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Market Information in Banking Supervision: the Role of Stress Test Design*

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Abstract

The Basel committee views market discipline as complementing banking supervision. This paper studies how supervisors should design stress tests when markets discipline banks via price signals their traded securities provide to bank creditors. We show that the optimal stress test is coarse and lenient. Speculators have incentives to identify bad banks that erroneously passed the test, which makes markets useful at reducing the type-2, but not the type-1, error of a stress test. Our results hold even when the supervisor can intervene directly based on private information. In the limit of costless supervisory interventions, the optimal stress test is uninformative.

Keywords: Feedback, market discipline, information design
JEL classification: G14, G28

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1 Introduction

The Basel Committee on Banking Supervision elevates *market discipline* to one of its three pillars of the prudential regulation of banks.[1] In this lies an acknowledgement that markets may achieve things that regulators cannot. Market discipline is often thought of as having two broad roles (see, e.g., Kwan, 2002): a direct role by restricting undeserving banks' ability to access capital, and an indirect, informational role.[2] In both cases, supervisors can presumably not easily replicate what markets achieve. A supervisor may find it costly to impose directly the penalties corresponding to the cost of modified funding conditions, and markets can provide information that supervisors cannot.[3] Market prices are free and forward-looking, generated by speculators with monetary incentives that are hard to replicate by supervisors facing up to increasingly complex banks (see Goldstein, 2023, and the citations therein, e.g., Stern, 2001). Unfortunately, the Basel Framework provides little help in identifying how supervision can be designed to leverage whatever it is that markets are good at. Pillar 3 largely reduces to a recommendation to improve disclosure, so as to facilitate direct market discipline. By this logic, more disclosure of supervisory information, including the disclosure of stress test results, fosters direct market discipline.[4] This leaves open the question of how the design and disclosure of stress test results affects jointly the direct *and* indirect role of market discipline. This paper explores the informational spillovers from supervisory information production, notably in the form of stress tests, to market discipline in its direct and indirect roles.

We provide a model in which banks can be of two, privately known types. Both types try to

[1]The other two pillars are, loosely speaking, capital requirements (see Flannery, 2014 and Ngambou, 2022 for overviews) and supervisory monitoring (see, among others, Colliard, 2019, and Carletti, Dell'Ariccia and Marquez, 2021).

[2]The distinction between direct and indirect market discipline is related to the *monitoring* and *influence* functions of markets, identified by Bliss and Flannery (2002). Market discipline can affect banks in a variety of other ways. See Flannery and Bliss (2019) for a detailed discussion and overview of research on market discipline.

[3]Acharya et al. (2014) and Haldane (2011) provide evidence that simple market capitalization based measures of bank health were better than regulatory measures at identifying banks that eventually experienced distress. Berger et al. (2000) show that both stock and bond prices are more accurate than supervisory assessments. As Flannery and Bliss (2019) argue: "We believe that market discipline can, potentially, complement and support official oversight of risky financial institutions, [...] by providing market signals that supervisors can use to motivate their own actions..."

[4]For example, Bernanke (2013) argues "[...] the disclosure of stress test results and assessments provides valuable information to market participants and the public [...] and promotes market discipline." As Flannery and Bliss (2019), however, point out "The appropriate relationship between market and regulatory discipline has never been fully developed - at least not in official documents."

raise funds to invest in a risky loan portfolio, but it is only efficient for a high-type bank to do so. The supervisor is concerned not to allow low-type banks to engage in wasteful risk taking.[5] A supervisor can generate noisy information about the bank type by studying its resilience to a stress test. The supervisor commits to a set of stress scenarios to which the bank is subjected and then publicly discloses the test results. This is followed by direct and indirect market discipline. Indirect market discipline, operating via informative price signals, is arguably most effectively exercised by traded claims, such as equity due to the ready availability of data and the liquidity of the underlying markets.[6] We model this by having a speculator decide how much costly information to produce, after having observed the stress test result. He then trades in the bank's shares which generates noisy price signals. Direct market discipline is mostly exercised by capital providers, such as short-term creditors, who need to renew their funding commitment frequently (see Kwan, 2002). We assume that the supervisor finds it too costly to intervene directly in funding or shutting down banks, an assumption we relax in an extension. She therefore relies on capital providers, such as uninsured depositors, to deny funding to low-type banks and provide funding to high type ones. Capital providers decide on the funding terms based on the information contained in the stress test result and the subsequent stock price signals. To summarize, we allow markets to complement banking supervision in two ways: indirectly, by producing information about banks and directly, by withdrawing funding from potentially undeserving banks, conditional on the available information.

We first study a setting in which the supervisor's cost of intervention is high, so she relies exclusively on market discipline to avoid low-type banks from getting access to funds that would then be squandered. We show that the optimal stress test design is a coarse *pass / fail* test that exhibits leniency. We demonstrate that stock markets are not equally good at identifying type-1 and type-2 errors of the stress test. Markets are better at identifying bad banks that were erroneously classified as good by the stress test (type-2 error), but they are less useful at identifying good banks that the stress test has mistakenly classified as bad (type-1 error).

[5]This way of modelling a low-type bank is consistent with regulators' fear that some banks will engage in zombie lending if they can raise funds. See Acharya et al., (2019) for empirical evidence on the prevalence of zombie lending.

[6]The literature has also highlighted the potential role of sub-ordinated bonds (for an excellent discussion of this point, see Flannery and Bliss, 2019). Our model is sufficiently stylized to make this distinction moot.

This asymmetric reaction is the result of the speculator's financial incentives to acquire and trade on information, depending on whether a bank did well or poorly in the stress test. Banks that fail a test are less likely to obtain funding, which reduces the scale of their operations, making information production less valuable. The opposite is true for banks that do well in the stress test: they are likely to get funding and have relatively larger operations, increasing the speculator's potential trading profits. A lenient stress test design makes it more likely that a bank passes the test. This has the advantage of improving the quality of the price signal upon which direct market discipline is based. However, distorting the test towards leniency also has a cost. Since the price signal is noisy, the positive stress test outcome will sometimes allow undeserving banks to obtain funding. Due to this trade-off, the optimal design features leniency, but not to the point of rendering the test uninformative. Other papers, discussed in more detail below, have shown that supervisors may employ lenient stress tests, either because they suffer from ex post forbearance, or are driven by a concern about inefficient bank runs. Our set-up features neither of these elements and is instead based on the interaction between the supervisor's direct information supply and the information environment in which market discipline operates.

We also show that a *pass/fail* structure of the test is optimal, although the supervisor could have chosen an arbitrarily granular test design. First, the optimal test, by virtue of being lenient, must be somewhat coarse. That is, for a *pass* test to be credible, banks with low(ish) resilience that would correspond to a marginal *fail* need to be lumped into the same test outcome as significantly more resilient banks. Second, a more granular information design reduces the average amount of information produced. Suppose the stress test had, in addition to a *fail* outcome, two *pass* levels, a moderate and a strong *pass*. Following a moderate *pass*, uninsured depositors would demand a fairly high interest rate. This dilutes equity, reduces trading profits and thereby dulls the speculator's information production incentives. It is therefore better to enlarge the moderate *pass* to include more resilient banks, which reduces the expected interest rate that banks in that category have to pay, thereby improving information production incentives.

Some papers have argued that stress tests should be lenient or kept confidential so as to prevent runs on banks (see Williams, 2017 and Bouvard, Chaigneau and de Motta, 2015).[7] At the same time, recent crisis episodes have shown that supervisors are also worried that sharp declines in a bank's stock price may itself trigger a run. Regulators have therefore at times prohibited short sales of bank stocks during crisis periods, somewhat in contradiction to their stated commitment to market discipline.[8] The question of how to design a stress test remains open, when the supervisor must, on the one hand, worry about potential crises in which fundamentally sound banks risk not being able to roll over their debt, and, on the other hand, wish to allow market discipline to operate. We extend our model to capture the possibility of a bank run or debt overhang. We do so by making creditors potentially deny funding to banks that the supervisor would like to see funded. In other words, for banks to obtain funding, creditors must have more positive beliefs about its type than the belief threshold the supervisor would apply. We show that, in a benchmark where there is no scope for information production by the speculator, debt overhang makes the optimal stress test more lenient.

Compared to this benchmark, the optimal stress test can be more lenient or more conservative when endogenous speculator information production is introduced. The argument for more leniency remains the same as before: it increases the likelihood of a *pass* with a resulting positive effect on information production. However, there is now also a counter-veiling effect. Since the benchmark *pass* grade is already quite lenient, and funds are provided by the market, the funding cost of a bank that passes the test is quite high. This dilutes equity holders and reduces information production, as mentioned above. In order to encourage information production, the supervisor may optimally apply a test that is more conservative than the benchmark to ensure that banks' funding costs are moderate with a correspondingly positive impact on market information and discipline. We show that the supervisor optimally sets a test more conservative than the benchmark when the gulf between social preferences and creditor preferences is particularly

[7] In these papers the supervisor's objective is for all banks to be funded, i.e., there is no role for market discipline.

[8] The Securities and Exchange Commission (SEC) imposed restrictions on short sales of bank stocks during the 2007-09 financial crisis, as did several European regulators during the 2011 sovereign debt crisis. Beber and Pagano (2013) show that short-sales bans slowed down price discovery in these markets. A ban on short sales can be justified by their potentially manipulative nature, as shown by Goldstein and Guembel (2008), Brunnermeier and Oehmke (2014) or Gao, Jiang and Lu (2024). We do not analyse this possibility here. See also Acharya, Gale and Yorulmazer (2011) for a detailed discussion of how rollover risks can lead to market breakdown.

wide.[9]

We also extend our model to allow for direct intervention by the supervisor, when the latter has noisy private information beyond the stress test result. A direct intervention can take two forms. Either the supervisor shuts down a bank, even though markets would be willing to fund it, or the supervisor provides public funding to a bank that is denied access to funding from private markets. We assume that intervention is costly so that, for a high enough cost, our extension nests the baseline model in which the supervisor never intervenes directly. When the cost of intervention is zero, direct market discipline loses its purpose and the optimal stress test degenerates to become completely uninformative. However, indirect market discipline remains useful: The supervisor, whose private information is noisy, can still learn from price signals. Since the supervisor's intervention decision is no longer directly related to the stress test outcome, passing the test ceases to be a pre-condition for stimulating information production by a speculator. In this case, the supervisor wants to make as little information available as possible, since any public information merely crowds out the speculator's private information. When the banking system is in good shape, such that creditors are willing to roll over debt in the absence of any information, the optimal stress test is informative and lenient for any strictly positive intervention cost, no matter how small. Intuitively, the supervisor wishes to enlist market discipline to save on her intervention cost. Since price signals are noisy, having some information from the stress test allows for more effective market discipline. When the banking system is in worse shape, such that creditors' default decision would be not to roll over debt in the absence of any information, a completely uninformative stress test is optimal for small, but strictly positive intervention costs and becomes lenient for higher intervention costs. The supervisor is more reluctant to design a highly lenient stress test, because under such a test, and given the poor health of the banking system, a *pass* grade is not sufficient for a bank to roll over its debt. To make it useful, the supervisor would have to raise the stress test's *pass* threshold to a high level. A *pass* is then so informative that it leaves little scope for an informational advantage to the speculator. At the same time, an uninformative test induces speculator information production at low levels of intervention costs, since the supervisor will

[9] It is conceivable that the supervisor would like to apply a harsher funding rule than credtiors, for example if there are default externalities. We analyse this case in Appendix A.

often intervene and fund the bank. Hence, when the intervention cost is low, it is better to induce speculator information production with a completely uninformative test, while an informative and lenient design becomes optimal when the intervention cost increases.

The remainder of the paper proceeds as follows. We provide a review of the literature in Section 2. Section 3 provides the description of the model, which is solved in Section 4 for the benchmark case without informational feedback from the stock market. Section 5 presents the main results on stress test design with market feedback. We extend the model to allow for debt overhang in Section 6 and introduce supervisor private information in Section 7. Section 8 concludes.

2 Related Literature

There has been considerable interest in recent years in the question how information conveyed by prices in secondary financial markets feeds back into real decisions (see Bond, Edmans and Goldstein, 2012 and Goldstein, 2023 for surveys). One application of that literature points to the importance of stock price information in guiding intervention decisions of regulators, for example, a supervisor who needs to decide whether to intervene in a troubled bank (Bond, Goldstein and Prescott, 2010, and Bond and Goldstein, 2015). The papers closest to ours are Bond and Goldstein (2015) and Siemroth (2019) who study the interaction of a regulator's information (including a decision to disclose such information) with information revealed by share prices, when that information is in turn used by the regulator. They show that more public information may crowd out private information as it reduces the informational advantage of speculators.[10] This effect is balanced by a crowding-in effect, as public information reduces the riskiness of speculators' trades, inducing them to take larger positions. Also related is Goldstein and Yang (2019) who study the interaction between public disclosure and market-based information in a context where the decision maker learns from both the public signal and market prices (unlike in Bond and Goldstein (2015) where the regulator has information regardless of whether or not it is made public). Goldstein and Yang (2019) focus on two dimensions of uncertainty and

[10]Recent empirical evidence by Heitz and Wheeler (2023) supports the notion that the information contained in stress tests does indeed crowd out information production by financial markets.

explore how disclosure affects the weight that traders put on one of the two private signals they possess. They show that when information is disclosed about the dimension of uncertainty that is relevant for the real decision, then this will reduce the weight that traders put on that dimension of their private signals. By crowding out information aggregation on the "useful" dimension, more public disclosure may reduce the overall amount of information relevant to the real decision.

Our focus is different in several respects. First, we focus on endogenous information production and not on the aggregation of an exogenous information endowment by speculators. Second, we model feedback from prices via a bank's access to funding. This is important because the bank's expected funding cost affects incentives for information production. Finally, we study information design in a way that allows us to identify leniency and coarseness as decision variables. The papers by Bond and Goldstein (2015), Goldstein and Yang (2019) and Siemroth (2019), share their focus on the *intensity* with which speculators trade on their private information. They, like many others, use variants of the Grossman-Stiglitz framework that assume normal distributions and thereby preserve the quasi-linearity of trades, which is a key property for tractability. That framework, however, has a very specific property: Residual uncertainty from the speculator's perspective is independent of the *realization* of the public signal. In this context, information design reduces to choosing the standard deviation of signal noise. This property makes the framework arguably less well suited to studying trade in non-linear claims such as highly leveraged bank equity. Quite plausibly, residual uncertainty is smaller for lower signal realizations, i.e., when the expected equity payoff is nearer the default region. This induces very different incentives to acquire information, depending on which part of the payoff distribution a speculator expects to navigate.[11] In our paper, the information production decision therefore depends sensitively on the *realization* of the public signal, with less information being produced following a negative public signal than following a positive one.

The effect that trading profits differ, depending on whether the outlook is positive or negative, is related to Dow, Goldstein and Guembel (2017) who show that speculators' information production may break down when firms' investment prospects are unfavorable. Such firms are

[11] The cost of moving away from the normal, quasi-linear framework, is that we cannot study the speculator's trading intensity, which is the main focus of the above papers.

unlikely to invest, which therefore undermines the incentive for speculators to produce information about those prospects. Deng and Shapiro (2024) identify feedback via consumer learning as a further channel that can affect the information sensitivity of a firm's shares, including a degenerate case where firm profits become independent of the underlying state of the world. In this paper we focus on the *ex ante* information design problem when information production in financial markets depends on the trader's belief about fundamentals. Moreover, the information environment is designed by a planner who cares about *ex ante* bank value, while market information is produced by stock traders. Since equity claims are protected by limited liability and diluted by the bank's fund-raising, this introduces a wedge between the payoffs that are relevant for the planner and the information producer. As such, stock traders have little incentive to produce information about banks with resilience levels close to and below the threshold where they can obtain funding. Those are, however, precisely the banks that the planner would most like to learn about. While we are by no means the first to point out that private incentives for information production differ from social value (see Paul, 1992, or Lenkey and Song, 2017, for a more recent example), we identify a new wedge between the two.

Davis and Gondhi (2024) analyse risk shifting with informational feedback from the stock market. Their focus is different from ours in that they explore how an agency conflict between debt and equity holders interacts with the endogenous information available in the stock market. They show that the relationship depends crucially on whether investment distortions are of a risk shifting or a debt-overhang type. As risk shifting in their model increases speculators' incentives to produce information, the feedback mechanism mitigates the inefficiency caused by the agency problem.

There are a number of papers that have studied whether stress test results should be disclosed, e.g., Bouvard, Chaigneau and de Motta (2015), and Leitner and Williams (2023) (see also Goldstein and Sapra, 2014 and Goldstein and Yang, 2017 for a more general discussion and review). Disclosure matters, as it may affect market discipline, the functioning of the interbank market, financial stability, bank lending behaviour and risk sharing. Our model can be re-interpreted as a disclosure choice: Since the supervisor relies on markets to discipline banks, all that matters is publicly available information. Our results thus suggest that partial disclosure dominates full disclosure of stress test results. In an extension we allow the supervisor to act, at a cost, on private information. This explicitly introduces a meaningful difference between what the supervisor knows (a bank's precise resilience level) and what she discloses (a stress test result which corresponds to a region in which the resilience level lies).

Some papers have modelled the disclosure choice as a Bayesian persuasion problem, that is, a supervisor chooses an information design to which she commits. One common theme among those papers is a supervisor's concern to design a stress test in a way to prevent bank runs (see Faria-e-Castro, Martinez and Philippon, 2017, or Williams, 2017). In line with Kamenica and Gentzkow (2011), this pushes the optimal experiment to be of a *pass/fail* nature, featuring a maximum of *pass* grades consistent with avoiding a run. The optimal stress test is lenient, in the sense that it admits type-2 errors (some low-type banks passing the test), but no type-1 errors (no high-type banks failing the test). Some papers extend the basic Bayesian-persuasion-*cum*-global-games approach, while remaining in relatively abstract settings that are not specifically geared towards modelling bank stress tests. Quigley and Walther (2023) study how a publicly disclosed stress test affects a bank's incentives to privately disclose verifiable information at a cost. They show that private disclosure may lead to unravelling, which the stress test can preempt by applying a richer message space than a simple *pass/fail*. Inostroza and Pavan (2023) look into robust information design in a global games framework with privately informed agents. The optimal policy coordinates all market participants on the same course of action, but without fully revealing the state. Under some conditions, the optimal policy is a *pass/fail* stress test.

Other papers add flesh to the Bayesian persuasion approach by modelling the details of financial frictions faced by banks. Faria-e-Castro, Martinez and Philippon (2017) show that the opacity implied by the test can generate an adverse selection cost at the fundraising stage. They investigate the optimal test design as a function of a country's fiscal capacity, when the regulator can trade off more transparency against the fiscal costs of guarantees that prevent bank runs. Goldstein and Leitner (2018) show that a more informative stress test may destroy insurance opportunities among banks. This potentially renders no disclosure of stress test results optimal. Inostroza (2023) studies stress testing with multiple audiences, such as short-term creditors and shareholders. He shows that the optimal policy is opaque when the bank has high-quality assets, and transparent when the bank has poor-quality assets. Full transparency is optimal because the complementarity in incentives to provide funds between the two types of capital providers generates a convexity in bank value as a function of the underlying fundamentals. Orlov, Zryumov and Skrzypacz (2023) show the optimality of *pass/fail* tests failing all weak and some strong banks in order to limit the stigma of failure. The optimal test is not fully informative, because banks are subject to a convex cost of distress, which renders bank value concave in its initial wealth. Fuchs, Fukuda and Neuhann (2024) study the interaction between *ex ante* rules and *ex post* disclosure. They show that regulation helps *ex ante* incentive provision, while *ex post* disclosure serves to provide insurance. Overall, the conclusions from these papers are quite nuanced as optimal stress test design depends sensitively on the precise financing frictions faced by banks.

Although we share with the above papers the feature that a supervisor chooses an information design and then commits to it, our focus is on optimal learning. As such, we assume that the supervisor is limited to noisy experiments, as in Parlatore and Philippon (2022). The choice of scenario adversity then affects the trade-off between type-1 and 2 errors of the experiment. A more adverse scenario increases the probability of a type-1 error (mistakenly classifying a good bank as bad) and reduces that of a type-2 error (mistakenly classifying a bad bank as good).

Shapiro and Zeng (2024) study the reputational implications of stress test design for a supervisor. A supervisor may design either a lenient or a tough stress test, depending on whether she wishes to build a reputation for being soft or tough. This approach is closer in spirit to Bouvard, Chaigneau and de Motta (2015) and Parlasca (2024) where, in contrast to Bayesian persuasion, the supervisor chooses information revelation strategically, *after* having become privately informed herself.

Our paper is also related to the literature on banking regulation which regards a moral hazard problem at the bank level as a central friction to address by regulation, for example, Bhattacharya (1982), Rochet (1992), Hellmann, Murdock and Stiglitz (2000), Gorton and Huang (2004), Morrison and White (2005), Calzolari and Lóránth (2011), Calzolari, Colliard, and Lóránth (2019) or Fecht, Inderst and Pfeil (2022). In Carletti, Dell'Ariccia and Marquez (2021) banks take too much risk in a laissez-faire equilibrium and supervision is designed to reduce their risk exposure. The supervisor monitors and learns about the amount of a bank's capital (and its portfolio) and can then intervene so as to reduce risk exposure. When an intervention occurs, shareholders are expropriated. Our model is similar in spirit, except that the supervisor finds it costly to intervene directly and therefore wishes to enlist market discipline. High leverage associated with a poorly capitalized bank could also lead to debt overhang, a problem addressed by Philippon and Schnabl (2013) who analyze the efficient design of a recapitalization when the regulator does not know the bank's type. We extend our model to analyse the possibility of debt overhang. Our central point on designing the supervisor's information and its interaction with market-based information and discipline is new to this literature.

Also somewhat related are models on the design of credit rating agencies' evaluation scheme. Goldstein and Huang (2020) predict that CRAs inflate ratings in a model where creditors' heterogeneous beliefs affect credit market conditions, which in turn generates a feedback loop from the CRA to the firm's actual investment decisions.[12] Apart from the difference in focus, Goldstein and Huang (2020) have in mind a CRA without commitment power over its rating announcements, so ratings are subject to ex-post opportunism by the CRA. Moreover, in their paper the issue of information production by speculators or other market participants does not arise, as creditors have an exogenous information endowment. Piccolo and Shapiro (2022) look at a CRA who is subject to a moral hazard problem in information production. Informative stock prices serve to mitigate the agency problem. Higher ratings precision reduces information production.[13]

3 Model Set-up

We begin with a brief overview of the model. There are five dates $t = 0, ..., 4$. At the initial date $t = 0$, a banking supervisor designs a stress test. The outcome of the stress test is publicly observable at date $t = 1$. Afterwards, a speculator decides how much effort to expend on information acquisition. At date $t = 2$ the speculator can trade in the bank's shares and prices

[12] Terovitis (2020) models a similar feedback loop from credit rating to project financing where managers have private information about the project quality.

[13] Note that private information in the loan market may also be transmitted through interest rates, which act to coordinate banks' actions in supplying credit to the real economy (see Shen, 2021).

are publicly observed. Then, at $t = 3$, providers of capital, such as uninsured depositors, choose whether and at what terms to roll over credit to the bank. If the bank can roll over the credit, it invests in a risky loan portfolio. Payoffs are realized at the final date $t = 4$.

We now describe the full model. There is a state of the world ω, which can take the values l and h with equal probability. The state ω realizes at $t = 0$, is unobservable and determines whether the bank is worth funding ($\omega = h$) or not ($\omega = l$). We can think of the model as applying to a single bank, or to many ex-ante identical banks. In the latter case, ω should be interpreted as specific to banks, i.e., we do not model learning about an aggregate shock from conducting stress tests across many banks (see Parlatore and Philippon, 2022, or Parlasca, 2024, for learning about aggregate shocks). Although the bank's true type ω is not directly observed, there is a learnable characteristic that is correlated with the bank's type ω, which we call the bank's resilience $s \in [0, 1]$. For analytical tractability, we assume

$$\begin{aligned} f(s|\omega = h) &= 2s, \\ f(s|\omega = l) &= 2(1 - s), \end{aligned} \tag{1}$$

with corresponding cumulative distributions $F_\omega(s) \equiv F(s|\omega)$. The supervisor designs a stress test at $t = 0$, which reveals information about the bank's resilience s.[14] Both the test design and its outcome are publicly observed. A stress test is formally defined as follows.

Definition 1 (Stress Test and Outcome). *A stress test is a partition $S = \{s_0, s_1, \ldots, s_n\}$ of the stress resilience space $[0, 1]$ with $0 = s_0 < s_1 < \cdots < s_n = 1$. An outcome of the stress test S is a public signal m_i for $s \in [s_{i-1}, s_i)$, $i \in \{1, \ldots, n\}$.*

Suppose, for example, that the supervisor chooses a partition $\{0, s_1, s_2, 1\}$ where $0 < s_1 < s_2 < 1$. This stress test can be interpreted as consisting of two scenarios s_1 and s_2. The supervisor can first apply the more benign stress scenario s_1 which will result in either a *pass* or a *fail*.[15] If the bank fails scenario s_1 the supervisor knows that the bank's underlying resilience

[14] Since the supervisor does not intervene directly, it does not matter whether she observes s or learns about s from the stress test. In Section 7 we allow the supervisor to intervene at a cost and observe s independently of the stress test design.

