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Optimal insurance for time-inconsistent agents

Frédéric Cherbonnier

IAST
INSTITUTE for ADVANCED STUDY in TOULOUSE

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Frédéric Cherbonnier * †

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Abstract: We examine the provision of insurance against non-observable liquidity shocks for time-inconsistent agents who can privately store resources. When lack of self-control is strong enough, optimal contracts are similar to individual financial accounts with remunerated savings and costly borrowing. The corresponding rate of return decreases with savings, which gives a theoretical rationale for pension accounts with decreasing incentive schemes, as implemented in most developed countries. Extending the model to an infinite horizon, we show that, in the presence of repeated shocks, optimal contracts lead to impoverishment almost surely. Usury laws, capping interest rates, worsen this tendency to over-indebtedness for consumers with low risk aversion. By contrast, hidden storage constrains resource allocation for time-consistent agents, so that optimal contracts induce them to accumulate wealth. Those results show how lack of self-control changes the nature of optimal savings and borrowing instruments, with normative implications in terms of tax policy and credit regulation.

Keywords: Time-inconsistency, self-control, mechanism design, insurance, over-indebtedness, retirement savings, consumer credit, credit regulation, saving incentives.

JEL classification numbers: C61, C63, C73, D82, E21, H21.

∗. Toulouse School of Economics and IEP Toulouse, Email: cherbfred@gmail.com, 1, Esplanade de l'Université, 31000 Toulouse

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1 Introduction

Consumers exposed to future preference shocks optimally smooth their consumption through contracts with financial intermediaries. The resulting flexibility in the timing of consumption, while desirable under the time-consistent preferences assumed in the literature on optimal insurance provision, however may be costly when consumers have present-biased preferences. As pointed out by Strotz (1955), Phelps and Pollak (1968), Laibson (1997) and the subsequent literature, consumers then value constraints on their future choices. The main purpose of this paper is to analyze optimal saving and borrowing instruments for a weak-willed agent, and to understand whether such contracts are able to stabilize debt and consumption when the agent faces repeated shocks. These issues are particularly relevant as time-inconsistency may induce excessive debt accumulation or undersavings, by either an individual or a governement.

We assume quasi-hyperbolic preferences and consider (initially), as in Diamond and Dybvig (1983), two periods of consumption, 1 and 2, and an initial period 0 at which the agent contracts with a competitive financial intermediary (who therefore maximizes his expected utility). The agent receives at date 1 a private taste shock θ that acts as a multiplicative factor on his utility. The first-best contract is an insurance mechanism that allocates higher consumption to "high types", i.e. individuals facing liquidity shocks in period 1. However, under asymmetric information, the agent may report untruthfully in period 1 a higher taste shock. In other words, a "low type" may masquerade as a "high type" so as to increase his utility. This agency problem is worsened by the lack of self-control, but the agent is aware of it, i.e. sophisticated, so that he prefers a contract which counteracts at least partly his lack of self-control by penalizing date-2 consumption when a high liquidity shock is reported in period 1. As in Jacklin (1987), the agent can secretly store money between periods at the same rate of return as the principal.We later extend this model to an infinite horizon with repeated shocks.

Our work unveils two marked differences depending on whether the agent is time-consistent or time-inconsistent. First, when the agent is time-inconsistent, there is still a role for multi-period contracts even when hidden storage is allowed. By contrast, a time-consistent agent would insure against unobservable shocks but the principal cannot provide such insurance if the agent can secretly store. Secondly, in the presence of repeated shocks, the optimal allocation leads almost surely a time-inconsistent agent to impoverishment whereas the time-consistent agent almost surely accumulates wealth indefinitely. Such differences arise from the fact that the lack of self-control changes the nature of the contract. Indeed, when the agent is time-consistent, the optimal contract would transfer budget at the benefit of high types, suffering from adverse liquidity shocks, as in a traditional insurance contract. However, when the agent can secretly store money, no insurance is feasible so that the constrained optimum is the laisser-faire situation with no budget transfer between types. In contrast, when the agent is time-inconsistent, a transfer of resources at the detriment of high types must be implemented to counter the tendency to overconsume. The optimal contract is thus like a classical financial accounts, with remunerated savings and costly borrowing. When lack of self-control is strong enough, this contract is robust to hidden storage and leads individuals to immerisation almost surely. Quite surprisingly, at the aggregate level, precautionary motives may dominate and induce the population of such agents to accumulate aggregate savings. More precisely, we show that this latter property holds when prudence is greater than twice the risk aversion or, equivalently, when the inverse of the derivative of agents' utility function is concave.This result may seem paradoxical, and reflects a strong increase in inequality : a vanishing fraction of individuals sees their wealth increase while the rest of the population sinks into poverty, and this accumulation of wealth among the best-offs is sufficient to more than compensate at the aggregate level for the excessive indebtedness of the rest of the population.

The optimal contract for a time-inconsistent agent is strictly concave. Put differently, the interest rate implicit in the optimal contract increases with the amount of borrowing and decreases with savings. This result has multiple policy implications. It supports the idea that progressive incentive schemes may be needed in order to cope with time-inconsistency. In particular, this analysis can be applied to the optimal design of a pension system when agents suffer from the temptation to overconsume. When transfers between types are not allowed, Amador et al. (2006) show that the optimal scheme relies on imposing a minimum level of savings. However, in most developed countries, contributions to pension schemes rather benefit from an incentive mechanism that gradually decreases with the amount of money invested. Indeed, retirement savings accounts generally provide for capped employer matching contributions, and benefit from deferred taxation up to a certain contribution limit. Our results show that these progressive incentive mechanisms are indeed part of the optimal commitment policy. This work admits reinterpretations beyond the realm of intertemporal consumption smoothing, to situations where a continuum of agents with heterogenous preferences do not internalize a positive externality generated by the consumption of a "social good" such as education, R&D or vaccines. The concavity of the optimal contract means in that case that more subsidies should be granted for the early efforts. In particular, this supports, as conjectured by Amador et al. (2006), paternalistic monetary incentives to schooling.

These results shed light on the process that leads to over-indebtedness and on policy measures that are supposed to prevent it. We show that high interest rates induce time-inconsistent consumers to accumulate unlimited debt, even if one may expect high rates to keep some of them away from temptation. In other words, high interest rates are not a strong enough commitment device to prevent impoverishment. Faced with the problem of over-indebtedness, many States intervene by capping eligible interest rates, but the effect of such usury laws on over-indebtedness is ambiguous: it reduces the financial burden caused by high interest rates but, at the same time, induces more borrowing and less precautionary savings. We show that capping interest rates does not systematically prevent the immerisation of time-inconsistent consumers, and even worsens this phenomenon for consumers with low risk aversion. Usury laws alone can be ineffective in this regard, which calls into question the choice of many developed countries to prohibit interest rates above a certain threshold.

2 Related literature

A new feature of the present study relative to the existing literature is that it allows arbitrary resource transfers. Indeed, our two-period model is very similar to the one considered by Amador et al. (2006), except that they consider a type-by-type budget constraint while we allow resources to be transferred across types. Also, the infinite horizon model resembles to that in Atkeson and Lucas Jr (1992), except that they limit the total consumption handed out in each period to a population of time-consistent households while we allow intertemporal transfers among time-inconsistent households. Those differences are crucial when analyzing the optimal tradeoff between flexibility and commitment: whereas additional resources must be targeted at agents facing a high liquidity shock, those agents must be latter penalized in order to induce truthful reporting. Hence, the constrained optimum requires transfer of resources among types and periods.

The natural benchmark is when the agent is time-consistent. Two different strands of litterature provide important insights on that situation, whether you consider taste shocks as in Diamond and Dybvig (1983) or revenue shocks as in Townsend (1982). According to these works, the agent insures against unobservable shocks by holding a multiperiod contract that may, as shown by Thomas and Worrall (1990), lead him almost surely to impoverishment in the presence of repeated shocks. However, if we assume, as in Jacklin (1987) or Cole and Kocherlakota (2001), that the agent can secretly store money between periods, no insurance can be provided anymore by the principal: the agent only borrows and saves using risk-free bonds and, as shown by Aiyagari (1994) and Chamberlain and Wilson (2000), almost surely accumulates precautionary savings indefinitely. The model developed in this paper contains the case of a time-consistent agent as a special case, and extends it to take into account present-biased preferences.

Our results show that Thomas and Worrall's impoverishment result still holds when hidden storage is allowed, as long as the lack of control is strong enough. This reconciles the initial view that hyperbolic discounting leads to undersavings (Laibson, 1997) with the result of Salanié and Treich (2006) showing that hyperbolic consumers may either oversave or undersave when knowing that they will suffer from lack of self-control in the future. In this latter work, sophisticated consumers can only alleviate their time-inconsistency by saving more. When such consumers have access to commitment devices, our work shows that a population of time-inconsistent households will on average accumulate aggregate debts or savings, depending on the same criteria of prudence identified by Salanie and Treich. However, those households will nevertheless almost surely become over-indebted whatever their prudence, whereas time-consistent consumers almost surely accumulate wealth, which is consistent with the initial intuition of Laibson.

This work is related to the large literature that deals with the policy implication of time-inconsistency. Regarding savings for retirement, we know since Diamond (1977) that a paternalistic intervention may be required to counter behavioral biases. As mentioned in the introduction, our work refines the result of Amador et al. (2006) by allowing transfer between types and across periods. This provides a theoretical justification for the saving accounts put in place in most developed countries, based on a decreasing incentive mechanism resulting from tax reliefs and employer matching contributions. Galperti (2015) addresses this issue when time inconsistency is unobservable. He allows also for transfer between types, but assumes that the second period's utility is linear, which implies that no tension initially arises between commitment and flexibility (as we show subsequently in proposition 1). Such tension occurs in his work once he assumes unobservable time-inconsistency, inducing adverse selection. In order to screen between time-consistent and time-inconsistent consumers, less flexibility must be provided to the latter.

Regarding consumer credit, the fact that present-biased individuals have significantly higher amounts of credit card debt have been observed empirically (Meier and Sprenger, 2010). However, the literature is quite scarce on how to counter the over-indebtedness that time-inconsistency is likely to generate. To our knowledge, our work is the first to address this issue from a theoretical point of view, and to analyze the effectiveness of two distinct approaches: self-control by means of commitment devices (generating high interest rates) and credit regulation with usury law (which instead lower interest rates). According to Glaeser and Scheinkman (1998), usury laws may reduce inequalities by limiting transfers between borrowers and lenders: our work question this analysis by showing that, in the end, such laws can worsen impoverishment and widen inequalities.

Our result matters also for a social planner who has a higher discount factor than individuals, because he values more future generations. Farhi and Werning (2007) have looked at such a problem except that, consistent with the framework of Atkeson and Lucas Jr (1992), they consider a pure redistributional issue, and thus do not allow transfer among generations. In Atkeson and Lucas's framework, the degree of inequality continually increases: individuals individuals almost surely become infinitely poor, whereas a vanishing fraction of the population still accumulates wealth. We obtain a similar result for time-inconsistent agents when transfer between types and across periods are allowed. By contrast, Farhi and Werning show that, when the social planner uses a more patient geometric discounting, a steady state solution exists with no one trapped at misery. More specifically, they consider an economy populated by a continuum of individuals who live for one period and are replaced by a single descendant in the next, and assume that the social planner values future generation, so that the social discount factor $\hat{\alpha}$ is higher than the private one α. In a two-period economy, this model is similar to ours. [1] Our work shows that, apart when utility functions are logarithmic, transfers among generations may enhance allocative efficiency. When there are more than two periods, the models differ in how the principal and the agent disagree on discounting (i.e. more or less patient geometric discounting, or geometric vs. quasi-hyperbolic discounting), and this is enough to induce very different long-term properties in the presence of repeated shocks. [2]

3 A model of insurance for time-inconsistent agents

We analyze the optimal [3] allocation of consumption in the presence of liquidity shocks when consumers are time-inconsistent, with a quasi-hyperbolic parameter β. Utility functions in the two periods of consumption ($t = 1, 2$) are u and w respectively. There is a single consumption good available in each period. We impose a subset of the Inada conditions,i.e. utility functions are smooth, increasing, strictly concave

1. The discounting rules are the same once we identify period 1 and period 2 consumption with parent's and child's consumption, and set $\hat{\alpha} = \delta$ and $\alpha = \beta\delta$. They focus partly on the case of logarithmic utility and we show that transfer across periods is not part of the solution in that case, so that the respective optimal allocations of the two models coincide exactly under that assumption and when considering only two periods.

2. In Farhi and Werning's work (resp. in our work) agents at period 1 put a geometric $\alpha^2 = \beta^2\delta^2$ (resp. a quasi-hyperbolic $\beta\delta^2$) weight on period-3 consumption.

3. Equilibrium contracts are socially optimal because we assume sophisticated present-biased agents and perfectly competitive financial markets.

and such that the limit of their derivative at 0 and ∞ are respectively ∞ and 0. The consumer faces in the first period a liquidity shock $\theta > 0$ in a bounded interval $\Theta = [\underline{\theta}, \bar{\theta}]$, with a distribution represented by a continuous distribution function $f(\theta)$. We assume $f > 0$ and denote by $F(\theta)$ the corresponding cumulative distribution function. The average shock is normalized to one, i.e. $\int \theta f(\theta) = 1$. The liquidity shock is privately observed at date 1 by the consumer, who contracts at date 0 with a competitive financial intermediary in order maximize his expected utility

$$\max_{c,k} \int_{\underline{\theta}}^{\bar{\theta}} (\theta u(c(\theta)) + w(k(\theta)))f(\theta) \qquad \text{(EU)}$$

where c and k denote the first and second period's consumptions respectively, subject to incentive compatibility constraints :

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'})) + \beta w(k(\theta^{'}))\ \forall \theta, \theta^{'} \qquad (IC)$$

and some budget constraint that remains to be specified. The financial intermediary can invest 1 at $t = 1$ to get at $t = 2$ return $\delta^{-1} \geq 1$. The allocation defines a type-contingent budget function $B(\theta) = c(\theta) + \delta k(\theta)$. In contrast with Amador et al. (2006) who consider a type-by-type budget constraint ($B(\theta) \leq y$), and with Atkeson and Lucas Jr (1992) who assume a fixed endowment in all periods (i.e. $\int c(\theta)f(\theta) \leq e_1$ and $\int k(\theta)f(\theta) \leq e_2$), we follow Diamond and Dybvig (1983) in considering a global budget constraint that allows transfer of resources between types and across periods:

$$\int (c(\theta) + \delta k(\theta))f(\theta) = \int B(\theta)f(\theta) \leq y \qquad (BC)$$

We consider in the following the optimization problem (M) maximizing the expected utility (EU) under the constraints (IC) and (BC). An allocation (or contract) $\{c(\theta), k(\theta)\}$ is said to be admissible if it satisfies the constraints.

The first-best contract, when taste shocks are not private information, equalizes the marginal utilities, i.e.

$$\theta u^{'}(c(\theta)) = \delta^{-1} w^{'}(k(\theta)) = \text{ constant } \forall \theta. \tag{1}$$

This is an insurance mechanism that allocates higher consumption to individuals facing liquidity shocks in period 1, while consumption in period 2 is type-independent. When shocks are privately observed, consumers are tempted to report a high shock and consume more immediately. In order to prevent that, lower consumption in period 2 must be allocated to reported high types.

Regarding the constrained optimum, a useful benchmark is when the second period's utility is linear, i.e. $w(k) = wk$. In this case, reallocating period-2 consumption among types has no impact on the expected utility. One can choose period-1 consumption to satisfy the condition (1) of the first-best contract, and pick period-2 consumption in order to respect the budget constraint (BC) subject to the incentive compatibility constraints (IC). It is then easy to see that the optimal contract is a first-best contract in which period-2 consumption is chosen such that $c(\theta)+\beta\delta k(\theta)$ is constant. It is like a financial contract in which additional resources, used in period 1 by the consumer facing a high liquidity shock θ, must be paid back with interest in period 2. More precisely, time-inconsistency can be perfectly corrected by a fixed increase in interest rate from $1/\delta - 1$ to $1/\beta\delta - 1$. In a consumption-savings framework, with a competitive intermediary providing credit cards, the increase of interest rate must be offset by a fixed increase in consumption or, equivalently, by lowering credit card's fixed fees. We get therefore the following proposition which is a generalization of the result obtained by DellaVigna and Malmendier (2004), who assume that both u and w are linear.

Proposition 1. *When the second period's utility function is linear, time-inconsistency has no impact on welfare, as it can be perfectly corrected by an increase of interest rate i.e. a return* $1/\beta\delta$ *above the technological rate of return* $1/\delta$.

However, as soon as the second period's utility is strictly concave, a lack of self-control makes it more difficult to induce truthful reporting: incentives must be strengthened and the constrained optimum moves away from the first-best contract when time-inconsistency increases:

Proposition 2. *When* w *is strictly concave, the expected utility provided by the optimal contract strictly decreases when time-inconsistency increases (i.e.* β *decreases).*

4 Optimal transfer of resources

The objective of this section is to understand how lack of self-control constrains the allocation of resources among types, and in particular what type of resource transfer among types is implemented by the optimal contract $\{c(\theta), k(\theta)\}$. We will see that, under fairly general assumptions, the budget function $B(\theta) = c(\theta) + \delta k(\theta)$ is monotonic. An increasing budget function means that a transfer of resource is made for the benefit of consumers suffering from adverse shocks, as in a traditional insurance contract. In contrast, a decreasing budget function $B(\theta)$ means that the money granted during a liquidity shock will have to be repaid later with additional interest charges. With a slight abuse of langage, we say that the contract is an "insurance" contract when the budget function is increasing, and a "credit" contract when it is decreasing. In practice, the latter contract is similar to a classic individual financial account offered by banks, with a return on savings and a charge on loans higher than the risk-free rate of return available on the market. An insurer's contract pools risks by transferring resources to consumers facing an adverse (liquidity) shock whereas, as shown by Diamond and Dybvig (1983), such a banking contract is another way to provide insurance liquidity to consumers.

Equivalently, the optimal allocation is characterized by the slope of the menu $\mathcal{K}$ defined by the relation $\mathcal{K} = k(c^{-1}(.))$ on the domain spanned by c. In the consumption-savings framework presented above, this slope reflects the marginal cost of borrowing as well as the marginal return on saving: if the consumer borrows and consumes at period 1 an additional dc, he will have to reimburse $-dk$ at the second period, and thus pay an interest rate equal to $-dk/dc - 1$. Accordingly, a contract is an "insurance" (resp. "credit") type contract iff the derivative of $\mathcal{K}$ is higher (resp. lower) than the slope $-1/\delta$ corresponding to the technological rate of return. When consumers are time-inconsistent with a second-period linear utility, we have seen with the Proposition 1 in section 2 that the slope of the curve $\mathcal{K}(c)$ is constant and equal to $-1/\beta\delta$: the consumer can perfectly correct his lack of self-control with a constant premium on his saving and borrowing interest rate. Thus, this constrained optimum is a so-called "credit" contract whereas, as seen in the previous section, the first best is an "insurance contract" that equalizes marginal utilities.

4.1 Optimal contract without hidden savings

We follow thereafter an intuitive approach, by using perturbations of the optimal allocation $\{u(\theta), w(\theta)\}$ that maximizes the Lagrangian associated with the budget constraint. Detailed proofs of the results are provided in the mathematical appendix. As usual, and with a slight abuse of notation, it is convenient to change variables from $c(\theta)$ and $k(\theta)$ to $\mathrm{U}(\theta) = u(c(\theta))$ and $\mathrm{W}(\theta) = w(k(\theta))$, and to consider the state variable $V(\theta) = V(\theta, \theta)$ where $V(\theta, \theta') = \theta \mathrm{U}(\theta') + \beta \mathrm{W}(\theta')$. When those functions are differentiable, the constraints (IC) are equivalent to $\mathrm{U}(\theta)$ non-decreasing and $\theta \mathrm{U}'(\theta) = -\beta \mathrm{W}'(\theta)$. In the following, we drop the constraint " $\mathrm{U}(\theta)$ non-decreasing ", and use simple perturbations that satisfy the relation

$$\theta \mathrm{U}'(\theta) = -\beta \mathrm{W}'(\theta). \tag{2}$$

First, we can get the so-called " transversality conditions " associated with the maximization problem by using perturbations at the two ends of the interval Θ. Indeed, when $\theta = \underline{\theta}$ or $\bar{\theta}$, we can slightly change the optimal allocation in an incentive compatible way by adding $\theta\epsilon$ to $\mathrm{W}(\theta)$ and substracting $\beta\epsilon$ from $\mathrm{U}(\theta)$ for ϵ infinitely small. The expected utility is then increased by $(1-\beta)\theta\epsilon$ whereas the overall budget increases by $\delta\theta\epsilon/w'(k(\theta)) - \beta\epsilon/u'(c(\theta))$. Those two terms must be equal, up to a positive multiplicative factor equal to the budget multiplier λ. More precisely, using the straightforward relation $B'(\theta)/\mathrm{U}'(\theta) = 1/u'(c(\theta)) - \delta\theta/\beta w'(k(\theta))$, the transversality conditions are:

$$\frac{B'(\theta)}{\mathrm{U}'(\theta)} = \lambda^{-1}(1-\beta^{-1})\theta \quad \text{for} \quad \theta = \underline{\theta}, \bar{\theta}. \tag{3}$$

Those conditions give us the marginal transfer of resources $dB(\theta)$ that must be implemented by the optimal contract at the two ends of the interval Θ in order to provide an additional unit of utility $d\mathrm{U}(\theta)$ to type $\theta + d\theta$. This transfer is zero when $\beta = 1$ and strictly negative when $\beta < 1$.

