stubborn type faces another stubborn type, his payoff is zero. Hence, there is a sense in which the stubborn type has a stronger incentive to make a demand that is "just compatible" with the opponent's demand. The rational type on the other hand never has a strict incentive to make a demand that is compatible with the opponent's demand - he can always get at least as much from conceding. This implies that types may separate by type in equilibrium. When they do, the stubborn type makes lower demands than the rational type given any positive ex ante probability of stubbornness.

The intuition for why the rational type may separate over arbitrarily many demands is as follows. I show that when the rational type separates over multiple demands, even the lowest separating demand is incompatible with any other demand made with positive probability by either type. Following the logic of Myerson (1991), when facing any incompatible demand (that is not a separating demand by the rational type), the rational type has to concede immediately. As a result, the precise value of his demand is irrelevant. When facing an incompatible demand that is a separating demand by the rational type, both players have revealed themselves as rational. In this case, there is no updating of beliefs on path. Thus, the war of attrition can last infinitely long: players simply keep conceding at a rate that keeps them indifferent between waiting for an instant and conceding now. This gives the rational type a payoff equivalent to the complement of his opponent's demand. Thus, even though his rate of concession depends on his own demand, his expected payoff is independent of it.

[35] Except for case (c), fixing a demand configuration for the rational type in the limit, the behavior of the stubborn type is not uniquely pinned down. Precise statements regarding the behavior of the stubborn type are omitted for ease of exposition, but a characterization is available on request. Given that the proofs are constructive, further details are provided in the Online Appendix.

6 Robustness

In this section, I discuss the robustness of the results to modeling choices. First, I briefly discuss asymmetric equilibria. Then, I cover sequential move bargaining and briefly discuss other asymmetries in model parameters. Finally, I return to the question of off-equilibrium-path beliefs when I discuss refinements.

6.1 Other equilibria

Throughout, I have focused on symmetric equilibria. It is clear that asymmetric equilibria exist. For instance, there exists an equilibrium with player i demanding α and player j demanding $1-\alpha$ for any $\alpha \in [0,1]$.[36] It remains an open question whether any feasible payoff can be While the differences in the two types of preferences over demands also impose some structure on the demand configurations that can arise in asymmetric equilibria, it remains an open question whether the strong predictions in terms of the size of the equilibrium support are robust to considering asymmetric equilibria.

6.2 Sequential move bargaining

Suppose that players make demands sequentially, rather than simultaneously; i.e., first, player 1 makes demand α_1, then player 2 makes demand α_2. If $\alpha_1 + \alpha_2 > 1$, a concession game starts as before. In this case, the symmetric pooling equilibria in pure strategies (i.e., one-demand equilibria) in the simultaneous move game remain equilibria in the sequential move game.[37] However, not surprisingly, the symmetric pooling equilibria in mixed strategies (in the simultaneous move game) are not robust to this change in the bargaining protocol. To see this, consider a simple example with two demands. In particular, suppose that

[36]Note that this implies that any division of surplus can be achieved in equilibrium. Hence, payoff multiplicity exists not only along the 45-degree line, but also along the diagonal. This implies that with a public randomization device, any feasible payoff, i.e., any payoff pair (v_1, v_2) such that $v_1 + v_2 \leq 1$ with $v_1 \geq 0$, $v_2 \geq 0$, can be achieved in equilibrium. It remains an open question whether this is the case when no randomization device is assumed.

[37]In fact, there exists a symmetric pooling equilibrium in pure strategies if and only if $\alpha = \{1/2, 1\}$ (as in the simultaneous move game). Note that with a public randomization device, we can already generate any expected payoff pair along the 45-degree line by using the public randomization device to determine which pure strategy equilibrium will be played.

player 1 randomizes over demands 1/3 and 3/4. If player 1 demands 1/3, then player 2 is strictly better off by demanding at least 2/3: if demanding 1/3, player 2 would receive 1/2; if demanding 2/3, player 2 would receive 2/3. Hence, the two players cannot be made indifferent over the two demands.

Of course, restricting attention to symmetric equilibria is somewhat unnatural when moves are sequential. If we allow for asymmetric equilibria, Folk-theorem-like payoff multiplicity arises (as in the simultaneous move game). In particular, fix any $\alpha_1 \in [0,1]$. Then, there exists an equilibrium where player 1 demands α_1 and player 2 demands $\alpha_2 = 1 - \alpha_1$ in the sequential move game. Player i's equilibrium payoff in such an equilibrium is simply given by α_i. Hence, there is a Folk-theorem-like payoff multiplicity in the sequential move game. A heuristic argument for the existence of such equilibria is as follows. Provided that player 1 places sufficiently high probability on player 2 being rational conditional on seeing an out-of-equilibrium demand, the rational type receives no more than $1 - \alpha_1$ by demanding more than $1 - \alpha_1$. The stubborn type will receive a payoff strictly less than $1 - \alpha_1$ if he demands more than $1 - \alpha_1$. Moreover, regardless of player 1's belief, player 2 does not want to make a demand less than $1 - \alpha_1$. Hence, player 2 has no incentive to deviate to an out-of-equilibrium demand. By an analogous argument, player 1 has no incentive to deviate.

Note that any such asymmetric pooling equilibrium in pure strategies satisfies refinements such as *D1*. In short, (i) regardless of the belief of player j, a rational player i is willing to make a demand higher than $\alpha_i = 1 - \alpha_j$; (ii) regardless of player j's belief, neither type of player i would be willing to make a demand less than $\alpha_i = 1 - \alpha_j$. Hence, there exists no belief of player j (and associated best response) that makes the stubborn type of player i willing to deviate from his equilibrium demand while the rational type of player i is not.

