On a Stationary Schrödinger Equation with Periodic Magnetic Potential

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Abstract

We prove existence results for a stationary Schrödinger equation with periodic magnetic potential satisfying a local integrability condition on the whole space using a critical value function.

Version 7

1 Introduction and main result

intro

We wish to investigate for which $\lambda > 0$ there is a weak solution to the stationary Schrödinger equation with magnetic potential:

$$\begin{cases} (-\mathrm{i}\,\nabla + A)^2 u + V(x)u = \lambda f(x,|u|)\dfrac{u}{|u|} \ \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

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where $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is the magnetic potential, $B = \operatorname{curl} A$ is the magnetic field, $V : \mathbb{R}^N \longrightarrow \mathbb{C}$, and $f : \mathbb{R}^N \times [0,\infty) \longrightarrow \mathbb{R}$ satisfy some suitable assumptions. Here, $\mathrm{i}^2 = -1$ and in what follows, unless specified, all functions are complex-valued ($H^1(\mathbb{R}^N) = H^1(\mathbb{R}^N;\mathbb{C})$, $L^p(\mathbb{R}^N) = L^p(\mathbb{R}^N;\mathbb{C})$, $\mathscr{D}(\mathbb{R}^N) = \mathscr{D}(\mathbb{R}^N;\mathbb{C})$, etc).

We make assumptions that insure the functional associated with (1.1) is invariant with respect to the transformations $u \longmapsto e^{\mathrm{i}\varphi_y} u(\,.\,+y)$, where φ_y is defined in (3.4) and $y \in \mathbb{Z}^N$. In [7], the authors stated that this set of transformations was a group of dislocations as defined in [9] which is false. In Section 3 we prove that the set D of such transformations is a set of dislocations permitting us to use the profile decomposition theorem [9, Theorem 3.1, p.62-63].

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Arioli and Szulkin [1] treated a similar problem with more general conditions on V (the spectrum of the operator $(-\mathrm{i}\,\nabla + A)^2 + V(x)$ can be negative), but they assume the Rabinowitz condition on the right hand side. We make less restrictive assumptions on the right hand side and introduce a parameter λ and an interval $I_\gamma = (M, \infty) \subset [0, \infty)$ such that for almost every $\lambda \in I_\gamma$ there is a solution to (1.1).

In Section 2 we show that if the magnetic potential $A \in L^N_{\mathrm{loc}}(\mathbb{R}^N)$ then $H^1_A(\mathbb{R}^N) = H^1(\mathbb{R}^N)$ where

$$H^1_A(\mathbb{R}^N) \overset{\text{def}}{=} \Big\{ u \in L^2(\mathbb{R}^N); \nabla u + \mathrm{i} A u \in L^2(\mathbb{R}^N) \Big\}. \tag{1.2}$$ eq:defH1A

In Section 3, we introduce the set of invariant dislocations acting on (1.1) and prove necessary results to the dislocation theorem in [9]. In Section 4 we prove a cocompactness result. In Section 5 we introduce a related critical value function the study of which allows us to obtain our main result.
Throughout this paper, we use the following notation. We denote by $\overline{z}$ the conjugate of the complex number z and by $\mathrm{Re}(z)$ its real part. By $\{Q_j\}_{j\geqslant 1}$ we will denote a countable covering of $\mathbb{R}^N \setminus \mathbb{Z}^N$ by open unit cubes, thus $\mathbb{R}^N = \bigcup_{j\geqslant 1} \overline{Q_j}$, and $Q = (0,1)^N$. For a Banach space X, we denote by $X^\star$ its topological dual and by $\langle\, .\, , .\rangle_{X^\star, X} \in \mathbb{R}$ the $X^\star - X$ duality product and for a Hilbert space H, its (real) scalar product will be denoted by $\langle ., .\rangle_H$. We denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1, \ldots, a_n$, write $C(a_1, \ldots, a_n)$ to indicate that the constant C continuously depends only on $a_1, \ldots, a_n$ (this convention also holds for constants which are not denoted by "C"). Finally, we denote by $2^\star = \frac{2N}{N-2}$ the critical exponent of the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^\star}(\mathbb{R}^N)$, with the convention that $2^\star = \infty$, if $N \leqslant 2$.

We shall make the following assumptions on $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$.

assA **Assumption 1.1.** Let $(e_1, \ldots, e_n)$ be the canonical basis of $\mathbb{R}^N$.

assA1 1. The magnetic potential $A : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ satisfies,

$$\begin{cases} A \in L^{N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^N) \text{ and } \alpha_A \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|A\|_{L^N(Q_j)} < \infty, \text{ for some } \varepsilon > 0, & \text{if } N \geqslant 3, \\ A \in L^{2+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2) \text{ and } \alpha_A \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|A\|_{L^{2+\varepsilon}(Q_j)} < \infty, \text{ for some } \varepsilon > 0, & \text{if } N = 2, \\ A \in L^2_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}) \text{ and } \alpha_A \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|A\|_{L^2(Q_j)} < \infty, & \text{if } N = 1. \end{cases} \tag{1.3}$$ A

assA2 2. A is a <u>$\mathbb{Z}^N$−periodic magnetic potential</u>:

$$\forall j \in [\![1, N]\!],\ \mathrm{curl}\, A(x + e_j) \overset{\mathscr{D}'(\mathbb{R}^N)}{=} \mathrm{curl}\, A(x), \tag{1.4}$$ curlA

where $\operatorname{curl} A \in \mathscr{M}_N\left(\mathscr{D}'(\mathbb{R}^N)\right)$ is the skew-symmetric, matrix-valued distribution with $A_{ij} = \partial_i A_j - \partial_j A_i$. Note that for $N = 1$, (1.4) is always satisfied.

rmkhypA2 **Remark 1.2.** It is easy to see that in Assumption 1.1, (1.4) is equivalent to the condition: for any $y \in \mathbb{Z}^N$, $\operatorname{curl} A(x+y) \overset{\mathscr{D}'(\mathbb{R}^N)}{=} \operatorname{curl} A(x)$. By Lemma 1.1 in Leinfelder [4], (1.4) is also equivalent to: for any $y \in \mathbb{Z}^N$, there exists $\varphi_y \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ ($\varphi_y \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, if $N = 1$) such that for almost every $x \in \mathbb{R}^N$, $A(x+y) = A(x) + \nabla\varphi_y(x)$.

assf **Assumption 1.3.** We will use the following assumptions on V, f, and f. Let $f : \mathbb{R}^N \times [0,\infty) \longrightarrow \mathbb{R}$ be such that $f(x,s)$ is measurable in x and continuous in s and let $F(x,u) \overset{\text{def}}{=} \int_0^u f(x,s)\mathrm{d}s$, for almost every $x \in \mathbb{R}^N$ and any $u \geqslant 0$.

assf1 1. For every $\varepsilon > 0$ and any $2 < p < 2^\star$, there is a $C_{\varepsilon,p}$ such that for almost every $x \in \mathbb{R}^N$ and any $s \geqslant 0$,

$$|f(x,s)| \leqslant \varepsilon(s + s^{2^\star-1}) + C_{\varepsilon,p}s^{p-1}, \tag{1.5}$$ eq-subcritical_bour

if $N \geqslant 3$ and

$$|f(x,s)| \leqslant \varepsilon s + C_{\varepsilon,p}s^{p_\varepsilon-1}, \tag{1.6}$$ eq-subcritical_1_2

if $N \leqslant 2$.