We can similarly derive the Euler-Lagrange equation. We consider the perturbation starting at $\theta \in \Theta$ defined by $\mathrm{W}^\epsilon(\tilde{\theta}) = \mathrm{W}(\tilde{\theta}) + \epsilon\theta \mathbb{I}_{\tilde{\theta} \geq \theta}$ and $\mathrm{U}^\epsilon(\tilde{\theta}) = \mathrm{U}(\tilde{\theta}) - \beta\epsilon\mathbb{I}_{\tilde{\theta} \geq \theta}$. This perturbation is incentive compatible, and equivalent to $k^\epsilon(\tilde{\theta}) \simeq k(\tilde{\theta}) + \epsilon\theta\mathbb{I}_{\tilde{\theta} \geq \theta}/w'(k(\tilde{\theta}))$ and $c^\epsilon(\tilde{\theta}) \simeq c(\tilde{\theta}) - \beta\epsilon\mathbb{I}_{\tilde{\theta} \geq \theta}/u'(c(\tilde{\theta}))$. The budget is raised by

$$\int_\theta^{\bar{\theta}} \left(-\frac{\beta}{u'(c(\tilde{\theta}))} + \frac{\delta\theta}{w'(k(\tilde{\theta}))} \right) \epsilon f(\tilde{\theta}) d\tilde{\theta}$$

whereas the variation of the expected utility is

$$\int_\theta^{\bar{\theta}} \left(\theta - \beta\tilde{\theta} \right) \epsilon f(\tilde{\theta}) d\tilde{\theta}.$$

Those two terms must be equal up to the budget multiplier λ. Differentiating twice this equation with respect to θ yields the Euler-Lagrange equation:

$$\frac{\partial}{\partial\theta}\left[f(\theta)\frac{\beta B'(\theta)}{\mathrm{U}'(\theta)}\right] = \frac{\delta f(\theta)}{w'(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial\theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right]. \tag{4}$$

By integrating this Euler-Lagrange equation, and using the tranversality conditions, we get as well the so-called inverse Euler equation (cf. mathematical appendix for a detailed proof). This relation reflects that one cannot improve the solution through a fixed transfer of utility from one period to the other,and can also be obtained with a perturbation that increases $w(\theta)$ by ϵ and decreases $u(\theta)$ by ϵ for all θ:

$$\lambda^{-1} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{\delta}{w'(k(\theta))} f(\theta)d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{u'(c(\theta))} f(\theta)d\theta. \tag{5}$$

When $\beta = 1$, the Euler-Lagrange equation can be rewritten as

$$\frac{\partial}{\partial\theta}\left[f(\theta)\frac{B'(\theta)}{\mathrm{U}'(\theta)}\right] = f(\theta) \times \left(\frac{\delta}{w'(k(\theta))} - \lambda\right)$$

whereas, when the distribution of shocks is uniform, the Euler-Lagrange equation becomes

$$\frac{\partial}{\partial\theta}\left[\beta\frac{B'(\theta)}{\mathrm{U}'(\theta)}\right] = \frac{\delta}{w'(k(\theta))} - \lambda^{-1}(2-\beta).$$

If $\mathrm{U}(\theta)$ is non decreasing, then $w'(k(\theta))$ is non-decreasing with expected value equal to $\delta\lambda^{-1}$, and we see that in the case $\beta = 1$ (resp. the case f uniform) the term $f(\theta)B'/\mathrm{U}'$ is an increasing then decreasing (resp. concave) function of θ. Besides, we know from the transversality conditions that $B'(\underline{\theta})$ and $B'(\bar{\theta})$ are both zero when $\beta = 1$, and both negative when $\beta < 1$.

Thus, when the distribution of shock f is uniform, the budget function $B(.)$ is increasing when $\beta = 1$, and that it is decreasing outside a sub-interval (where it is non-decreasing) when $\beta < 1$. It is also shown in the appendix that the constrained optimum tends to the optimal bunching solution (c^*, k^*) when β tends to zero. This bunching solution satisfies $u'(c^*) = \delta^{-1}w'(k^*)$. The derivative of the implementation curve $\mathcal{K}'(c(\theta)) = -\theta u'(c(\theta))/\beta w'(k(\theta))$ is then asymptotically equivalent to $-\theta/\beta\delta$ when β tends to zero. This implies that the budget function is decreasing when β is small enough.

We extend this result in the mathematical appendix to any distribution of shocks with a positive density $f(\theta)$ on the interval $[\underline{\theta}, \bar{\theta}]$. More precisely, we show the following proposition once we assume the existence of optimal solutions $\mathrm{U}(\theta)$ to the maximization problem that are piecewise-C^1: [4]

Proposition 3. *There exists a unique optimal solution to the maximization problem* (EU) *under constraints* (IC) *and* (BC)*, and it satisfies the following properties:*

(i) When $\beta = 1$, the optimum transfers resources to high types, i.e. the budget function $B(.)$ is increasing (insurance contract);

(ii) When β is small enough, the optimal allocation transfers resources to low types, i.e. the budget function is decreasing (credit contract), and the implementation curve $\mathcal{K}(c)$ is strictly concave.

Thus, when consumers are time-consistent, the constrained optimum induces a positive transfer of resources in favor of types who face high shocks, and is an 'insurance' contract, according to the terminology defined at the beginning of this section. By contrast, when lack of self-control is strong enough, a transfer of resources at the detriment of high types must be implemented: the optimal allocation is a 'credit contrat', in the sense defined previously, that allows individuals to borrow at a premium over the market rate.

Figure 1 illustrates Proposition 3 for a uniform distribution of shocks, and gives an estimate of the threshold value that induces a switch from insurance to credit. [5] This threshold appears to be quite high, close to 0.95, and much higher than empirically estimated values of time-inconsistency, such as $\beta = 0.7$ in Angeletos et al. (2001) or $\beta = 0.5$ in Laibson et al. (2007). This threshold cannot be explicited in general, but we provide an upper bound in subsection 5.2 when there are only two

4. This assumption is made in similar settings, such as in Athey et al. (2005) and Halac and Yared (2014). When the distribution of shock is uniform, existence and smoothness of the optimal solution are rigorously proven in the mathematical appendix. In that case, we show that the monotonicity constraint on $\mathrm{U}(\theta)$ is not binding. In the general case of a non-uniform shock, existence is trickier to prove because the monotonicity constraint can be binding. This cannot be achieved with the classic tools of optimal control theory, and has been addressed for some specific cases, cf.Rochet and Choné (1998) and Carlier (2001).

5. Those pictures are obtained by solving numerically the maximization problem when $u(c) = \frac{(1+c)^{1-\gamma}}{1-\gamma}$ and $w = \delta u$, with $\gamma = 7$ and $\delta = 0.957$ following ibid.

shocks. Figure 2 illustrates result (ii) of Proposition 3 when the distribution is a normal one. In that case, the constraint « $u(\theta)$ non-decreasing » (or, equivalently, $c(\theta)$ non decreasing) appears to be binding at the left side of the interval $[\underline{\theta}, \bar{\theta}]$. This confirms, as explained previously, that the corresponding problem is an optimal control problem for which admissible solutions are required to be convex, which requires an additional assumption of existence, and a more rigorous proof which is given in the mathematical appendix.

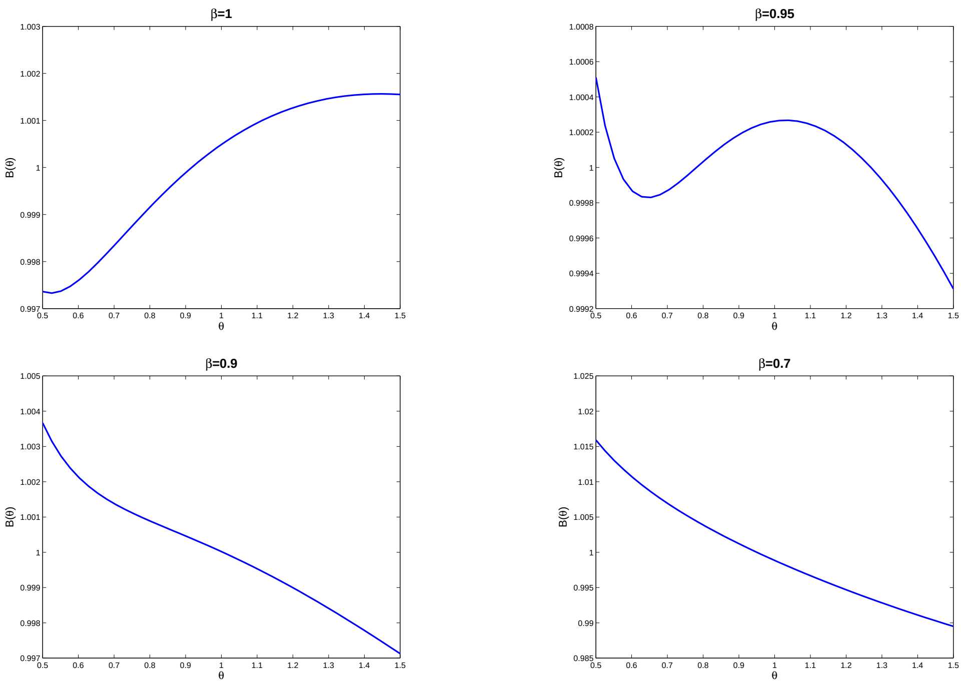

FIGURE 1 – Budget function $B(\theta)$ with a uniform distribution of shocks

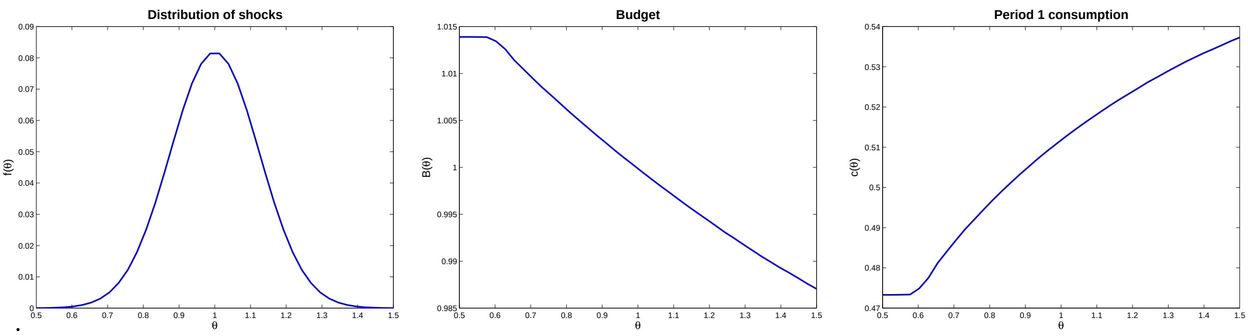

FIGURE 2 – Example in which the non-decreasing-consumption constraint is binding over some interval (normal distribution of shocks with $\beta = 0.7$)

4.2 Optimal contract with hidden savings

In the spirit of Jacklin (1987), it may be more realistic to assume that consumers can store money between periods at the same rate of return as the principal. We consider truthful mechanisms that satisfy a more stringent incentive compatibility constraint, with 'Hidden Savings':

$$\theta u(c(\theta))+\beta w(k(\theta)) \geq \theta u(c(\theta^{'})-\delta\Delta)+\beta w(k(\theta^{'})+\Delta) \ \forall\theta^{'}, \forall\Delta \in [0, c(\theta^{'})/\delta] \qquad \text{(ICHS)}$$

This new constraint implies that $B(\theta)$ is non-increasing: if $B(\theta^{'}) > B(\theta)$ for $\theta^{'} > \theta$, then type-θ would report untruthfully type-$\theta^{'}$ and save between periods $\Delta = (c(\theta^{'}) - c(\theta))/\delta$, and this incentive compatibility constraint does not hold. The contract must therefore be a 'credit contract': borrowing must be costly in order to prevent consumers from asking more money at period 1 and saving afterward.

The following proposition describes the optimal allocation for the two polar cases, time-consistent consumer and strongly hyperbolic consumer:

Proposition 4. *When $\beta = 1$, there is no transfer of resources among types: the optimal solution of the maximization problem (EU), under budget constraint (BC) and incentive compatibility constraint with hidden savings $(ICHS)$, is such that $B(\theta) = y$ for all $\theta \in \Theta$. When β is small enough, the hidden storage constraint is not binding so that the solution is the optimum defined in the Proposition 3 (credit contract).*

Thus, it would be optimal to provide more resources to a consumer facing high liquidity shocks but agents' ability to secretly store prevents that. When $\beta = 1$, no resources are transferred among types at the optimal allocation, which is equivalent to a bunching solution where the principal lends the maximum amount of money that the agent would be willing to borrow, with no premium on interest rates. This result extends Cole and Kocherlakota (2001) to the case of a continuum of liquidity shocks. When β is small enough, the optimal contract in the absence of hidden savings only allows individuals to borrow and to lend at some contractual interest rates high enough to deter agents to secretly store money. Figure 3 summarizes in a simple way those results (the middle graph is provided as an illustration but, depending on the distribution of shocks, may be more complex with multiple intervals where it is constant).

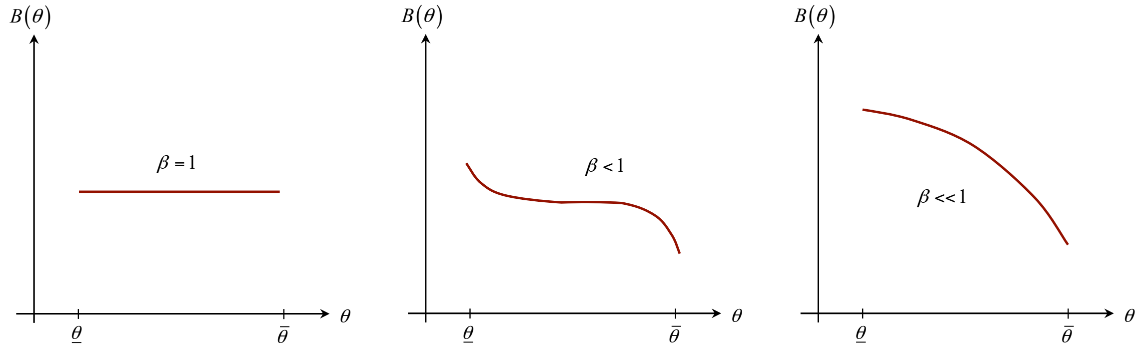

FIGURE 3 – Budget function $B(\theta)$ with hidden storage

4.3 Interpretation and policy implications

As in Amador et al. (2006), these results provide a better understanding of the optimal design of individual pension accounts. Households generally use dedicated retirement savings systems, while also being able to save in other financial instruments. This is the framework considered in the previous section with hidden savings. When agents are time-consistent, we show that no insurance is feasible and that the constrained optimum is the laisser-faire situation: households save and borrow freely at the market rate, without relying on a more sophisticated financial contract. On the other hand, when household are time-inconsistent enough, incentives are needed to penalize those who do not save enough and to reward prudent savers. The concavity of the implementation curve means that the incentive to invest in a retirement account must be stronger for the first dollar saved.

When transfers between types are not allowed, Amador et al. (2006) show that the optimal scheme relies on imposing a minimum level of savings. However, most pension plans do not impose such a constraint, but instead offer a decreasing incentive system, often with a maximum ceiling on the amounts of eligible savings.[6] For instance, regarding the US defined-contribution pension account 401(k) plans, employees can opt-out, are subject to maximum contribution limits, with capped employer matching

6. According to OECD (2019), all OCDE countries use progressive financial incentives (tax and non-tax incentives, usually a mix of deferred taxation and matching contribution, with a maximum cap) to encourage individuals to save for retirement, whereas minimum savings requirement are present in five countries (Iceland, Korea, Lithuania, New Zealand and Norway).

contributions, and benefit from the progressivity of taxation by defering pay until retirement. Federal employees benefit from a similar scheme (the Thrift Savings Plan) with a declining contribution from the employer: the first 3% of employee contribution is matched dollar-for-dollar, while the next 2% is matched at 50 cents on the dollar. We show that these pension systems correspond well in form and principle to the theoretically optimal design when agents suffer from the temptation to over-consume.

More generally, the concavity property means that the contractual interest rate on borrowing (resp. return on savings) increases with the amount of borrowing (resp. decreases with the amount of savings). The intuition behind this result is that, at the constrained optimum, the marginal utility is higher for consumers facing higher liquidity shocks. In order to induce truthful reports, this must be offset by a higher contractual interest rate, i.e. a higher penalty when an individual is under-saving or over-borrowing. This result applies also to consumer credit. In this case, the concavity of the optimal contract means that the interest rate should ideally increase with the size of the loan, regardless of the level of risk induced on the lender, in order to alleviate the behavioral bias. To analyze the implications in terms of consumer credit, a model with repeated borrowing is necessary. This is addressed in section 5.2.

These results admit reinterpretation beyond the realm of intertemporal consumption smoothing. The model considered in section 4.1 also captures the optimal Pigovian taxation of an agent who consumes two goods: an ordinary consumer good whose utility differs among agents, and a social good that generates a positive externality. Let's consider a population of agents indexed by θ whose utility is given by $\theta u(c) + \beta w(k)$, whereas the externality generated by the consumption of the good k is given by $(1-\beta)w(k)$. A social planner does internalize the externality, and thus maximizes the utilitarian welfare criterion $\int(\theta u(c) + w(k))f(\theta)$.[7] When $\beta = 1$, the contract $\mathcal{K}(c)$ describes the optimal taxation when the presence of non-observable

7. In this case, there is no intertemporal transfer, hence $\delta = 1$. If an agent consumes an additional fraction dk of the social good, the optimal contract induces him to consume $-dc = -dk/k'(c)$ less of the ordinary good. This reflects a marginal Pigovian subsidy equal to $-dc/dk - 1$.

shocks induces a need for redistribution, in the absence of externalities. We saw in section 4.1 that the slope of the budget function B is zero at $\theta = \underline{\theta}, \bar{\theta}$, a result similar to the classical Mirrleesian property of non-distortion at the top and at the bottom (Seade, 1977). When $\beta > 1$, the contract $\mathcal{K}(c)$ describes the corrective taxation implementing the optimal trade-off between the redistributive and the corrective objectives of taxation. The concavity of $\mathcal{K}(c)$, stated in Proposition 3, means that the optimal Pigovian subsidy should be decreasing per unit of social good when the externality is strong enough. This applies to environmental public policy, but also to paternalistic regulation. For instance, consider the case of low-altruism parents who put a low weight β on the education k of their own children. As conjectured by Amador et al. (2006), our result shows that a paternalistic government should provide schooling subsidies that decrease with the amount of educational expenses.

5 Asset accumulation and overindebtedness

5.1 Aggregate asset accumulation

Prudent consumers generally save money when they face uncertainty (Gollier, 2001). In a deterministic framework, time-inconsistency may induce under-saving (Phelps and Pollak, 1968). When a population of agents faces both uncertainty and inconsistency, the aggregate level of savings and debt reflects the combined effect of these two factors.

We investigate whether uncertainty induces a population of time-inconsistent households to save money or accumulate debts. We assume $w = \delta u$ so that, in the absence of shocks, consumption would be the same in both periods. Thus, in our framework, the intertemporal transfer $T = \int_{\underline{\theta}}^{\bar{\theta}} (c(\theta) - k(\theta)) f(\theta) d\theta$ reflects the aggregate level of net wealth (T positive means that households borrow money on average in order to face uncertainty). The following proposition provides answers to the above mentioned questions when β is small enough. This limit property is derived from the inverse Euler equation (5): consumption varies more at the second period than at the first period, and the result follows in the limit. We provide at the end of section 5.2 a more direct proof with an explicit condition for β small when there are only two discrete shocks.

Proposition 5. *When lack of self-control is strong enough, a population of hyperbolic consumers issue aggregate debt if* $1/u'$ *is convex, and accumulate (net) positive wealth if* $1/u'$ *is concave.*

In other words, precautionary motives dominate at the aggregate level if $1/u'$ is concave.[8] In this case, the intertemporal transfer T is negative, which reflects aggregate savings at period 1. This is reminiscent of the result of Rogerson (1985) on repeated moral hazard which states that the wages paid by a risk neutral principal to a population of risk averse agents either increase or decrease over time, depending on the curvature of the inverse of agents' marginal utility of income. It also echoes the result of Salanié and Treich (2006) who show that the same criteria determine whether time-inconsistent consumers under- or over-save when they know that they will suffer from a lack of self-control. In their three-period model, consumers can only alleviate their time-inconsistency by saving more at the first period whereas we allow here consumers to contract with a financial intermediary (as we will see in the next section, this result also extends to an infinite horizon model).

5.2 Overindebtedness

The previous section looked at the accumulation of debts and assets at the aggregate level of an entire population. In this section, we are concerned with long-run inequality and immerisation at the individual level. We follow assumptions used by part of the literature dealing with these questions (in particular Atkeson and Lucas Jr (1992) and Halac and Yared (2014)) that is, we consider two shocks $\theta_h > \theta_l$, and assume that consumers have a constant relative risk aversion.

