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par

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Titre :

Étude qualitative de quelques équations et systèmes dynamiques non-linéaires amortis

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Résumé

Des méthodes d'énergie adaptées permettent d'obtenir la localisation spatiale, l'extinction en temps fini et la propriété de temps d'attente de solutions d'équations aux dérivées partielles. Ces trois types de propriétés sont ainsi regroupés car les méthodes mathématiques pour y parvenir sont très proches. Les travaux présentés dans une grande partie de cette habilitation à diriger des recherches concernent les deux premières propriétés que l'on applique à des équations de Schrödinger (stationnaires et d'évolution) avec un terme d'amortissement. Tout d'abord, des théorèmes d'existence et/ou d'unicité sont démontrés. Puis, une étude qualitative des solutions est effectuée : phénomène de localisation, pour l'équation stationnaire et extinction en temps fini, pour l'équation d'évolution.

Une partie plus mince concerne la stabilisation en temps infinie de solutions des équations des ondes et des poutres à l'aide, également, d'un terme d'amortissement. Ce dernier permet d'obtenir l'extinction en temps infinie des solutions. On commence par établir une inégalité généralisée de Hölder. Puis, à l'aide de celle-ci, on donne la vitesse de convergence de l'énergie associée à chaque solution.

Une autre partie traite de l'étude d'un système gradient du second ordre. Ici encore, un terme d'amortissement est présent impliquant, sous des hypothèses adéquats, l'extinction en temps infinie des solutions. En déformant l'énergie totale du système et en utilisant l'inégalité de Kurdyka-Łojasiewicz, on montre que ce système gradient amorti du second ordre et les systèmes quasi-gradients sont de même nature. Par ailleurs, on donne les vitesses de convergence des solutions.

Dans une dernière partie, on s'intéresse à l'équation de Schrödinger dont la non-linéarité est critique pour la masse. On montre à l'aide d'une inégalité améliorée de Strichartz que, près du temps d'explosion, la masse de la solution se concentre dans une boule de rayon nulle.

Mots-clés : équation amortie, système dynamique dissipatif, système gradient, méthode d'énergie, solution à support compact, extinction en temps fini/infini, comportement asymptotique, stabilisation, existence globale, unicité, solution auto-semblable, explosion en temps fini, inégalité généralisée de Hölder, inégalité de Kurdyka-Łojasiewicz, inégalité améliorée de Strichartz.

Key words : damped equation, dissipative dynamical systems, gradient systems, inertial systems, compactly supported solution, finite time extinction, asymptotic behavior, stabilization, global existence, uniqueness, self-similar solution, finite time blow-up, generalized Hölder's inequality, Kurdyka-Łojasiewicz inequality, generalized Strichartz's estimate

Introduction

Chapitres 1–6 : Études spatiale et asymptotique pour des équations de Schrödinger non-linéaires amorties ([24, 25, 26, 27, 28, 29, 20]

Dans ces chapitres, on s'intéresse à des équations de Schrödinger avec un terme d'amortissement :

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} + \Delta u + a|u|^{-(1-m)}u = f(t,x), & \text{dans } \mathbb{R}\times\Omega, \\ u_{|\Gamma} = 0, & \text{dans } \mathbb{R}\times\Gamma, \\ u(0) = u_0, & \text{dans } \Omega, \end{cases} \tag{1}$$

pour l'équation d'évolution, et

$$\begin{cases} -\mathrm{i}\Delta u + a|u|^{-(1-m)}u + bu = F, & \text{dans } \Omega, \\ u_{|\Gamma} = 0, & \text{sur } \Gamma, \end{cases} \tag{2}$$

pour l'équation stationnaire. Ici, $(a,b)\in\mathbb{C}^2$, $0<m<1$, $\Omega\subseteq\mathbb{R}^N$ est un ouvert et les termes sources f et F, et la donnée initiale u_0 sont choisis dans un espace adéquat. Le but est de savoir s'il existe des solutions à support compact ou bien qui s'annulent en temps fini.

Chapitre 1 : Estimation et localisation du support pour l'équation stationnaire

Ce chapitre concerne l'équation (2). Des hypothèses sont faites pour obtenir l'existence et l'unicité de solution. On peut les formuler de la manière géométrique suivante.

Hypothèse d'Existence 1. Soit $(a,b)\in\mathbb{C}^2$. Alors $[a,b]\cap\mathbb{R}_-\times\mathrm{i}\{0\}=\emptyset$.

Hypothèses d'Unicité 2. Soit $(a,b)\in\mathbb{C}^2$. Alors $a\neq 0$, $\mathrm{Re}(a)\geqslant 0$ et $\overrightarrow{a}.\overrightarrow{b}\geqslant 0$ [1].

Formulées ainsi, l'hypothèse d'unicité implique l'hypothèse d'existence. Les résultats principaux de ce chapitre peuvent s'énoncer ainsi (Théorèmes 1.4.1 et 1.5.2).

1. ou de façon équivalente, $|\mathrm{mes}(\widehat{\overrightarrow{a},\overrightarrow{b}})|\leqslant\frac{\pi}{2}$ rad.

Théorème 3. *Soient $\Omega \subseteq \mathbb{R}^N$ un ouvert non vide et $0 < m < 1$. Si le couple (a, b) satisfait l'Hypothèse d'Existence 1, avec éventuellement $b = 0$, alors pour n'importe quel $F \in L^{\frac{m+1}{m}}(\Omega; \mathbb{C})$, l'équation (2) admet au moins une solution $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$. Si de plus l'Hypothèse d'Unicité 2 est vérifiée alors la solution est unique.*

L'existence s'obtient de la façon suivante. On commence par obtenir des estimations *a priori*. On approxime ensuite l'équation avec une suite de non-linéarités tronquées à l'aide du Théorème de point fixe de Schauder, puis on passe à la limite. L'unicité s'obtient en disant que, en gros, la non-linéarité est la différentielle d'une fonction convexe et est donc monotone. Concernant la compacité du support des solutions, le résultat est celui ci-dessous (Théorèmes 1.3.5 et 1.3.6).

Théorème 4. *Soient $\Omega \subseteq \mathbb{R}^N$ un ouvert non vide, $0 < m < 1$ et (a, b) satisfaisant l'Hypothèse d'Existence 1, avec éventuellement $b = 0$.*

1. *Soient $F \in L^{\frac{m+1}{m}}(\Omega; \mathbb{C})$ et $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ une solution quelconque de (2). Si F est à support compact et si $\|F\|_{L^{\frac{m+1}{m}}(\Omega)}$ est suffisamment petite alors u est à support compact et $\operatorname{supp} u \subset \Omega$.*
2. *Soient $F \in L^p(\mathbb{R}^N; \mathbb{C})$, pour un $p \in [1, \infty]$, et $u \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ une solution quelconque de (2). Si F est à support compact alors u est à support compacte.*

La démonstration du Théorème 4 est pour le moins technique et repose sur une méthode d'énergie. Elle fait appel, entre autres, à une inégalité de trace-interpolation (voir (1.7.12)). À l'aide des Théorèmes 3 et 4, on peut construire des solutions à support compact en espace pour l'équation (1) de la façon suivante. Soient $a \in \mathbb{C}$ tel que $a \notin \mathbb{R}_+$, $b \in \mathbb{R}_+^\star$, $0 < m < 1$ et $F \in L^{\frac{m+1}{m}}(\mathbb{R}^N; \mathbb{C})$. Soient alors $u_0 \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ une solution de (2) donnée par le Théorème 3, avec $-a$, au lieu de a et $-F$, au lieu de F. On pose pour tout $t \in \mathbb{R}$, $f(t) = Fe^{ibt}$ et $u(t) = u_0 e^{ibt}$. Alors on vérifie aisément que $u \in C^\infty\big(\mathbb{R}; H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)\big)$ est une solution de (1) pour un tel f. Le Théorème 4 donne alors le résultat suivant.

Corollaire 5. *Avec les hypothèses et notations ci-dessus, si F est à support compact alors pour tout $t \in \mathbb{R}$, $\operatorname{supp} u(t) = \operatorname{supp} F$.*

Chapitre 2 : Existence de solutions faibles pour des équations de Schrödinger stationnaires amorties

À ce stade, deux choses sont insatisfaisantes :

– le résultat de compacité pour les solutions de l'équation d'évolution est trop restrictif (Corollaire 5),

– l'hypothèse $F \in L^{\frac{m+1}{m}}$ est moins naturelle et plus restrictive que l'hypothèse $F \in L^2$ (au moins lorsque $|\Omega|$ est de mesure finie car dans ce cas, et puisque $0 < m < 1$, $L^{\frac{m+1}{m}}(\Omega) \subsetneq L^2(\Omega)$).

Pour ce dernier point, le choix est d'établir des résultats analogues au Chapitre 1, avec $F \in L^2$ (quitte à exclure la valeur $b = 0$) et d'y inclure la condition de Neumann homogène au bord. On s'intéresse donc également à,

$$\begin{cases} -\mathrm{i}\Delta u + a|u|^{-(1-m)}u + bu = F, & \text{dans } \Omega, \\ \dfrac{\partial u}{\partial \nu}_{|\Gamma} = 0, & \text{sur } \Gamma, \end{cases} \tag{3}$$

Évidemment, on considérera systématiquement dans ce cas que l'ouvert Ω est borné et de classe C^1.

Concernant le premier point, l'idée est de se concentrer sur les solutions auto-semblables. En effet, établir un résultat de compacité en espace des solutions pour l'équation (1) est à ce jour encore trop difficile. D'où le choix de regarder d'abord les solutions auto-semblables car on peut se ramener à une équation stationnaire de la façon (bien connue) suivante. Pour des raison d'homogénéité, f doit vérifier,

$$\forall \lambda > 0,\ f(t,x) = \lambda^{-\frac{2m}{1-m}} f(\lambda^2 t, \lambda x), \tag{4}$$

pour tout $t \geqslant 0$ et presque tout $x \in \mathbb{R}^N$. Des solutions auto-semblables u de (1) sont des solutions qui s'écrivent sous la forme,

$$u(t,x) = t^{\frac{1}{1-m}} U\left(\frac{x}{\sqrt{t}}\right), \tag{5}$$

pour tout $t \geqslant 0$ et presque tout $x \in \mathbb{R}^N$, où le profil U est solution de,

$$-\Delta U - a|U|^{-(1-m)}U - \frac{\mathrm{i}}{1-m}U + \frac{\mathrm{i}}{2}x.\nabla U = -f(1).$$

En effectuant le changement (également très connu) d'inconnue $g(x) = U(x)e^{-\mathrm{i}\frac{|x|^2}{8}}$, on se ramène à l'étude de,

$$-\Delta g - a|g|^{-(1-m)}g - \mathrm{i}\frac{N(1-m)+4}{4(1-m)}g - \frac{1}{16}|x|^2 g = -f(1)e^{-\mathrm{i}\frac{|x|^2}{8}}, \tag{6}$$

que l'on peut généraliser sous la forme,

$$-\Delta v + a|v|^{-(1-m)}v + bv + cV^2 v = H, \tag{7}$$

où V est un potentiel réel et c est un nombre complexe. En adaptant la démonstration du Théorème 3, on obtient les résultats suivants (Théorèmes 2.2.1, 2.2.4, 2.2.8, et 2.2.10).

Théorème 6. *Soient $\Omega \subseteq \mathbb{R}^N$ un ouvert non vide et $0 < m < 1$. Si le couple (a,b) satisfait l'Hypothèse d'Existence 1 alors pour n'importe quel $F \in L^2(\Omega;\mathbb{C})$, les équations (2) et (3) admettent au moins une solution $u \in H^1(\Omega) \cap L^{m+1}(\Omega)$. Si de plus l'Hypothèse d'Unicité 2 est vérifiée alors la solution est unique.*

Théorème 7. *Soient $\Omega \subseteq \mathbb{R}^N$ un ouvert non vide, $V \in L^\infty(\Omega;\mathbb{R})$, $0 < m < 1$ et a, b et c des nombres complexes tels que $\mathrm{Im}(a) < 0$, $\mathrm{Im}(b) < 0$ et $\mathrm{Im}(c) \leqslant 0$. Alors pour n'importe quel $H \in L^2(\Omega;\mathbb{C})$, l'équation (7) admet au moins une solution $v \in H^1(\Omega) \cap L^{m+1}(\Omega)$, avec Dirichlet homogène ou Neumann homogène comme condition au bord. Si de plus l'Hypothèse d'Unicité 2 est vérifiée et $\overrightarrow{a}.\overrightarrow{c} \geqslant 0$ alors la solution est unique.*

Théorème 8. *Soient les hypothèses et notations du Théorème 6. On suppose que F est à support compact. Si l'une des conditions ci-dessous est satisfaite alors n'importe quelle solution u est à support compact et* $\operatorname{supp} u \subset \Omega$.

1. $\Omega = \mathbb{R}^N$.
2. $u \in H_0^1(\Omega)$ *et* $\|F\|_{L^2(\Omega)}$ *est suffisamment petite.*
3. *L'Hypothèse d'Unicité 2 est vérifiée,* ${\dfrac{\partial u}{\partial \nu}}_{|\Gamma} = 0$ *et* $\|F\|_{L^2(\Omega)}$ *est suffisamment petite.*

Le résultat suivant permet d'obtenir des solutions pour l'équation (2), sans restrictions sur a, à l'aide de l'inégalité de Poincaré,

$$\forall u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} \leqslant C_{\mathrm{P}} \|\nabla u\|_{L^2(\Omega)}. \tag{8}$$

qui est valable dès que Ω est de mesure finie.

Théorème 9. *Soient $\Omega \subset \mathbb{R}^N$ un ouvert non vide de mesure finie, $0 < m < 1$ et $(a,b) \in \mathbb{C}^2$. Si $b \in \mathbb{R}_-^\star$ alors on suppose de plus que $bC_{\mathrm{P}}^2 > -1$, où C_{P} est la meilleure constante dans (8). Alors pour n'importe quel $F \in L^2(\Omega;\mathbb{C})$, l'équation (2) admet au moins une solution $u \in H_0^1(\Omega)$. Si de plus l'Hypothèse d'Unicité 2 est vérifiée alors la solution est unique.*

Chapitre 3 : Méthode d'énergie affinée pour la localisation du support de solutions d'équations de Schrödinger non-linéaires amorties

Bien que cités sous une forme globale, les théorèmes de localisation spatiale sont des résultats locaux. Ils se démontrent avec une méthode d'énergie issue du livre de Antontsev, Díaz et Shmarev [11]. Cette méthode est très bien adaptée pour les équations scalaires et les systèmes mais elle s'avère inapplicable, en tous les cas telle quelle, pour les équations complexes, même si celles-ci sont vues comme des systèmes d'équations scalaires en séparant la partie réelle de la partie imaginaire. Plutôt que d'adapter la méthode à chaque type d'équation comme cela est fait dans le Chapitre 1, le but est d'établir un critère qui engendrerait le phénomène de localisation voulu. Le résultat est alors le suivant (Théorème 3.2.1) et s'obtient en affinant la méthode initiale de [11].

Théorème 10. *Soit $0 < m < 1$. Alors il existe une constante $C = C(N,m) > 0$ ayant la propriété suivante: soient $x_0 \in \mathbb{R}^N$, $\rho_0 > 0$ et $u \in H^1_{\mathrm{loc}}\big(B(x_0,\rho_0)\big)$. Si l'on peut trouver des constantes $L, M > 0$ telle que pour presque tout $\rho \in (0,\rho_0)$,*

$$\|\nabla u\|^2_{L^2(B(x_0,\rho))} + L\|u\|^{m+1}_{L^{m+1}(B(x_0,\rho))} \leqslant M \left| \int_{\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|} \mathrm{d}\sigma \right|, \tag{9}$$

alors

$$u_{|B(x_0,\rho_{\max})} \equiv 0,$$

où

$$\rho_{\max}^{\nu} = \left(\rho_0^{\nu} - CM^2 \max\left\{1, \frac{1}{L^2}\right\} \max\left\{\rho_0^{\nu-1}, 1\right\} \times \min_{\tau \in \left(\frac{m+1}{2}, 1\right]} \left\{ \frac{E(\rho_0)^{\gamma(\tau)} \max\{b(\rho_0)^{\mu(\tau)}, b(\rho_0)^{\eta(\tau)}\}}{2\tau - (1+m)} \right\} \right)_+ ,$$

et,

$$E(\rho_0) = \|\nabla u\|^2_{L^2(B(x_0,\rho_0))}, \qquad b(\rho_0) = \|u\|^{m+1}_{L^{m+1}(B(x_0,\rho_0))},$$
$$k = 2(1+m) + N(1-m), \qquad \nu = \tfrac{k}{m+1} > 2,$$
$$\gamma(\tau) = \frac{2\tau - (1+m)}{k} \in (0,1), \quad \mu(\tau) = \frac{2(1-\tau)}{k}, \quad \eta(\tau) = \frac{1-m}{1+m} - \gamma(\tau) > 0.$$

pour tout $\tau \in \left(\frac{m+1}{2}, 1\right]$.

Outre la difficulté de montrer qu'une solution vérifie (9), il convient également de contrôler les différentes normes de celle-ci pour éviter que le rayon $\rho_{\max}$ soit nul. Ceci explique les hypothèses de petitesse sur F du Chapitre 1.

Chapitre 4 : Solutions auto-semblables

Nous avons maintenant les outils nécessaires pour construire des solutions auto-semblables à support compact. On montre que, sous des hypothèses adéquats, si g est une solution de (6) alors elle vérifie (9). En appliquant alors les Théorèmes 7 et 10 on obtient le résultat suivant (Théorème 4.1.2).

Théorème 11. *Soient* $0 < m < 1$, $a \in \mathbb{C}$ *tel que* $\mathrm{Im}(a) > 0$ *et* $f \in C\big((0,\infty); L^2(\mathbb{R}^N)\big)$ *satisfaisant* (4). *On suppose également que* $\operatorname{supp} f(1)$ *est compact. Si* $\|f(1)\|_{L^2(\mathbb{R}^N)}$ *est suffisamment petite alors il existe une solution auto-semblable* (*c'est-à-dire satisfaisant* (5)),

$$u \in C\big((0,\infty); H^2(\mathbb{R}^N)\big) \cap C^1\big((0,\infty); H^1(\mathbb{R}^N)\big) \cap C^2\big((0,\infty); L^2(\mathbb{R}^N)\big),$$

de (1) *telle que pour tout* $t > 0$, $\operatorname{supp} u(t)$ *est compact.*

Chapitres 5 et 6 : Extinction en temps fini pour des solutions d'équations de Schrödinger non-linéaires amorties

Dans ces deux chapitres, on étudie l'extinction en temps fini des solutions de (1). L'idée, qui est particulièrement simple et semble être due à Carles et Gallo [53] [2], et Carles et Ozawa [55] [3], est la suivante. Pour fixer les idées, supposons que $f \equiv 0$ dans (1). Si l'on multiplie l'équation (1) par $\overline{iu}$, que l'on intègre par parties et que l'on prend la partie réelle, il vient alors :

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2} + \mathrm{Im}(a)\|u(t)\|^{m+1}_{L^{m+1}} = 0. \tag{10}$$

2. dans le cas d'une variété compacte sans bord.
3. dans le cas de l'espace entier avec $N \leqslant 2$.

Il bien clair que si l'on veut avoir extinction en temps fini alors nécessairement, $\mathrm{Im}(a) > 0$. On utilise ensuite l'inégalité de Gagliardo-Nirenberg suivante :

$$\|u(t)\|_{L^2}^{2\frac{m+1}{2\theta_\ell}} \leqslant \|u(t)\|_{L^{m+1}}^{m+1} \|u(t)\|_{H^\ell}^{\frac{(m+1)(1-\theta)}{\theta_\ell}}, \tag{11}$$

où $\theta_\ell \in (0,1)$ est une constante connue. Ainsi, si la solution u est uniformément bornée dans H^ℓ alors on déduit de (10)–(11),

$$y' + Cy^\delta \leqslant 0,$$

avec $\delta = \frac{m+1}{2\theta_\ell}$, où $y(t) = \|u(t)\|_{L^2}^2$. Après intégration, on obtient alors le comportement asymptotique de u suivant les valeurs de δ.

- Si $\delta < 1$ alors $y(t)^{1-\delta} \leqslant (y(0)^{1-\delta} - Ct)_+$ et u s'annule au plus tard au temps $T_\star = C^{-1}y(0)^{1-\delta}$.
- Si $\delta = 1$ alors $y(t) \leqslant y(0)e^{-Ct}$.
- Si $\delta > 1$ alors $y(t)^{\delta-1} \leqslant y(0)^{\delta-1}(1+Ct)^{-1}$.

Ainsi pour obtenir extinction en temps fini, on doit avoir $\delta < 1$ ce qui s'avère être équivalent à la dimension $N = 1$ lorsque la solution en temps est dans H^1. Si l'on veut augmenter en dimension d'espace, la solution doit être alors plus régulière, disons uniformément bornée dans H^2. Dans ce cas, $\delta < 1$ lorsque $N \leqslant 3$. Étant donnée la non-linéarité, il n'est pas raisonnable d'espérer obtenir des solutions plus régulières que H^2, ce qui limite le résultat d'extinction en temps fini, tout du moins pour cette méthode, aux dimensions d'espace 1, 2 et 3.

Concernant l'existence de solutions, on utilise la théorie des opérateurs maximaux monotones dans L^2. La monotonie de l'opérateur $Au = -\mathrm{i}\Delta u - \mathrm{i}a|u|^{-(1-m)}u$, avec en gros $D(A) = H^2 \cap H_0^1$, repose sur l'inégalité,

$$\mathrm{Re}\left(-\mathrm{i}\,a\int\big(|u|^{m-1}u - |v|^{m-1}v\big)(\overline{u-v})\mathrm{d}x\right) \geqslant 0. \tag{12}$$

On voit alors apparaître alors une compétition entre les parties réelles et imaginaires de a et de $\int\big(|u|^{m-1}u - |v|^{m-1}v\big)(\overline{u-v})\mathrm{d}x$. Ce problème peut être réglé en utilisant un résultat de Liskevich and Perel′muter [132] (Lemme 6.4.3) ce qui réduit le choix pour a à,

$$a \in C(m) \overset{\text{déf}}{=} \Big\{z \in \mathbb{C};\ 2\sqrt{m}\,\mathrm{Im}(z) > (1-m)|\mathrm{Re}(z)|\Big\}. \tag{13}$$

L'existence de solutions est alors réduit à la surjectivité de l'opérateur $I + A$. La méthode diffère selon que l'on est dans un domaine borné ou dans tout l'espace.

Chapitre 5 : Le cas des domaines bornés

Lorsque Ω est borné la non-linéarité vérifie $|u|^m \in L^{\frac{2}{m}}(\Omega) \hookrightarrow L^2(\Omega)$ et l'on se retrouve dans l'espace de l'opérateur. La surjectivité de l'opérateur $I + A$ s'obtient alors par une méthode de perturbation. On peut ainsi prolonger les résultats de Carles and Gallo [53] (Théorèmes 5.4.3, 5.4.4 et 5.4.5).