assf2 2. The function f and electric potential $V : \mathbb{R}^N \longrightarrow \mathbb{C}$ are measurable and $\mathbb{Z}^N$-periodic, that is for almost every $(x,y) \in \mathbb{R}^N \times \mathbb{Z}^N$ and any $s \geqslant 0$, $f(x+y,s) = f(x,s)$ and $V(x+y) = V(x)$. We assume that

$$\nu \overset{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\mathbb{R}^N} \mathrm{Re}V(x) > 0. \tag{1.7}$$ V1

assf3 3. The electric potential $V : \mathbb{R}^N \longrightarrow \mathbb{C}$ satisfies,

$$\begin{cases} V \in L^{\frac{N}{2}}_{\mathrm{loc}}(\mathbb{R}^N) \text{ and } \alpha_V \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|V\|_{L^{\frac{N}{2}}(Q_j)} < \infty, & \text{if } N \geqslant 3, \\ V \in L^{1+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^2) \text{ and } \alpha_V \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|V\|_{L^{1+\varepsilon}(Q_j)} < \infty, \text{ for some } \varepsilon > 0, & \text{if } N = 2, \\ V \in L^1_{\mathrm{loc}}(\mathbb{R}) \text{ and } \alpha_V \overset{\text{def}}{=} \sup\limits_{j\in\mathbb{N}} \|V\|_{L^1(Q_j)} < \infty, & \text{if } N = 1. \end{cases} \tag{1.8}$$ V2

defsol **Definition 1.4.** We shall write that u is a *weak solution* of (1.1) if $u \in H^1(\mathbb{R}^N)$ and if u is satisfies (1.1) in $\mathscr{D}'(\mathbb{R}^N)$.

rmkdefsol **Remark 1.5.** The above definition makes sense. Indeed, we have for any $u \in H^1(\mathbb{R}^N)$,

$$(-\mathrm{i}\,\nabla + A)^2u = -\Delta u - \mathrm{i}u\nabla.A - 2\mathrm{i}A.\nabla u + |A|^2u.$$

Then $\Delta u \in H^{-1}(\mathbb{R}^N)$ and, by Assumption 1.1 and Hölder's inequality, $A.\nabla u, |A|^2 u \in L^1_{\rm loc}(\mathbb{R}^N)$. In addition, for any $\varphi \in \mathscr{D}(\mathbb{R}^N)$, $\nabla(u\overline{\varphi}) \in L^2(\mathbb{R}^N)$ with compact support, so that $u\nabla.A \in \mathscr{D}'(\mathbb{R}^N)$, and,

$$\langle \mathrm{i}u\nabla.A, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^N),\mathscr{D}(\mathbb{R}^N)} = -\mathrm{Re}\int\limits_{\mathbb{R}^N} \mathrm{i}A.\nabla(u\overline{\varphi})\mathrm{d}x. \qquad (1.9)$$

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Indeed, denoting by $(\rho_n)_{n\in\mathbb{N}}$ any standard sequence of mollifiers, one has

$$\begin{aligned}
&\langle \mathrm{i}u\nabla.A, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^N),\mathscr{D}(\mathbb{R}^N)} \\
&= \lim_{n\to\infty} \langle \mathrm{i}(\rho_n\star u)\nabla.A, \varphi\rangle_{\mathscr{D}'(\mathbb{R}^N),\mathscr{D}(\mathbb{R}^N)} = \lim_{n\to\infty} \langle \mathrm{i}\nabla.A, \overline{(\rho_n\star u)}\varphi\rangle_{\mathscr{D}'(\mathbb{R}^N),\mathscr{D}(\mathbb{R}^N)} \\
&= -\lim_{n\to\infty} \big\langle \mathrm{i}A, \nabla(\overline{(\rho_n\star u)}\varphi)\big\rangle_{\mathscr{D}'(\mathbb{R}^N),\mathscr{D}(\mathbb{R}^N)} = -\big\langle \mathrm{i}A, \nabla(\overline{u}\varphi)\big\rangle_{L^2(\mathbb{R}^N),L^2(\mathbb{R}^N)}.
\end{aligned}$$

Hence (1.9). In summary, if $u \in H^1(\mathbb{R}^N)$ then $(-\mathrm{i}\,\nabla + A)^2 u \in \mathscr{D}'(\mathbb{R}^N)$.

Below, the main result of this paper.

thm-main

Theorem 1.6. *Let Assumptions 1.1 and 1.3 be satisfied. Then equation (1.1) admits, at least, a non zero weak solution for almost every $\lambda > 0$ sufficiently large.*

2 Another definition of $H^1(\mathbb{R}^N)$

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Definition 2.1. Let A and V satisfy (1.3) and (1.7)–(1.8), respectively. We define $H^1_A(\mathbb{R}^N)$ by,

$$H^1_A(\mathbb{R}^N) = \Big\{u \in L^2(\mathbb{R}^N); \nabla u + \mathrm{i}Au \in L^2(\mathbb{R}^N)\Big\}.$$

We endow $H^1_A(\mathbb{R}^N)$ with the following scalar product and its corresponding norm,

$$\begin{aligned}
&\forall u, v \in H^1_A(\mathbb{R}^N),\ \langle u, v\rangle_{H^1_A(\mathbb{R}^N)} = \mathrm{Re}\int\limits_{\mathbb{R}^N} V u\overline{v}\mathrm{d}x + \mathrm{Re}\int\limits_{\mathbb{R}^N} (\nabla u + \mathrm{i}Au).(\overline{\nabla v + \mathrm{i}Av})\mathrm{d}x, \\
&\forall u \in H^1_A(\mathbb{R}^N),\ \|u\|^2_{H^1_A(\mathbb{R}^N)} = (u,u)_{H^1_A(\mathbb{R}^N)} = \mathrm{Re}\int\limits_{\mathbb{R}^N} V|u|^2\mathrm{d}x + \|\nabla u + \mathrm{i}Au\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}$$

making this space a real pre-Hilbert space, by (1.7) and Lemma 2.4 below.

rmkdefH1A

Remark 2.2. Below are some comments on the definition of the space $H^1_A(\mathbb{R}^N)$.

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1. If $u \in L^2(\mathbb{R}^N)$ then $\nabla u \in H^{-1}(\mathbb{R}^N)$ and $Au \in L^1_{\rm loc}(\mathbb{R}^N)$ (by Cauchy-Schwarz' inequality). So, the definition of $H^1_A(\mathbb{R}^N)$ makes sense and if $u \in H^1_A(\mathbb{R}^N)$ then $\nabla u \in L^1_{\rm loc}(\mathbb{R}^N)$.

rmkdefH1A2

2. In the literature (see for instance Sections 7.19–7.22, p.191–195, of Lieb and Loss [5]), the assumption on A is not $A \in L^N_{\rm loc}(\mathbb{R}^N)$ but merely $A \in L^2_{\rm loc}(\mathbb{R}^N)$. In this case, it can be shown

that $H^1_A(\mathbb{R}^N)$ is a Hilbert space having $\mathscr{D}(\mathbb{R}^N)$ as a dense subset. In addition, if $u\in H^1_A(\mathbb{R}^N)$ then $|u|\in H^1(\mathbb{R}^N)$ and the so-called *diamagnetic inequality* (2.1) below holds. Nevertheless, $H^1(\mathbb{R}^N)\not\subset H^1_A(\mathbb{R}^N)$ and $H^1_A(\mathbb{R}^N)\not\subset H^1(\mathbb{R}^N)$. However, when A has more local integrability then we have $H^1_A(\mathbb{R}^N)=H^1(\mathbb{R}^N)$ (see Theorem 2.3 below). Note that when $N=1$, then our assumption is $A\in L^2_{\mathrm{loc}}(\mathbb{R}^N)$ which is the same hypothesis that we usually find in the literature, and it seems that the fact $H^1_A(\mathbb{R})=H^1(\mathbb{R})$ was never remarked.

rmkdefH1A3 3. If $N\geqslant 2$ and if $A\in L^N_{\mathrm{loc}}(\Omega)$ ($A\in L^{2+\varepsilon}_{\mathrm{loc}}(\Omega)$ if $N=2$) then it can be shown that $H^1_A(\Omega)=H^1(\Omega)$ with equivalent norms for open bounded subsets Ω of $\mathbb{R}^N$ with smooth boundaries (see Lemma 2.3 in Arioli and Szulkin [1]). Actually, it can be shown that the same result holds true for $\Omega=\mathbb{R}^N$ with any $N\geqslant 1$ (see Theorem 2.3 below).