We thus consider a discrete-type model with repeated i.i.d. shocks in $\{\theta_l, \theta_h\}$ over an infinite horizon. The respective probabilities are π_l and π_h with $\pi_l + \pi_h = 1$, the

8. This condition can be interpreted in term of prudence. The measure of prudence P follows the definition of Kimball (1990), i.e. $P(c) = -u'''(c)/u''(c)$. The concavity of $1/u'$ is equivalent to $P \geq 2A$, where A is the Arrow-Pratt measure of risk aversion defined by $A(c) = -u''(c)/u'(c)$. This criteria is commonly used in classical risk theory, that shows that a (time-consistent) consumer facing risk accumulates money iff $P \geq 0$ when the risk cannot be avoided, or iff $P \geq 2A$ when he can adapt by choosing the amount of risk (Gollier, 2001).

average taste shock is equal to 1, that is $\pi_l\theta_l + \pi_h\theta_h = 1$. The felicity function is u in each period. We denote by $\hat{\theta}^t = (\theta^1, \theta^2, ..., \theta^t)$ the history of shocks up to time t. For any $x \in \{l, h\}$, with a slight abuse of notation, we set $(\hat{\theta}^t, \theta_x) = (\theta^1, \theta^2, ..., \theta^t, \theta_x)$ the history of shocks $\hat{\theta}^t$ enriched by the additional shock θ_x at the subsequent period. We use similar notation convention with $(\theta_x, \hat{\theta}^t)$ and $(\hat{\theta}^t, \theta_x, \hat{\theta}^{t'})$. An allocation of a given budget y is a function that maps shock's report $\hat{\theta}^t$ to consumption $c(\hat{\theta}^t)$, and such that the budget constraint (BC) is satisfied, that is $\Sigma_{t=0}^{\infty} E[\delta^t c(\hat{\theta}^t)]$ converges to a real number not greater than y. The optimal allocation maximizes the expectation of the discounted utility

$$V_\infty(y) = \max E[\sum_{t=0}^{\infty} \theta^t \delta^t u(c(\hat{\theta}^t))] \qquad (SP)$$

under the budget constraint (BC), and incentive compatibility constraints with hidden savings

$$\theta_x u(c(\hat{\theta}^t, \theta_x)) + \beta\delta U^0(\hat{\theta}^t, \theta_x) \geq \theta_x u(c(\hat{\theta}^t, \theta_{x'}) - \delta\epsilon) + \beta\delta U^\epsilon(\hat{\theta}^t, \theta_{x'}) \qquad \text{(ICHS2)}$$

for any non-negative integer t ($t = 0$ meaning that there is no past history $\hat{\theta}^t$), for any $\hat{\theta}^t$, any $x, x' \in \{l, h\}$ and $\epsilon \geq 0$, where $U^\epsilon(\hat{\theta}^t, \theta_x)$ denotes the highest future expected utility the consumer can get after period $t+1$ by first reporting $\hat{\theta}^t$ then θ_x, and consuming an additional ϵ at period $t+1$ and/or in subsequent periods (that is, by storing money at the same rate of return as the principal). [9]

In line with the dynamic programming literature, we call this maximization program the sequence problem (SP) and consider the corresponding functional equation

$$V_\infty = \max E[\theta_x u(c_x) + \delta V_\infty(k_x)] \qquad \text{(FE)}$$

where the maximum is taken over the two-period allocations $\{c_l, c_h\}$ and $\{k_l, k_h\}$ that satisfy the corresponding budget and incentive compatibility constraints (detailed

9. Thus, $U^0(\hat{\theta}^t, \theta_x)$ is simply the expected utility provided at the next period after reporting shock θ_x, that is $E[\sum_{i=0}^{\infty} \theta'^i \delta^i u(c(\hat{\theta}^t, \theta_x, \hat{\theta}'^i))]$. When $\epsilon > 0$, since we consider sophisticated consumers, we assume that they only consider future expected utilities that can be reached through hidden savings when taking into account their own lack of self-control. This means that the allocation that provides the expected utility $U^\epsilon(\hat{\theta}^t, \theta_x)$ must satisfy itself the corresponding incentive compatibility constraints.

therafter). We have assumed that the continuation value depends only on the current budget announcement, and we will show that this is indeed the case: we prove in this section that solutions to (SP) and (FE) coincide, in accordance with Bellman's principle of optimality, by showing that they are the fixed point of the one-step operator L providing the solution of the corresponding two-period maximization problem, that is $V_\infty = L(V_\infty) = \max E[\theta_i u(c_i) + \delta V_\infty(k_i)]$. For a given concave utility function v, this problem is similar to the one studied in the previous sections 3 and 4, but with two discrete shocks and $w(.) = \delta v(.)$. It is defined thus by the following maximization program.

$$L(v)(y) = \max \pi(\theta_h u(c_h) + \delta v(k_h)) + (1-\pi)(\theta_l u(c_l) + \delta v(k_l)) \qquad \text{(M2)}$$

where the first and second period's consumptions for type $x \in \{l, h\}$ are denoted respectively by c_x and k_x, subject to incentive compatibility constraints with hidden storage

$$\begin{cases} \theta_h u(c_h) + \beta\delta v(k_h) \geq \theta_h u(c_l - \delta\epsilon) + \beta\delta v(k_l + \epsilon), \forall \epsilon \in [0, c_l/\delta] & (IC_h) \\ \theta_l u(c_l) + \beta\delta v(k_l) \geq \theta_l u(c_h - \delta\epsilon) + \beta\delta v(k_h + \epsilon), \forall \epsilon \in [0, c_h/\delta] & (IC_l) \end{cases}$$

Note and subject to the budget constraint

$$\pi(c_h + \delta k_h) + (1-\pi)(c_l + \delta k_l) \leq y \qquad (BC2).$$

The following lemma shows that this model exhibits the same property as seen before in Proposition 4. It provides also (inverse) Euler equations, as well as an explicit threshold for the hyperbolic discounting factor β under which the budget function $B(\theta)$ is decreasing: [10]

Lemma 1. *In the two-type case with hidden storage, the optimal allocation is unique and satisfies the following properties:*

(i) When $\beta = 1$, no transfers of resources between types are needed, i.e. the budget function $B(\theta)$ is constant, and the allocation satisfies the Euler Equations

10. We consider here only the polar case $\beta = 1$ and $\beta < \theta_l/\theta_h$ for which clear-cut results regarding long-term properties can be obtained. The in-between case $\beta \in [\theta_l/\theta_h, 1)$ is discussed more in detail in the appendix. As in the continuum model considered previously, the hidden storage constraint is binding in this later case.

$\theta_x u^{'}(c_x) = v^{'}(k_x)$, so that

$$L^{'}(v)(y) = \pi\theta_h u^{'}(c_h) + (1-\pi)\theta_l u^{'}(c_l) = \pi v^{'}(k_h) + (1-\pi)v^{'}(k_l);$$

(ii) When $\beta < \frac{\theta_l}{\theta_h}$, there is a transfer of resources from high to low types, i.e. the budget function $B(\theta)$ is strictly decreasing, and the allocation satisfies the Inverse Euler Equation

$$\frac{1}{L^{'}(v)(y)} = \frac{\pi}{u^{'}(c_h)} + \frac{1-\pi}{u^{'}(c_l)} = \frac{\pi}{v^{'}(k_h)} + \frac{(1-\pi)}{v^{'}(k_l)};$$

(iii) when $\beta < \theta_l/\theta_h$ (resp. when $\beta = 1$), the optimum is also the solution of the similar problem in which the incentive compatibility constraint for high type is relaxed, and the hidden storage condition is removed (resp. replaced by non-increasing budget condition, i.e. $c_h + \delta k_h \leq c_l + \delta k_l$).

We now consider consumers with a constant relative risk aversion, i.e. with utility $u(z) = au_\gamma(z)$ where $u_\gamma(z) = z^{1-\gamma}/(1-\gamma)$ with $\gamma \neq 1$ and $a > 0$.[11] For any $a > 0$, the operator L transforms the function $au_\gamma(z)$ in another utility function with the same relative risk aversion. Thus, the function $L(au_\gamma(z))$ can be written as $f(a)u_\gamma(z)$ where $f(a)$ is a real number. It defines a function f that satisfies the following properties:

Lemma 2. *The function f has a unique fixed point a^* which is the limit of the sequence $\lim_{n\to\infty} f^n(a)$ for any $a > 0$.*

This gives the unique solution $a^* u_\gamma$ of the functional equation (FE). It is entirely defined by its value for a given budget y, that we normalize to 1, and by the corresponding two-period optimal allocation $\mathcal{A}^* = (c_x^*, k_x^*)_{x=l,h}$ for the problem $L(a^*u_\gamma)(1)$. Similarly, any solution of the infinite horizon problem (SP) is a CRRA function, entirely defined by its value for the budget $y = 1$ and the corresponding allocation. From the allocation $\mathcal{A}^*$, we build another one for (SP), that we note $\mathcal{A}^\infty = c^*(.)$, by setting

$$c^*(\hat{\theta}^t) = k^*_{\theta 1} k^*_{\theta 2} ... k^*_{\theta t-1} c^*_{\theta t}.$$

It remains to show that $\mathcal{A}^\infty$ is the unique optimal allocation solving the infinite horizon problem (SP) for $y = 1$, and that solutions to (FE) and (SP) coincide. In

11. For the sake of simplicity, we rule out in this section the case $\gamma = 1$ which is treated in the appendix, with identical results.

other words, Bellman's principle of optimality holds. This is a stated in the following lemma: [12]

Lemma 3. *The infinite horizon problem has a unique solution V_∞, corresponding to the allocation $\mathcal{A}^\infty$, and such that $V_\infty = L(V_\infty) = \lim_\infty L^n(u_\gamma) = a^* u_\gamma$.*

Let's now consider the long-term properties of the optimal infinite-horizon allocation. The remaining budgets after T periods are $k(\hat{\theta}^T) = \Pi_{t=1}^{t=T} k^*_{\theta^t}$. According to lemma 1, the marginal utility $V'_\infty(k(\hat{\theta}^T))$ (resp. the inverse of the marginal utility $1/V'_\infty(k(\hat{\theta}^T))$) follows a martingale defined by the Euler equation when $\beta = 1$ (resp. by the inverse Euler equation when $\beta < \beta_l/\beta_h$). These martingale properties are similar to the ones obtained in the literature. More precisely, the relation $E(1/V'_\infty) = 1/V'_\infty$ derived from the inverse Euler equation in the case $\beta \leq \theta_l/\theta_h$ corresponds to the case of incentive compatible contracts under information asymmetry. This relation is similar to the one obtained for instance by Thomas and Worrall (1990). The relation $E(V'_\infty) = V'_\infty$ obtained in the case $\beta = 1$ is similar to the martingales obtained in the context of self-insurance, as in Aiyagari (1994) and Chamberlain and Wilson (2000).

This allows to characterize the long-run property of individual wealth, with a striking difference between time-consistent and time-inconsistent agents.

Proposition 6. *When consumer are time-inconsistent enough ($\beta \leq \theta_l/\theta_h$), the optimal contract is a credit contract with costly borrowing, consumer's wealth becomes almost surely arbitrarily small, and inequality increases as an ever smaller fraction of the population still accumulates wealth. Conversely, when consumers are time-consistent ($\beta = 1$), there is no transfer of resources among types: households save and borrow freely at the market rate, which leads them to almost surely accumulate precautionary savings indefinitely.*

As explained by the literature mentioned in the previous paragraph, the long-term properties result from the fact that the optimal contract must spread continuation values in order to induce truthful reports. Because of the martingale properties,

12. Thereafter, with a slight abuse of notation, we identify solutions of (FE) and (SP) with the corresponding allocations for a budget normalized to one.

this spread of continuation values requires either a downward or an upward trend of the remaining budget. This can be classically obtained using Doob's martingale convergence theorem. We provide thereafter a more direct proof that clarifies this intuition and that will be useful in the next section. The remaining budget $k(\hat{\theta}^T)$ tends almost surely to 0 (resp. to ∞) iff $\ln(k(\hat{\theta}^T)) = \Sigma_{t=1}^{t=T} \ln(k^*_{\theta_t})$ tends almost surely to $-\infty$ (resp. to ∞). This is true iff $E[\ln(k^*_x)] = \pi \ln(k^*_h) + (1-\pi)\ln(k^*_l)$ is negative (resp. positive).[13] Let's set $h(z) = a^* u'_\gamma(z)$ when $\beta = 1$, and $h(z) = 1/(a^* u'_\gamma(z))$ when $\beta < \theta_l/\theta_h$. Let's set also $g(z) = \ln(h^{-1}(z))$. The martingale relation derived from the (inverse) Euler equations are equivalent to $h(1) = \pi h(k^*_h) + (1-\pi)h(k^*_l)$. In the case $\beta = 1$, we have $h(z) = a^* z^{-\gamma}$ and g is strictly convex, so that $0 = g(h(1)) < \pi \ln(k^*_h) + (1-\pi)\ln(k^*_l)$. Similarly, in the case $\beta < \theta_l/\theta_h$, we have $h(z) = a^* z^\gamma$, the inverse Euler equation is

$$\frac{\pi}{u'_\gamma(k^*_l)} + \frac{1-\pi}{u'_\gamma(k^*_h)} = \pi k^{*\gamma}_l + (1-\pi)k^{*\gamma}_h = 1 \qquad (6)$$

and the function g is strictly concave, so that we get the reverse inequality $E[\ln(k^*_x)] < 0$. This yields the result of proposition 6. The fact that, in the case $\beta < \theta_l/\theta_h$, an ever smaller fraction of the population still accumulates wealth results from $k^*_l > 1$. This is a straightforward consequence of the inverse Euler equation, since $k^*_l > k^*_h$.

Similarly, the previous section can be interpreted more explicitly in the case of discrete shocks by considering the expectation of $k(\hat{\theta}^T)$, which reflects the aggregate level of endowment at time T. It is equal to $(E[k^*_x])^T$, and tends to ∞ (resp. to zero) iff $\pi k^*_h + (1-\pi)k^*_l > 1$ (resp. < 1), which is a consequence of relation (6) when risk aversion is lower (resp. greater) than one. Thus, when $\beta < \theta_l/\theta_h$, and u is CRRA, this gives a more direct proof ot the results of proposition 5.

This explicit approach allows to quantify the long-term properties of the optimal allocation. Figure 4 presents such numerical results for two values of β.[14] The two curves under the x-axis describe the per-period average expected decrease in wealth. The two other curves, which cross the x-axis when relative risk aversion is equal to 1, describe the average aggregate savings per-period. It provides a numerical confirmation of the previous results: time-inconsistent consumers sink into over-indebtedness

13. This is a consequence of the central limit theorem that states the convergence of the binomial distribution to the normal distribution.

14. Parameters are $\theta_l = 0.8$, $\theta_h = 1.5$, $\delta = 0.957$ and $\pi_h = 2/7$ (so that $E[\theta] = 1$).

almost surely (proposition 6) but a population of such consumers accumulates a net positive wealth over time if they are prudent enough (proposition 5), the condition of prudence being equivalent here for a CRRA consumers to a relative risk aversion lower than 1. This simulation tends to indicates that the impoverishment trend is significant for consumers with a low risk aversion, and becomes quite negligible for consumers with a high risk aversion.

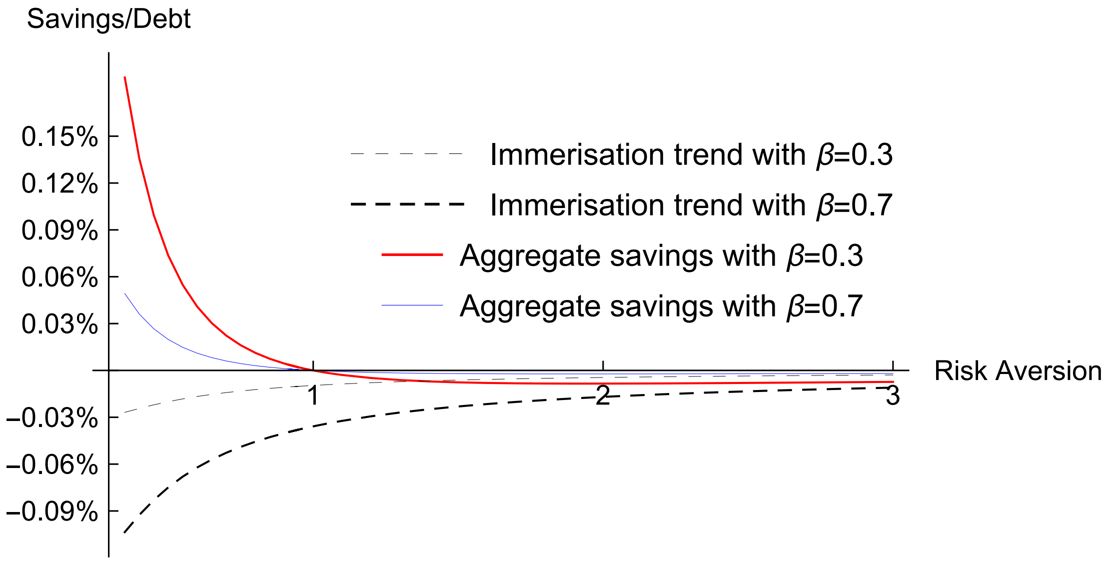

FIGURE 4 – Long-term behavior in terms of constant relative risk aversion

5.3 Usury laws

Faced with the problem of over-indebtedness, some States intervene for paternalistic reasons and implement usury laws that cap eligible interest rates. We discuss the rationale of such laws in more detail in the next section. Our target here is to answer the following question: is there a similar snowball effect leading to impoverishment when interest rates are capped, or does it prevent / limit it (which is apparently the expected effect of these public policies) ?

A priori, the effect on over-indebtedness is ambiguous: usury laws reduce the financial burden caused by high interest rates but, at the same time, lower rates induce more borrowing and less precautionary savings. Of course, in the extreme, if the usury rate is low enough, a credit contract cannot be provided and there is no more impoverishment as stated in the second part of proposition 6. To be relevant, the question must be formulated in a more precise way: what happens when one starts to constrain the interest rates with a usury constraint close to current caps (e.g. between 20% and 30% for personal loans) ?

This question can be addressed by extending the model developed previously. We need to restrict allocations so that, when asking more cash at a given period, the later repayment does not correspond to an implicite rate higher than the usury interest rate R_u. The implicit interest rate defined by the optimal two-period allocation is

$$\bar{R} = \frac{k_l^* - k_h^*}{c_h^* - c_l^*} - 1$$

where $\{c_x^*, k_x^*\}_{x=l,h}$ is the two-period fixed-point allocation $\mathcal{A}^*$ defined in section 5.2. In the two-period model, a usury law is then equivalent to imposing the following constraint

$$k_l - k_h < (1 + R_u)(c_h - c_l) \quad \text{(UL)}$$

where $R_u < \bar{R}$ is the usury interest rate.

We extend similarly the infinite-horizon model. More precisely, we require, for any t and any $\hat{\theta}^t$, that $k_l(\hat{\theta}^t) - k_h(\hat{\theta}^t) \leq (1 + R_u)(c(\hat{\theta}^t, \theta_h) - c(\hat{\theta}^t, \theta_l))$ where $k_x(\hat{\theta}^t) = \Sigma_{i=1}^{\infty} E_{\hat{\theta}^i}[\delta^{i-1} c(\hat{\theta}^t, \theta_x, \hat{\theta}^i)]$ is the expected value of the budget allocated to subsequent periods after the history $\{\hat{\theta}^t, \theta_x\}$. This extended infinite-horizon model is a priori more complex than the one without usury constraints because the optimal allocation may depend on the entire history of past reported shocks, and not only on the continuation budget. Indeed, usury laws prevent the principal from penalizing the debtor with a high interest rate. Instead, the principal may restrict future allocations - for instance, by providing less flexibility in the future to a consumer borrowing money today. Thus, the optimal allocation may depend on past reported shocks. Beside, it is not necessarily unique, since there is a priori an infinite number of ways of reducing the borrowing possibilities offered in the future so as to reduce the expected utility by a certain amount (in other words, there may not exist recursive Markov equilibria). More explicitly, this means that the optimal long-horizon allocation constrained by usury laws is not necessarily obtained by iterating a two-period allocation, as it was the case in the previous section regarding the optimal unconstrained allocation $\mathcal{A}_\infty$.

We show that, in this framework, the answer to the previous question is negative when the usury constraint is not too stringent: in that case, the optimal allocation depends only on the continuation budget and the immerisation trend still holds (and can even be worsened as we will see thereafter):

Lemma 4. *Note $\bar{R}$ the implicit interest rate of the optimal infinite-horizon contract for a time-inconsistent consumer (with $\beta < \theta_l/\theta_h$) in the absence of usury laws,*

(i) If the usury interest rate R_u is close enough to $\bar{R}$, the long-horizon problem constrained by the usury law has a unique solution, corresponding to an allocation $\mathcal{A}^\dagger_\infty = c^\dagger(.)$, which is a stationary equilibrium obtained by iterating over time a two-period allocation $\mathcal{A}^\dagger = \{c^\dagger_x, k^\dagger_x\}_{x=l,h}$, i.e. $c^\dagger(\hat{\theta}^t) = k^\dagger_{\theta^1} k^\dagger_{\theta^2} ... k^\dagger_{\theta^{t-1}} c^\dagger_{\theta^t}$ for any sequence of shocks $\hat{\theta}^t$.

(ii) When (i) is satisfied, the inverse of the marginal utility is a strict super martingale, and consumers accumulate almost surely unlimited debt, which is respectively equivalent to $E[1/u'_\gamma(k^\dagger_x)] < 1$ and $E[\ln(k^\dagger_x)] < 1$.

We provide in the mathematical appendix an explicit sufficient condition, i.e. a threshold $\tilde{R}$ such that lemma 4 holds when $R_u \geq \tilde{R}$. This condition is quite loose for a wide set of parameters. In the example illustrated thereafter, we have $\bar{R} = 50\%$, $\tilde{R} = 15\%$ and we choose $R_u = 20\%$.

When relative risk aversion is less than or equal to 1, we are able to prove that the immerisation trend is worsened, as stated in the following proposition. This result is not obvious a priori: usury constraints should intuitively induce a narrowing of the spread between the second-period remaining budgets, and a widening of the spread between the first-period budgets, with an ambiguous impact on immerisation. However, this comes up against the incentive compatibility constraint. One way to comply to both constraints is to transfer budget from the second to the first period. [15] In other words, shifting more consumption up-front (regardless of shocks) is a way to keep the incentives with a lower (implicit) contractual interest rate. This explains why the "snow-ball" effect of debt is worsened by usury laws.

Proposition 7. *Under assumption of lemma 4, adding a usury constraint worsens*

15. Because of the concavity of the utility function, adding $\epsilon > 0$ to c_l and c_h, and subtracting ϵ/δ to k_l and k_h, would decrease $u(c_h) - u(c_l)$ and increase $v(k_l) - v(k_h)$, which relaxes the incentive compatibility constraint for low types.

the immerisation trend, i.e. $E[\ln(k_x^\dagger)] < E[\ln(k_x^*)]$ *where* $k_x^\dagger$ *and* k_x^* *are the second-period optimal allocations for the problem with and without usury law respectively,*

(i) when relative risk aversion is equal to one (logarithmic utility);

(ii) when relative risk aversion is lower than one, and liquidity shocks are small enough.

Since expected shock is normalized to one, 'small enough liquidity shocks' means that θ_l and θ_h must be close enough to one. This is very relative since (ii) is satisfied in the example discussed thereafter and illustrated in Figure 6, based on $\theta_l = 0.8$ and $\theta_h = 1.5$.

The proof, illustrated in Figure 5, is slightly more complex than the intuition presented in the previous paragraph. This figure describes the locus of second-period consumptions $\{k_h, k_l\}$ corresponding to different constraints. Initial wealth is normalized to 1, utility is logarithmic, β is equal to $1/2$, and the other parameters are the same as in the example provided previously (cf. footnote 14). The point $\mathcal{P} = \{k_l^*, k_h^*\}$ corresponds to the optimal allocation in the absence of usury constraint, and the point $\mathcal{P}' = \{k_l^\dagger, k_h^\dagger\}$ to the optimal allocation with a usury constraint $R_u = 20\%$. The curve $\mathcal{C}_I$ describes the set of points that exhibits the same immerisation trend than the initial optimal allocation $\mathcal{A}^*$, i.e. such that $E[\ln(k_x)] = E[\ln(k_x^*)]$. The curve $\mathcal{C}_M$ corresponds to martingale condition (6) arising from the inverse Euler equation, that is $E[1/u'_\gamma(k_x)] = 1$. The curve $\mathcal{C}_F$ represents the locus of points corresponding to allocations satisfying the incentive compatibility constraint, as well as the budget condition and another relation derived from first order conditions satisfied by an optimal allocation (detailed in the appendix). Those curves intersect at $\mathcal{P}$ and we need to show that $\mathcal{P}'$ is strictly under the curve $\mathcal{C}_I$: this would mean that immerisation trend is worsened by the usury constraint. As stated in lemma 4, the inverse of the marginal utility is a strict super martingale, which is equivalent to $\mathcal{P}'$ strictly under the curve $\mathcal{C}_M$. We also show in the appendix that $\mathcal{P}'$ is under the curve $\mathcal{C}_F$, and that the curve $\mathcal{C}_F$ (resp. $\mathcal{C}_M$) is steeper (resp. less steeper) than the curve $\mathcal{C}_I$. The result follows.

