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Representation formulas for limit values of long run stochastic optimal controls

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Abstract

A classical problem in stochastic ergodic control consists of studying the limit behavior of the optimal value of a discounted integral in infinite horizon (the so called Abel mean of an integral cost) as the discount factor λ tends to zero or the value defined with a Cesaro mean of an integral cost when the horizon T tends to $+\infty$. We investigate the possible limits in the norm of uniform convergence topology of values defined through Abel mean or Cesaro means when $\lambda \to 0^+$ and $T \to +\infty$, respectively. Here we give two types of new representation formulas for the accumulation points of the values when the averaging parameter converges. We show that there is only one possible accumulation point which is the same for Abel means or Cesàro means. The first type of representation formula is based on probability measures on the product of the state space and the control state space, which are limits of occupational measures. The second type of representation formulas is based on measures which are the projection of invariant measure on the space of relaxed controls. We also give a result comparing the both sets of measures involved in the both classes of representation formulas. An important consequence of the representation formulas is the existence of the limit value when one has the equicontinuity property of Abel or Cesàro mean values. This is the case, for example, for nonexpansive stochastic control systems. In the end some insightful examples are given which make to better understand the results.

Keywords. Stochastic nonexpansivity condition; limit value; stochastic optimal control.
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1 Introduction

Let us consider a stochastic control system described by the following stochastic differential equation living in a compact subset Y of $\mathbf{R}^n$

$$
(1) \qquad \begin{cases} dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dW_s, \\ X_0 = x \in Y, \end{cases}
$$

where A is a compact metric space, $b : \mathbf{R}^n \times A \to \mathbf{R}^n$, $\sigma : \mathbf{R}^n \times A \to \mathbf{R}^{n\times d}$ are continuous functions satisfying suitable assumptions and the admissible control α is an A-valued $\mathbb{F}$-progressively measurable processes. Let $\mathcal{A}$ be the set of all A-valued $\mathbb{F}$-progressively measurable processes α. We denote by $t \mapsto X_t^{x,\alpha}$ the solution of (1) (whose existence and uniqueness is ensured by classical assumptions stated later on) and we suppose the invariance of Y with respect to (1).

Associated with $\lambda > 0$ and $T > 0$, we define the following value functions

$$
(2) \qquad V_\lambda(x) = \inf_{\alpha(\cdot)\in\mathcal{A}} E[\lambda \int_0^\infty e^{-\lambda s}\ell(X_s^{x,\alpha}, \alpha_s)ds],\ x \in \mathbf{R}^n, \qquad \text{(Abel mean)}
$$

and

$$
(3) \qquad U_T(x) = \inf_{\alpha(\cdot)\in\mathcal{A}} E[\frac{1}{T}\int_0^T \ell(X_s^{x,\alpha}, \alpha_s)ds],\ x \in \mathbf{R}^n, \qquad \text{(Cesàro mean)}
$$

where $\ell : \mathbf{R}^N \times A \mapsto \mathbf{R}$ is a bounded function.

The existence of the limit of V_λ as $\lambda \to 0^+$ and U_T as $T \to +\infty$ has been studied widely in the literature by several different methods. We refer the reader to Alvarez and Bardi [1], Arisawa and P.L. Lions [2], Buckdahn, Goreac and Quincampoix [7], Buckdahn, Quincampoix and Renault [10], Goreac [20], Li, Quincampoix and Renault [23], Quincampoix and Renault [27] (and also Renault [28] for an analogous problem in discrete time).

A large number of works concern the so called ergodic case. It consists of studying the case where the above limits exist and are independent of the initial condition x. This has been done by PDE techniques by imposing coercivity assumptions on the Hamiltonian (c.f., Alvarez and Bardi [1], Arisawa and P.L. Lions [2], P.L. Lions, Papanicolaou and Varadhan [25] and references therein); indeed, the value functions U_T and V_λ satisfy the suitable Hamilton-Jacobi-Bellman equation (in viscosity sense) for which one can hope to pass directly to the limit as $\lambda \to 0^+$ or U_T as $T \to +\infty$. The ergodic deterministic case has also been studied through direct methods by assuming suitable controllability assumptions in Artstein and Gaitsgory [3], Fathi [15], and so on.

There are several very basic examples in which the limit value exists but it depends on the initial condition. Indeed, if one considers the following two dimensional deterministic control system

$$
(4) \qquad \begin{cases} dX_t^1 &= \alpha(t)(1 - X_t^1)dt \\ dX_t^2 &= \alpha(t)^2(1 - X_t^1)dt \end{cases}, X_0 = (X_0^1, X_0^2) = x \in \mathbf{R}^2;
$$

with $A := [0, 1]$ and the integral cost $\ell(x) = 1 - x_1(1 - x_2)$, one can show directly that $Y := \{x = (x_1, x_2) \in [0, 1]^2 : x_1 \geq x_2\}$ is invariant for the above control system. Moreover, U_T and V_λ converge, as $T \to +\infty$ and $\lambda \to 0^+$, respectively, to $\widehat{V}(x_1, x_2) := x_2$ uniformly on Y (see, e.g., [27]).

The nonergodic case has been studied in Cannarsa and Quincampoix [12], Quincampoix and Renault [27] for deterministic control problems and in Buckdahn, Goreac and Quincampoix [7], Goreac and Serea [19], Goreac [20], Li and Zhao [24] for the stochastic control problems by imposing a suitable non-expansivity condition on the dynamics.

Here our approach is different: we divide the problem of the existence of the uniform limit $\lim_{\lambda\to 0^+} V_\lambda$ in two distinct questions: First is to show that $(V_\lambda)_{\lambda\geq 0}$ has at most one accumulation point (in the uniform convergence topology), second is to obtain the equicontinuity of the family $(V_\lambda)_{\lambda\geq 0}$. Obviously, Arzelà-Ascoli Theorem implies that if both above questions have a positive answer then the uniform limit $\lim_{\lambda\to 0^+} V_\lambda$ does exist. The present paper mainly focuses on the first question: we obtain that the unique accumulation point, if it exists, has a precise form which we give an explicit description. This is the main novelty of our work. The first type of representation formula is based on measures on the state space $Y \times A$ which are the limit of occupational measures. The second type of representation formula is based on the projection of invariant measures on the path space (the product of the set of trajectories $C([0,+\infty), Y)$ and the set of admissible controls). We also show that the same argument can be applied to $\lim_{T\to+\infty} U_T$.

For the first type of representation formula, we use and extend several results already obtained for limits of occupational measures in Bhatt and Borkar [5], Buckdahn, Goreac and Quincampoix [6], Borkar and Gaitsgory [11], Gaitsgory [16], Gaitsgory and Quincampoix [17, 18] for the deterministic case. For the second type of representation formula we use a formulation of stochastic optimal control problems in terms of relaxed control in El Karoui, Nguyen and Jeanblanc-Picqué [14], and Kushner [22]. This enables us to define suitable invariant probability measures on the path space, as well as their projection. The first type of representation formula can be regarded as a non-trivial extension of results obtained in [10] for deterministic case. It is worth pointing out that our present second type of representation formula is new not only for the stochastic control case, but also new for the deterministic case, which can be interpreted as a particular case of the stochastic one (with $\sigma \equiv 0$) in (1). Indeed, our result can cope with a rather general integral cost ℓ (while the results of [10] are only valid for an integral cost $\ell(x, a) = \ell(x)$ independent of the control). Both representation formulas suppose the existence of an invariant compact set Y for (1). The invariance of such Y can be characterized through several equivalent conditions (c.f., for instance, Buckdahn, Peng, Quincampoix and Rainer [9]).

Our representation formulas have important consequences. Since the representation formula for accumulation points of $(V_\lambda)_{\lambda>0}$ and of $(U_T)_{T>0}$ are the same, we deduce that if $\lim_{\lambda\to 0^+} V_\lambda(\cdot)$ and $\lim_{T\to+\infty} U_T(\cdot)$ exist in the topology of uniform convergence, then they necessarily coincide (compare this with the Abelian-Tauberian Theorems in Gaitsgory [16], Oliu-Barton and Vigeral [26]). We will also discuss some conditions insuring the equicontinuity of the families $(V_\lambda)_{\lambda>0}$ and of $(U_T)_{T>0}$, from which the existence of the limit value is deduced with the help of our representation formulas.

Our theoretical studies are completed with illustrating examples presented in the last section. The first example is very instructive. Indeed, although the standard assumptions of the coefficients of the dynamics and the running cost are satisfied, $(V_\lambda(.))_{\lambda>0}$ and $(U_T(.))_{T>0}$ do not have any accumulation point in the topology of uniform converge, as $\lambda \searrow 0$ and $T \to +\infty$, respectively, but they converge pointwise to some limit functions $V_0(.)$ and $U_\infty(.)$, and these pointwise limits differ. Later Example 3.6 of [7] (in [7] studied under the non-expansivity condition) is revisited. It is in particular shown that it reduces to an ergodic problem. By modifying this Example 3.6 we obtain a non-ergodic one.

Let us now describe how our paper is organized. After a preliminary section devoted to notations and basic facts, the second section introduces occupational measures and derives our first type of representation formula. We also discuss examples and the non-expansive condition. In the third section, after recalling several notions and results for relaxed control, we state and prove the second type of representation formula. Section 5 concerns the comparison between both types of representation formulas. Finally, Section 6 is devoted to the illustrating examples mentioned above, and in the Appendix (Section 7) the proof of Lemma 4.5 is given.

2 Preliminaries

For any metric space D, the set of Borel sets is denoted by $\mathcal{B}(D)$, while the notation $\Delta(D)$ stands for the associated set of Borel probability measures over D. The set of all continuous real-valued functions on D is denoted by $C(D)$. Finally, $|x|$ is the usual Euclidean norm of $x \in \mathbf{R}^n$ $(n \geq 1)$.

Let us consider the Brownian motion $W = (W(s),\ s \geq 0)$ as the d-dimensional coordinate process on the classical Wiener space $(\Omega, \mathcal{F}, P)$, where Ω is the set of continuous functions from $\mathbf{R}^+$ to $\mathbf{R}^d$ starting from 0 (i.e., $\Omega = C_0([0, +\infty); \mathbf{R}^d)$) endowed with the topology of uniform convergence on compacts, $\mathcal{F}$ is the completed Borel σ-algebra over Ω, P is the Wiener measure on $(\Omega, \mathcal{F})$: $W_s(\omega) = \omega_s,\ s \geq 0,\ \omega \in \Omega$. By $\mathbb{F} = \{\mathcal{F}_s,\ s \geq 0\}$ we denote the natural filtration generated by $\{W_s\}_{s\geq 0}$ and augmented by all P-null sets, i.e.,

$$\mathcal{F}_s = \sigma\{W_r, r \leq s\} \vee \mathcal{N}_P,\ s \geq 0,$$

where $\mathcal{N}_P$ is the set of all P-null subsets on Ω.

For our functions $b : \mathbf{R}^n \mapsto \mathbf{R}^n$, $\sigma : \mathbf{R}^n \times A \mapsto \mathbf{R}^{n\times d}$ we suppose the following standard conditions:

(H1) b and σ are uniformly continuous in (x, a);

(H2) $|\sigma(x, a) - \sigma(x', a)| \leq c|x - x'|$ for all $x, x' \in \mathbf{R}^n$, for all $a \in A$;

(H3) $|b(x, a) - b(x', a)| \leq c|x - x'|$ for all $x, x' \in \mathbf{R}^n$, for all $a \in A$,

where $c > 0$ is a given constant.

With above assumptions, for every $x \in \mathbf{R}^n$ and all admissible control $\alpha \in \mathcal{A}$, there exists a unique continuous $\mathbb{F}$-adapted solution $t \mapsto X_t^{x,\alpha}$ defined on $[0, +\infty)$. Furthermore, we consider a running cost function $\ell : \mathbf{R}^n \times A \mapsto \mathbf{R}$ satisfying

(H4) ℓ is continuous in (x, a) and there exists some $c > 0$ such that
$0 \leq \ell(x, a) \leq 1$, $|\ell(x, a) - \ell(x', a)| \leq c|x - x'|$, for all $x, x' \in \mathbf{R}^n$, and $a \in A$.

Then it is also standard that, for any $T > 0$ and any $\lambda > 0$, the value functions $U_T(\cdot)$ and $V_\lambda(\cdot)$ are continuous and bounded. Here we have supposed that $0 \leq \ell \leq 1$ only for the sake of simplicity; of course our approach remains valid for any bounded continuous function ℓ which is Lipschitz in x, uniformly with respect to a.

Now we suppose the following condition on the stochastic control system:

(H5) there exists a compact $Y \subset \mathbf{R}^n$ which is invariant for (1).

It is equivalent to say that, for all $x \in Y$ and any $\alpha \in \mathcal{A}$, $X_s^{x,\alpha} \in Y$, P-a.s., for all $s \geq 0$. This condition can be characterized through an equivalent condition on b, σ and Y (c.f., e.g., [9]).

To end with our preliminaries, we recall the definition of the second order differential operator $\mathcal{L}$ associated with (1). It is given by

$$(\mathcal{L}\phi)(x,a) = \langle b(x,a), \nabla\phi(x)\rangle + \frac{1}{2}\mathrm{tr}(\left(D^2\phi\sigma\sigma^*\right)(x,a)),\ x \in \mathbf{R}^n,\ a \in A,$$

where $\phi \in C^2(\mathbf{R}^n)$.

3 First type of representation formulas

The representation formula we will state later involves limits of discounted occupational measures which we want to introduce now.