thmHH **Theorem 2.3.** *Let A and V satisfy (1.3) and (1.7)–(1.8), respectively. Then,*

$$H^1_A(\mathbb{R}^N)=H^1(\mathbb{R}^N),$$

with equivalent norms and each term in the integrals of $\langle\,.\,,\,.\,\rangle_{H^1_A(\mathbb{R}^N)}$ belongs to $L^1(\mathbb{R}^N)$.

lemHH **Lemma 2.4.** *Let the assumptions of Theorem 2.3 be fulfilled. Then the following holds.*

lemHH1 *1. If $u\in H^1(\mathbb{R}^N)\cup H^1_A(\mathbb{R}^N)$ then $|u|\in H^1(\mathbb{R}^N)$, $(\nabla u+\mathrm{i}Au)\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and*

$$|\nabla|u|\,|\overset{a.e.}{\leqslant}|\nabla u+\mathrm{i}Au|. \tag{2.1}$$ diamineq

If $u\in H^1(\mathbb{R}^N)$ then $|\nabla|u|\,|\overset{a.e.}{\leqslant}|\nabla u|$.

lemHH2 *2. For any $u\in H^1(\mathbb{R}^N)\cup H^1_A(\mathbb{R}^N)$, $Au\in L^2(\mathbb{R}^N)$, $\sqrt{|V|}\,u\in L^2(\mathbb{R}^N)$, $\|Au\|_{L^2(\mathbb{R}^N)}\leqslant C\alpha_A\|\,|u|\,\|_{H^1(\mathbb{R}^N)}$ and $\|\sqrt{|V|}\,u\|_{L^2(\mathbb{R}^N)}\leqslant C\sqrt{\alpha_V}\,\|\,|u|\,\|_{H^1(\mathbb{R}^N)}$, where $C=C(N)$ ($C=C(N,\varepsilon)$, if $N=2$).*

lemHH3 *3. For any $u,v\in H^1(\mathbb{R}^N)$, $(Au).\nabla v\in L^1(\mathbb{R}^N)$, $|A|^2uv\in L^1(\mathbb{R}^N)$ and we have,*

$$\int_{\mathbb{R}^N}|(Au).\nabla v|\mathrm{d}x\leqslant C\alpha_A\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)},$$
$$\int_{\mathbb{R}^N}|A|^2|uv|\mathrm{d}x\leqslant C^2\alpha_A^2\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)},$$
$$\int_{\mathbb{R}^N}|V||uv|\mathrm{d}x\leqslant C^2\alpha_V\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)},$$

where the constant C is given by Property 2.

Proof. Let $u\in H^1(\mathbb{R}^N)\cup H^1_A(\mathbb{R}^N)$. The proof of 1 is well-known but for the sake of completeness, we recall the main steps. By 1 of Remark 2.2, $u\in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and $\nabla u+\mathrm{i}Au\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. It follows that $|u| \in W^{1,1}_{\rm loc}(\mathbb{R}^N)$ and $\nabla|u| \overset{\rm a.e.}{=} {\rm Re}\left(\frac{\overline{u}}{u}\nabla u\right)$ [1] (Theorem 6.17, p.152, in Lieb and Loss [5]). In particular, $|\nabla|u|\,| \overset{\rm a.e.}{\leqslant} |\nabla u|$, if $u \in H^1(\mathbb{R}^N)$. Since ${\rm Re}\left(\frac{\overline{u}}{|u|}(\nabla u + {\rm i}Au)\right) = {\rm Re}\left(\frac{\overline{u}}{|u|}\nabla u\right) \overset{\rm a.e.}{=} \nabla|u|$, one obtains (2.1). Now, both inequalities in 1 imply that $|u| \in H^1(\mathbb{R}^N)$. Let us prove 2. By the Sobolev embedding $H^1(Q_j) \hookrightarrow L^{2^\star}(Q_j)$ $(N \geqslant 3)$, there exists $C = C(N,|Q_j|)$ such that for any $j \in \mathbb{N}$, $\|u\|_{L^{2^\star}(Q_j)} \leqslant C\|\,|u|\,\|_{H^1(Q_j)}$. Actually, C does not depend on Q_j since for any $j \in \mathbb{N}$, $|Q_j| = 1$. It follows from Hölder's inequality that if $N \geqslant 3$,

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |Au|^2{\rm d}x\right) &= \sum_{j\in\mathbb{N}} \int_{Q_j} |Au|^2{\rm d}x \\
&\leqslant \sum_{j\in\mathbb{N}} \|A\|^2_{L^N(Q_j)}\|u\|^2_{L^{2^\star}(Q_j)} \\
&\leqslant C^2\alpha_A^2 \sum_{j\in\mathbb{N}} \|\,|u|\,\|^2_{H^1(Q_j)} \\
&= C^2\alpha_A^2\|\,|u|\,\|^2_{H^1(\mathbb{R}^N)}.
\end{aligned}$$

If $N = 2$ then the second line is replaced with $\sum_{j\in\mathbb{N}} \|A\|^2_{L^{2+\varepsilon}(Q_j)}\|u\|^2_{L^{\frac{2(2+\varepsilon)}{\varepsilon}}(Q_j)}$ and we use the embedding $H^1(Q_j) \hookrightarrow L^{\frac{2(2+\varepsilon)}{\varepsilon}}(Q_j)$, while if $N = 1$ then the second line is replaced with $\sum_{j\in\mathbb{N}} \|A\|^2_{L^2(Q_j)}\|u\|^2_{L^\infty(Q_j)}$ and we use the embedding $H^1(Q_j) \hookrightarrow L^\infty(Q_j)$. The estimate with V follows in the same way (formally, replace A with $\sqrt{|V|}$). Now, we turn to the proof of 3. Let $v \in H^1(\mathbb{R}^N)$. By Cauchy-Schwarz' inequality and 2 we have,

$$\begin{gathered}
\int_{\mathbb{R}^N} |(Au).\nabla v|{\rm d}x \leqslant \|Au\|_{L^2(\mathbb{R}^N)}\|\nabla v\|_{L^2(\mathbb{R}^N)} \leqslant C\alpha_A\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)}, \\
\int_{\mathbb{R}^N} |A|^2\,|uv|{\rm d}x \leqslant \|Au\|_{L^2(\mathbb{R}^N)}\|Av\|_{L^2(\mathbb{R}^N)} \leqslant C^2\alpha_A^2\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)}, \\
\int_{\mathbb{R}^N} |V||uv|{\rm d}x \leqslant \|\sqrt{|V|}\,u\|_{L^2(\mathbb{R}^N)}\|\sqrt{|V|}\,v\|_{L^2(\mathbb{R}^N)} \leqslant C^2\alpha_V\|u\|_{H^1(\mathbb{R}^N)}\|v\|_{H^1(\mathbb{R}^N)},
\end{gathered}$$

which completes the proof. □

Proof of Theorem 2.3. The last statement of the theorem is due to 3 of Lemma 2.4, once $H^1(\mathbb{R}^N) = H^1_A(\mathbb{R}^N)$ is proved.