Figure 6 illustrates what happens over time, with an additional example with relative risk aversion equal to 0.5. The shaded areas show the growth of inequalities, the upper (resp. lower) thin lines representing the evolution of wealth in the absence

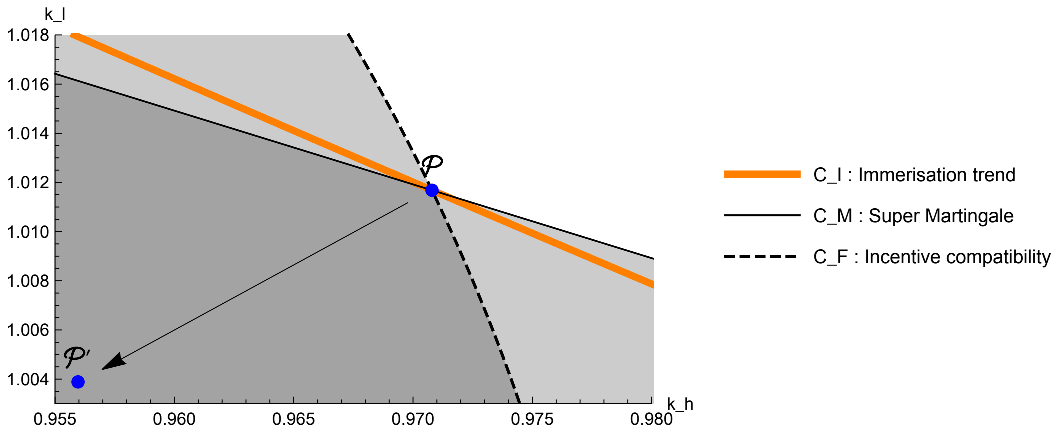

FIGURE 5 – Impact of usury law on second-period budget $\{k_l, k_h\}$. The curve C_I is slightly modified on the graph, with a less steep slope, in order to make visible the way those curves intersect.

of negative shocks (resp. the worst cases when the individual is submitted to repeated liquidity shocks). Solid (resp. dotted) lines correspond to the initial model without usury law (resp. with usury law). The thick (solid and dotted) lines show the immerisation trend in both cases (measure, as explained previously, by $E[\ln(k_x)]$). We see that it is very small when there is no usury law (about -0.02% per period when $\gamma = 1$), and becomes much stronger under such law (about -0.63% per period).

Thus, borrowers are less penalized by interest rates in the presence of usury laws, but this is not enough to prevent the snowball effect of over-indebtedness. Quite the contrary in this example, the graph shows that individuals then save less in the good states of nature (less precautionary savings when there is no liquidity shocks) and become more impoverished in the bad states (more borrowings in presence of an adverse liquidity shock). As explained at the end of section 5.2, it is also possible to quantify the aggregate saving behavior characterized by proposition 5. When utility is logarithmic, consistent with this proposition, there is no transfer across periods. In the presence of usury law, on the contrary, the population goes into debt over time: the aggregate debt increases by $+0.60\%$ per period. When $\gamma = 1/2$, usury laws induce a trend reversal, from an upward trend in total wealth ($+0.04\%$) to a decrease at a rate of 2%. However, as illustrated by the top dotted lines, an ever smaller fraction of the population still accumulates wealth.

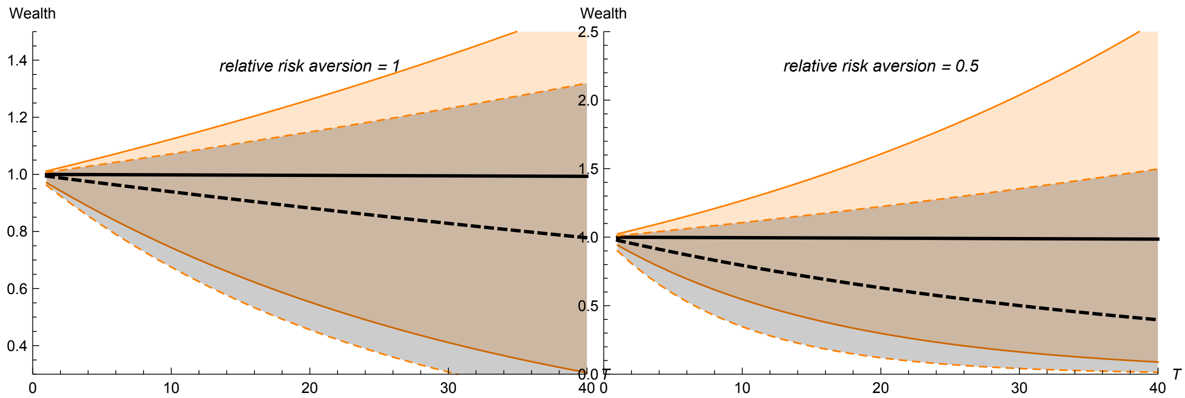

FIGURE 6 – Immerisation and growth of inequality with (dotted lines) and without (solid lines) usury law with $\beta = 0.5$ (and other parameters as stated in 14).

5.4 Interpretation and policy implications

By building a bridge between different known results and by extending them to time-inconsistent agents, this work shows that the lack of self-control radically change long-run trend of consumption and indebtedness. We know, in particular since Green (1987) and Thomas and Worrall (1990), that immerisation occurs when time-consistent agents undertake contractual arrangements to insure against liquidity or income shocks: consumer's wealth becomes almost surely arbitrarily small. However, as shown by Cole and Kocherlakota (2001), agents' ability to secretly store money prevents such incentive contracts. No insurance can be provided anymore: agents only borrow and save using risk-free bonds and, as shown by Aiyagari (1994) and Chamberlain and Wilson (2000), accumulate precautionary savings indefinitely.

Proposition 6 proves that, when agents are time-inconsistent enough, hidden storage does not constrain the optimal contract anymore, so that immerisation again prevails. This property holds even when the population of consumers accumulates aggregate savings over time, which is in particular the case for consumers with constant relative risk aversion lower than 1 according to proposition 5.[16] This result may seem

16. The relevance of this result depends on the level of risk aversion encountered in the population. Since we are considering here consumption and savings choice, the most relevant analysis in our case are the ones based on micro-level data describing household inter-temporal expenditure allocation. Those studies, like Blundell et al. (1994) and Attanasio and Weber (1993), estimate an average intertemporal elasticity

paradoxical - the corresponding case with time-consistent consumers was not considered by literature.[17] It reflects the fact that all those results hold almost surely. In particular, whatever the level of relative risk aversion, inequality increases: a vanishing fraction of individuals sees their wealth increase while the rest of the population sinks into poverty. In other words, when risk aversion is lower than one, the accumulation of wealth among the best-offs is sufficient to more than compensate at the aggregate level for the excessive indebtedness of the rest of the population.

Thus, although a high contractual interest rate serves as a commitment device for restraining time-inconsistent consumers, it leads to poverty almost surely. It also means that a sophisticated agent values the flexibility offered by a credit card enough that he prefers to choose such an instrument, even if the long-term consequences are impoverishment. This is not intuitive, and means that the negative consequences of impoverishment are far enough away that the benefits of flexibility in the shorter term outweigh them. What does this say about the behavior observed in practice? A large number of consumers frequently choose not to use a credit card: debit cards are indeed now used more often than credit cards at the point-of-sale in the United States according to Zinman (2009) and Borzekowski et al. (2008). Spending control, for consumers suffering from a present bias, was initially seen as a potential motive for giving up a credit card. Proposition 6 tends to refute this explanation - a sophisticated agent with quasi-hyperbolic preferences prefers to keep the credit card. This is consistent with the literature (ibid.), which now primarily explains the growth of debit card transactions (to the detriment of other payment methods) by non-behavioral reasons - in particular for purely pecuniary reasons.

Faced with the problem of over-indebtedness, many States intervene by capping eligible interest rates. Such usury laws are indeed present in many developed coun-

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of substitution for consumption slightly lower than one. The corresponding relative risk aversion is greater than one for the majority of the population, but a significant fraction appears to exhibit a risk aversion lower than unity.

17. Atkeson and Lucas Jr (1992) show that that immerisation occurs for for any CRRA consumers, including those with risk aversion lower than one, but when there is no money transfer across periods and no hidden savings. Thomas and Worrall (1990) assume that the utility function $u(c)$ satisfies $\inf u(c) = -\infty$ as well as $\sup u(c) < \infty$, which rules out CRRA with risk aversion lower than one.

tries: this is the case in Japan, in most European countries, and in the US for some types of loans.[18] However, the effect of those laws on over-indebtedness is ambiguous: they reduce the financial burden caused by high interest rates, but lower rate induces more borrowing and less precautionary savings. Proposition 7 shows that usury laws can be ineffective in preventing over-indebtedness, and even worsen this phenomenon, at least for consumers with low risk aversion. As explained in the previous section, this results from the complex trade-off between flexibility and commitment, reflected in the contract chosen by the sophisticated time-inconsistent consumer. Capping interest rates limits the incentives provided in the subsequent periods. It is then optimal to shift more budget for present consumption. This leads to less precautionary savings (in the good states of nature), more borrowing (in the face of an adverse liquidity shock) and, ultimately, more impoverishment.

Some important caveats must be considered when interpreting Proposition 7 in this way. First, usury laws are generally justified by a paternalistic argument whose underlying assumptions are not in our model, that is other cognitive biases such as consumers' naivety and financial illiteracy. Considering a naive rather than sophisticated agent would not fundamentally change our results.[19] However, usury laws can indeed prevent banks from taking advantage of their customers' ignorance, or even ban some forms of predatory lending as observed in the case of payday loans. Dasgupta and Mason (2020) shows that capping annual interest rates of payday loans at 36% in four US states led to a banishment of this industry.[20] Yet, they found

18. Usury laws are present in most European countries, but only a few have an explicit usury rate - France, Italy, Portugal and Switzerland according to Masciandaro (2001) whereas price caps on short-term credits have been reintroduced in the UK in 2014. In the US, credit cards were removed from usury law restrictions in a 1978 U.S. Supreme Court ruling, but high interest rate loans are still banned in 19 U.S. states for specific loans such as payday loans.

19. A naive consumer would become even more easily over-indebted, with and without usury laws, because of his lack of self control. This can be easily seen in our model under the assumption of proposition 6: a naive and time-inconsistent agent would choose the same contract as the time-consistent agent, and then report always high liquidity shocks, which leads him to over-indebtedness.

20. In the absence of usury laws, payday loans charge a very large 14-day fee,

no effect on bankruptcy: payday loan bans push borrowers into alternative forms of finance, with lower interest rates, but the debt spiral seems unchanged which is consistent with our results.[21] Second, and more importantly, as is done in the literature on this topic, our immerisation result assumes that the spiral of debt can drag individuals down to absolute misery. However, this is not realistic since there is implicitly a threshold of over-indebtedness which is never exceeded, because one can flee his country without repaying his debt or can benefit from a bankruptcy discharge order. We leave that for further study, but we conjoncture that properties similar to those stated in propositions 6 and 7 would still be valid in the presence of such a mechanism.[22]

With these caveats in mind, our result tends to show that usury laws are insufficient, and can even be counterproductive, to prevent over-indebtedness. What can be done otherwise to address over-indebtedness ? As mentioned in the previous paragraph, another approach is to strengthen personal insolvency laws in order make it easier for a debtor to qualify for a debt relief. However, it is less easy for a judge to hold banks accountable for the insolvency of one of their customer when the information on total household debt is not available to banks. This so-called "positive" information on debt is usually available through credit bureaus, in addition to "negative" information such as late or missed payments but, in a lot of countries, the collection of positive information requires the data subject's consent, and some developed countries collect only negative data - in particular Danemark, France, Japan and Spain according to Rothemund and Gerhardt (2011). Some countries, notably France and Japan, both have banned interest rates above a certain threshold (close to 20%) and forbid the collection of positive information on debt. Our work shows that this type of consumer credit regulation can be highly counterproductive.[23]

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equivalent to annual interest rates of over 400%.

21. Desai and Elliehausen (2017) find a similar result. According to Bhutta et al. (2016), consumers shift to other forms of credit with a lower but still very high interest rate (such as pawnshop loans whose average rate is about 250%).

22. This would mean, more realistically, that agents do not gradually sink into absolute misery but rather reach a level of over-indebtedness which leads to bankruptcy with some probability, and benefit from a debt relief at the expense of the lender.

23. A gradual tightening of usury laws in Japan between 1986 and 2010 did not

6 Conclusion

This paper shows how a lack of self-control changes the nature of optimal savings and borrowing instruments. When the agents can secretly save, the impoverishment result of Thomas and Worrall (1990) holds only when time-inconsistency is strong enough. By contrast, time-consistent agents accumulate wealth indefinitely. Paradoxically, the impoverishement of time-inconsistent agents holds even though, at the aggregate level, precautionary motives may induce the population as a whole to save money. Indeed, we show that if households are prudent enough, they accumulate aggregate savings despite their lack of self-control. This reflects a strong increase in inequality: the accumulation of wealth among the best-offs offsets the excessive indebtedness of the rest of the population.

This work also characterizes optimal commitment schemes, leading to two distinct policy recommendations. First, it shows that a progressive incentive system is always part of the optimum: the interest rate implicit in the optimal contract increases with borrowing and decreases savings. Applied to the optimal design of a pension scheme, this result supports incentive mechanisms that decrease with the amount of money invested (such as tax-favored retirement savings accounts with capped employer matching contributions, as is the case in most developed countries). As mentioned, this analysis can be applied as well to the Pigovian taxation of externalities, and support for instance schooling subsidies that decrease with the amount of educational expenses. Secondly, this analysis sheds light on the process that leads to overindebtedness, and on some mechanisms supposed to prevent it. It shows that neither commitment devices (such as high interest rates on credit cards,

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prevent the boom in money-lending and over-indebtedness in this country. The average level of outstanding debt per customer has been roughly halved at the end of this period, but the main driver was a limit on total borrowing set by the money lending law in 2006 (Gibbons, 2012). France's regulation still prevents control on total borrowing and manage instead over-indebtedness. Laws were passed in 2013 and 2018 to respectively protect the fraction of the population deemed to be financially fragile, then to facilitate debt renegotiation procedures. A high share of low income households are barred from commercial borrowing, following a credit card or current account delinquency.

or precautionary reserves), nor usury laws (which, at the opposite, lower interest rates on credit cards) suffice to prevent time-inconsistent households from accumulating unlimited debt. Usury laws even worsen this phenomenon for consumer with low risk aversion. To our knowledge, this is the first theoretical analysis of the impact on over-indebtedness of usury laws, which are still in force in many developed countries. An interesting direction for future research is to consider debt renegotiation process, that partly prevent consumers from sinking into complete misery. Also, the need for screening different types of consumers may modify the optimal contract. It would be worth studying if these considerations have an impact on the long-run properties discussed here.

The economic properties obtained in this work depend on some key parameters, in particular the degree of risk aversion, the level of time-inconsistency, the stringency of the constraint set by usury laws. Different forces are at play, and depending on the parameters involved, some outweigh others. We show that, for commonly accepted values of those key parameters, a clear conclusion can be obtained on these combined effects. An interesting direction for future research is to consider this from an empirical point of view. Households are in particular heterogeneous with respect to time-inconsistency and risk aversion. One can wonder whether these characteristics explain the heterogeneity empirically observed with regard to financial services usage and its consequence in term of credit card delinquency and personal bankruptcy.

7 Proof

Proof of Proposition 1: If $w(z) = \delta' z$, the budget constraint (BC) will bind so that the optimization problem (M) is equivalent to maximizing $\int_{\underline{\theta}}^{\bar{\theta}}(\theta u(c(\theta)) - \delta^{-1}\delta' c(\theta))f(\theta)$ i.e. $u'(c(\theta)) = \delta^{-1}\delta'/\theta$ whereas, as explained in section 4.1, the incentive compatibility contraints (IC) implies the equation $\theta u'(\theta)c'(\theta) = -\beta w'(\theta)k'(\theta)$, which is equivalent here to $k'(\theta) = -c'(\theta)/\beta\delta$, so that consumption at period 2 is equal to

$$k(\theta) = -\frac{c(\theta)}{\beta\delta} + \int_{\underline{\theta}}^{\bar{\theta}} \frac{1-\beta}{\delta\beta} c(\theta)f(\theta)d\theta + \delta^{-1}y$$

Therefore, when $\beta < 1$, the optimal allocation is the allocation that would be provided to a time-consistent consumer if the discount rate was distorted from δ to $\delta\beta$

(up to a fixed transfer of money balancing the global budget). □

Proof of Proposition 2: We show in what follows that, $\forall \beta < \beta'$, a consumer with hyperbolicity β' enjoys more utility on average at the constrained optimum than a consumer with hyperbolicity β. Let's consider the optimal allocation (c^*, k^*) for the consumer with hyperbolicity β, and define a new allocation (c, k) by $c(\theta) = c^*(\theta)$ and $k(\theta) = w^{-1}(\alpha w(k^*(\theta)) + a)$ where $\alpha = \beta/\beta'$ and where the constant a is chosen such that $\int (c(\theta) + \delta k(\theta)) f(\theta) = y$ or, equivalently, $\int k(\theta) f(\theta) = \int k^*(\theta) f(\theta)$. This new allocation satisfies the incentive compatibility constraint $\theta u'(\theta) c'(\theta) = -\beta' w'(\theta) k'(\theta)$ for a consumer with hyperbolicity β'. Therefore, in order to prove the proposition, it is enough to show that

$$\int (\theta u(c^*(\theta)) + w(k^*(\theta))) f(\theta) d\theta \leq \int (\theta u(c(\theta)) + w(k(\theta))) f(\theta) d\theta.$$

This inequality is true iff the constant a is greater than $\int (1-\alpha) w(k^*(\theta)) f(\theta) d\theta$ or, equivalently,

$$\int k^*(\theta) f(\theta) d\theta \geq \int w^{-1} \left(\alpha w(k^*(\theta)) + \int (1-\alpha) w(k^*(\tilde{\theta})) f(\tilde{\theta}) d\tilde{\theta} \right) f(\theta) d\theta$$

The right term of this inequality can be rewritten as

$$\int w^{-1} \left(\int \left(\alpha w(k^*(\theta)) + (1-\alpha) w(k^*(\tilde{\theta})) \right) f(\tilde{\theta}) d\tilde{\theta} \right) f(\theta) d\theta$$

and the result follows from the fact that w^{-1} is convex. This inequality is strict when w is non-linear on the domain spanned by $k^*(\theta)$. □

Proof of Proposition 3 and 4: The intuitive underpinnings and the main ingredients of the proof are given in the body of the text. A rigorous proof is given in Mathematical appendix. □

Proof of Proposition 5: We know from results of section 4 that the storage condition is not binding when β is small enough, and that the inverse Euler equation 5 is then verified:

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{\delta}{w'(k(\theta))} f(\theta) d\theta = \int_{\underline{\theta}}^{\bar{\theta}} \frac{1}{u'(c(\theta))} f(\theta) d\theta.$$

For any function twice differentiable almost everywhere, we have

$$\int_{\underline{\theta}}^{\bar{\theta}} g(c(\theta)) f(\theta) d\theta = g(\tilde{c}) + \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{c}}^{c(\theta)} g''(u)(c(\theta) - u) du f(\theta) d\theta$$

where $\tilde{c} = \int_{\underline{\theta}}^{\bar{\theta}} c(\theta) f(\theta) d\theta$. Indeed,

$$g(x) = g(c) + \int_c^x g^{'}(u)du = g(c) + g^{'}(x)x - g^{'}(c)c - \int_c^x ug^{''}(u)du$$

or equivalently

$$g(x) = g(c)+(x-c)g^{'}(c)+(g^{'}(x)-g^{'}(c))x-\int_c^x ug^{''}(u)du = g(c)+(x-c)g^{'}(c)+\int_c^x (x-u)g^{''}(u)du$$

and the result follows by choosing $c = \tilde{c}$. If we apply this property to the function $g = 1/u^{'}$, when $w = \delta u$, then

$$\frac{1}{u^{'}(\tilde{c})} - \frac{1}{u^{'}(\tilde{k})} = \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{k}}^{k(\theta)} g^{''}(u)(k(\theta)-u)duf(\theta)d\theta - \int_{\underline{\theta}}^{\bar{\theta}} \int_{\tilde{c}}^{c(\theta)} g^{''}(u)(c(\theta)-u)duf(\theta)d\theta$$

and, since $1/u^{'}$ is increasing, the transfer T is positive iff the left term of this equation is positive. When β tends toward zero, the optimal allocation tends to the bunching allocation $(\bar{c}, \bar{k})$ and the function $-\frac{k^{'}(\theta)}{c^{'}(\theta)}$ tends uniformly to ∞. Consequently, when β is small enough, the left term of the previous equation is positive if g is convex, negative if g is concave. Since the second derivative of $g = 1/u^{'}$ is $-u^{'''}/u^{'2}+2u^{''2}/u^{'3}$, the convexity of g is equivalent to $-u^{'''}/u^{''} \leq -2u^{''}/u^{'}$, i.e. the absolute prudence is lower than twice the absolute risk aversion. □

Proof of Lemma 1: This extension of proposition 4 to the case of two discrete shocks is proven in the mathematical appendix.

Proof of Lemma 2: We show in the following that the function f is convex when $\gamma < 1$ and concave when $\gamma > 1$. When $\gamma > 1$, f non-decreasing is then a consequence of the concavity of f and of the fact that f is strictly positive on $\mathcal{R}^+$. When $\gamma < 1$, the optimal allocation solution to the problem $L(au_\gamma)(1)$ is still a compatible allocation for the problem $L((a+\epsilon)u_\gamma)(1)$ for ϵ positive. This implies that $L((a+\epsilon)u_\gamma)(1) \geq L(au_\gamma)(1) + \epsilon\beta\delta(\pi_h u_\gamma(k_h) + \pi_l u_\gamma(k_l)) \geq L(au_\gamma(1))$ so that $f(a+\epsilon) \geq f(a)$, i.e. f is non-decreasing. In order to prove the lemma, it is then enough to show that, for a small enough (resp. large enough), we have $f(a) > a$ (resp. $f(a) < a$), with $\lim_{a\to+\infty} f(a) = +\infty$ and $\lim_{a\to 0} f(a) \in \mathcal{R}^{+*}$. This is quite straightforward, and results from the fact that lower (resp. upper) bound on $L(au_\gamma)$ can be obtained from the bunching allocation (resp. from the problem where the incentive compatibility constraints are relaxed) for which explicit formula can be easily obtained. In the case $\beta = 1$, the optimal solution is such that $\theta_x u^{'}(c_x) = au^{'}(k_x)$

and $c_x + \delta k_x = 1$. It is then easy to get $f(a) = E\left[(\theta_x^{1/\gamma} + \delta a^{1/\gamma})^\gamma\right]$ and the mentioned properties follow.

It remains to show the curvature property when $\beta < \theta_l/\theta_h$. It is enough to show that this is true locally. Folllowing lemma 1(iii), we consider the relaxed problem in which the hidden storage constraints and the incentive compatibility constraint for high types are removed. Let's choose a real $\tilde{a} > 0$, a small positive number ϵ and $\lambda \in [0,1]$. We set $a_0 = \tilde{a} - (1-\lambda)\epsilon$ and $a_1 = \tilde{a} + \lambda\epsilon$. Thus, $\tilde{a} = \lambda a_0 + (1-\lambda)a_1$. We prove in the following that $L(\tilde{a}u_\gamma)(1) \leq \lambda L(a_0 u_\gamma)(1) + (1-\lambda)L(a_1 u_\gamma)(1)$. Since, by definition, $f(a) = (1-\gamma)L(a.u_\gamma)(1)$, this implies the announced curvature properties of f. Let's consider an optimal allocations $(\tilde{c}_x, \tilde{k}_x)_{x=l,h}$ providing the maximal expected utility for the two-period problem $L(\tilde{a}u_\gamma)$ under the budget $y = 1$.