Théorème 12. *Soient Ω un ouvert borné régulier de $\mathbb{R}^N$, $0 < m < 1$, $a \in \overline{C(m)} \setminus \{0\}$ et*

$$f \in L^1_{\mathrm{loc}}\big([0,\infty); L^2(\Omega)\big).$$

Alors pour tout $u_0 \in L^2(\Omega)$, *il existe une unique solution,*

$$u \in C\big([0,\infty); L^2(\Omega)\big) \cap W^{1,1}_{\mathrm{loc}}\big([0,\infty); H^{-2}(\Omega)\big),$$

de (1). *De plus, on a les résultats suivants.*

1. *Si de plus* $f \in W^{1,1}\big((0,\infty); H^1_0(\Omega)\big)$ *et si* $u_0 \in H^1_0(\Omega)$ *alors,*

$$u \in C_{\mathrm{w,b}}\big([0,\infty); H^1_0(\Omega)\big) \cap W^{1,\infty}\big([0,\infty); H^{-1}(\Omega)\big).$$

2. *Si de plus* $f \in W^{1,1}\big((0,\infty); L^2(\Omega)\big)$ *et si* $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$ *alors,*

$$u \in C_{\mathrm{b}}\big([0,\infty); H^1_0(\Omega)\big) \cap C^1_{\mathrm{b}}\big([0,\infty); H^{-1}(\Omega)\big) \cap L^\infty\big((0,\infty); H^2(\Omega)\big) \cap W^{1,\infty}\big((0,\infty); L^2(\Omega)\big).$$

Concernant l'extinction en temps fini, nous avons le résultat suivant (Théorème 5.3.2).

Théorème 13. *Soient* Ω *un ouvert borné régulier de* $\mathbb{R}^N$ *avec* $N \leqslant 3$, $0 < m < 1$, $a \in \overline{C(m)} \setminus \{0\}$, $f \in W^{1,1}\big((0,\infty); L^2(\Omega)\big)$ *et* $u_0 \in H^1_0(\Omega)$. *Supposons que l'une des conditions ci-dessous soit satisfaite.*

1. $N = 1$ *et* $f \in W^{1,1}\big((0,\infty); H^1_0(\Omega)\big)$.
2. $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$.

Soit u *l'unique solution de* (1). *S'il existe* $T_0 \geqslant 0$ *tel que pour presque tout* $t > T_0$,

$$f(t) = 0,$$

alors il existe un temps fini $T_\star \geqslant T_0$ *pour lequel,*

$$\|u(t)\|_{L^2(\Omega)} = 0.$$

pour tout $t \geqslant T_\star$. *De plus, sous des hypothèses supplémentaires de décroissance de* $\|f(t)\|_{L^2(\Omega)}$ *sur l'intervalle* $[0, T_0]$, *on a* $T_\star = T_0$.

Chapitre 6 : Le cas de tout l'espace

Pour monter que $R(I+A) = L^2(\mathbb{R}^N)$, on procède comme suit. On doit montrer que les solutions de (2) sont dans $H^2(\mathbb{R}^N)$, ce qui revient à dire que $\Delta u \in L^2(\mathbb{R}^N)$ ou, de façon équivalente, que $u \in L^{2m}(\mathbb{R}^N)$. On commence par établir des estimations *a priori* fines des solutions (voir les Lemmes 6.4.2 et 6.4.4, ainsi que les figures p.111). Ensuite, on construit des solutions à support compact grâce au Théorème 6 et au point 2 du Théorème 4. Un argument de densité permet alors de conclure. On peut ainsi prolonger les résultats de Carles and Ozawa [55] (Théorèmes 6.2.4, 6.2.6 et 6.2.7).

Théorème 14. *Soient* $0 < m < 1$, $a \in C(m)$ *et*

$$f \in L^1_{\mathrm{loc}}\big([0,\infty); L^2(\Omega)\big).$$

Alors pour tout $u_0 \in L^2(\mathbb{R}^N)$, *il existe une unique solution,*

$$u \in C\big([0,\infty); L^2(\mathbb{R}^N)\big) \cap W^{1,1}\big([0,\infty); H^{-2}(\mathbb{R}^N) + L^{\frac{2}{m}}(\mathbb{R}^N)\big),$$

de (1). *De plus, on a les résultats suivants.*

1. *Si de plus* $f \in W^{1,1}\big((0,\infty);H^1(\mathbb{R}^N)\big)$ *et si* $u_0 \in H^1(\mathbb{R}^N)$ *alors,*

$$u \in C_{\mathrm{w,b}}\big([0,\infty);H^1(\mathbb{R}^N)\big) \cap W^{1,\infty}\big((0,\infty);H^{-1}(\mathbb{R}^N) + L^{\frac{2}{m}}(\mathbb{R}^N)\big).$$

2. *Si de plus* $f \in W^{1,1}\big((0,\infty);L^2(\mathbb{R}^N)\big)$ *et si* $u_0 \in H^2(\mathbb{R}^N)$ *alors,*

$$u \in C_{\mathrm{b}}\big([0,\infty);H^1(\mathbb{R}^N)\big) \cap L^{\infty}\big((0,\infty);H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)\big) \cap W^{1,\infty}\big((0,\infty);L^2(\mathbb{R}^N)\big).$$

Concernant l'extinction en temps fini, nous avons le résultat suivant (Théorème 6.3.1).

Théorème 15. *Avec* $a \in C(m)$*, le Théorème 13 est valable pour* $\Omega = \mathbb{R}^N$.

Chapitre 7 : Stabilisation de solutions d'équations amorties ([30])

Considérons l'équation des ondes amortie suivante :

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + a(x)u_t(t,x) = 0, & \text{avec } (t,x) \in (0,\infty)\times(0,1), \\ u(t,0) = u(t,1) = 0, & \text{avec } t \in [0,\infty), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), & \text{avec } x \in (0,1), \end{cases}$$

où $a \in L^\infty(0,1)$ est positive presque partout dans $(0,1)$. L'énergie associée à la solution est,

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|^2_{L^2(0,1)} + \|u_x(t)\|^2_{L^2(0,1)}\right),$$

et l'espace fonctionnel associée à cette énergie est $H^1_0(0,1)\times L^2(0,1)$. Il est facile de voir que lorsque le terme d'amortissement est absent ($a = 0$) alors l'énergie est constante, et que lorsque celui-ci est présent alors l'énergie décroît. Par ailleurs, on sait également montrer que si, pour une constante $a_0 > 0$, on a $a \geqslant a_0$, presque partout sur un sous-ensemble $I \subset (0,1)$ de mesure non nulle, alors l'énergie tend vers 0 (Haraux [95]). Ce terme permet donc de stabiliser l'énergie des solutions. Supposons maintenant que l'on soit capable d'établir l'*inégalité d'observabilité* suivante :

$$E(0) - E(T) \geqslant CE(0),$$

pour un temps $T > 0$ et une constante $C > 0$. Alors il est bien connu que dans ce cas, on a la décroissance exponentielle des solutions,

$$\forall t \geqslant 0,\ E(t) \leqslant CE(0)e^{-\omega t},$$

pour des constantes $C, \omega > 0$.

Considérons maintenant l'équation des ondes avec un amortissement plus faible,

$$u_{tt} - u_{xx} + \delta_a u_t(t,a) = 0, \quad (t,x) \in (0,\infty)\times(0,1), \tag{14}$$

où $a \in (0,1)$. L'inégalité d'observabilité que l'on peut obtenir est alors,

$$E(0) - E(T) \geqslant CE_-(0), \tag{15}$$

où E_- est une énergie faible, dans le sens où $E_- \leqslant E$. L'idée est de prendre les données initiales dans un espace plus réguliers, typiquement dans $[H^2(0,1) \cap H_0^1(0,1)] \times H_0^1(0,1)$. On a alors l'énergie forte associée E_+ et $E_-(0) \leqslant E(0) \leqslant E_+(0)$. On aimerait donc interpoler $E(0)$ entre $E_-(0)$ et $E_+(0)$ à l'aide d'un inégalité généralisée de Hölder du type,

$$1 \leqslant \Phi\left(\frac{E_-(0)}{E(0)}\right)\Psi\left(\frac{E_+(0)}{E(0)}\right),$$

de laquelle on obtiendrait,

$$E(0)\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{E_+(0)}{E(0)}\right)}\right) \leqslant E_-(0). \tag{16}$$

De (15) et (16), on aurait alors,

$$E(0)\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{E_+(0)}{E(0)}\right)}\right) \leqslant C(E(0) - E(T)).$$

De là, on serait capable d'obtenir la vitesse de convergence (Ammari and Tucsnak [8]),

$$\forall t \geqslant 0,\ E(t) \leqslant \frac{C}{\Psi^{-1}\left(\frac{1}{\Phi\left(\frac{1}{t+1}\right)}\right)}\|(u^0,u^1)\|^2_{H^2(0,1)\times H_0^1(0,1)}.$$

Dès lors que Φ etΨ peuvent être déterminées, la vitesse est explicite.

Les énergies faibles et fortes sont reliées à l'énergie d'origine par des poids ω_1 et ω_2. On fait les hypothèses suivantes. Soient $(\Omega, \mathscr{T}, \mu)$ un espace mesuré et $\omega_1, \omega_2 : \Omega \longrightarrow [0,\infty)$ deux poids μ–mesurables. On suppose qu'il existe deux fonctions concaves $\Phi, \Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ telles que pour presque tout $x \in \Omega$,

$$\Phi(\omega_1(x))\Psi(\omega_2(x)) \geqslant 1. \tag{17}$$

À l'aide de l'inégalité de Jensen, on peut alors démontrer le résultat suivant (Théorème 7.2.1).

Théorème 16. *Avec les hypothèses et notations ci-dessus, on a pour tout* $0 < p < \infty$ *et toute fonction* $f \in L^p(\Omega, \mathscr{T}, \mu)$ *non nulle,*

$$1 \leqslant \Phi\left(\frac{\displaystyle\int_\Omega |f|^p \omega_1 \mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right)\Psi\left(\frac{\displaystyle\int_\Omega |f|^p \omega_2 \mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right), \tag{18}$$

dès lors que $f \in L^p(\Omega, \mathscr{T}, \omega_1\mathrm{d}\mu) \cap L^p(\Omega, \mathscr{T}, \omega_2\mathrm{d}\mu)$.

Dans les applications, $\Omega = [1, +\infty)$, μ est la mesure de Lebesgue et les poids vérifient, entre autres, des hypothèses de convexité. On peut alors montrer l'existence de fonctions Φ et Ψ satisfaisant (18) (Théorème 7.2.2 et Lemme 7.2.6).

Théorème 17. *Soient* $\omega_1, \omega_2 : [1, +\infty) \longrightarrow [0, \infty)$ *deux poids convexes. On suppose que* ω_1 *est strictement décroissante avec* $\lim_{t \to +\infty} \omega_1(t) = 0$, *et que* ω_2 *est strictement croissante avec* $\lim_{t \to +\infty} \omega_2(t) = +\infty$. *On pose,*

$$\forall t \geqslant 1,\ \varphi(t) = \frac{\omega_1(t)}{t},$$

et,

$$\Phi = \frac{1}{\varphi^{-1}} \quad et \quad \Psi = \omega_2^{-1}.$$

Alors, $\Phi, \Psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *sont des fonctions concaves qui vérifient l'inégalité* (17). *En particulier,* Φ *et* Ψ *satisfont l'inégalité* (18).

La vitesse de convergence des solutions de (14) est connue pour $a \in \mathcal{S}$, où $\mathcal{S}$ est un sous-ensemble distinct de $(0,1) \cap \mathbb{Q}^c$, et ce résultat est dû Jaffard, Tucsnak and Zuazua [110, Theorem 3.3]). À l'aide de la méthode ci-dessus, on étend ces résultats à $(0,1) \cap \mathbb{Q}^c$ (Propositions 7.4.2.3 et 7.4.3.4, et Théorèmes 7.4.3.5 et 7.4.3.7), ce qui est optimal puisqu'il est connu que pour $(0,1) \cap \mathbb{Q}$, il existe des solutions dont l'énergie ne tend pas vers 0.

Chapitre 8 : Sur des systèmes gradients amortis ([22, 21])

Dans le chapitre, on souhaite étudier le comportement asymptotique des solutions d'un système gradient amorti du type “boule pesante”,

$$u''(t) + \gamma u'(t) + \nabla G(u(t)) = 0, \quad t \in \mathbb{R}_+. \tag{19}$$

On aimerait également faire le lien entre ce système et les systèmes quasi-gradients,

$$u'(t) + F\big(u(t)\big) = 0,\ t \in \mathbb{R}_+, \tag{20}$$

qui sont *a priori* de nature totalement différente. Ici, $\gamma > 0$ et $G \in C^2(\mathbb{R}^N; \mathbb{R})$. On rappelle que le système (20) est dit *quasi-gradient* sur un sous-ensemble fermé Γ de $\mathbb{R}^N$, s'il existe une fonction différentiable $E : \mathbb{R}^N \longrightarrow \mathbb{R}$ et $\alpha > 0$ tels que,

$$\big\langle \nabla E(u), F(u) \big\rangle \geqslant \alpha \, \|\nabla E(u)\| \, \|F(u)\|, \text{ pour tout } u \in \Gamma, \tag{21}$$

$$\operatorname{crit} E \cap \Gamma = F^{-1}(\{0\}) \cap \Gamma, \tag{22}$$

où $\operatorname{crit} E$ désigne l'ensemble des points critiques de E.

Le premier outil est l'inégalité de Kurdyka-Łojasiewicz ([133, 134, 124]). On dit alors que G est une *fonction KL en* $\overline{u} \in \mathbb{R}^N$ s'il existe une fonction concave $\varphi : [0, r_0) \longrightarrow \mathbb{R}_+$, dite *désingularisante*, telle que $\varphi(0) = 0$, $\varphi \in C([0, r_0)) \cap C^1(0, r_0)$, $\varphi' > 0$ sur $(0, r_0)$ et

$$\big\|\nabla(\varphi \circ |G(\,.\,) - G(\overline{u})|)(u)\big\| \geqslant 1,$$

pour tout u dans un voisinage de $\overline{u}$. Par exemple, les fonctions analytiques sont des fonctions KL. L'hypothèse KL permet d'assurer la convergence des solutions bornées des systèmes gradients (c'est-à-dire (20) avec $F = \nabla E$). En fait, on a le même résultat pour les systèmes quasi-gradients, comme le montre le résultat suivant (Théorème 8.3.1.2).

Théorème 18. *Soit* $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *une fonction localement Lipschitzienne définissant un champ de vecteurs quasi-gradient différentiable* E *sur* $\mathbb{R}^N$. *On suppose que* E *est une fonction KL. Soit* u *une solution de* (20). *Alors,*

1. *ou bien* $\|u(t)\| \xrightarrow{t\to\infty} \infty$,
2. *ou bien* u *converge vers un point singulier* u_∞ *de* F *lorsque* $t \longrightarrow \infty$.

Dans ce dernier cas, on a également $u' \in L^1\big((0,\infty);\mathbb{R}^N\big)$, $u'(t) \xrightarrow{t\to\infty} 0$ *et*

$$\forall t \geqslant 0,\ \|u(t) - u_\infty\| \leqslant \frac{1}{\alpha}\varphi\big(E(u(t)) - E(u_\infty)\big),$$

pour n'importe quelle fonction désingularisante φ *de* E *en* u_∞, *où* α *est la constante dans* (21).

Par ailleurs, les fonctions désingularisantes ont une vitesse d'explosion minimale à l'origine (Proposition 8.2.1.3 et Lemme 8.2.2.1).

Lemme 19. *Soit* $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ *une fonction analytique telle que* $G(0) = 0$, $\nabla G(0) = 0$ *et* 0 *n'est pas dans l'intérieur* $\operatorname{int}\operatorname{crit} G$ *de l'ensemble des points critiques de* G. *Puisque* G *est analytique, elle est KL. Soit alors* φ *une fonction désingularisante. Alors, il existe deux constantes* $c, \varepsilon > 0$ *telles que,*

$$\varphi'(t) \geqslant \frac{c}{\sqrt{t}}, \tag{23}$$

pour tout $t \in (0, \varepsilon)$.

Pour $G \in C^2(\mathbb{R}^N;\mathbb{R})$, on définit $\mathcal{F} : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ par

$$\mathcal{F}(u,v) = \big(-v,\ \gamma v + \nabla G(u)\big).$$

Alors le système (19) est équivalent à

$$U'(t) + \mathcal{F}\big(U(t)\big) = 0, \quad t \in \mathbb{R}_+, \ \text{ avec } \ U = (u,v).$$

Et finalement, en déformant l'énergie totale $E_T(u,v) \overset{\text{déf}}{=} \frac{1}{2}\|v\|^2 + G(u)$ du système (19), il s'avère que les systèmes gradients du second ordre sont des systèmes quasi-gradients (Propositions 8.3.3.1 et 8.3.3.3).

Proposition 20. *Soient* $G \in C^2(\mathbb{R}^N;\mathbb{R})$ *et* $\gamma > 0$. *Pour chaque* $\lambda > 0$, *on définit l'énergie déformée* $\mathcal{E}_\lambda \in C^1(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R})$ *par*

$$\mathcal{E}_\lambda(u,v) = E_T(u,v) + \lambda\langle \nabla G(u), v\rangle,$$

où $\langle\ .\ ,\ .\ \rangle$ *désigne le produit scalaire dans* $\mathbb{R}^N$. *Alors pour chaque* $R > 0$, *il existe* $\lambda_0 > 0$ *satisfaisant la propriété suivante. Pour tout* $\lambda \in (0, \lambda_0]$, *il existe* $\alpha > 0$ *tel que*

$$\big\langle \nabla\mathcal{E}_\lambda(u,v), \mathcal{F}(u,v)\big\rangle \geqslant \alpha\,\|\nabla\mathcal{E}_\lambda(u,v)\|\,\|\mathcal{F}(u,v)\|, \tag{24}$$

pour tout $(u,v) \in \overline{B}(0,R) \times \mathbb{R}^N$. *De plus,*

$$\operatorname{crit}\mathcal{E}_\lambda \cap \big(\overline{B}(0,R) \times \mathbb{R}^N\big) = \mathcal{F}^{-1}(\{0\}) \cap \big(\overline{B}(0,R) \times \mathbb{R}^N\big), \tag{25}$$

pour tout $\lambda \in [0, \lambda_0]$.
Enfin, si φ *vérifiant* (23) *désingularise* G *en* $\overline{u} \in \operatorname{crit} G$ *alors* φ *désingularise* $\mathcal{E}_\lambda$ *en* $(\overline{u}, 0)$, *pour tout* $\lambda \geqslant 0$ *suffisamment petit.*

Ainsi, grâce au Théorème 18 et la Proposition 20, nous sommes en mesure de déterminer la vitesse de convergence des solutions (Théorème 4.1.2). Par ailleurs, ces résultats étendent ceux de Haraux et Jendoubi [97].

Théorème 21. *Soient $G \in C^2(\mathbb{R}^N;\mathbb{R})$ et $(u_0,u_0') \in \mathbb{R}^N \times \mathbb{R}^N$. Soit $u \in C^2\big([0,\infty);\mathbb{R}^N\big)$ l'unique solution de (19) telle que $(u(0),u'(0)) = (u_0,u_0')$. On suppose que l'on a les hypothèses ci-dessous.*

1. *La trajectoire de u est bornée*: $\sup\limits_{t>0} \|u(t)\| < \infty$.
2. *G est une fonction KL et chaque fonction désingularisante vérifie (23).*

Alors on a les résultats suivants.

1. *$u' \in L^1\big((0,\infty);\mathbb{R}^N\big)$, $u'' \in L^1\big((0,\infty);\mathbb{R}^N\big)$ et u converge vers un point critique u_∞ de G.*
2. *Soit φ désingularisant G en u_∞. Alors il existe une constante $c > 0$ telle que pour tout $t \geqslant 0$,*

$$\|u(t) - u_\infty\| \leqslant c\nu(t),$$

où ν est solution de,

$$\nu'(t) + (\varphi^{-1})'(\nu(t)) = 0,$$

avec $\nu(0) > 0$.

Chapitre 9 : Phénomène de concentration de masse pour l'équation de Schrödinger non-linéaire dans le cas critique ([32])

Dans ce chapitre, on s'intéresse au comportement des solutions de l'équation (1) dans le cas critique pour la masse. On montre que si le temps d'existence est fini alors un phénomène de concentration de masse se produit. Plus précisément, on a le résultat suivant (Théorème 9.1.1).

Théorème 22. *Soient $a \in \mathbb{R} \setminus \{0\}$, $m = 1 + \dfrac{4}{N}$, $f = 0$, $u_0 \in L^2(\mathbb{R}^N) \setminus \{0\}$ et*

$$u \in C((-T_{\min},T_{\max});L^2(\mathbb{R}^N)) \cap L^{\frac{2(N+2)}{N}}_{\text{loc}}((-T_{\min},T_{\max});L^{\frac{2(N+2)}{N}}(\mathbb{R}^N)),$$

l'unique solution maximale de (1) telle que $u(0) = u_0$. Il existe $\varepsilon = \varepsilon(\|u_0\|_{L^2},N,|a|) > 0$ satisfaisant la propriété suivante. Si $T_{\max} < \infty$ alors

$$\limsup_{t \nearrow T_{\max}} \sup_{c\in\mathbb{R}^N} \int_{B(c,(T_{\max}-t)^{\frac{1}{2}})} |u(t,x)|^2 \mathrm{d}x \geqslant \varepsilon.$$

On un résultat analogue lorsque $T_{\min} < \infty$.

Le Théorème 22 a été établi dans le cas particulier de la dimension $N = 2$ par Bourgain [42]. Sa démonstration repose sur une inégalité de Strichartz plus fine démontrée en dimension 2 par Moyua, Vargas and Vega [139]. Pour démontrer le Théorème 22, on commence donc par généraliser l'inégalité améliorée de Strichartz pour n'importe quelle dimension (Théorème 9.1.2). L'outil majeur pour y parvenir est une inégalité de restriction bilinéaire due à Tao [166] (Théorème 9.2.1).

Pour chaque $j \in \mathbb{Z}$, on recouvre $\mathbb{R}^N$ de cubes dyadiques $\tau_k^j = \prod_{m=1}^{N} [k_m 2^{-j}, (k_m+1)2^{-j})$, où $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$. On pose : $f_k^j(x) = f \mathbb{1}_{\tau_k^j}(x)$. Soient $1 \leqslant p, q < \infty$. On définit l'espace suivant :

$$X_{p,q} = \Big\{ f \in L^p_{\text{loc}}(\mathbb{R}^N);\ \|f\|_{X_{p,q}} < \infty \Big\},$$

où

$$\|f\|_{X_{p,q}} = \left[\sum_{j \in \mathbb{Z}} 2^{j \frac{N}{2} \frac{2-p}{p} q} \sum_{k \in \mathbb{Z}^N} \|f_k^j\|^q_{L^p(\mathbb{R}^N)} \right]^{\frac{1}{q}}.$$

On vérifie que $(X_{p,q}, \| \, . \, \|_{X_{p,q}})$ est bien un espace de Banach. L'inégalité améliorée de Strichartz est la suivante (Théorèmes 9.1.2 et 9.1.4).