6.3 Type space

My model replaces the types in AG with behavioral types who can choose their initial demand. It is not difficult to instead *add* my types to the model of AG. Note that doing so is equivalent to taking the model of AG and restricting a small fraction of rational types to choose only their initial demand (as the stubborn types in this paper). With these perturbations, it can be shown that there is payoff multiplicity in equilibrium. For instance, suppose there is a single behavioral type à la AG who always demands $\alpha_2 > \frac{1}{2}$ and never accepts anything less. Each player is rational with probability $1-z_{AG}-z_S$, behavioral à la AG with probability z_{AG} and stubborn with probability z_S, for some small z_{AG}, z_S. Then there exists $\alpha > 1-\alpha_2$ such that for every $\alpha_1 \geq \alpha$, there exists an equilibrium where the rational and stubborn types randomize over α_1 and α_2. It can be shown that if z_{AG} and z_S vanish at the appropriate rates, the rational type's demand strategy converges to the probabilities stated in Proposition 3. This multiplicity is somewhat surprising because usually restricting players' strategies increases the predictive power of a model. Here, however, the set of outcomes achievable in equilibrium increases as we restrict the permissible strategies of a small fraction of rational types. The intuition for this is as follows: restricting the permissible strategies of a small fraction of rational types enlarges the type space that a rational player (whose permissible strategies were not restricted) can imitate. In other words, a richer set of types increases the set of strategies a rational player can use without revealing himself.

6.4 Model parameters

Allowing players to differ in (i) their ex ante probability of stubbornness or (ii) their patience does not affect the support of the demands and the probabilities of these demands. It does, however, affect players' payoffs. Everything else being equal, an increase in the ex ante probability of stubbornness of a player or similarly in his patience increases the player's payoff. This is analogous to the reasoning and the results in AG.

6.5 Refinement - Divinity

Recall that thus far, I have simply assigned probability 1 to any deviation coming from a rational type, and this deterred deviations.

The refinement *D1* is not defined for dynamic games beyond signaling games. However, first, note that, given the realized demands and associated beliefs, I can compute the expected payoff from the continuation game. Hence, I can associate to my game a corresponding game that ends once demands are chosen. This is the game to which I apply *D1*.

Loosely speaking, *D1* attaches probability 1 to the type with the strongest incentive to deviate to a given demand. More formally, denote the set of types by $\Theta = \{R, S\}$, where R stands for rational and S for stubborn. Let $u_i^*(\theta)$ be the equilibrium payoff of type

$\theta \in \{R, S\}$. Define $D(\theta, S, d)$ to be the set of mixed-strategy best responses (MBRs) F_2 to demand d and beliefs concentrated on S that make type θ strictly prefer d to his equilibrium strategy,

$$D(\theta, S, d) = \cup_{\mu:\mu(S|d)=1}\{F_2 \in MBR(\mu, d) \text{ s.t. } u_1^*(\theta) < u_1(d, F_1, \theta)\},$$

and let $D^0(\theta, S, d)$ be the set of mixed best responses that make type θ exactly indifferent. A type θ is deleted for demand d under criterion *D1* if there is a θ' such that

$$\{D(\theta, \Theta, d) \cup D^0(\theta, \Theta, d)\} \subset D(\theta', \Theta, d).$$

In other words, if the set of best responses (and associated beliefs about a player being stubborn conditional on d) for which a rational player benefits from deviating to d is strictly smaller than the set of best responses for which a stubborn player benefits from deviating to d, then *D1* assigns probability 0 to the deviation coming from a rational player.

Proposition 6. *Fix any $v \in [0, 1/2]$. Then there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, a symmetric equilibrium satisfying D1 exists such that the equilibrium payoff for a rational player is v.*

Proof. See Online Appendix. □

In short, the payoff multiplicity generated when perturbing the game with behavioral types who can choose their initial demand is robust to *D1*.

7 Conclusion

Models on reputational bargaining have introduced a perturbation with simple behavioral types as a way of refining payoff predictions for the rational type. I have shown that if we slightly relax the strategy restriction on behavioral types, perturbing the bargaining game with behavioral types does not refine the set of outcomes. One can interpret this result as saying that the nature of the perturbation must be analyzed carefully to derive some economically justifiable restrictions. The perturbation introduced here is particularly suited to situations where agents themselves choose what posture to portray, knowing full well it may be costly to back down from a posture once taken.

While this paper focuses on endogenizing behavioral types in a bargaining setting, the idea of endogenizing behavioral types applies more broadly. For instance, some agents may restrict attention to stationary strategies in a repeated game. Whatever drives their preference for this restriction does not mean that they do not choose optimally within the set of stationary strategies. There is a middle ground between rational and behavioral agents, and this paper is a first attempt to explore this territory in a well-known and tractable environment.

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Appendix

Proofs of Section 4

I prove a slightly stronger statement (Proposition 3.1, stated below), which apart from existence of pooling equilibria with two demands (Proposition 2), also states that any convergent sequence of such equilibria must converge to the limits stated. The proof relies on Lemmas 1 and 2, proved next and independently.

Proposition 3.1

(a) Let (z^n, r^n, s^n) be a convergent sequence of pooling equilibria with $|\operatorname{supp} r^n \cup \operatorname{supp} s^n| = 2$ and $\lim_{n\to\infty} z^n = 0$. Then there exist $a_1 \in (0, 1/2]$ and $a_2 \in (1 - a_1, 1]$ such that

$$\lim_{n\to\infty} \alpha_1^n = a_1, \quad \lim_{n\to\infty} \alpha_2^n = a_2. \tag{12}$$

Moreover, along any such sequence,

$$\lim_{n\to\infty} \begin{pmatrix} r_1^n \\ r_2^n \end{pmatrix} = \begin{cases} \frac{2(a_1+a_2-1)}{2a_2-1}, \\ \frac{1-2a_1}{2a_2-1}, \end{cases} \quad \text{and} \quad \lim_{n\to\infty} \begin{pmatrix} s_1^n \\ s_2^n \end{pmatrix} = \begin{cases} \frac{1-a_2}{2-a_1-a_2}, \\ \frac{1-a_1}{2-a_1-a_2}. \end{cases} \tag{13}$$

(b) For any $a_1 \in (0, 1/2]$ and $a_2 \in (1 - a_1, 1]$, there exists a sequence $z^n \to 0$ and a corresponding convergent sequence of pooling equilibria $(\alpha_1^n, \alpha_2^n, r^n, s^n)$ satisfying (12) and (13).