- Let $u \in H^1(\mathbb{R}^N)$. By Lemma 2.4, $Au \in L^2(\mathbb{R}^N)$ so that $\nabla u + {\rm i}Au \in L^2(\mathbb{R}^N)$ and $\sqrt{|V|}\,u \in L^2(\mathbb{R}^N)$. It follows that $u \in H^1_A(\mathbb{R}^N)$ and

$$\|\nabla u + {\rm i}Au\|_{L^2(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^N)} + C\alpha_A\|\,|u|\,\|_{H^1(\mathbb{R}^N)} \leqslant (C\alpha_A + 1)\|u\|_{H^1(\mathbb{R}^N)}.$$

Thus $H^1(\mathbb{R}^N) \hookrightarrow H^1_A(\mathbb{R}^N)$, since by Lemma 2.4, ${\rm Re}\int\limits_{\mathbb{R}^N} V|u|^2{\rm d}x \leqslant C^2\alpha_V\|u\|^2_{H^1(\mathbb{R}^N)}$.

- Let $u \in H^1_A(\mathbb{R}^N)$. By Lemma 2.4, $Au \in L^2(\mathbb{R}^N)$ so that $\nabla u = \big((\nabla u + {\rm i}Au) - {\rm i}Au\big) \in L^2(\mathbb{R}^N)$.

[1] $\nabla|u| = 0$, almost everywhere where $u = 0$.

It follows that $u \in H^1(\mathbb{R}^N)$ and by (2.1),

$$\begin{aligned}\|\nabla u\|_{L^2(\mathbb{R}^N)} &\leqslant \|\nabla u + \mathrm{i}Au\|_{L^2(\mathbb{R}^N)} + C\alpha_A \|\,|u|\,\|_{H^1(\mathbb{R}^N)} \\ &\leqslant \|\nabla u + \mathrm{i}Au\|_{L^2(\mathbb{R}^N)} + C\alpha_A \sqrt{\|u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla u + \mathrm{i}Au\|^2_{L^2(\mathbb{R}^N)}} \\ &\leqslant (C\alpha_A + 1)\|u\|_{H^1_A(\mathbb{R}^N)}.\end{aligned}$$

Hence $H^1_A(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$, since by (1.7), $\operatorname{Re} \int\limits_{\mathbb{R}^N} V|u|^2 \mathrm{d}x \geqslant \nu \|u\|^2_{L^2(\mathbb{R}^N)}$. □

rmkA **Remark 2.5.** Let $N \geqslant 3$. Note that in Theorem 2.3 the assumption $A \in L^{N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ is not needed but merely $A \in L^N_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$. It is needed in Lemmas 3.1–3.2 and so in Proposition 3.3 below.

3 The set of dislocations

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lemAA **Lemma 3.1.** *Let $\varepsilon > 0$ and let $A \in L^{N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ ($A \in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, if $N = 1$) satisfying (1.4). Then for any $y \in \mathbb{Z}^N$, there exists a unique continuous function $\psi_y \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ ($\psi_y \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, if $N = 1$) such that*

$$\psi_y(0) = 0, \tag{3.1}$$ lemAApsi0

$$\forall x \in \mathbb{R}^N,\ \psi_y(x-y) + \psi_{-y}(x) = \psi_y(-y) = \psi_{-y}(y), \tag{3.2}$$ psieven

$$A(x+y) = A(x) + \nabla\psi_y(x), \tag{3.3}$$ lemAApsi

for almost every $x \in \mathbb{R}^N$. In particular, $\psi_0 = 0$ over $\mathbb{R}^N$.

Proof. Let $y \in \mathbb{Z}^N$. Uniqueness for ψ_y comes from (3.1) and (3.3), once continuity is proved. By Remark 1.2 and the Sobolev embedding, there exists $\widetilde{\psi_y} \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ ($\widetilde{\psi_y} \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, if $N = 1$) satisfying (3.3) and continuous over $\mathbb{R}^N$. Setting $\psi_y = \widetilde{\psi_y} - \widetilde{\psi_y}(0)$, we see that ψ_y verifies all the desired properties, except (3.2). Notice that the function $x \longmapsto 0$ satisfies (3.3) for $y = 0$, so that $\psi_0 = 0$, by uniqueness. It remains to establish (3.2). Applying (3.3) with y at the point $x - y$ and a second time with $-y$, we obtain for almost every $x \in \mathbb{R}^N$,

$$A(x-y) = A(x) - \nabla\psi_y(x-y) = A(x) + \nabla\psi_{-y}(x).$$

It follows that there exists $c \in \mathbb{R}$ such that,

$$\forall x \in \mathbb{R}^N,\ \psi_y(x-y) + \psi_{-y}(x) = c.$$

Substituting first $x = 0$, then $x = y$ and using (3.1) we obtain (3.2). □

lemphi **Lemma 3.2.** *Let* $\varepsilon > 0$ *and let* $A \in L^{N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ ($A \in L^2_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, *if* $N = 1$) *satisfying* (1.4). *Let* $\big(\psi_y\big)_{y\in\mathbb{Z}^N}$ *be given by Lemma* 3.1. *For any* $y \in \mathbb{Z}^N$, *let* $\varphi_y \in W^{1,N+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R})$ ($\varphi_y \in H^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, *if* $N = 1$) *be defined by,*

$$\varphi_y \overset{def}{=} \psi_y - \frac{1}{2}\psi_y(-y), \tag{3.4}$$ defphi

Then $\varphi_y \in C(\mathbb{R}^N;\mathbb{R})$ *and verifies,*

$$\forall x \in \mathbb{R}^N,\ \varphi_y(x-y) + \varphi_{-y}(x) = 0, \tag{3.5}$$ lemphiequ

$$A(x+y) = A(x) + \nabla\varphi_y(x), \tag{3.6}$$ lemAphi

for almost every $x \in \mathbb{R}^N$. *Finally,* $\varphi_0 = 0$ *over* $\mathbb{R}^N$.

Proof. By Lemma 3.1 and (3.4), we only have to check (3.5). The result then comes from (3.4) and (3.2). □

Assume that A satisfies Assumption 1.1. For any $y \in \mathbb{Z}^N$, we define $g_y \in \mathscr{L}\big(H^1(\mathbb{R}^N)\big)$ as follows.

$$\begin{aligned} g_y : H^1(\mathbb{R}^N) &\longrightarrow H^1(\mathbb{R}^N)\\ u &\longmapsto e^{\mathrm{i}\varphi_y}u(\,\cdot\,+y), \end{aligned}$$

where φ_y is given by (3.4). Indeed, it is clear that $g_y : H^1(\mathbb{R}^N) \longrightarrow L^2(\mathbb{R}^N)$ is linear and continuous. In addition, for any $y \in \mathbb{Z}^N$ and $u \in H^1(\mathbb{R}^N)$,

$$\begin{aligned} &\nabla(g_yu) = \big(\nabla u(\,.\,+y) + \mathrm{i}u(\,.\,+y)\nabla\varphi_y\big)e^{\mathrm{i}\varphi_y},\\ &|\nabla\varphi_y|^2 \in L^{\frac{N}{2}}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}) \text{ and } |u(\,.\,+y)|^2 \in L^{\frac{2^\star}{2}}(\mathbb{R}^N;\mathbb{R}), \text{ if } N \geqslant 3,\\ &|\nabla\varphi_y|^2 \in L^{\frac{2+\varepsilon}{2}}_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}) \text{ and } |u(\,.\,+y)|^2 \in L^{\frac{2+\varepsilon}{\varepsilon}}(\mathbb{R}^2;\mathbb{R}), \text{ if } N = 2,\\ &|\nabla\varphi_y|^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{R}) \text{ and } |u(\,.\,+y)|^2 \in L^\infty(\mathbb{R};\mathbb{R}), \text{ if } N = 1. \end{aligned}$$

from which we deduce, with help of Lemma 2.4, Hölder's inequality and the Sobolev embedding, that $g_y : H^1(\mathbb{R}^N) \longrightarrow H^1(\mathbb{R}^N)$ is well-defined, linear and

$$\|\nabla(g_yu)\|_{L^2(\mathbb{R}^N)} \leqslant \|\nabla u\|_{L^2(\mathbb{R}^N)} + 2C\alpha_A\|\,|u|\,\|_{H^1(\mathbb{R}^N)} \leqslant C'\|u\|_{H^1(\mathbb{R}^N)}.$$