Théorème 23. *Soient $q = \frac{2(N+2)}{N}$ et $1 < p < 2$ tels que $\frac{N+3}{N+1} \frac{1}{q} + \frac{1}{p} < 1$. Pour chaque fonction g telle que $g \in X_{p,q}$ ou $\widehat{g} \in X_{p,q}$, on a*

$$\|\mathcal{T}(\, . \,)g\|_{L^q(\mathbb{R}^{N+1})} \leqslant C \min \left\{ \|g\|_{X_{p,q}}, \|\widehat{g}\|_{X_{p,q}} \right\}, \tag{26}$$

où $C = C(N,p)$. De plus, $L^2(\mathbb{R}^N) \hookrightarrow X_{p,q}$ avec $L^2(\mathbb{R}^N) \neq X_{p,q}$. Finalement, il existe $\mu \in \left(0, \frac{1}{p}\right)$ tel que pour tout $g \in L^2(\mathbb{R}^N)$, on ait

$$\|\mathcal{T}(\, . \,)g\|_{L^q(\mathbb{R}^{N+1})} \leqslant C \left[\sup_{(j,k) \in \mathbb{Z} \times \mathbb{Z}^N} 2^{j \frac{N}{2}(2-p)} \int_{\tau_k^j} |\widehat{g}(\xi)|^p \mathrm{d}\xi \right]^{\mu} \|g\|^{1-\mu p}_{L^2(\mathbb{R}^N)} \leqslant C \|g\|_{L^2(\mathbb{R}^N)}, \tag{27}$$

où $C = C(N,p)$ et $\mu = \mu(N,p)$.

Ensuite, grosso modo, grâce à (27), on se ramène à étudier le comportement d'un nombre fini de solutions de l'équation libre de Schrödinger [4] (Lemmes 9.3.1 et 9.3.3) et l'on démontre le Théorème 22.

4. c'est-à-dire (1) avec $a = 0$, et toujours $f \equiv 0$.

Chapitre 1

Localizing Estimates of the Support of Solutions of some Nonlinear Schrödinger Equations – The Stationary Case

with Jesús Ildefonso Díaz*

Abstract

The main goal of this paper is to study the nature of the support of the solution of suitable nonlinear Schrödinger equations, mainly the compactness of the support and its spatial localization. This question touches the very foundations underlying the derivation of the Schrödinger equation, since it is well-known a solution of a linear Schrödinger equation perturbed by a regular potential never vanishes on a set of positive measure. A fact, which reflects the impossibility of locating the particle. Here we shall prove that if the perturbation involves suitable singular nonlinear terms then the support of the solution is a compact set, and so any estimate on its spatial localization implies very rich information on places not accessible by the particle. Our results are obtained by the application of certain energy methods which connect the compactness of the support with the local vanishing of a suitable "energy function" which satisfies a nonlinear differential inequality with an exponent less than one. The results improve and extend a previous short presentation by the authors published in 2006.

1.1 Introduction

This paper deals with the study of the following stationary nonlinear Schrödinger equation (SNLS) with a complex singular potential

$$-\mathbf{i}\Delta \boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u} = \boldsymbol{F}(x), \text{ in } \Omega. \tag{1.1.1}$$

Here, $\Omega \subseteq \mathbb{R}^N$ is an open subset, $0 < m < 1$, and $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$. The interest of the consideration of this stationary problem is motivated not only in order to study the asymptotic states, when $t \longrightarrow \infty$, of

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the solutions of the associated evolution problem but also by the study of the so called standing waves of the evolution problem (1.1.2) below, with $\boldsymbol{b} \in \mathbf{i}\mathbb{R}$ in (1.1.1). Indeed, choosing arbitrarily $\boldsymbol{b} \in \mathbf{i}\mathbb{R}$ in (1.1.1) and setting for any $(t,x) \in \mathbb{R} \times \Omega$, $\boldsymbol{\varphi}(t,x) = \boldsymbol{u}(x)e^{\boldsymbol{b}t}$, if $\boldsymbol{u}$ is a solution to (1.1.1) then $\boldsymbol{\varphi}$ is a solution to

$$\begin{cases} \mathbf{i}\dfrac{\partial \boldsymbol{\varphi}}{\partial t} + \Delta \boldsymbol{\varphi} + \mathbf{i}\boldsymbol{a}|\boldsymbol{\varphi}|^{-(1-m)}\boldsymbol{\varphi} = \mathbf{i}\boldsymbol{F}(x)e^{\boldsymbol{b}t}, \text{ in } \mathbb{R} \times \Omega, \\ \boldsymbol{\varphi}_{|\partial\Omega} = \mathbf{0}, \text{ on } \mathbb{R} \times \partial\Omega, \\ \boldsymbol{\varphi}(0) = \boldsymbol{u}, \text{ in } \Omega. \end{cases} \tag{1.1.2}$$

The main goal of this paper is to study the nature of the support of the solution of (1.1.1) : mainly its compactness and localization. Let us mention that, in our opinion, this question touches the very foundations of the derivation of the Schrödinger equation. Indeed, one of the main modifications introduced by Quantum Mechanics, with respect Classical Mechanics, is the impossibility to localize the state (position and velocity) of a particle. The solution $\boldsymbol{u}(t,x)$ is related to the probability of finding the position and momentum of particle (see, e.g. the presentation made in the text book by Strauss [163]. It is well-known that in most of the different versions of the Schrödinger equations the corresponding solution never vanishes on a subset positive measure of the domain, which reflects the impossibility of localizing the particle as mentioned above. This is the case, for instance, in case of the linear Schrödinger equation and also for some nonlinear versions where the linear equation is perturbed by a nonlinear regular potential (see, for instance, the monographs of Sulem and Sulem [165] and Cazenave [57]).

The main goal of this work is to show that if the linear Schrödinger equation is perturbed with suitable singular nonlinear potentials, then the support of the solution becomes a compact set and so any estimate on its spatial localization implies very rich information on places which can not be occupied by the particle.

We point out that complex potentials with certain types of singularities arise in many different situations (see, for instance, in Brezis and Kato [47], LeMesurier [127] and Liskevitch and Stollmann [131], and the references therein). We also refer the reader to the survey Belmonte-Beitia [33] in which the author supplying many references to this type of equation and many other contexts such as : semiconductors, nonlinear optics, Bose-Einstein condensation, plasma physics, molecular dynamics. Special mention is paid in this paper to the so-called Gross-Pitaevskii (corresponding to $\boldsymbol{b} \neq \mathbf{0}$).

In this paper, we improve some of our previous results, outlined briefly in Bégout and Díaz [24]. Moreover, we include here new estimates and generalizations. We are aware of very few other results in the literature dealing with the support of solutions of nonlinear Schrödinger equations. For instance, Rosenau and Schochet [155] propose a (one-dimensional) quasilinear Schrödinger equation in order to get solutions with compact support for each t fixed. That equation and the techniques used in that paper are very different from the ones in the present work. Analogously, in a paper dated from 2008 ([116]), Kashdan and Rosenau consider the question of the existence (with some numerical experiments) of some special solutions : an one-dimensional travelling wave solution of soliton type $\boldsymbol{u}(t,x) = A(x-\lambda t)\exp\big(\mathbf{i}(\ell(x-\lambda t)+\omega t)\big)$, for the special case of $\boldsymbol{a} = \mathbf{i}\gamma$ (in problem (1.1.2)) and $m \in (0,1)$. They also consider the two-dimensional case (now with changing propagation directions). A nonlinear term (of cubic type) is added in their equation. Those interesting results are independent of our study which also applies in the presence of some additional nonlinear terms as in the above

mentioned reference.
A more restricted point of view was taken in the paper by Carles and Gallo [53] where the authors prove finite time stabilization for a linear Schrödinger equations perturbed with a suitable singular nonlinear potential. In their setting, they also prove some kind of compactness of the support of the solution by means of a different energy method, but in their case the compactness occurs merely in time and not in the spatial coordinates.
We also point out that different propagation effects have been intensively studied in the literature, but most of them are related to singularities, spectral and other properties (see, for instance, Jensen [115]). The question of the compactness of the support considered here is of very different nature.
In order to present our results, we shall start by indicating some very special cases which are consequences of more technical results stated later (see Theorem 1.2.1 below).

Theorem 1.1.1. *Let $0<m<1$, let $a\in\mathbb{R}\backslash\{0\}$ and let $b\in\mathbb{R}$, $b>0$. Let $\boldsymbol{F}\in\boldsymbol{L}^{\frac{m+1}{m}}(\mathbb{R}^N)$ with compact support. Then there exists a unique weak solution $\boldsymbol{u}\in\boldsymbol{H}^{\boldsymbol{1}}(\mathbb{R}^N)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\mathbb{R}^N)$ (see Definition 1.2.3 below) of the problem*

$$-\mathbf{i}\Delta\boldsymbol{u}+a|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}+\mathbf{i}b\boldsymbol{u}=\boldsymbol{F}(x),\ \text{in } \mathbb{R}^N.$$

In addition, $\boldsymbol{u}$ is compactly supported.

Theorem 1.1.2. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $a\in\mathbb{R}\setminus\{0\}$ and let $b\in\mathbb{R}$, $b>0$. Let $\boldsymbol{F}\in\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$ with compact support. Assume that $\boldsymbol{F}$ is small enough in $\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$. Then there exists a unique weak solution $\boldsymbol{u}\in\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$ (see Definition 1.2.3 below) of the problem*

$$\begin{cases}-\mathbf{i}\Delta\boldsymbol{u}+a|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}+\mathbf{i}b\boldsymbol{u}=\boldsymbol{F}(x),\ \text{in } \Omega,\\ \qquad\qquad\qquad\qquad\qquad\quad \boldsymbol{u}_{|\partial\Omega}=\boldsymbol{0},\ \text{on } \partial\Omega.\end{cases}$$

In addition, $\boldsymbol{u}$ is compactly supported in Ω.

We emphasize that no sign assumption has been made on a in the precedent statements. Much more general versions of our results are presented in the next section where we also include a detailed explanation of the notations used in this paper.

1.2 Notations and general versions of the main results

Before stating our main results we shall indicate here some of the notations used throughout. Bold symbols are used for complex mathematics objets. For a real number r, $r_+=\max\{0,r\}$ is the positive part of r. We write $\mathbf{i}^2=-1$. We denote by $\overline{\boldsymbol{z}}$ the conjugate of the complex number $\boldsymbol{z}$, by $\mathrm{Re}(\boldsymbol{z})$ its real part and by $\mathrm{Im}(\boldsymbol{z})$ its imaginary part. For $1\leqslant p\leqslant\infty$, p' is the conjugate of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. Let $j,k\in\mathbb{Z}$ with $j<k$. We then write $[\![j,k]\!]=[j,k]\cap\mathbb{Z}$. We denote by $\partial\Omega$ the boundary of a nonempty subset $\Omega\subseteq\mathbb{R}^N$, $\overline{\Omega}$ its closure, $\Omega^{\mathrm{c}}=\mathbb{R}^N\setminus\Omega$ its complement and $\omega\Subset\Omega$ means that $\omega\subset\Omega$ and that ω is a compact subset of $\mathbb{R}^N$. For an open subset $\Omega\subseteq\mathbb{R}^N$, the usual Lebesgue and Sobolev spaces are respectively denoted by $\boldsymbol{L}^{\boldsymbol{p}}(\Omega)=\boldsymbol{L}^{\boldsymbol{p}}(\Omega;\mathbb{C})$ and $\boldsymbol{W}^{\boldsymbol{m,p}}(\Omega)=\boldsymbol{W}^{\boldsymbol{m,p}}(\Omega;\mathbb{C})$ ($1\leqslant p\leqslant\infty$ and $m\in\mathbb{N}$), $\boldsymbol{H}^{\boldsymbol{m}}(\Omega)=\boldsymbol{W}^{\boldsymbol{m,2}}(\Omega;\mathbb{C})$, $\boldsymbol{H}^{\boldsymbol{m}}_{\boldsymbol{0}}(\Omega)=\boldsymbol{W}^{\boldsymbol{m,2}}_{\boldsymbol{0}}(\Omega;\mathbb{C})$ is the closure of $\mathscr{D}(\Omega)=\mathscr{D}(\Omega;\mathbb{C})$

under the $\boldsymbol{H^m}$-norm, and $\boldsymbol{H^{-m}}(\Omega)$ is its topological dual. $\boldsymbol{H^1_c}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{H^1}(\Omega); \operatorname{supp} \boldsymbol{u} \Subset \Omega\right\}$. $\boldsymbol{C}(\Omega) = \boldsymbol{C^0}(\Omega) = \boldsymbol{C}(\Omega;\mathbb{C}) = \boldsymbol{C^0}(\Omega;\mathbb{C})$ is the space of continuous functions from Ω to $\mathbb{C}$. For $k \in \mathbb{N}$, $\boldsymbol{C^k}(\Omega) = \boldsymbol{C^k}(\Omega;\mathbb{C})$ is the space of functions lying in $\boldsymbol{C}(\Omega;\mathbb{C})$ and having all derivatives of order lesser or equal than k belonging to $\boldsymbol{C}(\Omega;\mathbb{C})$. For $0 < \alpha \leqslant 1$ and $k \in \mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$, $\boldsymbol{C^{k,\alpha}_{\text{loc}}}(\Omega) = \boldsymbol{C^{k,\alpha}_{\text{loc}}}(\Omega;\mathbb{C}) = \left\{\boldsymbol{u} \in \boldsymbol{C^k}(\Omega;\mathbb{C}); \forall \omega \Subset \Omega,\ \sum\limits_{|\beta|=k} H^\alpha_\omega(D^\beta \boldsymbol{u}) < +\infty\right\}$, where $H^\alpha_\omega(\boldsymbol{u}) = \sup\limits_{\left\{\substack{(x,y)\in\omega^2 \\ x\neq y}\right.} \frac{|\boldsymbol{u}(x)-\boldsymbol{u}(y)|}{|x-y|^\alpha}$. The Laplacian in Ω is written $\Delta = \sum\limits_{j=1}^N \frac{\partial^2}{\partial x_j^2}$. For a functional space $\boldsymbol{E} \subset \boldsymbol{L^1_{\text{loc}}}(\Omega;\mathbb{C})$, we denote by $\boldsymbol{E_{\text{rad}}}$ the space of functions $\boldsymbol{f} \in \boldsymbol{E}$ such that $\boldsymbol{f}$ is spherically symmetric. For a Banach space E, we denote by $E^\star$ its topological dual and by $\langle\ .\ ,\ .\ \rangle_{E^\star,E} \in \mathbb{R}$ the $E^\star - E$ duality product. In particular, for any $\boldsymbol{T} \in \boldsymbol{L^{p'}}(\Omega)$ and $\boldsymbol{\varphi} \in \boldsymbol{L^p}(\Omega)$ with $1 \leqslant p < \infty$, $\langle \boldsymbol{T}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^{p'}}(\Omega),\boldsymbol{L^p}(\Omega)} = \operatorname{Re}\int\limits_\Omega \boldsymbol{T}(x)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x$. For $x_0 \in \mathbb{R}^N$ and $r > 0$, we denote by $B(x_0,r) = \{x \in \mathbb{R}^N;\ |x-x_0| < r\}$ the open ball of $\mathbb{R}^N$ of center x_0 and radius r, by $\mathbb{S}(x_0,r) = \{x \in \mathbb{R}^N;\ |x-x_0| = r\}$ its boundary and by $\overline{B}(x_0,r) = B(x_0,r) \cup \mathbb{S}(x_0,r)$ its closure. We also use the notation $B_\Omega(x_0,r) = \Omega \cap B(x_0,r)$. As usual, we denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1,\ldots,a_n$, write $C(a_1,\ldots,a_n)$ to indicate that the constant C continuously depends only on $a_1,\ldots,a_n$ (this convention also holds for constants which are not denoted by "C").

Let us return to equation (1.1.2). Note that no boundary condition is imposed since all the compact support results (which are due to Theorem 1.2.1 below) rest on the notion of local solution (Definition 1.2.3 below). If $\Omega \neq \mathbb{R}^N$, boundary conditions are necessary for establishing existence and uniqueness of global solutions of (1.1.1). For the purpose of clarity, we shall consider the Dirichlet case,

$$\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}, \text{ on } \partial\Omega, \tag{1.2.1}$$

rather than Neumann boundary condition, mixed boundary condition or another one. The choice of the boundary condition is motivated by the integration by parts relation $\langle \Delta u, v\rangle = -\langle \nabla u, \nabla v\rangle$.

Compactness, existence and uniqueness results will follow from assumptions on $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ stated below. Define the following subsets

$$\begin{cases} \mathbb{A} = \mathbb{C} \setminus \left\{\boldsymbol{z} \in \mathbb{C}; \operatorname{Re}(\boldsymbol{z}) = 0 \text{ and } \operatorname{Im}(\boldsymbol{z}) \leqslant 0\right\}, \\ \mathbb{B} = \mathbb{A} \cup \left\{\boldsymbol{0}\right\}. \end{cases}$$

Existence assumption. Let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfy

$$(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A} \times \mathbb{B} \quad \text{and} \quad \begin{cases} \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b}) \geqslant 0, \\ \text{or} \\ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b}) < 0 \text{ and } \operatorname{Im}(\boldsymbol{b}) > \dfrac{\operatorname{Re}(\boldsymbol{b})}{\operatorname{Re}(\boldsymbol{a})}\operatorname{Im}(\boldsymbol{a}). \end{cases} \tag{1.2.2}$$

Uniqueness assumption. Let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfy

$$\operatorname{Im}(\boldsymbol{a})\geqslant 0 \quad \text{and} \quad \begin{cases} \boldsymbol{a}\neq\boldsymbol{0} \ \text{ and } \ \operatorname{Re}(\boldsymbol{a}\overline{\boldsymbol{b}})\geqslant 0,\\ \text{or}\\ \boldsymbol{a}=\boldsymbol{0} \ \text{ and } \ \boldsymbol{b}\in\mathbb{B}. \end{cases} \tag{1.2.3}$$

For a geometric explanation of these hypotheses, see Section 1.6. For $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2), it will be convenient to introduce the following constants. Let $\delta>0$ be an arbitrarily chosen parameter.

$$A(\delta)=\frac{|\operatorname{Re}(\boldsymbol{a})|+|\operatorname{Im}(\boldsymbol{a})|+\delta}{|\operatorname{Re}(\boldsymbol{a})|}, \ \text{if } \operatorname{Re}(\boldsymbol{a})\neq 0, \tag{1.2.4}$$

$$B=\frac{|\operatorname{Re}(\boldsymbol{b})|+|\operatorname{Im}(\boldsymbol{b})|}{|\operatorname{Re}(\boldsymbol{b})|}, \ \text{if } \operatorname{Re}(\boldsymbol{b})\neq 0, \tag{1.2.5}$$

$$L=\begin{cases} \delta, & \text{if } \operatorname{Im}(\boldsymbol{a})<0 \ \text{ and } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0,\\ |\operatorname{Re}(\boldsymbol{a})|, & \text{if } \operatorname{Im}(\boldsymbol{a})=0, \ \operatorname{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0,\\ \operatorname{Im}(\boldsymbol{a}) & \text{if } \operatorname{Im}(\boldsymbol{a})>0 \ \text{ and } \ \operatorname{Im}(\boldsymbol{b})\geqslant 0,\\ \operatorname{Im}(\boldsymbol{a})-\dfrac{\operatorname{Re}(\boldsymbol{a})}{\operatorname{Re}(\boldsymbol{b})}\operatorname{Im}(\boldsymbol{b}), & \text{otherwise,} \end{cases} \tag{1.2.6}$$

$$M=\begin{cases} \max\big\{A(\delta),B\big\}, & \text{if } \operatorname{Im}(\boldsymbol{a})<0, \ \operatorname{Im}(\boldsymbol{b})<0 \ \text{ and } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0,\\ A(\delta), & \text{if } \operatorname{Im}(\boldsymbol{a})<0, \ \operatorname{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0,\\ 2 & \text{if } \operatorname{Im}(\boldsymbol{a})\geqslant 0, \ \operatorname{Im}(\boldsymbol{b})\geqslant 0 \ \text{ and } \ \big(\operatorname{Im}(\boldsymbol{a})>0 \ \text{ or } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0\big),\\ B & \text{if } \big(\operatorname{Im}(\boldsymbol{a})\geqslant 0 \ \text{ and } \ \operatorname{Im}(\boldsymbol{b})<0\big) \ \text{ or } \ \operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})<0. \end{cases} \tag{1.2.7}$$

Under hypothesis (1.2.2), one easily checks that $A(\delta)$, B, L and M are well defined and positive. The parameter δ may seem very mysterious but, actually, it is not. In order to obtain the crucial estimate (1.7.7), we apply Lemma 1.7.3 to (1.7.8) and (1.7.9). The hard case $\operatorname{Im}(\boldsymbol{a})<0$ can be treated in the following way. If $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})>0$ then we add the assumption $\operatorname{Im}(\boldsymbol{b})>\frac{\operatorname{Re}(\boldsymbol{b})}{\operatorname{Re}(\boldsymbol{a})}\operatorname{Im}(\boldsymbol{a})$. But when $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\leqslant 0$, if we do not want make an additional assumption on $\boldsymbol{a}$ and $\boldsymbol{b}$, we have to introduce a positive parameter δ in order to obtain a positive coefficient $L=L(\delta)$ in front of $\|\boldsymbol{u}\|_{\boldsymbol{L}^{m+1}(B(x_0,\rho))}^{m+1}$ (played by C_2 in Lemma 1.7.3). If we do not introduce this parameter (that is, if we choose $\delta=0$) then we get $L=0$ in (1.7.7) and we loose the effect of the nonlinearity (see Cases 5 and 6 in the proof of Lemma 1.7.3).

Numerical computations of stationary solutions are done in Bégout and Torri [31], while the evolution case and self-similar solutions are studied in Bégout and Díaz [23, 26], respectively. In this paper, we prove the results stated in Bégout and Díaz [24] and add some generalizations. This paper is concerned with the propagation of the support of $\boldsymbol{F}$ to the solution $\boldsymbol{u}$, and all these results are a consequence of the following theorem.