Proof of Proposition 3.1. The proof has the following steps. First, a pooling equilibrium only exists if (18)–(21) has a solution (Claim 2). Second, in any sequence of equilibria, $\mu_i \to 0$ for $i = 1, 2$ (Claim 3). Third, an equilibrium with support $\{\alpha_1, \alpha_2\}$ exists in the limit (Claim 4). Finally, I show that the system (18)–(21) can be solved locally around $z = 0$, with $q_i \in (0, 1)$, and $\mu_i \in (0, 1)$ for $i = 1, 2$ (Claim 5).

Claim 2. *There exists a pooling equilibrium with support $\{\alpha_1, \alpha_2\}$ only if the offers α_1 and α_2 along with probabilities q_1 and q_2, and positive numbers μ_1 and μ_2 solve (18)–(21).*

Proof. Fix $z > 0$, and an equilibrium, specifying $\{\alpha_1, \alpha_2\}$, $\mu_1, \mu_2 > 0$, and $q_1, q_2 > 0$. For $k = 1, 2$, define

$$
\begin{aligned}
v_k^r = & \sum_{\substack{i \text{ s.t.} \\ \alpha_i \leq 1-\alpha_k}} q_i \left(\frac{\alpha_k + 1 - \alpha_i}{2} \right) \\
& + \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i \left(\alpha_k \min \left\{ 0, 1 - \left(\frac{\mu_i}{\mu_k} \right)^{1-\alpha_i} \right\} + (1 - \alpha_i) \min \left\{ 1, \left(\frac{\mu_i}{\mu_k} \right)^{1-\alpha_i} \right\} \right),
\end{aligned} \tag{14}
$$

$$
v_k^s = v_k^r - \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1-\alpha_k}} q_i (1 - \alpha_i) \max \left\{ \mu_i^{\alpha_k}, \left(\frac{\mu_i}{\mu_k} \right)^{1-\alpha_i} \mu_k^{\alpha_k} \right\}. \tag{15}
$$

For a detailed derivation of these payoffs see the supplementary material on my website. For any $k, k' \leq K$, define

$$
\Delta_{k,k'}^r = v_k^r - v_{k'}^r, \tag{16}
$$

$$
\Delta_{k,k'}^s = v_k^s - v_{k'}^s. \tag{17}
$$

Given z and $\{\alpha_1, \ldots, \alpha_K\}$, define the following system in (q_i, μ_i), $i = 1, \ldots, K$:

$$
\Delta_{1,2}^r = 0, \tag{18}
$$

$$
\Delta_{1,2}^r - \Delta_{1,2}^s = 0, \ \forall k < K \tag{19}
$$

$$
\sum_{i=1}^{2} q_i \mu_i^{1-\alpha_i} = z, \text{ and} \tag{20}
$$

$$
\sum_{i=1}^{2} q_i = 1. \tag{21}
$$

Note that there are 4 equations (and as many variables). For a candidate equilibrium with support $\{\alpha_1, \alpha_2\}$, both types need to be indifferent over demands α_1 and α_2, with probabilities $q_i > 0$, given an ex ante probability of a player being stubborn, z. Equation (18) shows the difference in payoff for a rational type between making a demand of α_1 and making a demand of α_2, conditional on the opponent mixing over the offers α_1 and α_K. Hence, equation (19) ensures indifference of the rational type between any two offers, α_1 and α_2. In the same manner, equation (19) ensures indifference of the stubborn type between the two offers, simplified using the indifference of the rational type. Equation (21) ensures that the probabilities of being faced with a given offer add up to 1; and equation (20) ensures that the conditional probabilities of stubbornness, $\mu_i^{1-\alpha_i}$, are consistent with the ex ante probability of a player being stubborn, z.

Fix 2 demands (satisfying Lemmas 1 and 2). Suppose that for all $\bar{z} > 0$, there exists $z < \bar{z}$, such that there exist $q_i > 0$, and $\mu_i > 0$ for $i = 1, 2$ such that (z, α, q, μ) satisfies (18) to (21). Then there exists a sequence $(z^n, \alpha^n, q^n, \mu^n)_{n\in\mathbb{N}}$, with $\lim_{n\to\infty} z^n \to 0$, solving (18)–(21), such that it is *not* the case that $\alpha_1^n - \alpha_2^n \to 0$. Recall, that $\alpha^n, q^n, \mu^n \in [0, 1]$. Hence, without loss, assume that α^n, q^n and μ^n converge. By continuity, $(z = 0, \lim_{z\to0} \alpha, \lim_{z\to0} q, \lim_{z\to0} \mu)$ also solves (18)–(21). In the following, I drop the subscript n; limits are indicated explicitly by $\lim_{z\to0}$ throughout.