3.1 Discounted occupational measures

For any $x_0 \in Y$, $\lambda > 0$ and any $\alpha \in \mathcal{A}$ the associated discounted occupational measure $\gamma_\lambda^{x_0,\alpha}$ is defined as follows:

$$(5) \qquad \int_{Y\times A} \varphi d\gamma_\lambda^{x_0,\alpha} = \lambda E[\int_0^\infty e^{-\lambda s}\varphi(X_s^{x_0,\alpha}, \alpha_s)ds],\ \text{for any } \varphi \in C(Y\times A).$$

We observe that $\gamma_\lambda^{x_0,\alpha} \in \Delta(Y\times A)$ is a probability measure on $Y \times A$. The set of occupational measures is denoted by

$$\Gamma_\lambda(x_0) := \{\gamma_\lambda^{x_0,\alpha},\ \alpha \in \mathcal{A}\}.$$

Now we recall some basic facts on occupational measures [5, 6, 11, 17] which will be used frequently in the following.

First note that, for all $\gamma_\lambda^{x_0,\alpha} \in \Gamma_\lambda(x_0)$ and $\varphi \in C^2(Y)$,

$$(6) \qquad \int_{Y\times A} \left((\mathcal{L}\varphi)(x,a) + \lambda(\varphi(x_0) - \varphi(x))\right) d\gamma_\lambda^{x_0,\alpha}(x,a) = 0.$$

Indeed, by applying Itô's formula to $e^{-\lambda s}\varphi(X_s^{x_0,\alpha})$ on the interval $[0,T]$, we get

$$E[\varphi(X_T^{x_0,\alpha})e^{-\lambda T} - \varphi(x_0) + \int_0^T \{\lambda e^{-\lambda s}\varphi(X_s^{x_0,\alpha}) - e^{-\lambda s}(\mathcal{L}\varphi)(X_s^{x_0,\alpha}, \alpha_s)\}ds] = 0.$$

Then by taking the limit as $T \to \infty$, from the Lebesgue convergence theorem and (5) we get (6). This means that

$$\Gamma_\lambda(x_0) \subset W_\lambda(x_0),$$

where

$$W_\lambda(x_0) := \{\nu \in \Delta(Y\times A) : \int_{Y\times A} \left((\mathcal{L}\varphi)(x,a) + \lambda(\varphi(x_0) - \varphi(x))\right) d\nu = 0,\ \text{ for all } \varphi \in C^2(Y)\}.$$

Since the set $W_\lambda(x_0)$ is defined by linear equalities, it is convex. Moreover, by Prohorov's Theorem it is compact. Consequently, $co(\Gamma_\lambda(x_0)) \subset W_\lambda(x_0)$, where co denotes the closed convex hull. One can also prove that $co(\Gamma_\lambda(x_0)) = W_\lambda(x_0)$ (c.f., [6] and also Kurtz, Stockbridge [21]).

Now, if $\nu \in \Delta(Y \times A)$ is such that there exists sequences $\lambda_n \to 0^+$ and $\nu_n \in W_{\lambda_n}(x_0)$ with $\nu_n \rightharpoonup \nu$ (i.e., ν_n converges weakly to ν), then, obviously, ν belongs to the set

$$(7) \qquad \mathcal{W} := \{\nu \in \Delta(Y\times A) : \int_{Y\times A} (\mathcal{L}\varphi)(x,a)d\nu(x,a) = 0,\ \text{ for all } \varphi \in C^2(Y)\}.$$

Thus,

$$\limsup_{\lambda \to 0^+} co(\Gamma_\lambda(x_0)) \subset \mathcal{W},$$

where $\limsup_{\lambda \to 0^+} co(\Gamma_\lambda(x_0))$ denotes the set of accumulation points[1] of all sequences $\nu_n \in co(\Gamma_{\lambda_n}(x_0))$. The above inclusion has a kind of converse precisely stated in the following.

Lemma 3.1. *(Proposition 2 in [6]) We have*

$$\lim_{\lambda \to 0^+} d_H(co(\Gamma_\lambda(Y)), \mathcal{W}) = 0, \tag{8}$$

where d_H is the Hausdorff distance[2] associated with any distance d which is consistent with the weak convergence of measures in $\Delta(Y \times A)$, and $\Gamma_\lambda(Y) := \bigcup_{x_0 \in Y} \Gamma_\lambda(x_0)$.

Now we are ready to state our first main result in which the set $\mathcal{W}$ defined by (7) is involved.

3.2 Representation formula for Abel means

We first give the following definition.

Definition 3.2. *For all x_0 in Y we set*

$$u^*(x_0) := \sup\{w(x_0),\, w \in \mathcal{K}\}$$

where $\mathcal{K}$ denotes the set of all functions $w : Y \to [0,1]$ which are continuous and satisfy the following conditions:

i) $E[w(X_t^{x_0,\alpha})|\mathcal{F}_s] \geq w(X_s^{x_0,\alpha})$, P-a.s., for all $\alpha \in \mathcal{A}$, $0 \leq s \leq t$, (i.e., $(w(X_t^{x_0,\alpha}))_{t\geq 0}$ is an $(\mathbb{F}, P)$-submartingale, for all $\alpha \in \mathcal{A}$, $x_0 \in Y$).

ii) $\displaystyle\int_{Y\times A} w(x)d\mu(x,a) \leq \int_{Y\times A} \ell(x,a)d\mu(x,a)$, for all $\mu \in \mathcal{W}$.

The above definition enables us to give the following representation formula for accumulation points of Abel mean value functions.

Theorem 3.3. *We suppose that the assumptions (H1)-(H5) hold true. Then any accumulation point - in the uniform convergence topology - of $(V_\lambda(\cdot))_{\lambda>0}$ as $\lambda \to 0^+$, is equal to $u^*(\cdot)$.*

Before proving the theorem we state a useful auxiliary statement. It turns out to be crucial for our extension of the deterministic case of Theorem 3.4 in [10] to the stochastic one (Theorem 3.3). More precisely, we need to prove the following result.

Proposition 3.4. *We suppose that the assumptions (H1)-(H5) hold true. Then for all $\lambda > 0$ and $y \in Y$ we have*

$$V_\lambda(y) = \mathrm{essinf}_{\alpha\in\mathcal{A}} E[\lambda \int_s^\infty e^{-\lambda(t-s)} \ell(X_t^{s,y,\alpha}, \alpha_t)dt|\mathcal{F}_s], \; s \geq 0, \tag{9}$$

[1] The set $\limsup_{\lambda\to 0^+} co(\Gamma_\lambda(x_0))$ is the upper Kuratowski limit of $co(\Gamma_\lambda(x_0))$ (c.f., for instance, Aubin and Frankowska [4]).

[2] Recall that the Hausdorff distance d_H between two sets M_1 and M_2 is given by

$$d_H(M_1, M_2) = \max\{\sup_{\mu\in M_1} d(\mu, M_2),\, \sup_{\mu\in M_2} d(\mu, M_1)\},$$

where $d(\mu, M_i) := \inf\{d(\mu,\nu),\, \nu \in M_i\}$.

where

$$X_t^{s,y,\alpha} = y + \int_s^t b(X_r^{s,y,\alpha}, \alpha_r)dr + \int_s^t \sigma(X_r^{s,y,\alpha}, \alpha_r)dW_r,\ t \geq s(\geq 0),\ y \in Y. \tag{10}$$

We denote $X^{y,\alpha} := X^{0,y,\alpha}$.

Proof. Let us fix $\lambda > 0$, $y \in Y$ and $s \geq 0$. We denote by $\widehat{V}_\lambda(y)$ the right-hand side of equality (9). We will prove the theorem by the following three steps.
Step 1: The function $\widehat{V}_\lambda(y)$ is deterministic.

Indeed, let H denote the Cameron-Martin space of all absolutely continuous elements $h \in \Omega$ whose Radon-Nikodym derivative $\dot{h}$ belongs to $L^2([0,\infty), \mathbf{R}^d)$. For any $h \in H$, we define the mapping $\tau_h\omega := \omega + h,\ \omega \in \Omega$. Obviously, $\tau_h : \Omega \to \Omega$ is a bijection. Let $(s,x) \in [0,T] \times \mathbf{R}^n$ be arbitrarily fixed, and put $H_s = \{h \in H | h(\cdot) = h(\cdot \wedge s)\}$. For $h \in H_s$, the law of τ_h is given by $P \circ [\tau_h]^{-1} = \exp\{\int_0^s \dot{h}_r dW_r - \frac{1}{2}\int_0^s |\dot{h}_r|^2 dr\}P$. Using the fact that $P \circ [\tau_h]^{-1}$ and P are equivalent, with the same arguments as in Buckdahn and Li [8] we get $\widehat{V}_\lambda(y)(\tau_h) = \widehat{V}_\lambda(y)$, P-a.s., for any $h \in H_s$.

On the other hand, as $\widehat{V}_\lambda(y)$ is $\mathcal{F}_s$-measurable (a direct consequence of the definition of the essential infimum over a family of $\mathcal{F}_s$-measurable random variable, see, e.g., Dunford and Schwartz [13]), this equality extends to all $h \in H$, i.e.

$$\widehat{V}_\lambda(y)(\tau_h) = \widehat{V}_\lambda(y), P\text{-a.s., for any } h \in H.$$

Consequently, from Lemma 3.4 in [8], we know that $\widehat{V}_\lambda(y)$ is deterministic. Therefore, adapting the argument of [8] we also get

$$\widehat{V}_\lambda(y) = \inf_{\alpha \in \mathcal{A}} E[\lambda \int_s^\infty e^{-\lambda(t-s)} \ell(X_t^{s,y,\alpha}, \alpha_t)dt],\ s \geq 0. \tag{11}$$

Step 2: $V_\lambda(y) \geq \widehat{V}_\lambda(y)$.

Let us consider $\vartheta_s(\omega) := W_{\cdot+s}(\omega) - W_s(\omega),\ s \geq 0, \omega \in \Omega$, i.e., $\vartheta_s = \vartheta_s(\omega)$ is the translation operator associated with the Brownian motion W. Given $\alpha \in \mathcal{A}$ and an arbitrary element a_0 of A, let us define

$$\overline{\alpha}_t := \begin{cases} a_0,\ t \in [0,s), \\ \alpha_{t-s}(\vartheta_s) = \alpha_{t-s}(W_{\cdot+s} - W_s),\ t \geq s. \end{cases} \tag{12}$$

Here we have used that $\alpha \in \mathcal{A}$ can be identified in the $dtdP$-a.e. sense with a progressively measurable function $\alpha : \mathbf{R}_+ \times C([0,+\infty)) \mapsto A$ (recall that $(\Omega, \mathcal{F}, P)$ is the classical Wiener space).

Then, we have $\overline{\alpha} \in \mathcal{A}$ and

$$X_t^{y,\alpha}(\vartheta_s) = X_{s+t}^{s,y,\overline{\alpha}},\ t \geq 0,\ P\text{-a.s.} \tag{13}$$

Indeed, applying the operator ϑ_s to our equation (10) (with $s = 0$) yields

$$\begin{aligned} X_t^{y,\alpha}(\vartheta_s) &= y + \int_0^t b(X_r^{y,\alpha}(\vartheta_s), \alpha_r(\vartheta_s))dr + \int_0^t \sigma(X_r^{y,\alpha}(\vartheta_s), \alpha_r(\vartheta_s))dW_r(\vartheta_s) \\ &= y + \int_0^t b(X_r^{y,\alpha}(\vartheta_s), \overline{\alpha}_{r+s})dr + \int_0^t \sigma(X_r^{y,\alpha}(\vartheta_s), \overline{\alpha}_{r+s})dW_{r+s},\ t \geq 0. \end{aligned}$$

That is, $X_t^{y,\alpha}(\vartheta_s) = y + \int_s^{s+t} b(X_{r-s}^{y,\alpha}(\vartheta_s), \overline{\alpha}_r)dr + \int_s^{s+t} \sigma(X_{r-s}^{y,\alpha}(\vartheta_s), \overline{\alpha}_r)dW_r,\ t \geq 0$, and from the uniqueness of the solution, we get (13).

On the other hand, from the definition of $V_\lambda(y)$ we know that, for any $\epsilon > 0$, there exists $\alpha \in \mathcal{A}$ such that

$$\begin{aligned}
\epsilon + V_\lambda(y) &\geq E[\lambda \int_0^\infty e^{-\lambda t}\ell(X_t^{y,\alpha}, \alpha_t)dt] \\
&= E[\lambda \int_s^\infty e^{-\lambda(r-s)}\ell(X_{r-s}^{y,\alpha}, \alpha_{r-s})dr] \\
&= E[\lambda \left(\int_s^\infty e^{-\lambda(r-s)}\ell(X_{r-s}^{y,\alpha}, \alpha_{r-s})dr\right)(\vartheta_s)] \\
&= E[\lambda \int_s^\infty e^{-\lambda(r-s)}\ell(X_r^{s,y,\overline{\alpha}}, \overline{\alpha}_r)dr] \geq \widehat{V}_\lambda(y).
\end{aligned}$$

Here we have used that $P \circ [\vartheta_s]^{-1} = P$, (11) and (13). Since ϵ is arbitrary we get our desired relation $V_\lambda(y) \geq \widehat{V}_\lambda(y)$.

Step 3: $V_\lambda(y) \leq \widehat{V}_\lambda(y)$.