Theorem 1.2.1. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2), let $L>0$ be given by (1.2.6) and let $M>0$ be given by (1.2.7). There exists $C=C(N,m)>0$ satisfying the following property. Let $\boldsymbol{F}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, let $\boldsymbol{u}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be any local weak solution of (1.1.1) (see Definition 1.2.3 below), let $x_0\in\Omega$ and let $\rho_0>0$. If $\rho_0>\operatorname{dist}(x_0,\partial\Omega)$ then assume further that $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)$. If $\boldsymbol{F}_{|B_\Omega(x_0,\rho_0)}\equiv\boldsymbol{0}$ then $\boldsymbol{u}_{|B_\Omega(x_0,\rho_{\max})}\equiv\boldsymbol{0}$, where*

$$\rho_{\max}^\nu=\left(\rho_0^\nu-CM^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{\rho_0^{\nu-1},1\right\}\times\min_{\tau\in\left(\frac{m+1}{2},1\right]}\left\{\frac{E(\rho_0)^{\gamma(\tau)}\max\{b(\rho_0)^{\mu(\tau)},b(\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right\}\right)_+,\tag{1.2.8}$$

and where for any $\tau\in\left(\frac{m+1}{2},1\right]$,

$$E(\rho_0)=\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(B_\Omega(x_0,\rho_0))},\quad b(\rho_0)=\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,\rho_0))},\quad \gamma(\tau)=\tfrac{2\tau-(1+m)}{k}\in(0,1),$$
$$\mu(\tau)=\tfrac{2(1-\tau)}{k},\quad \eta(\tau)=\tfrac{1-m}{1+m}-\gamma(\tau)>0,\quad k=2(1+m)+N(1-m),$$
$$\nu=\tfrac{k}{m+1}>2.$$

Remark 1.2.2. If the solution is too "large", it may happen that $\rho_{\max}=0$ and so the above result is not consistent. A sufficient condition to observe a localizing effect is that the solution is small enough, in a suitable sense. We give two results in this direction. The first one (Theorem 1.3.3) pertains to the size of the solution, while the second one is concerned with the size of the external source $\boldsymbol{F}$ (Theorem 1.3.5), which seems to be more natural. In addition, Theorem 1.3.5 says where the support of the solutions is localized with respect to the support of the external source $\boldsymbol{F}$.

Now, we state the precise notion of solution.

Definition 1.2.3. Let $\Omega\subseteq\mathbb{R}^N$ be an open subset, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$, let $0<m<1$ and let $\boldsymbol{F}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. We say that $\boldsymbol{u}$ is a *local weak solution* of (1.1.1) if $\boldsymbol{u}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ and if $\boldsymbol{u}$ is a solution of (1.1.1) in $\mathscr{D}'(\Omega)$, that is

$$\langle-\mathbf{i}\Delta\boldsymbol{u}+\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}+\boldsymbol{b}\boldsymbol{u},\boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}=\langle\boldsymbol{F},\boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},\tag{1.2.9}$$

for any $\boldsymbol{\varphi}\in\mathscr{D}(\Omega)$.
We say that $\boldsymbol{u}$ is a *global weak solution* of (1.1.1) and (1.2.1) if $\boldsymbol{u}$ is a local weak solution of (1.1.1) and if furthermore $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$.
Let $\boldsymbol{z}\in\mathbb{C}\setminus\{\boldsymbol{0}\}$. Since $\big||\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}\big|=|\boldsymbol{z}|^m$, it is understood that $\big||\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}\big|=0$ when $\boldsymbol{z}=\boldsymbol{0}$.

Remark 1.2.4. Here are some comments about Definition 1.2.3.

1. For a global weak solution $\boldsymbol{u}$ of (1.1.1) and (1.2.1), the boundary condition $\boldsymbol{u}_{|\partial\Omega}=0$ is included in the assumption $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)$. On the contrary, the notion of local weak solution does not consider any boundary condition.
2. When $\boldsymbol{u}$ is a local weak solution of (1.1.1), we have $\nabla\boldsymbol{u}\in\boldsymbol{L^2_{\mathrm{loc}}}(\Omega)$, $\boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}\in\boldsymbol{L}^{\frac{m+1}{m}}_{\mathbf{loc}}(\Omega)$ and $\boldsymbol{b}\boldsymbol{u}\in\boldsymbol{L^2_{\mathrm{loc}}}(\Omega)$. Then $\Delta\boldsymbol{u}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ and equation (1.1.1) makes sense in $\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. Furthermore,

$\boldsymbol{L}^{\frac{m+1}{m}}_{\mathbf{loc}}(\Omega) \subset \boldsymbol{L}^{\mathbf{2}}_{\mathbf{loc}}(\Omega)$ and $\mathscr{D}(\Omega)$ is dense in $\boldsymbol{H}^{\mathbf{1}}_{\mathbf{c}}(\Omega)$. It follows from Sobolev's embedding that if $\boldsymbol{u}$ is a local weak solution of (1.1.1) then

$$\operatorname{Re}\int_\Omega \mathrm{i}\nabla\boldsymbol{u}(x).\overline{\nabla\boldsymbol{\varphi}(x)}\mathrm{d}x + \operatorname{Re}\int_\Omega \Big(\boldsymbol{a}|\boldsymbol{u}(x)|^{-(1-m)}\boldsymbol{u}(x) + \boldsymbol{b}\boldsymbol{u}(x)\Big)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x = \operatorname{Re}\int_\Omega \boldsymbol{F}(x)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x, \quad (1.2.10)$$

for any $\boldsymbol{\varphi} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{c}}(\Omega)$ with either $\operatorname{supp}\boldsymbol{\varphi}\cap\operatorname{supp}\boldsymbol{F} = \emptyset$ or $\boldsymbol{F} \in \boldsymbol{L}^{\frac{p}{p-1}}_{\mathbf{loc}}(\Omega)$, for some $1 \leqslant p \leqslant \infty$ if $N = 1$, $1 \leqslant p < \infty$ if $N = 2$ or $1 \leqslant p \leqslant \frac{2N}{N-2}$, if $N \geqslant 3$. For example, $p = m + 1$ is always an admissible value.

3. In the same way, by density of $\mathscr{D}(\Omega)$ in $\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{p}}(\Omega)$, for any $1 \leqslant p < \infty$, and in $\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$, if $\boldsymbol{u}$ is a global weak solution of (1.1.1) and (1.2.1) then (1.2.10) holds for any $\boldsymbol{\varphi} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$ with either $\operatorname{supp}\boldsymbol{\varphi}\cap\operatorname{supp}\boldsymbol{F} = \emptyset$ or $\boldsymbol{\varphi} \in \boldsymbol{L}^{\boldsymbol{p}}(\Omega)$ and $\boldsymbol{F} \in \boldsymbol{L}^{\frac{p}{p-1}}(\Omega)$, for some $1 \leqslant p < \infty$. In particular, if p is as in 2. of this remark with additionally $p \geqslant m+1$, then in view of $\boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega) \hookrightarrow \boldsymbol{L}^{\boldsymbol{p}}(\Omega)$, equation (1.1.1) makes sense in $\boldsymbol{H}^{-\mathbf{1}}(\Omega) + \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$ and (1.2.10) holds for any $\boldsymbol{\varphi} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)\cap\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$.

1.3 Spatial localization property

Theorem 1.3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$ and let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.2.2). Let $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$, let $\boldsymbol{u} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$ be any local weak solution of (1.1.1) (Definition 1.2.3), let $x_0 \in \Omega$ and let $\rho_1 > 0$. If $\rho_1 > \operatorname{dist}(x_0,\partial\Omega)$ then assume further that $\boldsymbol{u} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$. Then there exist $E_\star > 0$ and $\varepsilon_\star > 0$ satisfying the following property. Let $\rho_0 \in (0,\rho_1)$. If $\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^{\mathbf{2}}(B_\Omega(x_0,\rho_1))} < E_\star$ and*

$$\forall\rho \in (0,\rho_1),\ \|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B_\Omega(x_0,\rho))} \leqslant \varepsilon_\star(\rho-\rho_0)^p_+, \quad (1.3.1)$$

where $p = \frac{2(1+m)+N(1-m)}{1-m} > N+2$, then $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)} \equiv \mathbf{0}$. In other words, with the notation of Theorem 1.2.1, $\rho_{\max} = \rho_0$.

Remark 1.3.2. We may estimate $E_\star$ and $\varepsilon_\star$ as

$$E_\star = E_\star\left(\|\boldsymbol{u}\|^{-1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho_1))},\rho_1,\frac{\rho_0}{\rho_1},\frac{L}{M},N,m\right),$$
$$\varepsilon_\star = \varepsilon_\star\left(\|\boldsymbol{u}\|^{-1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho_1))},\frac{\rho_0}{\rho_1},\frac{L}{M},N,m\right),$$

where $L > 0$ and $M > 0$ are given by (1.2.4) and (1.2.7), respectively. The dependence on $\frac{1}{\delta}$ means that for any value δ small enough, $E_\star$ and $\varepsilon_\star$ are bounded from below.
Note that $p = \frac{1}{\gamma(1)}$, where γ is the function defined in Theorem 1.2.1.

Theorem 1.3.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $0 < m < 1$, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.2.2), let $L > 0$ be given by (1.2.6) and let $M > 0$ be given by (1.2.7). There exists*

$C=C(N,m)>0$ satisfying the following property. Let $\boldsymbol{F}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, let $\boldsymbol{u}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be any local weak solution of (1.1.1) (Definition 1.2.3), let $x_0\in\Omega$ and let $\rho_0>0$. If $2\rho_0>\mathrm{dist}(x_0,\partial\Omega)$ then assume further that $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)$. Finally, suppose $\boldsymbol{F}_{|B_\Omega(x_0,2\rho_0)}\equiv\boldsymbol{0}$, $\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,2\rho_0))}\leqslant 1$ and one of the two estimates (1.3.2) or (1.3.3) below is satisfied.

$$\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(B_\Omega(x_0,2\rho_0))}^{\frac{2(1-m)}{k}}\leqslant C(2^\nu-1)(1-m)M^{-2}\min\left\{1,L^2\right\}\min\left\{\frac{1}{2},\rho_0\right\}^{\nu-1}\rho_0, \tag{1.3.2}$$

$$\begin{cases}\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(B_\Omega(x_0,2\rho_0))}\leqslant 1,\\ \|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(B_\Omega(x_0,2\rho_0))}^{\frac{2s(m+1)}{k}}\leqslant C(2^\nu-1)(1-m-2s)M^{-2}\min\left\{1,L^2\right\}\min\left\{\frac{1}{2},\rho_0\right\}^{\nu-1}\rho_0,\end{cases} \tag{1.3.3}$$

for some $s\in\left(0,\frac{1-m}{2}\right)$, where the constants $k>\nu>2$ are given in Theorem 1.2.1. Then $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)}\equiv$ $\boldsymbol{0}$.

Remark 1.3.4. Note that in estimate (1.3.2), $\frac{2(1-m)}{k}=\frac{2}{p}$, where $p>N+2$ is given in Theorem 1.3.1.

Theorem 1.3.5. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2), let $L>0$ be given by (1.2.6) and let $M>0$ be given by (1.2.7). Then for any $\varepsilon>0$, there exists $\delta_0=\delta_0(\varepsilon,N,m,L,M)>0$ satisfying the following property. Let $\boldsymbol{F}\in\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)$ and let $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$ be any global weak solution of (1.1.1) and (1.2.1). If $\mathrm{supp}\,\boldsymbol{F}$ is a compact set and if $\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}\leqslant\delta_0$ then $\mathrm{supp}\,\boldsymbol{u}\subset\overline{\Omega}\cap\mathcal{O}(\varepsilon)$, where $\mathcal{O}(\varepsilon)$ is the open bounded set*

$$\mathcal{O}(\varepsilon)=\left\{x\in\mathbb{R}^N;\ \exists y\in\mathrm{supp}\,\boldsymbol{F}\ \text{ such that }\ |x-y|<\varepsilon\right\}.$$

In particular, if $\varepsilon>0$ is small enough then $\mathrm{supp}\,\boldsymbol{u}\subset\mathcal{O}(\varepsilon)\subset\Omega$.

We see that localization effect occurs under some smallness condition, either on the solution $\boldsymbol{u}$ (Theorem 1.3.3) or on the external source $\boldsymbol{F}$ (Theorem 1.3.5). When $\Omega=\mathbb{R}^N$, the phenomenon is simpler since localization effect is always observed, without any condition of the size, neither on the solution nor on the external source, as show the following result.

Theorem 1.3.6. *Let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2), let $\boldsymbol{F}\in\boldsymbol{L^p}(\mathbb{R}^N)$, for some $1\leqslant p\leqslant\infty$, and let $\boldsymbol{u}\in\boldsymbol{H^1}(\mathbb{R}^N)\cap\boldsymbol{L^{m+1}}(\mathbb{R}^N)$ be any global weak solution of (1.1.1). If $\mathrm{supp}\,\boldsymbol{F}$ is a compact set then $\mathrm{supp}\,\boldsymbol{u}$ is also compact.*

1.4 Existence and smoothness

In this section, we give an existence result of solutions for equation (1.1.1) (Theorem 1.4.1), some *a priori* bounds for the solutions of equation (1.1.1) (Theorem 1.4.4), which will be useful to establish our existence result, and a smoothness result for equation (1.1.1) (Proposition 1.4.5).

Theorem 1.4.1. *Let $\Omega\subseteq\mathbb{R}^N$ be a nonempty open subset, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2) and let $\boldsymbol{F}\in\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)$. Then equations (1.1.1) and (1.2.1) admits at least one global weak solution $\boldsymbol{u}\in\boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$. Furthermore, the following properties hold for any global weak solution $\boldsymbol{u}$ (except Property 3)).*

1) $\boldsymbol{u} \in \boldsymbol{W}^{\boldsymbol{2},\frac{m+1}{m}}_{\mathbf{loc}}(\Omega)$.
2) *Let* $\alpha \in (0, m]$. *If* $\boldsymbol{F} \in \boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$ *then* $\boldsymbol{u} \in \boldsymbol{C}^{\boldsymbol{2},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$.
3) *If* $\Omega = \left\{x \in \mathbb{R}^N;\ r < |x| < R\right\}$, *for some* $-\infty < r \leqslant r_+ < R \leqslant +\infty$, *and if* $\boldsymbol{F}$ *is spherically symmetric then there exists a spherically symmetric global weak solution* $\boldsymbol{u} \in \boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$ *of* (1.1.1) *and* (1.2.1). *For* $N = 1$, *this means that if* $\boldsymbol{F}$ *is an even* (*respectively, an odd*) *function on* $\Omega = (-R, -r) \cup (r, R)$ *then* $\boldsymbol{u}$ *is also an even* (*respectively, an odd*) *function.*

Remark 1.4.2. Assume $\boldsymbol{F}$ is spherically symmetric. Since we do not know, in general, if we have uniqueness of the solution, we are not able to show that any solution is radially symmetric. For a uniqueness result, see Theorem 1.5.2 below.

Remark 1.4.3. Assume $|\Omega| < \infty$. There exists $\varepsilon = \varepsilon(N) > 0$ such that for any $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$, $0 < m < 1$ and $\boldsymbol{F} \in \boldsymbol{L}^{\boldsymbol{2}}(\Omega)$, if $|\boldsymbol{b}||\Omega|^{\frac{2}{N}} < \varepsilon$ then equations (1.1.1) and (1.2.1) admits at least one global weak solution $\boldsymbol{u} \in \boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$. In addition, $\boldsymbol{u} \in \boldsymbol{H}^{\boldsymbol{2}}_{\mathbf{loc}}(\Omega)$. Finally, Properties 2) and 3) of Theorem 1.4.1 hold. For more details, see Bégout and Torri [31].

Theorem 1.4.4. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$, *let* $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.2.2), *let* $L > 0$ *be given by* (1.2.6), *let* $M > 0$ *be given by* (1.2.7) *and let* $\boldsymbol{F} \in \boldsymbol{L}^{\frac{m+1}{m}}(\Omega)$. *Let* $\boldsymbol{u} \in \boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m+1}}(\Omega)$ *be any global weak solution of* (1.1.1) *and* (1.2.1). *Then we have the following estimates.*

$$\|\nabla \boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)} + \|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} \leqslant M_0 \|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}, \tag{1.4.1}$$

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H^1_0}(\Omega)} + \|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)} \leqslant C\widetilde{M_0}\left(1 + \|\boldsymbol{F}\|^{\frac{\delta(m+1)}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}\right)\|\boldsymbol{F}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega)}, \tag{1.4.2}$$

where $M_0 = M\left(\frac{2M}{L}\right)^{\frac{1}{m}} \max\left\{1, \frac{2}{L}\right\}$, $\delta = \frac{2(1-m)}{(N+2)-m(N-2)}$, $\widetilde{M_0} = M_0(1 + M_0^\delta)$ *and* $C = C(N, m)$.

Proposition 1.4.5. *Let* $\boldsymbol{a} \in \mathbb{C}$, *let* $0 < m < 1$, *let* $\boldsymbol{V} \in \boldsymbol{L}^{\boldsymbol{r}}_{\mathbf{loc}}(\Omega; \mathbb{C})$, *for any* $1 < r < \infty$, *let* $\boldsymbol{F} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega; \mathbb{C})$ *and, for some* $\varepsilon > 0$, *let* $\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1+\varepsilon}}_{\mathbf{loc}}(\Omega; \mathbb{C})$ $\big(\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega; \mathbb{C})$ *suffices if* $\boldsymbol{V} \in \boldsymbol{L}^{\boldsymbol{\infty}}_{\mathbf{loc}}(\Omega; \mathbb{C})\big)$ *be a solution to*

$$-\Delta \boldsymbol{u} + \boldsymbol{V}\boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} = \boldsymbol{F}(x), \text{ in } \mathscr{D}'(\Omega). \tag{1.4.3}$$

Let $1 < q < \infty$ *and suppose* $\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{q}}_{\mathbf{loc}}(\Omega)$. *Then the following regularity results hold.*

1) *If for some* $p \in [q, \infty)$, $\boldsymbol{F} \in \boldsymbol{L}^{\boldsymbol{p}}_{\mathbf{loc}}(\Omega)$ *then* $\boldsymbol{u} \in \boldsymbol{W}^{\boldsymbol{2},\boldsymbol{p}}_{\mathbf{loc}}(\Omega)$.
2) *Let* $\alpha \in (0, m]$. *If* $(\boldsymbol{F}, \boldsymbol{V}) \in \boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega) \times \boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$ *then* $\boldsymbol{u} \in \boldsymbol{C}^{\boldsymbol{2},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$.

Remark 1.4.6. Since $0 < m < 1$ and $\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$, one has $\boldsymbol{L}^{\frac{1}{m}}_{\mathbf{loc}}(\Omega) \subset \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$ and so $|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$. In addition, from Hölder's inequality $\boldsymbol{V}\boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$ and it follows that $\Delta \boldsymbol{u} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$. In conclusion, equation (1.4.3) makes senses in $\boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$.

Remark 1.4.7. We only state a local smoothness result since we are interested by compactly supported solutions. In this case, global smoothness is immediate. Nevertheless, one may wonder what happens when a solution is not compactly supported. We use the notation of Proposition 1.4.5 and assume further that Ω is bounded [1] and has a $C^{1,1}$ boundary. Let the assumptions of Proposition 1.4.5 be fulfilled and let $\boldsymbol{u} \in \boldsymbol{L^q}(\Omega)$, for some $1 < q < \infty$, be a solution to (1.4.3) such that $\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace [2].

1. If for some $p \in [q, \infty)$, $\boldsymbol{F} \in \boldsymbol{L^p}(\Omega)$ and $\boldsymbol{V} \in \boldsymbol{L^r}(\Omega)$, $\forall r \in (1, \infty)$, then $\boldsymbol{u} \in \boldsymbol{W^{2,p}}(\Omega) \cap \boldsymbol{W_0^{1,p}}(\Omega)$. Indeed, recalling that if for some $1 < p < \infty$, a function $\boldsymbol{v} \in \boldsymbol{L^p}(\Omega)$ satisfies $\Delta\boldsymbol{v} \in \boldsymbol{L^p}(\Omega)$ and $\boldsymbol{v}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace[2] then $\boldsymbol{v} \in \boldsymbol{W^{2,p}}(\Omega) \cap \boldsymbol{W_0^{1,p}}(\Omega)$ (Grisvard [93], Corollary 2.5.2.2 p.131). We then apply the bootstrap method of the proof of Proposition 1.4.5 to prove the result, where we use the embedding $\boldsymbol{L^r}(\Omega) \hookrightarrow \boldsymbol{L^s}(\Omega)$, which holds for any $r \geqslant s$ (since Ω is bounded) and the global regularity result of Grisvard [93] (Corollary 2.5.2.2 p.131) in place of a local regularity result (Cazenave [58], Proposition 4.1.2 p.101–102).
2. Let $\alpha \in (0, m]$. If Ω has a $C^{2,\alpha}$ boundary and $(\boldsymbol{F}, \boldsymbol{V}) \in \boldsymbol{C^{0,\alpha}}(\overline{\Omega}) \times \boldsymbol{C^{0,\alpha}}(\overline{\Omega})$ then $\boldsymbol{u} \in \boldsymbol{C^{2,\alpha}}(\overline{\Omega}) \cap \boldsymbol{C_0}(\Omega)$ [3]. Indeed, it follows from the above remark that $\boldsymbol{u} \in \boldsymbol{W^{2,N+1}}(\Omega) \cap \boldsymbol{H_0^1}(\Omega)$ and by Sobolev's embedding, $\boldsymbol{u} \in \boldsymbol{C^{0,1}}(\overline{\Omega})$. Setting
$$\boldsymbol{f} = \boldsymbol{F}(x) - \boldsymbol{V}\boldsymbol{u} - \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u},$$
it then follow from equation (1.4.3) and estimate (1.8.5) below that $\boldsymbol{f} \in \boldsymbol{C^{0,\alpha}}(\overline{\Omega})$. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{C}} \stackrel{\text{def}}{=} \boldsymbol{C^{2,\alpha}}(\overline{\Omega}) \cap \boldsymbol{C_0}(\Omega)$ be a solution to
$$-\Delta\boldsymbol{w} = \boldsymbol{f}, \tag{1.4.4}$$
given by Gilbarg and Trudinger [90], Theorem 6.14 p.107. Since $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$ is also a solution to (1.4.4), uniqueness for equation (1.4.4) holds in $\boldsymbol{H_0^1}(\Omega)$ (Lax-Milgram's Theorem) and $\boldsymbol{\mathcal{C}} \subset \boldsymbol{H_0^1}(\Omega)$, we conclude that $\boldsymbol{u} = \boldsymbol{v}$ and so $\boldsymbol{u} \in \boldsymbol{\mathcal{C}}$.

We end this section by giving a result for the evolution equation (in a particular case).