[15] The *pass/fail* nature of the response to an individual scenario is only for illustrative purposes. Since a stress test is an arbitrarily granular partition, the overall stress test can be much more nuanced than a simple *pass/fail*.

is quite weak ($s < s_1$). If the bank passes scenario s_1 the supervisor can apply the more adverse scenario $s_2 > s_1$. If the bank passes scenario s_1 but fails scenario s_2 the supervisor knows that the bank's resilience s is in the interval $[s_1, s_2)$. If, however, the bank passes both scenarios, the supervisor knows that resilience is $s \geq s_2$. Note that from (1) it follows that all banks, even $\omega = l$ types, pass the most lenient stress scenario 0 and all banks, including the $\omega = h$ types, fail the most adverse scenario, given by 1.

Note that our definition of stress test S is quite flexible. In particular, since the supervisor can run as many scenarios as she wishes at no cost, we allow for complete learning of s, i.e. fully granular grades ($n \to \infty$). Alternatively, the stress test may provide no information about s ($n = 1$), or learning s noisily (n finite).[16] Nevertheless, we impose two notable restrictions. First, we require the stress test to be monotone, i.e. two disjoint intervals cannot produce the same test outcome. This assumption is motivated by the fact that the stress test consists of a sequential application of stress scenarios. This way of learning is plausible and rules out that an observer may believe that resilience can be high or low, but not in the middle. Second, in some of the Bayesian persuasion literature, the supervisor can condition the public signal m_i directly on the state of the world ω, such that the full revelation of the true state is possible. We rule this out by assuming that a bank's resilience s is itself only a noisy signal of ω. This is intended to capture real world limitations to how informative stress tests can be.[17]

Using (1) it is easy to show that a stress test S induces a distribution of posterior beliefs

$$\nu_i \equiv \Pr(m_i) = s_i - s_{i-1}, \tag{2}$$

$$\mu_i \equiv \Pr(\omega = h|m_i) = \frac{s_{i-1} + s_i}{2}, \tag{3}$$

satisfying Bayes-plausibility, i.e. $\sum_{i=1}^{n} \nu_i \mu_i = \Pr(\omega = h) = 1/2$, with $\sum_{i=1}^{n} \nu_i = 1$.

[16] Strictly speaking, with n being an integer, the stress test cannot fully reveal s which is a real number. However, since the limiting case is not materially affected by this distinction, we prefer to avoid complicating the notation in a way that would be required to formally take on board this point.

[17] As Leitner and Yilmaz (2019) argue, more intense monitoring by the supervisor may lead to a reduction in the informativeness of the bank's internal model. This puts a limit on how much a supervisor can learn, even if the supervisor's monitoring technology could be arbitrarily precise. Parlatore and Philippon (2022) study the design of stress test scenarios as an optimal learning problem when a supervisor receives noisy signals from multiple banks in response to the application of a stress scenario.

At date 1 the outcome of the stress test m_i is publicly observed. The speculator then chooses how much private information to acquire about the underlying state ω.[18] Information acquisition generates a signal $z \in \{l, h, \varnothing\}$. The signal is fully informative ($z = \omega$) with probability σ and uninformative ($z = \varnothing$) otherwise. The speculator can choose σ, i.e., how much information to acquire, subject to a cost $\frac{1}{2}\tau\sigma^2$ (with $\tau > 0$) this incurs. We assume throughout that τ is large enough to ensure that the optimal $\sigma \leq 1$.

At $t = 2$ the speculator can trade. His order is denoted $x_I \in \mathbb{R}$. In addition to the speculator, there is a noise trader who either buys or sells with equal probability a number of units that we normalize to one. The noise trader's order is denoted $x_U \in \{-1, 1\}$. The speculator and the noise trader both submit their market order to a market maker, who can observe each order, but not its originator, i.e., orders are anonymous. Formally, we let order flow X be random, taking either the value $X = (x_I, x_U)$ or $X = (x_U, x_I)$ with equal probability. The market maker sets a price that allows him to make zero profits in expectation on any trades he makes out of his inventory. That is, like in a standard Kyle (1985) model, the market maker sets the price equal to the expected value of a share, conditional on the information contained in the order flow.

At date 3 the bank can make an investment of 1 in a risky loan portfolio. For a type $\omega = h$ bank the loan portfolio returns, at date 4, R with probability p and 0 otherwise. Assume $pR > 1$. A type l bank has returns R with probability p_l and zero otherwise. For simplicity, we set $p_l = 0$. Suppose investing generates a (small) private benefit for the banker and the social planner does not care about the banker's private benefit. The bank thus invests whenever it can, regardless of its type.[19] If the bank does not invest, it has a value that we normalize to 0.[20]

In order to capture capital market discipline, assume that the bank has internal funds normalized to 1 and short-term creditors who have a total claim of 1 coming to maturity at date 3. If the creditors do not roll over their loans, the bank has to pay out its internal funds and cannot

[18] Although we do not explicitly endogenize the timing of information acquisition, it is clearly optimal for the speculator to wait until after he observes the stress test result. Doing so allows him to condition the amount of costly information acquisition on the information contained in the stress test.

[19] This is a simple way of modeling excessive risk taking or over-investment. For our purposes it does not matter whether the bank knows its own type as long as the low type banker cannot be prevented contractually from engaging in excessive risk-taking.

[20] An alternative interpretation is that all banks have a brick-and-mortar line of activities which has zero net present value, but only high type banks have access to an additional, positive NPV project. Under this interpretation, even low type banks deserve to operate the brick-and-mortar business, but they should be prevented from expanding into further activities.

invest. If the creditors roll over their loans, the bank can use its internal funds for investment in risky lending. Note that it would make no difference if we assumed instead that the bank does not have any of its own funds and needs to raise 1 from outside providers of capital. The two are equivalent, because the bank could pay off the old creditors using internal funds and then raise funds from fresh creditors for the investment. Assume that the bank can invest if and only if it secures private funding. Hence, we rule out any direct capital injections by the supervisor. We relax this assumption in Section 7.

After the stress test and the bank's share price have been observed, the bank can make a take-it-or-leave-it offer asking creditors to roll over their loans at a gross interest rate r.[21] If creditors do not roll over their loans, the bank is forced to pay out 1 to creditors and does not invest.

The timing is summarized in Figure 1.

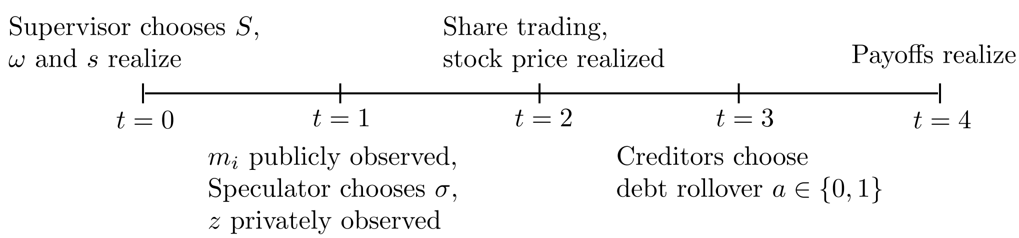

Figure 1: Timing

The supervisor cares about the total value created from the bank's activity. We denote by V the bank's expected payoffs at date 4, net of any amount invested at date 3.[22] This payoff depends on both the state of the world ω and an action $a \in \{0, 1\}$, denoting the creditors' rollover decision. We set $a = 1$ if creditors roll over their debt and $a = 0$ if they do not. We can therefore write the supervisor's payoff V_ω^a as follows:

$$V_h^1 = pR - 1, \tag{4}$$

$$V_h^0 = 0, \tag{5}$$

[21] We do not introduce any frictions in negotiations with multiple creditors, so we can think of the bank rolling over debt with a single creditor who is subject to a break-even constraint.

[22] Since capital providers break even in expectation, the supervisor's objective is the same as when maximizing the joint payoffs of the bank and its providers of capital.

$$V_l^1 = -1, \tag{6}$$

$$V_l^0 = 0. \tag{7}$$

Since $V_h^1 > V_h^0$ and $V_l^0 > V_l^1$, it is socially optimal to allow a high type bank to invest, while it is optimal not to provide funds to a low type bank. The same two inequalities imply that $V_h^1 - V_h^0 > 0 > V_l^1 - V_l^0$ which in turn implies $V_h^1 - V_l^1 > V_h^0 - V_l^0$. That is, the action $a = 0$ reduces the variability of bank value compared to $a = 1$. As such, we can think of $a = 0$ more broadly as an action that leads to the reduction in risk, be it downsizing / failing to expand the bank's operations, or a direct intervention to reduce the bank's risk exposure (see also Carletti, Dell'Ariccia and Marquez, 2021 or the discussion of *market influence* in Flannery and Bliss (2019)).

4 Optimal Stress Test without Price Signals

This section develops the benchmark in which stock markets provide no information, so the capital providers can only condition their roll-over decision on the stress test result. Denote by μ the belief that the state is $\omega = h$, conditional on all publicly available information. In this section, public information is limited to the stress test result, while the next section allows for an additional endogenous signal stemming from noisy stock prices. It will be useful to define belief thresholds for which the supervisor prefers for the bank to be able to invest. The supervisor prefers the bank to continue if

$$\mu V_h^1 + (1-\mu) V_l^1 \geq \mu V_h^0 + (1-\mu) V_l^0. \tag{8}$$

Defining $\Delta V_h \equiv V_h^1 - V_h^0$ and $\Delta V_l \equiv V_l^0 - V_l^1$, this can be re-written as

$$\mu \geq \frac{\Delta V_l}{\Delta V_l + \Delta V_h}. \tag{9}$$

or, using the definitions of ΔV_h and ΔV_l, as

$$\mu \geq \mu^* \equiv \frac{1}{pR}. \tag{10}$$

Consider next the bank's funding problem. Good and bad banks will try to roll over debt and invest, but only good banks repay with probability p. Creditors' participation constraint therefore depends on their belief μ as follows:

$$\mu p r \geq 1. \tag{11}$$

Since the bank can make a take-it-or-leave-it offer, the interest rate is set at

$$r = \frac{1}{\mu p}. \tag{12}$$

For a debt roll-over to be feasible, we also require

$$r \leq R. \tag{13}$$

The constraints (12) and (13) together imply that a debt roll-over is only feasible if creditors are sufficiently optimistic they are lending to a high-type bank:

$$\mu \geq \widehat{\mu} \equiv \frac{1}{pR}. \tag{14}$$

Note that $\mu^* = \widehat{\mu}$. Hence, the supervisor's preferred action is also the one implemented via market discipline. We relax this assumption in Section 6 and Appendix A.[23]

The supervisor's stress test design problem is potentially quite complex as the partition describing the test can be arbitrarily granular. In the benchmark, the problem simplifies considerably, because the stress test induces a single action a which can take only two values, 0 or 1. From (3) we know that beliefs μ_i, induced by stress test outcome m_i are increasing in i.

[23]In Section 6 we allow for debt overhang, so that some bank types are unable to raise funds, although it would be socially optimal to do so, i.e., $\mu^* < \hat{\mu}$. In Appendix A we allow for default externalities, which implies that the supervisor would prefer to apply a harsher continuation threshold than that of capital providers ($\mu^* > \hat{\mu}$). In both cases, we show the robustness of our main result, namely an optimal distortion toward leniency.

We can thus collect all partitions that induce action $a = 0$ into one message, and all those that induce $a = 1$ into another message. The stress test can thus be described by a single cut-off s_1 such that for a test result m_1 (i.e., $s < s_1$), we have $\mu_1 < \widehat{\mu}$ and $a = 0$. If the test result is m_2 (i.e., $s \geq s_1$), then $\mu_2 \geq \widehat{\mu}$ and $a = 1$.[24] The supervisor's objective function is thus given by

$$
\begin{aligned}
v(s_1) &\equiv \frac{1}{2}\Big(F_h(s_1)V_h^0 + F_l(s_1)V_l^0 + \big(1 - F_h(s_1)\big)V_h^1 + \big(1 - F_l(s_1)\big)V_l^1\Big) \\
&= \frac{1}{2}\Big(s_1^2 V_h^0 + (2s_1 - s_1^2)V_l^0 + (1 - s_1^2)V_h^1 + \big(2(1 - s_1) - (1 - s_1^2)\big)V_l^1\Big),
\end{aligned}
\tag{15}
$$

and the stress test design problem is

$$
\begin{aligned}
&\max_{s_1}\ v(s_1) \\
&\text{s.t. } \mu_1 < \widehat{\mu}, \\
&\qquad \mu_2 \geq \widehat{\mu},
\end{aligned}
\tag{16}
$$

where from (3), $\mu_1 = \frac{s_1}{2}$ and $\mu_2 = \frac{1+s_1}{2}$. Denote by s_N the solution to (16), that is, the optimal cut-off in the no-feedback benchmark.

Lemma 1. *Without information acquisition by the speculator, the optimal stress test is a binary partition with a passing threshold $s_N = \mu^*$.*

Proof. See Appendix B. □

The optimal stress test simplifies to a *pass/fail* experiment. Failing the test (message m_1), shows that the bank's resilience level s is below $s_N = \widehat{\mu} = \mu^*$, which means that market discipline bites and the bank cannot roll over its debt. The opposite happens if the bank passes the test (message m_2). Note that the optimal cut-off s_N depends on the relative costs of making type-1 and type-2 errors. Define a type-1 error as denying funds to a good bank. The benefit of avoiding this error is ΔV_h.[25] Under a type-2 error a bad bank can roll over debt and invest. Avoiding a type-2 error has a benefit of ΔV_l. If a type-2 error is relatively more costly, then μ^* increases

[24] Since μ_1 tends to zero for s_1 close to 0 and μ_2 tends to 1 for s_1 close to 1, it is always possible to choose an s_1 that induces actions that are contingent on the stress test result. It is also optimal to do so because a stress test that never affects a would throw away useful information.

[25] Denying funding to a high type bank leads to a reduction in credit supply to the real economy. See Acharya et al. (2018) for the declined lending by stress-tested banks in the US and Ahmed and Calice (2023) for the UK banks that failed the stress tests.

(see (9)). This in turn corresponds to an increase in the optimal *pass* threshold s_N: The test becomes more conservative.

In what follows we will use $s_N = \mu^*$ as the reference level of how lenient / conservative a stress test should be and we will be interested in how the possibility of generating informative price signals affects the optimal stress test design compared to this benchmark.

5 Optimal Stress Test with Feedback from Stock Prices

Before fully characterizing the stress test design problem with an active speculator, we need to calculate the profits the speculator can reap from acquiring and trading on private information. For this, we need to determine the fundamental value of the shares, which depends on the underlying state ω and on the bank's access to capital, captured by a. If the bank rolls over its debt, equity value will depend on the interest rate r, which is a function of beliefs μ. We denote the underlying equity value by $E_\omega^a(\mu)$.

If the bank raises funds ($a = 1$) the required repayment is $r = \frac{1}{\mu p}$ (see (12)). The expected equity value of a high type bank which can roll over its debt is therefore

$$E_h^1(\mu) = p(R - r) = \frac{1}{\widehat{\mu}} - \frac{1}{\mu}. \tag{17}$$

Note that $E_h^1(\mu)$ is increasing in μ, because equity is more valuable when the bank can roll over debt at a lower interest rate. This in turn happens when the creditors have more positive beliefs (higher μ) about the bank's type. The equity value is zero in all other states, either because the bank fails to roll over its debt ($E_h^0 = E_l^0 = 0$), or because the bank rolls over debt but wastes the funds on a bad investment ($E_l^1 = 0$).[26] We can now state the speculator's trading strategy and resulting profits, $\pi(\mu)$, conditional on the belief μ induced by a stress test.

[26]The assumption that equity is wiped out following a bank's failure to secure funding is stronger than strictly necessary, but significantly simplifies the exposition. What is crucial for our mechanism to work, is that the traded claim (be it equity or subordinated bonds) becomes less responsive to the true state of the world when the bank fails to raise funds ($a = 0$). Note that this property may hold more generally since $a = 0$ is defined as a risk-reducing action ($V_h^1 - V_l^1 > V_h^0 - V_l^0$). Here we make the simplifying assumption that security payoffs following $a = 0$ do not depend on ω at all. Note that this can be the case in practice, even if equity retains a positive value. For example, if $a = 0$ corresponded to a liquidation or a forced takeover by another bank one could have $E_h^0 = E_l^0 > 0$. In that case, security payoffs do not depend on ω following $a = 0$ and a speculator cannot benefit from acquiring private information about ω.

Lemma 2. *The speculator's optimal trading strategy is*

$$x(z) = \begin{cases} 1 & \text{if } z = h \\ 0 & \text{if } z = \varnothing \\ -1 & \text{if } z = l. \end{cases}$$

For a given amount σ of information produced, the speculator's trading profits $\pi(\mu)$ are

$$\pi(\mu) = \begin{cases} \sigma\,\mu(1-\mu)\left(\dfrac{1}{\widehat{\mu}} - \dfrac{1}{\mu}\right) & \text{if } \mu \geq \widehat{\mu} \\ 0 & \text{if } \mu < \widehat{\mu}. \end{cases} \tag{18}$$

Proof. See Appendix B. □

Note that the speculator's trading profits are zero if the stress test induces a belief $\mu < \widehat{\mu}$. To see why, consider possible order flows and associated trading profits. If the speculator acquires information and trades on it, order flow can either reveal or hide his direction of trade. When order flow is $(-1,-1)$ or $(1,1)$, the speculator's direction of trade is fully revealed. Since the speculator only buys if $\omega = h$ and only sells when $\omega = l$, these orders fully reveal ω. The market maker then sets a price that fully reflects ω and the speculator therefore cannot make a trading profit. An order flow of $(-1,1)$ or $(1,-1)$ does not reveal the speculator's order. The market maker learns nothing and sets a correspondingly uninformative price. Hence, the roll-over creditors do not learn anything from market prices, and stick to the belief μ induced by the outcome of the stress test. When that outcome is so negative as to deny the bank access to funds ($\mu < \widehat{\mu}$), the bank's equity value drops to zero, regardless of the bank's true type. Since the equity valuations no longer depend on the true state ω, the speculator cannot benefit from learning and trading on knowledge of ω.

When the belief μ induced by the stress test is high enough to allow the bank to roll over its debt ($\mu \geq \widehat{\mu}$), trading profits are hump-shaped. As μ becomes very large, everyone, including the market maker, is confident that the bank is of a high type. This leaves little scope for the speculator to benefit from acquiring private information, which, with a high likelihood, will simply confirm the public belief. The speculator can benefit most from acquiring private information, when doing so confers a significant informational advantage. This is the case when the stress test is least conclusive, i.e., when possible resilience levels are intermediate (μ close to $\frac{1}{2}$). Moreover, as μ drops, the bank will have to roll over debt at less favorable terms, leaving less value for equity holders. This makes it less attractive to speculate on the bank's stock. When $\mu = \widehat{\mu}$, trading profits drop to zero, because rolling over debt is so expensive as to reduce equity value to zero. Overall, trading profits are maximized at $\mu = \frac{1+\widehat{\mu}}{2} \in \left(\frac{1}{2}, 1\right)$.

From the expression for trading profits (18) we can take the first-order condition with respect to σ to find the optimal amount of information acquired by the speculator:

$$\sigma(\mu) = \begin{cases} \frac{1}{\tau}\,\mu(1-\mu)\left(\dfrac{1}{\widehat{\mu}} - \dfrac{1}{\mu}\right) & \text{if } \mu \geq \widehat{\mu} \\ 0 & \text{if } \mu < \widehat{\mu}. \end{cases} \tag{19}$$

In what follows, we assume

$$\tau > \frac{1}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^2, \tag{20}$$

which ensures that information acquisition in (19) is a non degenerate probability.

We can now express the supervisor's problem in a simplified manner.

Lemma 3. *The supervisor solves the following stress test design problem:*

$$\begin{aligned} &\max_S V(S) = v(s_1) + \Sigma(S) \\ &\text{s.t. } \mu_1 < \widehat{\mu} \\ &\qquad \mu_2 \geq \widehat{\mu}, \end{aligned} \tag{21}$$

where $v(s_1)$ is defined in (15) and

$$\Sigma(S) \equiv \frac{1}{2\tau}\sum_{i=2}^{n} \nu_i\,\mu_i(1-\mu_i)^2\left(\frac{1}{\widehat{\mu}} - \frac{1}{\mu_i}\right). \tag{22}$$

Proof. See Appendix B. □

Lemma 3 states that the supervisor's objective function can be decomposed into two parts.

The first part, $v(s_1)$ consists of the supervisor's expected payoff, when the only source of information is the stress test. This corresponds to the payoff under the benchmark in Section 4. The second part, $\Sigma(S)$, consists of the additional payoff from an improved allocation of capital when the stock market provides useful information. Note that the choice of s_1 also enters this second part via its effect on ν_2 and μ_2 (see (2) and (3)).

We can now state one of our key results.

Proposition 1. *With information acquisition by the speculator, the optimal stress test is a binary partition with a single passing threshold, s_F, where s_F is the unique s_1 that solves*

$$s_1 = \mu^* - \frac{1}{4\tau}(1 - s_1)^2 \left(s_1 - \frac{3\mu^* - 1}{2} \right). \tag{23}$$

The test is lenient ($s_F < \mu^*$) and informative ($s_F > 0$).

Proof. See Appendix B. □

When market information matters, the stress test is optimally distorted toward leniency ($s_F < s_N = \mu^*$). That is, it awards *pass* grades to some banks that would have failed the test in the benchmark without market feedback.

Why does the supervisor wish to apply a more lenient *pass* threshold? The answer is that, by virtue of being lenient, the stress test generates more *pass* grades. Since a *pass* grade is a precondition for the speculator to acquire information, leniency encourages the production of market information. This information helps creditors in their roll-over decision. In particular, some banks that would be marginal *fail* under the benchmark (banks with resilience levels $s \in [s_F, \widehat{\mu})$) are worth investigating further before denying them access to capital. That way, some of them will be identified as high type banks who were unlucky to have a marginally substandard resilience level. When their stock price holds up after the stress test announcement, creditors will be willing to roll over their debt, which is efficient. Of course, some of them will see their stock price drop and be denied funding. While this is also efficient, it does not constitute an improvement compared to the situation where the stress test was not distorted towards leniency: those banks would have failed the benchmark stress test and thus also have been denied funding.

But leniency also has a cost: it allows some banks to roll over debt and invest, although this is socially inefficient. This is the case of a low type bank that marginally passes the lenient test, for which the stock price fails to adjust downwards. This allows such a bank to inefficiently roll over debt, which would have been prevented under a less lenient stress test. The trade-off between the direct information value of the stress test, and its role in encouraging information production by the speculator implies that the optimal cut-off is determined as an internal solution, $s_F \in (0, \mu^*)$.

Note that our model encompasses the corner solutions of a completely uninformative stress test ($s_1 \in \{0, 1\}$), which is just like not conducting a stress test at all. If the test was uninformative and $\hat{\mu} > \frac{1}{2}$, there would never be any information production by the speculator. An informative test then leads to crowding-in of speculator information. In this case it is obviously optimal to have an informative test. When $\hat{\mu} \leq \frac{1}{2}$, and the stress test is uninformative, the speculator would always produce some information. This is, however, not efficient, because costly information gets produced even about banks that have a resilience level that pins down their type with high precision. Instead, it is better to have the speculator produce a lot of information about some banks and little (or no) information about others, rather than an intermediate amount of information about all banks. By rendering the stress test informative, the supervisor can boost the speculator's information production for the banks that pass the test. This is because the bank's debt roll-over is less dilutive when creditors are more optimistic about the bank's type. An informative stress test changes the allocation of information production incentives across bank types, but in general does not neatly map into crowding out (or crowding in) as in some of the literature (see Goldstein and Yang, 2017).

As a thought experiment it is instructive to consider a modification of our model whereby it is easy to induce speculator information production. Suppose τ was very small, but positive such that the speculator would always produce the maximum amount of information ($\sigma = 1$) as long as he anticipates the bank to roll over its debt following an uninformative stock price. An informative test would then clearly lead to crowding out when $\mu^* \leq \frac{1}{2}$: with an uninformative test, the speculator would produce a maximum amount of information on all banks. Following an informative test, the speculator would produce no information for banks that failed the test, and (still) produce the maximum amount for those that pass it. That is, there is an overall reduction in information production. An uninformative test, however, would not be optimal even in this case. That is because the speculator's information gets impounded into the price with noise. Even if an uninformative test were to lead to maximum information production about all banks, the capital providers will roll over debt for many undeserving banks: When the stock price is uninformative and the bank has a low resilience level, which remains unknown since the stress test is uninformative, a bank gets funded, but should not. It is therefore optimal to reveal very low resilience levels, i.e., make the test informative and lenient, even if it is very easy to induce information production by speculators.

Another result from Proposition 1 is that the optimal test retains its *pass/fail* nature. This no longer follows immediately from the binary nature of the capital providers' decision ($a \in \{0, 1\}$), because the stress test also affects information production which is a continuous choice variable. The mechanism described before points to a robust reason why the test must be coarse on a sub-interval around μ^*: Leniency aims to encourage information production for some banks with resilience levels below μ^*. But for the capital market to fund such banks, they must be lumped into the same category as banks with a resilience level $s > \widehat{\mu} = \mu^*$. In other words, for a *pass* category to contain sub-standard banks ($s < \widehat{\mu}$) *and* be credible, it must also contain a sufficient number of above-standard banks.