Let $\alpha \in \mathcal{A}$, and identify α with a progressively measurable functional α of ω such that $\alpha_t = \alpha_t(W)$, dtdP-a.e. We define

$$\alpha_t^{\widetilde{\omega}} := \alpha_{t+s}(\widetilde{\omega}_{.\wedge s} + W_{(.-s)^+}),\ \widetilde{\omega} \in C_0([0,s]),\ t \geq 0.$$

Notice that $\alpha^{\widetilde{\omega}} \in \mathcal{A}$ and has the following properties:

(i) $\alpha_t^{\widetilde{\omega}}(\vartheta_s) = \alpha_{s+t}(\widetilde{\omega}_{.\wedge s} + (W_{.\vee s} - W_s))$;
(ii) $\alpha_t^{\widetilde{\omega}}(\vartheta_s)|_{\widetilde{\omega}=W_{.\wedge s}} = \alpha_{s+t}(W_{.\wedge s} + (W_{.\vee s} - W_s)) = \alpha_{s+t}(W) = \alpha_{s+t},\ t \geq 0$;
(ii) $X_r^{y,\alpha^{\widetilde{\omega}}}(\vartheta_s)|_{\widetilde{\omega}=W_{.\wedge s}} = X_{s+r}^{s,y,\alpha},\ r \geq 0$.

Hence, from the definitions of $V_\lambda(y)$ and $\widehat{V}_\lambda(y)$ we have

$$\begin{aligned}
&E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{s,y,\alpha}, \alpha_t)dt|\mathcal{F}_s] \\
&= E[\lambda \int_0^\infty e^{-\lambda r}\ell(X_{s+r}^{s,y,\alpha}, \alpha_{s+r})dr|\mathcal{F}_s] \\
&= E[\lambda \int_0^\infty e^{-\lambda r}\ell(X_r^{y,\alpha^{\widetilde{\omega}}}(\vartheta_s), \alpha_r^{\widetilde{\omega}}(\vartheta_s))dr|\mathcal{F}_s]|_{\widetilde{\omega}=W_{.\wedge s}} \\
&= E[\lambda \left(\int_0^\infty e^{-\lambda r}\ell(X_r^{y,\alpha^{\widetilde{\omega}}}, \alpha_r^{\widetilde{\omega}})dr\right)(\vartheta_s)]|_{\widetilde{\omega}=W_{.\wedge s}} \\
&\qquad\qquad\qquad\qquad \text{(because } X_r^{y,\alpha^{\widetilde{\omega}}}(\vartheta_s), \alpha_r^{\widetilde{\omega}}(\vartheta_s) \text{ are independent of } \mathcal{F}_s) \\
&= E[\lambda \int_0^\infty e^{-\lambda r}\ell(X_r^{y,\alpha^{\widetilde{\omega}}}, \alpha_r^{\widetilde{\omega}})dr]|_{\widetilde{\omega}=W_{.\wedge s}} \geq V_\lambda(y),
\end{aligned}$$

and, from the arbitrariness of α, we get $\widehat{V}_\lambda(y) \geq V_\lambda(y)$. The proof is complete. □

Proof. (of Theorem 3.3) Let us consider any accumulation point u of $(V_\lambda(\cdot))_{\lambda>0}$. Then, along a subsequence, V_λ converges uniformly to u as $\lambda \to 0^+$. In order to simplify the notations, let us suppose that $V_\lambda \to u$, as $\lambda \to 0^+$.

Step 1: $u^*(x_0) \geq u(x_0)$.

For proving this result, it suffices to show that $u \in \mathcal{K}$. Now we begin to prove it.

i) From Proposition 3.4 and the continuity of V_λ we deduce by a standard argument that, for all $\lambda > 0$ and $\alpha \in \mathcal{A}$

$$\begin{aligned} V_\lambda(X_s^{x_0,\alpha}) &= \operatorname{essinf}_{\widetilde{\alpha}\in\mathcal{A}} E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{s,X_s^{x_0,\alpha},\widetilde{\alpha}},\widetilde{\alpha}_t)dt|\mathcal{F}_s] \\ &= \operatorname{essinf}_{\widetilde{\alpha}\in\mathcal{A}} E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{x_0,\alpha\oplus\widetilde{\alpha}},\widetilde{\alpha}_t)dt|\mathcal{F}_s], P\text{-a.s.}, \end{aligned} \tag{14}$$

where

$$(\alpha\oplus\widetilde{\alpha})_t = \begin{cases} \alpha_t, & t\in[0,s) \\ \widetilde{\alpha}_t, & t\geq s. \end{cases} \in \mathcal{A}.$$

Furthermore, it is standard to prove that (see, for example, the proof of (3.22) in [8]), for any $\epsilon > 0$ there exists $\widetilde{\alpha}^\epsilon \in \mathcal{A}$ with $\widetilde{\alpha}^\epsilon_r = \alpha_r$, drd$P$-a.e. on $[0,s]\times\Omega$, such that

$$V_\lambda(X_s^{x_0,\alpha}) \geq E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{x_0,\widetilde{\alpha}^\epsilon},\widetilde{\alpha}^\epsilon_t)dt|\mathcal{F}_s] - \epsilon.$$

Therefore, for any $0\leq r\leq s$,

$$\begin{aligned} &E[V_\lambda(X_s^{x_0,\alpha})|\mathcal{F}_r] \geq E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{x_0,\widetilde{\alpha}^\epsilon},\widetilde{\alpha}^\epsilon_t)dt|\mathcal{F}_r] - \epsilon \\ &= E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{r,y,\widetilde{\alpha}^\epsilon},\widetilde{\alpha}^\epsilon_t)dt|\mathcal{F}_r]|_{y=X_r^{x_0,\alpha}} - \epsilon \\ &\geq E[\lambda \int_r^\infty e^{-\lambda(t-r)}\ell(X_t^{r,y,\widetilde{\alpha}^\epsilon},\widetilde{\alpha}^\epsilon_t)dt|\mathcal{F}_r]|_{y=X_r^{x_0,\alpha}} - \lambda\int_r^s e^{-\lambda(t-s)}dt - \epsilon \\ &\geq V_\lambda(X_r^{x_0,\alpha}) - (1-e^{-\lambda(r-s)}) - \epsilon,\ P\text{-a.s.} \end{aligned}$$

Then taking $\epsilon\to 0$ and $\lambda\to 0$, we get

$$E[u(X_s^{x_0,\alpha})|\mathcal{F}_r] \geq u(X_r^{x_0,\alpha}),\ P\text{-a.s., for any } \alpha\in\mathcal{A},\ s\geq r\geq 0.$$

That is, u satisfies (i) of $\mathcal{K}$.

ii) Now we continue with proving that u satisfies ii) of the definition of $\mathcal{K}$. Let us consider any $0<\lambda'<\lambda$ and the discounted occupational measure $\gamma_{\lambda'}^{x_0,\alpha}\in\Delta(Y\times A)$ associated with $(x_0,\alpha)\in Y\times\mathcal{A}$ and $\lambda'>0$. Taking into account that

$$V_\lambda(X_s^{x_0,\alpha}) \leq E[\lambda \int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{s,X_s^{x_0,\alpha},\alpha},\alpha_t)dt|\mathcal{F}_s],$$

a forward computation combined with Fubini's theorem and a change of variables yields

$$\begin{aligned} &\int_{Y\times A} V_\lambda d\gamma_{\lambda'}^{x_0,\alpha} = \lambda' E[\int_0^\infty e^{-\lambda' s}V_\lambda(X_s^{x_0,\alpha})ds] \\ &\leq \lambda' E[\int_0^\infty e^{-\lambda' s}\lambda\int_s^\infty e^{-\lambda(t-s)}\ell(X_t^{s,X_s^{x_0,\alpha},\alpha},\alpha_t)dtds] \\ &= \lambda'\lambda E[\int_0^\infty e^{-\lambda t}\int_0^t e^{-(\lambda'-\lambda)s}ds\,\ell(X_t^{x_0,\alpha},\alpha_t)dt] \\ &= \lambda'\lambda E[\int_0^\infty e^{-\lambda t}\frac{1}{\lambda-\lambda'}(e^{-(\lambda'-\lambda)t}-1)\ell(X_t^{x_0,\alpha},\alpha_t)dt] \\ &= \frac{\lambda'\lambda}{\lambda-\lambda'}E[\int_0^\infty (e^{-\lambda' t}-e^{-\lambda t})\ell(X_t^{x_0,\alpha},\alpha_t)dt] \\ &\leq \frac{\lambda'\lambda}{\lambda-\lambda'}E[\int_0^\infty e^{-\lambda' t}\ell(X_t^{x_0,\alpha},\alpha_t)dt] \\ &= E[\lambda'\int_0^\infty e^{-\lambda' t}\ell(X_t^{x_0,\alpha},\alpha_t)dt] + \frac{\lambda'}{\lambda-\lambda'}E[\lambda'\int_0^\infty e^{-\lambda' t}\ell(X_t^{x_0,\alpha},\alpha_t)dt] \\ &\leq \int_{Y\times A}\ell d\gamma_{\lambda'}^{x_0,\alpha} + \frac{\lambda'}{\lambda-\lambda'}. \end{aligned}$$

Notice that the set $\Gamma_{\lambda'}(Y)$ introduced in Lemma 3.1 is just the set of all discounted occupational measures $\gamma_{\lambda'}^{x_0,\alpha} \in \Delta(Y \times A)$ with $(x_0, \alpha) \in Y \times \mathcal{A}$. Consequently,

$$\int_{Y \times A} V_\lambda d\gamma \leq \int_{Y \times A} \ell d\gamma + \frac{\lambda'}{\lambda - \lambda'}, \quad \text{for all } \gamma \in \Gamma_{\lambda'}(Y).$$

Then, using Lemma 3.1 for $\in \mathcal{W}$ defined in (7), we see that taking the limit as $\lambda' \to 0^+$ yields

$$\int_{Y \times A} V_\lambda d\gamma \leq \int_{Y \times A} \ell d\gamma, \quad \text{for all } \gamma \in \mathcal{W}.$$

Finally, using our assumption of the beginning of the proof that V_λ converges uniformly to u as $\lambda \to 0^+$, we obtain

$$\int_{Y \times A} u d\gamma \leq \int_{Y \times A} \ell d\gamma, \quad \text{for all } \gamma \in \mathcal{W},$$

and this is just condition ii) in the definition of $\mathcal{K}$.

Step 2: $u^*(x_0) \leq u(x_0)$.

Let us fix any $w \in \mathcal{K}$ and prove that $w(x_0) \leq u(x_0)$.

For an arbitrarily given but fixed $\varepsilon > 0$ there exists an ε-optimal control $\alpha^\varepsilon \in \mathcal{A}$ such that

$$V_\lambda(x_0) \leq \int_{Y \times A} \ell d\gamma_\lambda^{x_0,\alpha^\varepsilon} = E[\lambda \int_0^\infty e^{-\lambda s} \ell(X_s^{x_0,\alpha^\varepsilon}, \alpha_s^\epsilon) ds] \leq V_\lambda(x_0) + \epsilon. \tag{15}$$

By Prokohov's Theorem, as $\lambda \to 0^+$, $\gamma_\lambda^{x_0,\alpha^\varepsilon}$ converges weakly along a subsequence to some measure $\gamma \in \Delta(Y \times A)$. Once again we suppose for simplicity of notation that $\gamma_\lambda^{x_0,\alpha^\varepsilon} \rightharpoonup \gamma$. By taking the limit $\lambda \to 0^+$, we deduce from (15) that

$$\int_{Y \times A} \ell d\gamma \leq u(x_0) + \varepsilon. \tag{16}$$

Moreover, since $\gamma_\lambda^{x_0,\alpha^\varepsilon} \in \Gamma_\lambda(x_0) \subset co\Gamma_\lambda(Y)$, Lemma 3.1 implies that $\gamma \in \mathcal{W}$. Consequently, from condition ii) of Definition 3.2 we have

$$\int_{Y \times A} w d\gamma \leq \int_{Y \times A} \ell d\gamma. \tag{17}$$

On the other hand, since w satisfies condition i) of Definition 3.2, we also have

$$w(x_0) = \lambda \int_0^\infty e^{-\lambda s} w(x_0) ds \leq \lambda \int_0^\infty e^{-\lambda s} E[w(X_s^{x_0,\alpha^\epsilon})] ds = \int_{Y \times A} w d\gamma_\lambda^{x_0,\alpha^\epsilon}.$$

Hence, letting λ tend to 0^+, this gives

$$w(x_0) \leq \int_{Y \times A} w d\gamma. \tag{18}$$

Finally, combining (16), (17) and (18), we obtain

$$w(x_0) \leq \varepsilon + u(x_0),$$

which is just the wished conclusion, recalling that $\varepsilon > 0$ has been chosen arbitrarily. □

Notice that our method allows also to treat the limit value for Cesàro means.

3.3 Representation formula for Cesàro means

Theorem 3.5. *We suppose that the assumptions (H1)-(H5) hold true. Then any accumulation point - in the uniform convergence topology - of $(U_T(\cdot))_{T>0}$ as $T \to +\infty$, is equal to $u^*(\cdot)$.*

The proof is similar to that of Theorem 3.3, so we omit it. We just emphasize that instead of using Lemma 3.1 we need the following result.