Corollary 1.4.8. *Let $0 < m < 1$, let $(\boldsymbol{\lambda}, b) \in \mathbb{C} \times \mathbb{R}$ satisfying $\boldsymbol{\lambda} \neq \boldsymbol{0}$ and $b \geqslant 0$. If $\mathrm{Im}(\boldsymbol{\lambda}) = 0$ then assume further $\mathrm{Re}(\boldsymbol{\lambda}) \leqslant 0$. Finally, let $\boldsymbol{F} \in \boldsymbol{C^{0,m}}(\mathbb{R}^N)$ be compactly supported. Then there exists a solution $\boldsymbol{u} \in \boldsymbol{C^\infty}\big(\mathbb{R}; \boldsymbol{C_b^{2,m}}(\mathbb{R}^N)\big)$ to*
$$\begin{cases} \mathbf{i}\dfrac{\partial \boldsymbol{u}}{\partial t} + \Delta\boldsymbol{u} + \boldsymbol{\lambda}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} = \boldsymbol{F}(x)e^{\mathbf{i}bt}, & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ \boldsymbol{u}(0) = \boldsymbol{\varphi}, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.4.5}$$
given by
$$\forall (t,x) \in \mathbb{R} \times \mathbb{R}^N, \ \boldsymbol{u}(t,x) = \boldsymbol{\varphi}(x)e^{\mathbf{i}bt}, \tag{1.4.6}$$

1. Actually, assumptions on Ω we use in this remark are $\partial\Omega$ bounded and $|\Omega| < \infty$. But these two conditions imply that Ω is bounded.

2. Let $\boldsymbol{T} : \boldsymbol{u} \longrightarrow \left\{\boldsymbol{\gamma u}, \boldsymbol{\gamma}\frac{\partial \boldsymbol{u}}{\partial \nu}\right\}$ be the trace function defined on $\mathscr{D}(\overline{\Omega})$, let $1 < p < \infty$ and let $\boldsymbol{X_p}(\Omega) = \big\{\boldsymbol{u} \in \boldsymbol{L^p}(\Omega); \Delta\boldsymbol{u} \in \boldsymbol{L^p}(\Omega)\big\}$. By density of $\mathscr{D}(\overline{\Omega})$ in $\boldsymbol{X_p}(\Omega)$, $\boldsymbol{T}$ has a continuous and linear extension from $\boldsymbol{X_p}(\Omega)$ into $\boldsymbol{W^{-\frac{1}{p},p}}(\partial\Omega) \times \boldsymbol{W^{-1-\frac{1}{p},p}}(\partial\Omega)$ (Hörmander [107], Theorem 2 p.503 ; Lions and Magenes [129], Lemma 2.2 and Theorem 2.1 p.147 ; Lions and Magenes [130], Propositions 9.1, Proposition 9.2 and Theorem 9.1 p.82 ; Grisvard [93], p.54). Since $\boldsymbol{u} \in \boldsymbol{L^q}(\Omega)$, it follows from equation (1.4.3) and Hölder's inequality that $\boldsymbol{u} \in \boldsymbol{X_p}(\Omega)$, for any $1 < p < q$. Then "$\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}$ in the sense of the trace" makes sense and means that $\boldsymbol{\gamma u} = \boldsymbol{0}$.

3. For $k \in \mathbb{N}_0$ and $0 < \alpha \leqslant 1$, $\boldsymbol{C^{k,\alpha}}(\overline{\Omega}) = \Big\{\boldsymbol{u} \in \boldsymbol{C^k}(\overline{\Omega};\mathbb{C}); \sum\limits_{|\beta|=k} H_\Omega^\alpha(D^\beta\boldsymbol{u}) < +\infty\Big\} \subset \boldsymbol{W^{k,\infty}}(\Omega)$ (since Ω is bounded) and $\boldsymbol{C_0}(\Omega) = \big\{\boldsymbol{u} \in \boldsymbol{C}(\overline{\Omega}); \forall x \in \partial\Omega, \ \boldsymbol{u}(x) = \boldsymbol{0}\big\}$.

where $\boldsymbol{\varphi} \in \boldsymbol{C}^{\mathbf{2},\boldsymbol{m}}_{\mathbf{b}}(\mathbb{R}^N)$ *is a solution compactly supported of*

$$-\Delta\boldsymbol{\varphi} - \boldsymbol{\lambda}|\boldsymbol{\varphi}|^{-(1-m)}\boldsymbol{\varphi} + b\boldsymbol{\varphi} = -\boldsymbol{F}(x), \ \text{in } \mathbb{R}^N, \tag{1.4.7}$$

given by Theorem 1.4.1*. Furthermore, for any* $t \in \mathbb{R}$, $\operatorname{supp} \boldsymbol{u}(t)$ *is compact.*

1.5 Uniqueness

Theorem 1.5.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$*, let* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2 \setminus \{(\mathbf{0},\mathbf{0})\}$ *satisfying* (1.2.3) *and let* $\boldsymbol{F_1},\boldsymbol{F_2} \in \boldsymbol{L}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$ *be such that* $\boldsymbol{F_1} - \boldsymbol{F_2} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$*. Let* $\boldsymbol{u_1},\boldsymbol{u_2} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m}+\mathbf{1}}(\Omega)$ *be two global weak solutions of*

$$-\mathbf{i}\Delta\boldsymbol{u_1} + \boldsymbol{a}|\boldsymbol{u_1}|^{-(1-m)}\boldsymbol{u_1} + \boldsymbol{b}\boldsymbol{u_1} = \boldsymbol{F_1}(x), \ \text{in } \Omega, \tag{1.5.1}$$

$$-\mathbf{i}\Delta\boldsymbol{u_2} + \boldsymbol{a}|\boldsymbol{u_2}|^{-(1-m)}\boldsymbol{u_2} + \boldsymbol{b}\boldsymbol{u_2} = \boldsymbol{F_2}(x), \ \text{in } \Omega, \tag{1.5.2}$$

respectively. We have the following estimates.

$$\begin{cases} \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant \dfrac{|\boldsymbol{a}|}{\operatorname{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right)}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}, & \text{if } \boldsymbol{a} \neq \mathbf{0} \text{ and } \operatorname{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) > 0, \\ \|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant \dfrac{1}{b_0}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}, & \text{if } \boldsymbol{a} = \mathbf{0}, \end{cases} \tag{1.5.3}$$

where $b_0 = |\operatorname{Re}(\boldsymbol{b})|$*, if* $\operatorname{Re}(\boldsymbol{b}) \neq 0$ *and* $b_0 = \operatorname{Im}(\boldsymbol{b})$*, if* $\operatorname{Re}(\boldsymbol{b}) = 0$*. If* $\boldsymbol{a} \neq \mathbf{0}$ *and* $\operatorname{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) = 0$ *then assume further that* $\boldsymbol{u_1},\boldsymbol{u_2} \in \boldsymbol{L}^{\boldsymbol{\infty}}(\Omega)$*. Then there exists a positive constant* $C = C(N,m)$ *such that*

$$\|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}(\Omega)} \leqslant C\frac{\left(\|\boldsymbol{u_1}\|_{\boldsymbol{L^\infty}(\Omega)} + \|\boldsymbol{u_2}\|_{\boldsymbol{L^\infty}(\Omega)}\right)^{1-m}}{|\boldsymbol{a}|}\|\boldsymbol{F_1} - \boldsymbol{F_2}\|_{\boldsymbol{L^2}(\Omega)}. \tag{1.5.4}$$

Theorem 1.5.2. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$*, let* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.2.3) *and let* $\boldsymbol{F} \in \boldsymbol{L}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$*. Then equations* (1.1.1) *and* (1.2.1) *admit at most one global weak solution* $\boldsymbol{u} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m}+\mathbf{1}}(\Omega)$*.*

Corollary 1.5.3. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$*, let* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A} \times \mathbb{B}$ *satisfying* (1.2.3) *and let* $\boldsymbol{F} \in \boldsymbol{L}^{\frac{\boldsymbol{m}+\mathbf{1}}{\boldsymbol{m}}}(\Omega)$*. Then equations* (1.1.1) *and* (1.2.1) *admit a unique global weak solution* $\boldsymbol{u} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m}+\mathbf{1}}(\Omega)$*. Furthermore, this solution satisfies Properties* $1) - 3)$ *of Theorem* 1.4.1*.*

Corollary 1.5.4. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset, let* $0 < m < 1$ *and let* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ *satisfying* (1.2.3)*. Then the problem*

$$\begin{cases} -\mathbf{i}\Delta\boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u} = \mathbf{0}, \ \text{in } \Omega, \\ \boldsymbol{u} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m}+\mathbf{1}}(\Omega), \end{cases}$$

has for unique solution $\boldsymbol{u} \equiv \mathbf{0}$*.*

Corollary 1.5.5. *Let* $0 < m < 1$*, let* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{A} \times \mathbb{B}$ *satisfying* (1.2.3) *and let* $\boldsymbol{F} \in \boldsymbol{C}^{\mathbf{0},\boldsymbol{m}}(\mathbb{R}^N)$ *be compactly supported. Then there exists a unique solution* $\boldsymbol{u} \in \boldsymbol{C}^{\mathbf{2},\boldsymbol{m}}_{\mathbf{b}}(\mathbb{R}^N)$ *of* (1.1.1) *and* (1.2.1) *compactly supported. If furthermore* $\boldsymbol{F}$ *is spherically symmetric then* $\boldsymbol{u}$ *is also spherically symmetric. For* $N = 1$*, this means that if* $\boldsymbol{F}$ *is an even (respectively, an odd) function then* $\boldsymbol{u}$ *is also an even (respectively, an odd) function.*

1.6 Pictures

In this section, we give some geometric interpretation of the values of $\boldsymbol{a}$ and $\boldsymbol{b}$. For convenience, we repeat the hypotheses (1.2.2) and (1.2.3). We recall that,

$$\begin{cases} \mathbb{A} = \mathbb{C} \setminus \big\{ \boldsymbol{z} \in \mathbb{C}; \mathrm{Re}(\boldsymbol{z}) = 0 \text{ and } \mathrm{Im}(\boldsymbol{z}) \leqslant 0 \big\}, \\ \mathbb{B} = \mathbb{A} \cup \big\{ \boldsymbol{0} \big\}. \end{cases}$$

For existence of solutions to problem (1.1.1) and (1.2.1), we suppose $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$ satisfies

$$(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{A} \times \mathbb{B} \quad \text{ and } \quad \begin{cases} \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) \geqslant 0, \\ \text{ or} \\ \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) < 0 \ \text{ and } \ \mathrm{Im}(\boldsymbol{b}) > \dfrac{\mathrm{Re}(\boldsymbol{b})}{\mathrm{Re}(\boldsymbol{a})}\mathrm{Im}(\boldsymbol{a}), \end{cases} \tag{1.6.1}$$

while for uniqueness, we assume

$$\mathrm{Im}(\boldsymbol{a}) \geqslant 0 \quad \text{ and } \quad \begin{cases} \boldsymbol{a} \neq \boldsymbol{0} \ \text{ and } \ \mathrm{Re}(\boldsymbol{a}\overline{\boldsymbol{b}}) \geqslant 0, \\ \text{ or} \\ \boldsymbol{a} = \boldsymbol{0} \ \text{ and } \ \boldsymbol{b} \in \mathbb{B}. \end{cases} \tag{1.6.2}$$

Existence. Condition (1.6.1) may easily be interpreted in this way : if $\boldsymbol{b} \neq \boldsymbol{0}$ the one requires that $[\boldsymbol{a}, \boldsymbol{b}] \cap \mathscr{B} = \emptyset$, where $\mathscr{B}$ is the geometric representation of $\mathbb{A}^c$. See Figures 1.1 and 1.2 below.
Uniqueness. The second condition of (1.6.2) is trivial. Indeed, $\boldsymbol{b}$ can be chosen anywhere in the complex plane, except on the half-axis where $\mathrm{Im}(\boldsymbol{z}) < 0$. Let us consider the first condition. We first choose $\boldsymbol{a} \in \mathbb{C} \setminus \{\boldsymbol{0}\}$ such that $\mathrm{Im}(\boldsymbol{a}) \geqslant 0$, and we choose $\boldsymbol{b}$ with respect to $\boldsymbol{a}$. We see $\boldsymbol{a}$ and $\boldsymbol{b}$ as vectors of $\mathbb{R}^2$. Then we write, $\overrightarrow{a} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{a}) \\ \mathrm{Im}(\boldsymbol{a}) \end{pmatrix}$, $\overrightarrow{b} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{b}) \\ \mathrm{Im}(\boldsymbol{b}) \end{pmatrix}$ and we have

$$\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) = \mathrm{Re}(\boldsymbol{a})\mathrm{Re}(\boldsymbol{b}) + \mathrm{Im}(\boldsymbol{a})\mathrm{Im}(\boldsymbol{b}) = \overrightarrow{a}.\overrightarrow{b}, \tag{1.6.3}$$

where . denotes the scalar product between two vectors of $\mathbb{R}^2$. Then the condition $\mathrm{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right) \geqslant 0$ is equivalent to $\left|\angle(\overrightarrow{a}, \overrightarrow{b})\right| \leqslant \dfrac{\pi}{2}\mathrm{rad}$ (see Figure 1.3 below).

Remark 1.6.1. Let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{C}^2$. Thanks to (1.6.3), the following assertions are equivalent.

1) $(\boldsymbol{a}, \boldsymbol{b})$ satisfies (1.6.1)–(1.6.2) (or (1.2.2)–(1.2.3)).
2) $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{A} \times \mathbb{B}$ satisfies (1.6.2) (or (1.2.3)).
3) $\Big((\boldsymbol{a}, \boldsymbol{b})$ satisfies (1.6.2)$\Big)$, $\Big(\boldsymbol{a} \neq \boldsymbol{0}\Big)$ and $\Big(\mathrm{Im}(\boldsymbol{a}) = \mathrm{Re}(\boldsymbol{b}) = 0 \implies \mathrm{Im}(\boldsymbol{b}) \geqslant 0\Big)$.

In other words, when $\mathrm{Im}(\boldsymbol{a}) \neq 0$, uniqueness hypothesis (1.6.2) implies existence hypothesis (1.6.1) (see Figure 1.4 below).

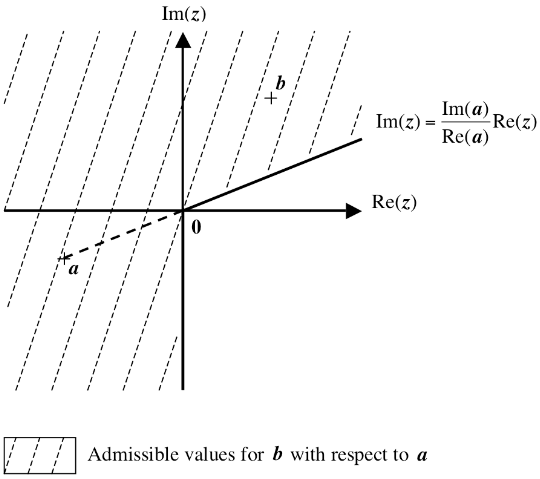

FIGURE 1.1 – Existence, choice of $\boldsymbol{b}$

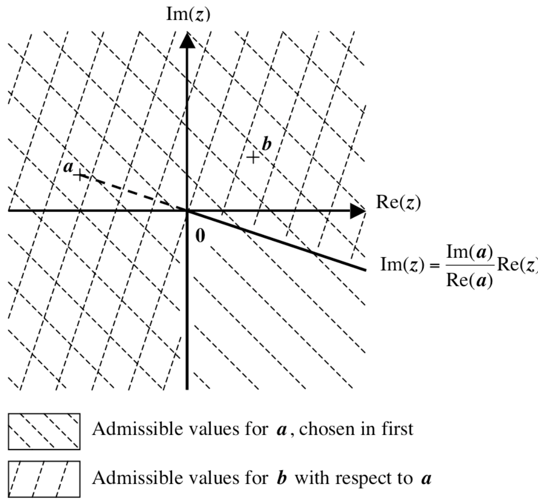

FIGURE 1.2 – Existence, choice of $\boldsymbol{a}$ and $\boldsymbol{b}$

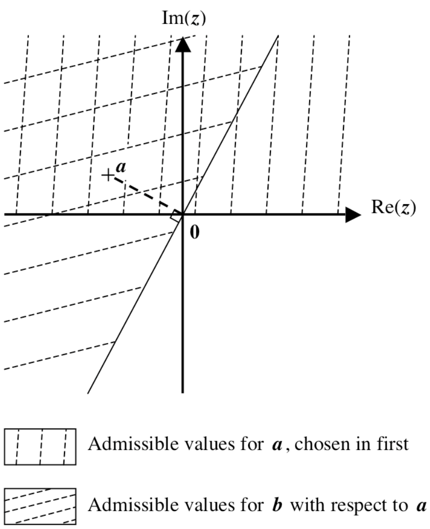

FIGURE 1.3 – Uniqueness

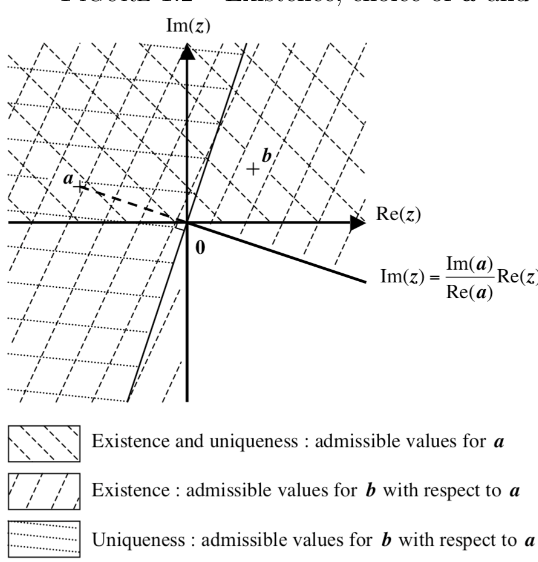

FIGURE 1.4 – Uniqueness implies existence

1.7 Proofs of the localization properties

In this Section, we prove Theorems 1.2.1, 1.3.1, 1.3.3, 1.4.4, 1.3.5 and 1.3.6. We recall some useful Gagliardo-Nirenberg's and Young's inequalities.

Proposition 1.7.1. *Let* $\Omega \subseteq \mathbb{R}^N$ *be a nonempty open subset and let* $0 \leqslant p \leqslant 1$. *Then, there exists a* *positive constant $C=C(N)$ such that*

$$\forall \boldsymbol{u}\in \boldsymbol{H_0^1}(\Omega)\cap \boldsymbol{L^{p+1}}(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}\leqslant C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{N(1-p)}{(N+2)-p(N-2)}}\|\boldsymbol{u}\|_{\boldsymbol{L^{p+1}}(\Omega)}^{\frac{2(1+p)}{(N+2)-p(N-2)}}, \tag{1.7.1}$$

$$\forall \boldsymbol{u}\in \boldsymbol{H_0^1}(\Omega)\cap \boldsymbol{L^1}(\Omega),\ \|\boldsymbol{u}\|_{\boldsymbol{L^{p+1}}(\Omega)}^{p+1}\leqslant C\|\nabla\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{2pN}{N+2}}\|\boldsymbol{u}\|_{\boldsymbol{L^1}(\Omega)}^{\frac{(N+2)-p(N-2)}{N+2}}. \tag{1.7.2}$$

Note that C does not depend on Ω.

Lemma 1.7.2. *For any real $x\geqslant 0$, $y\geqslant 0$, $\varepsilon>0$ and $p>1$, one has*

$$xy\leqslant \frac{1}{p'}\varepsilon^{p'}x^{p'}+\frac{1}{p}\varepsilon^{-p}y^p. \tag{1.7.3}$$

Lemma 1.7.3. *Let $(\boldsymbol{a},\boldsymbol{b})\in\mathbb{C}^2$ satisfying (1.2.2) and let C_0, C_1, C_2, C_3 be four nonnegative real numbers satisfying*

$$\left|C_1+\operatorname{Im}(\boldsymbol{a})C_2+\operatorname{Im}(\boldsymbol{b})C_3\right|\leqslant C_0, \tag{1.7.4}$$

$$\left|\operatorname{Re}(\boldsymbol{a})C_2+\operatorname{Re}(\boldsymbol{b})C_3\right|\leqslant C_0. \tag{1.7.5}$$

Then one has

$$0\leqslant C_1+LC_2\leqslant MC_0, \tag{1.7.6}$$

where the positive constants L and M are defined by (1.2.6) and (1.2.7), respectively.

Proof. We split the proof in 6 cases. Let $\delta>0$.
Case 1. $\operatorname{Im}(\boldsymbol{a})>0$ and $\operatorname{Im}(\boldsymbol{b})\geqslant 0$.
Then (1.7.6) follows from (1.7.4).
Case 2. $\operatorname{Im}(\boldsymbol{a})=0$, $\operatorname{Im}(\boldsymbol{b})\geqslant 0$ and $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0$.
We compute (1.7.4) $+\operatorname{sign}(\operatorname{Re}(\boldsymbol{a}))$(1.7.5) and then obtain (1.7.6).
Case 3. $\operatorname{Im}(\boldsymbol{a})\geqslant 0$, $\operatorname{Im}(\boldsymbol{b})<0$ and $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0$.
We compute (1.7.4) $+\frac{|\operatorname{Im}(\boldsymbol{b})|}{\operatorname{Re}(\boldsymbol{b})}$(1.7.5) and then obtain (1.7.6).
Case 4. $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})<0$.
If $\operatorname{Im}(\boldsymbol{b})=0$ then (1.2.2) implies $\operatorname{Im}(\boldsymbol{a})>0$, which falls into the scope of Case 1. So we may assume $\operatorname{Im}(\boldsymbol{b})\neq 0$. We compute (1.7.4) $-\frac{\operatorname{Im}(\boldsymbol{b})}{\operatorname{Re}(\boldsymbol{b})}$(1.7.5) and then obtain (1.7.6).
Case 5. $\operatorname{Im}(\boldsymbol{a})<0$, $\operatorname{Im}(\boldsymbol{b})\geqslant 0$ and $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})\geqslant 0$.
We compute (1.7.4) $+\frac{|\operatorname{Im}(\boldsymbol{a})|+\delta}{\operatorname{Re}(\boldsymbol{a})}$(1.7.5) and then obtain (1.7.6).
Case 6. $\operatorname{Im}(\boldsymbol{a})<0$, $\operatorname{Im}(\boldsymbol{b})<0$ and $\operatorname{Re}(\boldsymbol{a})\operatorname{Re}(\boldsymbol{b})>0$.
We compute (1.7.4) $+\max\left\{\frac{|\operatorname{Im}(\boldsymbol{a})|+\delta}{|\operatorname{Re}(\boldsymbol{a})|},\frac{|\operatorname{Im}(\boldsymbol{b})|}{|\operatorname{Re}(\boldsymbol{b})|}\right\}$(1.7.5). We then obtain (1.7.6).
This ends the proof. □

Proof of Theorems 1.2.1 and 1.3.1. In order to establish our result in all cases of (1.2.2), we will adopt the proofs of Theorem 2.1 p.12–18 and Theorem 3.2 p.28–30 of Antontsev, Díaz and Shmarev [11], which has to be adapted. We denote by σ the surface measure on a sphere, $\rho_2=\rho_0$, if we are concerned by Theorem 1.2.1 and $\rho_2=\rho_1$, if we are concerned by Theorem 1.3.1. Assume we have either $\rho_2<\operatorname{dist}(x_0,\partial\Omega)$ $\big(\iff \overline{B}(x_0,\rho_2)\subset\Omega\big)$ or $\rho_2>\operatorname{dist}(x_0,\partial\Omega)$. The remaining case $\rho_2=\operatorname{dist}(x_0,\partial\Omega)$ $\big(\iff B(x_0,\rho_2)\subset\Omega$ and $\partial\Omega\cap\mathbb{S}(x_0,\rho_2)\neq\emptyset\big)$, will be treated at the end of the proof[4]. If $\rho_2 > \operatorname{dist}(x_0, \partial\Omega)$, we have $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$. So we may define $\widetilde{\boldsymbol{u}} \in \boldsymbol{H_0^1}\big(\Omega \cup B(x_0, \rho_2)\big)$ satisfying $\widetilde{\boldsymbol{u}}_{|\Omega} \in \boldsymbol{H_0^1}(\Omega)$, by setting $\widetilde{\boldsymbol{u}} = \boldsymbol{u}$, in Ω and $\widetilde{\boldsymbol{u}} = \boldsymbol{0}$, in $\Omega^c \cap B(x_0, \rho_2)$. Then $\nabla\widetilde{\boldsymbol{u}} = \nabla\boldsymbol{u}$, almost everywhere in Ω and $\nabla\widetilde{\boldsymbol{u}} = \boldsymbol{0}$, almost everywhere in $\Omega^c \cap B(x_0, \rho_2)$. Still if $\rho_2 > \operatorname{dist}(x_0, \Omega)$, we denote by $\widetilde{\boldsymbol{F}}$ the extension of $\boldsymbol{F}$ by $\boldsymbol{0}$ in $\Omega^c \cap B(x_0, \rho_2)$. We now proceed with the proof in 7 steps.