Note also that it is sub-optimal to introduce further sub-categories of a *pass* test. With a single *pass* category and a lenient cut-off $s_F < \mu^*$, the induced belief μ_2^F is below that which generates maximum information production ($\mu_2^F < \tilde{\mu} = \frac{\mu^*+1}{2}$). Suppose now that s_F was kept unchanged, but a second *pass* grade with a cut-off $s_2 \in (\mu^*, 1)$ was introduced. There would thus be two possible *pass* grades m_2 and m_3 with corresponding induced beliefs which we denote by μ_2' and μ_3', respectively. By construction, we have $\mu_2' < \mu_2^F < \tilde{\mu}$, and $\mu_3' > \tilde{\mu}$. As a result, information production following either *pass* grade m_2 or m_3 would drop. Intuitively, after a moderate *pass* (m_2) the interest rate of roll-over debt increases, which dulls information production incentives. After a strong *pass* m_3 there is little uncertainty over the bank's true type. This reduces the speculator's potential advantage from becoming privately informed, decreasing information acquisition. Hence, splitting a single *pass* grade in two will reduce information production and is therefore sub-optimal.

Going back to the broader question of the role of market discipline in supporting banking supervision, our analysis reveals the following insight. Markets, via the information they provide, can help reduce type-2 errors. That is, if a bad bank slips through the supervisor's net (by passing a stress test), it will be subject to market scrutiny and possibly "disciplined" by being denied funding. This provides a reason to allow more banks to slip through the net, i.e., to be lenient. On the other hand, markets are not good at reducing type-1 errors. Banks that are caught in the supervisor's net (by failing the stress test), will not be subject to market scrutiny. Any mistake made in the supervisory process will thus not be corrected. Given the asymmetry in the way that (indirect) market discipline operates, a lenient stress test design is optimal. This contrasts with the literature, which has mainly associated leniency with a misalignment of the supervisor's objective and the objective of the recipients of the stress test result. For example, in Bouvard et al. (2015), Williams (2017), Goldstein and Leitner (2018), Parlasca (2024) and Shapiro and Zeng (2024), the supervisor may wish to hide information from markets to avoid a bank run.

In addition to identifying a new channel via which stress tests matter, our theory also provides new policy implications. The improvement in the information environment from stress test leniency only accrues to banks whose shares are publicly traded. Our theory therefore implies that publicly listed banks should be subject to more lenient stress tests than otherwise equivalent privately held banks. To the extent that regulation does not explicitly distinguish between publicly listed and privately held banks in the stress test design, this would lead to sub-optimal supervision. Ignoring the impact of stress test design on the quality of price signals, would lead to stress tests that are too adverse, reducing the information in the price signals available to creditors in banks who do poorly in the test.

The following proposition describes how the optimal degree of leniency is affected by model parameters.

Proposition 2. *At the optimum, the stress test is more lenient, i.e. s_F decreases, when:*

- *the high type bank's expected returns are higher (p or R increase),*
- *information acquisition is less expensive (τ decreases).*

Proof. See Appendix B. □

In general, the optimal *pass* threshold s_F is directly affected by changes in μ^* and indirectly by the optimal extent of distorting s_1 away from μ^*. In developing the intuition for the comparative statics, we will make use of this distinction.

Consider a reduction in the cost of information acquisition, τ. Note that τ has no direct effect on μ^*. It does, however, have an effect on s_F: When financial markets can cheaply acquire information about the bank's fundamentals, private information becomes more precise, and the supervisor obtains more benefit from distorting the stress test towards more leniency.

By contrast, changes in p and R have a direct effect on the relative costs of type-1 and type-2 errors and thereby on μ^*. In addition, they have an indirect effect on the optimal extent of distorting s_1 away from μ^*. As p and R increase, the cost of a type-1 error increases, as denying funds to a good bank becomes more costly. A bank is therefore allowed to continue for a lower resilience level, i.e., μ^* decreases and so does s_F. In addition, there is an indirect effect. Higher financial returns of the good bank increase the information sensitivity of the equity claim and thus the speculator's incentives to acquire information. As a consequence, market information becomes more precise, increasing the benefit of distorting the stress test towards more leniency. The direct effect of an increase in p and R on μ^* and the indirect effect thus work in the same direction towards more leniency.

5.1 Social versus Private Value of Information

It is instructive to ask how much information the supervisor would acquire if she had access to the speculator's information technology and could make the collected information public so as to allow market discipline to be based on it. We do not consider this possibility throughout the paper, because we want to capture the notion, advanced by regulators, that financial markets can generate information that supervisors cannot. However, since the speculator's incentives are not aligned with the supervisor's, this raises the question of what distortion the misalignment may create. In particular, one may ask how the belief μ affects the supervisor's value of producing additional information. This depends on whether a bank can roll over its debt when $z = \varnothing$, which in turn depends on whether μ is above or below μ^*. When $\mu < \mu^*$, and $z = \varnothing$, creditors do not roll over debt ($a = 0$). Hence, the supervisor's expected payoff is

$$E(V) = \mu(\sigma V_h^1 + (1-\sigma)V_h^0) + (1-\mu)V_l^0 - \frac{\tau}{2}\sigma^2. \tag{24}$$

When $\mu \geq \mu^*$, the expected payoff is instead

$$E(V) = \mu V_h^1 + (1-\mu)(\sigma V_l^0 + (1-\sigma)V_l^1 - \frac{\tau}{2}\sigma^2. \tag{25}$$

Taking the first-order condition in each of the two regions gives us the supervisor's optimal amount of information production, as a function of belief μ, denoted by $\sigma_S(\mu)$:

$$\sigma_S(\mu) = \begin{cases} \frac{1}{\tau}(1-\mu)\Delta V_l & \text{if } \mu \geq \mu^* \\ \frac{1}{\tau}\mu\Delta V_h & \text{if } \mu < \mu^*. \end{cases} \tag{26}$$

In the above expression, $\sigma_S(\mu)$ is continuous and maximized at $\mu = \mu^*$. This is intuitive. At μ^*, the expected cost of making a type-1 or a type-2 error is the same. That is precisely when additional information is most valuable. In sharp contrast, the speculator's incentives to acquire information are zero at the belief $\mu = \mu^*$. This is because at the corner $\mu = \mu^*$ the bank needs to raise funds at the least favorable conditions: creditors only roll over their debt at such a high interest rate that nothing is left for equity holders. The speculator therefore cannot make a trading profit, even if the bank can roll over its debt. There is therefore a wedge between the social and the private value of information. Viewing the problem from this angle gives us a further understanding of the main result on the optimal leniency of the stress test: The social value of information is highest for banks with levels of resilience around the threshold μ^*. However, these are precisely the banks for which the speculator's private value of information is particularly low. The supervisor therefore induces information production about these banks by lumping them into the same stress test result as those banks for which the speculator has a strong information production incentive.

5.2 Empirical Implications

While our analysis is largely normative in nature, our theory has a number of empirical implications. In particular, our theory predicts that the amount of informed trading in a bank's shares depends on how a bank performs in the stress test. We would expect there to be less informed trading following the announcement that a bank failed a stress test, compared to when it passes it. Some papers have shown that there is abnormal trading volume after the announcement of stress test results (see Flannery, Hirtle and Kovner, 2017 among others). This is indicative of informed trading activity, although not proof of it. More direct measures of informed trading have been developed in the market micro-structure literature, for example, bid-ask spreads, the probability of informed trading (PIN) (see Easley, Kiefer and O'Hara, 1997) or Multimarket Information Asymmetry (MIA) (see Johnson and So, 2018). It would thus be possible to estimate such microstucture-based measures for banks after the announcement of stress tests and check whether they are lower for banks that did poorly in the test compared to those that did well. There is no empirical research to date that conducts such an analysis.

A few papers have looked into abnormal returns following stress test announcements. If the stress test conveys information, one would expect prices to increase if the stress test result is better than expected and decrease if the opposite is true. Petrella and Resti (2013) and Morgan, Peristiani and Savino (2014) provide evidence supporting this hypothesis using stock price reactions to the first stress tests after the 2008-09 crisis. The findings remain similar for the more recent, regular stress tests implemented by both the ECB and the US Federal Bank (See Ahnert et al., 2020, for example). Since the event date (announcement of stress test results) is known in advance, one would, however, expect there to be no abnormal returns on average after the announcement. As pointed out by Flannery, Hirtle and Kovner (2017), this is a direct implication of market efficiency. It should thus be true in our model, but is not specific to it. As Flannery, Hirtle and Kovner (2017) argue, a more relevant metric is $|\text{CAR}|$, the absolute value of cumulative abnormal returns, which should be higher after the announcement date for banks that are subject to a stress test compared to banks that are not. Flannery, Hirtle and Kovner (2017) do not condition specifically on the stress test outcome and find that $|\text{CAR}|$ is

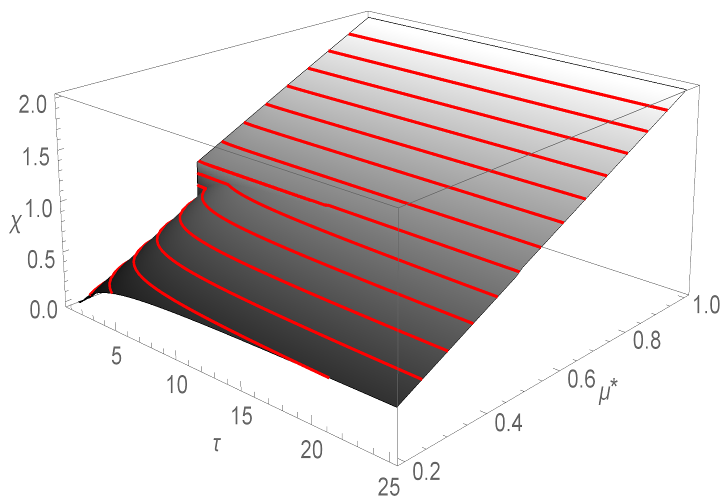

Figure 2: Difference between |CAR| for Tested and Non-Tested Institutions: $\chi(\tau, \mu^*)$

indeed higher for the sample of tested banks compared to non-tested banks.[27] We can compute a comparable metric implied by our model, by calculating the ex-ante expected |CAR| as a function of the stress test design. One can think of the banks that were not stress tested as the limit case in our model where the stress test is uninformative. Figure 2 depicts the difference between |CAR| for tested and non-tested institutions, $\chi(\tau, \mu^*)$, using the optimal design for tested institutions and an uninformative design for non-tested institutions (see Appendix C for detailed computations). This speaks directly to the additional price changes induced by information production following the stress test announcement, which is positive for reasonable parameter values, as found by Flannery, Hirtle and Kovner (2017).

6 Recapitalization under Debt Overhang ($\mu^* < \widehat{\mu}$)

So far, we have analysed the case where the supervisor's preferred course of action coincided with that implemented by capital providers ($\mu^* = \widehat{\mu}$). In practice, supervisors often worry about disclosing negative news for fear of tightening financial constraints for sound banks, including the extreme case of provoking a bank run. Similarly, banks may be unable to access private

[27]Note that from a theoretical perspective it is not entirely clear whether |CAR| should increase for tested banks. The test has two implications for |CAR|. First, the information contained in the announcement has a direct effect on prices. Second, information production and trade depend on the public information revealed (or not) through the test. Hence, non-tested banks could have a higher |CAR| if markets produced significantly more private information about them.

funding due to a debt overhang problem (see Philippon and Schnabl, 2013). In our framework, this corresponds to the case $\mu^* < \widehat{\mu}$. Banks with observed resilience levels $s \in [\mu^*, \widehat{\mu})$ will not be able to fund their activities, although the supervisor would like them to.

One way, among others, to illustrate this case is by modifying our micro-foundation to allow for debt overhang, which drives a wedge between the efficient investment choice and that actually implemented by the capital market.[28] To adapt our model, suppose banks have an existing $t = 1$ level of senior debt $D < R$ outstanding, which cannot be renegotiated. D could capture the amount of insured deposits, for example. The level of pre-existing debt does not affect the desirability of investing, so μ^* remains unchanged. In order to abstract away from complications arising with multiple classes of pre-existing creditors, we no longer interpret the funding stage as a roll-over of existing debt. Suppose instead that the bank needs to raise 1 from new providers of capital who are junior to depositors.[29] The participation constraint of capital providers, such as uninsured creditors, changes to

$$\mu p(r - D) \geq 1. \tag{27}$$

Together with the feasibility constraint $r \leq R$ this pins down a new belief threshold given by

$$\widehat{\mu} = \frac{1}{p(R - D)} > \mu^*. \tag{28}$$

In order not to burden the notation, we continue to refer to the threshold defined in (28) simply as $\widehat{\mu}$. Taking the special case $D = 0$ gets us back to the definition of $\widehat{\mu}$ from Section 4. We begin by clarifying how the benchmark is affected by debt overhang.

[28] An alternative would be to micro-found the belief threshold based on a coordination problem among multiple uninsured depositors who may run on the bank, as in Bouvard et al. (2015). Applying a global game refinement generates a unique belief cut-off, much like what we have. One shortcoming of the bank-run approach as typically implemented is that uninsured depositors' claims are assumed to be fixed. Hence, equityholders' payoffs depend only on whether a run occurs, but not on the beliefs when a run does not occur. In our setting creditors demand a higher interest rate when they extend credit at more pessimistic beliefs.

[29] With debt overhang, a junior creditor who needs to roll over debt faces a different outside option than an outside provider of capital, since the former depends on the bank's liquidation value. In order to keep the treatment simple, we do not introduce this complication here. In general, if the bank has a large liquidation value, it may be the case that markets are willing to fund it, even though the planner would prefer a liquidation, i.e., $\mu^* > \widehat{\mu}$. While conceivable, this case appears of lesser concern in practice. We provide an analysis of this case in Appendix A.

Lemma 4. *Without information acquisition by the speculator, the optimal stress test is a binary partition with passing threshold* $s_N^D \equiv \max\{2\widehat{\mu} - 1, \mu^*\}$.

Proof. See Appendix B. $\square$

Lemma 4 shows that the stress test has a cut-off at μ^*, just like before. Since $\mu^* < \widehat{\mu}$ this test is lenient from the capital provider's point of view: The supervisor labels banks as a *pass* when she thinks they should have access to capital, not when capital markets would like them to. From the perspective of capital providers, the supervisor is too generous with *pass* grades. Note that a cut-off at $s_N^D = \mu^*$ only implements the supervisor's preferred outcome, if it induces a sufficiently positive belief such that the capital market is willing to provide funds. When μ^* is too low, a *pass* grade may no longer allow the bank to raise funds. In this case, the supervisor chooses a threshold s_N^D at the lowest level that still ensures that a *pass* grade allows the corresponding banks to access capital markets.[30]

Proposition 3. *With information acquisition by the speculator, the optimal stress test is a binary partition with passing threshold*

$$s_F^D \equiv \max\{2\widehat{\mu} - 1, \widetilde{s}\} \tag{29}$$

where $\widetilde{s}$ *is the unique* s_1 *that solves*

$$s_1 = \mu^* - \frac{\mu^*}{\widehat{\mu}} \frac{1}{4\tau} (1 - s_1)^2 \left(s_1 - \frac{3\widehat{\mu} - 1}{2} \right). \tag{30}$$

The stress test is:

- *neutral, i.e.* $s_F^D = s_N^D$*, when* $\mu^* \in (0, \mu^\circ]$ *or* $\mu^* = \frac{3\widehat{\mu}-1}{2}$*,*
- *conservative, i.e.* $s_F^D > s_N^D$*, when* $\mu^* \in \left(\mu^\circ, \frac{3\widehat{\mu}-1}{2}\right)$*,*

[30] This observation is akin to the finding in Williams (2017), or Bouvard et al. (2015). In their model, the supervisor would like all banks to be able to access capital markets, corresponding to the case $\mu^* = 0$. When $\widehat{\mu} \leq \frac{1}{2}$, then $s_N^D = \mu^* = 0$, is optimal as a completely uninformative test allows all banks to access capital markets. An uninformative test is like no information disclosure in Bouvard et al. (2015). When $\widehat{\mu} > \frac{1}{2}$, an uninformative test would result in no bank being able to access capital markets. In that case, it is better to raise the threshold just enough to allow the banks that receive a *pass* grade to get funding. That threshold is given by s_1 such that $\widehat{\mu} = \frac{s_1+1}{2}$.

- *lenient, i.e.* $s_F^D < s_N^D$*, when* $\mu^* \in \left(\frac{3\widehat{\mu}-1}{2}, \widehat{\mu}\right)$.

where

$$\mu^\circ \equiv \frac{2\widehat{\mu}-1}{1+\frac{1}{2\tau}\frac{(1-\widehat{\mu})^3}{\widehat{\mu}}}. \tag{31}$$

Proof. See Appendix B. □

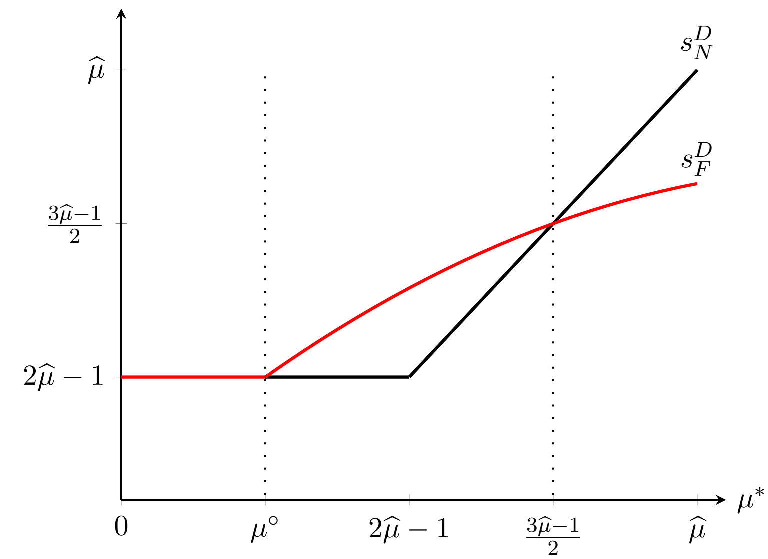

Figure 3: Passing Threshold of the Optimal Stress when $\mu^* < \widehat{\mu}$.
Benchmark s_N^D (in black) and Model with Feedback from Stock Prices s_F^D (in red).

Figure 3 summarizes Proposition 3 by depicting the cutoff of the optimal stress test with feedback from stock prices and in the benchmark without such feedback. The key takeaways from Proposition 3 are as follows. When μ^* is close to $\widehat{\mu}$, the optimal stress test is distorted towards leniency ($s_F^D < \mu^*$). This is just an extension of the result in Proposition 1, where we have already shown the optimality of leniency. As μ^* drops, distorting towards leniency becomes increasingly unattractive as the cut-off s_1 moves further and further below the cut-off $\widehat{\mu}$, making the recapitalization ever more expensive. This in turn reduces the trading profits a speculator can make. In order to provide sufficient incentives for the speculator to acquire information, the recapitalization must not be too expensive, requiring a high enough threshold s_F^D to make the *pass* grade convey sufficiently good news. Hence, as μ^* drops, the supervisor lowers s_F^D less than one for one and eventually the test becomes conservative ($s_N^D < s_F^D$). In other words, conservatism serves to make the financing terms at the recapitalization stage adequately attractive to preserve incentives for the speculator to acquire information about stock values. This result is reminiscent of Orlov et al. (2023) who show that the optimal stress test admits false-negatives. In their paper, banks that fail the test must recapitalize. By making the test more conservative, failing it conveys less negative information, allowing such banks to recapitalize at more favorable terms. In our case, a more conservative test conveys more positive information for a bank that passes it, allowing the latter to raise funds at a lower cost. One key difference is that in Orlov et al. (2023) a more dilutive recapitalization constitutes an inefficient allocation of capital and thus reduces welfare. In contrast, in our model, any dilution at the capital-raising stage redistributes wealth from high to low type banks, but is not inefficient per se. Nevertheless, dilution matters because it affects the quality of the price signal.

Finally, as μ^* drops even further, s_F^D will reach the lower bound given by $2\widehat{\mu} - 1$. At this point, passing the test becomes such a weak signal that the recapitalization will be so expensive as to undermine the speculator's information acquisition incentives. Although the market signal disappears at this point, the stress test still generates market discipline by directly providing information to capital providers. This improves resource allocation just like in the benchmark of Lemma 4.

The following comparative statics hold.

Proposition 4. *At the optimal passing threshold s_F^D decreases, when:*

- *the high type bank's financial returns are higher (p and R increase),*
- *the level of outstanding debt is lower (D decreases).*

When the stress test is lenient (conservative) the optimal passing threshold s_F^D increases (decreases) in τ. When the stress test is neutral the passing threshold s_F^D is independent of τ.

Proof. See Appendix B. □

First, consider the case where a *pass* grade conveys sufficiently positive news to render the debt rollover constraint non-binding ($s_F^D > 2\widehat{\mu} - 1$). An increase in p and R reduces the optimal pass threshold as in the model without debt overhang (see Proposition 1): higher financial returns for the good bank increase the cost of a type-1 error (direct effect) and increase the speculator's incentives to acquire information (indirect effect). By contrast, an increase in the debt exposure of the bank, D, has no direct effect on μ^* but reduces the optimal pass threshold s_F^D through its indirect effect on the speculator's incentives to acquire information. A higher level of outstanding debt destroys value to equity holders only in the state where the bank is able to repay such debt ($\omega = h$). As a consequence, the equity claim becomes less information sensitive, depressing the speculator's profits. This reduces information acquisition and the information content of the price signal, which weakens the supervisor's motive to distort the stress test towards leniency. Interestingly, a reduction in the cost of information acquisition, τ, affects the optimal *pass* threshold differently, depending on whether the test is lenient or conservative. This happens because a reduction in τ makes market information more precise, and thus amplifies the supervisor's motives toward leniency/conservatism.

Finally, consider the case where μ^* and therefore s_F^D are so low that any further reduction in the cut-off would undermine the possibility to recapitalize the bank. In this case, the *pass* threshold is pinned down by the belief threshold that induces funding provision, $\widehat{\mu}$. Nevertheless, the qualitative effects of changes in p, R, and D remain the same: higher financial returns of the good bank and lower debt exposure relax the funding constraint reducing $\widehat{\mu}$ and hence s_F^D. Since the speculator makes no profits and does not acquire information, the *pass* threshold is independent of the cost of information acquisition, τ.

7 Supervisor Private Information

So far, we have assumed that the supervisor does not directly intervene in banks, be it to fund those unable to roll over their debt, or to restrict the activities of banks that have access to private funding. Whether banks are able to invest in a risky project depends entirely on their ability to roll over their debt, i.e., on market discipline. Going back to the baseline model without debt overhang ($\mu^* = \widehat{\mu}$), the lack of direct intervention by the supervisor can be justified in several ways. First, when $\mu^* = \widehat{\mu}$, creditors implement the supervisor's preferred action. If they have the same information as the supervisor, there is no need for the latter to intervene. Second, even if the supervisor had private information, she might not intervene if there is a sufficiently high cost of doing so. The appeal to market discipline as a pillar in the Basel framework is an implicit acknowledgement of the practical relevance of such costs. For example, a supervisor may find it costly to inject public funds into banks that are unable to roll over their debt, because of tax distortions (e.g., White and Yorulmazer, 2014, Faria-e-Castro et al., 2017, or Shapiro and Zeng, 2024). Moreover, the supervisor may suffer from forbearance or be reluctant to restrict the activities of banks that can access credit markets (Martynova et al., 2022).

In this section we extend our analysis to the case where the supervisor has private information and can intervene at a cost, including the corner of a zero cost. Suppose parameter values are such that $\mu^* = \widehat{\mu}$, i.e., we are back to the base-line model.[31] The timing is as follows. Like before, assume that the supervisor designs a stress test at date 0. Subsequently, the stress test result is publicly observed. The supervisor simultaneously and privately observes the bank's true resilience level s. Since the supervisor's information is independent of the stress test design, the latter is just a disclosure strategy to which the supervisor commits.[32] At date 1, after having observed the stress test result, the speculator decides whether to become informed and trade. After a trade has taken place and prices are observed, the supervisor can take an action $a_s \in \{\varnothing, 0, 1\}$. If the supervisor does nothing ($a_s = \varnothing$), the bank approaches creditors and tries to roll over its debt. Assume that creditors observe a_s. When the supervisor chooses not to intervene, creditors update their beliefs and funding conditions are set accordingly. If the supervisor intervenes ($a_s \in \{0, 1\}$), she incurs a cost $\delta \geq 0$ and can either provide public funding to the bank ($a_s = 1$),[33] or shut it down ($a_s = 0$). As before, the intervention $a_s = 0$ should be thought of more broadly as an action that reduces the bank's risk exposure, but for simplicity we just refer to it as shutting down of the bank. If the supervisor provides public funding ($a_s = 1$), we assume that she does so at an interest rate that allows her to break even, conditional on her

[31] Studying the case $\mu^* = \widehat{\mu}$ allows us to focus on what we view as the core focus of this paper, namely the information spillovers from stress tests to market discipline. If we had $\mu^* < \widehat{\mu}$, the supervisor might want to intervene even without private information, simply because market discipline is too tough. This case (without stock price feedback) has been studied by Faria-e-Castro et al. (2017) among others. For analytical clarity and tractability, we focus only on the first mechanism.

[32] This distinguishes our approach from Bouvard et al. (2015) and Parlasca (2024) who analyze the signalling game where the supervisor has private information *before* deciding a disclosure strategy or stress test design.

[33] One may associate this to the practice of the Emergency Liquidity Assistance (ELA) that the ECB may offer to a financially distressed bank, such as in the case of Greek banks in 2014.

private signal s.[34] To keep the treatment simpler, we restrict the stress test design problem to a *pass/fail* test, i.e., there is a single cut-off $s_1 \in [0, 1]$.