Lemma 3.6. *(c.f., [11]) Recalling the notations introduced for Lemma 3.1, we have*

$$\lim_{T\to+\infty} d_H(co(N_T(Y)),\ \mathcal{W}) = 0, \tag{19}$$

where

$$N_T(Y) = \{\, \nu_T^{x_0,\alpha} : x_0 \in Y,\ \alpha \in \mathcal{A} \,\}$$

and for any $x_0 \in Y$ and any $\alpha \in \mathcal{A}$ the occupational measure $\nu_T^{x_0,\alpha} \in \Delta(Y \times A)$ is defined as follows:

$$\int_{Y\times A} \varphi d\nu_T^{x_0,\alpha} = \frac{1}{T} E[\int_0^T \varphi(X_s^{x_0,\alpha}, \alpha_s)ds], \text{ for any } \varphi \in C(Y \times A). \tag{20}$$

3.4 Non-expansivity condition

Let us introduce the following nonexpansivity condition:

$$\text{(NE)} \quad \begin{cases} \text{There exists a constant } c \in \mathbf{R}_+ \text{ such that, for any } x, \widetilde{x} \in Y, a \in A,\ \exists\ \widetilde{a} \in A \text{ with:} \\ \text{(i) } \nabla N^2(x-\widetilde{x})(b(x,a) - b(\widetilde{x},\widetilde{a})) + \\ \quad \frac{1}{2}Tr[(\sigma(x,a) - \sigma(\widetilde{x},\widetilde{a}))'[\nabla^2(N^2)(x-\widetilde{x})](\sigma(x,a) - \sigma(\widetilde{x},\widetilde{a}))] \le 0; \\ \text{(ii) } |\ell(x,a) - \ell(\widetilde{x},\widetilde{a})| \le c|x - \widetilde{x}|; \end{cases}$$

where N is a norm on $\mathbf{R}^n$ such that $x \mapsto N^2(x)$ is of class C^2. Observe that when ℓ is Lipschitz and independent of the a variable, the second condition is automatically satisfied.

Proposition 3.7. *We suppose that the assumptions (H1)-(H5) and nonexpansivity condition (NE) hold true. Then*

$$\lim_{\lambda\to 0^+} V_\lambda = u^* = \lim_{T\to\infty} U_T$$

in the sense of the topology of uniform convergence in $C(Y)$.

Proof. We only give the proof for the Cesàro mean. Because N and the usual Euclidean norm are equivalent, there exists some $\eta > 0$ with $N(\cdot) \le \eta|\cdot|$. Take $T > 0$, $\epsilon > 0$, $x,\ \widetilde{x} \in Y$ and $\alpha \in \mathcal{A}$. From the same arguments as those in the proof for Lemma 3 in [7] one can derive from condition (NE) that there exists $\widetilde{\alpha} \in \mathcal{A}$ such that

$$\begin{aligned} &E[N(X_t^{x,\alpha} - X_t^{\widetilde{x},\widetilde{\alpha}})] \le N(x-\widetilde{x}) + \varepsilon \le \eta|x-\widetilde{x}| + \varepsilon, \text{ for any } t \in [0,T], \\ &|\frac{1}{T}E\int_0^T (\ell(X_t^{x,\alpha}, \alpha_t) - \ell(X_t^{\widetilde{x},\widetilde{\alpha}}, \widetilde{\alpha}_t))dt| \le c\eta|x-\widetilde{x}| + \varepsilon. \end{aligned}$$

From above one can deduce that

$$|U_T(x) - U_T(\widetilde{x})| \le c\eta|x-\widetilde{x}|, \text{ for any } x,\ \widetilde{x} \in Y.$$

On the other hand, as $0 \le \ell \le 1$, we also have $0 \le U_T \le 1$. So $(U_T)_{T>0}$ is equicontinuous and equibounded. By Arzelà-Ascoli Theorem, for any subsequence of U_T as $T \to +\infty$, there is an accumulation point, and so using Theorem 3.5, we see that U_T converges uniformly to u^*, as $T \to +\infty$. □

Observe that Example (4) satisfies the above nonexpansivity condition with the usual Euclidean norm.

We consider the following elementary deterministic two dimensional example ($\delta > 0$ is fixed and $\ell(x,a) := \ell(x)$ is assumed to be Lipschitz)

$$\begin{cases} y_1'(t) = y_2(t), \\ y_2'(t) = -\delta y_1(t) + \alpha(t),\ \alpha(t) \in A := [0,1]. \end{cases}$$

One can observe that the above non-expansivity condition fails for the usual Euclidean norm. Nevertheless, it holds true (by taking $\widetilde{a} = a$) for the norm

$$N(x_1, x_2) := \sqrt{x_1^2 + \frac{1}{\delta} x_2^2}.$$

Hence by using Proposition 3.7 for this example one can deduce the existence of a limit value.

We refer the reader to Section 3.6 in [7], for a nontrivial example in the stochastic case where the nonexpansivity condition holds true.

4 Second type of representation formulas

The second type of representation formulas involves suitably defined invariant measures of the dynamical system (1) and the notion of relaxed controls. We recall some basic facts.

4.1 Relaxed control

We now recall some necessary results concerning relaxed controls. This part is mainly taken from [14]. Recall that $\mathcal{C} := C(\mathbf{R}_+; Y)$ endowed with the norm $||x(.)|| := \sup_{t \in \mathbf{R}_+} \left(e^{-Mt}|x(t)|\right)$, $x(.) \in \mathcal{C}$ ($M > 0$ arbitrary but fixed), is a complete normed space. We introduce the space $\mathcal{V}$ of generalized actions, i.e., the set of all positive Radon measures q on $\mathbf{R}_+ \times A$ whose projection on $\mathbf{R}_+$ is the Lebesgue measure: $q(dsda) = dsq(s,da)$, where $q(s,da)$ is a measurable Markov kernel: $q(s,.) \in \Delta(A)$, $s \geq 0$ and $s \mapsto q(s,B)$ is Borel measurable for all $B \in \mathcal{B}(A)$.

It is worth pointing out that any Borel function $\alpha : \mathbf{R}_+ \to A$ can be embedded in $\mathcal{V}$: $q(dsda) = ds\delta_{\alpha_s}(da)$. It is well-known that $\mathcal{V}$ equipped with the weak $*$ topology[3] is a compact metrizable space.

Definition 4.1. *(c.f., [14]) The probability measure $Q \in \Delta(\mathcal{C} \times \mathcal{V})$ is a relaxed control (we write $Q \in \mathcal{R}(\mathcal{C} \times \mathcal{V})$), if and only if there exists*

- *A filtered probability space $(\Omega', \mathcal{F}', \mathbf{F}'(= (\mathcal{F}'_t)_{t \geq 0}), P')$;*
- *A continuous Y-valued $\mathbf{F}'$-adapted process $X = (X_t)_{t \geq 0}$ defined over Ω';*
- *A $\mathcal{V}$-valued random variable q whose restriction $I_{(0,t]}q$ is $\mathcal{F}'_t$-measurable, for all $t \geq 0$,*

such that

i) $C_t(f, X, q) := f(X_t) - f(X_0) - \displaystyle\int_0^t \int_A \mathcal{L}f(X_r, a)q(drda)$, $t \geq 0$, *is an $(\mathbf{F}', P')$-martingale, for*

[3] In this topology, $q_m \rightharpoonup q$ iff $\int_{\mathbf{R}_+ \times A} f dq_m \to \int_{\mathbf{R}_+ \times A} f dq$, for all $f : \mathbf{R}_+ \times A \to \mathbf{R}$ bounded, Borel measurable and with compact support, such that $f(t,.)$ is continuous, for all $t \in \mathbf{R}_+$.

all $f \in C_b^2(\mathbf{R}^n)$[4]*;*
ii) *The probability* Q *is the law of* (X, q) *with respect to* P'*:* $Q = P'_{(X,q)}$.

For any $x \in X$ *we also define* $\mathcal{R}_x(\mathcal{C} \times \mathcal{V})$ *as the subset of all* $Q \in \mathcal{R}(\mathcal{C} \times \mathcal{V})$ *with* $Q(\{\zeta_0 = x\}) = 1$ *(Recall that* $(\zeta(.), \theta)$ *is the coordinate process on the canonical space* $\mathcal{C} \times \mathcal{V}$*).*

Remark 4.2. *(1) Let us consider the canonical space* $\widehat{\Omega} = \mathcal{C} \times \mathcal{V}$, $\widehat{\mathcal{F}} = \mathcal{B}(\mathcal{C} \times \mathcal{V})$*; and the coordinates* $\zeta_t(x(\cdot)) := x(t)$, $x \in \mathcal{C}, t \geq 0$, $\theta(q) := q$, $q \in \mathcal{V}$*. We take the canonical filtration* $\widehat{\mathcal{F}}_t := \sigma\{\zeta_s, I_{(0,s]}\theta,\ s \in [0,t]\}$, $t \geq 0$*. Then:* $Q \in \Delta(\mathcal{C} \times \mathcal{V})$ *belongs to* $\mathcal{R}(\mathcal{C} \times \mathcal{V})$ *if and only if for all* $f \in C_b^2(R^n)$, $C_t(f, \zeta, \theta)$, $t \geq 0$*, is an* $(\widehat{\mathbf{F}}, Q)$*-martingale.*

(2) $\mathcal{R}(\mathcal{C} \times \mathcal{V})$ *is a convex, compact metrizable space (c.f., Proposition 3.4 in [14]).*

The following result says that classical optimal control problems can be equivalently formulated using relaxed controls.

Proposition 4.3. *(c.f., Theorem 4.11 in [14], also see [22]) Consider* $H \in \mathcal{C}(Y)$, $\phi \in C(Y \times A)$ *and* $T > 0$*. Then*

$$\begin{aligned}
&\inf_{\alpha \in \mathcal{A}} E\left[H(X_T^{x,\alpha}) + \int_0^T \phi(X_s^{x,\alpha}, \alpha_s) ds\right] \\
&= \inf_{Q \in \mathcal{R}_x(\mathcal{C} \times \mathcal{V})} \int_{\mathcal{C} \times \mathcal{V}} \left(H(x(T)) + \int_0^T \int_A \phi(x(s), a) q(dsda)\right) Q(dx(\cdot)dq).
\end{aligned}$$

4.2 Projected invariant measures

Following [10], we define now a suitable notion of invariant probability measure on the space of trajectories and controls.

Definition 4.4. *We define the three following notions*

- *We say that a relaxed control* $Q \in \mathcal{R}(\mathcal{C} \times \mathcal{V})$ *is invariant (for the canonical path process* $(\zeta(x(\cdot)) = x(\cdot), \theta(q) = q)$, $(x(\cdot), q) \in \mathcal{C} \times \mathcal{V}$*) if and only if for all* $t \geq 0$, $\varphi \in C(\mathcal{C} \times \mathcal{V})$,

  $$\int_{\mathcal{C} \times \mathcal{V}} \varphi(x(t + .), q(t + .)) Q(dx(\cdot)dq) = \int_{\mathcal{C} \times \mathcal{V}} \varphi(x(.), q) Q(dx(\cdot)dq),$$

  where $q(t + .) = (dsq(t + s, da)) \in \mathcal{V}$*.*

  We denote $\mathcal{R}_{inv}(\mathcal{C} \times \mathcal{V})$ *be the set of invariant relaxed controls.*

- *Let an invariant relaxed control* $Q \in \mathcal{R}_{inv}(\mathcal{C} \times \mathcal{V})$ *be given. The "projected measure"* μ_Q *is the probability measure in* $\Delta(Y \times A)$ *such that for all* $\varphi \in C(Y \times A)$,

  $$\int_{Y \times A} \varphi(x, a) \mu_Q(dxda) = \int_{\mathcal{C} \times \mathcal{V}} \big(\int_0^1 \int_A \varphi(x(s), a) q(s, da) ds\big) Q(dx(.)dq). \tag{21}$$

- *We denote by* $\mathcal{M}$ *the set of probability measures which are the projection of an invariant relaxed control:*

  $$\mathcal{M} := \left\{\mu_Q \in \Delta(Y \times A),\ Q \in \mathcal{R}_{inv}(\mathcal{C} \times \mathcal{V})\right\}.$$

[4] $C_b^2(\mathbf{R}^n)$ denotes the space of all twice continuously differentiable functions $f : \mathbf{R}^n \to \mathbf{R}$ with bounded first and second order derivatives.

Observe that for defining the projected measure μ_Q we have taken in the right-hand side of (21) the average of the time interval $[0,1]$. In fact, the definition does not change if we define μ_Q by an averaging on any other time interval $[0,T]$. This is due to the following lemma. Its proof is postponed to an appendix to maintain the flow of the paper.

Lemma 4.5. *Let $Q \in \mathcal{R}_{inv}(\mathcal{C}\times\mathcal{V})$. Then, for all $g \in C(Y\times A)$, $f \in C(Y)$, $T>0$, we have*
i) $\displaystyle\int_{\mathcal{C}\times\mathcal{V}} \Big(\frac{1}{T}\int_0^T\int_A g(x(s),a)q(s,da)ds\Big)Q(dx(.)dq) = \int_{Y\times A} g d\mu_Q,$
ii) $\displaystyle\int_{\mathcal{C}\times\mathcal{V}} f(x(0))Q(dx(.)dq) = \int_{Y\times A} f(x)\mu_Q(dxda).$

Now we are ready to state our second main result.

4.3 Second representation formula for Cesàro means

To give the representation formula we need the following definition.

Definition 4.6. *For every $x\in Y$ we set*

$$\widehat{u}(x) := \sup_{h\in\mathcal{H}} h(x)$$

where $\mathcal{H}$ denotes the set of all functions $h\in C(Y,[0,1])$ which satisfy
i) $\displaystyle\int_{\mathcal{C}\times\mathcal{V}} h(x(T))Q(dx(\cdot)dq) \geq h(x)$, *for all $T\geq 0$, $x\in Y$, $Q\in\mathcal{R}_x(\mathcal{C}\times\mathcal{V})$;*
ii) $\displaystyle\int_{Y\times A} h(x)\mu(dxda) \leq \int_{Y\times A} \ell(x,a)\mu(dxda)$, *for all $\mu\in\mathcal{M}$.*

The following theorem, our main result in this section, can be considered as the continuous-time analogue to Theorem 2.10 in Renault and Venel [29].