In other words, if the system has a solution for small enough z, then $\alpha_1 \neq \alpha_2$. □

Note that by Lemmas 1 and 2, I can write (18) and (19) as:

$$q_1\left(\alpha_1 - \frac{1}{2}\right) + q_2\left(\alpha_1 + \alpha_2 - 1\right)\left(1 - \left(\frac{\mu_2}{\mu_1}\right)^{1-\alpha_2}\right) = 0, \text{ and} \tag{22}$$

$$q_1\left(1 - \alpha_1\right)\mu_1^{\alpha_2} - q_2\left(1 - \alpha_2\right)\mu_2^{1-\alpha_2}\left(\mu_1^{\alpha_1+\alpha_2-1} - \mu_2^{2\alpha_2-1}\right) = 0. \tag{23}$$

Claim 3. *For (18)–(21) to be satisfied,* $\lim_{z\to0} \mu_i = 0$ *for* $i = 1, 2$.

Proof. By (20) and (21), either $\lim_{z\to0} q_i = 0$ or $\lim_{z\to0} \mu_i = 0$ for $i = 1, 2$. Moreover, if $\lim_{z\to0} q_i = 0$, then $\lim_{z\to0} \mu_j = 0$. Recall that by Lemma 1, $\mu_2 < \mu_1$, $\forall z > 0$. Hence, by (20), it follows that $\lim_{z\to0} \mu_2 = 0$. If $\lim_{z\to0} \mu_2 = 0$, then (18) can only be satisfied if $\lim_{z\to0} \mu_1 = 0$: if $\lim_{z\to0} q_1 = 0$, then it must be that $\lim_{z\to0} l_{2,1} = 1$, and hence, $\lim_{z\to0} \mu_1 = 0$. Therefore, $\lim_{z\to0} \mu_i = 0$ for $i = 1, 2$. □

NB. Recall that by Lemma 1 , in order for (18) to be satisfied it must be that $\mu_{k+1} \leq \mu_k$, $\forall k$, $\forall z > 0$. Hence, all ratios $\frac{\mu_i}{\mu_k}$ and $\frac{\mu_i}{\mu_{k+1}}$ in (18) and (19) are bounded above by 1. Hence, without loss, assume that these ratios converge. Call the ratios $l_{i,k}$ and $l_{i,k+1}$.

Claim 4. *The system (18)–(21) has a solution in the limit, with*

$$\lim_{z\to0} r_1 = \lim_{z\to0} q_1 = \frac{2\left(\alpha_1 + \alpha_2 - 1\right)}{2\alpha_2 - 1}, \text{ and} \tag{24}$$

$$\lim_{z\to0} s_1 = \frac{1 - \alpha_2}{2 - \alpha_1 - \alpha_2}. \tag{25}$$

Proof. I first reduce the system (20)–(23) to two equations. Then I use Taylor approximations to derive (24) and (25). Using (21), I can replace q_2 by $1-q_1$ in (22). I can then solve (22) for q_1 as a function of μ_1 and μ_2 only:

$$q_1 = \frac{2\left(\alpha_1+\alpha_2-1\right)\left(1-l_{2,1}^{1+\alpha_2}\right)}{\left(2\alpha_2-1\right)-2\left(\alpha_1+\alpha_2-1\right)l_{2,1}^{1+\alpha_2}}. \tag{26}$$

I can then replace q_2 and q_1 (using (26)) in (23) and (20). I can write the stubborn type's indifference, (23), as:

$$\frac{\left(1-2\alpha_1\right)\left(1-\alpha_2\right)\left(\mu_2^{1-\alpha_2}\mu_1^{\alpha_1-1}-l_{2,1}^{\alpha_2}\right)+2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_2-1\right)\left(l_{2,1}^{1-\alpha_2}-1\right)}{\mu_1^{-\alpha_2}\left(2\left(\alpha_1+\alpha_2-1\right)l_{2,1}^{1-\alpha_2}-\left(2\alpha_2-1\right)\right)}=0. \tag{27}$$

I can then show that

$$\lim_{z\to 0}\frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}}=\frac{2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_2-1\right)}{\left(1-2\alpha_1\right)\left(1-\alpha_2\right)}. \tag{28}$$

More precisely,

$$\mu_1=\left(\frac{\left(1-2\alpha_1\right)\left(1-\alpha_2\right)}{2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_2-1\right)}\right)^{\frac{1}{1-\alpha_1}}\mu_2^{\frac{1-\alpha_2}{1-\alpha_1}}+\mathcal{O}\left(\mu_2^{\frac{1-\alpha_2}{1-\alpha_1}\left(1+\alpha_2-\alpha_1\right)}\right). \tag{29}$$

To derive (28) and (29), note that for (27) to be satisfied either

$$\lim_{z\to 0}l_{2,1}=K,\quad \text{or}\quad \lim_{z\to 0}\frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}}=K,$$

where K is some positive constant. If $\lim_{z\to 0}l_{2,1}=K$, then $\lim_{z\to 0}\frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}}\to\infty$, and hence, (27) cannot be satisfied. If $\lim_{z\to 0}\frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}}=K$, then $\lim_{z\to 0}l_{2,1}=0$. Hence, we can solve (27) for K:

$$K=\frac{2\left(1-\alpha_1\right)\left(\alpha_1+\alpha_2-1\right)}{\left(1-2\alpha_1\right)\left(1-\alpha_2\right)}, \tag{30}$$

and (28) follows. Using Taylor approximation, I can then derive (29). Using (29), I can rewrite (20) and (26) as

$$q_1=\frac{2\left(\alpha_1+\alpha_2-1\right)}{2\alpha_2-1}-k_1\mu_2^{\frac{\left(1+\alpha_2\right)\left(1-\alpha_1\right)-\left(1-\alpha_2\right)^2}{1-\alpha_1}}+\mathcal{O}\left(\mu_2^{\frac{2\left(2\alpha_2-\alpha_1-\alpha_2^2\right)}{1-\alpha_1}}\right), \tag{31}$$

$$z=\frac{\left(1-2\alpha_1\right)\left(2-\alpha_1-\alpha_2\right)}{\left(1-\alpha_1\right)\left(2\alpha_2-1\right)}\mu_2^{1-\alpha_2}+\mathcal{O}\left(\mu_2^{\frac{1-2\alpha_1+\alpha_2\left(2-\alpha_2\right)}{1-\alpha_1}}\right), \tag{32}$$

where

$$k_1 = \left(\frac{2(\alpha_1+\alpha_2-1)}{2\alpha_2-1}\right)^2 \left(\frac{1-2\alpha_1}{2(\alpha_1+\alpha_2-1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} \left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{\frac{1-\alpha_2}{1-\alpha_1}}.$$