We slightly modify the allocation $(\tilde{c}_x, \tilde{k}_x)$ in order to obtain admissible allocations for the problems $L(a_1 u_\gamma)(1)$ and $L(a_0 u_\gamma)(1)$. More precisely, we consider allocations defined by $c_x = \tilde{c}_x + \delta\epsilon'$ and $k_x = \tilde{k}_x - \epsilon'$ where ϵ' is small, for $x = k, l$. Such modification leaves the budget unchanged. These new allocations are admissible for the problem $L(a.u_\gamma)(1)$, where a is a positive real number, with a binding incentive compatibility constraint (IC_l) iff $h(\epsilon') = a$ where

$$h(\epsilon') = \frac{\theta_l(u_\gamma(\tilde{c}_h + \delta\epsilon') - u_\gamma(\tilde{c}_l + \delta\epsilon'))}{\beta\delta(u_\gamma(\tilde{k}_l - \epsilon') - u_\gamma(\tilde{k}_h - \epsilon'))}$$

The function h is smooth and equal to $\tilde{a}$ at $\epsilon' = 0$. Besides, since u_γ is concave, $c_h > c_l$ and $k_l > k_h$, we have $h'(0) < 0$. Thus, we can locally invert this function and set $\epsilon'_i = h^{-1}(a_i)$ for $i = 0, 1$. The corresponding allocations, defined by $k_x^i = \tilde{k}_x - \epsilon'_i$ and $c_x^i = \tilde{c}_x + \delta\epsilon'_i$, are respectively admissible allocation for the problems $L(a_i u_\gamma)(1)$ for $i = 0, 1$. This implies

$$\lambda L(a_0 u_\gamma)(1) + (1-\lambda)L(a_1 u_\gamma)(1) \geq \lambda E[\theta_x u_\gamma(c_x^0) + \delta a_0 u_\gamma(k_x^0)] + (1-\lambda)E[\theta_x u_\gamma(c_x^1) + \delta a_1 u_\gamma(k_x^1)]$$

It remains to show that the right part of this inequality is greater than $L(\tilde{a}u_\gamma)(1)$ when ϵ is small enough. With a slight abuse of notation, we note $O(\epsilon)$ any function which is bounded in the neighborhood of zero. The first-order Taylor expansion of h^{-1} is $h^{-1}(\tilde{a} + \epsilon) = \epsilon/h'(0) + \epsilon^2 O(\epsilon)$. Similarly, we can then write $\epsilon'_0 = -(1-\lambda)\epsilon/h'(0) + \epsilon^2 O(\epsilon)$ and $\epsilon'_1 = \lambda\epsilon/h'(0) + \epsilon^2 O(\epsilon)$. The first order approximation of the expected utility provided by the allocation $\{c_x^i, k_x^i\}$ of the problem $L(a_i u_\gamma)(1)$ is

$$E[\theta_x u_\gamma(\tilde{c}_x + \delta\epsilon'_i) + \delta a_i u_\gamma(\tilde{k}_x - \epsilon'_i)] = E[\theta_x u_\gamma(\tilde{c}_x) + \delta a_i u_\gamma(\tilde{k}_x)] + \delta\epsilon'_i E[\theta_x u'_\gamma(\tilde{c}_x) - a_i u'_\gamma(\tilde{k}_x)] + \epsilon^2 O(\epsilon)$$

which implies, since $\lambda\epsilon_0' + (1-\lambda)\epsilon_1' = \epsilon^2 O(\epsilon)$,

$$\lambda L(a_0 u_\gamma)(1) + (1-\lambda) L(a_1 u_\gamma)(1) \geq L(\tilde{a} u_\gamma)(1) - \delta E[u_\gamma'(\tilde{k}_x)](\underbrace{\lambda\epsilon_0' a_0 + (1-\lambda)\epsilon_1' a_1}_{=\lambda(1-\lambda)\epsilon^2/h'(0)}) + \epsilon^2 O(\epsilon).$$

The desired inequality when ϵ is small enough then follows from $h'(0) < 0$. We have thus proven the curvature property of the function f. □

Proof of Lemma 3: Using lemma 1(iii), classical reasoning detailed in the mathematical appendix implies $L(V_\infty) \geq V_\infty \geq a^* u_\gamma$. If we set $V_\infty = a_\infty u_\gamma$, these inequalities are equivalent to $f(a_\infty) \geq a_\infty \geq a^*$ when $\gamma < 1$, and the reverse inequality when $\gamma > 1$. Properties of the function f, stated in lemma 2, implies then $a_\infty = a^*$, so that $\lim_\infty L^n(V_\infty) = V_\infty = a^* u_\gamma$. The unicity of the two-period optimal allocation, stated in lemma 1, implies that the infinite horizon problem has a unique solution that coincides with $\mathcal{A}^\infty$. This proves the lemma. □

Proof of lemma 4: One of the key ingredient of the proof is to slightly modify the one-step operator L considered in section 5.2 by allowing the possibility for the principal to penalize the high-type utility at the second period (while keeping the same high type's second-period budget). More precisely, we consider two-period allocations defined by consumptions $\{c_x, k_x\}_{x=l,h}$ and a non-negative penalty $\{z_k\}$ for the second period utility of the high type consumer, and the one-step operator $\bar{L}$ that solves the maximization problem on this set of allocations

$$\bar{L}(v)(y) = \max \pi(\theta_h u_\gamma(c_h) + \delta(v(k_h) - z_k)) + (1-\pi)(\theta_l u_\gamma(c_l) + \delta v(k_l)) \qquad (M3)$$

such that $z_k \geq 0$, and under the budget constraint (BC), the usury constraint (UL) and the following incentive compatibility constraints (enriched with possible penalization):

$$\begin{cases} \theta_h u_\gamma(c_h) + \beta\delta(v(k_h) - z_k) \geq \theta_h u_\gamma(c_l - \delta\epsilon) + \beta\delta v(k_l + \epsilon), \forall\epsilon \in [0, c_l/\delta] & (IC_h') \\ \theta_l u_\gamma(c_l) + \beta\delta v(k_l) \geq \theta_l u_\gamma(c_h - \delta\epsilon) + \beta\delta(v(k_h + \epsilon) - z_k), \forall\epsilon \in [0, c_h/\delta] & (IC_l') \end{cases}$$

This translates into a two-period model the possibility for the principal, mentioned in section 5.3, to penalize the borrower by offering him less flexibility later. We show in the mathematical appendix that, when R_u is close enough to $\bar{R}$, there is no such penalization (both in the two-period and the infinite-horizon optimal allocations), the optimal contract constrained by the usury law depends only on the last reported shock and the remaining budget, and the properties stated in lemma 1, 2 and 3 of the previous section hold with a slight difference: the inverse Euler equation is a strict inequality, so that the inverse of the marginal utility is a strict super martingale. Consequently, the arguments developed at the end of section 5.2 can be carried out: the immerisation property still holds. □

Proof of proposition 7: We prove thereafter the case $\gamma = 1$ with $u = \ln(z)$. The case $\gamma < 1$ is proven in the mathematical appendix using a similar method. When the function v is also logarithmic, i.e. of the form $v(z) = A\ln(z) + B$ with $A > 0$, it is easy to see that any optimal allocation of the two-period problems $L(v)$ and $\bar{L}(v)$ has a similar form, that there are unique fixed points V^* and $\bar{V}^*$ for these two operators, which are

$$V^*(z) = \frac{1}{1-\delta}\ln(z) + \frac{L(\tilde{v})(1)}{1-\delta} \quad \text{and} \quad \bar{V}^*(z) = \frac{1}{1-\delta}\ln(z) + \frac{\bar{L}(\tilde{v})(1)}{1-\delta}$$

with $\tilde{v}(z) = \ln(z)/(1-\delta)$, and that Lemma 1 and 3 still hold. We can then extend easily the results presented in the core text to the case with logarithmic utility.

It remains to show that the optimum constrained by the usury law exhibits a faster immerisation trend, which is equivalent to $E\ln(k_x^\dagger) < E\ln(k_x^*)$ as explained at the end of section 5.2. Inverse Euler equations established in lemma 1 and lemma 4 implies $\pi k_h^\dagger + (1-\pi)k_l^\dagger < \pi k_h^* + (1-\pi)k_l^* = 1$. Let's consider the graphs $\mathcal{C}_I$ and $\mathcal{C}_M$ of functions f_I and f_M defined respectively by the conditions $\pi\ln(x) + (1-\pi)\ln(f_I(x)) = E\ln(k_x^*)$ and $\pi x + (1-\pi)f_M(x) = 1$. Let's also consider the graph $\mathcal{C}_F$ that represents for all k_h the solutions k_l of the system of equation defined by budget constraint (BC), the incentive compatibility constraint (IC), and one of the first order conditions derived in the mathematical appendix - cf. relation (19) -, i.e. the system of equations [24]

$$\begin{cases} \pi c_h + (1-\pi)c_l = 1 - \pi\delta k_h - (1-\pi)\delta k_l & \text{(BC)} \\ \theta_l(\ln(c_h) - \ln(c_l)) = \beta\delta(1-\delta)^{-1}(\ln(k_l) - \ln(k_h)) & \text{(IC)} \\ \theta_l k_h - \beta(1-\delta)^{-1}c_h + \theta_h\beta - \theta_l = 0 & \text{(FOC)} \end{cases}$$

This is illustrated in the figure 5 presented in the core text. Those curves intersect at $\mathcal{P} = \{k_l^*, k_h^*\}$. The immerisation trend is worsened, when the usury constraint R_u

24. It is easy to see that $\mathcal{C}_F$ is smooth, with a most a unique k_h for a given k_l, and at most two k_l for a given k_h. We are considering this curve locally around $\mathcal{P}$.

is added, iff $\mathcal{P}' = \{k_l^\dagger, k_h^\dagger\}$ is strictly under the graph $\mathcal{C}_I$. This is a consequence of the following lemma proven in the mathematical appendix. □

Lemma 5. *In the neighborhood of $\mathcal{P}$,*

(i) The graph $\mathcal{C}_M$ is a curve that is decreasing with a less steeper slope than $\mathcal{C}_I$;

(ii) The graph $\mathcal{C}_F$ is a curve that is decreasing with a steeper slope than $\mathcal{C}_I$;

(iii) The point $\mathcal{P}'$ defined by the usury-constrained optimal allocation is strictly under the curves $\mathcal{C}_M$ and $\mathcal{C}_F$.

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8 Online mathematical appendix (not for publication)

8.1 Supplementary material for section 4

8.1.1 Reformulation of the optimal control problem

We recall that the optimization problem (M) defined in the section 3 is to maximize the expected utility

$$\max_{c,k} \int_{\underline{\theta}}^{\bar{\theta}} (\theta u(c(\theta)) + w(k(\theta)))f(\theta) \qquad \text{(EU)}$$

subject to a budget constraint that allows transfer of resources among types and across periods

$$\int (c(\theta) + \delta k(\theta))f(\theta) = \int B(\theta)f(\theta) \leq y \qquad (BC)$$

and to incentive compatibility constraints

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'})) + \beta w(k(\theta^{'}))\ \forall \theta, \theta^{'} \qquad (IC)$$

that may be enriched, as in subsection 4.2, in order to allow for hidden savings

$$\theta u(c(\theta)) + \beta w(k(\theta)) \geq \theta u(c(\theta^{'}) - \delta\Delta) + \beta w(k(\theta^{'}) + \Delta)\ \forall \theta^{'}, \forall \Delta \qquad \text{(HS)}$$

We note $\text{U}(\theta) = u(c(\theta))$ and $\text{W}(\theta) = w(k(\theta))$, and consider the state variable $V(\theta) = \theta\text{U}(\theta) + \beta\text{W}(\theta)$. It is easy to check that the incentive compatibility constraints imply that U is non-decreasing and W is non-increasing. Besides, those functions are bounded since they are defined over a closed interval. From the envelope property (Milgrom and Segal, 2002), we know that the function $\max_{\theta'} \theta\text{U}(\theta^{'}) + \beta\text{W}(\theta^{'})$ is absolutely continuous, and left- and -right-hand differentiable with derivative $\lim_{\theta^{\pm}} \text{U}(\theta^*(\theta))$ where $\theta^*(\theta) \in$ argmax $\theta\text{U}(\theta^{'}) + \beta\text{W}(\theta^{'})$. The conditions "$V$ absolutely continuous with left- and -right-hand derivatives such that $V^{'\pm}(\theta) = \lim_{\theta^{\pm}} \text{U}(\theta)$" together with the condition "U non-decreasing" are then necessary conditions for incentive compatibility. It is straightfoward that these conditions are also sufficient and, when U and W are differentiable, that the first condition is equivalent to $\theta\text{U}^{'}(\theta) = -\beta\text{W}^{'}(\theta)$.

The maximization problem can then be written as a classical constrained optimization problem on the space X of convex and absolutely continuous real-valued functions V on the interval $[\underline{\theta}, \bar{\theta}]$. Since $\text{W}(\theta) = \beta^{-1}V(\theta) - \beta^{-1}\theta\text{U}(\theta)$, it consists in finding the maximum $U^* = \max U(V)$ where

$$U(V) = \int_{\underline{\theta}}^{\bar{\theta}} f_0(\theta, V(\theta), V^{'}(\theta))d\theta \tag{7}$$

and

$$f_0(\theta, x, \text{U}) = (\beta^{-1}x + (1-\beta^{-1})\theta\text{U})f(\theta) \tag{8}$$

under the budget constraint

$$\int_{\underline{\theta}}^{\bar{\theta}} b(\theta, x, \text{U})d\theta \leq y \tag{9}$$

with $b(\theta, x, \text{U}) = B(\theta)f(\theta) = (C(u)+\delta K(\beta^{-1}x - \beta^{-1}\theta\text{U}))f(\theta)$ where C and K are the inverse of the strictly concave utility functions u and w. This maximization problem is convex, since f_0 and g are respectively linear and convex in x and U.

Instead of the hidden storage condition (HS) defined at the beginning of section 4.2, we consider a local condition that is a consequence of (HS) when the control function $\text{U}(\theta)$ is increasing and differentiable. Hidden storage implies $B(\theta)$ non increasing. Since

$$B'(\theta) = \Big(C'(\text{U}(\theta)) - \beta^{-1}\theta\delta K^{'}(\text{W}(\theta))\Big)\,\text{U}^{'}(\theta),$$

this requires

$$h_\theta(x(\theta), \text{U}(\theta)) \geq 0 \qquad \text{with} \qquad h_\theta(x, \text{U}) = \beta^{-1}\delta\theta K^{'}(\beta^{-1}x - \beta^{-1}\theta\text{U}) - C^{'}(\text{U}). \tag{10}$$

In order to take into account the budget constraint, we need to add one dimension to the state variable and consider the function $x(\theta) = (x_1(\theta), x_2(\theta))$ with $x_1(\theta) = V(\theta)$ and $x_2(\theta) = -\int_{\underline{\theta}}^{\theta} B(\tilde{\theta})f(\tilde{\theta})d\tilde{\theta}$. Since $B(\theta) = C(V^{'}(\theta)) + \delta K(\beta^{-1}V(\theta) - \beta^{-1}\theta V'(\theta))$, the control function is

$$\mathcal{F}(\theta, x, \text{U}) = (\text{U}, -g(\theta, x_1, \text{U}))$$

where $g(\theta, x_1, \text{U}) = f(\theta) \times (C(\text{U}) + \delta K(\beta^{-1}x_1 - \beta^{-1}\theta\text{U})$.

We first consider the corresponding relaxed problem in which we rule out "U non-decreasing" constraint and further assume that the control U is bounded. What precedes define an optimal control problem denoted by (OC1), which is to maximize expected utility $\int_{\underline{\theta}}^{\bar{\theta}} f_0(\theta, x_1(\theta), \text{U}(\theta))d\theta$ under the control $x^{'}(\theta) = \mathcal{F}(\theta, x(\theta), u(\theta))$. This problem has furthermore the initial constraint $x_2(\underline{\theta}) = 0$ and terminal constraint

$x_2(\bar{\theta}) \geq -y$ whereas x_1 is free at the end-times $\underline{\theta}$ and $\bar{\theta}$). Besides, we restrict here the control, so that $\text{U} \in [\underline{u}, \bar{u}]$, which is equivalent to add the two additional constraints $u \geq \underline{u}$ and $u \leq \bar{u}$. We denote also by (OC1') this optimal control problem enriched with the mixed constraint (10). Those two problems are defined on the space of absolutely continuous function, that is the Sobolev space $W^{1,1}(\Theta)$ with $\Theta = (\underline{\theta}, \bar{\theta})$. However, when u is bounded, the state variable V has bounded variations, and the budget constraint translates into an upper bound on V. Besides, we can restrain our analysis to admissible solutions such that $U(V) \geq U(V_0)$ where V_0 is the full-bunching admissible solution, and this last condition translates into a lower bound on V. Thus, both the control and the state variable belongs to a fixed bounded region, as well as the resources used at period one $C(V'(\theta))$ and period two $K(\beta^{-1}V(\theta) - \beta^{-1}\theta V'(\theta))$. Therefore, admissible solutions belong to the Sobolev space $W^{1,\infty}(\Theta)$, i.e. the variable state v must be Lipschitz continuous with a bounded Lipschitz constant (and thus almost everywhere differentiable with a derivative $u = v'$ that belongs to the Lebesgue space $L^{\infty}(\Theta)$).

In problem (OC1) and (OC1'), the "non-decreasing" condition on the utility $\text{U}(\theta)$ is relaxed and $\text{U}(\theta)$ is bounded. If we restrict to finding solutions with a piecewise differentiable and non-decreasing utility $\text{U}(\theta)$, we can consider a second optimal control problem, denoted by (OC2), by adding a third dimension to the state variable and consider the function $x(\theta) = (x_1(\theta), x_2(\theta), x_3(\theta))$ with $x_2(\theta) = \text{U}(\theta)$ and $x_3(\theta) = -\int_{\underline{\theta}}^{\theta} B(\tilde{\theta}) f(\tilde{\theta}) d\tilde{\theta}$. The optimal control problem is to maximize $\int_{\underline{\theta}}^{\bar{\theta}} f_0(\theta, x_1(\theta), x_2(\theta)) d\theta$ under the control $x'(\theta) = \tilde{\mathcal{F}}(\theta, x(\theta), v(\theta))$ with a control function defined by $\tilde{\mathcal{F}}(\theta, x, u) = (x_2, v, -g(\theta, x_1, x_2))$. The terminal conditions are the same as before (initial and terminal constraints on x_3, and no end-times constraints on x_1 and x_2), without the previous constraint $\text{U} \in [\underline{u}, \bar{u}]$ but with instead the additional constraint on the control $v \geq 0$. We denote also by (OC2') this optimal control problem enriched with the mixed constraint (10).

8.1.2 Existence and of necessary conditions for a uniform distribution

The case of a uniform distribution of shocks is treated by first considering bounded solutions of the relaxed maximization problem, that is the optimal control (OC1) and (OC1') defined in the previous section. This is necessary in order to prove the existence of an optimum, because usual arguments (such as the coercivity condition of Tonellli's existence theorem) do not apply. We then show thereafter that these bounds can be chosen wide enough so as to be not binding, and that the constraint "U non-decreasing" is not binding as well.

Under those assumptions, the existence of an optimal solution is relatively straightforward. Since the closed unity ball of $W^{1,\infty}(\Theta)$ is compact,[25] if we consider a sequence of admissible solutions V_n whose expected utilities $U(V_n)$ converges to the upper bound U^*, we can extract a subsequence that converges uniformly to another solution V^*. This solution is admissible, because f_0 is linear and g convex, and provides the maximal expected utility U^*. The necessary conditions are described in the following lemma:

Lemma 6. *The maximization program (M) with a bounded period-*1 *utility* $\text{U} \in [\underline{u}, \bar{u}]$ *and without the "non-decreasing" condition on* U *has an optimum which satisfies the following necessary conditions:*

(i) The control U *is continuous on* $[\underline{\theta}, \bar{\theta}]$ *and* C^∞ *on the open subset* $\text{U}^{-1}]\underline{u}, \bar{u}[$*, where it satisfies the Euler-Lagrange equation* (4)*, that is*

$$\frac{\partial}{\partial \theta}\left[f(\theta)\frac{\beta B'(\theta)}{\text{U}'(\theta)}\right] = \frac{\delta f(\theta)}{w'(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial \theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right];$$

(ii) Transversality conditions: if $\text{U}(\underline{\theta})$ *(resp.* $\text{U}(\bar{\theta})$*) belongs to the open subset* $]\underline{u}, \bar{u}[$*, the following relation holds for* $\theta = \underline{\theta}$ *(resp.* $\theta = \bar{\theta}$*)*

$$\frac{1}{u'(c(\theta))} - \frac{\delta\theta}{\beta w'(k(\theta))} = \lambda^{-1}(1-\beta^{-1})\theta; \tag{11}$$

(iii) Inverse Euler equation:

$$\lambda^{-1} = \int \frac{\delta f(\theta)}{w'(k(\theta))} d\theta$$

and, when the bounds $\underline{u}$ *and* $\bar{u}$ *are not binding,*

$$\lambda^{-1} = \int_{\underline{\theta}}^{\bar{\theta}} \frac{f(\theta)}{u'(c(\theta))} d\theta$$

---

25. More precisely, is it compact for the weak topology, but Mazur's lemma allows us to replace our sequence of admissible solutions by a sequence of convex combinations such that we have uniform convergence of the sequence and point-wise convergence almost everywhere of its derivative, cf. Brezis corollary 3.8 p 61. Also, the mixed constraints 10 can be easily rewritten as $\text{U} \leq l_\theta(x_1)$ where l_θ is a smooth function, so weak convergence implies that they hold almost everywhere for the solution V^*.