Theorem 4.7. *We suppose that assumptions* (H1)-(H5) *hold true. Then any accumulation point - in the uniform convergence topology - of $(U_T)_{T>0}$, as $T\to+\infty$, is equal to $\widehat{u}$.*

Remark 4.8. *In view of the result of Section 3 we have also that $\widehat{u}=u^*$.*

Proof. Let $u\in C(Y)$ be any accumulation point of $(U_T)_T$, as $T\nearrow+\infty$, i.e., $U_T\to u$ in $C(Y)$ along some subsequence of $T\nearrow+\infty$. To simplify the notation we suppose $U_T\to u$. Fix $x\in Y$.
Step 1: $\widehat{u}(x)\geq u(x)$.

Obviously, for proving this, it suffices to show that $u\in\mathcal{H}$. We first show that u satisfies i) in the definition of $\mathcal{H}$.

Let $t>0$ and $\alpha\in\mathcal{A}$. Analogously to Proposition 3.4 one has for all $y\in Y$,

$$U_T(y) = \operatorname{essinf}_{\alpha'\in\mathcal{A}} E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{t,y,\alpha'},\alpha'_s)ds \,\Big|\, \mathcal{F}_t\right],\ P\text{-a.s.}$$

Thus, using the continuity of $U_T(.)$, we can substitute $y=X_t^{x,\alpha}$,

$$\begin{aligned} U_T(X_t^{x,\alpha}) &= \operatorname{essinf}_{\alpha'\in\mathcal{A}} E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{t,X_t^{x,\alpha},\alpha'},\alpha'_s)ds \,\Big|\, \mathcal{F}_t\right] \\ &= \operatorname{essinf}_{\alpha'\in\mathcal{A}} E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{x,\alpha\oplus\alpha'},(\alpha\oplus\alpha')_s)ds \,\Big|\, \mathcal{F}_t\right], \end{aligned}$$

where $(\alpha \oplus \alpha') := \alpha I_{[0,t]} + \alpha' I_{(t,+\infty)} \in \mathcal{A}$. Since, for all $\varepsilon > 0$ we can construct an ε-optimal control $\alpha^\varepsilon \in \mathcal{A}$ with $\alpha_s^\varepsilon = \alpha_s$, $dsdP$-a.e. on $[0,t] \times \Omega$, such that

$$U_T(X_t^{x,\alpha}) \geq E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{x,\alpha^\varepsilon}, \alpha_s^\varepsilon)ds \,\middle|\, \mathcal{F}_t\right] - \varepsilon, \;\; P\text{-a.s.},$$

we can conclude that

$$E\left[U_T(X_t^{x,\alpha})\right] = \inf_{\alpha' \in \mathcal{A}} E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{x,\alpha\oplus\alpha'}, (\alpha\oplus\alpha')_s)ds\right].$$

Consequently, recalling that $0 \leq \ell \leq 1$,

$$\begin{aligned} E\left[U_T(X_t^{x,\alpha})\right] &= \inf_{\alpha' \in \mathcal{A}} E\left[\frac{1}{T}\int_t^{T+t} \ell(X_s^{x,\alpha\oplus\alpha'}, (\alpha\oplus\alpha')_s)ds\right] \\ &\geq \inf_{\alpha' \in \mathcal{A}} E\left[\frac{1}{T}\int_0^{T} \ell(X_s^{x,\alpha\oplus\alpha'}, (\alpha\oplus\alpha')_s)ds\right] - \frac{t}{T} \geq U_T(x) - \frac{t}{T}. \end{aligned}$$

Passing to the limit as $T \nearrow +\infty$ we get $E\left[u(X_t^{x,\alpha})\right] \geq u(x)$, and from the arbitrariness of $\alpha \in \mathcal{A}$ combined with Proposition 4.3 it follows that

$$\inf_{Q \in \mathcal{R}_x(\mathcal{C}\times\mathcal{V})} \int_{\mathcal{C}\times\mathcal{V}} u(x(T))Q(dx(\cdot)dq) = \inf_{\alpha\in\mathcal{A}} E\left[u(X_t^{x,\alpha})\right] \geq u(x).$$

Now we will prove that u satisfies ii) of $\mathcal{H}$.

Let $Q \in \mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V})$. We disintegrate as follows

$$Q = \int_Y \mu_Q^1(dx)Q_x \tag{22}$$

where $\mu_Q^1 = \mu_Q(\cdot \times A)$ is the first margin of the projected measure μ_Q of the invariant relaxed control Q and $Q_x := Q\{\,\cdot \mid \zeta_0 = x\}$, $x \in Y$, is the regular conditional probability of Q knowing ζ_0. Observe that such regular conditional measure exists, because $\mathcal{C}\times\mathcal{V}$ is a separable complete metrizable space ($\mathcal{C}$ is known to be separable and so is $\mathcal{V}$, see Theorem 2.3 in [14]).

We also note that with $Q \in \mathcal{R}(\mathcal{C}\times\mathcal{V})$ also $Q_x \in \mathcal{R}_x(\mathcal{C}\times\mathcal{V})$, $x \in Y$.

Indeed, as $Q \in \mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V}) \subset \mathcal{R}(\mathcal{C}\times\mathcal{V})$, following the proof of Theorem 2.5 in [14] we construct a uniformly continuous matrix $\gamma = \gamma(x,m), (x,m) \in Y \times \Delta(A)$ which vanishes when m is a Dirac measure, as well as an $(\mathbf{F}', P')$-Brownian motion (W', B') on a suitable filtered probability space $(\Omega', \mathcal{F}', \mathbf{F}' = (\mathcal{F}'_t)_{t\geq 0}, P')$ such that

i) $Q = P'_{(X,q)}$, where (X,q) is $\mathbf{F}'$-adapted and obeys the equation;

ii) $dX_t = b(X_t,q_t)dt + \sigma(X_t,q_t)dW'_t + \gamma(X_t,q_t)dB'_t$, $t \geq 0$, where for any $m \in \Delta(A)$,

$$\begin{aligned} &b(x,m) := \int_A b(x,a)m(da),\; \sigma(x,m) := \int_A \sigma(x,a)m(da); \\ &\gamma\gamma^*(x,m) := \int_A \sigma\sigma^*(x,a)m(da) - \Big(\int_A \sigma(x,a)m(da)\Big)\Big(\int_A \sigma(x,a)m(da)\Big)^*. \end{aligned}$$

Under $P'\{\,\cdot \mid X_0 = x\}$ the above stochastic differential equation ii) has the initial condition $X_0 = x$, and as (W', B') is an $(\mathbf{F}', P')$-Brownian motion independent of X_0, we see also that $Q_x = P'\{(X,q) \in \;\cdot \mid X_0 = x\} \in \mathcal{R}_x(\mathcal{C}\times V)$.

Then, as $Q_x \in \mathcal{R}_x(\mathcal{C}\times V)$, $x\in Y$, from Proposition 4.3

$$U_T(x) \le \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{T}\int_0^T \int_A \ell(x(s),a)q(s,da)ds\right) Q_x(dx(\cdot)dq),\ x\in Y.$$

Thus, by integrating with respect to μ_Q^1 and using the invariance property of $Q=\int_Y \mu_Q^1(dx)Q_x$ we obtain

$$\begin{aligned}
&\int_{Y\times A} U_T(x)\mu(dxda) = \int_Y U_T(x)\mu_Q^1(dx)\\
&\le \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{T}\int_0^T \int_A \ell(x(s),a)q(s,da)ds\right) Q(dx(\cdot)dq)\\
&= \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^1 \int_A \ell(x(s),a)q(s,da)ds\right) Q(dx(\cdot)dq) = \int_{Y\times A} \ell(x,a)\mu(dxda).
\end{aligned}$$

(Recall also Lemma 4.5).

Passing to the limit as $T\nearrow +\infty$ we get

$$\int_{Y\times A} u(x)\mu(dxda) \le \int_{Y\times A} \ell(x,a)\mu(dxda),$$

i.e., u satisfies ii) of $\mathcal{H}$.

We have proved that $u\in\mathcal{H}$, consequently $u\le\widehat{u}$.

Step 2: $\widehat{u}(x)\le u(x)$.

By the very definition of $\widehat{u}$ it suffices to show that $u(x)\ge h(x)$, $x\in Y$, for all $h\in\mathcal{H}$. Let $h\in\mathcal{H}$ and $\varepsilon>0$ be fixed.

Due to Proposition 4.3 for all $T>0$, there is some $Q^T\in\mathcal{R}_x(\mathcal{C}\times\mathcal{V})$ such that

$$U_T(x)+\varepsilon \ge \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{T}\int_0^T \int_A \ell(x(s),a)q(s,da)ds\right) Q^T(dx(\cdot)dq).$$

Let $t\ge 0$. Given any $Q\in\mathcal{R}_x(\mathcal{C}\times\mathcal{V})$, we define the shifted probability measure $Q[t]\in\Delta(\mathcal{C}\times\mathcal{V})$ as follows

$$(23)\qquad \int_{\mathcal{C}\times\mathcal{V}} \varphi(x(\cdot),q)Q[t](dx(\cdot)dq) = \int_{\mathcal{C}\times\mathcal{V}} \varphi(x(t+\cdot),q(t+\cdot))Q(dx(\cdot)dq),\ \varphi\in C(\mathcal{C}\times\mathcal{V}),$$

where $q(t+\cdot) := (dsq(t+s,\cdot))$.

We claim that $Q[t]\in\mathcal{R}(\mathcal{C}\times\mathcal{V})$.

Indeed, using the canonical space $\big(\widehat{\Omega}=\mathcal{C}\times V,\ \widehat{\mathcal{F}}=\mathcal{B}(\mathcal{C}\times V)\big)$ we have

i) $\zeta^t(s) := \zeta(t+s)$, $\theta^t(q) := (dsq(t+s,\cdot)), q\in V$: the time shifted coordinates on $(\widehat{\Omega},\widehat{\mathcal{F}},Q)$;
ii) $\widehat{\mathcal{F}}^t_s := \widehat{\mathcal{F}}_{t+s}, s\ge 0$: the associated time-shifted filtration;
iii) $C_s(f,\zeta^t,\theta^t)$: the time-shift of the process $C_s(f,\zeta,\theta)$, $s\ge 0$.

Then $C_s(f,\zeta^t,\theta^t)$, $s\ge 0$, is an $(\widehat{\mathbf{F}}^t,Q)$-martingale, i.e., $Q[t]\in\mathcal{R}(\mathcal{C}\times\mathcal{V})$, which proves our claim.

We now introduce the occupation measure $R^T\in\Delta(\mathcal{C}\times\mathcal{V})$ by setting, for all $\varphi\in C(\mathcal{C}\times\mathcal{V})$,

(24)
$$\begin{aligned}
&\int_{\mathcal{C}\times\mathcal{V}} \varphi(x(\cdot),q)R^T(dx(\cdot)dq) = \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{T}\int_0^T \varphi(x(t+\cdot),q(t+\cdot))dt\right) Q^T(dx(\cdot)dq) =\\
&\frac{1}{T}\int_0^T \left(\int_{\mathcal{C}\times\mathcal{V}} \varphi(x(\cdot),q)Q^T[t](dx(\cdot)dq)\right) dt = \int_{\mathcal{C}\times\mathcal{V}} \varphi(x(\cdot),q)\left(\frac{1}{T}\int_0^T Q^T[t]dt\right) dx(\cdot)dq.
\end{aligned}$$

Since $Q^T[t]$ belongs to the compact and convex set $\mathcal{R}(\mathcal{C}\times V)$, we deduce that $R^T\in\mathcal{R}(\mathcal{C}\times V)$ and that R^T converges - up to a subsequence - to some $R\in\mathcal{R}(\mathcal{C}\times V)$.

We claim that $R\in\mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V})$.

Indeed, for any fixed $\varphi\in C(\mathcal{C}\times\mathcal{V}),\ t\geq 0$, we have

$$
\begin{aligned}
&|\int_{\mathcal{C}\times\mathcal{V}}\varphi(x(t+\cdot),q(t+\cdot))R^T(dx(\cdot)dq)-\int_{\mathcal{C}\times\mathcal{V}}\varphi(x(\cdot),q)R^T(dx(\cdot)dq)|\\
&=|\int_{\mathcal{C}\times\mathcal{V}}\big((\frac{1}{T}\int_0^T\varphi(x(t+s+\cdot),q(t+s+\cdot))ds)-(\frac{1}{T}\int_0^T\varphi(x(s+\cdot),q(s+\cdot))ds)\big)Q^T(dx(\cdot)dq)\big|\\
&\quad+|\int_{\mathcal{C}\times\mathcal{V}}(\frac{1}{T}\int_t^{T+t}\varphi(x(s+\cdot),q(s+\cdot))ds\big)Q^T(dx(\cdot)dq)\\
&\qquad-\int_{\mathcal{C}\times\mathcal{V}}(\frac{1}{T}\int_0^T\varphi(x(s+\cdot),q(s+\cdot))ds\big)Q^T(dx(\cdot)dq)\big|\\
&\leq\frac{2t}{T}\sup_{(x(\cdot),q)\in\mathcal{C}\times\mathcal{V}}|\varphi(x(\cdot),q)|\longrightarrow 0,\quad\text{as } T\to\infty.
\end{aligned}
$$

Consequently,

$$
\int_{\mathcal{C}\times\mathcal{V}}\varphi(x(t+\cdot),q(t+\cdot))R(dx(\cdot)dq)=\int_{\mathcal{C}\times\mathcal{V}}\varphi(x(\cdot),q)R(dx(\cdot)dq).
$$

This proves our claim $R\in\mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V})$.