It follows that for any $y \in \mathbb{Z}^N$, $g_y \in \mathscr{L}\big(H^1(\mathbb{R}^N)\big)$ with $\|g_y\|_{\mathscr{L}(H^1(\mathbb{R}^N))}$ independent of y. Let

$$D \overset{\text{def}}{=} \big\{g_y; y \in \mathbb{Z}^N\big\}. \tag{3.7}$$ D

propDD1 **Proposition 3.3.** *Let D be defined by (3.7). Then D is a set of unitary operators on $H^1(\mathbb{R}^N)$ with respect to the norm $\| \,.\, \|_{H^1_A(\mathbb{R}^N)}$ defined in Definition 2.1. In addition,*

$$g_0 = \mathrm{Id}, \tag{3.8}$$ propDD1-0

$$g_y^{-1} = g_{-y}, \tag{3.9}$$ propDD1-1

$$\langle g_y u, g_y v\rangle_{H^1_A(\mathbb{R}^N)} = \langle u, v\rangle_{H^1_A(\mathbb{R}^N)}, \tag{3.10}$$ propDD1-2

for any $y \in \mathbb{Z}^N$ and $u, v \in H^1(\mathbb{R}^N)$.

Proof. Recall that D is set of bounded linear operators on $H^1(\mathbb{R}^N)$. By Lemma 3.2, $\varphi_0 = 0$ so that $g_0 = \mathrm{Id}$. Let $y \in \mathbb{Z}^N$ and let $u \in H^1(\mathbb{R}^N)$. For almost every $x \in \mathbb{R}^N$, one has,

$$g_y\big(g_{-y}u\big)(x) = e^{\mathrm{i}\varphi_y(x)}\big(g_{-y}u\big)(x+y) = e^{\mathrm{i}\varphi_y(x)}e^{\mathrm{i}\varphi_{-y}(x+y)}u(x) = u(x),$$

where we have used (3.5) in the last equality. Still with (3.5), we show that $g_{-y}\big(g_y u\big) = u$. It follows that g_y is invertible and $g_y^{-1} = g_{-y}$. Now, let $v \in H^1(\mathbb{R}^N)$. By a straightforward calculation and with help of (3.5) again and (3.6), we obtain

$$\langle u, g_y^\star v\rangle_{H^1_A(\mathbb{R}^N)} \stackrel{\text{def}}{=} \langle g_y u, v\rangle_{H^1_A(\mathbb{R}^N)} = \langle u, g_y^{-1} v\rangle_{H^1_A(\mathbb{R}^N)},$$

so that, $g_y^\star = g_y^{-1}$ which concludes the proof. □

lemD **Lemma 3.4.** *Let $(y_k)_k \subset \mathbb{Z}^N$. Then,*

$$g_{y_k} \longrightarrow 0 \iff |y_k| \xrightarrow{k\to\infty} \infty.$$

Moreover if $g_{y_k} \not\longrightarrow 0$ then $\big(g_{y_k}\big)_k$ admits a constant subsequence.

Proof. Let $(y_k)_k \subset \mathbb{Z}^N$.

Step 1: If $\liminf\limits_{k\to\infty} |y_k| < \infty$ then $(y_k)_k$ admits a constant subsequence.

Indeed, if $\liminf\limits_{k\to\infty} |y_k| < \infty$ then $(y_k)_k$ admits a bounded subsequence, from which we extract a convergent subsequence $\big(y_{k_\ell}\big)_\ell$. Since $\big(y_{k_\ell}\big)_\ell$ converges in $\mathbb{Z}^N$, Step 1 follows.

Step 2: Proof of $\Longrightarrow$.

We show the contraposition. Assume that $\liminf\limits_{k\to\infty} |y_k| < \infty$. By Step 1, there exists $\big(y_{k_\ell}\big)_\ell \subset (y_k)_k$ such that for any $\ell \in \mathbb{N}$, $y_{k_\ell} = y_{k_1}$. Let $v \in H^1(\mathbb{R}^N) \setminus \{0\}$ and $u = g_{y_{k_1}}^{-1} v$. It follows that,

$$\forall \ell \in \mathbb{N},\ \langle g_{y_{k_\ell}} u, v\rangle_{H^1_A(\mathbb{R}^N)} = \|v\|^2_{H^1_A(\mathbb{R}^N)} > 0,$$

and so, $g_{y_k} \not\longrightarrow 0$.

Step 3: Proof of $\Longleftarrow$.

Assume $|y_k| \xrightarrow{k\to\infty} \infty$. Let $\varphi,\psi \in \mathscr{D}(\mathbb{R}^N)$. Then for any $k\in\mathbb{N}$ large enough, $\operatorname{supp}(g_{y_k}\varphi)\cap\operatorname{supp}\psi=\emptyset$, so that,

$$\langle g_{y_k}\varphi,\psi\rangle_{H^1_A(\mathbb{R}^N)} \xrightarrow{k\to\infty} 0. \tag{3.11}$$ prooflemD

Let $u,v\in H^1(\mathbb{R}^N)$. Let $\varepsilon>0$. By density and Theorem 2.3, there exists $(\varphi_n)_n,(\psi_n)_n \subset \mathscr{D}(\mathbb{R}^N)$ such that, $\varphi_n \xrightarrow[n\to\infty]{H^1_A(\mathbb{R}^N)} u$ and $\psi_n \xrightarrow[n\to\infty]{H^1_A(\mathbb{R}^N)} v$. Let $n_0\in\mathbb{N}$ be such that,

$$\|v\|_{H^1_A(\mathbb{R}^N)}\|u-\varphi_{n_0}\|_{H^1_A(\mathbb{R}^N)} + \|\varphi_{n_0}\|_{H^1_A(\mathbb{R}^N)}\|v-\psi_{n_0}\|_{H^1_A(\mathbb{R}^N)} \leqslant \varepsilon,$$

for any $n\geqslant n_0$. We then infer with help of (3.10), that for any $k\in\mathbb{N}$,

$$\begin{aligned}
&|\langle g_{y_k}u,v\rangle_{H^1_A}| \leqslant |\langle g_{y_k}(u-\varphi_{n_0}),v\rangle_{H^1_A}| + |\langle g_{y_k}\varphi_{n_0},v-\psi_{n_0}\rangle_{H^1_A}| + |\langle g_{y_k}\varphi_{n_0},\psi_{n_0}\rangle_{H^1_A}|\\
&\leqslant \|v\|_{H^1_A}\|u-\varphi_{n_0}\|_{H^1_A} + \|\varphi_{n_0}\|_{H^1_A}\|v-\psi_{n_0}\|_{H^1_A} + |\langle g_{y_k}\varphi_{n_0},\psi_{n_0}\rangle_{H^1_A}|\\
&\leqslant \varepsilon + |\langle g_{y_k}\varphi_{n_0},\psi_{n_0}\rangle_{H^1_A}|.
\end{aligned}$$

By (3.11), if follows that: $\limsup\limits_{k\to\infty}|(g_{y_k}u,v)_{H^1_A(\mathbb{R}^N)}| \leqslant \varepsilon$. Since $\varepsilon>0$ is arbitrary, we then get that for any $u,v\in H^1(\mathbb{R}^N)$, $\big(g_{y_k}u,v\big)_{H^1_A(\mathbb{R}^N)} \xrightarrow{k\to\infty} 0$, which is the desired result.

Step 4: If $g_{y_k} \not\rightharpoonup 0$ then $\big(g_{y_k}\big)_k$ admits a constant subsequence.