Step 1. Let L and M be the constants defined by (1.2.6) and (1.2.7), respectively. For almost every $\rho \in (0, \rho_2)$,

$$\|\nabla\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))} + L\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))} \leqslant MI(\rho) + MJ(\rho), \tag{1.7.7}$$

where $I(\rho) = \left|\displaystyle\int_{\mathbb{S}(x_0,\rho)} \widetilde{\boldsymbol{u}}\overline{\nabla\widetilde{\boldsymbol{u}}}.\frac{x - x_0}{|x - x_0|}\mathrm{d}\sigma\right|$ and $J(\rho) = \displaystyle\int_{B(x_0,\rho)} |\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x$. Moreover, $I \in L^1(0, \rho_2)$ and $J \in L^\infty(0, \rho_2)$.

From Hölder's inequality, the above discussion and Sobolev's embedding,

$$\begin{gathered}
\|I\|_{L^1(0,\rho_2)} \leqslant \|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{H^1}(B(x_0,\rho_2))} < \infty, \\
\|J\|_{L^\infty(0,\rho_2)} \leqslant \|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho_2))}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^{m+1}}(B(x_0,\rho_2))} < \infty.
\end{gathered}$$

Let $\rho \in (0, \rho_2)$ For any $n \in \mathbb{N}$, $n > \frac{1}{\rho}$, we define the cutoff function $\psi_n \in W^{1,\infty}(\mathbb{R})$ by

$$\forall t \in \mathbb{R},\ \psi_n(t) = \begin{cases} 1, & \text{if } |t| \in \left[0, \rho - \frac{1}{n}\right], \\ n(\rho - |t|), & \text{if } |t| \in \left(\rho - \frac{1}{n}, \rho\right), \\ 0, & \text{if } |t| \in [\rho, \infty), \end{cases}$$

and we set for almost every $x \in \Omega \cup B(x_0, \rho_2)$, $\boldsymbol{\varphi_n}(x) = \psi_n(|x - x_0|)\widetilde{\boldsymbol{u}}(x)$. If $\rho_2 < \operatorname{dist}(x_0, \partial\Omega)$ then $\operatorname{supp}\boldsymbol{\varphi_n} \subseteq \overline{B}(x_0, \rho) \subset \Omega$ and so $\boldsymbol{\varphi_n} \in \boldsymbol{H_c^1}(\Omega)$. If $\rho_2 > \operatorname{dist}(x_0, \partial\Omega)$ then $\boldsymbol{\varphi_n}_{|\Omega} \in \boldsymbol{H_0^1}(\Omega)$ and $\operatorname{supp}\boldsymbol{\varphi_n} \subseteq \overline{\Omega} \cap \overline{B}(x_0, \rho)$. It follows from Definition 1.2.3 and Remark 1.2.4, 2. and 3., that $\boldsymbol{\varphi} = \mathbf{i}\boldsymbol{\varphi_n}_{|\Omega}$ is an admissible test function and so

$$\begin{aligned}
\mathrm{Re}\int\limits_{B(x_0,\rho)} &\psi_n(|x - x_0|)\left(|\nabla\widetilde{\boldsymbol{u}}|^2 - \mathbf{i}\boldsymbol{a}|\widetilde{\boldsymbol{u}}|^{m+1} - \mathbf{i}\boldsymbol{b}|\widetilde{\boldsymbol{u}}|^2\right)\mathrm{d}x \\
&= -\mathrm{Re}\int\limits_{B(x_0,\rho)} \psi_n'(|x - x_0|)\overline{\widetilde{\boldsymbol{u}}}\nabla\widetilde{\boldsymbol{u}}.\frac{x - x_0}{|x - x_0|}\mathrm{d}x + \mathrm{Im}\int\limits_{B(x_0,\rho)} \psi_n(|x - x_0|)\widetilde{\boldsymbol{F}}\overline{\widetilde{\boldsymbol{u}}}\mathrm{d}x.
\end{aligned}$$

4. For simplicity, we assume that $\partial\Omega \neq \emptyset$. Otherwise, we have $\Omega = \mathbb{R}^N$ and we only have to treat the first case : $\overline{B}(x_0, \rho_2) \subset \Omega$.

Introducing the spherical coordinates (r,σ), we get

$$\left|\operatorname{Re}\int\limits_{B(x_0,\rho)}\psi_n(|x-x_0|)\left(|\nabla\widetilde{\boldsymbol{u}}|^2-\mathbf{i}\boldsymbol{a}|\widetilde{\boldsymbol{u}}|^{m+1}-\mathbf{i}\boldsymbol{b}|\widetilde{\boldsymbol{u}}|^2\right)\mathrm{d}x\right|$$
$$=\left|\operatorname{Re}\left(n\int\limits_{\rho-\frac{1}{n}}^{\rho}\left(\int\limits_{\mathbb{S}(x_0,r)}\overline{\widetilde{\boldsymbol{u}}}\nabla\widetilde{\boldsymbol{u}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right)\mathrm{d}r\right)+\operatorname{Im}\int\limits_{B(x_0,\rho)}\psi_n(|x-x_0|)\widetilde{\boldsymbol{F}}\overline{\widetilde{\boldsymbol{u}}}\mathrm{d}x\right|$$
$$\leqslant n\int\limits_{\rho-\frac{1}{n}}^{\rho}I(r)\mathrm{d}r+\int\limits_{B(x_0,\rho)}\psi_n(|x-x_0|)|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x.$$

We now let $n\nearrow\infty$. Using the Lebesgue's dominated convergence Theorem and recalling that $I\in L^1(0,\rho_2)$, we obtain

$$\left|\|\nabla\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))}+\operatorname{Im}(\boldsymbol{a})\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))}+\operatorname{Im}(\boldsymbol{b})\|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))}\right|\leqslant I(\rho)+J(\rho).\tag{1.7.8}$$

Proceeding as above with $\boldsymbol{\varphi}=\boldsymbol{\varphi_{n}}_{|\Omega}$, we get

$$\left|\operatorname{Re}(\boldsymbol{a})\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))}+\operatorname{Re}(\boldsymbol{b})\|\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))}\right|\leqslant I(\rho)+J(\rho).\tag{1.7.9}$$

Then Step 1 follows from (1.7.8), (1.7.9) and Lemma 1.7.3.
Let us recall and introduce some notations. Let $\tau\in\left(\frac{m+1}{2},1\right]$ and let $\rho\in(0,\rho_2)$. We set

$$E(\rho)=\|\nabla\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))},\qquad b(\rho)=\|\widetilde{\boldsymbol{u}}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))},\qquad \delta=\tfrac{k}{2(1+m)},$$
$$\theta=\tfrac{(1+m)+N(1-m)}{k}\in(0,1),\qquad \ell=\tfrac{1}{\theta(1+m)},\qquad \gamma(\tau)=\tfrac{2\tau-(1+m)}{k}\in(0,1),$$
$$\mu(\tau)=\tfrac{2(1-\tau)}{k},\qquad \eta(\tau)=\tfrac{1-m}{1+m}-\gamma(\tau)>0.$$

Step 2. $E\in W^{1,1}(0,\rho_2)$, for a.e. $\rho\in(0,\rho_2)$, $E'(\rho)=\|\nabla\widetilde{\boldsymbol{u}}\|^2_{\boldsymbol{L^2}(\mathbb{S}(x_0,\rho))}$ and

$$0\leqslant E(\rho)+b(\rho)\leqslant CL_1ME'(\rho)^{\frac{1}{2}}\left(E(\rho)^{\frac{1}{2}}+\rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta}b(\rho)^{\frac{1-\theta}{m+1}}$$
$$+(2L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))},\tag{1.7.10}$$

where $C=C(N,m)$ and $L_1=\max\left\{1,\frac{1}{L}\right\}$.

We have the identity $E(\rho)=\displaystyle\int_0^\rho\left(\int_{\mathbb{S}(x_0,r)}|\nabla\widetilde{\boldsymbol{u}}|^2\mathrm{d}\sigma\right)\mathrm{d}r$. Since the mapping $r\longmapsto\displaystyle\int_{\mathbb{S}(x_0,r)}|\nabla\widetilde{\boldsymbol{u}}|^2\mathrm{d}\sigma$ lies in $L^1(0,\rho_2)$, E is absolutely continuous on $(0,\rho_2)$. We then get the first part of the claim and we only have to establish (1.7.10). Let $\rho\in(0,\rho_2)$. It follows from Cauchy-Schwarz's inequality that

$$I(\rho)\leqslant\|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^2}(\mathbb{S}(x_0,\rho))}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^2}(\mathbb{S}(x_0,\rho))}=E'(\rho)^{\frac{1}{2}}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^2}(\mathbb{S}(x_0,\rho))}.\tag{1.7.11}$$

We recall the interpolation-trace inequality (see Corollary 2.1 in Díaz and Véron [78]. Note there is a misprint : δ has to be replaced with $-\delta$).

$$\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^2}(\mathbb{S}(x_0,\rho))}\leqslant C\left(\|\nabla\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^2}(B(x_0,\rho))}+\rho^{-\delta}\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L^{m+1}}(B(x_0,\rho))}\right)^{\theta}\|\widetilde{\boldsymbol{u}}\|^{1-\theta}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))},\tag{1.7.12}$$

where $C=C(N,m)$. Putting together (1.7.7), (1.7.11) and (1.7.12), we obtain,

$$E(\rho)+b(\rho)\leqslant CL_1ME'(\rho)^{\frac{1}{2}}\left(E(\rho)^{\frac{1}{2}}+\rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta}b(\rho)^{\frac{1-\theta}{m+1}}+L_1M\int\limits_{B(x_0,\rho)}|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x. \tag{1.7.13}$$

Applying Young's inequality (Lemma 1.7.2) with $x=\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}$, $y=\|\widetilde{\boldsymbol{u}}\|_{\boldsymbol{L}^{m+1}(B(x_0,\rho))}$, $\varepsilon=\left(\frac{2L_1M}{m+1}\right)^{\frac{1}{m+1}}$ and $p=m+1$, we get

$$\int\limits_{B(x_0,\rho)}|\widetilde{\boldsymbol{F}}(x)\overline{\widetilde{\boldsymbol{u}}(x)}|\mathrm{d}x\leqslant\frac{m}{m+1}\left(\frac{2L_1M}{m+1}\right)^{\frac{1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}+\frac{1}{2L_1M}b(\rho), \tag{1.7.14}$$

for any $\rho\in(0,\rho_2)$. Putting together (1.7.13) and (1.7.14), we obtain (1.7.10). Hence Step 2.
Step 3. Let C_0 be the constant in (1.7.10). For any $\tau\in\left(\frac{m+1}{2},1\right]$ and for a.e. $\rho\in(0,\rho_2)$,

$$C_0L_1ME'(\rho)^{\frac{1}{2}}\left(E(\rho)^{\frac{1}{2}}+\rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta}b(\rho)^{\frac{1-\theta}{m+1}}\leqslant\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{2}}\left(E(\rho)+b(\rho)\right)^{\frac{\gamma(\tau)+1}{2}}, \tag{1.7.15}$$

where $K_1(\tau)=CL_1^2M^2\max\left\{\rho_2^{\nu-1},1\right\}\max\{b(\rho_2)^{\mu(\tau)},b(\rho_2)^{\eta(\tau)}\}$ and $C=C(N,m)$.
Let $\tau\in\left(\frac{m+1}{2},1\right]$ and let $\rho\in(0,\rho_2)$. A straightforward calculation yields

$$\begin{aligned}&\left(E(\rho)^{\frac{1}{2}}+\rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)b(\rho)^{\frac{1-\theta}{\theta(m+1)}}\\&=E(\rho)^{\frac{1}{2}}b(\rho)^{\frac{1-\theta}{\theta(m+1)}}+\rho^{-\delta}b(\rho)^{\frac{1}{\theta(m+1)}}\\&=E(\rho)^{\frac{1}{2}}b(\rho)^{\tau(1-\theta)\ell}b(\rho)^{(1-\tau)(1-\theta)\ell}+\rho^{-\delta}b(\rho)^{\frac{1}{2}+\tau(1-\theta)\ell}b(\rho)^{\ell-\tau(1-\theta)\ell-\frac{1}{2}}\\&\leqslant2\rho^{-\delta}\max\left\{\rho_2^{\delta},1\right\}K_2(\tau)^{\frac{1}{\theta}}\left(E(\rho)+b(\rho)\right)^{\frac{1}{2}+\tau(1-\theta)\ell},\end{aligned}$$

where $K_2^2(\tau)=\max\{b(\rho_2)^{\mu(\tau)},b(\rho_2)^{\eta(\tau)}\}$. Hence (1.7.15) with $K_1(\tau)=4C_0^2L_1^2M^2K_2^2(\tau)\max\left\{\rho_2^{\nu-1},1\right\}$.
Step 4. For any $\tau\in\left(\frac{m+1}{2},1\right]$ and for a.e. $\rho\in(0,\rho_2)$,

$$0\leqslant E(\rho)^{1-\gamma(\tau)}\leqslant K_1(\tau)\rho^{-(\nu-1)}E'(\rho)+(4L_1M)^{\frac{(m+1)(1-\gamma(\tau))}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{(m+1)(1-\gamma(\tau))}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}. \tag{1.7.16}$$

Putting together (1.7.10) and (1.7.15), and applying again Young's inequality (1.7.3) with $p=\frac{2}{\gamma(\tau)+1}$, $\varepsilon=(\gamma(\tau)+1)^{\frac{\gamma(\tau)+1}{2}}$, $x=\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{2}}$ and $y=(E(\rho)+b(\rho))^{\frac{\gamma(\tau)+1}{2}}$, we obtain

$$\begin{aligned}&E(\rho)+b(\rho)\\&\leqslant\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{2}}\left(E(\rho)+b(\rho)\right)^{\frac{\gamma(\tau)+1}{2}}+(2L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))},\\&\leqslant C\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}}+\frac{1}{2}(E(\rho)+b(\rho))+(2L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))},\end{aligned}$$

where $C=\frac{p-1}{p}\varepsilon^{\frac{p}{p-1}}=C(N,m)$. Changing, if needed, the constant C in the definition of $K_1(\tau)$, we obtain

$$E(\rho)+b(\rho)\leqslant\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}}+(4L_1M)^{\frac{m+1}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{m+1}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}.$$

Raising both sides of the above inequality to the power $1-\gamma(\tau)$ and recalling that $\big(1-\gamma(\tau)\big)\in(0,1)$, we obtain (1.7.16).

Step 5. Let $\alpha\in(0,\rho_0]$. If $E(\alpha)=0$ then $\boldsymbol{u}_{|B_\Omega(x_0,\alpha)}\equiv\boldsymbol{0}$.

From our hypothesis, $E'=0$ on $(0,\alpha)$. Furthermore, $\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\alpha))}=0$ $\big($from assumption of Theorem 1.2.1 or (1.3.1)$\big)$. It follows from Step 2 and continuity of b that $b(\alpha)=0$. Hence Step 5 follows.

Step 6. Proof of Theorem 1.2.1.

Thus $\rho_2=\rho_0$ and $\|\widetilde{\boldsymbol{F}}\|_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho_0))}=0$. For any $\tau\in\left(\frac{m+1}{2},1\right]$, set $r(\tau)^\nu=\left(\rho_0^\nu-\nu\frac{K_1(\tau)E(\rho_0)^{\gamma(\tau)}}{\gamma(\tau)}\right)_+$ and let $\rho_{\max}=\max\limits_{\tau\in\left(\frac{m+1}{2},1\right]}r(\tau)$. Note that definition of $\rho_{\max}$ coincides with (1.2.8). Let $\tau\in\left(\frac{m+1}{2},1\right]$. We claim that $E(r(\tau))=0$. Otherwise, $E(r(\tau))>0$ and so $E>0$ on $[r(\tau),\rho_0)$. From (1.7.16), one has (we recall that $\gamma(\tau)-1<0$),

$$\text{for a.e. } \rho\in(r(\tau),\rho_0),\ K_1(\tau)E'(\rho)E(\rho)^{\gamma(\tau)-1}\geqslant\rho^{\nu-1}. \tag{1.7.17}$$

We integrate this estimate between $r(\tau)$ and ρ_0. We obtain

$$\nu\frac{K_1(\tau)}{\gamma(\tau)}\left(E(\rho_0)^{\gamma(\tau)}-E(r(\tau))^{\gamma(\tau)}\right)\geqslant\rho_0^\nu-r^\nu(\tau).$$

By definition of $r(\tau)$, this gives $E(r(\tau))\leqslant 0$. A contradiction, hence the claim. In particular, $E(\rho_{\max})=0$. It follows from Step 5 that $\boldsymbol{u}_{|B_\Omega(x_0,\rho_{\max})}\equiv 0$, which is the desired result. It remains to treat the case where $\rho_0=\operatorname{dist}(x_0,\partial\Omega)$. We proceed as follows. Let $n\in\mathbb{N}$, $n\geqslant\frac{1}{\rho_0}$. We work on $B\left(x_0,\rho_0-\frac{1}{n}\right)$ instead of $B(x_0,\rho_0)$ and apply the above result. Thus $\boldsymbol{u}_{|B(x_0,\rho^n_{\max})}\equiv\boldsymbol{0}$, where $\rho^n_{\max}$ is given by (1.2.8) with $\rho_0-\frac{1}{n}$ in place of ρ_0. We then let $n\nearrow\infty$ which leads to the result. This finishes the proof of Theorem 1.2.1.

Step 7. Proof of Theorem 1.3.1.

We have $\rho_2=\rho_1$. Let $\gamma=\gamma(1)$ and set for any $\rho\in[0,\rho_1]$, $F(\rho)=(4L_1M)^{\frac{(m+1)(1-\gamma)}{m}}\|\widetilde{\boldsymbol{F}}\|^{\frac{(m+1)(1-\gamma)}{m}}_{\boldsymbol{L}^{\frac{m+1}{m}}(B(x_0,\rho))}$ and $K=K_1(1)\rho_0^{-(\nu-1)}$. Let $E_\star=\left(\frac{\gamma}{2K}(\rho_1-\rho_0)\right)^{\frac{1}{\gamma}}$ and $\varepsilon_\star=\frac{1}{2^{p'}(4L_1M)^{\frac{m+1}{m}}}\left(\frac{\gamma}{2K}\right)^p$. Note that $p=\frac{1}{\gamma}$. Assume now $E(\rho_1)<E_\star$. Applying Step 4 with $\tau=1$, one has for a.e. $\rho\in(\rho_0,\rho_1)$,

$$-KE'(\rho)+E(\rho)^{1-\gamma}\leqslant F(\rho). \tag{1.7.18}$$

Let define the function G by

$$\forall\rho\in[0,\rho_1],\ G(\rho)=\left(\frac{\gamma}{2K}(\rho-\rho_0)_+\right)^{\frac{1}{\gamma}}. \tag{1.7.19}$$

Then $G(\rho_1)=E_\star$, $G\in C^1([0,\rho_1];\mathbb{R})$ $\left(\text{since } \frac{1}{\gamma}>2\right)$ and G satisfies

$$\forall\rho\in[0,\rho_1],\ -KG'(\rho)+\frac{1}{2}G(\rho)^{1-\gamma}=0, \tag{1.7.20}$$

$$E(\rho_1)<G(\rho_1). \tag{1.7.21}$$

Finally and recalling that $\gamma=\frac{1}{p}$, from our hypothesis (1.3.1) and (1.7.19), one has

$$\forall\rho\in(0,\rho_1),\ F(\rho)\leqslant\frac{1}{2}\left(\frac{\gamma}{2K}(\rho-\rho_0)_+\right)^{\frac{1-\gamma}{\gamma}}=\frac{1}{2}G(\rho)^{1-\gamma}. \tag{1.7.22}$$

Putting together (1.7.18), (1.7.22) and (1.7.20), one obtains

$$-KE'(\rho)+E(\rho)^{1-\gamma}\leqslant -KG'(\rho)+G(\rho)^{1-\gamma},\ \text{for a.e. } \rho\in(\rho_0,\rho_1). \tag{1.7.23}$$

Now, we claim that for any $\rho\in[\rho_0,\rho_1)$, $E(\rho)\leqslant G(\rho)$. Indeed, if the claim does not hold, it follows from (1.7.21) and continuity of E and G that there exist $\rho_\star\in(\rho_0,\rho_1)$ and $\delta\in(0,\rho_\star-\rho_0]$ such that

$$E(\rho_\star)=G(\rho_\star), \tag{1.7.24}$$

$$E(\rho)>G(\rho),\ \forall\rho\in(\rho_\star-\delta,\rho_\star). \tag{1.7.25}$$

It follows from (1.7.23) and (1.7.25) that for a.e. $\rho\in(\rho_\star-\delta,\rho_\star)$, $G'(\rho)<E'(\rho)$. But, with (1.7.24), this implies that for any $\rho\in(\rho_\star-\delta,\rho_\star)$, $G(\rho)>E(\rho)$, which contradicts (1.7.25), hence the claim. It follows that $0\leqslant E(\rho_0)\leqslant G(\rho_0)=0$. We deduce with help of the Step 5 that $\boldsymbol{u}_{|B_\Omega(x_0,\rho_0)}\equiv\boldsymbol{0}$, which is the desired result. It remains to treat the case where $\rho_1=\operatorname{dist}(x_0,\partial\Omega)$. We proceed as follows. Assume $E(\rho_1)<E_\star$. Then there exists $\varepsilon>0$ small enough such that $\rho_0<\rho_1-\varepsilon$ and $E(\rho_1)<E_\star(\varepsilon)$, where $E_\star(\varepsilon)=\left(\frac{\gamma}{2K}(\rho_1-\rho_0-\varepsilon)\right)^{\frac{1}{\gamma}}$. Since $\varepsilon_\star$ is a non increasing function of ρ_1, we do not need to change its definition. Estimates (1.7.18)–(1.7.23) holding with $\rho_1-\varepsilon$ in place of ρ_1, it follows that $E(\rho_0)=0$ and we finish with the help of Step 5. This ends the proof of Theorem 1.3.1. □