Now we consider the projected measure $\mu_R\in\Delta(Y\times A)$ of R. Since $R\in\mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V})$, it holds $\mu_R\in\mathcal{M}$, and, thus, as $h\in\mathcal{H}$,

$$
\int_{Y\times A}h(x)\mu_R(dxda)\leq\int_{Y\times A}\ell(x,a)\mu_R(dxda).
$$

But h also satisfies i) of the definition of $\mathcal{H}$: As $Q^T\in\mathcal{R}_x(\mathcal{C}\times\mathcal{V})$,

$$
\int_{\mathcal{C}\times\mathcal{V}}h(x(s))Q^T(dx(\cdot)dq)\geq h(x),\ s\geq 0.
$$

Thus,

$$
\begin{aligned}
h(x)&\leq\frac{1}{T}\int_0^T\left(\int_{\mathcal{C}\times\mathcal{V}}h(x(s))Q^T(dx(\cdot)dq)\right)ds\\
&=\int_{\mathcal{C}\times\mathcal{V}}\left(\frac{1}{T}\int_0^T h(x(s))ds\right)Q^T(dx(\cdot)dq)=\int_{\mathcal{C}\times\mathcal{V}}h(x(0))R^T(dx(\cdot)dq).
\end{aligned}
$$

Hence, as $h\circ\zeta_0\in C(\mathcal{C}\times\mathcal{V})$, passing to the limit in the above inequality yields

$$
h(x)\leq\int_{\mathcal{C}\times\mathcal{V}}(h\circ\zeta_0)(x(\cdot))R(dx(\cdot)dq)=\int_{Y\times A}h(x)\mu_R(dxda)\leq\int_{Y\times A}\ell(x,a)\mu_R(dxda),\tag{25}
$$

where the later inequality has used again that $h\in\mathcal{H}$.

On the other hand, as $0\leq\ell\leq 1$, we have

$$
\begin{aligned}
U_T(x)+\varepsilon&\geq\int_{\mathcal{C}\times\mathcal{V}}\left(\frac{1}{T}\int_0^T\int_A\ell(x(t),a)q(t,da)dt\right)Q^T(dx(\cdot)dq)\\
&\geq\int_{\mathcal{C}\times\mathcal{V}}\frac{1}{T}\int_0^1\int_s^{T+s}\int_A\ell(x(t),a)q(t,da)dtdsQ^T(dx(\cdot)dq)-\frac{1}{T}\\
&=\int_{\mathcal{C}\times\mathcal{V}}\frac{1}{T}\int_0^1\int_0^T\int_A\ell(x(t+s),a)q(t+s,da)dtdsQ^T(dx(\cdot)dq)-\frac{1}{T}.
\end{aligned}
$$

Thus, from Fubini's Theorem, and with the notation

$$\psi(x(\cdot), q) := \int_0^1 \int_A \ell(x(s), a)q(s, da)ds,\ (x(\cdot), q) \in \mathcal{C} \times \mathcal{V},$$

(observe that $\psi \in C(\mathcal{C} \times \mathcal{V})$) we have

$$\begin{aligned}
&U_T(x) + \varepsilon \\
&\geq \int_{\mathcal{C}\times\mathcal{V}} \frac{1}{T} \int_0^1 \int_0^T \int_A \ell(x(t+s), a)q(t+s, da)dtdsQ^T(dx(\cdot)dq) - \frac{1}{T} \\
&= \int_{\mathcal{C}\times\mathcal{V}} \frac{1}{T} \int_0^T \Big(\int_0^1 \int_A \ell(x(t+s), a)q(t+s, da)ds \Big) dtQ^T(dx(\cdot)dq) - \frac{1}{T} \\
&= \int_{\mathcal{C}\times\mathcal{V}} \frac{1}{T} \int_0^T \psi(x(t+\cdot), q(t+\cdot))dtQ^T(dx(\cdot)dq) - \frac{1}{T} \\
&= \int_{\mathcal{C}\times\mathcal{V}} \psi(x(\cdot), q)R^T(dx(\cdot)dq) - \frac{1}{T}.
\end{aligned}$$

Passing to the limit $R^T \rightharpoonup R$, $U_T \to u$ as $T \nearrow \infty$, we get

$$\begin{aligned}
&u(x) + \varepsilon \geq \int_{\mathcal{C}\times\mathcal{V}} \psi(x(\cdot), q)R(dx(\cdot)dq) \\
&= \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^1 \int_A \ell(x(s), a)q(s, da)ds \right) R(dx(\cdot)dq) = \int_{Y\times A} \ell(x, a)\mu_R(dxda).
\end{aligned}$$

In view of (25), the above inequality implies that $u(x) \geq h(x)$, since $\varepsilon > 0$ was chosen arbitrarily. The proof is complete. □

4.4 Second Representation formula for Abel means

Using similar techniques for the proof one obtain the following result.

Theorem 4.9. *We suppose that assumptions* (H1)-(H5) *hold true. Then any accumulation point - in the uniform convergence topology - of $(V_\lambda)_{\lambda>0}$, as $\lambda \to 0^+$, is equal to $\widehat{u}$.*

Of course one can also use the second type of representation formulas to obtain the existence of a limit value as in Proposition 3.7.

5 Comparison between both types of representation formulas

For the both representations formulas of the Sections 3 and 4 two different sets of measures $\mathcal{M}$ and $\mathcal{W}$ are used.

Theorem 5.1. *We suppose that the assumptions* (H1)-(H5) *hold true. Then*

$$\mathcal{M} = \mathcal{W}.$$

Proof. We will first prove that $\mathcal{M} \subset \mathcal{W}$. Fix $\mu \in \mathcal{M}$ and let $g \in C^2(Y)$. There exists $Q \in \mathcal{R}_{\text{inv}}(\mathcal{C} \times \mathcal{V})$ such that μ is the projected invariant measure of Q, namely $\mu = \mu_Q$. As in (22) we disintegrate Q

$$Q = \int_Y \mu_Q^1(dx)Q_x.$$

In view of Lemma 4.5, applying Proposition 4.3 to $H := 0$ and $\phi := \mathcal{L}g(x,a)$ yields

$$\begin{aligned}&\int_{Y\times A}\mathcal{L}g(x,a)\mu(dxda) = \int_{\mathcal{C}\times\mathcal{V}}\Big(\int_0^1\int_A \mathcal{L}g(x(s),a)q(s,da)ds\Big)Q(dx(.)dq)\\ &= \int_Y\int_{\mathcal{C}\times\mathcal{V}}\Big(\int_0^1\int_A \mathcal{L}g(x(s),a)q(s,da)ds\Big)Q_x(dx(.)dq)\mu_Q^1(dx)\\ &\geq \inf_{\alpha\in\mathcal{A}}\frac{1}{T}\int_Y E[\int_0^T \mathcal{L}g(X_s^{x,\alpha},\alpha_s)ds]\mu_Q^1(dx) = \inf_{\alpha\in\mathcal{A}}\frac{1}{T}\int_Y E[g(X_T^{x,\alpha})-g(x)]\mu_Q^1(dx),\end{aligned}$$

where the latter relation follows from the Itô Formula. As g is bounded, the above term vanishes as $T\to\infty$.

Thus, $\int_{Y\times A}\mathcal{L}g(x,a)\mu(dxda) \geq 0$, and using again Proposition 4.3 but now $H := 0$ and $\phi := -\mathcal{L}g(x,a)$, we deduce

$$\int_{Y\times A}\mathcal{L}g(x,a)\mu(dxda) = 0.$$

Since the above equality is valid for all $g \in C^2(Y)$ we obtain $\mu \in \mathcal{W}$, which is our wished conclusion.

Let us prove now $\mathcal{M} \supset \mathcal{W}$. Fix $\mu \in \mathcal{W}$. By Lemma 3.6, there exists sequences $N_n, T_n \to +\infty$ and $x_n^i \in Y$, $\alpha_n^i \in \mathcal{A}$, $\pi_n^i \in [0,1]$, $i = 1,2\ldots N_n$ with $\sum_{i=1}^{N_n}\pi_i^n = 1$ such that $\sum_{i=1}^{N_n}\pi_i^n\nu_{T_n}^{x_n,\alpha_n}$ converges weakly to μ. Fix $\varepsilon > 0$ and $\varphi \in C(Y\times A)$. By Proposition 4.3 there exists $Q^{T_n} \in \mathcal{R}_{x_n}(\mathcal{C}\times\mathcal{V})$ such that

$$\frac{1}{T_n}\left(\int_0^{T_n}\int_A \varphi(x(s),a)q(dsda)\right)Q^{T_n}(dx(\cdot)dq) \leq \inf_{\alpha\in\mathcal{A}}\frac{1}{T_n}E\left[\int_0^{T_n}\varphi(X_s^{x_n,\alpha},\alpha_s)ds\right]+\varepsilon. \tag{26}$$

For any $t > 0$ one can associate to Q^{T_n} a probability measure $Q^{T_n}[t]$ defined by (23) and an occupational measure R^{T_n} defined by (24). From Prohorov's Theorem, R^{T_n} converges weakly to some R along a a subsequence. By the same arguments as those used in the proof of Theorem 4.7, we know that $R \in \mathcal{R}_{\text{inv}}(\mathcal{C}\times\mathcal{V})$. So (26) yields

$$\int_{Y\times A}\varphi d\nu_{T_n}^{x_n^i,\alpha_n^i}+\varepsilon \geq \int_{\mathcal{C}\times\mathcal{V}}\varphi(x(\cdot),q)R^{T_n}(dx(\cdot)dq) = \int_{Y\times A}\varphi d\mu_{R^{T_n}},\ i = 1,2\ldots N_n.$$

Up to a subsequence $\mu_{R^{T_n}}$ weakly converge to some $\widehat{\mu} \in \mathcal{M}$ because $\mathcal{M}$ is compact. Thus taking the convex combination $\sum_{i=1}^{N_n}\pi_i^n$ and passing to the limit in the above inequality we have

$$\int_{Y\times A}\varphi d\mu + \varepsilon \geq \int_{Y\times A}\varphi d\widehat{\mu}.$$

Replacing φ by $-\varphi$ and taking into account that ε is arbitrary we get

$$\int_{Y\times A}\varphi d\mu = \int_{Y\times A}\varphi d\widehat{\mu},\ \forall\varphi \in C(Y\times A)$$

and so $\widehat{\mu} = \mu \in \mathcal{M}$ our wished conclusion. The proof is complete.

□

6 Illustrating Examples

We begin with discussing an example in which the assumptions (H1)-(H5) are satisfied but not the nonexpansive condition, and we will show for that this example neither the Cesàro nor the Abel means U_T ($T \to +\infty$) and V_λ ($\lambda \searrow 0$), respectively, have an accumulation point (and, hence, there is no limit) in the topology of uniform convergence. However, we will see that U_T and V_λ converge pointwise as $T \to +\infty$ and $\lambda \searrow 0$, but their limits differ.

Example A: An example without accumulation point

Let us consider the following deterministic example with the dynamics

$$\left\{ \begin{array}{l} X_1'(t) = X_2(t) \min\{1, 4 - X_1(t)\},\ X_1(0) \in \mathbf{R}_+, \\ X_2'(t) = \alpha(t)\ \min\{1, 4 - X_1(t)\},\ X_2(0) \in \mathbf{R}_+. \end{array} \right. \tag{27}$$

Here the control $\alpha \in \mathcal{A}$ lives in $A := [0,1]$. As $0 \le X_1(0) \le 4,\ X_2(0) \ge 0$, it can be easily checked that both $X_1'(t) \ge 0$ and $X_2'(t) \ge 0$, $t \ge 0$. Moreover, $X_2(t)X_2'(t) = X_1'(t)\alpha(t) \le X_1'(t)$ for all $t \ge 0$, i.e., $\frac{1}{2}X_2^2(t) \le \frac{1}{2}X_2^2(0) - X_1(0) + X_1(t)$, $t \ge 0$. This proves that the compact set $Y = \{x = (x_1, x_2) \in \mathbf{R}_+^2 : x_1 \le 4, x_2 \le \sqrt{2x_1}\}$ is invariant. Thus, as the conditions $(H1) - (H5)$ are satisfied by our example, Theorem 5.1 allows to conclude that $\mathcal{W} = \mathcal{M}$. But, however, for our example we have:

Proposition 6.1. *Neither the Abel mean $V_\lambda(.)$ nor the Cesàro mean $U_T(.)$ have an accumulation point in the topology of the uniform convergence as $\lambda \searrow 0$ and $T \to +\infty$, respectively.*

Proof. Let us denote by $\mathcal{M}_Y = \{\mu_Y, \mu \in \mathcal{M}\} \subset \Delta(Y)$ the set of first marginals of projected invariant relaxed controls. Notice that if $X_1(t) < 3$, we have $X_1''(t) = X_2'(t) = \alpha(t)$, i.e., $\alpha(t)$ and $X_2(t)$ represent the acceleration and the speed of $X_1(t)$, respectively.