Now assume that $g_{y_k} \not\rightharpoonup 0$. By Steps 2–3, this means $\liminf\limits_{k\to\infty}|y_k|<\infty$, and we conclude with help of Step 1. □

propDD2 **Proposition 3.5.** *Let D be defined by (3.7). Then D is a set of dislocations on $(H^1(\mathbb{R}^N),\|.\|_{H^1_A(\mathbb{R}^N)})$.*

Proof. By Proposition 3.1 p.61 in Fieseler and Tintarev [9], it is sufficient to show that if $(y_k)_k\subset\mathbb{Z}^N$ is such that $g_{y_k}\not\rightharpoonup 0$ then g_{y_k} has a strongly convergence subsequence. This is a consequence of Lemma 3.4. □

4 Cocompactness

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Let D be defined as in Section 3.

thmcoc **Theorem 4.1.** *Let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^N)$. Let $p\in(2,2^\star)$ ($p\in(2,\infty)$ if $N=2$, $p\in(2,\infty]$ if $N=1$). Then we have the following result.*

$$u_k \xrightarrow{D} 0 \iff u_k \xrightarrow[k\to\infty]{L^p(\mathbb{R}^N)} 0.$$

Proof. Let $(u_k)_{k\in\mathbb{N}}$ a bounded sequence in $H^1(\mathbb{R}^N)$ be such that $u_k \xrightarrow{D} 0$. Let p be as in the theorem with $p<\infty$. We claim that,

$$\forall k\in\mathbb{N},\ \exists y_k\in\mathbb{Z}^N \text{ such that } \sup_{y\in\mathbb{Z}^N}\int\limits_{Q-y}|u_k|^p\mathrm{d}x=\int\limits_{Q}|g_{y_k}u_k|^p\mathrm{d}x. \tag{4.1}$$ demthmcoc

Indeed, if $\sup\limits_{y\in\mathbb{Z}^N}\int\limits_{Q-y}|u_k|^p\mathrm{d}x=0$, there is nothing to prove. If $\sup\limits_{y\in\mathbb{Z}^N}\int\limits_{Q-y}|u_k|^p\mathrm{d}x=\delta>0$ then if the supremum in y was not a maximum then there would be an infinite number of $y\in\mathbb{Z}^N$ such that $\int\limits_{Q-y}|u_k|^p\mathrm{d}x>\frac{\delta}{2}$, contradicting the fact that $(u_k)_k$ is bounded in $H^1(\mathbb{R}^N)$.
By the Sobolev embedding $H^1(Q)\hookrightarrow L^p(Q)$ and translation, there exists $C>0$ such that for any $k\in\mathbb{N}$ and $y\in\mathbb{Z}^N$, $\|u_k\|^2_{L^p(Q-y)}\leqslant C\|u_k\|^2_{H^1(Q-y)}$. Multiplying the both sides by $\|u\|^{p-2}_{L^p(Q-y)}$, we get

$$\int\limits_{Q-y}|u_k|^p\mathrm{d}x\leqslant C\|u_k\|^2_{H^1(Q-y)}\left(\int\limits_{Q-y}|u_k|^p\mathrm{d}x\right)^{\frac{p-2}{p}}.$$

Summing over $y\in\mathbb{Z}^N$, we obtain for any $k\in\mathbb{N}$,

$$\|u_k\|^p_{L^p(\mathbb{R}^N)}\leqslant C\|u_k\|^2_{H^1(\mathbb{R}^N)}\sup_{y\in\mathbb{Z}^N}\left(\int\limits_{Q-y}|u_k|^p\mathrm{d}x\right)^{\frac{p-2}{p}}.$$

For any $k\in\mathbb{N}$, let $y_k\in\mathbb{Z}^N$ be given by (4.1). Noticing that $\sup\limits_{k\in\mathbb{N}}\|u_k\|_{H^1(\mathbb{R}^N)}<\infty$, we infer from the compactness of the Sobolev embedding $H^1(Q)\hookrightarrow L^p(Q)$ that

$$\forall k\in\mathbb{N},\ \|u_k\|^p_{L^p(\mathbb{R}^N)}\leqslant C\|g_{y_k}u_k\|^{p-2}_{L^p(Q)}\xrightarrow{k\to\infty}0,$$

since $g_{y_k}u_k\rightharpoonup 0$ in $H^1_{\mathrm{w}}(\mathbb{R}^N)$. When $N=1$ and $p=\infty$, we use the above result and Gagliardo-Nirenberg's inequality to see that,

$$\|u_k\|_{L^\infty(\mathbb{R})}\leqslant C\|u_k\|^{\frac{2}{3}}_{L^4(\mathbb{R})}\|u_k\|^{\frac{1}{3}}_{H^1(\mathbb{R})}\leqslant C\|u_k\|^{\frac{2}{3}}_{L^4(\mathbb{R})}\xrightarrow{k\to\infty}0.$$

To prove the converse assume that for some $p\in(2,2^\star)$ ($p\in(2,\infty)$ if $N=2$, $p\in(2,\infty]$ if $N=1$), $u_k\xrightarrow[k\to\infty]{L^p(\mathbb{R}^N)}0$. Note that if $N=1$ and $p=\infty$ then,

$$\|u_k\|^2_{L^4(\mathbb{R})}\leqslant\|u_k\|_{L^2(\mathbb{R})}\|u_k\|_{L^\infty(\mathbb{R})}\leqslant C\|u_k\|_{L^\infty(\mathbb{R})}\xrightarrow{k\to\infty}0.$$

So we may assume that $p<\infty$. Let $(g_k)_k\in D$. Since for any $k\in\mathbb{N}$, $\|g_ku_k\|_{L^p(\mathbb{R}^N)}=\|u_k\|_{L^p(\mathbb{R}^N)}$ and $\|g_ku_k\|_{H^1_A(\mathbb{R}^N)}=\|u_k\|_{H^1_A(\mathbb{R}^N)}$ by (3.10), we obtain that for some $\left(g_{k_\ell}\right)_\ell\subset(g_k)_k$ and $u\in H^1(\mathbb{R}^N)$,

$$g_ku_k\longrightarrow 0,\ \text{in } L^p(\mathbb{R}^N),\ \text{as } k\to\infty,$$
$$g_{k_\ell}u_{k_\ell}\rightharpoonup u,\ \text{in } H^1_{\mathrm{w}}(\mathbb{R}^N),\ \text{as } \ell\to\infty.$$

In particular, both convergences hold in $\mathscr{D}'(\mathbb{R}^N)$ so that $u = 0$ and $g_k u_k \xrightarrow{H^1_w} 0$, for the whole sequence $(g_k u_k)_k$. This concludes the proof. □

5 An associated critical value function and proof of the main result

lue_function

Let

$$\psi(u) \overset{\text{def}}{=} \int_{\mathbb{R}^N} F(x, |u|)\mathrm{d}x. \tag{5.1}$$

eq-definition_psi

The functional ψ is of class $C^1(H^1(\mathbb{R}^N);\mathbb{R})$, $\psi'(u) = f(\,.\,, |u|)\frac{u}{|u|}$ and ψ and ψ' are bounded on bounded sets [2, Proposition 3.2.5, p.60]. We note also that by compact Sobolev embeddings, if $(u_k)_k \subset H^1(\mathbb{R}^N)$ and $u_k \xrightarrow{H^1_w} u$ then $\psi'(u_k) \xrightarrow{H^{-1}_w} \psi'(u)$ since $\mathscr{D}(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$. If $(u_k)_k \subset H^1(\Omega) \bigcap H^1(\mathbb{R}^N)$ where $\Omega \subset \mathbb{R}^N$ is bounded then $\psi(u_k) \to \psi(u)$.