Proof of Theorem 1.3.3. Let $C_0=C_0(N,m)$ be the constant in estimate (1.2.8) given by Theorem 1.2.1. We then choose $C=C_0^{-1}$ in (1.3.2) and (1.3.3). Using the notations of Theorem 1.2.1 and its proof, we define for any $\tau\in\left(\frac{m+1}{2},1\right]$,

$$r(\tau)^\nu=\left((2\rho_0)^\nu-C_0M^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{(2\rho_0)^{\nu-1},1\right\}\right.$$
$$\left.\times\frac{E(2\rho_0)^{\gamma(\tau)}\max\{b(2\rho_0)^{\mu(\tau)},b(2\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right)_+,$$

and recall that $\rho_{\max}=\max\limits_{\tau\in\left(\frac{m+1}{2},1\right]}r(\tau)$. Assume (1.3.2) holds. Then $\rho_{\max}\geqslant\rho_1(1)\geqslant\rho_0$ and it follows from (1.2.8) of Theorem 1.2.1 that $b(\rho_0)=0$. Now assume (1.3.3) holds. Since $E(2\rho_0)\leqslant1$, $b(2\rho_0)\leqslant1$ and $0<\mu(\tau)<\eta(\tau)<1$, for any $\tau\in\left(\frac{m+1}{2},1\right)$, it follows from definitions of ρ_1 and $\rho_{\max}$, that

$$\rho_{\max}^\nu\geqslant\rho_1^\nu(1-s)\geqslant(2\rho_0)^\nu-C_0M^2\min\{1,L^2\}\frac{\max\{(2\rho_0)^{\nu-1},1\}}{1-m-2s}b(2\rho_0)^{\mu(1-s)}\geqslant\rho_0^\nu.$$

By (1.2.8) of Theorem 1.2.1, $b(\rho_0)=0$. This concludes the proof. □

Proof of Theorem 1.4.4. By Definition 1.2.3 and of Remark 1.2.4, 3., we can choose $\boldsymbol{\varphi}=\mathbf{i}\boldsymbol{u}$ and $\boldsymbol{\varphi}=\boldsymbol{u}$ in (1.2.10). We then obtain,

$$\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)}+\operatorname{Im}(\boldsymbol{a})\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}+\operatorname{Im}(\boldsymbol{b})\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)}=\operatorname{Im}\int\limits_\Omega\boldsymbol{F}\overline{\boldsymbol{u}}\mathrm{d}x,$$

$$\operatorname{Re}(\boldsymbol{a})\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}+\operatorname{Re}(\boldsymbol{b})\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)}=\operatorname{Re}\int\limits_\Omega\boldsymbol{F}\overline{\boldsymbol{u}}\mathrm{d}x.$$

Applying Lemma 1.7.3, these estimates yield,

$$\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(\Omega)}+L\|\boldsymbol{u}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega)}\leqslant M\int\limits_\Omega|\boldsymbol{F}|\,|\boldsymbol{u}|\,\mathrm{d}x. \tag{1.7.26}$$

We apply Young's inequality (1.7.3) with $x=|\boldsymbol{F}|$, $y=|\boldsymbol{u}|$, $\varepsilon=\left(\frac{2M}{(m+1)L}\right)^{\frac{1}{m+1}}$ and $p=m+1$. With (1.7.26), we get

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^2+\frac{L}{2}\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)}^{m+1}\leqslant M\left(\frac{2M}{L}\right)^{\frac{1}{m}}\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}^{\frac{m+1}{m}},$$

from which we deduce (1.4.1). Finally, applying Gagliardo-Nirenberg's inequality (1.7.1), with $p=m$, and Young's inequality (1.7.3), with $p=\frac{4+N(1-m)}{N(1-m)}$ and $\varepsilon=1$, one obtains

$$\|\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{2\frac{(N+2)-m(N-2)}{4+N(1-m)}}\leqslant C\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{2N(1-m)}{4+N(1-m)}}\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)}^{\frac{4(1+m)}{4+N(1-m)}}\leqslant C\left(\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^2+\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)}^{m+1}\right),$$

and finally

$$\|\boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^2\leqslant C\left(\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^2+\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)}^{m+1}\right)^{\delta+1}, \tag{1.7.27}$$

where $\delta=\frac{2(1-m)}{(N+2)-m(N-2)}$. Estimate (1.4.2) then follows from (1.4.1) and (1.7.27). □

Proof of Theorem 1.3.5. Let C be the constant given by Theorem 1.3.3 and let $\varepsilon>0$. Set $K=\operatorname{supp} F$ and $K(\varepsilon)=\overline{\mathcal{O}(\varepsilon)}$. We would like to apply Theorem 1.3.3 with $\rho_0=\frac{\varepsilon}{4}$. By (1.4.1) of Theorem 1.4.4, there exists $\delta_0=\delta_0(\varepsilon,N,m,L,M)>0$ such that if $\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}\leqslant\delta_0$ then $\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\Omega)}\leqslant 1$ and

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(\Omega)}^{\frac{2(1-m)}{k}}\leqslant C2^{-2\nu}(2^\nu-1)(1-m)M^{-2}\min\{1,L^2\}\min\{2,\varepsilon\}^{\nu-1}\varepsilon. \tag{1.7.28}$$

We recall that the distance between two closed sets $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^N$ with one of them compact is defined by

$$\operatorname{dist}(\mathcal{A},\mathcal{B})=\min_{(x,y)\in\mathcal{A}\times\mathcal{B}}|x-y|$$

and that

$$\operatorname{dist}(\mathcal{A},\mathcal{B})>0\iff \mathcal{A}\cap\mathcal{B}=\emptyset.$$

Let $x_0\in\overline{K(\varepsilon)^{\mathrm{c}}}$. Let $y\in\overline{B}\left(x_0,\frac{\varepsilon}{2}\right)$ and let $z\in K$. By definition of $K(\varepsilon)$, $\operatorname{dist}(\overline{K(\varepsilon)^{\mathrm{c}}},K)=\varepsilon$. We then have

$$\varepsilon=\operatorname{dist}(\overline{K(\varepsilon)^{\mathrm{c}}},K)\leqslant|x_0-z|\leqslant|x_0-y|+|y-z|\leqslant\frac{\varepsilon}{2}+|y-z|.$$

Taking the minimum on $(y,z)\in\overline{B}\left(x_0,\frac{\varepsilon}{2}\right)\times K$, we get

$$\frac{\varepsilon}{2}\leqslant\operatorname{dist}\left(\overline{B}\left(x_0,\frac{\varepsilon}{2}\right),K\right),$$

which means that $\overline{B}\left(x_0,\frac{\varepsilon}{2}\right)\cap K=\emptyset$, for any $x_0\in\overline{K(\varepsilon)^{\mathrm{c}}}$. By (1.7.28), $\boldsymbol{u}$ satisfies (1.3.2) with $\rho_0=\frac{\varepsilon}{4}$ and we deduce that for any $x_0\in\overline{K(\varepsilon)^{\mathrm{c}}}$, $\boldsymbol{u}_{|\Omega\cap B\left(x_0,\frac{\varepsilon}{4}\right)}\equiv\boldsymbol{0}$ (Theorem 1.3.3). Let $n\in\mathbb{N}$. By compactness, $\overline{K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}}\cap\overline{B}(0,n)$ may be covered by a finite number of balls $B\left(x_0,\frac{\varepsilon}{4}\right)$ with $x_0\in\overline{K(\varepsilon)^{\mathrm{c}}}$. Thus for any $n\in\mathbb{N}$, $\boldsymbol{u}_{|\Omega\cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}\cap B(0,n)}\equiv\boldsymbol{0}$. It follows that $\boldsymbol{u}=\boldsymbol{0}$ almost everywhere on

$$\bigcup_{n\in\mathbb{N}}\left(\Omega\cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}\cap B(0,n)\right)=\Omega\cap K\left(\frac{7\varepsilon}{8}\right)^{\mathrm{c}}.$$

This means that $\operatorname{supp} \boldsymbol{u} \subset \overline{\Omega} \cap K\left(\frac{7\varepsilon}{8}\right) \subset \overline{\Omega} \cap \mathcal{O}(\varepsilon)$. Finally, since K is a compact set, Ω is open and $K \subset \Omega$, it follows that if ε is small enough then $\mathcal{O}(\varepsilon) \subset \Omega$. This ends the proof. □

Proof of Theorem 1.3.6. Let L, M and C be the constants given by (1.2.6), (1.2.7) and Theorem 1.3.3, respectively. We would like to apply Theorem 1.3.3 with $\rho_0 = 1$. Since $\boldsymbol{F}$ is compactly supported and $\boldsymbol{u} \in \boldsymbol{H^1}(\mathbb{R}^N) \cap \boldsymbol{L^{m+1}}(\mathbb{R}^N)$, there exists $R > 1$ such that $\operatorname{supp} \boldsymbol{F} \subset B(0, R-1)$,

$$\|\boldsymbol{u}\|_{\boldsymbol{L^{m+1}}(\{|x|>R-1\})} \leqslant 1 \quad \text{and} \quad \|\nabla \boldsymbol{u}\|^{\frac{2(1-m)}{k}}_{\boldsymbol{L^2}(\{|x|>R-1\})} \leqslant C 2^{1-\nu}(2^\nu - 1)(1-m)M^{-2}\min\{1, L^2\}.$$

Let $x_0 \in \mathbb{R}^N$ be such that $|x_0| \geqslant R+1$. Then $\overline{B}(x_0, 2) \cap \operatorname{supp} \boldsymbol{F} = \emptyset$ and, with help of the above estimate, $\boldsymbol{u}$ satisfies (1.3.2) with $\rho_0 = 1$. It follows from Theorem 1.3.3 that $\boldsymbol{u}_{|B(x_0,1)} \equiv \boldsymbol{0}$. For each integer $n \geqslant 2$, define the compact set C_n by

$$C_n = \left\{ x \in \mathbb{R}^N;\ R + \frac{1}{n} \leqslant |x| \leqslant R + n - \frac{1}{n} \right\}.$$

By compactness, C_n may be covered by a finite number of balls $B(x_0, 1)$, where $R+1 \leqslant |x_0| \leqslant R+1+n$. Thus for any $n \in \mathbb{N}$, $\boldsymbol{u}_{|C_n} \equiv \boldsymbol{0}$. It follows that $\boldsymbol{u} = \boldsymbol{0}$ almost everywhere on

$$\bigcup_{n \geqslant 2} C_n = \Big\{ x \in \mathbb{R}^N;\ |x| > R \Big\}.$$

Then $\operatorname{supp} \boldsymbol{u} \subset \overline{B}(0, R)$, which is the desired result. □

1.8 Proofs of the existence and smoothness results

In this Section, we prove Proposition 1.4.5, Theorem 1.4.1 and 1.4.8.

Proof of Proposition 1.4.5. By Remarks 1.4.6, equation (1.4.3) makes senses in $\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$.

Proof of Property 1). Let $1 < q \leqslant p < \infty$. Assume $\boldsymbol{F} \in \boldsymbol{L^p_{\mathrm{loc}}}(\Omega)$ and $\boldsymbol{u} \in \boldsymbol{L^q_{\mathrm{loc}}}(\Omega)$ is a solution to (1.4.3). For $r \in (1, \infty)$, r^- denotes any real in $(1, r)$. Assume $\boldsymbol{v} \in \boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega)$, for some $1 < r < \infty$, is a solution of (1.4.3). It follows that $|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v} \in \boldsymbol{L^{\frac{r^-}{m}}_{\mathrm{loc}}}(\Omega)$ and since $0 < m < 1$, $\boldsymbol{L^{\frac{r^-}{m}}_{\mathrm{loc}}}(\Omega) \subset \boldsymbol{L^r_{\mathrm{loc}}}(\Omega)$. So by (1.4.3) and Hölder's inequality, $\boldsymbol{V}\boldsymbol{v} \in \boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega)$ and so $\Delta \boldsymbol{v} \in \boldsymbol{L^{\min\{r^-, p\}}_{\mathrm{loc}}}(\Omega)$. Furthermore, if for some $1 < r < \infty$, $\boldsymbol{v} \in \boldsymbol{L^r_{\mathrm{loc}}}(\Omega; \mathbb{C})$ and $\Delta \boldsymbol{v} \in \boldsymbol{L^r_{\mathrm{loc}}}(\Omega; \mathbb{C})$ then $\boldsymbol{v} \in \boldsymbol{W^{2,r}_{\mathrm{loc}}}(\Omega; \mathbb{C})$ (see for instance Cazenave [58], Proposition 4.1.2 p.101–102). We then have shown the following property. Let $1 < r < \infty$.

$$\boldsymbol{u} \in \boldsymbol{L^{r^-}_{\mathrm{loc}}}(\Omega) \implies \boldsymbol{u} \in \boldsymbol{W^{2,\min\{r^-, p\}}_{\mathrm{loc}}}(\Omega). \tag{1.8.1}$$

Now, we proceed to the proof of Property 1) in 2 cases.

Case 1. $\left(\frac{N}{2} \leqslant q \leqslant p\right)$ or $\left(q < \frac{N}{2} \text{ and } q \leqslant p \leqslant \frac{Nq}{N-2q}\right)$.

It follows from (1.8.1), applied with $r = q$, that $\boldsymbol{u} \in \boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega)$. In one hand, if $q < \frac{N}{2}$ then $\boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega) \subset \boldsymbol{L^{p^-}_{\mathrm{loc}}}(\Omega)$. It follows from (1.8.1) (applied with $r = p$) and Sobolev's embedding that $\boldsymbol{u} \in \boldsymbol{L^{p+\delta}_{\mathrm{loc}}}(\Omega)$, for $\delta \in (0, 1)$ small enough. On the other hand, if $q \geqslant \frac{N}{2}$ then $\boldsymbol{W^{2,q^-}_{\mathrm{loc}}}(\Omega) \subset \boldsymbol{L^{p+1}_{\mathrm{loc}}}(\Omega)$. So in both cases, $\boldsymbol{u} \in \boldsymbol{L^{p+\delta}_{\mathrm{loc}}}(\Omega)$. Applying (1.8.1) with $r = p + \delta$, we then obtain $\boldsymbol{u} \in \boldsymbol{W^{2,p}_{\mathrm{loc}}}(\Omega)$.

Case 2. $1<q<p$, $q<\frac{N}{2}$ and $\frac{Nq}{N-2q}<p$.
We recall that if $1<r<\frac{N}{2}$ then Sobolev's embedding is

$$\boldsymbol{W}^{\boldsymbol{2},\boldsymbol{r}^-}_{\mathbf{loc}}(\Omega)\subset\boldsymbol{L}^{\boldsymbol{s}^-}_{\mathbf{loc}}(\Omega),\ \text{for any }\ 1\leqslant s<\infty \text{ such that }\ \frac{1}{s}\geqslant\frac{1}{r}-\frac{2}{N}. \tag{1.8.2}$$

Since $\frac{Nq}{N-2q}<p$, we may define the smallest integer $n_0\geqslant 2$ such that $\frac{1}{q}-\frac{2n_0}{N}<\frac{1}{p}$. We then set

$$\frac{1}{p_{n_0}}=\begin{cases}\frac{1}{p+1}, & \text{if } \frac{1}{q}-\frac{2n_0}{N}\leqslant 0,\\ \frac{1}{q}-\frac{2n_0}{N}, & \text{if } \frac{1}{q}-\frac{2n_0}{N}>0,\end{cases}$$

in order to have $p<p_{n_0}<\infty$. Finally, define the n_0 real $(p_n)_{n\in[\![0,n_0-1]\!]}$ by $p_0=q$ and

$$\forall n\in[\![0,n_0-1]\!],\ \frac{1}{p_n}=\frac{1}{p_0}-\frac{2n}{N}.$$

It follows that for any $n\in[\![1,n_0-1]\!]$, $q\leqslant p_{n-1}<p_n\leqslant p<p_{n_0}<\infty$ and

$$\forall n\in[\![1,n_0]\!],\ \frac{1}{p_n}\geqslant\frac{1}{p_{n-1}}-\frac{2}{N}. \tag{1.8.3}$$

From (1.8.1)–(1.8.3) applied n_0 times (and recalling that $p<p_{n_0}<\infty$), we then obtain $\boldsymbol{u}\in\boldsymbol{W}^{\boldsymbol{2},\boldsymbol{p}}_{\mathbf{loc}}(\Omega)$. This ends the proof of Property 1).
Proof of Property 2). We recall the following Sobolev's embedding and estimate.

$$\boldsymbol{W}^{\boldsymbol{2},\boldsymbol{N+1}}_{\mathbf{loc}}(\Omega)\subset\boldsymbol{C}^{\boldsymbol{1},\frac{\boldsymbol{1}}{\boldsymbol{N+1}}}_{\mathbf{loc}}(\Omega)\subset\boldsymbol{C}^{\boldsymbol{0},\boldsymbol{1}}_{\mathbf{loc}}(\Omega), \tag{1.8.4}$$

$$\forall(\boldsymbol{z_1},\boldsymbol{z_2})\in\mathbb{C}^2,\ \Big||\boldsymbol{z_1}|^{-(1-m)}\boldsymbol{z_1}-|\boldsymbol{z_2}|^{-(1-m)}\boldsymbol{z_2}\Big|\leqslant 5|\boldsymbol{z_1}-\boldsymbol{z_2}|^m. \tag{1.8.5}$$

Assume further that $(\boldsymbol{F},\boldsymbol{V})\in\boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)\times\boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$, for some $\alpha\in(0,m]$. In particular, $\boldsymbol{V}\in\boldsymbol{L}^{\boldsymbol{\infty}}_{\mathbf{loc}}(\Omega)$ and by Property 1), $\boldsymbol{u}\in\boldsymbol{W}^{\boldsymbol{2},\boldsymbol{N+1}}_{\mathbf{loc}}(\Omega)$. It follows from (1.8.4), (1.8.5) and (1.4.3) that $|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}\in\boldsymbol{C}^{\boldsymbol{0},\boldsymbol{m}}_{\mathbf{loc}}(\Omega)$ and so $\Delta\boldsymbol{u}\in\boldsymbol{C}^{\boldsymbol{0},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$. Thus $\boldsymbol{u}\in\boldsymbol{C}^{\boldsymbol{2},\boldsymbol{\alpha}}_{\mathbf{loc}}(\Omega)$ (Theorem 9.19 p.243–244 in Gilbarg and Trudinger [90]). This concludes the proof of the proposition. □

Proof of Theorem 1.4.1. Let L and M be the constants given by (1.2.6) and (1.2.7), respectively. We proceed in 4 steps.
Step 1. Let $\Omega\subset\mathbb{R}^N$ be an open bounded subset and let $\boldsymbol{g}\in\boldsymbol{L}^{\boldsymbol{2}}(\Omega)$. Then there exists a unique solution $\boldsymbol{u}\in\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$ of

$$-\Delta\boldsymbol{u}=\boldsymbol{g},\ \text{ in } \boldsymbol{L}^{\boldsymbol{2}}(\Omega). \tag{1.8.6}$$

Moreover, there exists a positive constant $C=C(|\Omega|,N)$ such that

$$\big\|(-\Delta)^{-1}\boldsymbol{g}\big\|_{\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)}\leqslant C\|\boldsymbol{g}\|_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega)},\ \forall\boldsymbol{g}\in\boldsymbol{L}^{\boldsymbol{2}}(\Omega). \tag{1.8.7}$$

In particular, the mapping $(-\Delta)^{-1}:\boldsymbol{L}^{\boldsymbol{2}}(\Omega)\longrightarrow\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$ is linear continuous.
Existence and uniqueness come from Lax-Milgram's Theorem where the bounded coercive bilinear form a on $\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)\times\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$ and the bounded linear functional L on $\boldsymbol{H}^{\boldsymbol{-1}}(\Omega)$ are defined by

$$a(\boldsymbol{u},\boldsymbol{v})=\operatorname{Re}\int_\Omega\nabla\boldsymbol{u}(x).\overline{\nabla\boldsymbol{v}(x)}\mathrm{d}x\quad\text{ and }\quad\langle L,\boldsymbol{v}\rangle_{\boldsymbol{H}^{\boldsymbol{-1}},\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}}=\operatorname{Re}\int_\Omega\boldsymbol{v}(x)\overline{\boldsymbol{g}(x)}\mathrm{d}x,$$

respectively. Note that a is coercive due to Poincaré's inequality. Taking the $\boldsymbol{H^{-1}} - \boldsymbol{H_0^1}$ duality product of equation (1.8.6) with $\boldsymbol{u}$ and applying Poincaré's inequality, we obtain estimate (1.8.7) and so continuity of $(-\Delta)^{-1}$.