We introduce now the potential $q(x) = {x_2}^2 - 3x_1$, $x = (x_1, x_2) \in \mathbf{R}_+^2$. As $2X_2(t)X_2'(t) - 3X_1'(t) = 2X_1'(t)\alpha(t) - 3X_1'(t) \le -X_1'(t)$, $t \ge 0$, $q(X(t))$ is non-increasing in t. Hence, if the probability law μ of $X(0)$ on Y belongs to $\mathcal{M}$, we must have $q(X(t)) = q(X(0))$, $t \ge 0$, but this is equivalent with $X(t) = X(0)$, $t \ge 0$. As a consequence we see that a probability μ on Y belongs to $\mathcal{M}_Y$ if and only if its support is included in the set of possible rest points of Y:

$$Z := ([0,4] \times \{0\}) \cup (\{4\} \times [0, 2\sqrt{2}]).$$

Let us consider a cost function ℓ which only depends on the first coordinate of the state process. For $x = (x_1, x_2) \in \mathbf{R}_+^2$ we put:

$$\ell(x, \alpha) = \ell(x) = \ell(x_1) = \left\{ \begin{array}{ccc} 1 - x_1 & \text{if} & x_1 \in [0,1] \\ 0 & \text{if} & x_1 \in [1,2] \\ x_1 - 2 & \text{if} & x_1 \in [2,3] \\ 1 & \text{if} & x_1 \in [3,4] \end{array} \right.$$

Recall the definition of u^* (Definition 3.2) and that of $\widehat{u}$ (Definition 4.6). We have here $\hat{u}(x_0) = u^*(x_0) = \sup\{w(x_0), w \in \mathcal{K}\}$, where $\mathcal{K}$ is the set of continuous functions $w : Y \to [0,1]$ such that, for all admissible control, $(w(X(t)))$, $t \ge 0$, is a submartingale (or, as there is no stochastic integral in the dynamics, for all deterministic admissible control, $(w(X(t)))$, $t \ge 0$, is non decreasing) and which satisfy: $w(x) \le \ell(x)$ for each $x \in Z$. So $u^*(x_1, 0) \le l(x_1)$, for $x_1 \in [0,4]$.

Let $w \in \mathcal{K}$. Then, from the definition of ℓ, $w(x_1, 0) = 0$ for $x_1 \in [1,2]$. Let us consider as initial condition $X(0) = (x_1, 0)$, for $x_1 \in [0,1]$. Then, by choosing the constant control

$\alpha(t) = \varepsilon$ with $\varepsilon > 0$ small, we arrive at $t = \sqrt{2/\varepsilon}$ at the position $X(t) = (x_1 + 1, \sqrt{2\varepsilon})$, so by continuity and monotonicity of w we obtain that $w(x_1, 0) = 0$ for x_1 in $[0, 1]$. Thus, thanks to the arbitrariness of $w \in \mathcal{K}$, we see that $u^*(x_1, 0) = 0$, for all x_1 in $[0, 2]$.

Let us consider now $\delta > 0$. The function w, defined by $w(x_1, x_2) = \min\{1, x_2/\delta\}$, $(x_1, x_2) \in Y$, belongs to $\mathcal{K}$. As $w \le u^* \le 1$, it follows from the arbitrariness of $\delta > 0$ that $u^*(x_1, x_2) = 1$ if $x_2 > 0$. Finally considering w such that $w(x_1, x_2) = 0$ if $x_1 \le 2$ and $w(x_1, x_2) = l(x_1)$ if $x_1 \in [2, 4]$ defines a function in $\mathcal{K}$ and gives that $u^*(x_1, 0) = l(x_1)$ for $x_1 \ge 2$. Indeed, $(x_1, 0) \in Z$, and on Z it holds $w \le u^* \le \ell$.

Resuming our above computation, we have $u^*(x) = 1$ if $x_2 > 0$, $u^*(x_1, 0) = 0$ if $x_1 \le 2$, and $u^*(x_1, 0) = l(x_1)$ for $x_1 \ge 2$. This shows in particular that the function u^* is discontinuous, and so it cannot be a uniform limit of a subsequence of Abel means (V_λ) $(\lambda \searrow 0)$. Consequently, from the Theorems 3.3, 3.5, 4.7 and 4.9 we obtain that the value functions (V_λ) and $(U_T)_T$ do not have any accumulation point in the topology of the uniform convergence, as $\lambda \searrow 0$ and $T \to +\infty$. The proof is complete. $\square$

As we do not have any accumulation point for the Cesàro and the Abel means for our example, the question raises, if these means have pointwise limits. Indeed, we have the following result:

Proposition 6.2. *The Abel and the Cesàro means $(V_\lambda(.))_{\lambda>0}$ and $(U_T(.))_{T>0}$ have pointwise limits $V_0(.)$ and $U_\infty(.)$ as $\lambda \searrow 0$ and $T \to +\infty$, respectively. Moreover, these pointwise limits are discontinuous, they do not coincide and they are different from $u^*(x_0) = \sup\{w(x_0), w \in \mathcal{K}\}$. In particular, we have*

$$\begin{aligned} U_T(0,0) &\xrightarrow[T\to\infty]{} \beta := \sqrt{5} - 2 \simeq 0.236, \\ V_\lambda(0,0) &\xrightarrow[\lambda\to 0]{} \beta^* := \min\{1 - \tfrac{1}{\gamma}(1 - e^{-\gamma} - e^{-2\gamma} + e^{-3\gamma}), \gamma > 0\} \simeq 0.449. \end{aligned}$$

Proof. 1) If one starts from a position $X(0) = x = (x_1, x_2)$ with $x_2 > 0$, we have $X_1(t) \ge t\, x_2$ as long as $X_1(t) \le 3$. Thus, $l(X_1(t)) = 1$ for $t \ge 3/x_2$, and it follows that $V_0(x) = U_\infty(x) = 1$. On the contrary, if we start from a position $X(0) = x = (x_1, 0)$ with $x_1 \in [1, 2]$, the optimal control for the Abel and the Cesàro mean cost functional is $\alpha(t) = 0$, $t \ge 0$, and we get that $V_0(x) = U_\infty(x) = 0$. In particular we see that the pointwise limits $V_0(.)$ and $U_\infty(.)$ are discontinuous.

It remains to show the existence of the pointwise limits of $V_\lambda(x)$ and $U_T(x)$ for $x = (x_1, 0)$ with $x_1 \in [0, 1)$. The argument is essentially the same as that in the case $x = (0, 0)$, and so we restrict the proof to the latter case:

2) We first show that $U_T(0, 0) \ge \beta$, for all $T > 0$. It is enough to restrict attention to trajectories which are not constant and such that $\alpha = 0$ as soon as $X_1(.) \ge 1$. We consider any such trajectory $X = (X(t))$ starting from $(0, 0)$. Let us define T_1 (respectively, T_2, T_3) as the time such that $X_1(T_1) = 1$ (respectively, $X_1(T_2) = 2$, $X_1(T_3) = 3$). For all $t \ge T_1$ we have $X_1(t) = 1 + (t - T_1)X_2(T_1)$. Recall that $t \to \ell(X(t)) = 1 - X_1(t)$ is first decreasing on the time interval $[0, T_1]$, then identically equal to 0 on $[T_1, T_2]$, and increasing on $[T_2, T_3]$: $\ell(X(t)) = X_1(t) - 2$, and it is identically equal to 1 on $[T_3, +\infty)$. Hence, defining $c > 0$ and $S > 0$ by setting

$$c = \min_{T>0} \frac{1}{T} \int_0^T \ell(x(t))dt = \frac{1}{S} \int_0^S \ell(x(t))dt,$$

we deduce from the above described behaviour of the graph of $\ell(X(.))$ that $S \in (T_2, T_3)$, and by minimality, $\ell(X(S)) = c \in (0,1)$. Consequently, $X_1(S) = 2 + c$, and from

$$\begin{array}{rcl} 2 = X_1(T_2) & = & 1 + (T_2 - T_1)X_2(T_1), \\ 2 + c = X_1(S) & = & 1 + (S - T_1)X_2(T_1), \end{array}$$

we deduce that

$$\frac{S - T_2}{S - T_1} = \frac{c}{1 + c}.$$

Directly from the definition of c and S we have

$$S \times c = \int_0^{T_1} l(x(t))dt + \int_{T_1}^{T_2} l(x(t))dt + \int_{T_2}^{S} l(x(t))dt.$$

Moreover, from the definition of the T_i's and that of ℓ it follows that $\int_{T_1}^{T_2} l(x(t))dt = 0$, and since the speed of $X(.)$ is constant on $[T_2, S]$, $\int_{T_2}^{S} l(x(t))dt = (S - T_2)\frac{c}{2}$. Moreover, $\int_0^{T_1} l(X(t))dt \geq \frac{T_1}{2}$ since the speed $X_1'(t) = X_2(t)$ is increasing on $[0, T_1]$. This yields $Sc \geq \frac{1}{2}T_1 + \frac{c^2}{2(1+c)}(S - T_1)$. Finally, as the speed $X_1'(t) = X_2(t)$ is increasing also on the larger interval $[0, S]$, we have $\frac{T_1}{S} \geq \frac{X_1(T_1)}{X_1(S)} = \frac{1}{2+c}$. Combining this latter relation with the preceding estimate for Sc gives:

$$c \geq \frac{c^2 + 1}{2(2 + c)} \geq \beta,$$

where $\beta = \sqrt{5} - 2$ is just the minimum of the function $f(v) = (v^2 + 1)/(2(2 + v)), v > 0$. Now we observe that

$$\begin{array}{rcl} \inf_{T>0} U_T(0,0) & = & \inf\limits_{T>0} \inf\limits_{\alpha} \frac{1}{T} \int_0^T \ell(X_1^\alpha(t))dt \\ & = & \inf\limits_{\alpha} \big(\inf\limits_{T>0} \frac{1}{T} \int_0^T \ell(X_1^\alpha(t))dt \big), \end{array}$$

where the infinmum is taken over all deterministic admissible controls α, and $X^\alpha(.)$ denotes the dynamics controlled by α. Given any $\varepsilon > 0$, the control α with which we have worked in the above computations can be chosen ε-optimal, and then we have

$$\inf_{T>0} U_T(0,0) + \varepsilon \geq \inf_{T>0} \frac{1}{T} \int_0^T \ell(X_1(t))dt = c \geq \beta.$$

Considering the arbitrariness of $\varepsilon > 0$, this is just what we had to prove in this step.

3) Let us show now that $\limsup_{T\to+\infty} U_T(0,0) \leq \beta$. We fix $T > \sqrt{2\sqrt{5}}$ and consider the trajectory $(X(t))$ starting at $(0,0)$ and with the control $\alpha(t) = 1$ for $0 \leq t \leq s$, and $\alpha(t) = 0$ for $t \geq s$, where $s > 0$ is chosen so that $X_1(T) = 2 + \beta$, i.e., $s = T - \sqrt{T^2 - 2\sqrt{5}}\,(< 1)$ (obviously, the choice of the control α depends on T). Then using the definition of ℓ, we get

$$\int_0^T \ell(X(t))dt = \int_0^s (1 - t^2/2)dt + \int_s^{1/s+s/2} (1 - st - s^2/2)dt + \int_{2/s+s/2}^{T} (st - s^2/2 - 2)dt$$

(indeed, as $X_1(s) = \frac{1}{2}s^2 < \frac{1}{2} < 1$, $0 < s < T_1 < T_2 < T < T_3$, and a straight-forward computation shows that $X_1(t) = \frac{1}{2}t^2$, $t \in [0, s]$; $X_1(t) = st - \frac{1}{2}s^2$, $t \geq s$; and $T_1 = \frac{1}{2}s + \frac{1}{s}$ and $T_2 = \frac{1}{2}s + \frac{2}{s}$), and since $s \searrow 0$ while $1/(sT) \to 1/\sqrt{5}$, as $T \to +\infty$, we obtain that $U_T(0,0) \leq \frac{1}{T}\int_0^T l(x(t))dt \xrightarrow[T\to\infty]{} \beta$.

4) It remains to make the proof for the limit behaviour of $V_\lambda(0,0)$ as $\lambda \searrow 0$. We first show that $V_\lambda(0,0) \geq \beta^*$, for all $\lambda > 0$. By approximation we can consider without loss of generality a trajectory $X(.)$ starting from $X(0) = (0,0)$ and governed by a non constant control α such that $\alpha = 0$ as soon as $X_1(t) \geq 1$. We define T_1 such that $X_1(T_1) = 1$, and we put $x_2 := x_2(T_1)$. Observe that $x_2 \in (0, \sqrt{2}]$. The λ-discounted cost associated with this α is:

$$\gamma_\lambda = \int_0^\infty \lambda e^{-\lambda t}\ell(X_1(t))dt = (1 - e^{-\lambda T_1})w_\lambda + e^{-\lambda T_1}V_\lambda(1, x_2),$$

where

$$w_\lambda = \frac{1}{1 - e^{-\lambda T_1}} \int_0^{T_1} \lambda e^{-\lambda t}(1 - x_1(t))dt.$$

Indeed, observe that for the starting point $(1, x_2)$ the control identically equal to zero is optimal. We consider now the trajectory $\hat{X}(.)$ starting at $(0, x_2)$ with constant control $\hat{\alpha}(t) = 0$, $t \geq 0$, so that it reaches the state $(1, x_2)$ at time $1/x_2$. Its λ-discounted cost is:

$$\hat{\gamma}_\lambda = \int_0^\infty \lambda e^{-\lambda t}\ell(\hat{X}_1(t))dt = (1 - e^{-\lambda/x_2})\hat{w}_\lambda + e^{-\lambda/x_2}V_\lambda(1, x_2),$$

with

$$\hat{w}_\lambda = \frac{1}{1 - e^{-\lambda/x_2}} \int_{t=0}^{1/x_2} \lambda e^{-\lambda t}(1 - tx_2)dt.$$

We notice that $X_1(t) = \int_0^t X_2(s)ds \leq x_2 t$, $t \in [0, T_1]$, i.e., $1/x_2 (= X_1(T_1)/x_2) \leq T_1$ and $1 - x_1(t) \geq 1 - tx_2$, $t \in [0, T_1]$. As, moreover, $1 - tx_2$ is decreasing in t, we can compare the averages w_λ and $\hat{w}_\lambda$, and we obtain $w_\lambda \geq \hat{w}_\lambda$. Let us put $\gamma = \lambda/x_2 > 0$. Then, as the function $\varphi(r) = e^{-\gamma} + \frac{1}{2}\gamma(1 + e^{-\gamma}) - 1$, $r \geq 0$, is increasing in r and takes in $r = 0$ the value 0, we get $\varphi(\gamma) \geq 0$. Thus,

$$\hat{w}_\lambda = \frac{\gamma - 1 + e^{-\gamma}}{\gamma(1 - e^{-\gamma})} \geq \frac{1}{2}.$$

Furthermore, we have

$$V_\lambda(1, x_2) = \frac{1}{\gamma}e^{-\gamma}(1 - e^{-\gamma}) \text{ and } \hat{\gamma}_\lambda = 1 - \frac{1}{\gamma}(1 - e^{-\gamma} - e^{-2\gamma} + e^{-3\gamma}).$$

Both formulas are the result of a direct computation; that for $V_\lambda(1, x_2)$ is obtained by the fact that for the starting point $(1, x_2)$ the control $\bar{\alpha} = 0$ is optimal, and for the associated state process $\bar{X}(.)$ it holds $V_\lambda(1, x_2) = \int_0^{+\infty} \lambda e^{-\lambda t}\ell(\bar{X}_1(t))dt$. The explicit computation of the latter expression yields the above formula.