Let $S_t \overset{\text{def}}{=} \{u \in H^1_A(\mathbb{R}^N); \|u\|^2_{H^1_A(\mathbb{R}^N)} = t\}$, $B_t \overset{\text{def}}{=} \{u \in H^1_A(\mathbb{R}^N); \|u\|^2_{H^1_A(\mathbb{R}^N)} \leqslant t\}$,

$$\gamma(t) \overset{\text{def}}{=} \sup_{u \in S_t} \psi(u), \tag{5.2}$$

eq-critical_value_f

and $\Sigma_t \overset{\text{def}}{=} \{u \in S_t; \psi(u) = \gamma(t)$. Furthermore let

$$I_\gamma \overset{\text{def}}{=} \left(2 \inf_{t \neq s} \frac{\gamma(t) - \gamma(s)}{t - s}, 2 \sup_{t \neq s} \frac{\gamma(t) - \gamma(s)}{t - s}\right) \tag{5.3}$$

eq-interval

and

$$G_\rho(u) \overset{\text{def}}{=} \frac{\rho}{2}\|u\|^2_{H^1_A(\mathbb{R}^N)} - \psi(u). \tag{5.4}$$

eq-Grho

Note that if we find a $w_\rho \in H^1_A(\mathbb{R}^N)$ such that $G'_\rho(w_\rho) = 0$ then w_ρ is a weak solution to (1.1) with $V \equiv 1$ and $\lambda = 1/\rho$.

lem51 **Lemma 5.1.** *Assume 1 of Assumption 1.3. Then $\gamma(t)$ is locally Lipschitz continuous and nondecreasing in t. For every $\alpha \in [0, t]$*

$$\gamma(\alpha) + \gamma(t - \alpha) \leqslant \gamma(t). \tag{5.5}$$

eq-lions_split_inec

Proof. Let $u \in H^1_A(\mathbb{R}^N)$ and $\theta > 0$. Let $(v_k)_{k\in\mathbb{N}} \subset S_1$ be such that $v_k \xrightarrow{H^1_w} 0$ and $\operatorname{supp} v_k \subset Q$. Then $\psi(u + \theta v_k) \longrightarrow \psi(u)$ and $\|u + \theta v_k\|^2_{H^1_A(\mathbb{R}^N)} \longrightarrow \|u\|^2_{H^1_A(\mathbb{R}^N)} + \theta$.
Let $(u_k)_{k\in\mathbb{N}} \subset S_t$ be a maximizing sequence of $\gamma(t)$. Since $\mathscr{D}(\mathbb{R}^N)$ is dense in $H^1_A(\mathbb{R}^N)$ we may find $(y_k)_k \subset \mathbb{R}^N$, with $\lim_{k\to\infty} |y_k| = \infty$, such that $\psi(u + u_k(\cdot + y_k)) \longrightarrow \psi(u) + \gamma(t)$.

Since ψ' is bounded on bounded sets, [2, Proposition 3.2.5, p.60] we conclude that for $u \in B_t$,

$$\langle \psi'(u), u \rangle_{H^{-1},H^1} \leqslant C_t.$$

The result is now a consequence of [6, Theorem 2.1]. $\square$

PS_sequences **Lemma 5.2.** *Assume 1 of Assumption 1.3. Then for every $\rho \in I_\gamma$ either there is a $t_0 \geqslant 0$ such that a maximizing sequence of $\psi(u)$ in S_{t_0} is a minimizing sequence for $G_\rho(u)$ or $G_\rho(u)$ has mountain pass geometry and there is a critical sequence sequence $(u_k)_k \subset H^1_A(\mathbb{R}^N)$, satisfying*

$$\begin{cases} G_\rho(u_k) \longrightarrow c > 0, \\ G'_\rho(u_k) \xrightarrow{H^{-1}(\mathbb{R}^N)} 0. \end{cases} \tag{5.6}$$

eq-PS

Proof. The proof of [6, Theorem 2.15] can be adapted to prove Lemma 5.2.

Let

$$\rho \in I_\gamma \tag{5.7}$$

eq-rho_in_I_gamma

and

$$\Gamma_\rho(t) \stackrel{\text{def}}{=} \frac{\rho}{2} t - \gamma(t). \tag{5.8}$$

eq-definition_Gamma

Then $\Gamma_\rho(t)$ is not monotone increasing. Indeed, if so then for $t_1 < t_2$ we would have

$$\frac{\rho}{2} t_1 - \gamma(t_1) \leqslant \frac{\rho}{2} t_2 - \gamma(t_2)$$

which implies

$$(\gamma(t_2) - \gamma(t_1))/(t_2 - t_1) \leqslant \rho/2$$

contradicting (5.7). Similarly $\Gamma_\rho(t)$ is not monotone decreasing. Therefore $\Gamma_\rho(t)$ admits either a local minimum or a global maximum. If t_0 is a local minimum of $\Gamma(t)$, then since $G_\rho(u) \geqslant \Gamma_\rho(\|u\|^2)$, if $(u_k)_k \subset S_{t_0}$ is a maximizing sequence of $\psi(u)$ $G_\rho(u_k)$ converges towards a local minimum of $G_\rho(u)$.

If $\Gamma(t)$ does not admit a local minimum, then it admits a positive global maximum at a point $t_0 > 0$ with $c \stackrel{\text{def}}{=} \Gamma_\rho(t_0) > 0$. We have $G_\rho(0) < c$, and for all $u \in S_{t_0}$, $G_\rho(u) \geqslant \Gamma_\rho(t_0) = c$. However we can find a $t_1 > t_0$ and a $\delta > 0$ such that $\Gamma_\rho(t_1) \leqslant c - \delta$. It follows from the definition of $\gamma(t)$ that there is a $u_1 \in S_{t_1}$ such that $G_\rho(u_1) < c - \delta/2$. Thus G_ρ has mountain pass geometry. $\square$

BPS_solution **Lemma 5.3.** *Assume 1 of Assumption 1.1. Suppose 1 of Assumption 1.3. Then the existence of a bounded sequence $(u_k)_{k\in\mathbb{N}} \subset H^1_A(\mathbb{R}^N)$ satisfying (5.6) with $\rho > 0$ implies the existence of a $w_\rho \in H^1_A(\mathbb{R}^N) \setminus \{0\}$ such that $G'_\rho(w_\rho) = 0$.*

Proof. Let $\rho > 0$ and let $(u_k)_{k\in\mathbb{N}} \subset H^1_A(\mathbb{R}^N)$ be a bounded sequence satisfying (5.6). The sequence $u_k \not\xrightarrow[k\to\infty]{H^1_A(\mathbb{R}^N)} 0$ because $c > 0$ and $G(0) = 0$. Thus we may assume that, up to a subsequence that we still denote by $(u_k)_{k\in\mathbb{N}}$, $\|u_k\|^2_{H^1_A(\mathbb{R}^N)} \longrightarrow t > 0$. It follows from (5.6) that $\langle G'_\rho(u_k), u_k\rangle_{H^{-1},H^1} \longrightarrow 0$. If $u_k \xrightarrow{D} 0$ then 1 of Assumption 1.3 and Theorem 4.1 imply $\langle \psi'(u_k), u_k\rangle_{H^{-1},H^1} \longrightarrow 0$, which implies that $\langle G'_\rho(u_k), u_k\rangle_{H^{-1},H^1} \longrightarrow \rho t \neq 0$, a contradiction. Theorem 2.3 and Proposition 3.5 imply we can use [9, Theorem 3.1, p.62-63] to assert the existence of $(w^{(n)})_n \subset H^1_A(\mathbb{R}^N)$, $(g_k^{(n)})_{k,n} \subset D$ and $\mathbb{D} \subset \mathbb{N}$ such that

$$g_k^{(n)^{-1}} u_k \rightharpoonup w^{(n)}, \tag{5.9}$$ eq-beavis_weak_limi

$$g_k^{(n)^{-1}} g_k^{(m)} \rightharpoonup 0 \text{ for } n \neq m, \tag{5.10}$$ eq-beavis_asymptoti

$$\sum_{n\in\mathbb{D}} \|w^{(n)}\|_{H^1_A(\mathbb{R}^N)} \leqslant t, \tag{5.11}$$ eq-beavis_norm_boun

$$u_k - \sum_{n\in\mathbb{D}} g_k^{(n)} w^{(n)} \xrightarrow{D} 0. \tag{5.12}$$ eq-beavis_D-weak_li