To derive (31), note that I can write $l_{2,1}^{1-\alpha_2}$ as

$$l_{2,1}^{1-\alpha_2} = \left(\frac{(1-2\alpha_1)(1-\alpha_2)}{2(1-\alpha_1)(\alpha_1+\alpha_2-1)}\right)^{-\frac{1-\alpha_2}{1-\alpha_1}} \mu_2^{\frac{(1+\alpha_2)(1-\alpha_1)-(1-\alpha_2)^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{\frac{2\left(2\alpha_2-\alpha_1-\alpha_2^2\right)}{1-\alpha_1}}\right).$$

Using (32), and recalling that $s_1 = \frac{\mu_1^{1-\alpha_1} q_1}{z}$, I can now write s_1 as a function of μ_2 only:

$$s_1 = \frac{1-\alpha_2}{2-\alpha_1-\alpha_2} - k_2 \mu_2^{\frac{(1+\alpha_2)(1-\alpha_1)-(1-\alpha_2)^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{\frac{1-2\alpha_1+2\alpha_2-\alpha_2^2}{1-\alpha_1}-1+\alpha_2}\right), \tag{33}$$

where

$$k_2 = \left(\frac{(1-2\alpha_1)(1-\alpha_2)}{2(\alpha_1+\alpha_2-1)(1-\alpha_1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} \left(\frac{2(\alpha_1+\alpha_2-1)(1-\alpha_1)}{(2\alpha_2-1)(2-\alpha_1-\alpha_2)}\right).$$

Hence,

$$\lim_{z\to 0} r_1 = \frac{2(\alpha_1+\alpha_2-1)}{2\alpha_2-1}, \text{ and} \tag{34}$$

$$\lim_{z\to 0} s_1 = \frac{1-\alpha_2}{2-\alpha_1-\alpha_2}. \tag{35}$$

□

Claim 5. *The system (18)–(21) can be solved locally around $z=0$, with $s_1 \in (0,1)$, $r_1 \in (0,1)$.*

Proof. As before, I replace q_2 by $1-q_1$ in equations (20), (22) and (23) (using (21)). In analogue to before, I then solve (22) for q_1 as a function of μ_1 and μ_2 only:

$$q_1 = \frac{2(\alpha_1+\alpha_2-1)\left(1-\left(\frac{\mu_2}{\mu_1}\right)^{1+\alpha_2}\right)}{(2\alpha_2-1)-2(\alpha_1+\alpha_2-1)\left(\frac{\mu_2}{\mu_1}\right)^{1+\alpha_2}}. \tag{36}$$

Using this, I can then use (20) to solve for μ_2 as a function of z and μ_1:

$$\mu_2 = \mu_1 \left(\frac{2(\alpha_1+\alpha_2-1)\mu_1^{1-\alpha_1} - (2\alpha_2-1)z +}{2(\alpha_1+\alpha_2-1)\left(\mu_1^{1-\alpha_1}-z\right)-(1-2\alpha_1)\mu_1^{1-\alpha_2}}\right)^{\frac{1}{1-\alpha_2}}. \tag{37}$$

Hence, I can express (23) as a function of μ_1 and z only. Let me introduce two auxiliary variables, p and u, where

$$p = z^{\frac{\alpha_1 - \alpha_2(1-\alpha_1) + 2\alpha_2^2}{(1-\alpha_1)(1-\alpha_2)}}, \text{ and} \tag{38}$$

$$u = \mu_1^{1-\alpha_1} z^{-1} - \frac{(1-\alpha_2)(2\alpha_2 - 1)}{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}. \tag{39}$$

Using the Implicit Function Theorem one can derive:

$$\left.\frac{dp}{du}\right|_{(p,u)=(0,0)} = \frac{(2-\alpha_1-\alpha_2)}{1-\alpha_1}\left(\frac{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}{(1-\alpha_2)(2\alpha_2-1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} > 0. \tag{40}$$

I can rewrite (23) as a function of p and u, using (38) and (39). Denote this new function $\Delta^s_{p,u}$. Taking derivatives w.r.t. p and u, evaluating these derivatives at $p = u = 0$, and rearranging, I get (40), which is clearly finite and positive:

$$\begin{aligned}\left.\frac{dp}{du}\right|_{(p,u)=(0,0)} &= -\left.\frac{\partial \Delta^s_{p,u}/\partial u}{\partial \Delta^s_{p,u}/\partial p}\right|_{(p,u)=(0,0)} \\ &= \frac{(2-\alpha_1-\alpha_2)}{1-\alpha_1}\left(\frac{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}{(1-\alpha_2)(2\alpha_2-1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}}.\end{aligned} \tag{41}$$

Hence, the system (18)–(21) can be solved locally around $z = 0$ when $K = 2$, with $r_1 \in (0,1)$, and $s_1 \in (0,1)$. □