(iv) There is no money burning, i.e. the budget constraint (9) *is an equality.*

Proof of Lemma 6: Let's consider an optimal solution $(x^*(\theta), \text{U}^*(\theta))$ of the problem (OC1). According to the theorem 15 (p 396) of Seierstad and Sydsaeter (1986), there exist a number $p_0 = 0$ or 1, and a vector of absolutely continuous functions $p(\theta) = (p_1(\theta), p_2(\theta)))$, such that (p_0, p_1, p_2) is never zero, and such that the generalized Lagrangian defined by

$$L(\theta, x, \text{U}, p) = p_0 f_0(\theta, x_1, \text{U}) + p_1(\theta)\text{U} - p_2(\theta) g(\theta, x_1, \text{U})$$

satisfies the Euler equations $dp_i(\theta)/d\theta = -\partial L^*/\partial x_i$ for $i = 1, 2$, and the maximum principle $\partial L^*/\partial \text{U} = 0$ when $\text{U}^* \in]\underline{u}, \bar{u}[$ and $\partial L^*/\partial \text{U} \leq 0$ (resp. ≥ 0) if $\text{U}^* = \underline{u}$ (resp. $\text{U}^* = \bar{u}$).[26] Here * denotes evaluation at $(x^*(\theta), \text{U}^*(\theta))$. All those equations hold almost everywhere. The transversality conditions are $p_1(\bar{\theta}) = p_1(\underline{\theta}) = 0$, and $p_2(\bar{\theta}) \geq 0$ with equality if $x_2(\bar{\theta}) > -y$. The second Euler equation implies that $p_2(\theta)$ is a non negative constant that we denote p_2. We must have $p_0 = 1$. Otherwise, since $\partial g/\partial x_1 > 0$, the first Euler equation implies that p_1 is non-decreasing and must consequently be zero to satisfy the transversality condition, so that $p_2 = 0$ as well, which is impossible. Lastly, if $p_2 = 0$, the first Euler equation implies that $dp_1(\theta)/d\theta = -\beta^{-1} f(\theta)$ which would contradict the transversality equations. Consequently, $p_2 > 0$ and there is no money burning, i.e. $x_2(\bar{\theta}) = -y$.

We now prove the regularity properties of the optimal solution. Let's set $\tilde{L} = f_0 - p_2 g$. The first Euler equation is equivalent to $\dot{p}_1(\theta) = -\partial \tilde{L}^*/\partial x_1$ and the maximum principle can be rewritten as $p_1(\theta) = -\partial \tilde{L}^*/\partial \text{U}$ when $\text{U}^* \in]\underline{u}, \bar{u}[$. Thus, on any open subset I of Θ such that $\text{U}^* \in]\underline{u}, \bar{u}[$, we have the classical Euler-Lagrange equation

$$\left(\frac{\partial}{\partial x_1} - \frac{d}{d\theta}\frac{\partial}{\partial \text{U}}\right)\tilde{L}^* = 0$$

which implies that $\partial \tilde{L}^*/\partial \text{U}$ is continuous, differentiable with respect to θ, and equal up to a constant to the integrand of $\partial \tilde{L}^*/\partial x_1$. Besides, the second order Legendre condition is strictly satisfied, i.e. $\partial^2 \tilde{L}/\partial^2 \text{U} = -p_2 f(\theta)[C''(\text{U}) + \delta\theta^2\beta^{-2}K''(\beta^{-1}x_1 - \beta^{-1}\theta\text{U})] < 0$. Thus, using the implicit function theorem and the continuity of the

26. Formally, it would be necessary to add two Lagrange multipliers in order to take into account the constraint $\text{U} \in [\underline{u}, \bar{u}]$ but this translates directly into the conditions set out above, as is formulated in theorem 8 of Seierstad and Sydsaeter (1986).

state variable, we can invert locally the function $\partial\tilde{L}/\partial\text{U}$, and prove by recurrence that $\text{U}^*(\theta)$ is smooth on I. For the same reason, U cannot make any jump from $\text{U} \in]\underline{u}, \bar{u}[$ to $\bar{u}$ (resp. to $\underline{u}$) since the function $\partial L^*/\partial\text{U}$ would jump from zero to a negative (resp. positive) value, which would contradict the maximum principle (and similarly, there can't be any jump between $\underline{u}$ and $\bar{u}$). Therefore, the function U is continuous.

Lastly, the necessary conditions can be expressed in terms of the consumption $c(.)$ and $w(.)$ that are respectively defined by the relations $u(c(\theta)) = \text{U}(\theta)$ and $\beta w(k(\theta)) = x_1(\theta) - \theta\text{U}(\theta)$. Then, the first derivatives with respect to x_1 and u are

$$\begin{cases} \tilde{L}^*_{x_1} \equiv \frac{\partial \tilde{L}^*}{\partial x_1} = \frac{\partial L^*}{\partial x_1} = \beta^{-1} f(\theta)(1 - p_2\delta K'(\beta^{-1}x_1^*(\theta) - \beta^{-1}\theta\text{U}(\theta)) = \beta^{-1}f(\theta)(1 - \frac{p_2\delta}{w'(k(\theta))}) \\ \tilde{L}^*_u \equiv \frac{\partial \tilde{L}^*}{\partial u} = (1-\beta^{-1})\theta f(\theta) - p_2 f(\theta)[C'(\text{U}(\theta)) - \delta\beta^{-1}\theta K'(\beta^{-1}x_1(\theta) - \beta^{-1}\theta\text{U}(\theta)] \end{cases}$$

and the latter relation can be rewritten as

$$\tilde{L}^*_u = (1-\beta^{-1})\theta f(\theta) - p_2 f(\theta)\left(\frac{1}{u'(c(\theta))} - \frac{\delta\theta}{\beta w'(k(\theta))}\right).$$

Transversality conditions $p_1(\theta) = 0$ at $\underline{\theta}, \bar{\theta}$ imply that integrating the first Euler equation leads to $\int L^*_{x_1} = 0$ which is equivalent to the left part of the inverse Euler equation (5). Besides, the transversality conditions presented in Lemma 6(ii) result directly from the maximum principle, that is $p_1(\theta) = -\tilde{L}^*_u$ when $\text{U} \in]\underline{u}, \bar{u}[$. Lastly, on intervals where U is smooth, the Euler-Lagrange equation $d\tilde{L}^*_u/d\theta = \tilde{L}^*_{x_1}$ is equivalent to

$$\frac{\partial}{\partial\theta}\left[\frac{\beta f(\theta)}{u'(c(\theta))} - \frac{\delta\theta f(\theta)}{w'(k(\theta))}\right] = \frac{\delta f(\theta)}{w'(k(\theta))} - \lambda^{-1}\frac{\partial}{\partial\theta}\left[F(\theta) + (1-\beta)\theta f(\theta)\right]$$

with $\lambda = p_2$. We have $\theta u'(c(\theta))c'(\theta) = -\beta w'(k(\theta))k'(\theta)$ so that $B'(\theta)/\text{U}'(\theta) = 1/u'(c(\theta)) - \delta\theta/\beta w'(k(\theta))$, and the Euler-Lagrange equation is also equivalent to the relation (4). We have also

$$\theta f(\theta) - f(\theta)\frac{\lambda}{u'(c(\theta))} = \theta\tilde{L}^*_{x_1} + \tilde{L}^*_u.$$

When the bounds $\underline{u}$ and $\bar{u}$ are not binding, the maximum principle and the transversality conditions imply respectively that this is equal to the derivative $d(\theta\tilde{L}^*_u)/d\theta$ and that its integral over the whole interval is zero, which is equivalent to the second part of the inverse Euler equation

$$\int_{\underline{\theta}}^{\bar{\theta}} \frac{f(\theta)}{u'(c(\theta))} d\theta = \lambda^{-1}.$$

We have therefore proved the necessary conditions of Lemma 6. □

8.1.3 Proof of proposition 3 when f is uniform

We first show that the necessary conditions described in lemma 6 imply that neither the "non-decreasing" condition nor the (large enough) bounds on the control are binding. Properties (i) and (ii) of Proposition 3 follow from the arguments given in the core text, whereas the property (iii) is proved at the end of this section.

Proof of Proposition 3 (the "non-decreasing" constraint is not binding): The relation (2) holds on intervals where $u \in]\underline{u}, \bar{u}[$ and implies that $B^{'}(\theta)/\mathrm{U}^{'}(\theta) = 1/u^{'}(c(\theta)) - \delta\theta/\beta w^{'}(k(\theta))$, so that we can rewrite Euler-Lagrange equation (4), when the distribution $f(\theta)$ is uniform, as

$$\mathrm{U}^{'}(\theta) \times \left[-\frac{u^{''}(c(\theta))}{u^{'3}(c(\theta))} - \frac{\delta\theta^2 w^{''}(k(\theta))}{\beta^2 w^{'3}(k(\theta))}\right] = \frac{2}{\beta} \times \left[\frac{\delta}{w^{'}(k(\theta))} - \lambda^{-1}(1 - \frac{\beta}{2})\right] \qquad (12)$$

The left term of this equation is non negative iff $\mathrm{U}(\theta)$ is non-decreasing or, equivalently, iff the term $\delta/w^{'}(k(\theta))$ is non-increasing. According to the left part of the inverse Euler equation (5), the right term of this equation is non negative iff $\delta/w^{'}(k(\theta))$ has a value greater or equal to its means time $1 - \frac{\beta}{2}$. This implies that the continuous function $\delta/w^{'}(k(\theta))$ can never take a value under its means time $1 - \frac{\beta}{2}$ (since it would stay there and increase up to this value), and is always non-increasing. Thus, the control $\mathrm{U}(\theta)$ is an increasing function. □

Proof of Proposition 3 (the optimal solution is bounded): We prove that the optimal solution is bounded, as well as the Lagrange multiplier λ, and that those bounds do not depend on the parameter β. According to the previous paragraph, when the distribution of shocks is uniform, c is non-decreasing and k is non-increasing. Therefore, if we denote by θ_0 (resp. θ_1) the infinum (resp. the supremum) of θ such that $u(\theta) \in]\underline{u}, \bar{u}[$, we have $u \in]\underline{u}, \bar{u}[$ on the interval $I =]\theta_0, \theta_1[$. On this interval, according to Lemma 6(i), the Euler-Lagrange equation (4) holds and can be rewritten as

$$\frac{\partial}{\partial\theta}\left[\frac{\beta}{u^{'}(c(\theta))} - \frac{\delta\theta}{w^{'}(k(\theta))}\right] = \frac{\delta}{w^{'}(k(\theta))} - \lambda^{-1}(2 - \beta).$$

We set $\underline{c} = c(\underline{\theta})$, $\bar{c} = c(\bar{\theta})$, $\underline{k} = k(\bar{\theta})$ and $\bar{k} = k(\underline{\theta})$. The integral over I of the left part of the previous relation is

$$\frac{\beta}{u^{'}(\bar{c})} + \frac{\delta\theta_0}{w^{'}(\bar{k})} - \frac{\beta}{u^{'}(\underline{c})} - \frac{\delta\theta_1}{w^{'}(\underline{k})}$$

whereas, according to the inverse Euler equation, the integral of the right part is

$$-\lambda^{-1}(1-\beta)(\bar{\theta}-\underline{\theta}) - (\theta_0 - \underline{\theta})\left(\frac{\delta}{w^{'}(\bar{k})} - \lambda^{-1}(2-\beta)\right) - (\bar{\theta} - \theta_1)\left(\frac{\delta}{w^{'}(\underline{k})} - \lambda^{-1}(2-\beta)\right)$$

We know from the previous paragraph that $\delta/w'(\bar{k}) \geq \lambda^{-1} \geq \delta/w'(\underline{k}) \geq \lambda^{-1}(1-\beta/2)$, so that we can rewrite this to get the following inequality

$$\frac{1}{u'(\bar{c})} \leq \lambda^{-1} \times \left((\bar{\theta}+\theta_1)/2 - \theta_0\right) + \frac{1}{u'(\underline{c})} \tag{13}$$

The budget constraint provides an upper bound on $\underline{c}$ and $\underline{k}$, thus an upper bound on $1/u'(\underline{c})$ and $1/w'(\underline{k})$. The previous inequalities provides therefore upper bounds on c and λ^{-1} which are independent of $\bar{u}$ and β, so that we can get $\theta_1 = \bar{\theta}$ by choosing a sufficiently high $\bar{u}$.

Let's now consider the lower bound. According to Lemma 6(ii), the second transversality condition is an equality, so that we can rewrite the integral of the Euler-Lagrange equation presented in the previous paragraph as

$$\frac{\beta}{u'(\underline{c})} = \left(\frac{\delta}{w'(\bar{k})} - \lambda^{-1}\right) \times (2\theta_0 - \underline{\theta}) + \lambda^{-1}\beta\theta_0 \tag{14}$$

which implies $1/u'(\underline{c}) \geq \lambda^{-1}\theta_0$. If $\underline{c}$ were arbitrarily small, so would be λ^{-1} , and also both $\bar{c}$ and $\bar{k}$ according respectively to relation (13) and (14). This can not be since we must at least get the expected utility obtained with the bunching solution. We get therefore a lower bound on $\underline{c}$ that does not depend on β. We can therefore choose a lower bound that is not binding, and infer then a lower bound on λ^{-1} from the Inverse Euler Equation established in Lemma 6(iii). □

Proof of Proposition 3(iii) (β small): It is shown in the previous paragraph that, when β varies, consumptions $c(.)$ and $k(.)$, as well as the budget multiplier λ and the derivative $u'(c(.))$, remain in compact subsets of $\mathcal{R}^{+*}$. Let's consider a sequence of $\beta_n \to 0$ such that the corresponding parameter λ_n of the optimal solution (c_n, k_n) converges to a real number λ. The transversality conditions imply that the decreasing function $k_n(\theta)$ tends uniformly to a constant $k > 0$ such that $w'(k) = \delta\lambda$. The relation (2), that is $\theta \mathrm{U}'(\theta) = -\beta \mathrm{W}'(\theta)$, implies that the sequence of function $c_n(.)$ tends to a constant c as well. Those solutions satisfy the inverse Euler equation 5 which coincides, at the limit, with the Euler equation $1/u'(c) = \delta/w'(k)$. In other words, the optimal solution converges, when β tends to zero, to the optimal bunching solution. Besides, using relation (2) again, we can write budget variations as

$$B'(\theta) = c'(\theta) + \delta k'(\theta) = c'(\theta) \times \left(1 - \frac{\delta\theta u'(c(\theta))}{\beta w'(k(\theta))}\right)$$

and it is easy to see that it becomes always negative when β is small enough.

It remains to show the concavity of the the implementation curve $\mathcal{K}(c)$, which is defined on the domain spanned by c by the relation $\mathcal{K}(.) = k(c^{-1}(.))$. Its derivative is

$$\mathcal{K}'(c(\theta)) = k'(\theta)/c'(\theta) = -\frac{\theta u'(c(\theta))}{\beta w'(k(\theta))}.$$

From the proof of Proposition 3(iii), we know that this derivative is asymptotically equivalent to $-\theta/\beta\delta$ when β tends to zero. The second derivative is

$$\mathcal{K}''(c(\theta)) \times c'(\theta) = -\frac{u'(c(\theta))}{\beta w'(k(\theta))} - \frac{\theta u''(c(\theta))c'(\theta)}{\beta w'(k(\theta))} + \frac{\theta u'(c(\theta))w''(k(\theta))k'(\theta)}{\beta w'^2(k(\theta))}.$$

It is straightforward from relation (12) that $c'(\theta)/\beta$ tends uniformly to zero when β tends to zero, and this property holds also for $k'(\theta)$ according to relation (2). This implies that $c'\mathcal{K}''$ is asymptotically equivalent to $-1/\beta\delta$ when β tends to zero, which implies $\mathcal{K}'' < 0$. □

8.1.4 General proof of proposition 3

In the following, we relax the assumption of a uniform distribution of shocks, consider any type of distribution with a positive density $f(\theta)$ on the interval $[\underline{\theta}, \bar{\theta}]$, and assume the existence of piecewise-C^1-optimal solutions.

Proof of Proposition 3: We consider the optimal control problem (OC2) and assume, as formulated in proposition 3, the existence of an optimal allocation which is piecewise-C^1. We note this solution $x^*(\theta) = (x_1^*(\theta), x_2^*(\theta), x_3^*(\theta))$ and follow a similar reasonement to that of the previous section, using again the theorem 15 of Seierstad and Sydsaeter (1986): accordingly, there exist a number $p_0 = 0$ or 1, and a vector of absolutely continuous functions $p(\theta) = (p_1(\theta), p_2(\theta)), p_3(\theta))$, such that (p_0, p_1, p_2, p_3) is never zero and the generalized Lagrangian defined by

$$L(\theta, x, v, p) = p_0 f_0(\theta, x_1, x_2) + p_1(\theta)x_2 + p_2(\theta)v - p_3(\theta)g(\theta, x_1, x_2)$$

satisfies the Euler equations $dp_i(\theta)/d\theta = -\partial L^*/\partial x_i$ for $i = 1, 2, 3$ where * denotes evaluation at $(x^*(\theta), v^*(\theta))$. The maximum principle is $p_2(\theta) = \partial L^*/\partial v = 0$ when $v^* > 0$ and $p_2(\theta) = \partial L^*/\partial v \leq 0$ if $v^* = 0$. All those equations hold almost everywhere. The transversality conditions are $p_1(\bar{\theta}) = p_1(\underline{\theta}) = p_2(\bar{\theta}) = p_2(\underline{\theta}) = 0$, and $p_3(\bar{\theta}) \geq 0$ with equality if $x_3(\bar{\theta}) > -y$. The same arguments as before imply that $p_0 = 1$, that

$p_3(\theta)$ is a positive constant that we denote p_3 (i.e. there is no money burning) and, if we set $\tilde{L} = f_0 - p_3 g$, we have again, on any open subset I of Θ such that $v^* > 0$, the classical Euler-Lagrange equation

$$\left(\frac{\partial}{\partial x_1} - \frac{d}{d\theta}\frac{\partial}{\partial x_2}\right)\tilde{L}^* = 0$$

that implies the regularity properties of the optimal solution (that is, the function $\text{U}(\theta)$ is smooth on any such open subset). As before, those necessary conditions can be expressed in terms of consumptions functions $c(.)$ and $k(.)$, defined by the relations $u(c(\theta)) = \text{U}(\theta)$ and $w(k(\theta)) = x_1(\theta) - \theta x_2(\theta)$(we omit again thereafter the superscript *). The constraint $v \geq 0$ implies that those functions are respectively non-decreasing and non-increasing. The first Euler equation can be rewritten as

$$\dot{p}_1(\theta) = -\partial\tilde{L}^*/\partial x_1 = -\beta^{-1} f(\theta)(1 - \frac{p_3\delta}{w'(k(\theta))})$$

The transversality conditions $p_1(\theta) = 0$ at $\underline{\theta}, \bar{\theta}$ implies $\int L^*_{x_1} = 0$ so that $p_3^{-1} = \int \frac{\delta f(\theta)}{w'(k(\theta))} d\theta$. As in the previous section, we set $\lambda = p_3$. Besides, since $w'(k(\theta))$ is non decreasing, the function $p_1(\theta)$ is non-decreasing then non-increasing, and consequently is non negative.

When $\beta = 1$, we have $\partial\tilde{L}^*/\partial x_2 = -p_3 f(\theta) B'(\theta)/\text{U}'(\theta)$. On interval where the solution is non constant ($v > 0$), the Maximum Principle and the second Euler equation imply $p_1(\theta) = -\partial\tilde{L}^*/\partial x_2$, so that $B'(\theta)$ is non-negative. We have thus proved property (i) of proposition 3.

Regarding the case $\beta < 1$, the proof is similar to the one given in the previous section using the following lemma, proven at the end of this section:

Lemma 7. *When β tends to zero, solutions of the optimal control problem (OC2) tend uniformly to the optimal bunching solution defined by defined by $\delta u'(c) = w'(k)$ and $c + \delta k = y$.*

In order to follow the proof given previously, we just need to check that $c'(\theta)/\beta$ still tends uniformly to zero when β tends to zero. In the general case, relation (12) becomes

$$u'(c(\theta))\frac{c'(\theta)}{\beta}\left(-\frac{\beta^2 u''(c(\theta))}{u'^3(c(\theta))} - \frac{\delta\theta^2 w''(k(\theta))}{w'^3(k(\theta))}\right)$$
$$= \frac{2\delta}{w'(k(\theta))} - \frac{2-\beta}{\lambda} - \frac{f'(\theta)}{f(\theta)}\left(\frac{1-\beta}{\lambda}\theta + \frac{\beta}{u'(c(\theta))} - \frac{\delta\theta}{w'(k(\theta))}\right)$$

which yields the desired property since $\frac{\delta\theta}{w'(k(\theta))}$ uniformly tends to $\lambda^{-1} = \int \frac{\delta f(\theta)}{w'(k(\theta))} d\theta$.□

Proof of Lemma 7: Let θ_1 be the infinum of $\theta \in [\underline{\theta}, \bar{\theta}]$ such that $v^*(\theta) > 0$. Then, from the first Euler equation and the transversality condition

$$p_1(\theta_1) = p_1(\theta_1) - p_1(\underline{\theta}) = -\int_{\underline{\theta}}^{\theta_1} \partial\tilde{L}^*/\partial x_1 = -\beta^{-1}(1 - \frac{\delta\lambda}{w'(k(\theta_1))})F(\theta_1)$$

whereas the Maximum Principle and the second Euler equation implies

$$p_1(\theta_1) = -\partial\tilde{L}^*/\partial x_2(\theta_1) = -(1-\beta^{-1})\theta_1 f(\theta_1) + \lambda f(\theta_1)\left(\frac{1}{u'(c(\theta_1))} - \frac{\delta\theta_1}{\beta w'(k(\theta_1))}\right)$$

so that

$$0 \leq (\frac{\delta}{w'(k(\underline{\theta}))} - \lambda^{-1})(F(\theta_1) + \theta_1 f(\theta_1)) = \beta f(\theta_1)(\frac{1}{u'(c(\underline{\theta}))} - \theta_1\lambda^{-1}) \leq \frac{\beta f(\theta_1)}{u'(c(\underline{\theta}))}.$$

The budget condition implies that $c(\underline{\theta}) \leq y$ so that $1/u'(c(\underline{\theta})) \leq 1/u'(y)$. Thus, when β tends to zero, both $\delta/w'(k(\underline{\theta})) - \lambda^{-1}$ and β/λ tends to zero. Let's now consider θ_2 be the supremum of $\theta \in [\underline{\theta}, \bar{\theta}]$ such that $v^*(\theta) > 0$. From the first Euler equation and the transversality condition, we obtain similarly

$$p_1(\theta_2) = -(p_1(\bar{\theta}) - p_1(\theta_2)) = \int_{\theta_2}^{\bar{\theta}} \partial\tilde{L}^*/\partial x_1 = \beta^{-1}(1 - \frac{\delta\lambda}{w'(k(\theta_2))})(1 - F(\theta_2))$$

whereas the Maximum Principle and the second Euler equation implies

$$p_1(\theta_2) = -\partial\tilde{L}^*/\partial x_2(\theta_2) = -(1-\beta^{-1})\theta_2 f(\theta_2) + \lambda f(\theta_2)\left(\frac{1}{u'(c(\theta_2))} - \frac{\delta\theta_2}{\beta w'(k(\theta_2))}\right)$$

and, combining those two relations, we obtain (since $k(\bar{\theta}) = k(\theta_2)$)

$$(\lambda^{-1} - \frac{\delta}{w'(k(\bar{\theta}))})(1 - F(\theta_2) - \theta_2 f(\theta_2)) = \beta f(\theta_2)(\frac{1}{u'(c(\theta_2))} - \theta_2\lambda^{-1}) \geq -\beta f(\theta_2)\theta_2\lambda^{-1}$$

The Inverse Euler equation $\lambda^{-1} = \int \frac{\delta f(\theta)}{w'(k(\theta))} d\theta$ implies that

$$0 \leq (\lambda^{-1} - \frac{\delta}{w'(k(\bar{\theta}))})(1 - F(\theta_2)) \leq (\frac{\delta}{w'(k(\underline{\theta}))} - \lambda^{-1})F(\theta_2)$$

and those terms tends to zero when β tends to zero. It is then easy to see that the previous inequality implies as well that $(\lambda^{-1} - \frac{\delta}{w'(k(\bar{\theta}))})\theta_2 f(\theta_2)$ tends to zero when β tends to zero. Since f has a strictly positive lower bound by assumption, this implies that $\delta/w'(k(\theta)) - \lambda^{-1}$ tends uniformly to zero when β tends to zero, and this is also the case with regard to the variation of the function $\delta/w'(k(\theta))$. Since $E[k(\theta)]$ is bounded by the budget constraint, this implies as well that the variation of the function $k(\theta)$ tends to zero when β tends to zero. Besides, it is quite straightforward that the period-2 consumption in the optimal solution cannot tends to zero. Indeed, if $\lim_{0^+} w(k) = -\infty$, the expected utility tends to $-\infty$ as well. Otherwise, if $\lim_{0^+} w(k) = w(0) \in \mathcal{R}$, then period-1 consumption will tend to the constant $u(y)$, and this solution will provide less expected utility than the optimal bunching solution defined in the lemma. Thus, when β tends to zero, the optimal solution $\{c(\theta), k(\theta)\}$ tends uniformly to the bunching solution. $\square$

8.1.5 Proof of proposition 4

Once again, we consider in the following any type of distribution with a positive density $f(\theta)$ on the interval $[\underline{\theta}, \bar{\theta}]$, and assume the existence of piecewise-C^1-optimal solutions for the optimal control problem considered.