The bank's expected value depends on the cut-off s_1 via the beliefs and actions that the stress test induces. The speculator needs to form beliefs about the likelihood and direction of the supervisor's future direct interventions. In particular, it may happen that under a lenient test design, the supervisor chooses to unwind a bank, even though it passed a test. Conversely, under a conservative test, the supervisor may fund a bank that failed the test. For which resilience levels s a supervisor intervenes depends on the cost δ of such an action. We can thus distinguish the following regions.

Region a(i): $s_1 < 2\mu^* - 1$. The cut-off is so low, i.e., the stress test so lenient, that in the absence of a price signal, creditors are not willing to roll over debt, even if the bank passes the test. This happens when $\mu_2 = \frac{s_1+1}{2} < \mu^*$, i.e., when $s_1 < 2\mu^* - 1$. This region is empty if $\mu^* < \frac{1}{2}$. The bank can roll over its debt following a positive signal from the stock price. Following an uninformative stock price, creditors would not roll over their claims, but the supervisor would intervene and provide funding if

$$sV_h^1 + (1-s)V_l^1 - \delta \geq sV_h^0 + (1-s)V_l^0,$$

i.e., when

$$s \geq \bar{s} \equiv \frac{\Delta V_l + \delta}{\Delta V_l + \Delta V_h} = \mu^*(1+\delta). \tag{32}$$

Region a(ii): $s_1 \in [2\mu^* - 1, \mu^*(1-\delta))$. In this region, the stress test is sufficiently informative to allow a bank to roll over debt (in the absence of a price signal) if it passes the test, but not if it fails it, i.e., $\mu_1 < \mu^* \leq \mu_2$. Since $s_1 < \mu^*$, the supervisor never wants to fund a bank that failed the stress test. However, the supervisor may wish to intervene by shutting down the bank, even though it passed the test. Following message m_2, and an uninformative stock price, the creditors would roll over their debt, while, the supervisor, knowing s, would prefer to

[34] In principle the supervisor could provide funding at a subsidized rate. This may be undesirable if the cost of public funds is high. Moreover, a supervisor may be reluctant, for political reasons, to undercut private providers of capital for the benefit of leaving a rent to the bank. Finally, the assumption enables a clearer comparison to the main model since the only margin is *when* a bank can raise funds, but not whether the funding is subsidized. The supervisor cannot provide funds at a higher rate than the competitive one, since her willingness to do so signals to the market that $s \geq \widehat{\mu}$, in which case creditors will choose to roll over debt at competitive terms.

intervene by shutting down the bank ($a_s = 0$), if

$$sV_h^0 + (1-s)V_l^0 - \delta > sV_h^1 + (1-s)V_l^1.$$

The supervisor would thus intervene and shut down the bank when

$$s < \underline{s} \equiv \frac{\Delta V_l - \delta}{\Delta V_l + \Delta V_h} = \mu^*(1-\delta). \tag{33}$$

Region p: $s_1 \in [\mu^*(1-\delta), \mu^*(1+\delta)]$. In this region, the supervisor is always passive since the intervention cost is higher than the expected benefit of intervening.

Region b(i): $s_1 \in (\mu^*(1+\delta), 2\mu^*)$. The test is conservative. Absent a price signal, the creditors are willing to roll over debt for a bank that passed the test, but not for a bank that failed it, i.e., $\mu_1 < \mu^* \leq \mu_2$. The supervisor is willing to intervene and fund a bank that has failed the test if the resilience level is $s > \bar{s}$. The supervisor never shuts down a bank that passed the test since $s_1 > \mu^*$.

Region b(ii): $s_1 \in [2\mu^*, 1]$. In this region s_1 is so high, i.e., the test so conservative, that even a bank that failed the test would be able to roll over its debt. This region is empty if $\mu^* > \frac{1}{2}$. The supervisor intervenes to unwind a bank if $s < \underline{s}$.

Note that the regions a(ii) and b(i), in which the supervisor intervenes with a positive probability, are non-empty only if $\delta \leq \min\{1, \frac{1}{\mu^*} - 1\}$. For higher values of δ the supervisor never intervenes and the analysis of Section 5 is directly applicable.

The supervisor's objective function depends on the region in which s_1 is located. Lemma 6 in Appendix B provides the full details and Figure 5, also in Appendix B, depicts the objective function for some parameter values. We denote by s_F^P the optimal cut-off in the supervisor private-information case, when there is feedback from stock prices.

Proposition 5. *When the supervisor can intervene according to her private information s, the optimal stress test is uninformative (i.e. $s_F^P = 0$) when intervention is costless ($\delta = 0$). When intervention is costly ($\delta > 0$) and $\mu^* \leq \frac{1}{2}$, the optimal stress test is informative and lenient with* *passing threshold*

$$s_F^P = \begin{cases} s_1^{a(ii)} & \text{if } s_1^{a(ii)} < \mu^*(1-\delta) \\ s_F & \text{if } s_1^{a(ii)} \geq \mu^*(1-\delta), \end{cases} \tag{34}$$

where $s_1^{a(ii)} \equiv \delta \frac{1-\frac{\sigma_2}{2}}{\frac{\sigma_2}{2}(pR-1)}$ *with* σ_2 *defined in (62), and* s_F *solves (23). When* $\mu^* > \frac{1}{2}$ *and* δ *is high enough, the optimal stress test is informative and lenient with passing threshold* $s_1 = s_F$.

Proof. See Appendix B. □

Proposition 5 shows robustness of leniency when a privately informed supervisor can directly intervene in banks. Leniency only breaks down in the limit case when the supervisor's intervention cost is $\delta = 0$, in which case the optimal stress test is completely uninformative. When $\delta = 0$, the supervisor always intervenes, based on her private information, and direct market discipline becomes superfluous. Effectively, feedback from prices to real decisions now only acts via direct learning by the supervisor (as in Bond and Goldstein, 2015). The bank's ability to continue and its funding conditions therefore do not depend on the stress test outcome and a negative test result no longer undermines the speculator's information production incentives.[35] These are now maximized when the market maker remains completely in the dark about the bank's type, that is, when the stress test is uninformative ($s_1 = 0$).

Consider now what happens when the intervention cost is positive but very small. We need to distinguish between the cases $\mu^* \leq \frac{1}{2}$ and $\mu^* > \frac{1}{2}$. When $\mu^* \leq \frac{1}{2}$ the default is for banks about which nothing is known to be funded by the market. In this case, it is optimal to set $s_1 > 0$ for any $\delta > 0$. For a small positive s_1, banks with very low resilience will fail the stress test and not get funded by the market. This allows the supervisor to save on the intervention cost for banks with very low resilience levels, i.e., those for which the supervisor is in any case very confident that they should be shut down.[36] For the large majority of banks that pass the test, there will be market information available. In those cases where the market price is uninformative, but the supervisor observes a resilience level $s \in (s_1, \underline{s})$, she can intervene by shutting down the bank. In setting the stress test cut-off s_1 the supervisor trades off the cost of

[35] Note that $\delta = 0$ also eliminates the dilution effect that a more lenient test may have at the roll-over stage. This is because the bank is recapitalized by the supervisor at conditions that entirely depend on her private observation of the bank's resilience, decoupling funding conditions from the stress test result.

[36] Note that a bank with resilience level $s = 0$ is certain to be a low type.

direct interventions against the loss of market information. The optimum is thus a lenient yet informative test for any $\delta > 0$.

When $\mu^* > \frac{1}{2}$ banks do not get funded in the absence of further information. The supervisor thus needs to intervene by funding banks. When moving from $s_1 = 0$ to a small $s_1 > 0$ the market still does not fund a bank even if it passes the test. The supervisor therefore does not benefit from a reduction in intervention costs. Setting s_1 close to 1 would achieve that as banks that pass the test would now get funded, even in the absence of a price signal. It would, however, undermine the speculator's information production incentives: for the few banks that pass the test, the speculator stands to gain little informational advantage, while most of the banks that fail will end up being shut down (only those with $s \in (\bar{s}, s_1)$ will be rescued by the supervisor). It is therefore better to leave the test uninformative for small but positive values of δ, and then jump to the usual lenient test, once δ becomes large enough so that the supervisor prefers to refrain from direct interventions.

To summarize, by allowing the supervisor to learn from stock prices and intervene at a cost, we bridge the gap between two sets of models. First, there are papers that focus on a supervisor learning from stock prices (e.g., Bond and Goldstein, 2015 and Siemroth, 2019). Others study the implication of information design on access to capital (e.g., Inostroza, 2023, Orlov, Zryumov and Skrzypacz, 2023 or Fuchs, Fukuda and Neuhann, 2024). In this extension we allow for an interplay between these mechanisms: stress tests directly affect funding conditions, but to the extent that the supervisor can intervene at a cost, supervisory learning from stock prices becomes a more important determinant of the information design problem. While the precise stress test design depends on model parameters, we identify leniency as a robust feature.

8 Conclusion

This paper models the link between bank stress test design and market discipline. It allows markets to play an indirect role by generating useful price signals about bank fundamentals, and a direct role by providing funding at terms that are sensitive to the stress test results as well as stock price signals. We show that markets are useful at reducing type-2 errors, that is, identifying bad banks that did well in a stress test. Markets are less good at reducing type-1 errors, i.e., providing funds to good banks that did poorly in the stress test. The supervisor optimally distorts the test to be lenient, because this improves market discipline. We extend the model to allow for non-trivial interplay between direct intervention by the supervisor and market discipline. The supervisor prefers to save on direct interventions as they are costly, and therefore enlists market discipline. The optimal stress test is still distorted towards leniency, and it may degenerate into a completely uninformative test when the intervention cost becomes negligible.

Although our model is set up to address the design of bank stress tests, we believe the underlying information design problem is pertinent in other contexts. For example, a credit rating agency needs to decide on a rating system, keeping in mind that this may have an impact on the information that speculators subsequently produce about the issuing firm. Similarly, there is a degree of freedom in setting up accounting rules so that a firm's financial health can appear better or worse (e.g., marking to market versus historical value rules, loan-loss accounting rules etc.). Little is understood about how such rules interact with other sources of information, particularly those contained in stock prices. Our paper proposes a tractable model that can be used in future research to address these questions.

Finally, our model makes empirical predictions concerning the information content of bank stock prices, depending on the stress test design and outcome. These remain to be tested in future research.

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Appendix A: Default Externalities ($\mu^* > \widehat{\mu}$)

So far, we considered the case where the supervisor would finance more banks than the capital providers are inclined to support ($\mu^* \leq \widehat{\mu}$). In principle, it could be the case that the social value of liquidating a bank is higher than internalized by the bank's capital providers and the supervisor is more inclined than the market to cease the bank's operations. This might be the case, for example, if a bank's default generates negative externalities, either on other banks or the real economy. This corresponds to the case $\mu^* > \widehat{\mu}$: banks with resilience levels $s \in [\widehat{\mu}, \mu^*)$ would be able to raise funds, although the supervisor would like to prevent that.

To capture this situation, we modify our baseline micro-foundation in Section 4 by introducing a social cost c of defaulting. A default occurs when the bank raises funds and invests, but then generates a zero cash flow from the investment. Capital providers do not internalize the default externality so that $\widehat{\mu}$ remains unchanged. For a given belief μ the supervisor prefers for a bank to invest ($a = 1$) if

$$\mu(p(R-1) + (1-p)(-1-c)) + (1-\mu)(-1-c) \geq 0. \tag{35}$$

Hence, the supervisor prefers $a = 1$ for beliefs

$$\mu \geq \mu^* \equiv \frac{1+c}{p(R+c)} > \widehat{\mu}. \tag{36}$$

We start by providing the optimal stress test in the absence of stock market signals.

Lemma 5. *Without information acquisition by the speculator, the optimal stress test is a binary partition with passing threshold* $s_N^L \equiv \min\{2\widehat{\mu}, \mu^*\}$.

Proof. See Appendix B. ☐

The test is a *pass/fail* experiment just like in previous cases. Whenever the market implements the supervisor's preferred course of action, the optimal cut-off is set at $s_N^L = \mu^*$ to minimize type-1 and type-2 errors. When μ^* is too high, even a *fail* grade conveys sufficiently positive news for capital providers to extend credit to the bank. Therefore, the pass threshold is optimally adjusted downward to $s_N^L = 2\widehat{\mu}$, ensuring the credibility of the *fail* grade and dissuading the market from extending credit.

The following proposition describes the optimal stress test design with feedback from stock prices.

Proposition 6. *With information acquisition by the speculator, the optimal stress test is as follows:*

- *if* $\mu^* \in (\widehat{\mu}, \underline{\mu})$*, where* $\underline{\mu}$ *is defined in (84), it is a binary partition with passing threshold* $s_F^L = \widetilde{s}$*, where* $\widetilde{s}$ *is the unique* s_1 *that solves (30);*
- *if* $\mu^* \in [\underline{\mu}, 1)$ *and* $\widehat{\mu} < 2/5$ *it contains two coarse buckets followed by granular grades for resilience levels above the buckets, where the threshold that separates the two buckets is*

$$s_{1,F}^L \equiv \min\left\{2\widehat{\mu}, s^\dagger\right\}, \tag{37}$$

  where $s^\dagger$ *is the unique* s_1 *that solves (86), while the threshold that separates the upper bucket from the granular grades is*

$$s_{2,F}^L \equiv 1 - \frac{1}{2}\left(s_{1,F}^L - \widehat{\mu}\right); \tag{38}$$

- *if* $\mu^* \in [\underline{\mu}, 1)$ *and* $\widehat{\mu} \geq 2/5$*, there exists a threshold* $\overline{\mu}$ *(defined in (88)) such that if* $\mu^* < \overline{\mu}$ *the optimal stress test is as described above and if* $\mu^* \geq \overline{\mu}$ *the test contains only one coarse bucket followed by granular grades, where the threshold that separates the coarse bucket from the granular grades is*

$$s_{F,g}^L \equiv \min\left\{2\widehat{\mu}, s^\ddagger\right\}, \tag{39}$$

  where $s^\ddagger$ *is the unique* s_1 *that solves (89).*

Proof. See Appendix B. □

The stress test is a lenient binary partition when the supervisor's and the capital providers' preferences about debt roll-over decisions are sufficiently aligned, i.e. when μ^* is close to $\widehat{\mu}$.

The advantages of leniency are once again rooted in the additional information generated by the speculator, which in turn guides funding decisions more effectively. As μ^* rises, the optimal passing threshold follows suit to minimize the test's statistical errors, but remains lenient to incentivize information production by the speculator. As the passing threshold rises, the *pass* grade progressively conveys better news to speculators by encompassing banks of increasingly superior quality. Consequently, residual uncertainty following a *pass* diminishes, reaching a point where speculators' incentives to acquire information begin to decline. This happens when s_1 surpasses $\widehat{\mu}$, as the *pass* grade induces beliefs $\mu_2 = \frac{s_1+1}{2}$ while information acquisition is maximized at $\mu = \frac{\widehat{\mu}+1}{2}$. At this point, it is optimal to exclude highly sound banks from the *pass* grade by setting s_2 below 1. This maintains residual uncertainty following a *pass* without the need to lower s_1 far below the benchmark s_F^L. By doing so, the supervisor improves both direct and indirect market discipline by maintaining the speculator's incentives to gather information while optimizing resource allocation in the absence of market signals. Surprisingly, resilience levels above s_2 are fully disclosed. Why does the supervisor not lump them into a unique, coarse grade m_3? Refining the coarse grade m_3 by splitting it into two grades $m_3^{'}$ and $m_3^{''}$ increases information production following $m_3^{'}$ and reduces it following $m_3^{''}$, as the latter message constitutes more conclusive news. Since information production has value only when a poorly capitalized bank passes the test, an event that is less and less likely for higher resilience levels, refining the grade increases the ex-ante level of information acquisition.

As μ^* continues to increase, the passing threshold will eventually reach its upper bound defined by $2\widehat{\mu}$. At this point, the *fail* grade represents excessively positive news, to the extent that any further increase in the passing threshold would prompt capital providers to finance the bank after a *fail*. To ensure direct market discipline and efficient resource allocation, it is optimal to maintain the passing threshold at a constant level. Figure 4 depicts the cutoffs of the optimal stress test with feedback from stock prices and in the benchmark.

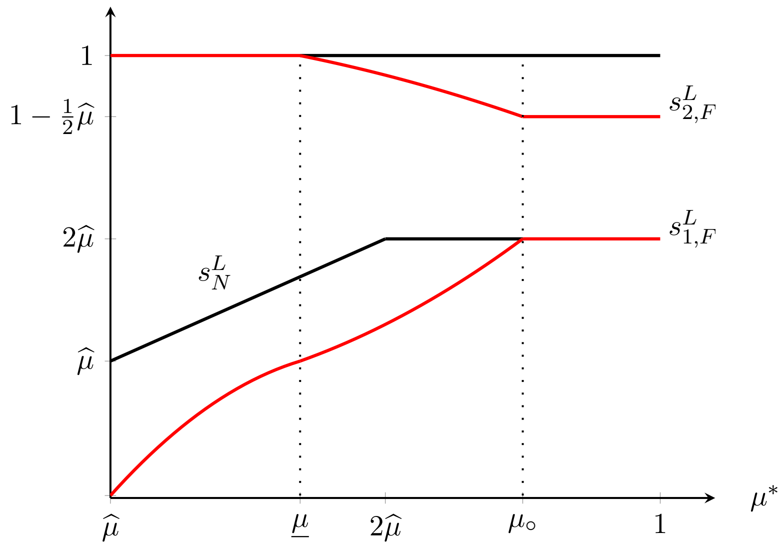

Figure 4: Thresholds of the Optimal Stress when $\mu^* > \widehat{\mu}$, and $\widehat{\mu} < 2/5$.
Benchmark $(s^L_N, 1)$ (in black), and Model with Feedback from Stock Prices $(s^L_{1,F}, s^L_{2,F})$ (in red).

Appendix B: Proofs

Proof of Lemma 1. Ignoring the constraints in problem (16), the FOC for s_1 is :

$$v'(s_1) = 1 - \frac{s_1}{\mu^*} = 0,$$

and is solved for $s_1 = \mu^*$. Since $\widehat{\mu} = \mu^*$ the constraints are satisfied at the unconstrained optimum. $\square$

Proof of Lemma 2. Consider the trading game at date 2 and let μ be the public posterior beliefs (that $\omega = h$) after a generic stress test result. Consider the following speculator's equilibrium trading strategy:

$$x_I(z) = \begin{cases} 1 & \text{if } z = h \\ 0 & \text{if } z = \varnothing \\ -1 & \text{if } z = l. \end{cases}$$

First, we determine the price $p(X)$ chosen by the market maker as a function of the order X. Notice that any order $x_I \notin \{-1, 1\}$ fully reveals the speculator's order who can therefore not make a trading profit. When the speculator is active, we can restrict attention to orders of size $x_I \in \{-1, 1\}$. When $X = (1, 1)$ the market maker infers that the informed speculator submitted a buy order and hence must have received a private signal $z = h$. It follows that the market will fund the bank ($a = 1$) at date 3. The market maker sets a price that reflects the speculator's private information and the supervisor's intervention decision, i.e. $P(1, 1) = E_h^1(1)$. Similarly, when $X = (-1, -1)$ the low state is revealed, the market maker sets $P(-1, -1) = E_l^0 = 0$. When $X \in \{(1, -1), (-1, 1)\}$ the order flow allows no inference over the speculator's private information and the market maker's posterior belief therefore remains equal to the prior, μ. When $X \in \{(1, 0), (-1, 0)\}$ the market maker understands that the speculator received an uninformative signal $z = \varnothing$ and abstained from trading. Again, the market maker does not update from the prior. It follows that, for $X \in \{(1, -1), (-1, 1), (1, 0), (-1, 0)\} \equiv X_\varnothing$, i) at date 3, the market will make the funding decision contingent on the outcome of the stress test alone, as the equity price reveals no additional information; and ii) the market maker sets a price $P(X) = \mu \max\{E_h^1(\mu), 0\}$ (since when $E_h^1(\mu) < 0$ the capital providers will not fund the bank). Thus, the price schedule is

$$P(X) = \begin{cases} E_h^1(1) & \text{if } X = (1, 1) \\ \mu \max\{E_h^1(\mu), 0\} & \text{if } X \in X_\varnothing \\ 0 & \text{if } X = (-1, -1). \end{cases}$$

Next, we compute the speculator's profits from the proposed trading strategy. If $z = h$, the speculator submits a buy order ($x_I = 1$). With probability 1/2 the liquidity trader trades in the same direction ($x_U = 1$ and $X = (1, 1)$) and the speculator's private information is revealed to the market maker (and the investors) who then sets a price equal to $E_h^1(1)$. The speculator's trading profits are nil, $E_h^1(1) - P(1, 1) = 0$. With probability 1/2 the liquidity trader trades in the opposite direction ($x_U = -1$ and $X = (-1, 1)$), the speculator retains his private information and makes profits equal to $\max\{E_h^1(\mu), 0\} - P(-1, 1) = (1 - \mu) \max\{E_h^1(\mu), 0\}$. It follows that, given $z = h$ the speculator's expected trading profits are $\frac{1}{2}(1 - \mu) \max\{E_h^1(\mu), 0\}$. If $z = l$, the speculator submits a sell order ($x_I = -1$) and, by the same reasoning trading profits are $\frac{1}{2}\mu \max\{E_h^1(\mu), 0\}$. If $z = \varnothing$ the speculator abstains from trading and makes profits equal to 0.

In summary, the expected trading profits are

$$\pi(z,\mu) = \begin{cases} \frac{1}{2}(1-\mu)\max\{E_h^1(\mu),0\} & \text{if } z = h \\ 0 & \text{if } z = \varnothing \\ \frac{1}{2}\mu\max\{E_h^1(\mu),0\} & \text{if } z = l \end{cases}$$

Taking expectations over z, we get the expected equilibrium trading profits after the realization of the stress test result:

$$\begin{aligned}\pi(\mu) &= \mu\,\sigma\left(\frac{1}{2}(1-\mu)\max\{E_h^1(\mu),0\}\right) + (1-\mu)\sigma\left(\frac{1}{2}\mu\max\{E_h^1(\mu),0\}\right) \\ &= \sigma\mu(1-\mu)\max\{E_h^1(\mu),0\}.\end{aligned}$$

We show that the proposed trading strategy is indeed optimal. Consider trading after a test result inducing a belief $\mu \geq \widehat{\mu}$. Given $z = h$ it is optimal to buy: abstaining from trading yields profits equal to $0 < \frac{1}{2}(1-\mu)E_h^1(\mu)$ and selling yields profits $\frac{1}{2}(\mu E_h^1(\mu) - E_h^1(\mu)) + \frac{1}{2}(0 - 0) = -\frac{1}{2}(1-\mu)E_h^1(\mu) < 0$. Given $z = l$ it is optimal to sell: abstaining from trading yields profits equal to $0 < \frac{1}{2}\mu E_h^1(\mu)$. If the speculator buys instead, the order flow will be either $X = (1,1)$ or $X = (-1,1)$. In either case, the bank gets funded and expected trading profits are $\frac{1}{2}(0 - E_h^1(1)) + \frac{1}{2}(0 - \mu E_h^1(\mu)) < 0$. If $z = \varnothing$ and the speculator buys, the bank will be funded and trading profits will be $\frac{1}{2}(\mu E_h^1(1) - E_h^1(1)) + \frac{1}{2}(\mu E_h^1(\mu) - \mu E_h^1(\mu)) < 0$. If the speculator sells instead, the order flow can be $X = (-1,1)$, in which case the bank obtains funding and has equity value $\mu E_h^1(\mu)$. Since this is equal to the price paid in this state, profits are zero. Instead, order flow may be $X = (-1,-1)$. The price will now be zero, the bank will not be funded and its equity value zero, yielding again zero trading profits.