Hypothesis 1 of Assumption 1.3 and equations (5.9), (5.10) and the fact that the functional $G_\rho(u)$ is invariant with respect to D implies that $\langle \psi(u_k), u_k\rangle_{H^{-1},H^1} = \sum_{n\in\mathbb{D}} \left\langle \psi(w^{(n)}), w^{(n)}\right\rangle_{H^{-1},H^1} + o(1)$. If all the $w^{(n)}$ were zero, then $\langle \psi(u_k), u_k\rangle_{H^{-1},H^1} \longrightarrow 0$ a contradiction (as above). Therefore there is at least one nonzero $w^{(n)}$ which we call w_ρ. From (5.9) and the invariance of G_ρ with respect to D, we may assume that $u_k \rightharpoonup w_\rho$, in $H^1_{\mathrm{w}}(\mathbb{R}^N)$. We conclude from (5.6) that $G'_\rho(u_k) \longrightarrow \rho w_\rho - \psi'(w_\rho) = 0$, in $\mathscr{D}'(\mathbb{R}^N)$. □

tions_Igamma **Corollary 5.4.** *For almost every $\rho \in I_\gamma$, either there is a $u_\rho \in H^1_A(\mathbb{R}^N) \setminus \{0\}$ such that $G'_\rho(u_\rho) = 0$, so that u_ρ is a weak solution to (1.1) or there is a $t_0 \geqslant 0$ such that a maximizing sequence of $\psi \in \S_t$ is a minimizing sequence for $G_\rho(u)$.*

Proof. Let $\rho > 0$, let $(u_k)_k$ be a critical sequence of G_ρ and let $\rho_k \searrow \rho$. If $\|u_k\|_{H^1_A(\mathbb{R}^N)} \longrightarrow \infty$ then since $G_\rho(u_k) \longrightarrow c$, it follows that $\psi(u_k) \longrightarrow \infty$. On the other hand, if $(u_k)_{k\in\mathbb{N}}$ is bounded then it follows that there is an $M > 0$ such that $\psi(u_k) \geqslant -M$. Dividing G_ρ by ρ, the functional is of the form: $\frac{1}{2}\|u\|^2_{H^1_A(\mathbb{R}^N)} - \lambda\psi(u)$, where $\lambda = \rho^{-1}$. Since the first term does not depend on λ we can apply [3, Theorem 2.1] (see also [8]) and conclude that the set of ρ for which the critical sequence (5.6) is unbounded has measure 0. The assertion now follows from Lemmas 5.2 and 5.3. □

k-chaque_rho **Remark 5.5.** If $\gamma(t)$ is differentiable then there is a solution for every $\rho \in I_\gamma$ which can be obtained by a maximizing sequence of $\psi(u)$ in some S_t [6, Theorem 2.1].

Proof of Theorem 1.6. We prove the result in the case $N \geqslant 3$. The proof when $N \leqslant 2$ is similar. Let Assumptions 1.1 and 1.3 be verified. Let $\varepsilon > 0$. We compute, with help of Lemma 5.1, Sobolev's embedding and Theorem 2.3,

$$
\begin{aligned}
0 &\leqslant \limsup_{t\searrow 0} \frac{\gamma(t)}{t} = \limsup_{t\searrow 0} \sup_{u\in S_t} \frac{1}{t}\int_{\mathbb{R}^N} F(x,|u|)\mathrm{d}x \\
&\leqslant \limsup_{t\searrow 0} \sup_{u\in S_t} \left[\frac{1}{t}\varepsilon \int_{\mathbb{R}^N} (|u|^2 + |u|^{2^\star})\mathrm{d}x + \frac{C_\varepsilon}{t}\int_{\mathbb{R}^N} |u|^{p_\varepsilon}\mathrm{d}x\right] \\
&\leqslant \limsup_{t\searrow 0} \left(\varepsilon \sup_{u\in S_1}\int_{\mathbb{R}^N} |u|^2\mathrm{d}x + \varepsilon t^{\frac{2^\star}{2}-1} \sup_{u\in S_1}\int_{\mathbb{R}^N} |u|^{2^\star}\mathrm{d}x + C_\varepsilon t^{\frac{p_\varepsilon}{2}-1}\sup_{u\in S_1}\int_{\mathbb{R}^N}|u|^{p_\varepsilon}\mathrm{d}x\right) \\
&\leqslant \varepsilon.
\end{aligned}
$$

Since ε is arbitrary, we can conclude that

$$\gamma'(0) = 0. \tag{5.13}$$

eq-gamma'(0)

It follows from Lemma 5.1 that $I_\gamma = (0, \sup_{t\neq s}\frac{\gamma(t)-\gamma(s)}{t-s})$. Let $\rho > 0$ and suppose that $G_\rho(u)$ does not have mountain pass geometry. Then from the proof of Lemma 5.2 we see that $\Gamma_\rho(t)$ has a local minimum. Let $t_0 \overset{\text{def}}{=} \inf\{t|\Gamma(t)\text{is a local minimum}\}$. If $\gamma(t)$ is differentiable at t_0, then since $\gamma(t)$ is locally Lipschitz, t_0 is a local minimum of $\Gamma_\rho(t)$ and $\rho/2 = \gamma'(t_0)$. From (5.13), we see that $\Gamma'_\rho(0) = \rho/2 \neq 0$ so $t_0 > 0$. Let $(u_k)_k \subset S_{t_0}$ be a maximizing sequence of $\psi(u)$. From [9, Theorem 3.1, p.62-63] we again assert the existence of $(w^{(n)})_n \subset H^1_A(\mathbb{R}^N)$, $(g_k^{(n)})_{k,n} \subset D$ and $\mathbb{D} \subset \mathbb{N}$ such that Equations (5.9) (5.11), (5.10), and (5.12) are verified. From (5.10), (5.11), and Theorem 4.1 we obtain that $\gamma(t_0) = \lim_k \psi(u_k) = \sum_k \psi(w^{(n)})$. □

Remark 5.6. We conclude with some remarks:

1. If there is an $M > 0$ such that $F(x,s) \geqslant s^{2+\varepsilon}$ for $s > M$, then there is a solution to (1.1) for almost every $\lambda > 0$ because one can prove that $\lim_{t\to\infty}\gamma(t)/t = \infty$.

2. From Remark 5.5 we see that if $F(x,s)$ is a finite sum of homogeneous terms, then $\gamma(t)$ is differentiable and there is a solution for every $\rho \in I_\gamma$.

References

[1] G. Arioli and A. Szulkin. A semilinear Schrödinger equation in the presence of a magnetic field. *Arch. Ration. Mech. Anal.*, 170(4):277–295, 2003.

[2] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics*. New York University Courant Institute of Mathematical Sciences, New York, 2003.

[3] L. Jeanjean and J. F. Toland. Bounded Palais-Smale mountain-pass sequences. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(1):23–28, 1998.

[4] H. Leinfelder. Gauge invariance of Schrödinger operators and related spectral properties. *J. Operator Theory*, 9(1):163–179, 1983.

[5] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.

[6] I. Schindler. A value function and applications to translation-invariant semilinear elliptic equations on unbounded domains. *Differential Integral Equations*, 8(4):813–828, 1995.

[7] I. Schindler and K. Tintarev. A nonlinear Schrödinger equation with external magnetic field. *Rostock. Math. Kolloq.*, (56):49–54, 2002.

[8] K. Tintarev. Mountain pass and impasse for nonsmooth functionals with constraints. *Nonlinear Anal.*, 27(9):1049–1054, 1996.

[9] K. Tintarev and K.-H. Fieseler. *Concentration compactness*. Imperial College Press, London, 2007. Functional-analytic grounds and applications.