Proof of Proposition 4 (β small): The proof follows easily from the limit property of the optimal solution when β tends to zero expressed in lemma 7. Indeed, if we note $U(\theta, \theta', \epsilon) = \theta u(c(\theta') - \epsilon) + \beta w(k(\theta') + \delta^{-1}\epsilon)$, the hidden storage condition is then equivalent to $U(\theta, \theta', \epsilon) \leq U(\theta, \theta, 0)$ for all $\theta', \theta \in \Theta$ and $\epsilon \in [0, c(\theta')]$. Any admissible solution satisfies the incentive compatibility constraint $U(\theta, \theta', 0) \leq U(\theta, \theta, 0)$. Therefore, a sufficient condition for the hidden storage constraint to hold is that $U(\theta, \theta', \epsilon)$ is non-increasing with respect to ϵ. Its derivative is $-\theta u'(c(\theta') - \epsilon) + \beta\delta^{-1} w'(k(\theta') + \epsilon)$ which is negative when β is small enough according to lemma 7. $\square$

Proof of Proposition 4 ($\beta = 1$): Let's consider the maximization problem (OC2') enriched with the mixed constraint (10), that is $h_\theta(x_1(\theta), x_2(\theta)) \geq 0$ with $h_\theta(x_1, x_2) = \beta^{-1}\delta\theta K'(\beta^{-1}x_1 - \beta^{-1}\theta x_2) - C'(x_2)$. We apply again the theorem of Seierstad and Sydsaeter (1986) mentioned above. The only difference here is that we need to consider one more Lagrange multiplier, i.e. a bounded, measurable and non negative function $q(\theta)$ such that $q(\theta)h_\theta(x_1^*(\theta), x_2^*(\theta))$ is always zero, and modify accordingly the generalized Lagrangian L which becomes

$$L(\theta, x, v, p) = p_0 f_0(\theta, x_1, x_2) + p_1(\theta)x_2 + p_2(\theta)v - p_3(\theta)g(\theta, x_1, x_2) + q(\theta)h_\theta(x_1(\theta), x_2(\theta))$$

At the optimal allocation, it satisfies the Euler equations $dp_i(\theta)/d\theta = -\partial L^*/\partial x_i$ for $i = 1, 2, 3$ where * denotes evaluation at $(x^*(\theta), v^*(\theta))$. The maximum principle is $p_2(\theta) = \partial L^*/\partial v = 0$ when $v^* > 0$ and $p_2(\theta) = \partial L^*/\partial v \leq 0$ if $v^* = 0$. As before, all those equations hold almost everywhere. Transversality conditions are $p_1(\bar{\theta}) = p_1(\underline{\theta}) = p_2(\bar{\theta}) = p_2(\underline{\theta}) = 0$, and $p_3(\bar{\theta}) \geq 0$ with equality if $x_3(\bar{\theta}) > -y$. When $\beta = 1$, we have $f_0(\theta, x_1, x_2) = \beta^{-1} x_1 f(\theta)$ so the first Euler equation is

$$p_1'(\theta) = -p_0 \beta^{-1} f(\theta) + p_3 \frac{\partial g}{\partial x_1}(\theta, x_1^*(\theta), x_2^*(\theta)) - q(\theta) \frac{\partial h_\theta}{\partial x_1}(x_1^*(\theta), x_2^*(\theta))$$

which implies, since $\partial h_\theta / \partial x_1 > 0$, that $p_3 > 0$. Using $\frac{\partial g}{\partial x_2}(\theta, x_1(\theta), x_2(\theta)) = -h_\theta(x_1(\theta), x_2(\theta)) f(\theta)$, the second Euler equation can be rewritten as

$$p_2'(\theta) = -p_1(\theta) - p_3 h_\theta(x_1^*(\theta), x_2^*(\theta)) f(\theta) - q(\theta) \frac{\partial h_\theta}{\partial x_2}(x_1^*(\theta), x_2^*(\theta)).$$

We want to show that $h(\theta) \equiv h_\theta(x_1^*(\theta), x_2^*(\theta))$ is always zero. We use the fact that when $p_2(\tilde{\theta}) < 0$ for some $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}[$, then $h(\theta) > 0$ for θ close to and higher than $\tilde{\theta}$. Indeed, it implies $v^* = 0$ locally around a neighborhood of $\tilde{\theta}$, $x_2^*(\theta)$ is constant and

$$h'(\theta) = \beta^{-1} \delta K'(\beta^{-1} x_1 - \beta^{-1} \theta x_2) > 0.$$

We first show that $h(\bar{\theta})$= 0. When $p_1(\theta) = 0$, since $\partial h_\theta / \partial x_2 < 0$, the second Euler equation implies that $p_2'(\theta) < 0$ is equivalent to $h_\theta(x_1(\theta), x_2(\theta)) > 0$. At $\theta = \bar{\theta}$, we have $p_1(\bar{\theta}) = p_2(\bar{\theta}) = 0$, and if $p_2'(\bar{\theta})$ was negative, then p_2 would be positive for θ close to and lower than $\bar{\theta}$, which would contradict the maximum principle. Therefore $p_2'(\bar{\theta}) \geq 0$, so that $h(\bar{\theta})$ is indeed zero.

Let's now assume that $h(\tilde{\theta}) > 0$ for some $\tilde{\theta} \in [\underline{\theta}, \bar{\theta}[$. Since h is increasing when $v^* = 0$ and since h must decrease in order to take value zero at $\theta = \bar{\theta}$, we can choose $\tilde{\theta}$ so that we have both $h > 0$ and $v^* > 0$ on a local neighborhood around $\tilde{\theta}$. In that case, $p_2 = q = 0$ and the second Euler equation requires p_1 negative. Since $p_1(\underline{\theta}) = 0$, it implies that $\tilde{\theta} > \underline{\theta}$ and that there exist $\theta_0 \in]\underline{\theta}, \tilde{\theta}[$ such that $p_1(\theta_0) = 0$ and $p_1'(\theta_0) \leq 0$. This leads to a contradiction. Indeed, the last term $-q.\partial h_\theta / \partial x_1$ in the first Euler equation is non-positive whereas the first two terms are equal to $r(\theta) f(\theta) \beta^{-1}$ where $r(\theta) = -p_0 + p_3 \delta K'(\beta^{-1} x_1^*(\theta) - \beta^{-1} \theta x_2^*(\theta))$ whose derivative $r'(\theta) = -p_3 \beta^{-1} \theta v^*(\theta) \delta K''(\beta^{-1} x_1^*(\theta) - \beta^{-1} \theta x_2^*(\theta))$ is negative. The multiplier p_1 must be able to increase again in order to reach zero at $p_1(\bar{\theta})$. This implies $r(\theta_0) > 0$, so that $q(\theta_0) > 0$ and $h(\theta_0) = 0$. But the second Euler equation then implies $p_2'(\theta_0) > 0$ since $\partial h_\theta / \partial x_2 < 0$. This requires $p_2(\theta_0) < 0$ and $v = 0$ in order to respect the Maximum principle. As shown previously, h must be strictly positive in a local neighborhood of θ_0 and, in particular, $h(\theta_0) > 0$, which is contradictory. Therefore we have proved by contradiction that $h = 0$, and consequently $B(\theta)$ is constant. □

8.2 Supplementary material for sections 5.2 and 5.3

Proof of Lemma 1 (two-type model): We first consider the problem without the hidden storage constraint, and look at the more general problem at the end. The maximization problem satisfies some basic properties proved thereafter:

— Existence of optima: the budget constraint provides upper bounds for consumptions and, if $\lim u(0^+) = -\infty$, one can find a lower bound $\epsilon > 0$ for consumptions once we restrict to allocations providing a minimum budget. Thus, admissible solutions belong to a compact space, and consequently, optima exists within this space.

— The two inequalities $k_l \geq k_h$ and $c_h \geq c_l$ are strict. If it were not the case, both would be equalities in order to comply to the incentive constraints. In this case, if we increase slightly c_h, and decrease slightly c_l while maintaining the budget constant, then the objective function will increase in proportion since $\theta_h > \theta_l$. On the contrary, if we increase slightly k_l, and decrease slightly k_h while maintaining the budget constant, then the variation of the objective function will be dominated by a constant factor times the square of the decrease of k_l (and negative since the utility function w is strictly concave). If we rewrite the incentive constraints as $\theta_h(u(c_h) - u(c_l)) \geq \delta\beta(v(k_l) - v(k_h)) \geq \theta_l(u(c_h) - u(c_l))$, it is easy to see that we can combine those two approaches in order to improve the expected utility while maintaining the incentive constraints valid.

— No money burning: the budget is entirely used at the optimum, i.e. (BC) is an equality. Otherwise, it is possible to increase c_h and c_l while keeping $u(c_h) - u(c_l)$ constant, thus improving the expected utility and still satisfying the incentive compatibility constraints.

— Unicity of the optimum: this is easily seen by considering the utility of consumptions instead: the objective and the incentive compatibility constraints are then linear whereas the budget constraint is a ceiling on a strictly convex function of those variables. The fact that there is no money burning at an optimal solution implies the unicity since otherwise one would be able to combine optimal solutions to find another one with money burning.

— At the optimum, the constraint (IC_l) is an equality and (IC_h) is a strict inequality. Indeed, if the incentive constraint (IC_l) were a strict inequality, then it would be possible to increase k_h and decrease k_l while maintaining constant $\pi k_h + (1-\pi)k_l$ so that both the (BC) and the (IC) are still valid. Since $v^{'}(k_h) > v^{'}(k_l)$, this improve the objective function. The constraint (IC_l) is thus an equality and, since $\theta_h > \theta_l$, the second constraint (IC_h) is slack.

— The solution of the problem coincides with the solution of the relaxed problem where the constraint (IC_h) is removed. Indeed, the previous arguments can be used to show that such allocation also satisfy $k_l \geq k_h$ and $c_h \geq c_l$, and that the constraint (IC_l) is binding, which implies that (IC_h) is also satisfied.

We can then reduce the maximization problem to the following problem:

$$\max \pi(\theta_h u(c_h) + \delta v(k_h)) + (1-\pi)(\theta_l u(c_l) + \delta v(k_l))$$

subject to budget and incentive compatibility constraint:

$$\begin{cases} \theta_l u(c_h) + \beta\delta v(k_h) \leq \theta_l u(c_l) + \beta\delta v(k_l) & (IC_l) \\ \pi(c_h + \delta k_h) + (1-\pi)(c_l + \delta k_l) \leq y & (BC) \end{cases}$$

If we introduce lagrangian multipliers λ_1 and λ_2 for the two constraints, the first order conditions are given by the following system of equations

$$\begin{cases} \pi\theta_h u^{'}(c_h) - \lambda_1\theta_l u^{'}(c_h) - \lambda_2\pi = 0 \\ \pi\delta v^{'}(k_h) - \lambda_1\beta\delta v^{'}(k_h) - \delta\lambda_2\pi = 0 \\ (1-\pi)\theta_l u^{'}(c_l) + \lambda_1\theta_l u^{'}(c_l) - (1-\pi)\lambda_2 = 0 \\ (1-\pi)\delta v^{'}(k_l) + \lambda_1\beta\delta v^{'}(k_l) - (1-\pi)\delta\lambda_2 = 0 \end{cases}$$

It is easy to recombine those equations to see that they are equivalent to the inverse Euler equation

$$\lambda_2^{-1} = \frac{\pi}{u^{'}(c_h)} + \frac{1-\pi}{u^{'}(c_l)} = \frac{\pi}{v^{'}(k_h)} + \frac{1-\pi}{v^{'}(k_l)} \tag{15}$$

and the following relation

$$\lambda_1 = \frac{(1-\pi)\lambda_2}{\theta_l u^{'}(c_l)} - (1-\pi) = \frac{(1-\pi)\lambda_2}{\beta v^{'}(k_l)} - \frac{1-\pi}{\beta} \tag{16}$$

Let's show that the function $B(\theta)$ is non-decreasing when $\beta = 1$. The two previous relations can be combined to obtain

$$\frac{\beta}{\theta_l u^{'}(c_l)} = \frac{1}{v^{'}(k_l)} - (1-\beta) \times \left(\frac{\pi}{v^{'}(k_h)} + \frac{1-\pi}{v^{'}(k_l)}\right) = \frac{1}{v^{'}(k_l)} - (1-\beta)\lambda_2^{-1} \tag{17}$$

which implies, since $k_l > k_h$,

$$\beta v^{'}(k_l) \leq \theta_l u^{'}(c_l) \leq v^{'}(k_l). \tag{18}$$

These inequalities are strict except if $\beta = 1$. The binding incentive constraint (IC_l) implies

$$\frac{\delta\beta}{\theta_l} = \frac{u(c_h) - u(c_l)}{v(k_l) - v(k_h)} \leq \frac{c_h - c_l}{k_l - k_h}\frac{u^{'}(c_l)}{v^{'}(k_l)} \leq \frac{1}{\theta_l}\frac{c_h - c_l}{k_l - k_h}$$

so $c_l + \beta\delta k_l \leq c_h + \beta\delta k_h$. Thus, when $\beta = 1$, we have $c_h + \delta k_h \geq c_l + \delta k_l$ i.e. the function $B(\theta)$ is non-decreasing.

Let's now show that, when $\beta < \theta_l/\theta_h$, the function $B(\theta)$ is decreasing. Relations (17) and (15) can be combined in order to get

$$\frac{\beta}{u^{'}(c_h)} = \frac{\theta_l}{v^{'}(k_h)} + (\beta\theta_h - \theta_l) \times \left(\frac{\pi}{v^{'}(k_h)} + \frac{1-\pi}{v^{'}(k_l)}\right) = \frac{\theta_l}{v^{'}(k_h)} + \lambda_2^{-1}(\beta\theta_h - \theta_l) \tag{19}$$

which implies that $\theta_l u^{'}(c_h) - \beta v^{'}(k_h)$ has the same sign as $\theta_l - \beta\theta_h$ and, since $k_l > k_h$, that $\theta_h u^{'}(c_h) > v^{'}(k_l)$. The incentive constraint (IC_l) then implies also, when $\beta < \theta_l/\theta_h$,

$$\frac{\delta\beta}{\theta_l} = \frac{u(c_h) - u(c_l)}{v(k_l) - v(k_h)} > \frac{c_h - c_l}{k_l - k_h}\frac{u^{'}(c_h)}{v^{'}(k_h)} > \frac{c_h - c_l}{k_l - k_h}\frac{\beta}{\theta_l} \tag{20}$$

and thus $c_h - c_l < \delta(k_l - k_h)$, i.e. $B(\theta)$ is decreasing.

We now consider the problem with the additional constraint of a non-increasing budget, i.e. $c_h + \delta k_h \leq c_l + \delta k_l$ (a weaker constraint than the hidden storage condition). The previous results imply that this constraint is binding when $\beta = 1$ i.e. there is no transfer of resources among types. It is then easy to check that the first-best solution, defined by $\theta_x u^{'}(c_x) = v^{'}(k_x)$ for $x = l, h$, satisfies the incentive constraint and the no storage condition. Let's call this allocation $\mathcal{A}_1$. It satisfies the Euler equation, i.e. the derivative of the utility delivered by this allocation, with respect to the budget, is equal to $U^{'}(y) = \pi v^{'}(k_h) + (1-\pi)v^{'}(k_l) = \pi u^{'}(c_h) + (1-\pi)u^{'}(c_l)$.

Lastly, it remains to check that the solutions considered here satisfy the hidden storage condition. We saw before that

$$\frac{\theta_h u'(c_l)}{\beta v'(k_l)} > \frac{\theta_l u'(c_l)}{\beta v'(k_l)} \geq 1$$

both in the general case and regarding the allocation $\mathcal{A}_1$. Thus an individual getting the low-type allocation has no incentive to store money. On the other hand, what happen if a low type individual falsely reports and gets the high type allocation? He could then store money in order to maximize his expected utility $\theta_l u(c_h - \delta\epsilon) + \delta\beta v(k_h + \epsilon)$. This is not optimal when $\beta = 1$ since $\mathcal{A}_1$ is the first best allocation with no budget transfer between types. This not optimal either when $\beta < \theta_l/\theta_h$, since we saw previously that $\theta_l u^{'}(c_h) > \beta v^{'}(k_h)$ in that case. Thus, the hidden storage condition is not binding in this latter case, and is equivalent to a non-decreasing budget condition in the case $\beta = 1$.[27] □

<u>*Proof of Lemma 3 (infinite-horizon problem):*</u> Proof of $L(V_\infty) \geq V_\infty \geq a^* u_\gamma$:

— In order to show $L(V_\infty(y)) \geq V_\infty(y)$, let's consider an optimal allocation $c(.)$ for the infinite-horizon problem that delivers the expected utility $V_\infty(y)$.[28] We define a two-period allocation by setting $c_x = c(\theta_x)$ and $k_x = E[\Sigma_{i=1}^{\infty} \delta^{i-1} c(\theta_x, \hat{\theta}^i)]$ for $x = l, h$. Since the allocation $c(\theta_x, \hat{\theta}^t)$ is also an admissible allocation for the infinite-horizon problem with budget k_x, it provides an utility $U^0(\theta_x) = E[\Sigma_{i=1}^{\infty} \delta^{i-1} u(c(\theta_x, \hat{\theta}^i))]$ lower than $V_\infty(k_x)$, so there is a value $k_x' \leq k_x$ such that $U^0(\theta_x) = \bar{V}_\infty(k_x')$. We set $c_l' = c_l$ and $c_h' = c_h$. The allocation $\{c_x', k_x'\}$ is an admissible allocation for the problem $L(V_\infty)(y')$ with $y' = E[c_x' + \delta k_x'] \leq y$. Indeed, according to lemma 1(iii), we just need to check the incentive compatibility constraint for low type and the decreasing budget condition, which is straightforward. Thus, we obtain a two-period allocation which is admissible for the problem $L(V_\infty)(y')$ with $y' \leq y$, and delivers an expected utility greater than $V_\infty(y)$. This implies the desired inequality.

— The inequality $V_\infty \geq a^* u$ results from the fact that the infinite allocation $\mathcal{A}_\infty$ is an admissible allocation for the problem $V_\infty(1)$. Indeed, it is easy to see that

27. If we set $\tilde{\beta} = \theta_l(u(c_h) - u(c_l))/(w(k_l) - w(k_h))$ with the consumption values defined by the allocation $\mathcal{A}_1$, it is easy to see that $\mathcal{A}_1$ is still the optimal allocation (under hidden storage constraint) when $\beta \in [\tilde{\beta}, 1]$, so that the budget function is constant in these cases (no transfer between types). When $\beta \in]\theta_l/\theta_h, \tilde{\beta}[$, allocation $\mathcal{A}_1$ is not admissible, and relation (19) shows that the hidden storage constraint is binding at the optimal allocation.

28. The same argument used at the beginning of the proof of Lemma 1 (coupled with the Tychonov's theorem) implies that admissible solutions of the infinite-horizon problems belong to a compact space, so that optimal solutions exist, and that there is "no money burning" as well.

this allocation delivers $a^* u_\gamma(1)$ as expected utility and that the total budget used by this allocation is equal to one.[29] It remains to check that the incentive compatibility constraints ($ICHS2$) are satisfied. At period t after the history of shocks $\{\hat{\theta}^{t-1}, \theta_x\}$, the expected value of the budget allocated to subsequent periods is $k_x(\hat{\theta}^{t-1}) = k^*_{\theta 1} k^*_{\theta 2} ... k^*_{\theta t-1} k^*_{\theta_x}$ and the expected utility delivered is equal to $a^* u_\gamma(k_x(\hat{\theta}^{t-1}))$. When $\epsilon = 0$, the incentive compatibility constraints are then a direct consequence of the incentive compatibility constraints of the two-period allocation $\mathcal{A}^*$. When $\epsilon > 0$, as mentioned in footnote 9, the allocation of resources with hidden savings must still be incentive compatible. Thus, when one transfers ϵ savings to subsequent periods after history $\{\hat{\theta}^{t-1}, \theta_x\}$, the expected utility cannot be greater than $a^* u_\gamma(k_x(\hat{\theta}^{t-1}) + \delta\epsilon)$. The incentive compatibility constraints with hidden storage follows. □

Proof of lemma 4 (Usury laws and penalization): We recall that we are under the assumption $\beta < \theta_l/\theta_h$. As previously, we rule out the case $\gamma = 1$ which satisfies the same properties and is treated explicitly in the proof of proposition 7. When the usury constraint is binding, the problem defined by $\bar{L}$ is not convex anymore and there can be multiple optimal solutions. The following lemma extends lemma 1 to the operator $\bar{L}$. The property (iv) is a technical condition that is used to ensure the validity of the numerical simulations presented in the core text.