Consider trading after a test result inducing a belief $\mu < \widehat{\mu}$. If the speculator sells, he always gets a price of zero, and there will never be funding so the equity value will also be zero. Hence, selling yields zero profits. If the speculator buys, with probability $\frac{1}{2}$ order flow will be $X = (-1,1)$ in which case the price is zero, there will be no funding, and equity value will also be zero. With equal probability, the order flow will be $X = (1,1)$, and the price equals $E_h^1(1)$ while the equity value would be lower and equal to $\mu E_h^1(1)$. If the speculator deviated to purchasing information and learned $\omega = h$, he makes zero trading profits. Hence, the deviation generated a loss, net of the information acquisition cost. If the speculator deviated to buying without a positive signal, the expected value of equity is below $E_h^1(\mu)$ so the speculator makes a loss. $\square$

Proof of Lemma 3. Consider all the outcomes m_i that induce beliefs $\mu_i < \widehat{\mu}$. All these m_i result in no information production by the speculator ($\sigma(\mu_i) = 0$) and no funding provision by the market ($a(\mu_i) = 0$). Hence, we can pool all these potential signals in a unique signal m_1 for all $s \in [0, s_1)$ with $\mu_1 < \widehat{\mu}$. The corresponding expected value for the supervisor generated by the outcome m_1 is

$$\begin{aligned} g(s_1) &= \frac{1}{2}\Big(\Pr(m_1|\omega = h)V_h^0 + \Pr(m_1|\omega = l)V_l^0\Big) \\ &= \frac{1}{2}\Big(s_1^2 V_h^0 + (2s_1 - s_1^2)V_l^0\Big). \end{aligned}$$

Now, consider all the outcomes m_i for $i \in \{2, 3, \ldots, n\}$. These outcomes induce posterior beliefs $\mu_i \geq \widehat{\mu}$ (otherwise we could have pooled the signal m_i with m_1). If, at date 2, the order flow is uninformative the market invests ($a(\mu_i) = 1$) at date 3. However, since the outcome induces a positive level $\sigma(\mu_i)$ of information production by the speculator, if the order flow reveals that $\omega = l$, the market does not invest at date 3 and chooses $a = 0$. If the state is $\omega = l$, order flow reveals it with probability $\frac{1}{2}\sigma(\mu_i)$. The corresponding expected bank value generated by some outcome m_i is

$$\begin{aligned} f(s_{i-1}, s_i) =& \frac{1}{2}\left(\Pr(m_i|\omega = h)V_h^1 + \Pr(m_i|\omega = l)\left(V_l^1 + \frac{1}{2}\sigma(\mu_i)\Delta V_l\right)\right) \\ =& \frac{1}{2}\left((s_i^2 - s_{i-1}^2)V_h^1 + \big(2(s_i - s_{i-1}) - (s_i^2 - s_{i-1}^2)\big)\left(V_l^1 + \frac{1}{2}\sigma(\mu_i)\Delta V_l\right)\right). \end{aligned}$$

The ex-ante expected value of the bank for a given stress test S can be written as

$$V(S) = g(s_1) + \sum_{i=2}^{n} f(s_{i-1}, s_i). \tag{40}$$

Note that, the second term in (40) is a telescoping sum where

$$\sum_{i=2}^{n}(s_i^2 - s_{i-1}^2) = (1 - s_1^2),$$
$$\sum_{i=2}^{n}\big(2(s_i - s_{i-1}) - (s_i^2 - s_{i-1}^2)\big) = 2(1 - s_1) - (1 - s_1^2).$$

It follows that the objective function can be written as

$$\begin{aligned} V(S) = &\frac{1}{2}\Big(s_1^2 V_h^0 + (2s_1 - s_1^2)V_l^0 + (1 - s_1^2)V_h^1 + \big(2(1 - s_1) - (1 - s_1^2)\big)V_l^1\Big) + \\ &+ \frac{1}{2}\sum_{i=2}^{n}\big(2(s_i - s_{i-1}) - (s_i^2 - s_{i-1}^2)\big)\frac{1}{2}\sigma(\mu_i)\Delta V_l \\ = &\, v(s_1) + \frac{1}{2}\sum_{i=2}^{n}(s_i - s_{i-1})\left(1 - \frac{s_{i-1} + s_i}{2}\right)\sigma(\mu_i)\Delta V_l, \end{aligned}$$

where $v(s_1)$ is defined in (15). By applying the definitions in equations (2), (3) and (19) we obtain the objective function in (22). □

Proof of Proposition 1. We first introduce some notation. Let

$$S(a, b) \equiv \{a = s_0, s_1, \ldots, s_{n-1}, s_n = b\}$$

be a partition of the interval $[a, b] \subset \mathbb{R}$ such that $a = s_0 < s_1 < s_2 \cdots < s_{n-1} < s_n = b$. In our application we will have $0 \le a < b \le 1$ so that $S(a, b)$ can be thought of as a partition of a subspace of $[0, 1]$. Let $\mathcal{S}(a, b)$ be the set of all possible partitions $S(a, b)$ over the interval $[a, b]$. Let $\overline{S}(a, b)$ be the finest partition in $\mathcal{S}[a, b]$, i.e. such that $n \to \infty$; and let $\underline{S}(a, b)$ be the coarsest partition in $\mathcal{S}[a, b]$, i.e. such that $n = 1$. Lastly, for some function $f : [a, b] \to \mathbb{R}$, we define

$$R\big(f, S(a, b)\big) \equiv \sum_{i=1}^{n}(s_i - s_{i-1})f\left(\frac{s_{i-1} + s_i}{2}\right)$$

as the midpoint Riemann sum of f with respect to the partition $S(a, b)$. In what follows, we will use the following properties of the midpoint Riemann sum (see, e.g., Davis and Rabinowitz (1984) p. 54):

- if f is convex over $[a,b]$ then $R(f,\overline{S}(a,b)) \geq R(f,S(a,b))$, $\forall\, S(a,b) \in \mathcal{S}(a,b)$;

- if f is concave over $[a,b]$ then $R(f,\underline{S}(a,b)) \geq R(f,S(a,b))$, $\forall\, S(a,b) \in \mathcal{S}(a,b)$.

We proceed in 3 steps. Step 1 establishes the general structure of the stress test. Step 2 simplifies the objective function in problem (21) and writes it as a function of two thresholds (s_1, s_2). Finally, Step 3 determines the optimal thresholds.

Step 1 (General Structure). Fix the optimal s_1 and assume it is interior, i.e. $s_1 \in (0, s_2)$ (this will be true at the optimum), consider a subset $[s_1, 1] \subseteq [0,1]$ and define the function $\widehat{\Sigma}(s) : [s_1, 1] \to \mathbb{R}$ as

$$\widehat{\Sigma}(s) = \frac{1}{2\tau} s(1-s)^2 \left(\frac{1}{\mu^*} - \frac{1}{s}\right) = \frac{1}{2\tau}(1-s)^2\left(\frac{s}{\mu^*} - 1\right).$$

Consider a partition $S(s_1, 1) \equiv \{s_1, s_2, s_3, \ldots, s_{n-2}, s_{n-1}, 1\}$ and note that the stress test design problem (21), with $\widehat{\mu} = \mu^*$, is equivalent to

$$\begin{aligned} \max_{S(s_1,1)} \;& R(\widehat{\Sigma}, S(s_1,1)) \\ \text{s.t. } & \frac{s_1 + s_2}{2} \geq \mu^* \\ & \frac{0 + s_1}{2} < \mu^*. \end{aligned} \tag{41}$$

The first and second derivative of $\widehat{\Sigma}(s)$ are, respectively:

$$\widehat{\Sigma}'(s) = \frac{1}{2\tau}\left[-2(1-s)\left(\frac{s}{\mu^*} - 1\right) + (1-s)^2 \frac{1}{\mu^*}\right],$$
$$\widehat{\Sigma}''(s) = \frac{1}{2\tau}\left[2\left(\frac{s}{\mu^*} - 1\right) - 4(1-s)\frac{1}{\mu^*}\right],$$

so that $\widehat{\Sigma}(s)$ attains its maximum at $s = \frac{1}{3} + \frac{2}{3}\mu^*$ and it is concave over $\left[s_1, \frac{2}{3} + \frac{1}{3}\mu^*\right]$ and convex over $\left(\frac{2}{3} + \frac{1}{3}\mu^*, 1\right]$. For some j, we have $S(s_1,1) = S(s_1, s_j) \cup S(s_j, 1)$ and $R(\widehat{\Sigma}, S(s_1,1)) = R(\widehat{\Sigma}, S(s_1, s_j)) + R(\widehat{\Sigma}, S(s_j, 1))$. For the properties of the midpoint Riemann sum introduced above, the partition that maximizes $R(\widehat{\Sigma}, S(s_1,1))$ is $S(s_1, 1) = \underline{S}(s_1, s_j) \cup \overline{S}(s_j, 1)$ for some optimally chosen s_j, that is the partition that solves problem (41) has a unique coarse sub-interval $[s_1, s_j]$ with $s_1 < s_j$ and a collection of infinitesimally small sub-intervals over the interval $[s_j, 1]$. Relabelling $s_j = s_2$, we have that the optimal stress test has two coarse messages ($m_0 = s \in [0, s_1)$ and $m_1 = s \in [s_1, s_2)$) and a set of granular grades for $s \in [s_2, 1]$.

Step 2 (Simplifying the Objective Function). Step 1 allows us to write the objective $V(S)$ as a function of the thresholds (s_1, s_2) only. Note that if the stress test S has fully granular grades for $s \in [s_2, 1]$ we have $\nu_i = ds$ and $\mu_i = s$ for $i \in \{3, \ldots, n\}$ where $n \to \infty$. It follows that the objective function in problem (21) reduces to:

$$V(s_1, s_2) = v(s_1) + \frac{1}{2\tau}\left(\nu_2\mu_2(1-\mu_2)^2\left(\frac{1}{\mu^*} - \frac{1}{\mu_2}\right) + \int_{s_2}^{1} s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\mathrm{d}s\right).$$

Adding and subtracting $\int_{s_1}^{s_2} s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\mathrm{d}s$ on the right-hand side we get

$$V(s_1, s_2) = v(s_1) + \frac{1}{2\tau}\left(\int_{s_1}^{1} s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\mathrm{d}s + A(s_1, s_2)\right)$$

where

$$\begin{aligned}
A(s_1, s_2) &\equiv \nu_2\mu_2(1-\mu_2)^2\left(\frac{1}{\mu^*} - \frac{1}{\mu_2}\right) - \int_{s_1}^{s_2} s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\mathrm{d}s,\\
&= \int_{s_1}^{s_2}\left[\mu_2(1-\mu_2)^2\left(\frac{1}{\mu^*} - \frac{1}{\mu_2}\right) - s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\right]\mathrm{d}s,\\
&= \int_{s_1}^{s_2}\left[\frac{1}{\mu^*}\Big(\mu_2(1-\mu_2)^2 - s(1-s)^2\Big) - \Big((1-\mu_2)^2 - (1-s)^2\Big)\right]\mathrm{d}s,\\
&= \left(\frac{1}{\mu^*}\frac{1}{12}(s_2 - s_1)^3(2 - 3\mu_2)\right) - \left(-\frac{1}{12}(s_2 - s_1)^3\right).
\end{aligned}$$

It follows that the objective function simplifies to:

$$V(s_1, s_2) = v(s_1) + \frac{1}{2\tau}\left(\int_{s_1}^{1} s(1-s)^2\left(\frac{1}{\mu^*} - \frac{1}{s}\right)\mathrm{d}s + \frac{1}{12}(\nu_2)^3\left(\frac{1}{\mu^*}(2 - 3\mu_2) + 1\right)\right), \tag{42}$$

where $v(s_1)$ is defined in (15), while $\nu_2 = s_2 - s_1$ and $\mu_2 = \frac{s_1 + s_2}{2}$, as defined in (2) and (3), and the stress test design problem in (21) simplifies to:

$$
\begin{aligned}
&\max_{s_1,s_2} V(s_1,s_2)\\
&s.t.\ \mu_1 < \mu^*\\
&\qquad \mu_2 \geq \mu^*.
\end{aligned}
\tag{43}
$$

Step 3 (Optimal Thresholds). Neglecting the constraints in problem (43), the FOCs are:

$$
\begin{aligned}
\frac{\partial V}{\partial s_1} &= v'(s_1) + \frac{1}{2\tau}\left[-s_1(1-s_1)^2\left(\frac{1}{\mu^*}-\frac{1}{s_1}\right) - \frac{3}{12}(\nu_2)^2\left(\frac{1}{\mu^*}(2-3\mu_2)+1\right) - \frac{3}{24}(\nu_2)^3\frac{1}{\mu^*}\right] = 0,\\
\frac{\partial V}{\partial s_2} &= \frac{1}{2\tau}\left[\frac{3}{12}(\nu_2)^2\left(\frac{1}{\mu^*}(2-3\mu_2)+1\right) - \frac{3}{24}(\nu_2)^3\frac{1}{\mu^*}\right] = 0,
\end{aligned}
$$

and after simple algebra they simplify to:

$$
\begin{aligned}
\frac{\partial V}{\partial s_1} &= v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\mu^*}-1\right) - (\nu_2)^2\frac{1-\frac{1}{2}(s_2-\mu^*)-s_1}{2\mu^*}\right] = 0,\\
\frac{\partial V}{\partial s_2} &= \frac{1}{2\tau}(\nu_2)^2\frac{1-\frac{1}{2}(s_1-\mu^*)-s_2}{2\mu^*} = 0.
\end{aligned}
\tag{44}
$$

First, we show that when the constraints in problem (43) are slack the optimal stress test is a binary partition, $s_2 = 1$; we show that the partition features leniency, $s_1 < \mu^*$; and we provide the equation that implicitly defines the optimal s_1. Note that the second equation in (44) has two solutions: $s_2 = s_1$ and $s_2 = 1 - \frac{1}{2}(s_1 - \mu^*)$. The first solution is a stationary point where $\frac{\partial V}{\partial s_1} = \frac{\partial^2 V}{\partial s_1^2} = 0$ but is not a maximum. The second solution, $s_2 = 1 - \frac{1}{2}(s_1 - \mu^*)$, is greater than 1, if $s_1 < \mu^*$.

Suppose the optimal $s_1 < \mu^*$ and therefore the optimal $s_2 = 1$. If $s_2 = 1$, the first-order condition with respect to s_1 becomes:

$$
\begin{aligned}
\left.\frac{\partial V}{\partial s_1}\right|_{s_2=1} &= v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\mu^*}-1\right) - (1-s_1)^2\frac{1-\frac{1}{2}(1-\mu^*)-s_1}{2\mu^*}\right]\\
&= 1 - \frac{s_1}{\mu^*} + \frac{1}{2\tau}(1-s_1)^2\frac{\frac{3\mu^*-1}{2}-s_1}{2\mu^*} = 0,
\end{aligned}
\tag{45}
$$

which is equivalent to (23). The first-order condition (45) has a unique solution in $s_1 \in [0,1]$, since the left-hand side of (the second line of) equation (45) is strictly decreasing. To see this differentiate it with respect to s_1 to get:

$$-\frac{1}{\mu^*} - \frac{3}{4\mu^*\tau}(1-s_1)(\mu^* - s_1)$$

which is strictly negative if

$$\tau > \frac{3}{4}(1-s_1)(s_1-\mu^*).$$

This inequality is satisfied even when the right-hand side takes the highest possible value, i.e. $\frac{3}{4}\left(\frac{1-\mu^*}{2}\right)^2$ (attained at $s_1 = (1+\mu^*)/2$), since we have

$$\tau > \frac{1}{\mu^*}\left(\frac{1-\mu^*}{2}\right)^2 > \frac{3}{4}\left(\frac{1-\mu^*}{2}\right)^2$$

by assumption (20). Finally, using (45) and substituting $s_1 = \mu^*$ it is immediate that the resulting expression is negative, hence the solution to (45) features $s_1 < \mu^*$. Similarly, substituting $s_1 = 0$ yields a positive expression, implying that the solution to (45) features $s_1 > 0$.

To rule out a maximum where $\mu^* < s_1 < s_2 < 1$, substitute the candidate interior optimum $s_2 = 1 - \frac{1}{2}(s_1 - \mu^*)$ into the first expression of (44) to get

$$\frac{\partial V}{\partial s_1} = v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\mu^*}-1\right) - \left(1-\frac{1}{2}(s_1-\mu^*)-s_1\right)^2 \frac{1-\frac{1}{2}(1-\frac{1}{2}(s_1-\mu^*)-\mu^*)-s_1}{2\mu^*}\right]. \tag{46}$$

Note that $v'(s_1) < 0$ for $s_1 > \mu^*$. It is therefore sufficient to show that the expression in square brackets is negative. We have

$$-(1-s_1)^2\left(\frac{s_1}{\mu^*}-1\right) - \left(1-\frac{1}{2}(s_1-\mu^*)-s_1\right)^2 \frac{1-\frac{1}{2}(1-\frac{1}{2}(s_1-\mu^*)-\mu^*)-s_1}{2\mu^*} < 0,$$

if and only if

$$2(1-s_1)^2(s_1-\mu^*) + \frac{1}{2}\left(1-s_1-\frac{1}{2}(s_1-\mu^*)\right)^3 > 0.$$

By our hypothesis that $s_1 > \mu^*$ it follows that $2(1-s_1)^2(s_1-\mu^*) > 0$. We now show that the second addend is also positive. The term is decreasing in s_1, so showing that it is positive for the highest admissible value for s_1 is sufficient. Note that the constraint $s_2 > s_1$ with $s_2 = 1 - \frac{1}{2}(s_1 - \mu^*)$ implies $s_1 < \frac{2+\mu^*}{3}$. For $s_1 = \frac{2+\mu^*}{3}$, the term $\left(1 - s_1 - \frac{1}{2}(s_1 - \mu^*)\right)$ is equal to

$$1 - \frac{2+\mu^*}{3} - \frac{1}{2}\left(\frac{2+\mu^*}{3} - \mu^*\right) = 0.$$

Hence, we have $\forall s_1 \in \left(\mu^*, 1 - \frac{1}{2}(s_1 - \mu^*)\right)$,

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1 > \mu^* \\ s_2 = 1 - \frac{1}{2}(s_1 - \mu^*)}} < 0,$$

hence the optimal s_1 must be below μ^*.

Lastly, we show that the constraints in problem (21) are satisfied at the unconstrained optimum. The first constraint $\mu_1 = (0 + s_1)/2 < \mu^*$ is satisfied since we know $s_1 < \mu^*$. The second constraint, $\mu_2 = (s_1 + 1)/2 \geq \mu^*$, is satisfied if $s_1 \geq 2\mu^* - 1$. Consider the following derivative:

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1 = 2\mu^* - 1 \\ s_2 = 1}} = 1 - \frac{2\mu^* - 1}{\mu^*} + \frac{1}{2\tau}\frac{(1-\mu^*)^3}{\mu^*}.$$

Note that $\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1 = 2\mu^* - 1 \\ s_2 = 1}} > 0$ (and so $s_1 > 2\mu^* - 1$) since $\mu^* < 1$. Thus, we have $\mu_2 > \mu^*$ and all the constraints are satisfied at the optimum.

It follows that the optimal stress test is a binary partition with $s_1 < \mu^*$. □

Proof of Proposition 2. The optimal cutoff s_F is implicitly defined as the solution to equation (23), which can be rearranged as

$$\mu^* - s_F + \frac{1}{4\tau}K(s_F, \mu^*) = 0, \tag{47}$$

where

$$K(s_F, \mu^*) \equiv (1 - s_F)^2\left(\frac{3\mu^* - 1}{2} - s_F\right).$$

Since the s_F that solves equation (47) is in the interval $\left(\frac{3\mu^*-1}{2}, \mu^*\right)$ we have that $K(s_F, \mu^*) < 0$.

In what follows, we will also use the fact that

$$k(s_F, \mu^*) \equiv \frac{\partial K}{\partial s_F} = 2(1 - s_F)(-1)\left(\frac{3\mu^* - 1}{2} - s_F\right) + (1 - s_F)^2(-1)$$
$$= -3(1 - s_F)(\mu^* - s_F) < 0.$$

Employing the implicit function theorem, we totally differentiate equation (47) with respect to τ and μ^* to get:

$$\frac{\partial s_F}{\partial \tau} = \frac{\frac{1}{4\tau^2} K(s_F, \mu^*)}{\frac{1}{4\tau} k(s_F, \mu^*) - 1} > 0$$
$$\frac{\partial s_F}{\partial \mu^*} = \frac{1 + \frac{1}{4\tau} \frac{\partial K}{\partial \mu^*}}{1 - \frac{1}{4\tau} k(s_F, \mu^*)} > 0,$$

where the last line follows from the fact that $\frac{\partial K}{\partial \mu^*} = (1 - s_F)^2 \frac{3}{2} > 0$. □

Proof of Lemma 4. Neglecting the constraints in problem (16), the FOC for s_1 is :

$$v'(s_1) = 1 - \frac{s_1}{\mu^*} = 0,$$

and is solved for $s_1 = \mu^*$. The constraint $\mu_1 = s_1/2 < \widehat{\mu}$ is satisfied for $s_1 = \mu^*$ since $\mu^* < \widehat{\mu}$. The constraint $\mu_2 \geq \widehat{\mu}$ is satisfied at the unconstrained optimum when $(\mu^* + 1)/2 \geq \widehat{\mu}$ and is binding otherwise. When the constraint is slack, the optimum is $s_1 = \mu^*$. When it is binding, s_1 is chosen to satisfy the constraint, i.e. $s_1 = 2\widehat{\mu} - 1$. Thus, the optimal stress test is a binary partition with cutoff $s_1 = \max\{2\widehat{\mu} - 1, \mu^*\}$. □

Proof of Proposition 3. The proof proceeds as in Proposition 1 until Step 3, and the stress test design problem can be written as:

$$\begin{aligned} &\max_{s_1, s_2} V(s_1, s_2) \\ &s.t.\ \mu_1 < \widehat{\mu} \\ &\qquad \mu_2 \geq \widehat{\mu}, \end{aligned} \tag{48}$$

where

$$V(s_1, s_2) = v(s_1) + \frac{1}{2\tau}\left(\int_{s_1}^{1} s(1-s)^2\left(\frac{1}{\widehat{\mu}} - \frac{1}{s}\right)\mathrm{d}s + \frac{1}{12}(\nu_2)^3\left(\frac{1}{\widehat{\mu}}(2 - 3\mu_2) + 1\right)\right), \tag{49}$$

and $v(s_1)$ is defined in (15), while $\nu_2 = s_2 - s_1$ and $\mu_2 = \frac{s_1+s_2}{2}$, as defined in (2) and (3).

Neglecting the constraints in problem (48), the FOCS, after some algebra, simplify to:

$$\begin{aligned}\frac{\partial V}{\partial s_1} &= v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) - (\nu_2)^2\frac{1 - \frac{1}{2}(s_2 - \widehat{\mu}) - s_1}{2\widehat{\mu}}\right] = 0,\\ \frac{\partial V}{\partial s_2} &= \frac{1}{2\tau}(\nu_2)^2\frac{1 - \frac{1}{2}(s_1 - \widehat{\mu}) - s_2}{2\widehat{\mu}} = 0.\end{aligned} \tag{50}$$

First, we show that when the constraints in problem (48) are slack the optimal stress test is a binary partition, $s_2 = 1$, and we provide the equation that implicitly defines the optimal s_1. Note that the second equation has two solutions: $s_2 = s_1$ and $s_2 = 1 - \frac{1}{2}(s_1 - \widehat{\mu})$. The first solution is a stationary point where $\frac{\partial V}{\partial s_1} = \frac{\partial^2 V}{\partial s_1^2} = 0$ but is not a maximum. The second solution, $s_2 = 1 - \frac{1}{2}(s_1 - \widehat{\mu})$, is greater than 1, since at the optimum we have $s_1 < \widehat{\mu}$.

Suppose the optimal $s_1 < \widehat{\mu}$ and therefore the optimal $s_2 = 1$. If $s_2 = 1$, the first-order condition with respect to s_1 becomes:

$$\begin{aligned}\left.\frac{\partial V}{\partial s_1}\right|_{s_2=1} &= v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) - (1-s_1)^2\frac{1 - \frac{1}{2}(1 - \widehat{\mu}) - s_1}{2\widehat{\mu}}\right]\\ &= 1 - \frac{s_1}{\mu^*} + \frac{1}{2\tau}(1-s_1)^2\frac{\frac{3\widehat{\mu}-1}{2} - s_1}{2\widehat{\mu}} = 0,\end{aligned} \tag{51}$$

which is equivalent to (30). The first-order condition (51) has a unique solution in $s_1 \in [0, 1]$, since the left-hand side of (the second line of) equation (51) is strictly decreasing. To see this differentiate it with respect to s_1 to get:

$$-\frac{1}{\mu^*} - \frac{3}{4\widehat{\mu}\tau}(1-s_1)(\widehat{\mu} - s_1)$$

which is strictly negative if

$$\tau > \frac{3}{4}\frac{\mu^*}{\widehat{\mu}}(1-s_1)(s_1 - \widehat{\mu}).$$

This inequality is satisfied even when the right-hand side takes the highest possible value, i.e. $\frac{3}{4}\frac{\mu^*}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^2$ (attained at $s_1 = (1+\widehat{\mu})/2$), since we have

$$\tau > \frac{1}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^2 > \frac{3}{4}\frac{\mu^*}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^2$$

by assumption (20). Finally, using (51) and substituting $s_1 = \widehat{\mu}$ it is immediate to show that the resulting expression is negative, hence the solution to (51) features $s_1 < \widehat{\mu}$. Similarly, substituting $s_1 = 0$ yields a positive expression, implying that the solution to (51) features $s_1 > 0$.

To rule out a maximum where $\widehat{\mu} < s_1 < s_2 < 1$, substitute the candidate interior optimum $s_2 = 1 - \frac{1}{2}(s_1 - \widehat{\mu})$ into the first expression of (51) to get

$$\frac{\partial V}{\partial s_1} = v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) - \left(1 - \frac{1}{2}(s_1 - \widehat{\mu}) - s_1\right)^2 \frac{1 - \frac{1}{2}(1 - \frac{1}{2}(s_1 - \widehat{\mu}) - \widehat{\mu}) - s_1}{2\widehat{\mu}}\right]. \tag{52}$$

This expression is similar to (46) with the exception that $\widehat{\mu}$ replaces μ^* in the expression in square brackets. Hence, the proof proceeds as in Proposition 1, and this yields that the optimal s_1 must be below $\widehat{\mu}$.

Second, we show that the constraints in problem (48) are satisfied at the unconstrained optimum when $\mu^* \geq \mu^\circ$ where

$$\mu^\circ \equiv \frac{2\widehat{\mu} - 1}{1 + \frac{1}{2\tau}\frac{(1-\widehat{\mu})^3}{\widehat{\mu}}}. \tag{53}$$

The first constraint $\mu_1 = (0+s_1)/2 < \widehat{\mu}$ is satisfied since we know $s_1 < \widehat{\mu}$. The second constraint, $\mu_2 = (s_1+1)/2 \geq \widehat{\mu}$, is satisfied if $s_1 \geq 2\widehat{\mu} - 1$. Consider the following derivative:

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1 = 2\widehat{\mu}-1 \\ s_2 = 1}} = 1 - \frac{2\widehat{\mu} - 1}{\mu^*} + \frac{1}{2\tau}\frac{(1-\widehat{\mu})^3}{\widehat{\mu}}.$$

Note that $\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1 = 2\widehat{\mu}-1 \\ s_2 = 1}} \geq 0$ (and so $s_1 \geq 2\widehat{\mu} - 1$) if and only if $\mu^* \geq \mu^\circ$. Thus, we have that the constraint $\mu_2 \geq \widehat{\mu}$ is satisfied when $\mu^* \geq \mu^\circ$ and binding otherwise.