Now, if $\hat{w}_\lambda \geq V_\lambda(1, x_2)$, we obtain $\gamma_\lambda \geq (1 - e^{-\lambda T_1})\hat{w}_\lambda + e^{-\lambda T_1}V_\lambda(1, x_2) \geq \hat{\gamma}_\lambda \geq \beta^*$ (Recall that by definition β^* is just the minimum of the latter expression taken over $\gamma > 0$). On the other hand, if $\hat{w}_\lambda < V_\lambda(1, x_2)$, we see easily that $\gamma_\lambda \geq (1 - e^{-\lambda T_1})\hat{w}_\lambda + e^{-\lambda T_1}V_\lambda(1, x_2) \geq \hat{w}_\lambda \geq \frac{1}{2} \geq \beta^*$.

From the arbitrariness of the above control process α which allows to approximate $V_\lambda(0,0)$ we can conclude that also $V_\lambda(0,0) \geq \beta^*$.

5) Finally, we shall still show that $\limsup_{\lambda \to \infty} V_\lambda(0,0) \leq \beta^*$.

For this we introduce γ^* achieving the minimum in the definition of β^* (numerically $\gamma^* \simeq 0.87$), and given any $\lambda \in (0, \sqrt{6}\gamma^*)$, we consider the state process $X(.)$ starting at $X(0) = (0,0)$ with control $\alpha(t) = 1$ for $t \leq x_2^* := \frac{\lambda}{\gamma^*}$, and with the control $\alpha(t) = 0$ for $t \geq x_2^*$. Then, $X_1(t) = \frac{1}{2}t^2$, $t \in [0, x_2^*]$; $X_1(t) = x_2^* t - \frac{1}{2}(x_2^*)^2$, $t \geq x_2^*$, and $T_i = \frac{1}{2}x_2^* + i/x_2^*$, $i = 1, 2, 3$. Consequently, the induced cost is:

$$\begin{aligned} V_\lambda(0,0) \quad \leq \quad & \int_0^{+\infty} \lambda e^{-\lambda t}\ell(X_1(t))dt = \int_0^{x_2} \lambda e^{-\lambda t}(1 - \frac{1}{2}t^2)dt + \int_{x_2}^{\frac{1}{x_2} + \frac{x_2}{2}} \lambda e^{-\lambda t}(1 - tx_2 + \frac{1}{2}x_2^2)dt \\ & + \int_{\frac{2}{x_2} + \frac{x_2}{2}}^{\frac{3}{x_2} + \frac{x_2}{2}} \lambda e^{-\lambda t}(tx_2 - \frac{1}{2}x_2^2 - 2)dt + \int_{\frac{3}{x_2} + \frac{x_2}{2}}^{+\infty} \lambda e^{-\lambda t}dt, \end{aligned}$$

and a straight-forward computation shows that the right-hand side converges to $1-\frac{1}{\gamma^*}(1-e^{-\gamma^*}-e^{-2\gamma^*}+e^{-3\gamma^*})=\beta^*$ when λ tends to 0. This proves that $\limsup_{\lambda\to\infty}V_\lambda(0,0)\le\beta^*$. $\square$

Example B: A simple stochastic control example

We revisit Example 3.6 in [7]. Let the control state space be $A=[0,1]$. With an admissible control process $\alpha\in\mathcal{A}$ (taking its values in A) we associate the state process $X^\alpha(t)=X(t)=(X_1(t),X_2(t))$, $t\ge 0$, with the dynamics

$$\left\{\begin{array}{l} dX_s=b(X_s,\alpha_s)ds+\sigma(X_s,\alpha_s)dW_s,\\ X_0=x\in\mathbf{R}^2,\end{array}\right. \tag{28}$$

where

$$b(x,\alpha)=\left(\begin{array}{c}-\frac{\alpha^2}{2}x_1+\alpha x_2\\ -\alpha x_1-\frac{\alpha^2}{2}x_2\end{array}\right),\ \sigma(x,\alpha)=\left(\begin{array}{c}\alpha x_2\\ -\alpha x_1\end{array}\right),\ x=(x_1,x_2)\in\mathbf{R}^2.$$

Here the solution of the dynamics is explicitly given by, for $t\ge 0$:

$$X_t^\alpha=\left(\begin{array}{cc}\cos A_t^\alpha & \sin A_t^\alpha\\ -\sin A_t^\alpha & \cos A_t^\alpha\end{array}\right)\cdot x,\ \text{where } A_t^\alpha=\int_0^t\alpha_s(ds+dW_s),\ t\ge 0.$$

As with respect to the Euclidean norm $|X_t^\alpha|=|x|$, $t\ge 0$, the compact set $Y=\{y\in\mathbf{R}^2,|y|=|x|\}$ is invariant. We consider a Lipschitz cost function $\ell:Y\to[0,1]$. The hypotheses (H1)–(H5) are satisfied and also the non expansivity condition (NE) holds true. Hence by Proposition 3.7 and Theorem 4.7, U_T and V_λ uniformly converge to $u^*=\widehat{u}$, as $T\to+\infty$ and $\lambda\searrow 0$, respectively.

The functions $u^*=\sup\{w,w\in\mathcal{K}\}$ and $\widehat{u}=\sup\{w,w\in\mathcal{H}\}$ are defined in Definitions 3.2 and 4.6, respectively, where $\mathcal{K}$ and $\mathcal{H}$ are subsets of $C(Y,[0,1])$. Choosing $\alpha=0$ shows that any point in Y is a rest point of the dynamics, so condition ii) of Definition 3.2 (and Definition 4.6, respectively) is equivalent to $w(y)\le\ell(y)$, $y\in Y$. The problem is ergodic here, and condition i) in the both definitions is equivalent to w being constant. Consequently, we obtain that the limit of the uniform convergence of U_T ($T\to+\infty$) and of V_λ ($\lambda\searrow 0$) is the constant: $\min_{y\in Y}\ell(y)$.

Example C: A non-ergodic example

Let us now modify Example B by removing the integral with respect to the Brownian motion W. Keeping $A=[0,1]$ and the drift coefficient from Example B,

$$b(x,\alpha)=-\frac{\alpha^2}{2}\left(\begin{array}{c}x_1\\ x_2\end{array}\right)+\alpha\left(\begin{array}{c}x_2\\ -x_1\end{array}\right),\ x=(x_1,x_2)\in\mathbf{R}^2,$$

we associate with any admissible control $\alpha\in\mathcal{A}$ the state process $X^\alpha(.)=X(.)=(X_1(.),X_2(.))$ with the dynamics

$$\left\{\begin{array}{l} dX_s=b(X_s,\alpha_s)ds,\ s\ge 0,\\ X_0=x\in\mathbf{R}^2.\end{array}\right. \tag{29}$$

Denoting again by $|.|$ the Euclidean norm and by $\langle .,.\rangle$ the canonical scalar product in $\mathbf{R}^2$, we have for all y, z in $\mathbf{R}^2$, $\alpha\in A=[0,1]$: $2\langle y-z,b(y,\alpha)-b(z,\alpha)\rangle=-\alpha^2|y-z|^2\le 0$.

Hence, the non expansivity condition (NE) holds true, and as $d(|X^\alpha(t)|^2)=-\alpha(t)^2|X^\alpha(t)|^2\le 0$, $t\ge 0$, $\alpha\in\mathcal{A}$, we have the invariance of $Y=\{y\in\mathbf{R}^2,|y|\le|x|\}$. Remark that unlike Example B, there is no non-empty compact strict subset of Y which is invariant with respect to the control dynamics, and in particular the set $\{y\in\mathbf{R}^2,|y|=|x|\}$ is not invariant here. As in Example B we consider a Lipschitz running cost $\ell:Y\to[0,1]$. As the hypotheses (H1)–(H5) are satisfied, the functions U_T and V_λ converge uniformly on Y as $T\to+\infty$ and $\lambda\searrow 0$, respectively, and the limit function is $Y\ni y\to u^*(y)=\widehat{u}(y)=\sup\{w(y),w\in\mathcal{K}\}=\sup\{w(y),w\in\mathcal{H}\}$. It can be checked easily that here $\mathcal{K}=\mathcal{H}$ is the set of all continuous functions $w:Y\to[0,1]$ such that

i) $w(y) = w(|y|)$ only depends on the norm of y, and ii), as any $y \in Y$ is a rest point for the control $\alpha = 0$, $w(y) \leq \ell(y)$. Hence $u^*(x) = \widehat{u}(x) = \min\{\ell(y), |y| = |x|\}$, $x \in Y$.

Assume for instance that the cost function is given by $\ell(y) = \frac{1}{2}(1+y_1)$, so that the limit value satisfies $u^*(x) = \frac{1}{2}(1-|x|)$. Fix the initial state as $x = (1,0)$. Then $u^*(x) = 0 = l((-1,0))$. The controller wants to go from $(1,0)$ as close as possible with trajectory to $(-1,0)$, in order to stay then there forever. But the unique trajectory starting at $(1,0)$ and remaining in the set of states $\{y \in Y : u^*(y) = 0\}$ is constant: $X(t) = (1,0)$, for all $t \geq 0$, and it bears the highest possible cost. Thus, a good control α necessarily makes that $X^\alpha(.)$ enters the set $\{y \in Y : u^*(y) > 0\}$. From the properties of u^* as limit we have that, for any ε *strictly* positive, there is a control $\alpha^\varepsilon \in \mathcal{A}$ such that the associated trajectory $X^{\alpha^\varepsilon}(.)$ with $X^{\alpha^\varepsilon}(0) = (1,0)$ has the property that $\lim_{t\to\infty} \ell(X^\varepsilon(t)) \leq \varepsilon$.

7 Appendix

Proof. (of Lemma 4.5) Let $Q \in \mathcal{R}_{\text{inv}}(\mathcal{C} \times \mathcal{V})$. We start by proving the part i) of the lemma. Fix $T > 0$ and $g \in C(Y \times A)$. Let $m \geq 1$ be an arbitrary integer. Then there exists an integer $k_m > 0$ such that $\frac{k_m}{m} \leq T < \frac{k_m+1}{m}$. Thus,

$$\begin{aligned}
&\int_{Y\times A} g\, \mu_Q(dxda) = \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^1 \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq) \\
&= \sum_{j=0}^{m-1} \int_{\mathcal{C}\times\mathcal{V}} \left(\int_{j/m}^{(j+1)/m} \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq) \\
&= \sum_{j=0}^{m-1} \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^{1/m} \int_A g(x(s+\frac{j}{m}),a)q(s+\frac{j}{m},da)ds \right) Q(dx(\cdot)dq) \\
&= m \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^{1/m} \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq).
\end{aligned}$$

With an obvious analogous argument,

$$\begin{aligned}
&\int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^1 \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq) \\
&= m \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^{1/m} \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq) \\
&= \frac{m}{k_m} \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^{k_m/m} \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq) \\
&\longrightarrow \frac{1}{T} \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^{T} \int_A g(x(s),a)q(s,da)ds \right) Q(dx(\cdot)dq),
\end{aligned}$$

as $m \to +\infty$, by the bounded convergence theorem.

ii) Let $f \in C(Y)$. From i), for all $t > 0$,

$$\begin{aligned}
&\int_{X\times A} f(x)\mu_Q(dxda) = \int_{\mathcal{C}\times\mathcal{V}} \left(\int_0^1 \int_A f(x(s))q(s,da)ds \right) Q(dx(\cdot)dq) \\
&= \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{t} \int_0^t \int_A f(x(s))q(s,da)ds \right) Q(dx(\cdot)dq) = \int_{\mathcal{C}\times\mathcal{V}} \left(\frac{1}{t} \int_0^t f(x(s))ds \right) Q(dx(\cdot)dq) \\
&\longrightarrow \int_{\mathcal{C}\times\mathcal{V}} f(x(0))Q(dx(\cdot)dq),
\end{aligned}$$

as $t \searrow 0$. Also here we have used the bounded convergence theorem. The proof is complete. $\square$

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