□

Proofs of Section 5.1

For the proofs of Lemma 1 and 2 that follow it is useful to introduce some notation. Define for $i = 1,2$, $\mathcal{W}(\alpha_i) = \{\alpha_j | \mu_i(\alpha_i) \leq \mu_j(\alpha_j), \alpha_i + \alpha_j > 1\}$, and $\mathcal{S}(\alpha_i) = \{\alpha_j | \mu_i(\alpha_i) > \mu_j(\alpha_j), \alpha_i + \alpha_j > 1\}$. In a candidate pooling equilibrium, the payoff to the rational type of player 2 from demanding α_2 is:

$$\begin{aligned} v_2^r(\alpha_2) = &\int_{\underline{\alpha}}^{1-\alpha_2} \frac{1-\alpha_i+\alpha_2}{2} dG(\alpha_i) \\ &+ \int_{1-\alpha_2}^{\bar{\alpha}} \left(\alpha_2 - (\alpha_i + \alpha_2 - 1)\min\left\{\left(\frac{\mu(\alpha_i)}{\mu(\alpha_2)}\right)^{1-\alpha_i}, 1\right\}\right) dG(\alpha_i),\end{aligned} \tag{42}$$

where $\bar{\alpha}$ denotes the highest demand made by player 1 wpp; $\underline{\alpha}$ denotes the lowest demand made by player 1 wpp; and $G(\alpha_i)$ is the cdf over offers by player 1.

Similarly, I can write the payoff of a stubborn player 2 demanding α_2 in a candidate pooling equilibrium as:

$$v_2^s(\alpha_2) = v_2^r(\alpha_2) - \int_{1-\alpha_2}^{\bar{\alpha}} (1-\alpha_i)\mu_i^{\alpha_2} \max\left\{1, \left(\frac{\mu_2}{\mu_i}\right)^{\alpha_i+\alpha_2-1}\right\} dG(\alpha_i). \tag{43}$$

Equivalently, for the rational and stubborn type of player 1. Using (42),(43), given $z > 0$, an equilibrium with support $\mathcal{C}$ requires $\forall \alpha, \alpha' \in \mathcal{C}$, and $j = 1, 2$,

$$v_j^r(\alpha) - v_j^r(\alpha') = 0, \tag{44}$$

$$v_j^s(\alpha) - v_j^s(\alpha') = 0, \tag{45}$$

$$G(\bar{\alpha}) = 1, \text{ and} \tag{46}$$

$$\int_{\mathcal{C}} \mu_j(\alpha_i)^{1-\alpha_i} g_j(\alpha_i) d\alpha_i = z, \tag{47}$$

with $g_j(\alpha_i), \mu_j(\alpha_i) \in [0, 1]$.

Proof of Lemma 1. Note first that by definition of strength, any separating offer by the rational type has strength 0 and any separating offer by the stubborn type has strength 1. By straightforward reasoning, the highest separating offer by the stubborn type must be below the lowest pooling offer (otherwise, the rational type would have an incentive to deviate to the separating offer by the stubborn type); and the lowest separating offer by the rational type must be above the highest pooling offer (otherwise, the rational type would have an incentive to deviate to the highest pooling offer – he is more likely conceded to, and receives more conditional on being conceded to). As a result, I will focus on pure pooling equilibria from now on.

Suppose the strength of player 2, $\mu_2(\alpha_2)$, was not decreasing in α_2.

Case 1: Suppose there exist α_2' and α_2'' with $\alpha_2' < \alpha_2''$, such that $\mathcal{W}(\alpha_2') = \mathcal{W}(\alpha_2'')$, and $\mathcal{S}(\alpha_2') = \mathcal{S}(\alpha_2'')$.

(a) Suppose that $\exists \alpha_i \in \mathcal{S}(\alpha_2')$. Recall that if $\alpha_1 \in \mathcal{S}(\alpha_2)$, then $\alpha_2 + \alpha_1 > 1$, and $\frac{\mu_1(\alpha_1)}{\mu_2(\alpha_2)} < 1$. This implies that (i) fixing strength, the rational player's payoff $v_2^r(\alpha_2)$, as defined in (42), is increasing in α_2; and (ii) fixing the offer α_2, $v_2^r(\alpha_2)$ is increasing in the strength, $\mu_2(\alpha_2)$. Evaluating (42) at $\alpha_2 = \alpha_2'$ and $\alpha_2 = \alpha_2''$, it then follows that

$\mu(\alpha_2') \leq \mu_2(\alpha_2'') \Rightarrow v_2^r(\alpha_2') < v_2^r(\alpha_2'')$. Hence, for $v_2^r(\alpha_2') = v_2^r(\alpha_2'')$, it is necessary that $\mu(\alpha_2') > \mu_2(\alpha_2'')$.

(b) Suppose that $\mathcal{S}(\alpha_2') = \emptyset$. Note that if $\exists \alpha_i \leq 1 - \alpha_2'$, then

$$\int_{\underline{\alpha}}^{1-\alpha_2'} \frac{1 - \alpha_i + \alpha_2'}{2} dG(\alpha_i) < \int_{\underline{\alpha}}^{1-\alpha_2''} \frac{1 - \alpha_i + \alpha_2''}{2} dG(\alpha_i).$$

Hence, for $v_2^r(\alpha_2'') = v_2^r(\alpha_2')$, it is necessary that $\alpha_i \in \mathcal{W}(\alpha_2') = \mathcal{W}(\alpha_2'')$, $\forall \alpha_i$.

Case 2: Suppose there exist α_2' and α_2'' with $\alpha_2' < \alpha_2''$, such that $\mathcal{W}(\alpha_2') \neq \mathcal{W}(\alpha_2'')$, or $\mathcal{S}(\alpha_2') \neq \mathcal{S}(\alpha_2'')$, or both.