Lastly, we show that when the constraint $\mu_2 \geq \widehat{\mu}$ is binding the optimal stress test is a binary partition with cutoff $s_1 = 2\widehat{\mu} - 1$. When the constraint is binding we have $s_1 = 2\widehat{\mu} - s_2$. Plugging this in the first-order derivative of s_2 we get

$$\left.\frac{\partial V}{\partial s_2}\right|_{s_1 = 2\widehat{\mu} - s_2} = \frac{1}{2\tau}(s_2 - \widehat{\mu})^2 \frac{2 - \widehat{\mu} - s_2}{\widehat{\mu}} \geq 0 \quad \text{for } s_2 \leq 2 - \widehat{\mu}.$$

Hence the objective function $V(2\widehat{\mu} - s_2, s_2)$ is weakly increasing in s_2 over the entire domain so that the optimum is $s_2 = 1$.

It follows that the optimal stress test is a binary partition with cutoff given by (29). □

Proof of Proposition 4. When the constraint is not binding, the optimal cutoff is $s_F^D = \widetilde{s}$ and is implicitly defined as the solution to equation (30), which can be rearranged as

$$\mu^* - \widetilde{s} + \frac{1}{4\tau} H(\widetilde{s}, \mu^*, \widehat{\mu}) = 0, \tag{54}$$

where

$$H(\widetilde{s}, \mu^*, \widehat{\mu}) \equiv (1 - \widetilde{s})^2 \frac{\mu^*}{\widehat{\mu}} \left(\frac{3\widehat{\mu} - 1}{2} - \widetilde{s} \right).$$

The sign of $H(\widetilde{s}, \mu^*, \widehat{\mu})$ depends on whether the stress test is lenient or conservative, in particular, $H(\widetilde{s}, \mu^*, \widehat{\mu}) > 0$ if and only if $\mu^* < \frac{3\widehat{\mu}-1}{2}$. To see this note that from Proposition 3 we have that $\mu^* < \frac{3\widehat{\mu}-1}{2}$ implies that $\widetilde{s} > \mu^*$, hence the $\widetilde{s}$ that solves (54) has to be such that $H(\widetilde{s}, \mu^*, \widehat{\mu}) > 0$. Similarly, $\mu^* > \frac{3\widehat{\mu}-1}{2}$ implies that $\widetilde{s} < \mu^*$ and $H(\widetilde{s}, \mu^*, \widehat{\mu}) < 0$.

In what follows, we make use of the fact that:

$$\begin{aligned} h(\widetilde{s}, \mu^*, \widehat{\mu}) \equiv \frac{\partial H}{\partial \widetilde{s}} &= 2(1 - \widetilde{s})(-1) \frac{\mu^*}{\widehat{\mu}} \left(\frac{3\widehat{\mu} - 1}{2} - \widetilde{s} \right) + (1 - \widetilde{s})^2 \frac{\mu^*}{\widehat{\mu}} (-1) \\ &= -3(1 - \widetilde{s}) \frac{\mu^*}{\widehat{\mu}} (\widehat{\mu} - \widetilde{s}) < 0, \end{aligned} \tag{55}$$

since we have $\widetilde{s} < \widehat{\mu}$ from Proposition 3.

To do comparative statics, we express equation (54) in terms of the primitives (p, R, D) to get:

$$\frac{1}{pR} - \widetilde{s} + \frac{1}{4\tau} H(\widetilde{s}, p, R, D) = 0, \tag{56}$$

where

$$H(\widetilde{s}, p, R, D) = (1-\widetilde{s})^2 \left(\frac{3}{2} \frac{1}{pR} - \left(\frac{1}{2} + \widetilde{s} \right) \left(1 - \frac{D}{R} \right) \right),$$

and

$$\begin{aligned} \frac{\partial H}{\partial p} &= -(1-\widetilde{s})^2 \frac{3}{2} \frac{1}{p^2 R} < 0, \\ \frac{\partial H}{\partial R} &= -(1-\widetilde{s})^2 \left(\frac{3}{2} \frac{1}{pR^2} + \left(\frac{1}{2} + \widetilde{s} \right) \frac{D}{R^2} \right) < 0, \\ \frac{\partial H}{\partial D} &= (1-\widetilde{s})^2 \left(\frac{1}{2} + \widetilde{s} \right) \frac{1}{R} > 0. \end{aligned} \tag{57}$$

Employing the implicit function theorem, we totally differentiate equation (56) with respect to (p, R, D) to get:

$$\begin{aligned} \frac{\partial \widetilde{s}}{\partial p} &= \frac{\frac{1}{4\tau} \frac{\partial H}{\partial p} - \frac{1}{p^2 R}}{1 - \frac{1}{4\tau} h(\widetilde{s}, \mu^*, \widehat{\mu})} < 0, \\ \frac{\partial \widetilde{s}}{\partial R} &= \frac{\frac{1}{4\tau} \frac{\partial H}{\partial R} - \frac{1}{pR^2}}{1 - \frac{1}{4\tau} h(\widetilde{s}, \mu^*, \widehat{\mu})} < 0, \\ \frac{\partial \widetilde{s}}{\partial D} &= \frac{\frac{1}{4\tau} \frac{\partial H}{\partial D}}{1 - \frac{1}{4\tau} h(\widetilde{s}, \mu^*, \widehat{\mu})} > 0, \end{aligned}$$

where the inequalities follow from (55) and (57). Doing the same for τ we get:

$$\frac{\partial \widetilde{s}}{\partial \tau} = -\frac{\frac{1}{4\tau^2} H(\widetilde{s}, \mu^*, \widehat{\mu})}{1 - \frac{1}{4\tau} h(\widetilde{s}, \mu^*, \widehat{\mu})},$$

where $\frac{\partial \widetilde{s}}{\partial \tau} < 0$ if and only if $\mu^* < \frac{3\widehat{\mu}-1}{2}$ (as this implies $H(\widetilde{s}, \mu^*, \widehat{\mu}) > 0$), since $h(\widetilde{s}, \mu^*, \widehat{\mu}) < 0$ by (55).

When the constraint is binding, so that $s_F^D = 2\widehat{\mu} - 1$, the comparative statics are driven by $\widehat{\mu} = 1/p(R-D)$, and are thus the same as above. □

Lemma 6. *When the supervisor can intervene according to her private information s, she solves the following stress test design problem:*

$$\max_{s_1 \in [0,1]} V(s_1)$$

where

$$V(s_1) = \begin{cases} V_{a(i)}(s_1) & \textit{if } s_1 \in \big[0, 2\mu^* - 1\big), \\ V_{a(ii)}(s_1) & \textit{if } s_1 \in \big[2\mu^* - 1, \mu^*(1-\delta)\big), \\ V_p(s_1) & \textit{if } s_1 \in \big[\mu^*(1-\delta), \mu^*(1+\delta)\big], \\ V_{b(i)}(s_1) & \textit{if } s_1 \in \big(\mu^*(1+\delta), 2\mu^*\big), \\ V_{b(ii)}(s_1) & \textit{if } s_1 \in \big[2\mu^*, 1\big]. \end{cases} \tag{58}$$

and $V_{a(i)}$, $V_{a(ii)}$, V_p, $V_{b(i)}$, $V_{b(ii)}$ *are defined in (59)-(70).*

Proof of Lemma 6. The supervisor's value function depends on i) the default action taken by the market without a stock market signal, and, ii) whether the supervisor intervenes or not to revert the market's action. Accordingly, we distinguish several cases and first state the relevant functions $V(s_1)$ for each region. The proof of the expression (59)-(70) follows further below.

- region a(i), $s_1 \in [0, 2\mu^* - 1)$. Note that this region is non-empty only if $\mu^* > \frac{1}{2}$. Expected bank value is

$$\begin{aligned} V_{a(i)}(s_1) =& \frac{V_h^1 + V_l^1}{2} - \frac{1}{2}\Delta V_h \left(\frac{\sigma_2}{2} s_1^2 + \left(1 - \frac{\sigma_2}{2}\right)\bar{s}^2\right) \\ &+ \frac{1}{2}\Delta V_l \left(\frac{\sigma_2}{2} + \left(1 - \frac{\sigma_2}{2}\right)(2\bar{s} - \bar{s}^2)\right) - \delta\left(1 - \frac{\sigma_2}{2}\right)(1 - \bar{s}), \end{aligned} \tag{59}$$

  where σ_2 is the information produced by the speculator if the bank passes the test, given by

$$\sigma_2 = \frac{1}{4\tau}\left(1 - \bar{s}^2\right)\left(pR - \frac{1}{\frac{1+\bar{s}}{2}}\right). \tag{60}$$

  The speculator does not produce information if the bank fails the test.

- region a(ii), $s_1 \in [2\mu^* - 1, \mu^*(1-\delta))$.

$$\begin{aligned} V_{a(ii)}(s_1) =& \frac{V_h^1 + V_l^1}{2} - \frac{1}{2}\Delta V_h \left(\frac{\sigma_2}{2} s_1^2 + \left(1 - \frac{\sigma_2}{2}\right)\underline{s}^2\right) \\ &+ \frac{1}{2}\Delta V_l \left(\frac{\sigma_2}{2} + \left(1 - \frac{\sigma_2}{2}\right)(2\underline{s} - \underline{s}^2)\right) - \delta\left(1 - \frac{\sigma_2}{2}\right)(\underline{s} - s_1). \end{aligned} \tag{61}$$

  There is no information acquisition following a *fail* result, and information acquisition following a *pass* is given by

$$\sigma_2 = \frac{1}{4\tau}\left(1-\underline{s}^2\right)\left(pR - \frac{1}{\frac{1+\underline{s}}{2}}\right). \tag{62}$$

Note that σ_2 from (62) and (60) are the same. This can be shown using the expressions for $\underline{s}, \bar{s}$ and $\mu^* = \frac{1}{pR}$.

- region p, $s_1 \in [\mu^*(1-\delta), \mu^*(1+\delta))$;

$$\begin{aligned} V_p(s_1) =& \frac{V_h^1 + V_l^1}{2} - \frac{1}{2}s_1^2 \Delta V_h \\ &+ \frac{1}{2}\Delta V_l \left(\frac{\sigma_2}{2} + \left(1 - \frac{\sigma_2}{2}\right)(2s_1 - s_1^2)\right). \end{aligned} \tag{63}$$

There is no information acquisition following a *fail* result, and information acquisition following a *pass* is given by

$$\sigma_2 = \frac{1}{4\tau}\left(1-s_1^2\right)\left(pR - \frac{1}{\frac{1+s_1}{2}}\right). \tag{64}$$

- region b(i), $s_1 \in [\mu^*(1+\delta), 2\mu^*)$.

$$\begin{aligned} V_{b(i)}(s_1) =& \frac{V_h^1 + V_l^1}{2} - \frac{1}{2}\Delta V_h \left(1 - \frac{\sigma_1}{2}\right)\bar{s}^2 \\ &+ \frac{1}{2}\Delta V_l \left(\frac{\sigma_1}{2}(2s_1 - s_1^2) + \left(1 - \frac{\sigma_1}{2}\right)(2\bar{s} - \bar{s}^2)\right) + \frac{1}{2}\Delta V_l \frac{\sigma_2}{2}(1-s_1)^2 \\ &- \delta\left(1 - \frac{\sigma_1}{2}\right)(s_1 - \bar{s}). \end{aligned} \tag{65}$$

Since the bank is sometimes funded even when it fails the test, the speculator acquires information σ_1 following m_1 (and σ_2 following m_2):

$$\sigma_1 = \frac{1}{2\tau}\left(s_1^2 - \bar{s}^2\right)\frac{1 - \frac{s_1}{2}}{s_1}\left(pR - \frac{1}{\frac{s_1+\bar{s}}{2}}\right), \tag{66}$$

$$\sigma_2 = \frac{1}{4\tau}\left(1-s_1^2\right)\left(pR - \frac{1}{\frac{1+s_1}{2}}\right). \tag{67}$$

- region b(ii), $s_1 \in [2\mu^*, 1]$. Note that this region is non-empty only if $\mu^* < \frac{1}{2}$. We again get information acquisition following both a *fail* and a *pass* result.

$$\begin{aligned} V_{b(ii)}(s_1) =& \frac{V_h^1 + V_l^1}{2} - \frac{1}{2}\Delta V_h \left(1 - \frac{\sigma_1}{2}\right) \underline{s}^2 \\ &+ \frac{1}{2}\Delta V_l \left(\frac{\sigma_1}{2}(2s_1 - s_1^2) + \left(1 - \frac{\sigma_1}{2}\right)(2\underline{s} - \underline{s}^2)\right) + \frac{1}{2}\Delta V_l \frac{\sigma_2}{2}(1 - s_1)^2 \\ &- \delta \left(1 - \frac{\sigma_1}{2}\right) \underline{s}, \end{aligned} \tag{68}$$

where

$$\sigma_1 = \frac{1}{2\tau}\left(s_1^2 - \underline{s}^2\right) \frac{1 - \frac{s_1}{2}}{s_1} \left(pR - \frac{1}{\frac{s_1 + \underline{s}}{2}}\right), \tag{69}$$

$$\sigma_2 = \frac{1}{4\tau}\left(1 - s_1^2\right)\left(pR - \frac{1}{\frac{1+s_1}{2}}\right). \tag{70}$$

Note that σ_1 from (69) and (66) are identical. This can be seen by using the expressions for $\underline{s}, \bar{s}$ and μ^*. Moreover, σ_2 in (70) and (67) are also identical. Finally, note also that on the interval $s_1 \in [\bar{s}, 1]$, σ_1 is maximized at $s_1 = 1$, σ_2 is maximized at $s_1 = \bar{s}$ and that $\sigma_1(s_1 = 1) = \sigma_2(s_1 = \bar{s})$. We define $\sigma \equiv \sigma_1(s_1 = 1)$, given by

$$\sigma = \frac{1}{2\tau}(1 - \bar{s})\left(\frac{1 + \bar{s}}{2\mu^*} - 1\right). \tag{71}$$

It is straightforward to verify that $V(s_1)$ is continuous over the entire interval $s_1 \in [0, 1]$ and that $V(s_1 = 0) = V(s_1 = 1)$.

Figure 5 depicts the objective function $V(s_1)$ for some parameter values. We now prove the expressions (59)-(70).

Region a(i): $s_1 \in [0, 2\mu^* - 1)$, non-empty only if $2\mu^* - 1 > 0$. Since $s_1 < 2\mu^* - 1$, we have $\mu_1 < \mu_2 < \mu^*$. Without information from the share price, neither the regulator nor the capital provider will act to fund the bank if the stress test generates message m_1. There is hence no information production by the speculator. Creditors do not roll over their debt even after a *pass* result, unless the stock price reveals $\omega = h$. Following a test outcome m_2, the supervisor funds

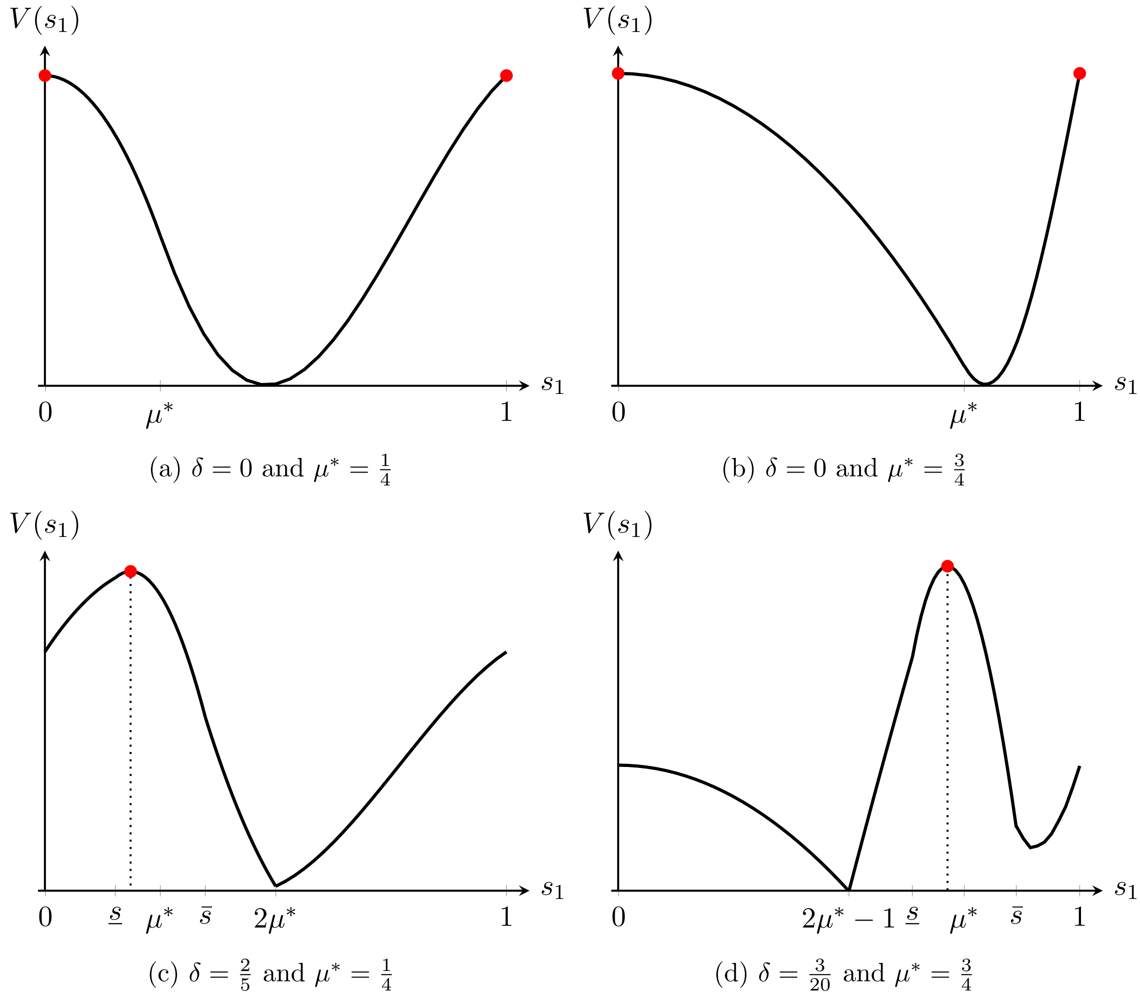

Figure 5: Objective Function $V(s_1)$ and $\max_{s_1} V(s_1)$ (in red).

the bank if the stock price is uninformative and $s > \bar{s}$. This then provides incentives to the speculator to acquire information and trade following m_2.

Given the funding rate $r = \frac{1}{ps}$ the equity value given $\omega = h$ and $a = 1$ is

$$E_h^1(s) = pR - \frac{1}{s}.$$

We have $\Pr(\omega = h, s|m_2) = \Pr(\omega = h|s)f(s|m_2) = \frac{s}{1-s_1}$ and hence, the market maker sets the price when the order flow is uninformative following m_2 at

$$P(m_2) = \int_{\bar{s}}^{1} \left(pR - \frac{1}{s}\right) \frac{s}{1-s_1} \mathrm{d}s,$$
$$= \frac{1}{2}\frac{1-\bar{s}^2}{1-s_1} E_h^1,$$

where

$$E_h^1 \equiv pR - \frac{1}{\frac{\bar{s}+1}{2}}.$$

The trading profit is therefore

$$E(\pi) = \frac{\sigma_2}{2}\left[\int_{\bar{s}}^{1} \frac{s}{1-s_1}\left(E_h^1(s) - P(m_2)\right)\mathrm{d}s + \frac{1}{2}\frac{\bar{s}^2 - s_1^2}{1-s_1}(-P(m_2)) + \frac{1}{2}\frac{(1-s_1)^2}{1-s_1}P(m_2)\right] - \frac{\tau}{2}\sigma_2^2,$$
$$= \frac{\sigma_2}{2}\left[\frac{1}{2}\frac{1-\bar{s}^2}{1-s_1}\left(E_h^1 - P(m_2)\right) + \frac{1}{2}\frac{\bar{s}^2 - s_1^2}{1-s_1}(-P(m_2)) + \frac{1}{2}\frac{(1-s_1)^2}{1-s_1}P(m_2)\right] - \frac{\tau}{2}\sigma_2^2,$$

which can be simplified to

$$E(\pi) = \frac{\sigma_2}{4}(1-\bar{s}^2)E_h^1 - \frac{\tau}{2}\sigma^2$$

Taking the first order condition, we get the amount of information acquisition at the optimum, following message m_2, given by (60). The supervisor's expected payoff is thus

$$V_{a(i)}(s_1) = \frac{1}{2}(1-\bar{s}^2)\left[\frac{\sigma_2}{2}V_h^1 + \left(1 - \frac{\sigma_2}{2}\right)\left(V_h^1 - \delta\right)\right] + \frac{1}{2}(1-\bar{s})^2\left(\frac{\sigma_2}{2}V_l^0 + \left(1 - \frac{\sigma_2}{2}\right)\left(V_l^1 - \delta\right)\right)$$
$$+ \frac{1}{2}\left[\bar{s}(2-\bar{s}) - s_1(2-s_1)\right]V_l^0 + \frac{1}{2}(\bar{s}^2 - s_1^2)\left(\frac{\sigma_2}{2}V_h^1 + \left(1 - \frac{\sigma_2}{2}\right)V_h^0\right)$$
$$+ \frac{1}{2}\left[\left(2s_1 - s_1^2\right)V_l^0 + s_1^2 V_h^0\right].$$

After some calculations, we get (59).

Region a(ii): $s_1 \in [\max\{0, 2\mu^* - 1\}, \underline{s})$, where $\underline{s} = \mu^*(1-\delta)$. We have $\mu_1 < \mu^* \leq \mu_2$ so that, in the absence of an informative stock price, creditors roll over their debt following a *pass* but not a *fail* result. If $s < s_1$ there is no information production and the bank will fail to raise funds from private markets or the supervisor. Following a *pass* result, capital providers are willing to fund the bank, unless the stock price signal reveals $\omega = l$. The supervisor does not intervene, unless the bank passes the test, the stock price is uninformative and $s \in [s_1, \underline{s})$. In that case, the supervisor shuts down the bank ($a_s = 0$). Overall, this yields the following expected payoff to the supervisor:

$$
\begin{aligned}
V_{a(ii)}(s_1) = \frac{1}{2}\Big[(1-\underline{s}^2)V_h^1 + (\underline{s}^2 - s_1^2)\Big[\frac{\sigma_2}{2}V_h^1 + (1-\frac{\sigma_2}{2})(V_h^0 - \delta)\Big] + s_1^2 V_h^0\Big] \\
+ \frac{1}{2}\left[1-(2\underline{s}-\underline{s}^2)\right]\Big[\frac{\sigma_2}{2}V_l^0 + (1-\frac{\sigma_2}{2})V_l^1)\Big]] \\
+ \frac{1}{2}\Big[\left[2\underline{s}-\underline{s}^2-(2s_1-s_1^2)\right]\Big[\frac{\sigma_2}{2}V_l^0 + (1-\frac{\sigma_2}{2})(V_l^0-\delta)\Big] + (2s_1 - s_1^2)V_l^0\Big].
\end{aligned}
$$

After some calculations, we get the expression in (61).

Next, we need to determine prices and trading profits in order to determine the relation between s_1 and the amount of information σ_2 produced by the speculator. When m_2 is observed and the order is uninformative, the market maker does not know whether the bank will be able to continue. This depends on whether s is above or below $\underline{s}$. Denote by $\mu_\omega^a(m_2) = Pr(\omega, a|m_2)$. Here a denotes whether the firm is able to continue ($a = 1$) or not ($a = 0$). To ease the exposition we do not introduce extra notation to distinguish between the case where the supervisor is decisive (unwinds before the bank can try to raise capital), or the providers of capital are not willing to fund the bank. For the sake of brevity, we drop the function argument m_2 from $\mu_\omega^a(m_2)$. Beliefs following m_1 do not require a similar distinction since the supervisor and capital providers agree that the optimal action in this case is $a = 0$ and the speculator produces no information.

$$
\begin{aligned}
\mu_h^1 &= \frac{1}{2}\frac{1-\underline{s}^2}{1-s_1}, \\
\mu_h^0 &= \frac{1}{2}\frac{\underline{s}^2 - s_1^2}{1-s_1}, \\
\mu_l^1 &= \frac{1}{2}\frac{1-\left(2\underline{s}-\underline{s}^2\right)}{1-s_1}, \\
\mu_l^0 &= \frac{1}{2}\frac{2\underline{s}-\underline{s}^2-(2s_1-s_1^2)}{1-s_1}.
\end{aligned}
$$

The speculator's expected trading profits, after having observed m_2 and acquired information

σ_2 are

$$E(\pi) = \frac{\sigma_2}{2}\left[\mu_h^1(E_h^1 - P) + \mu_h^0(-P) + (\mu_l^1 + \mu_l^0)P\right] - \frac{\tau}{2}\sigma_2^2.$$

Since the market maker learns nothing if the order flow is uninformative, and since $E_h^0 = E_l^1 = E_l^0 = 0$, the price is

$$P(m_2) = \mu_h^1 E_h^1$$

where E_h^1 is computed below. The expression for trading profits can be simplified to

$$E(\pi) = \sigma_2 E_h^1 \mu_h^1 (1 - \mu_h^1 - \mu_h^0) - \frac{\tau}{2}\sigma_2^2.$$

To calculate E_h^1 note that the bank only gets to approach capital markets if the supervisor has allowed it to go ahead. This reveals to the capital providers that $s \geq \underline{s}$. Hence, at this point, the capital providers believe $Pr(\omega = h) = \frac{1+\underline{s}}{2}$. Since the interest rate is set as a function of the belief μ at $r = \frac{1}{p\mu}$, we get

$$E_h^1 = pR - \frac{1}{\frac{1+\underline{s}}{2}} = pR - \frac{2}{1+\underline{s}}.$$

From this, we can calculate overall trading profits as

$$E(\pi) = \frac{\sigma_2}{2}\left(pR - \frac{2}{1+\underline{s}}\right)\left(1 - \underline{s}^2\right)\frac{1 - \frac{1+s_1}{2}}{1 - s_1} - \frac{\tau}{2}\sigma_2^2,$$

which can be simplified to

$$E(\pi) = \frac{\sigma_2}{4}\left(pR - \frac{2}{1+\underline{s}}\right)\left(1 - \underline{s}^2\right) - \frac{\tau}{2}\sigma_2^2.$$

Taking the first order condition with respect to σ_2 we get the expression in (62).