Lemma 8. *Any optimum of the problem defined by the operator $\bar{L}$ satisfies the following property (assuming $\beta < \theta_l/\theta_h$):*

(i) When the usury constraint is binding (strict inverse Euler inequality),

$$\frac{\pi}{u'_\gamma(c_h)} + \frac{1-\pi}{u'_\gamma(c_l)} > \frac{1}{\bar{L}'(v)(1)} > \frac{\pi}{v'(k_h)} + \frac{(1-\pi)}{v'(k_l)};$$

(ii) There is no penalization ($z_k = 0$) at optimal allocations when the usury rate R_u is close enough to the implicit interest rate $\bar{R}$ (defined at the optimal allocation obtained when there is no usury law);

29. Convergence results from the fact, shown at the beginning of the proof of lemma 1, that there is no money burning (i.e. $E[c^*_x + \delta k^*_x] = 1$) and that the budget is not entirely used in the second period (i.e. $\delta(\pi_l k^*_l + \pi_h k^*_h) < 1$). The expected utility a^*_∞ delivered by $\mathcal{A}_\infty$ for a budget $y = 1$ then satisfies the relation $a^*_\infty = E[\theta_x u_\gamma(c_x)] + \delta E[k_x^{1-\gamma} a^*_\infty]$, which is also satisfied by $a_* u_\gamma(1)$. Consequently $a^*_\infty = a_* u_\gamma(1)$.

(iii) The optimum considered in (ii) is then unique and is also the solution of the similar problem in which the incentive compatibility constraint for high type and the hidden storage condition for low type are relaxed;

(iv) A sufficient condition to have no penalization as in (ii) is

$$R_u > \tilde{R} \equiv (1 + \theta_l/\beta - \theta_h - \delta)/\delta \tag{21}$$

This lemma is proven thereafter. It is then easy to show that, when the condition (ii) of the previous lemma is satisfied, the properties stated in lemmas 2 and 3 still hold, i.e. the function $\bar{f}$ defined by $\bar{f}(a) = \bar{L}(au(z))(1)/u(1)$ has a unique fixed point $a^\dagger$, and the corresponding two-period allocation $\mathcal{A}^\dagger$ induces an infinite allocation which corresponds to the optimal solution, noted $\bar{V}_\infty$, of the infinite-horizon problem constrained by the usury law. Indeed, the convexity property of function f are shown in the proof of lemma 2 by using slight modifications of an allocation that leave unchanged the usury constraint (UL). Similarly, the limit property of f are obtained by considering two allocations that satisfy this constraint (since we have $k_l = k_h$ for both of them). Lastly, regarding lemma 3, the proof must be slightly modified in order to prove the inequality $\bar{L}(\bar{V}_\infty) \geq \bar{V}_\infty$. More precisely, we need to follow the same approach, starting for an optimal allocation providing expected utility $\bar{V}_\infty(y)$, and to build a two-period admissible allocation for the problem $\bar{L}(V_\infty(y'))$ for a budget $y' \leq y$. Using the same notation, we can replace k_l by the lower value k'_l since it still satisfy the usury constraint. We may have $k'_h < k_h$ but cannot replace k_h by k'_h without violating the usury constraint. Instead, we can consider the allocation with second-period consumption for high type equal to k_h and a penalty equal to $z_h = \bar{V}_\infty(k_h) - \bar{V}_\infty(k'_h)$. Since we restrict here to the case $\beta < \theta_l/\theta_h$, we just need to check the incentive compatibility constraint for low type without hidden storage, according to lemma 8(iii). Thus, if we keep $c'_l = c_l$ and $c'_h = c_h$, we obtain a two-period allocation that is admissible for the problem $\bar{L}(V_\infty(y'))$ with $y' = E[c'_x + \delta k'_x]$. The desired inequality $\bar{L}(\bar{V}_\infty) \geq \bar{V}_\infty$ follows. Similar arguments then allow us to conclude by showing that $\bar{V}_\infty$ corresponds to the fixed point of $\bar{L}$.

Proof of lemma 8 (two-type model with usury constraint and penalization): We follow the same approach and notation as when proving lemma 1. Let's first consider the basic property of optimal solutions of this maximization problem when the hidden storage constraints are relaxed. Existence follows from the same argument of compacity. The incentive compatibility constraints with no hidden storage can be rewritten as $\theta_l(u(c_h)-u(c_l)) \leq \beta\delta(v(k_l)-v(k_h)+z_h) \leq \theta_h(u(c_h)-u(c_l))$ which implies $c_h \geq c_l$. This latter inequality is strict, otherwise we can improve the allocation by slightly increasing c_h and decreasing c_l while maintaining the budget constant (the usury and the incentive compatibility constraints are then satisfied once z_k is slightly increased as well). We have also $k_l > k_h$. Otherwise, it would imply $z_k > 0$ and we could similarly slightly increase k_l and decrease k_h while maintaining the budget constant and still satisfying the usury constraint. Then, a slight decrease of z_k is necessary to satisfy the incentive compatibility constraints, and the expected utility is strictly improved which yields a contradiction. Also, there is no money burning otherwise, since utility is CRRA, one could improve the allocation by multiplying all consumptions by a scaling factor. The same arguments used in the proof of lemma 1 show that the constraint (IC_l') is an equality, that (IC_h') is a strict inequality, and that the optimal allocations of this problem coincide with the optimal allocations of the similar problem in which (IC_h') is relaxed.

We can then reduce the problem to the maximization (M3) under the budget constraint (BC), the incentive compatibility constraint (IC_l'), the usury constraint (UL) and a positive penalization $z_k \geq 0$. We need to add two non-negative multipliers λ_3 and λ_4 for the two latter constraints. The usury constraint can be rewritten as $c_h - c_{\geq}\delta'(k_l - k_h)$ with $\delta' = 1/(1+R_u)$. The first order equations are then

$$\begin{cases} \pi\theta_h u'(c_h) - \lambda_1\theta_l u'(c_h) - \lambda_2\pi + \lambda_3 = 0 \\ \pi\delta v'(k_h) - \lambda_1\beta\delta v'(k_h) - \delta\lambda_2\pi + \delta'\lambda_3 = 0 \\ (1-\pi)\theta_l u'(c_l) + \lambda_1\theta_l u'(c_l) - (1-\pi)\lambda_2 - \lambda_3 = \\ (1-\pi)\delta v'(k_l) + \lambda_1\beta\delta v'(k_l) - (1-\pi)\delta\lambda_2 - \delta'\lambda_3 = 0 \\ -\pi\delta + \beta\delta\lambda_1 + \lambda_4 = 0 \end{cases}$$

and this is equivalent to the two following sequences of relations

$$1 = \frac{\lambda_2\pi - \lambda_3}{u'(c_h)} + \frac{(1-\pi)\lambda_2 + \lambda_3}{u'(c_l)} = \frac{\lambda_2\pi - \delta'\delta^{-1}\lambda_3}{v'(k_h)} + \frac{(1-\pi)\lambda_2 + \delta'\delta^{-1}\lambda_3}{v'(k_l)} \tag{22}$$

$$\lambda_1 = \frac{(1-\pi)\lambda_2 + \lambda_3}{\theta_l u'(c_l)} - (1-\pi) = \frac{(1-\pi)\lambda_2 + \delta'\delta^{-1}\lambda_3}{\beta v'(k_l)} - (1-\pi)/\beta = \pi/\beta - \lambda_4/\beta\delta \tag{23}$$

The inverse Euler inequality (property (i) of lemma 8) results directly from relation (22): Combined with $k_h < k_l$ and $c_h > c_l$, it implies

$$\frac{\pi}{u'(c_h)} + \frac{(1-\pi)}{u'(c_l)} \geq \lambda_2^{-1} \geq \frac{\pi}{v'(k_h)} + \frac{(1-\pi)}{v'(k_l)}$$

which is strict when the usury constraint is binding, i.e. when $\lambda_3 > 0$ (as before, we have $\lambda_2 = \bar{L}'(v)(1)$ since the Lagrangian λ_2 is the budget multiplier). Also, one can combine relations (22) and (23) in order to obtain a variant of the first order equation (19) obtained in the case with no usury constraint:

$$\frac{\beta}{u'(c_h)} = \frac{\theta_l}{v'(k_h)} + \lambda_2^{-1}(\beta\theta_h - \theta_l) - \lambda_2^{-1}\lambda_3\pi^{-1}\left(\theta_l\frac{\delta'\delta^{-1}}{v'(k_h)} - \frac{\beta}{u'(c_h)}\right) \tag{24}$$

We now prove property (iv) of lemma 8 by looking at what happen when there is penalization at the optimal allocation. When $z_k > 0$, we must have $\lambda_4 = 0$ and $\lambda_1 = \pi/\beta$. Relations (22) and (23) can also be combined to get the additional relations

$$\beta\frac{\lambda_2\pi - \lambda_3}{u'(c_h)} = (\beta\theta_h - \theta_l)\pi + \theta_l\lambda_4/\delta \quad \text{and} \quad \frac{\lambda_2\pi - \delta'\delta^{-1}\lambda_3}{v'(k_h)} = \lambda_4/\delta \tag{25}$$

so that, when $z_k > 0$, we have $\lambda_2\pi = \delta'\delta^{-1}\lambda_3$ which implies, using relation (22), that $\lambda_2 = v'(k_l)$. It provides the two following relations

$$\theta_l u'(c_l) = \frac{1 - \pi + \pi\delta\delta'^{-1}}{\pi/\beta + 1 - \pi}\lambda_3 \quad \text{and} \quad u'(c_h) = \frac{\delta\delta'^{-1} - 1}{\theta_l/\beta - \theta_h}\lambda_3 \tag{26}$$

These relations implies, since $c_h > c_l$, that

$$(1 + \pi(\delta\delta'^{-1} - 1))(1/\beta - \theta_h/\theta_l) > (\delta\delta'^{-1} - 1)(\pi/\beta + 1 - \pi)$$

which can be rewritten as $1 + \theta_l/\beta - \theta_h > \delta\delta'^{-1} = \delta(1 + R_u)$. Reversely, this means that when $R_u > \tilde{R} = (1 + \theta_l/\beta - \theta_h - \delta)/\delta$, we cannot have $z_k > 0$.[30]

Let's then prove property (ii) of lemma 8, i.e. there is no penalization when R_u is close enough to $\bar{R}$. We use a reductio ad absurdum argument: let's assume

30. This implies also, in accordance with our previous results, that there is never penalization when the usury rate is not binding (since we have then $\lambda_3 = 0$).

instead that there is a sequence of optimal allocations with strict penalization, for a sequence of usury rate R_n converging to $\bar{R}$. We can easily see that the sequence of corresponding parameters belong to a compact space. Indeed, consumptions are bounded by the budget constraint. Besides, they are strictly greater than a positive number independent of the usury rate. Otherwise, using the explicit formulas given in the previous paragraph when $z_k > 0$, all consumptions allocated by such optimal solutions would be together arbitrarily close to zero, and the budget constraint would not be binding anymore. The relations obtained in this case also imply that the Lagrangians λ_2 and λ_3 belong to a compact subset of $\mathcal{R}^{+*}$. Consequently, we can extract a subsequence such that the consumptions and all the parameters of the first order conditions converge. This defines an allocation that is admissible and which satisfies the first order allocation with $\lambda_3 > 0$. This allocation is thus optimal for the problem at $R_u = \bar{R}$. However, when $R_u = \bar{R}$, problems with or without the usury constraint coincide, the optimal allocation is unique and with no penalization, i.e. $\lambda_3 = 0$ which yields a contradiction

A consequence of this result is that, when R_u is close enough to $\bar{R}$, the general properties of the optimal allocation when there is no usury constraint are still valid. In particular, the inequalities (18) and $\theta_l u'(c_h) > \beta v'(k_h)$ (also shown in the proof of Lemma 1 when $\beta < \theta_l/\theta_h$) are still valid. Using the same arguments as in lemma 1, it implies that the budget function is decreasing and that the hidden storage constraint is not binding. Lemma 8(iii) follows. □

Proof of lemma 5 (properties of the curves $\mathcal{C}_M$ and $\mathcal{C}_F$ when $\gamma = 1$): Part (i) of the lemma is a consequence of $k_l^* > k_h^*$ since $f_I'(x) = -\pi f_I(x)/(1-\pi)x$ and $f_M'(x) = -\pi/(1-\pi)$. Besides, as shown in lemma 8, the inverse Euler equation is a strict inequality when there is a usury constraint. This property means here that $\pi k_h^\dagger + (1-\pi)k_l^\dagger < 1$, which implies that $\mathcal{P}'$ lies strictly under the curve $\mathcal{C}_M$.

In order to address part (ii) of the lemma, we obtain the slope of the curve $\mathcal{C}_F$ at $\mathcal{P}$ by differentiating the system of equations that defines it (with a slight abuse of notation, we remove thereafter the * to lighten notation in the following paragraph)

$$\begin{cases} \pi dc_h + (1-\pi)dc_l = -\pi\delta dk_h - (1-\pi)\delta dk_l & (a) \\ \theta_l dc_h/c_h - \theta_l dc_l/c_l = \beta\delta(1-\delta)^{-1}(dk_l/k_l - dk_h/k_h) & (b) \\ \beta(1-\delta)^{-1}dc_h = \theta_l dk_h & (c) \end{cases}$$

We eliminate dc_l by combining the two first relations and, using the last relation, obtain the slope of the curve $\mathcal{C}_F$ at $\mathcal{P}$:

$$\frac{dk_l}{dk_h} = -\frac{(1-\pi)\beta\delta(1-\delta)^{-1}/k_h + \theta_l\pi\delta/c_l + ((1-\pi)\theta_l/c_h + \pi\theta_l/c_l)\theta_l(1-\delta)/\beta}{(\theta_l/c_l - \beta(1-\delta)^{-1}/k_l)\,(1-\pi)\delta}$$

which is negative according to relation (18): the curve $\mathcal{C}_F$ is decreasing locally around $\mathcal{P}$. It remains to prove that it is more decreasing at $\mathcal{P}$ than $\mathcal{C}_I$, which is equivalent to $dk_l/dk_h < -\pi k_l/(1-\pi)k_h$. Using the previous relation and the fact that, at the optimal solution, the inverse Euler equation implies $E[c_x] = 1-\delta$ and $E[k_x] = 1$ (no aggregate transfer across periods), this condition can be rewritten as

$$k_l - 1 = \pi(k_l - k_h) < \frac{\beta c_l}{\theta_l(1-\delta)} + \frac{\theta_l(1-\delta)^2 k_h}{\delta\beta c_h}.$$

This inequality is always satisfied, according to the first order condition (17) which can be rewritten, in the case of logarithmic utility, as $k_l = 1 - \beta + \frac{\beta c_l}{\theta_l(1-\delta)}$.

Lastly, we need to show that $\mathcal{P}' = (k_h^\dagger, k_l^\dagger)$ is under the curve $\mathcal{C}_F$. This results from the fact that this point is on a curve $\mathcal{C}'_F$ defined by a similar system of equations with budget constraint (BC) and incentive compatibility constraint (IC), and a slightly different first order condition. We use relation (24) instead, obtained in the proof of Lemma 8 , which gives

$$\theta_l k_h^\dagger - \beta(1-\delta)^{-1}c_h^\dagger + \beta\theta_h - \theta_l = \lambda_3\pi^{-1}\left(\theta_l\frac{\delta'\delta^{-1}}{v'(k_h^\dagger)} - \frac{\beta}{u'_\gamma(c_h^\dagger)}\right) < 0 \qquad (27)$$

where λ_3 is the Lagrangian multiplier associated to the usury law, and $v = \bar{V}_\infty$. The fact that the left term is negative results from relation (20), which holds when R_u is close enough to $\bar{R}$, and which can be rewritten as

$$\frac{u'_\gamma(c_h^*)}{v'(k_h^*)} < \frac{k_l^* - k_h^*}{c_h^* - c_l^*}\frac{\delta\beta}{\theta_l} = (1+\bar{R})\frac{\delta\beta}{\theta_l}$$

where $\bar{\delta} = 1/(1+\bar{R}) = (c_h^* - c_l^*)/(k_l^* - k_h^*)$.

Let's then prove that $\mathcal{C}'_F$ is indeed locally "under" the curve $\mathcal{C}_F$. For any point $\{k_h, k_l\}$ on $\mathcal{C}_F$ close to $\mathcal{P}$, for $\epsilon > 0$ small, we can divide k_l and k_h by $1+\epsilon$, and find an $\epsilon' > 0$ such that, when multiplying c_l and c_h by $1+\epsilon'$, both the budget and the incentive constraint are still respected. Thus, we end up with another point such that the left part of relation (27) is negative. It is easy to see that one can choose ϵ so that this negative value is exactly the value at the right of relation (27). The new point then belongs to $\mathcal{C}'_F$. Consequently, we can move from $\mathcal{C}_F$ to $\mathcal{C}'_F$ by decreasing both k_l and k_h. Since the curve $\mathcal{C}'_F$ is locally decreasing around $\mathcal{P}'$, it implies that, locally around the point $\mathcal{P}$ and $\mathcal{P}'$, the curve $\mathcal{C}'_F$ is under the curve $\mathcal{C}_F$. The property follows. □

Proof of proposition 7(ii) (impact of usury law when $\gamma < 1$): The method is similar to the proof of part (i) of this proposition, except that we consider more generally curves $\mathcal{C}_F(a, \alpha)$ defined implicitly by three equations - the first one being derived from the first order conditions (19) and (24):

$$\begin{cases} \frac{\theta_l}{u'_\gamma(k_h)} - \frac{\beta a}{u'_\gamma(c_h)} + \theta_h\beta - \theta_l = \alpha & \text{(EQ1)} \\ \pi c_h + (1-\pi)c_l = 1 - \pi\delta k_h - (1-\pi)\delta k_l & \text{(BC)} \\ \theta_l(u_\gamma(c_h) - u_\gamma(c_l)) = \beta\delta a(u_\gamma(k_l) - u_\gamma(k_h)) & \text{(IC)} \end{cases}$$

Once again, it is easy to see that this system of equations has at most two real solutions k_l for a given k_h, and at most one solution k_h for a given k_l.[31] We need first to compare the position of the curves $\mathcal{C}_F(a^*, 0)$ and $\mathcal{C}_F(a^\dagger, \alpha)$ where a^* (resp. $a^\dagger$) is the fixed point of the function associated to the operator L (resp. the operator $\bar{L}$), and $\alpha < 0$ corresponds to the value given by the relation (24).

One must to notice that, adding a usury constraint decreases the expected value, i.e. $\bar{L} \leq L$, and that, when $\gamma < 1$, it implies $a^\dagger < a^*$. An important difference between the case $\gamma < 1$ and the case $\gamma > 1$ is that, in the latter case, we have $a^\dagger > a^*$ so that, as shown in the next paragraph, the position of the curve $\mathcal{C}_F(a^*, 0)$ with respect to $\mathcal{C}_F(a^\dagger, \alpha)$ is ambiguous.

Under the asssumption $\gamma < 1$, we need then to show that, for a value a slightly

31. Indeed, if k_l is fixed, then c_h (resp. c_l) is an increasing (a decreasing) function of k_h, and the (IC) can be rewritten with the left part (resp. the right part) being an increasing (resp. a decreasing) function of k_h. If k_h is fixed, then c_h is given, and the relation (IC) can be rewritten with the left part (resp. the right part) being a convex (resp. a concave) function of k_l.

lower than a^*, and α small and negative, the curve $\mathcal{C}_F(a, \alpha)$ is under the curve $\mathcal{C}_F(a^*, 0)$ in a neighborhood of $\mathcal{P}$. Let's choose k_h close to k_h^*, and consider the points (k_h, k_l) and $(\tilde{k}_h, \tilde{k}_l)$, with $\tilde{k}_h = k_h$, respectively on the curve $\mathcal{C}_F(a^*, 0)$ and $\mathcal{C}_F(a, \alpha)$. Those curves are defined by a system of equations that implicitly associates to those two points a unique set of allocations, respectively (c_h, c_l, k_h, k_l) and $(\tilde{c}_h, \tilde{c}_l, \tilde{k}_h, \tilde{k}_l)$. Let's note $dc_x = \tilde{c}_x - c_x$ and $dk_x = \tilde{k}_x - k_x$ the small variations from the first to the second point (for $x = k, l$). We need to show $\tilde{k}_l < k_l$ that is, $dk_l < 0$.

— If α becomes slightly negative, then c_h increases according to the relation (EQ1), i.e. $dc_h > 0$. Relations (BC) and (IC) implies respectively $dc_l + \delta dk_l < 0$ and

$$\theta_l u_\gamma'(c_l) dc_l + \beta\delta a u_\gamma'(k_l) dk_l > 0$$

We know from relation (18) that $\beta a^* u_\gamma^{'}(k_l) \leq \theta_l u_\gamma^{'}(c_l)$ so that $dc_l > 0$ and $dk_l < 0$.

— If a is slightly lowered, then c_h increases according to the relation (EQ1), i.e. $dc_h > 0$. We then have similarly $dc_l + \delta dk_l < 0$ and

$$\theta_l u_\gamma'(c_l) dc_l + \beta\delta a u_\gamma'(k_l) dk_l = \theta_l u_\gamma'(c_h) dc_h - \beta\delta(u_\gamma(k_l) - u_\gamma(k_h)) da > 0$$

which implies as well $dk_l < 0$.

It remains to show that the curve $\mathcal{C}_F(a^*, 0)$ is steeper than the curve $\mathcal{C}_I$ at $\mathcal{P}$. In order to do that, we differentiate the system of equation defining this curve, using the fact that, since u_γ is CRRA, the derivative of $1/u_\gamma'$ is proportionate to $1/u_\gamma$:

$$\begin{cases} \beta a^* u_\gamma(k_h) dc_h = \theta_l u_\gamma(c_h) dk_h \\ \pi dc_h + (1-\pi) dc_l = -\pi\delta dk_h - (1-\pi)\delta dk_l \\ \theta_l u_\gamma'(c_h) dc_h - \theta_l u_\gamma'(c_l) dc_l = \beta\delta a^* (u_\gamma'(k_l) dk_l - u_\gamma'(k_h) dk_h) \end{cases}$$

These relations can be combined in order to obtain the slope of the curve

$$\frac{dk_h}{dk_l} = -\frac{\left(\theta_l u_\gamma'(c_l) - \beta v'(k_l)\right)(1-\pi)}{\pi\theta_l u_\gamma'(c_l) + (1-\pi)\beta v'(k_h) + \theta_l((1-\pi)u_\gamma'(c_h) + \pi u_\gamma'(c_l))\theta_l u_\gamma(c_h)/\beta\delta v(k_h)}$$

where $v = a^* u_\gamma$. This slope is negative according to relation (18). It remains to prove that $dk_h/dk_l > -(1-\pi)k_h/\pi k_l$. A sufficient condition is

$$1 - \frac{\beta v'(k_l)}{\theta_l u_\gamma'(c_l)} = \frac{\theta_l u_\gamma'(c_l) - \beta v'(k_l)}{\theta_l u_\gamma'(c_l)} < \frac{k_h}{k_l}$$

and, according to relation (18), the left term is lower than $1-\beta$. When θ_l and θ_h tends to 1, the optimal solution tends to the solution with no shocks, in which $k_l = k_h$. The inequality is then satisfied as soon as $k_h/k_l > 1 - \beta$. □