(a) Suppose first that (i) $\exists \alpha_i$ such that $\alpha_i < 1 - \alpha_2'$, and $\alpha_i \in \mathcal{W}(\alpha_2'')$; and (ii) that $\forall \alpha_j \neq \alpha_i$, $\alpha_j \in \mathcal{W}(\alpha_2') \iff \alpha_j \in \mathcal{W}(\alpha_2'')$, and $\alpha_j \in \mathcal{S}(\alpha_2') \iff \alpha_j \in \mathcal{S}(\alpha_2'')$. Then, evaluating (42) at $\alpha_2 = \alpha_2'$ and $\alpha_2 = \alpha_2''$, it is clear that $v_2^r(\alpha_2'') > v_2^r(\alpha_2')$. Hence, if $\exists \alpha_i$ such that $\alpha_i < 1 - \alpha_2'$, and $\alpha_i \in \mathcal{W}(\alpha_2'')$, then there must exist $\alpha_j \in \mathcal{S}(\alpha_2') \setminus \mathcal{S}(\alpha_2'')$. But this implies $\mu_2(\alpha_2') > \mu_2(\alpha_2'')$.

(b) Suppose that $\exists \alpha_i$ such that $\alpha_i = 1 - \alpha_2'$, and $\alpha_i \in \mathcal{W}(\alpha_2'')$, and that $\forall \alpha_j \neq \alpha_i$, $\alpha_j \in \mathcal{W}(\alpha_2') \iff \alpha_j \in \mathcal{W}(\alpha_2'')$, and $\alpha_j \in \mathcal{S}(\alpha_2') \iff \alpha_j \in \mathcal{S}(\alpha_2'')$. By Case 2(a), it must be that $\alpha_i = \underline{\alpha}$ (otherwise, there must exist $\alpha_j \in \mathcal{S}(\alpha_2') \setminus \mathcal{S}(\alpha_2'')$.). However, since $\alpha_2' < \alpha_2''$, for $v_2^r(\alpha_2') = v_2^r(\alpha_2'')$, it is necessary that $\mathcal{S}(\alpha_2') = \mathcal{S}(\alpha_2'') = \emptyset$.

(c) Suppose finally that (i)$\exists \alpha_i \in \mathcal{S}(\alpha_2'') \setminus \mathcal{S}(\alpha_2')$, and (ii) that $\forall \alpha_j \neq \alpha_i$, $\alpha_j \in \mathcal{W}(\alpha_2') \iff \alpha_j \in \mathcal{W}(\alpha_2'')$, and $\alpha_j \in \mathcal{S}(\alpha_2') \iff \alpha_j \in \mathcal{S}(\alpha_2'')$. Then, evaluating (42) at $\alpha_2 = \alpha_2'$ and $\alpha_2 = \alpha_2''$, it is clear that $v_2^r(\alpha_2'') > v_2^r(\alpha_2')$. Hence, $\mathcal{S}(\alpha_2'') \setminus \mathcal{S}(\alpha_2') = \emptyset$. This implies that if there exists $\alpha_i < 1 - \alpha_2'$, then $\mu_2(\alpha_2') > \mu_2(\alpha_2'')$. If there does not exist $\alpha_i < 1 - \alpha_2'$, then see Case 2 (b).

Hence, either (i) $\alpha_i \in \mathcal{W}(\alpha_2') = \mathcal{W}(\alpha_2'')$, $\forall \alpha_i$ (which implies $\underline{\alpha} > \frac{1}{2}$ and $\mu(\alpha_i') = \mu(\alpha_i'')$ $\forall \alpha_i', \alpha_i''$; or (ii) $\underline{\alpha} = 1 - \alpha_2'$, $\underline{\alpha} \in \mathcal{W}(\alpha_2'')$ and $\alpha_i \in \mathcal{W}(\alpha_2') = \mathcal{W}(\alpha_2'') \setminus \underline{\alpha}$, $\forall \alpha_i > \underline{\alpha}$; or (iii) $\mu_2(\alpha_2') > \mu_2(\alpha_2'')$. Note that (i) implies that $\alpha_2' > 1 - \underline{\alpha}$, and that $\mu(\alpha_2') = \mu(\alpha_2'')$. Hence, $\mu_2(\alpha_2)$ is strictly decreasing in α_2 unless $\alpha_2 \geq 1 - \underline{\alpha}$. $\square$

NB: We will see in Lemma 2, that (i) cannot be, since it must be that $\underline{\alpha} \leq 1/2$. For (ii) it must be that $\alpha_2' = 1 - \underline{\alpha}$ and $\alpha_2'' = \bar{\alpha}$ by Lemma 2, $\underline{\alpha} + \bar{\alpha} > 1$). This then implies $\mu_2(\bar{\alpha}) = \mu_2(1 - \underline{\alpha})$. Hence, $\mu_2(\alpha_2)$ is strictly decreasing in α_2 unless $\alpha_2 = 1 - \underline{\alpha}$, in which case $\mu_2(1 - \underline{\alpha}) = \mu_2(\bar{\alpha})$.

Proof of Lemma 2, Part 1. Suppose not; i.e., suppose that there exists a pooling offer which is compatible with every pooling offer made by the opponent wpp. Then the payoff to a rational and stubborn type from making this offer is identical. This would then have to be true for every other offer made wpp. When facing an incompatible demand, the rational type has the option value of concession, while the stubborn type does not. Hence, there could be no offer higher than 1/2. But if there is no offer higher than 1/2, then both types would want to demand at least 1/2. Hence, there would not be multiple offers being made wpp. Therefore, every pooling offer must be incompatible with at least one pooling offer made by the opponent wpp. $\square$

Proof of Lemma 2, Part 2. The proof is divided into two parts: I first focus on the difference between two pooling demands; then I turn to the difference between a separating demand (by the stubborn type) and a pooling demand.