Region p: $s_1 \in [\underline{s}, \bar{s})$, where $\underline{s} = \mu^*(1-\delta)), \bar{s} = \mu^*(1+\delta)$. We have $\mu_1 < \mu^* \leq \mu_2$ so that creditors roll over their debt following a *pass* but not a *fail* result. The supervisor is always passive: Following a *pass* result, $s > \underline{s}$ so the supervisor does not want to intervene with $a_s = 0$. Following a *fail* result, we always have $s < \bar{s}$ and hence the supervisor does not want to fund the bank. Since the supervisor is completely passive, the game proceeds as in the baseline model, and the supervisor's value function is given by Lemma 3. By setting $s_2 = 1$ and after some simplifications we get the value function in (63) and the amount of information acquisition in (64).

Region b(i): $s_1 \in [\bar{s}, \min\{2\mu^*, 1\})$. We have $\mu_1 < \mu^* < \mu_2$. If the bank fails the test, but the supervisor observes $s \in [\bar{s}, s_1]$, she funds the bank at rate $r = \frac{1}{ps}$. Since the bank gets funded with positive probability following a test outcome m_1, the speculator may now acquire information even after such a comparatively unfavorable test result, i.e., in spite of an induced belief $\mu_1 < \mu^*$. Denote by σ_1, the amount of information produced by the speculator, following m_1, and similarly, σ_2 denotes the information acquired following m_2.

We proceed to calculate trading profits following m_1. As before, the equity value is only non-zero when $\omega = h$ and $a = 1$. Given the funding costs $r = \frac{1}{ps}$, the equity value is

$$E_h^1(s) = pR - \frac{1}{s}.$$

We have $\Pr(\omega = h, s|m_1) = f(s|m_1)\Pr(\omega = h|s) = \frac{s}{s_1}$ and hence, the market maker sets the price when order flow is uninformative following m_1 at

$$\begin{aligned} P(m_1) &= \int_{\bar{s}}^{s_1} \left(pR - \frac{1}{s}\right) \frac{s}{s_1} \mathrm{d}s, \\ &= \frac{1}{2}\frac{s_1^2 - \bar{s}^2}{s_1} E_h^1, \end{aligned}$$

where

$$E_h^1 \equiv pR - \frac{1}{\frac{\bar{s}+s_1}{2}}.$$

From this we can calculate trading profits:

$$\begin{aligned} E(\pi) &= \frac{\sigma_1}{2}\left[\int_{\bar{s}}^{s_1} \frac{s}{s_1}\left(E_h^1(s) - P(m_1)\right)\mathrm{d}s + \frac{1}{2}\frac{\bar{s}^2}{s_1}(-P(m_1)) + \frac{1}{2}\frac{s_1(2-s_1)}{s_1}P(m_1)\right] - \frac{\tau}{2}\sigma_1^2, \\ &= \frac{\sigma_1}{2}\left[\frac{1}{2}\frac{s_1^2 - \bar{s}^2}{s_1}\left(E_h^1 - P(m_1)\right) + \frac{1}{2}\frac{\bar{s}^2}{s_1}(-P(m_1)) + \frac{1}{2}\frac{s_1(2-s_1)}{s_1}P(m_1)\right] - \frac{\tau}{2}\sigma_1^2. \end{aligned}$$

After some simplifications and taking the first-order condition with respect to σ_1, we find the expression in (66).

Trading profits and therefore information acquisition following m_2 can be calculated as in the baseline model. Using the function (18) derived previously and setting $\mu = \mu_2 = \frac{1+s_1}{2}$ we get the expression in (67). The supervisor's expected payoff is given by

$$\begin{aligned}V_{b(i)}(s_1) = \frac{1}{2}\Big[(1-s_1^2)V_h^1 + (s_1^2 - \bar{s}^2)\left(V_h^1 - (1-\frac{\sigma_1}{2})\delta\right) + \bar{s}^2\left(\frac{\sigma_1}{2}V_h^1 + \left(1-\frac{\sigma_1}{2}\right)V_h^0\right)\Big] \\ + \frac{1}{2}\Big[(1-s_1(2-s_1))\left(\frac{\sigma_2}{2}V_l^0 + (1-\frac{\sigma_2}{2})V_l^1\right)\Big] \\ + \frac{1}{2}\Big[(s_1(2-s_1) - \bar{s}(2-\bar{s}))\left(\frac{\sigma_1}{2}V_l^0 + (1-\frac{\sigma_1}{2})(V_l^1 - \delta)\right) + \bar{s}(2-\bar{s})V_l^0\Big].\end{aligned}$$

and after some simplification, this can be written as the expression in (65).

Region b(ii): $s_1 \in [2\mu^*, 1]$, non-empty only if $2\mu^* < 1$. In this region $\mu^* \leq \mu_1 < \mu_2$ so that capital providers are willing to fund the bank even if it fails the test (unless the stock price reveals $\omega = l$). The supervisor intervenes and sets $a_s = 0$ if she observes $s < \underline{s}$. After the message m_2 the game proceeds as in the baseline model: the speculator's information acquisition after m_2 is given by (70) and in the absence of a revealing stock price the market invests.

Following the message m_1, if the stock price reveals nothing, the supervisor may intervene depending on the realization of her private signal. When the market sees no intervention it realizes that $s \in [\underline{s}, s_1)$ and chooses the interest rate accordingly, leading to an equity value of

$$E_h^1 = pR - \frac{1}{\frac{\underline{s}+s_1}{2}}.$$

We can determine the speculator's and market maker's beliefs over the state and the expected supervisor's decision, $\mu_\omega^{a_s} = \Pr(\omega, a_s|m_1) = \Pr(\omega|a_s)\Pr(a_s|m_1)$. These are:

$$\begin{aligned}
\mu_h^1 &= \frac{\underline{s}+s_1}{2}\frac{s_1-\underline{s}}{s_1} \\
\mu_h^0 &= \frac{\underline{s}}{2}\frac{\underline{s}}{s_1} \\
\mu_l^1 &= \left(1-\frac{\underline{s}+s_1}{2}\right)\frac{s_1-\underline{s}}{s_1} \\
\mu_l^0 &= \left(1-\frac{\underline{s}}{2}\right)\frac{\underline{s}}{s_1}.
\end{aligned}$$

The market maker chooses a price

$$P = \mu_h^1 E_h^1,$$

and the speculator's expected profits are

$$\begin{aligned}
E(\pi) &= \frac{\sigma_1}{2}\left[\mu_h^1(E_h^1-P)+\mu_h^0(-P)+(\mu_l^1+\mu_l^0)P\right]-\frac{1}{2}\tau\sigma_1^2 \\
&= \sigma_1\mu_h^1(1-\mu_h^1-\mu_h^0)E_h^1-\frac{1}{2}\tau\sigma_1^2.
\end{aligned}$$

By taking the first-order condition with respect to σ_1 we get the expression in (69).

The supervisor's objective function is:

$$\begin{aligned}
V_{b(ii)}(s_1) =& \frac{1}{2}\left[\underline{s}^2\left(\frac{\sigma_1}{2}V_h^1+\left(1-\frac{\sigma_1}{2}\right)(V_h^0-\delta)\right)+\left((s_1^2-\underline{s}^2)+(1-s_1^2)\right)V_h^1\right] \\
&+\frac{1}{2}\left[(2\underline{s}-\underline{s}^2)\left(\frac{\sigma_1}{2}V_l^0+\left(1-\frac{\sigma_1}{2}\right)(V_l^0-\delta)\right)\right] \\
&+\frac{1}{2}\left[\left(2(s_1-\underline{s})-(s_1^2-\underline{s}^2)\right)\left(\frac{\sigma_1}{2}V_l^0+\left(1-\frac{\sigma_1}{2}\right)V_l^1\right)\right] \\
&+\frac{1}{2}\left[\left(2(1-s_1)-(1-s_1^2)\right)\left(\frac{\sigma_2}{2}V_l^0+\left(1-\frac{\sigma_2}{2}\right)V_l^1\right)\right].
\end{aligned}$$

This can be simplified to get (68). □

Lemma 7. *The function $V(s_1)$ defined in (58) attains a local maximum on the interval $s_1 \in [0, \bar{s}]$ at*

$$s_1 = s_F^P \equiv \begin{cases} s_1^{a(ii)} & \text{if } s_1^{a(ii)} < \mu^*(1-\delta) \\ s_F & \text{if } s_1^{a(ii)} \geq \mu^*(1-\delta), \end{cases} \in [0,\mu^*), \tag{72}$$

where $s_1^{a(ii)} \equiv \delta\frac{1-\frac{\sigma_2}{2}}{\frac{\sigma_2}{2}(pR-1)}$ with σ_2 defined in (62), and s_F solves (23).

Proof. When $\mu^* \leq \frac{1}{2}$, region a(i) is empty. For s_1 in region a(ii), the supervisor's objective is given by (61). It is straightforward to show that

$$\frac{\partial V_{a(ii)}}{\partial s_1} = \delta\left(1 - \frac{\sigma_2}{2}\right) - s_1 \frac{\sigma_2}{2} \Delta V_h. \tag{73}$$

Note that $\frac{\partial V_{a(ii)}}{\partial s_1}$ is strictly positive at $s_1 = 0$ and $\delta > 0$. Moreover, $V_{a(ii)}(0) = V_{b(ii)}(1)$. Hence, for any $\delta > 0$ the optimal $s_1 \in (0, 1)$. If s_1 increases so that we are in region p, we know from Proposition 1 that $V_p(s_1)$ attains a maximum at $s_1 < \mu^*$, as we have $\frac{\partial V_p}{\partial s_1} < 0$ at $s_1 = \mu^*$. Hence, $V(s_1)$ must attain a local maximum between 0 and μ^*.

The following can be said about whether the local maximum lies in region a(ii) or region p. The maximum of $V_{a(ii)}(s_1)$ is reached at

$$s_1 = \delta \frac{1 - \frac{\sigma_2}{2}}{\frac{\sigma_2}{2}(pR - 1)} \equiv s_1^{a(ii)}, \tag{74}$$

where σ_2 is defined in (62). If $s_1^{a(ii)} < \underline{s} = \mu^*(1 - \delta)$, then $V(s_1)$ reaches a local maximum in region a(ii). Moreover, using the expressions for $V_{a(ii)}(s_1)$ and $V_p(s_1)$, it can be shown that at the point $s_1 = \underline{s}$ we have $\frac{\partial V_{a(ii)}}{\partial s_1} < \frac{\partial V_p}{\partial s_1}$. Hence, if $\frac{\partial V_p}{\partial s_1} < 0$ at the corner $s_1 = \underline{s}$ of region p, then the local maximum is in region a(ii). If $s_1^{a(ii)} > \underline{s}$ we have that $V_{a(ii)(s_1)}$ is strictly increasing on the entire region a(ii) and the local maximum lies in region p. There can be cases where $\frac{\partial V_{a(ii)}}{\partial s_1} < 0$ for the corner $s_1 = \underline{s}$ in region a(ii), but $\frac{\partial V_p(s_1)}{\partial s_1} > 0$ for the corner $s_1 = \underline{s}$ in region p. In that case, $V(s_1)$ has two local maxima on $[0, \mu^*]$.

When $\mu^* > \frac{1}{2}$, the relevant regions are a(i) to b(i). $V_{a(i)}(s_1)$ is decreasing on the entire region a(i), as can be seen by calculating its derivative:

$$\frac{\partial V_{a(i)}(s_1)}{\partial s_1} = -s_1 \frac{\sigma_2}{2} \Delta V_h, \tag{75}$$

which is negative. Moreover, we know from before that for s_1 in regions a(ii)-p, the local maximum is below μ^*. It then follows that the local maximum over regions a(i)-p is either at the corner $s_1 = 0$, or is interior and given by s_F^P.

□

Lemma 8. *For $\mu^* \leq \frac{1}{2}$, s_F^P is a global maximum.*

Proof. The proof of Lemma 8 follows from the following two claims.

Claim 1. *On regions b(i)-b(ii), the objective function $V(s_1)$ attains a maximum at one of its corners $s_1 \in \{\bar{s}, 1\}$.*

Claim 2. *For $\delta > 0$ we have $V(s_F^P) > \max\{V(\bar{s}), V(1)\}$. For $\delta = 0$ we have $s_F^P = 0$ and $V(0) = V(1) > V(\mu^*)$.*

Proof of Claim 1: The proof proceeds by constructing a straight line connecting the extreme points $(\bar{s}, V(\bar{s}))$ and $(1, V(1))$ of the value function $V(s_1)$ and then showing that $V(s_1)$ lies below that line for all $s_1 \in (\bar{s}, 1)$. First, consider region b(i), i.e. $s_1 \in [\bar{s}, 2\mu^*)$. We want to show that the straight line connecting $(\bar{s}, V_{b(i)}(\bar{s}))$ to $(1, V_{b(ii)}(1))$ lies above $V_{b(i)}(s_1)$ for all $s_1 \in [\bar{s}, 2\mu^*)$. Note that $V_{b(i)}(s_1) > V_{b(ii)}(s_1)$ if and only if $s_1 < 2\mu^*$ so that the line connecting $(\bar{s}, V_{b(i)}(\bar{s}))$ to $(1, V_{b(ii)}(1))$ lies above the line $\widehat{V}_{b(i)}(s_1)$ connecting $(\bar{s}, V_{b(i)}(\bar{s}))$ to $(1, V_{b(i)}(1))$. We prove the stronger claim that the line $\widehat{V}_{b(i)}(s_1)$ lies above $V_{b(i)}(s_1)$ for all $s_1 \in [\bar{s}, 2\mu^*)$. The line $\widehat{V}_{b(i)}(s_1)$ is defined as

$$\widehat{V}_{b(i)}(s_1) = V_{b(i)}(\bar{s}) + \frac{V_{b(i)}(1) - V_{b(i)}(\bar{s})}{1 - \bar{s}}(s_1 - \bar{s}).$$

This can be re-written as

$$\widehat{V}_{b(i)}(s_1) = \frac{1 - s_1}{1 - \bar{s}} V_{b(i)}(\bar{s}) + \frac{s_1 - \bar{s}}{1 - \bar{s}} V_{b(i)}(1).$$

We then get $\widehat{V}_{b(i)}(s_1) \geq V_{b(i)}(s_1)$ if

$$\Delta V_h \left(\sigma \frac{s_1 - \bar{s}}{1 - \bar{s}} - \sigma_1(s_1)\right) \bar{s}^2 + 2\delta(\sigma - \sigma_1(s_1))(s_1 - \bar{s}) - \Delta V_l \sigma_2(s_1)(1 - s_1)^2$$
$$+ \Delta V_l \left(\sigma(1 - \bar{s})^2 + \sigma_1(s_1)(1 - s_1)^2 - \sigma_1(s_1)(1 - \bar{s})^2\right) \geq 0,$$

where σ is given by (71) and $\sigma_1(s_1)$ and $\sigma_2(s_1)$ by (66) and (67), respectively.

Using

$$(1-\bar{s})^2 = (1-\bar{s})^2 - (1-s_1)^2 + (1-s_1)^2,$$

this can be re-written as

$$\Delta V_h \left(\sigma \frac{s_1 - \bar{s}}{1-\bar{s}} - \sigma_1(s_1) \right) \bar{s}^2 + 2\delta(\sigma - \sigma_1(s_1))(s_1 - \bar{s})$$
$$+ \Delta V_l \left((\sigma - \sigma_1(s_1)) \left[(1-\bar{s})^2 - (1-s_1)^2 \right] + (\sigma - \sigma_2(s_1))(1-s_1)^2 \right) \geq 0. \quad (76)$$

We then make use of $\Delta V_l = 1$, $\Delta V_h = \frac{1}{\mu^*} - 1$ and

$$\begin{aligned} \sigma - \sigma_1(s_1) &= \frac{1}{2\tau} \frac{1-s_1}{s_1} \left(\frac{s_1(1-s_1)}{2\mu^*} + s_1 - \mu^*(1-\delta^2) \right), \\ \sigma \frac{s_1 - \bar{s}}{1-\bar{s}} - \sigma_1(s_1) &= -\frac{1}{2\tau} \frac{(s_1-\bar{s})(1-s_1)}{s_1} \frac{s_1 - 2\mu^*(1-\delta)}{2\mu^*}, \\ \sigma - \sigma_2(s_1) &= \frac{1}{2\tau}(s_1 - \bar{s}) \left(\frac{s_1 + \bar{s}}{2\mu^*} - 1 \right). \end{aligned} \quad (77)$$

Since $s_1 \geq \bar{s}$ and $\sigma \geq \sigma_1(s_1)$ and $\sigma \geq \sigma_2(s_1)$, the last line of (76) is positive. A sufficient condition then is that the first line is also positive.

Using (77) the first line is positive if

$$\frac{(s_1-\bar{s})(1-s_1)}{s_1} \left(\mu^*(1-\mu^*)(1+\delta)^2 \frac{2\mu^*(1-\delta) - s_1}{2\mu^*} + 2\delta \left[\frac{s_1(1-s_1)}{2\mu^*} + s_1 - \mu^*(1-\delta^2) \right] \right) \geq 0.$$

The factor $\frac{(s_1-\bar{s})(1-s_1)}{s_1} \geq 0$. The remainder of the expression is a negative quadratic function. It is therefore sufficient to check that at its borders $s_1 = \bar{s}$ and $s_1 = 2\mu^*$, the terms in brackets is positive. At $s_1 = \bar{s}$, the expression becomes

$$\frac{1}{2}\mu^*(1-\mu^*)(1+\delta)^2(1-\delta) - \mu^*(1-\mu^*)\delta(1+\delta)^2 + \delta(1+\delta) - \mu^*\delta(1-\delta^2),$$

which can also be written as

$$\frac{1}{2}\mu^*(1-\mu^*)(1+\delta)^2(1-\delta) + \delta(1+\delta) \left[1 - \mu^*(1-\mu^*)(1+\delta) - \mu^*(1-\delta) \right]. \quad (78)$$

Since $\mu^* \le \frac{1}{2}$ we have $\mu^*(1-\mu^*) \le \frac{1}{4}$ and $1+\delta \le 2$ so that $\mu^*(1-\mu^*)(1+\delta) \le \frac{1}{2}$. Moreover, $\mu^*(1-\delta) \le \frac{1}{2}$ and hence (78) is positive.

At $s_1 = 2\mu^*$ we get

$$\delta(2 - 2\mu^*(1-\delta^2) - \mu^*(1-\mu^*)(1+\delta)^2),$$

which is also positive. It follows that the line connecting $V_{b(i)}(\bar{s})$ to $V_{b(i)}(1)$, lies above $V_{b(i)}(s_1)$ for all $s_1 \in [\bar{s}, 2\mu^*]$, implying that the line connecting $V_{b(i)}(\bar{s})$ to $V_{b(ii)}(1)$, also lies above $V_{b(i)}(s_1)$ for all $s_1 \in [\bar{s}, 2\mu^*]$.

Now, consider region b(ii), i.e. $s_1 \in [2\mu^*, 1]$. We want to show that $V_{b(ii)}(1) \ge V_{b(ii)}(s_1)$ for all $s_1 \in [2\mu^*, 1]$. Since, $V(s_1)$ is continuous across regions, this is enough to conclude the proof of the claim.

Using (68) we can write

$$V_{b(ii)}(1) = \frac{V_h^1 + V_l^1}{2} - \delta\left(1 - \frac{\sigma}{2}\right)\underline{s} - \frac{1}{2}\Delta V_h\left(1 - \frac{\sigma}{2}\right)\underline{s}^2 + \frac{1}{2}\Delta V_l\left(\frac{\sigma}{2} + \left(1 - \frac{\sigma}{2}\right)\left(2\underline{s} - \underline{s}^2\right)\right).$$

We can then write the condition $V_{b(ii)}(1) - V_{b(ii)}(s_1) \ge 0$ as follows:

$$\begin{aligned}\frac{1}{4}\Delta V_h\underline{s}^2\left(\sigma - \sigma_1(s_1)\right) + \frac{1}{2}\delta\underline{s}(\sigma - \sigma_1(s_1)) - \frac{1}{4}\Delta V_l\sigma_2(s_1)(1-s_1)^2 \\ + \frac{1}{4}\Delta V_l\left(\sigma(1-\underline{s})^2 + \sigma_1(s_1)\left[(1-s_1)^2 - (1-\underline{s})^2\right]\right) \ge 0,\end{aligned}$$

where σ is given by (71) and $\sigma_1(s_1)$ and $\sigma_2(s_1)$ by (69) and (70), respectively.

Using $(1-\underline{s})^2 = (1-\underline{s})^2 - (1-s_1)^2 + (1-s_1)^2$, this can be re-written as

$$\left(\Delta V_h\underline{s}^2 + 2\delta\underline{s}\right)(\sigma - \sigma_1(s_1)) + \Delta V_l(\sigma - \sigma_2(s_1)) + \Delta V_l(\sigma - \sigma_1(s_1))\left((1-\underline{s})^2 - (1-s_1)^2\right) \ge 0.$$

Note that $\sigma \ge \sigma_1(s_1)$ and $\sigma \ge \sigma_2(s_1)$. Moreover, $(1-\underline{s})^2 \ge (1-s_1)^2$ since $s_1 \ge 2\mu^* > \underline{s}$. Hence, the inequality holds. It follows that on regions b(i)-b(ii), $V(s_1)$ attains a maximum at one of its corners.

Proof of Claim 2: For $\delta > 0$, we have that $V(s_F^P) > V(0)$ since we know from Lemma 7 that $V(s_1)$ is increasing at $s_1 = 0$. Since $V(0) = V(1)$, it follows that $V(s_F^P) > V(1)$. Moreover, we know that $V(s_F^P) > V(\mu^*) > V\big(\mu^*(1+\delta)\big)$, where the last inequality follows from $V_p(s_1)$ being decreasing for $s_1 \geq \mu^*$. Hence, for $\delta > 0$, s_F^P is the global maximum. For $\delta = 0$, we have $s_F^P = 0$, region p disappears, and $V(0) > V(\mu^*)$ since $V_{a(ii)}(s_1)$ is decreasing over region a(ii) (see (73)). Hence, $s_F^P = 0$ is the global maximum when $\delta = 0$. $\square$

Lemma 9. *For $\mu^* > \frac{1}{2}$, when $\delta = 0$ the global maximum is $s_1 = 0$ and there exist a $\widetilde{\delta}$ such that $s_1 = s_F^P$ is a global maximum for $\delta \geq \widetilde{\delta}$.*

Proof. The following two claims establish that the local maximum is also the global maximum.

Claim 3. *On region b(i), the objective function $V(s_1)$ attains a maximum at one of its corners $s_1 \in \{\bar{s}, 1\}$.*

Claim 4. *There exists a $\widetilde{\delta}$, such that $V(\mu^*) > V(1)$ for $\delta \geq \widetilde{\delta}$. For $\delta = 0$, we have $V(0) > V(\mu^*)$.*

Since $\arg\max V_p(s_1) < \mu^*$, and $V(s_F^P) > V(\mu^*)$ claims 3 and 4 imply that for $\delta \geq \widetilde{\delta}$, s_F^P is the global maximum.

Proof of Claim 3: We want to show that the straight line passing from $V_{b(i)}(\mu^*(1+\delta)) = V_{b(i)}(\bar{s})$ and $V_{b(i)}(1)$ lies above $V_{b(i)}(s_1)$ for all $s_1 \in [\bar{s}, 1]$. Define the straight line by $\widehat{V}_{b(i)}(s_1)$. The maximum of $\widehat{V}_{b(i)}(s_1)$ is obviously attained at either of the two corners $s_1 = \bar{s}$ or $s_1 = 1$. Moreover, if $\widehat{V}_{b(i)}(s_1) \geq V_{b(i)}(s_1)$ for $s_1 \in [\bar{s}, 1]$, knowing that by construction $\widehat{V}_{b(i)}(s_1) = V(s_1)$ at the corners $s_1 = \bar{s}$ and $s_1 = 1$, then $V_{b(i)}(s_1)$ must also attain its maximum at either of the two corners.

First, construct the function $\widehat{V}_{b(i)}(s_1)$. Denoting by

$$T \equiv \frac{V_1^h + V_1^l}{2} - \frac{\bar{s}^2}{2}\Delta V_h + \frac{\bar{s}(2-\bar{s})}{2}\Delta V_l,$$

we can write

$$V_{b(i)}(s_1) = T + \frac{\sigma_1(s_1)}{4}\left(\bar{s}^2\Delta V_h + (1-\bar{s})^2\Delta V_l\right) \\ + \frac{\sigma_2(s_1)-\sigma_1(s_1)}{4}(1-s_1)^2\Delta V_l - \delta\left(1-\frac{\sigma_1(s_1)}{2}\right)(s_1-\bar{s}),$$

where $\sigma_1(s_1)$ and $\sigma_2(s_1)$ are defined in (66) and(67), respectively. The linear function is given by

$$\widehat{V}_{b(i)}(s_1) = V_{b(i)}(\bar{s}) + \frac{V_{b(i)}(1) - V_{b(i)}(\bar{s})}{1-\bar{s}}(s_1-\bar{s}).$$

It is useful to note that $\sigma_1(s_1=\bar{s}) = 0$ and $\sigma_2(s_1=\bar{s}) = \sigma$, where σ is given by (71). Moreover, $\sigma_1(s_1=1) = \sigma$ and $\sigma_2(s_1=1) = 0$. We can then write

$$\widehat{V}_{b(i)}(s_1) = T + \frac{\sigma}{4}(1-\bar{s})^2\Delta V_l + \left[\frac{\sigma}{4}\frac{\bar{s}^2}{1-\bar{s}}\Delta V_h - \delta\left(1-\frac{\sigma}{2}\right)\right](s_1-\bar{s}).$$

The inequality $\widehat{V}_{b(i)}(s_1) \geq V_{b(i)}(s_1)$ can be written as:

$$(\sigma-\sigma_1(s_1))(1-\bar{s})^2\Delta V_l - (\sigma_2(s_1)-\sigma_1(s_1))(1-s_1)^2\Delta V_l \\ + \left(\sigma\frac{s_1-\bar{s}}{1-\bar{s}} - \sigma_1(s_1)\right)\bar{s}^2\Delta V_h + 2(\sigma-\sigma_1(s_1))(s_1-\bar{s})\delta \geq 0. \quad (79)$$

We can re-write the first line as follows

$$(\sigma-\sigma_1(s_1))(1-\bar{s})^2\Delta V_l - (\sigma_2(s_1)-\sigma_1(s_1))(1-s_1)^2\Delta V_l \\ = (\sigma-\sigma_2(s_1))(1-s_1)^2\Delta V_l + (\sigma-\sigma_1(s_1))(s_1-\bar{s})\left(2(1-s_1)+(s_1-\bar{s})\right)\Delta V_l.$$

Substituting this expression into (79) and simplifying yields

$$\begin{aligned}\frac{1}{2\tau}\frac{(1-s_1)(s_1-\bar{s})}{s_1}&\left[\frac{s_1-\mu^*(1-\delta)}{2\mu^*}s_1(1-s_1)\right]\\ &+\frac{1}{2\tau}\frac{(1-s_1)(s_1-\bar{s})}{s_1}\left[-\mu^*(1-\mu^*)(1+\delta)^2\frac{s_1-2\mu^*(1-\delta)}{2\mu^*}\right]\\ &+\frac{1}{2\tau}\frac{(1-s_1)(s_1-\bar{s})}{s_1}\left[2(1-s_1)\left(\frac{s_1(1-s_1)}{2\mu^*}+s_1-\mu^*(1-\delta^2)\right)\right]\\ &+\frac{1}{2\tau}\frac{(1-s_1)(s_1-\bar{s})}{s_1}\left[(s_1-\bar{s}+2\delta)\frac{s_1(1-s_1)}{2\mu^*}+(s_1-\bar{s}+2\delta)(s_1-\mu^*(1-\delta^2))\right]\geq 0.\end{aligned}$$

This can be re-written as

$$\begin{aligned}s_1(1-s_1)\frac{s_1-\mu^*(1-\delta)}{2\mu^*}+2(1-s_1)\left[\frac{s_1(1-s_1)}{2\mu^*}+s_1-\mu^*(1-\delta^2)\right]+\frac{s_1(1-s_1)}{2\mu^*}(s_1-\bar{s}+2\delta)\\ +\frac{1}{2}(1-\mu^*)(1+\delta)^2(2\mu^*(1-\delta)-s_1)+(s_1-\bar{s}+2\delta)(s_1-\mu^*(1-\delta^2))\geq 0.\end{aligned}$$

Since in region b(i), $s_1 \geq \mu^*(1+\delta)=\bar{s}$, we know that the first line is non-negative. A sufficient condition is thus that the second line is also non-negative. Re-writing the second line, this reduces to the quadratic equation

$$(s_1-\bar{s})^2+b(s_1-\bar{s})+c\geq 0,$$

where

$$\begin{aligned}b &\equiv \frac{1}{2}\left(\mu^*(1+\delta)(1+3\delta)-(1-\delta)^2\right)\\ c &\equiv \frac{1}{2}\mu^*(1+\delta)\left((1-\delta)^2+\mu^*(1+\delta)(3\delta-1)\right),\end{aligned}$$

which is always non-negative if $b^2-4c\leq 0$. This last inequality, after simple algebra, can be re-written as:

$$\Delta(\mu^*)\equiv\frac{(1-\delta)^2}{4}\left(\Big(3(1+\delta)\mu^*\Big)^2-(1+\delta)(10+6\delta)\mu^*+(1-\delta)^2\right)\leq 0. \tag{80}$$

Note that $\Delta(\mu^*)$ is convex so that if (80) is satisfied at either extremes of μ^*, then it also holds for any interior μ^*. We know that $\mu^* \in [1/2, 1/(1+\delta))$ since for region b(ii) to exists we also need $\bar{s} = \mu^*(1+\delta) < 1$. Consider first the lower bound and note that:

$$\Delta\left(\frac{1}{2}\right) = \frac{(1-\delta)^2}{4}\left(\frac{9}{4}(1+\delta)^2 - (1+\delta)(5+3\delta) + (1-\delta)^2\right)$$
$$= \frac{(1-\delta)^2}{4}\left(-(1+\delta)\frac{3}{4} - 2 + \frac{(1-\delta)^2}{1+\delta}\right) < 0,$$

since $(1-\delta)^2/(1+\delta) < 1 < 2$. Consider next the upper bound:

$$\Delta\left(\frac{1}{1+\delta}\right) = \frac{(1-\delta)^2}{4}\left(9 - (10+6\delta) + (1-\delta)^2\right)$$
$$= \frac{(1-\delta)^2}{4}\left(\delta(\delta-8)\right) < 0,$$

since $\delta \leq 1$. It follows that (80) is satisfied, implying that $\widehat{V}_{b(i)}(s_1) \geq V_{b(i)}(s_1)$ for every $s_1 \in [\mu^*(1+\delta), 1]$, i.e. that on region b(i), the objective $V(s_1)$ attains a maximum at one of its corners.

Proof of Claim 4: Using the corresponding expressions, the condition $V_p(\mu^*) > V_{b(i)}(1)$ can be written as

$$-\frac{1}{2}\Delta V_h(\mu^*)^2 + \frac{1}{2}\Delta V_h(1-\frac{\sigma}{2})\bar{s}^2 + \frac{1}{2}\Delta V_l\left(\frac{\hat{\sigma}_2}{2} + (1-\frac{\hat{\sigma}_2}{2})(2\mu^* - (\mu^*)^2)\right)$$
$$-\frac{1}{2}\Delta V_l\left(\frac{\sigma}{2} + (1-\frac{\sigma}{2})(2\bar{s} - \bar{s}^2)\right) + \delta(1-\frac{\sigma}{2})(1-\bar{s}) > 0,$$

where $\sigma = \sigma_1(s_1 = 1)$ given in (71) and $\hat{\sigma}_2 = \sigma_2(s_1 = \mu^*) \geq \sigma$. The inequality can be rewritten as

$$\frac{1}{2}\left(1-\frac{\sigma}{2}\right)h(\delta) > \frac{1}{2}\Delta V_h\frac{\sigma}{2}(\mu^*)^2 + \frac{1}{2}\Delta V_l(1-\mu^*)^2\left(1-\frac{\hat{\sigma}_2}{2}\right), \tag{81}$$

where

$$h(\delta) \equiv \Delta V_h(\bar{s}^2 - (\mu^*)^2) + \Delta V_l(1-\bar{s})^2 + 2\delta(1-\bar{s}).$$

Note that σ is a decreasing function of δ while $\hat{\sigma}_2$ is independent of δ. Hence the right-hand side of inequality (81) is decreasing in δ. Moreover, $h(\delta)$ can be simplified to yield

$$h(\delta) = 1 - \mu^*(1-\mu^*) + 2\delta - \mu^*(1+\delta)^2.$$

Note that $h(\delta)$ reaches its maximum at $\delta = \frac{1}{\mu^*} - 1$ and is increasing for lower values of δ. Furthermore, since we require $\bar{s} = \mu^*(1+\delta) \leq 1$, the maximum admissible value of δ when $\mu^* > \frac{1}{2}$, is just $\delta = \frac{1}{\mu^*} - 1$. Hence, the left-hand side of inequality (81) is increasing in δ.

For the special case $\delta = 0$, we have $\sigma = \hat{\sigma}_2$ and inequality (81) reduces to

$$\frac{1}{2}\left(1 - \frac{\hat{\sigma}_2}{2}\right)(1-\mu^*)^2 > \frac{1}{2}\mu^*(1-\mu^*)\frac{\hat{\sigma}_2}{2} + \frac{1}{2}\left(1 - \frac{\hat{\sigma}_2}{2}\right)(1-\mu^*)^2,$$

which is violated. At the upper bound of δ we have $\bar{s} = 1$ or, equivalently $\delta = \frac{1}{\mu^*} - 1$. Using the observation that $\sigma(\bar{s} = 1) = 0$, inequality (81) can be written as

$$\frac{1}{2}\left(1 - \mu^*(1-\mu^*) + 2\left(\frac{1}{\mu^*} - 1\right) - \frac{1}{\mu^*}\right) > \frac{1}{2}(1-\mu^*)^2\left(1 - \frac{\hat{\sigma}_2}{2}\right),$$

or, equivalently, as

$$\frac{1}{2}(1-\mu^*)^2\frac{1+\mu^*}{\mu^*} > \frac{1}{2}(1-\mu^*)^2\left(1 - \frac{\hat{\sigma}_2}{2}\right),$$

which always holds as $\frac{1+\mu^*}{\mu^*} > 1 > 1 - \frac{\hat{\sigma}_2}{2}$. Hence, there is a threshold value of δ, denoted $\widetilde{\delta}$, such that for any δ above the threshold $V(\mu^*) > V(1)$.

Hence, for δ above $\widetilde{\delta}$ we have $V(s_F^P) > V(\mu^*) > V(1) = V(0)$, where the first inequality follows from Lemma 7. Moreover, we have that $V(s_F^P) > V(\mu^*) > V(\mu^*(1+\delta))$ where the second inequality follows from $V_p(s_1)$ being decreasing for $s_1 \geq \mu^*$. For $\delta = 0$, region p disappears, and $V(0) > V(\mu^*)$ since $V_{a(i)}(s_1)$ is decreasing over region a(i) (see 75). □

Proof of Proposition 5. The proof of Proposition 5 follows from Lemma 7, Lemma 8 and Lemma 9. □

Proof of Lemma 5. The stress test design problem is given by (16) but, to ensure that an optimum exists, the constrained set changes to $\mu_1 \leq \widehat{\mu}$ and $\mu_2 > \widehat{\mu}$, that is we want to solve:

$$
\begin{aligned}
&\max_{s_1} v(s_1) \\
&\text{s.t. } \mu_1 \leq \widehat{\mu}, \\
&\qquad \mu_2 > \widehat{\mu},
\end{aligned} \tag{82}
$$

where $v(s_1)$ is defined in (15). Neglecting the constraints in problem (82), the FOC for s_1 is :

$$
v'(s_1) = 1 - \frac{s_1}{\mu^*} = 0,
$$

and is solved for $s_1 = \mu^*$. The constraint $\mu_2 = (s_1 + 1)/2 > \widehat{\mu}$ is satisfied for $s_1 = \mu^*$ since $\widehat{\mu} < \mu^*$. The constraint $\mu_1 = s_1/2 \leq \widehat{\mu}$ is satisfied at the unconstrained optimum when $\mu^*/2 \leq \widehat{\mu}$ and is binding otherwise. When the constraint is slack the optimum is $s_1 = \mu^*$, when it is binding, s_1 is chosen to satisfy the constraint, i.e. $s_1 = 2\widehat{\mu}$. Thus, the optimal stress test is a binary partition with cutoff $s_1 = \min\{2\widehat{\mu}, \mu^*\}$. □

Proof of Proposition 6. When $\mu^* > \widehat{\mu}$, the stress test design problem is given by (21) but, to ensure that an optimum exists, the constrain set changes to $\mu_1 \leq \widehat{\mu}$ and $\mu_2 > \widehat{\mu}$. Hence, the stress test design problem is

$$
\begin{aligned}
&\max_{s_1, s_2} V(s_1, s_2) \\
&s.t. \ \mu_1 < \widehat{\mu} \\
&\qquad \mu_2 \geq \widehat{\mu},
\end{aligned} \tag{83}
$$

where $V(s_1, s_2)$ is defined in (49), $v(s_1)$ is defined in (15), while $\nu_2 = s_2 - s_1$ and $\mu_2 = \frac{s_1+s_2}{2}$, as defined in (2) and (3).

The proof proceeds as in Proposition 1 until Step 3, and the FOCs are the ones in (50). We replicate the FOCs here for ease of exposition:

$$
\begin{aligned}
\frac{\partial V}{\partial s_1} &= v'(s_1) + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) - (\nu_2)^2 \frac{1 - \frac{1}{2}(s_2 - \widehat{\mu}) - s_1}{2\widehat{\mu}}\right] = 0, \\
\frac{\partial V}{\partial s_2} &= \frac{1}{2\tau}(\nu_2)^2 \frac{1 - \frac{1}{2}(s_1 - \widehat{\mu}) - s_2}{2\widehat{\mu}} = 0.
\end{aligned}
$$

First, we show that when $\mu^* < \underline{\mu}$, where

$$\underline{\mu} \equiv \frac{\widehat{\mu}}{1 - \frac{1}{\tau}\frac{1}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^3}, \tag{84}$$

the optimal stress test is a binary partition, $s_2 = 1$, and we provide the equation that implicitly defines the optimal s_1. Suppose $s_2 > s_1$, the second equation is solved for $s_2 = 1 - \frac{1}{2}(s_1 - \widehat{\mu})$. If at the optimum we have $s_1 < \widehat{\mu}$, then it is optimal to set $s_2 = 1$. To determine when this is the case, we evaluate the sign of the following derivative:

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1=\widehat{\mu} \\ s_2=1-\frac{1}{2}(s_1-\widehat{\mu})}} = 1 - \frac{\widehat{\mu}}{\mu^*} - \frac{1}{2\tau}\frac{(1-\widehat{\mu})^3}{4\widehat{\mu}}.$$

We have that $\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1=\widehat{\mu} \\ s_2=1-\frac{1}{2}(s_1-\widehat{\mu})}} < 0$ when $\mu^* < \underline{\mu}$. It follows that for these parameter values, we have $s_1 < \widehat{\mu}$ and $s_2 = 1$. The optimal s_1 solves:

$$\begin{aligned}\left.\frac{\partial V}{\partial s_1}\right|_{s_2=1} &= 1 - \frac{s_1}{\mu^*} + \frac{1}{2\tau}\left[-(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) - (1-s_1)^2\frac{1-\frac{1}{2}(1-\widehat{\mu}) - s_1}{2\widehat{\mu}}\right] \\ &= 1 - \frac{s_1}{\mu^*} + \frac{1}{2\tau}(1-s_1)^2\frac{\frac{3\widehat{\mu}-1}{2} - s_1}{2\widehat{\mu}} = 0,\end{aligned} \tag{85}$$

and thus is equal to $\widetilde{s}$, the unique solution to (30). The constraints are satisfied. The constraint $\mu_1 = (0 + s_1)/2 \leq \widehat{\mu}$ is satisfied since at the optimum we have $s_1 < \widehat{\mu}$. The constraint $\mu_2 = (s_1 + s_2) > \widehat{\mu}$ is also satisfied since at the optimum we have $s_1 > 2\widehat{\mu} - 1$. To see this note that:

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1=2\widehat{\mu}-1 \\ s_2=1}} = 1 - \frac{2\widehat{\mu}-1}{\mu^*} + \frac{1}{2\tau}\frac{(1-\widehat{\mu})^3}{\widehat{\mu}} > 0,$$

since we have that $1 - (2\widehat{\mu} - 1)/\mu^* > 0$ as $\mu^* > \widehat{\mu} > 2\widehat{\mu} - 1$, and the other term is also positive.

We now distinguish two cases when $\mu^* > \underline{\mu}$: Case A, where $\widehat{\mu} < 2/5$, and Case B, $\widehat{\mu} \geq 2/5$. Consider Case A. We show that when $\mu^* > \underline{\mu}$ the optimal stress test contains two buckets with fully granular grades for resilience levels above the buckets, i.e. $s_2 < 1$. From the discussion above we already know that for these parameter values, we have $s_1 > \widehat{\mu}$ and $s_2 = 1 - \frac{1}{2}(s_1 - \widehat{\mu})$.

When the constraints are satisfied, the optimal optimal s_1 solves:

$$\left.\frac{\partial V}{\partial s_1}\right|_{s_2=1-\frac{1}{2}(s_1-\widehat{\mu})} = 1-\frac{s_1}{\mu^*}-\frac{1}{2\tau}\left[(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}}-1\right)+\frac{(1-\frac{3}{2}s_1+\frac{1}{2}\widehat{\mu})^3}{4\widehat{\mu}}\right]=0. \tag{86}$$

The constraints are indeed satisfied when

$$\mu^* \leq \frac{2\widehat{\mu}}{1-\frac{1}{2\tau}\left((1-2\widehat{\mu})^2+\frac{(1-\frac{5}{2}\widehat{\mu})^3}{4\widehat{\mu}}\right)} \equiv \mu_\circ. \tag{87}$$

To see this note that the constraint $\mu_2 = (s_1+s_2)/2 > \widehat{\mu}$ is satisfied since $s_2 \geq s_1 > \widehat{\mu}$, while the constraint $\mu_1 = s_1/2 \leq \widehat{\mu}$ is satisfied if $s_1 \leq 2\widehat{\mu}$ and this is the case when

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1=2\widehat{\mu}\\ s_2=1-\frac{1}{2}(s_1-\widehat{\mu})}} = 1-\frac{2\widehat{\mu}}{\mu^*}+\frac{1}{2\tau}\left((1-2\widehat{\mu})^2+\frac{(1-\frac{5}{2}\widehat{\mu})^3}{4\widehat{\mu}}\right) \leq 0,$$

which is equivalent to (87).

We can also verify that at the optimum $s_2 = 1-\frac{1}{2}(s_1-\widehat{\mu}) > s_1$. This happens when $s_1 < \frac{2}{3}+\frac{1}{3}\widehat{\mu}$, which is the case when

$$\left.\frac{\partial V}{\partial s_1}\right|_{\substack{s_1=\frac{2}{3}+\frac{1}{3}\widehat{\mu}\\ s_2=1-\frac{1}{2}(s_1-\widehat{\mu})}} = 1-\frac{\frac{2}{3}+\frac{1}{3}\widehat{\mu}}{\mu^*}-\frac{1}{2\tau}\frac{2}{27}\frac{(1-\widehat{\mu})^3}{\widehat{\mu}} < 0,$$

or equivalently, when

$$\mu^* < \frac{\dfrac{2}{3}+\dfrac{1}{3}\widehat{\mu}}{1-\dfrac{1}{\tau}\dfrac{1}{\widehat{\mu}}\left(\dfrac{1-\widehat{\mu}}{3}\right)^3} \equiv \overline{\mu}. \tag{88}$$

One can show that this condition is milder than (87) when $\widehat{\mu} < 2/5$ and $\tau > \frac{1}{\widehat{\mu}}\left(\frac{1-\widehat{\mu}}{2}\right)^2$, as by assumption (20).

When the constraint $\mu_1 \leq \widehat{\mu}$ is binding, i.e. (87) is not satisfied, it is optimal to choose s_1 to satisfy the constraint, $s_1 = 2\widehat{\mu}$, and $s_2 = 1-\frac{1}{2}\widehat{\mu}$. Note that, once again, $s_2 > s_1$ if $\widehat{\mu} < 2/5$.

Consider now Case B. From the discussion above, we have that $s_1 < s_2$ when (88) is satisfied and the optimal stress test is as discussed above. When (88) is not satisfied, a coarse pass grade is not optimal anymore and thus we have $s_2 = s_1$. Plugging $s_2 = s_1$ in the FOC for s_1 we get the equation that implicitly defines the optimal s_1:

$$\left.\frac{\partial V}{\partial s_1}\right|_{s_2=s_1} = 1 - \frac{s_1}{\mu^*} - \frac{1}{2\tau}(1-s_1)^2\left(\frac{s_1}{\widehat{\mu}} - 1\right) = 0. \tag{89}$$

The constraint $\mu_1 = s_1/2 \leq \widehat{\mu}$ is satisfied when

$$\mu^* \leq \frac{2\widehat{\mu}}{1 - \frac{1}{2\tau}(1-2\widehat{\mu})^2} \tag{90}$$

since this implies that

$$\left.\frac{\partial V}{\partial s_1}\right|_{s_2=s_1=2\widehat{\mu}} = 1 - \frac{2\widehat{\mu}}{\mu^*} - \frac{1}{2\tau}(1-2\widehat{\mu})^2 \leq 0. \tag{91}$$

When the constraint is binding, it is optimal to choose s_1 and s_2 to satisfy the constraint, $s_1 = s_2 = 2\widehat{\mu}$. □

Appendix C: Computations of $\chi(\tau, \mu^*)$ for Section 5.2

Denote by r_{t+1} the returns around the stress test, and by p_t and p_{t+1} the bank's equity prices before and after the stress test. The bank's absolute cumulative abnormal returns when the bank is subject to an informative stress test with passing threshold s_1 is:

$$|\text{CAR}|(s_1) \equiv \mathbb{E}\big(\big|r_{t+1} - \mathbb{E}(r_{t+1})\big|\big) = \mathbb{E}\left(\left|\frac{p_{t+1} - \mathbb{E}(p_{t+1})}{p_t}\right|\right) = \mathbb{E}\left(\left|\frac{p_{t+1} - \mathbb{E}(p_{t+1})}{\mathbb{E}(p_{t+1})}\right|\right),$$

where the last equality follows from the efficient-market hypothesis, i.e. the pre-test price incorporates future information, $p_t = \mathbb{E}(p_{t+1})$.

For a bank subject to a stress test with passing threshold s_1 the bank's expected equity price is:

$$\mathbb{E}(p_{t+1}) = (1-s_1)\left[\mu_2\sigma(\mu_2)\frac{1}{2}E_h^1(1) + \left(1 - \frac{1}{2}\sigma(\mu_2)\right)\mu_2 E_h^1(\mu_2)\right] \equiv p(s_1). \tag{92}$$

With probability s_1 the bank fails the test, there is no information acquisition and no funding provision, so that the price is equal to 0. With the residual probability $(1-s_1)$, the bank passes the test. The realized price is $E_h^1(1)$ whenever the bank is financially sound and the speculator's positive signal is revealed to the market maker. Conversely, when the bank is insolvent and a negative signal is revealed to the market maker, the price is equal to 0. Finally, when the order flow is uninformative, or when the speculator observes no signal, the price is $\mu_2 E_h^1(\mu_2)$. Note that if the stress test is informative ($\mu_2 \geq \widehat{\mu}$), information acquisition $\sigma(\mu_2)$ and the expected price $p(s_1)$ are positive numbers. Following a similar reasoning we can compute $|\text{CAR}|(s_1)$:

$$\begin{aligned} |\text{CAR}|(s_1) = {} & (1-s_1)\mu_2\sigma(\mu_2)\frac{1}{2}\left|\frac{E_h^1(1)-p(s_1)}{p(s_1)}\right| + \\ & + (1-s_1)\left(1-\frac{1}{2}\sigma(\mu_2)\right)\left|\frac{\mu_2 E_h^1(\mu_2)-p(s_1)}{p(s_1)}\right| + \\ & + \left((1-s_1)(1-\mu_2)\sigma(\mu_2)\frac{1}{2}+s_1\right)\left|\frac{0-p(s_1)}{p(s_1)}\right|. \end{aligned} \tag{93}$$

For institutions that are subject to a stress test, we compute the $|\text{CAR}|$ at the optimal passing threshold $s_F(\tau,\mu^*)$ derived in Proposition 1 (equation (23)), while for non-tested institutions, we consider the $|\text{CAR}|$ implied by an uninformative stress test. In the latter case, the price and its expectation differ depending on whether the speculator acquires information at the prior beliefs (in which case the expected price is $\mathbb{E}(p_{t+1}) = p(0)$), or not ($\mathbb{E}(p_{t+1}) = 0$). The difference between the $|\text{CAR}|$ for tested and non-tested institution is:

$$\chi(\tau,\mu^*) = \begin{cases} |\text{CAR}|\big(s_F(\tau,\mu^*)\big) - 0 & \text{if } \frac{1}{2} < \mu^* \\ \\ |\text{CAR}|\big(s_F(\tau,\mu^*)\big) - |\text{CAR}|\big(0\big) & \text{if } \frac{1}{2} \geq \mu^*. \end{cases} \tag{94}$$