Pooling demands: Suppose the set of compatible demands is constant between two pooling demands α and α', with $\alpha < \alpha'$. Suppose further that (44) is satisfied for all $\alpha, \alpha' \in \mathcal{C}$. Then for $j = 2$, I can write (45) as

$$0 = -\int_{1-\alpha}^{\bar{\alpha}} (1 - \alpha_i) \cdot \left(\mu_1(\alpha_i)^{\alpha} \max\left\{1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)}\right)^{\alpha_i + \alpha - 1}\right\} - \mu_1(\alpha_i)^{\alpha'} \max\left\{1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)}\right)^{\alpha_i + \alpha' - 1}\right\}\right) dG(\alpha_i). \tag{48}$$

(a) Suppose $\mathcal{S}(\alpha) = \mathcal{S}(\alpha') = \emptyset$. Then $\forall \alpha_i > 1 - \alpha$,

$$\max\left\{1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)}\right)^{\alpha_i + \alpha - 1}\right\} = 1.$$

Similarly, for α', since $\mu_2(\alpha) \geq \mu_2(\alpha')$. Clearly, $\mu_1(\alpha_i)^{\alpha} > \mu_1(\alpha_i)^{\alpha'}$. Therefore, if $\mathcal{S}(\alpha) = \mathcal{S}(\alpha') = \emptyset$, the RHS of (48) is strictly negative, and hence, (48) cannot be satisfied.

(b) Suppose $\mathcal{W}(\alpha) = \mathcal{W}(\alpha') = \emptyset$. Then $\forall \alpha_i > 1 - \alpha$,

$$\max\left\{1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha'-1}\right\} = \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha'-1}, \tag{49}$$

and similarly for α. Hence, using (49), I can simplify the RHS of (48) to:

$$-\int_{1-\alpha}^{\bar{\alpha}} (1-\alpha_i) \cdot \mu_1(\alpha_i)^{1-\alpha_i} \left(\mu_2(\alpha)^{\alpha+\alpha_i-1} - \mu_2(\alpha')^{\alpha'+\alpha_i-1}\right) dG(\alpha_i). \tag{50}$$

By Lemma 1, $\mu_2(\alpha) \geq \mu_2(\alpha')$. Moreover, note that $0 < \alpha + \alpha_i - 1 < \alpha' + \alpha_i - 1 < 1$. Hence, (50) is strictly negative, and thus, (48) cannot be satisfied.

(c) Suppose $\exists \alpha_i \in \mathcal{S}(\alpha) \setminus \mathcal{S}(\alpha')$. Then

$$\max\left\{1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha-1}\right\} = \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha-1} \quad \text{and}$$

$$\max\left\{1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha'-1}\right\} = 1.$$

But

$$-g(\alpha_i)(1-\alpha_i) \cdot \left(\mu_1(\alpha_i)^{\alpha} \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)}\right)^{\alpha_i+\alpha-1} - \mu_1(\alpha_i)^{\alpha'}\right) < 0 \tag{51}$$

since $\mu_2(\alpha) > \mu_1(\alpha_i)$ and $\alpha < \alpha_i \leq \alpha'$. Hence, (48) cannot be satisfied.

(d) By Lemma 1, $\mathcal{S}(\alpha') \setminus \mathcal{S}(\alpha) = \emptyset$.

Therefore, if for some $\alpha, \alpha' \in \mathcal{C}$, $\nexists \alpha'' \in \mathcal{C}$ such that $\alpha \leq 1 - \alpha'' < \alpha'$, then (48) cannot be satisfied. Hence, for all $\alpha, \alpha' \in \mathcal{C}$, there exists $\alpha'' \in \mathcal{C}$ such that $\alpha \leq 1 - \alpha'' < \alpha'$.

Note that it follows that the lowest pooling demand is compatible with all but the highest pooling demand.

Semiseparating demand: Consider a candidate semiseparating equilibrium with K offers, with multiple separating offers. If the set of compatible offers is non-decreasing between any two separating offers, then the stubborn type strictly prefers the higher offer: any separating offer will be conceded to immediately by the rational type, regardless of its value (if less than 1). Hence, the higher offer yields a strictly higher payoff regardless of the offer made by the opponent.

Consider a candidate semiseparating equilibrium with support $\mathcal{C}$, with one separating offer, α_1, by the stubborn type. Suppose $\nexists \alpha_i$ such that $1-\alpha_2 < \alpha_i \leq 1-\alpha_1$, where α_2 is some pooling demand. Suppose further that the stubborn type is indifferent between the separating demand α_1 and the pooling demand α_2. Then the rational type does not want to deviate to α_1 iff:

$$\Delta^r_{1,2} - \Delta^s_{1,2} \leq 0$$

However,

$$\Delta^r_{1,2} - \Delta^s_{1,2} = \int_{1-\alpha}^{\bar{\alpha}} {\mu_i}^{1-\alpha_i}(1-\alpha_i)(1-\mu_2^{\alpha_2+\alpha_i-1})dG(\alpha_i) > 0.$$

Hence, if the stubborn type is indifferent between α_1 and α_2, the rational type strictly prefers α_1.

□

Proof of Lemma 3.

1. If the highest separating demand by the stubborn type was not lower than the lowest demand assigned positive probability by the rational type, the rational type would have an incentive to deviate from this lowest demand to the separating demand by the stubborn type.

2. If the highest pooling demand was not lower than the lowest separating demand by the rational type, the rational type would have an incentive to deviate from the lowest separating demand to the highest pooling demand.

3. Given Lemma 2 and Lemma 3.1, if there are multiple separating demands by the stubborn type, any such demand is compatible with the highest pooling demand. Conditional on facing a compatible demand, the payoff from making the higher separating

demand is strictly higher. Conditional on facing an incompatible demand, it comes from a rational type and hence such a demand concedes immediately. It follows that when making a separating demand, the payoff to the stubborn type from facing an incompatible demand is at least as big as from facing a compatible demand. Hence, the stubborn type strictly prefers the higher separating demand.

$\square$