Step 2. Let $\Omega \subset \mathbb{R}^N$ be an open bounded subset, let $0 < m < 1$, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ and let $\boldsymbol{F} \in \boldsymbol{L^2}(\Omega)$. For each $\ell \in \mathbb{N}$, define $\boldsymbol{f_\ell} = \boldsymbol{g_\ell} - \mathbf{i}\boldsymbol{F}$, where

$$\forall \boldsymbol{v} \in \boldsymbol{L^2}(\Omega),\ \boldsymbol{g_\ell}(\boldsymbol{v}) = \begin{cases} \mathbf{i}\boldsymbol{a}|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v} + \mathbf{i}\boldsymbol{b}\boldsymbol{v}, & \text{if } |\boldsymbol{v}| \leqslant \ell, \\ \mathbf{i}\boldsymbol{a}\ell^m \dfrac{\boldsymbol{v}}{|\boldsymbol{v}|} + \mathbf{i}\boldsymbol{b}\ell\dfrac{\boldsymbol{v}}{|\boldsymbol{v}|}, & \text{if } |\boldsymbol{v}| > \ell. \end{cases} \tag{1.8.8}$$

Then for any $\ell \in \mathbb{N}$, there exists at least one solution $\boldsymbol{u_\ell} \in \boldsymbol{H_0^1}(\Omega)$ of

$$-\Delta \boldsymbol{u_\ell} = \boldsymbol{f_\ell}\left(\boldsymbol{u_\ell}\right),\ \text{in } \boldsymbol{L^2}(\Omega).$$

It is clear that $(\boldsymbol{f_\ell})_{\ell\in\mathbb{N}} \subset \boldsymbol{C(L^2(\Omega);L^2(\Omega))}$. With the help of Step 1 and the continuous and compact embedding $\boldsymbol{i} : \boldsymbol{H_0^1}(\Omega) \hookrightarrow \boldsymbol{L^2}(\Omega)$, we may define a continuous and compact sequence of mappings $(\boldsymbol{T_\ell})_{\ell\in\mathbb{N}}$ of $\boldsymbol{H_0^1}(\Omega)$ as follows. For any $\ell \in \mathbb{N}$, set

$$\begin{array}{rccccccc} \boldsymbol{T_\ell} : \boldsymbol{H_0^1}(\Omega) & \overset{\boldsymbol{i}}{\hookrightarrow} & \boldsymbol{L^2}(\Omega) & \overset{\boldsymbol{f_\ell}}{\longrightarrow} & \boldsymbol{L^2}(\Omega) & \xrightarrow{(-\Delta)^{-1}} & \boldsymbol{H_0^1}(\Omega) \\ \boldsymbol{v} & \longmapsto & \boldsymbol{i}(\boldsymbol{v}) = \boldsymbol{v} & \longmapsto & \boldsymbol{f_\ell}(\boldsymbol{v}) & \longmapsto & (-\Delta)^{-1}(\boldsymbol{f_\ell})(\boldsymbol{v}) \end{array}$$

Let $\ell \in \mathbb{N}$. Let C be the constant in (1.8.7) and set $R = C(|\boldsymbol{a}| + |\boldsymbol{b}| + 1)\Big(2\ell|\Omega|^{\frac{1}{2}} + \|\boldsymbol{F}\|_{\boldsymbol{L^2}(\Omega)}\Big)$. Let $\boldsymbol{v} \in \boldsymbol{H_0^1}(\Omega)$. It follows from (1.8.7) that

$$\begin{aligned} \|\boldsymbol{T_\ell}(\boldsymbol{v})\|_{\boldsymbol{H_0^1}(\Omega)} &= \left\|(-\Delta)^{-1}(\boldsymbol{f_\ell})(\boldsymbol{v})\right\|_{\boldsymbol{H_0^1}(\Omega)} \leqslant C\|\boldsymbol{f_\ell}(\boldsymbol{v})\|_{\boldsymbol{L^2}(\Omega)} \\ &\leqslant C(|\boldsymbol{a}| + |\boldsymbol{b}| + 1)\Big((\ell^m + \ell)|\Omega|^{\frac{1}{2}} + \|\boldsymbol{F}\|_{\boldsymbol{L^2}(\Omega)}\Big) \leqslant R. \end{aligned}$$

Hence, $\boldsymbol{T_\ell}\left(\boldsymbol{H_0^1}(\Omega)\right) \subset \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$, where $\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R) = \Big\{\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega); \|\boldsymbol{u}\|_{\boldsymbol{H_0^1}(\Omega)} \leqslant R\Big\}$. In a nutshell, $\boldsymbol{T_\ell}$ is a continuous and compact mapping from $\boldsymbol{H_0^1}(\Omega)$ into itself, $\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$ is a bounded, closed and convex subset of $\boldsymbol{H_0^1}(\Omega)$ and $\boldsymbol{T_\ell}\Big(\overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)\Big) \subset \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$. By the Schauder's fixed point Theorem, $\boldsymbol{T_\ell}$ admits at least one fixed point $\boldsymbol{u_\ell} \in \overline{\boldsymbol{B}}_{\boldsymbol{H_0^1}}(0,R)$. Hence Step 2 follows.

Step 3. Let be the hypotheses of the theorem. Assume further that Ω is bounded. Then equation (1.1.1) admits at least one solution $\boldsymbol{u} \in \boldsymbol{H_0^1}(\Omega)$.

In other words, we have to solve

$$-\Delta \boldsymbol{u} = \boldsymbol{f}(\boldsymbol{u}),\ \text{in } \boldsymbol{L^2}(\Omega), \tag{1.8.9}$$

where $\boldsymbol{f} = \boldsymbol{g} - \mathbf{i}\boldsymbol{F}$ and for any $\boldsymbol{v} \in \boldsymbol{L^2}(\Omega)$, $\boldsymbol{g}(\boldsymbol{v}) = \mathbf{i}\boldsymbol{a}|\boldsymbol{v}|^{-(1-m)}\boldsymbol{v} + \mathbf{i}\boldsymbol{b}\boldsymbol{v}$. Let $(\boldsymbol{F^k})_{k\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ be such that $\boldsymbol{F^k} \xrightarrow[k\to\infty]{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)} \boldsymbol{F}$ and for any $k \in \mathbb{N}$, $\|\boldsymbol{F^k}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)} \leqslant 2\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}$. Let $\boldsymbol{g_\ell}$ be defined by (1.8.8) and set for any $(k,\ell) \in \mathbb{N}^2$, $\boldsymbol{f_\ell^k} = \boldsymbol{g_\ell} - \mathbf{i}\boldsymbol{F^k}$. For any $(k,\ell) \in \mathbb{N}^2$, let $\boldsymbol{u_\ell^k} \in \boldsymbol{H_0^1}(\Omega)$ be a solution of

$$-\Delta \boldsymbol{u_\ell^k} = \boldsymbol{f_\ell}(\boldsymbol{u_\ell^k}),\ \text{in } \boldsymbol{L^2}(\Omega), \tag{1.8.10}$$

given by Step 2. We take the $\boldsymbol{H^{-1}}-\boldsymbol{H_0^1}$ duality product of equation (1.8.10) with $\boldsymbol{u_\ell^k}$ first and $\mathbf{i}\boldsymbol{u_\ell^k}$ second. Applying Lemma 1.7.3, we then get for any $(k,\ell)\in\mathbb{N}^2$,

$$\|\nabla\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^2}(\Omega)}^2+L\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^{m+1}}(\{|\boldsymbol{u_\ell^k}|\leqslant\ell\})}^{m+1}+L\ell^m\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^1}(\{|\boldsymbol{u_\ell^k}|>\ell\})}$$
$$\leqslant M\int_\Omega|\boldsymbol{F^k}||\boldsymbol{u_\ell^k}|\left(\chi_{\{|\boldsymbol{u_\ell^k}|\leqslant\ell\}}+\chi_{\{|\boldsymbol{u_\ell^k}|>\ell\}}\right)\mathrm{d}x.$$

Applying Young's inequality (1.7.3) to the first term on the right-hand side and the Hölder's inequality to the second term of the right-hand side, we arrive to the following estimate.

$$2\|\nabla\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^2}(\Omega)}^2+L\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^{m+1}}(\{|\boldsymbol{u_\ell^k}|\leqslant\ell\})}^{m+1}+2\|\boldsymbol{u_\ell^k}\|_{\boldsymbol{L^1}(\{|\boldsymbol{u_\ell^k}|>\ell\})}\big(L\ell^m-M\|\boldsymbol{F^k}\|_{\boldsymbol{L^\infty}(\Omega)}\big)$$
$$\leqslant M\left(\frac{2M}{L}\right)^{\frac{1}{m}}\|\boldsymbol{F^k}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}^{\frac{m+1}{m}}\leqslant C\|\boldsymbol{F}\|_{\boldsymbol{L^{\frac{m+1}{m}}}(\Omega)}^{\frac{m+1}{m}}. \tag{1.8.11}$$

For any $k\in\mathbb{N}$, there exists $\ell_k\in\mathbb{N}$ large enough such that $L\ell_k^m-M\|\boldsymbol{F^k}\|_{\boldsymbol{L^\infty}(\Omega)}\geqslant1$. Moreover, Ω being bounded, we have $\boldsymbol{L^{m+1}}(\Omega)\hookrightarrow\boldsymbol{L^1}(\Omega)$. So $\big(\nabla\boldsymbol{u_{\ell_k}^k}\big)_{k\in\mathbb{N}}$ and $(\boldsymbol{u_{\ell_k}^k})_{k\in\mathbb{N}}$ are bounded in $\boldsymbol{L^2}(\Omega)$ and $\boldsymbol{L^1}(\Omega)$, respectively. It follows from Gagliardo-Nirenberg's inequality (1.7.2) (applied with $p=1$), that $(\boldsymbol{u_{\ell_k}^k})_{k\in\mathbb{N}}$ is also bounded in $\boldsymbol{L^2}(\Omega)$ and so in $\boldsymbol{H_0^1}(\Omega)$. Finally, by Rellich-Kondrachov's Theorem, there exists a subsequence $(\boldsymbol{u_{\varphi(n)}^n})_{n\in\mathbb{N}}$ of $(\boldsymbol{u_{\ell_k}^k})_{k\in\mathbb{N}}$ and $h\in L^2(\Omega;\mathbb{R})$, such that

$$\boldsymbol{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{\boldsymbol{L^2}(\Omega)}\boldsymbol{u}, \tag{1.8.12}$$
$$\boldsymbol{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}\boldsymbol{u}, \tag{1.8.13}$$
$$\left|\boldsymbol{u_{\varphi(n)}^n}\right|\leqslant h,\ \text{for any } n\in\mathbb{N},\ \text{a.e. in }\Omega, \tag{1.8.14}$$

By (1.8.13) and (1.8.14),

$$\boldsymbol{g_{\varphi(n)}}\left(\boldsymbol{u_{\varphi(n)}^n}\right)\chi_{\left\{|\boldsymbol{u_{\varphi(n)}^n}|\leqslant\varphi(n)\right\}}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}\boldsymbol{g}(\boldsymbol{u}),$$
$$\forall n\in\mathbb{N},\ \left|\boldsymbol{g_{\varphi(n)}}\left(\boldsymbol{u_{\varphi(n)}^n}\right)\right|\leqslant C(h^m+h)\in\boldsymbol{L^1}(\Omega),\ \text{a.e. in }\Omega.$$

It follows from the dominated convergence Theorem that

$$\boldsymbol{g_{\varphi(n)}}\left(\boldsymbol{u_{\varphi(n)}^n}\right)\chi_{\left\{|\boldsymbol{u_{\varphi(n)}^n}|\leqslant\varphi(n)\right\}}\xrightarrow[n\to\infty]{\boldsymbol{L^1}(\Omega)}\boldsymbol{g}(\boldsymbol{u}). \tag{1.8.15}$$

In addition, by (1.8.12) and Hölder's inequality,

$$\left\|\boldsymbol{g_{\varphi(n)}}\left(\boldsymbol{u_{\varphi(n)}^n}\right)\chi_{\left\{|\boldsymbol{u_{\varphi(n)}^n}|>\varphi(n)\right\}}\right\|_{\boldsymbol{L^1}(\Omega)}\leqslant\frac{C}{\varphi(n)}\left(\|\boldsymbol{u_{\varphi(n)}^n}\|_{\boldsymbol{L^{m+1}}(\Omega)}^{m+1}+\|\boldsymbol{u_{\varphi(n)}^n}\|_{\boldsymbol{L^2}(\Omega)}^2\right)\xrightarrow{n\to\infty}0. \tag{1.8.16}$$

Putting together (1.8.15) and (1.8.16), we obtain

$$\boldsymbol{g_{\varphi(n)}}\left(\boldsymbol{u_{\varphi(n)}^n}\right)\xrightarrow[n\to\infty]{\boldsymbol{L^1}(\Omega)}\boldsymbol{g}(\boldsymbol{u}). \tag{1.8.17}$$

Since $\boldsymbol{F^n} \xrightarrow{n\to\infty} \boldsymbol{F}$ in $\boldsymbol{L}^{\frac{m+1}{m}}(\Omega) \hookrightarrow \boldsymbol{L^1}(\Omega)$, we deduce with help of (1.8.12) and (1.8.17) that

$$\Delta \boldsymbol{u}^{\boldsymbol{n}}_{\boldsymbol{\varphi(n)}} \xrightarrow[n\to\infty]{\boldsymbol{H^{-2}}(\Omega)} \Delta \boldsymbol{u}, \tag{1.8.18}$$

$$\boldsymbol{f}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}^{\boldsymbol{n}}_{\boldsymbol{\varphi(n)}}\right) \xrightarrow[n\to\infty]{\boldsymbol{L^1}(\Omega)} \boldsymbol{f}(\boldsymbol{u}). \tag{1.8.19}$$

By (1.8.10), we have for any $n \in \mathbb{N}$, $-\Delta \boldsymbol{u}^{\boldsymbol{n}}_{\boldsymbol{\varphi(n)}} = \boldsymbol{f}^{\boldsymbol{n}}_{\boldsymbol{\varphi(n)}}\left(\boldsymbol{u}^{\boldsymbol{n}}_{\boldsymbol{\varphi(n)}}\right)$, in $\boldsymbol{L^2}(\Omega)$. Estimates (1.8.18) and (1.8.19) allow to pass in the limit in this equation in the sense of $\mathscr{D}'(\Omega)$. This means that $\boldsymbol{u} \in \boldsymbol{H^1_0}(\Omega)$ is a solution of (1.8.9) and since $\boldsymbol{f}(\boldsymbol{u}) \in L^2(\Omega)$, equation (1.8.9) makes sense in $\boldsymbol{L^2}(\Omega)$.

Step 4. Conclusion. Under the hypotheses of the theorem, equation (1.1.1) admits at least one solution $\boldsymbol{u} \in \boldsymbol{H^1_0}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$ and Properties 1)–3) of the theorem hold.

For any $n \in \mathbb{N}$, we write $\Omega_n = \Omega \cap B(0,n)$. Let $n_0 \in \mathbb{N}$ be large enough to have $\Omega_{n_0} \neq \emptyset$. For each $n > n_0$, let $\boldsymbol{u_n} \in \boldsymbol{H^1_0}(\Omega_n)$ be any solution of (1.1.1) in Ω_n given by Step 3, with the external source $\boldsymbol{F_n} = \boldsymbol{F}_{|\Omega_n}$. We define $\widetilde{\boldsymbol{u_n}} \in \boldsymbol{H^1_0}(\Omega)$ by extending $\boldsymbol{u_n}$ by $\boldsymbol{0}$ in $\Omega \cap B(0,n)^{\mathrm{c}}$. Then $\nabla\widetilde{\boldsymbol{u_n}} = \nabla\boldsymbol{u_n}$, almost everywhere in Ω_n and $\nabla\widetilde{\boldsymbol{u_n}} = \boldsymbol{0}$, almost everywhere in $\Omega \cap B(0,n)^{\mathrm{c}}$. It follows from (1.4.2) of Theorem 1.4.4 that $(\boldsymbol{u_n})_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{H^1_0}(\Omega_n) \cap \boldsymbol{L^{m+1}}(\Omega_n)$, or equivalently, $(\widetilde{\boldsymbol{u_n}})_{n\in\mathbb{N}}$ is bounded in $\boldsymbol{H^1_0}(\Omega) \cap \boldsymbol{L^{m+1}}(\Omega)$. Up to a subsequence, that we still denote by $(\widetilde{\boldsymbol{u_n}})_{n\in\mathbb{N}}$, there exists $\boldsymbol{u} \in \boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$ such that $\widetilde{\boldsymbol{u_n}} \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H^1_{\mathrm{w}}}(\Omega)$, as $n \longrightarrow \infty$, and $\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L^{m+1}_{\mathrm{loc}}}(\Omega)} \boldsymbol{u}$. Let $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$. Since $\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L^{m+1}_{\mathrm{loc}}}(\Omega)} \boldsymbol{u}$, we have $|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}} \xrightarrow[n\to\infty]{\boldsymbol{L}^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega)} |\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}$, and in particular

$$\lim_{n\to\infty} \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} = \langle \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}. \tag{1.8.20}$$

Recalling that $\boldsymbol{u} \in \boldsymbol{H^1_0}(\Omega)$ and $\widetilde{\boldsymbol{u_n}} \rightharpoonup \boldsymbol{u}$ in $\boldsymbol{H^1_{\mathrm{w}}}(\Omega)$, as $n \longrightarrow \infty$, we get with help of (1.8.20),

$$\begin{aligned}\lim_{n\to\infty} \Big(\langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} &+ \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} \\ &+ \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)}\Big) = \langle -\mathbf{i}\Delta\boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}.\end{aligned} \tag{1.8.21}$$

Let $n_1 > n_0$ be large enough to have $\operatorname{supp}\boldsymbol{\varphi} \subset \Omega_{n_1}$. Using the basic properties of $\widetilde{\boldsymbol{u_n}}$ described as above and the fact $\boldsymbol{u_n}$ is a solution of (1.1.1) in Ω_n, we obtain for any $n > n_1$, $\boldsymbol{\varphi}_{|\Omega_n} \in \mathscr{D}(\Omega_n)$ and

$$\begin{aligned} 0 &= \langle -\mathbf{i}\Delta\boldsymbol{u_n} + a|\boldsymbol{u_n}|^{-(1-m)}\boldsymbol{u_n} + \boldsymbol{b}\boldsymbol{u_n} - \boldsymbol{F_n}, \boldsymbol{\varphi}_{|\Omega_n}\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)} \\ &= \big\langle \mathbf{i}\nabla\boldsymbol{u_n}, \nabla\big(\boldsymbol{\varphi}_{|\Omega_n}\big)\big\rangle_{\boldsymbol{L^2}(\Omega_n),\boldsymbol{L^2}(\Omega_n)} + \big\langle \boldsymbol{a}|\boldsymbol{u_n}|^{-(1-m)}\boldsymbol{u_n}, \boldsymbol{\varphi}_{|\Omega_n}\big\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega_n),\boldsymbol{L^{m+1}}(\Omega_n)} \\ &\quad + \langle \boldsymbol{b}\boldsymbol{u_n}, \boldsymbol{\varphi}_{|\Omega_n}\rangle_{\boldsymbol{L^2}(\Omega_n),\boldsymbol{L^2}(\Omega_n)} - \big\langle \boldsymbol{F_n}, \boldsymbol{\varphi}_{|\Omega_n}\big\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega_n),\boldsymbol{L^{m+1}}(\Omega_n)} \\ &= \langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} + \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} \\ &\quad + \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} - \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}, \end{aligned}$$

from which we deduce

$$\begin{aligned} &\langle \mathbf{i}\nabla\widetilde{\boldsymbol{u_n}}, \nabla\boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} + \langle \boldsymbol{a}|\widetilde{\boldsymbol{u_n}}|^{-(1-m)}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)} + \langle \boldsymbol{b}\widetilde{\boldsymbol{u_n}}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L^2}(\Omega),\boldsymbol{L^2}(\Omega)} \\ &= \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\boldsymbol{L}^{\frac{m+1}{m}}(\Omega),\boldsymbol{L^{m+1}}(\Omega)}, \end{aligned} \tag{1.8.22}$$

for any $n > n_1$. Passing to the limit in (1.8.22), we get with (1.8.21),

$$\langle -\mathrm{i}\Delta \boldsymbol{u} + \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{b}\boldsymbol{u}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \boldsymbol{F}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \ \forall \boldsymbol{\varphi} \in \mathscr{D}(\Omega),$$

which is the desired result. Properties 1) and 2) follow from Proposition 1.4.5. Finally, if $\boldsymbol{F}$ is spherically symmetric then $\boldsymbol{u}$, obtained as a limit, is also spherically symmetric. Indeed, we replace all the functional spaces $\boldsymbol{E}$ with $\boldsymbol{E_{\mathrm{rad}}}$ and we follow the above proof step by step. For $N = 1$, this includes the case where $\boldsymbol{F}$ is an even function. Finally, if $\boldsymbol{F}$ is an odd function, it is sufficient to work with the space $\boldsymbol{E_{\mathrm{odd}}} = \{\boldsymbol{v} \in \boldsymbol{E};\ \boldsymbol{v} \text{ is odd}\}$ in place of $\boldsymbol{E}$. Hence Property 3). □

Proof of Corollary 1.4.8. Let the assumptions of the corollary be satisfied. Let $\boldsymbol{a} = -\mathrm{i}\boldsymbol{\lambda}$, $\boldsymbol{b} = \mathrm{i}b$ and $\boldsymbol{G} = -\mathrm{i}\boldsymbol{F}$. Then $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{A} \times \mathbb{B}$ satisfies (1.2.2) and we may apply Theorem 1.4.1 and Theorem 1.3.6 to find a solution $\boldsymbol{\varphi} \in \boldsymbol{C}^{\boldsymbol{2},\boldsymbol{m}}_{\mathbf{b}}(\mathbb{R}^N)$ of (1.1.1) compactly supported for such $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{G}$. It follows that $\boldsymbol{\varphi}$ is a solution to (1.4.7). A straightforward calculation show that $\boldsymbol{u}$ defined by (1.4.6) is a solution to (1.4.5). This ends the proof. □

1.9 Proofs of the uniqueness results

In this Section, we prove Theorems 1.1.1, 1.1.2, 1.5.1 and 1.5.2, and Corollaries 1.5.3, 1.5.4 and 1.5.5. Let $0 < m \leqslant 1$. Set for any $\boldsymbol{z} \in \mathbb{C}$, $\boldsymbol{f}(\boldsymbol{z}) = |\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}$, where it is understood that $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{0}$. The proof of Theorem 1.5.1 relies on the two following lemmas.

Lemma 1.9.1. *Let $0 < m \leqslant 1$. Then there exists a positive constant C such that*

$$\forall (\boldsymbol{z_1}, \boldsymbol{z_2}) \in \mathbb{C}^2,\ \mathrm{Re}\Big(\big(\boldsymbol{f}(\boldsymbol{z_1}) - \boldsymbol{f}(\boldsymbol{z_2})\big)(\overline{\boldsymbol{z_1} - \boldsymbol{z_2}})\Big) \geqslant C\frac{|\boldsymbol{z_1} - \boldsymbol{z_2}|^2}{(|\boldsymbol{z_1}| + |\boldsymbol{z_2}|)^{1-m}},$$

as soon as $|\boldsymbol{z_1}| + |\boldsymbol{z_2}| > 0$.

Proof. We denote by $|\ .\ |_2$ the Euclidean norm in $\mathbb{R}^2$. From Lemma 4.10, p.264 of Díaz [73], there exists a positive constant C such that

$$\Big(|X|_2^{-(1-m)}X - |Y|_2^{-(1-m)}Y\Big).(X - Y) \geqslant C\frac{|X - Y|_2^2}{(|X|_2 + |Y|_2)^{1-m}},$$

for any $(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2$ satisfying $|X|_2 + |Y|_2 > 0$. We apply this lemma with $X = \begin{pmatrix} \mathrm{Re}(\boldsymbol{z_1}) \\ \mathrm{Im}(\boldsymbol{z_1}) \end{pmatrix}$ and $Y = \begin{pmatrix} \mathrm{Re}(\boldsymbol{z_2}) \\ \mathrm{Im}(\boldsymbol{z_2}) \end{pmatrix}$. Note that $|X|_2 = |\boldsymbol{z_1}|$, $|Y|_2 = |\boldsymbol{z_2}|$ and $|X - Y|_2 = |\boldsymbol{z_1} - \boldsymbol{z_2}|$. The result follows from a direct calculation. □

Corollary 1.9.2. *Let $0 < m \leqslant 1$. Then,*

$$\mathrm{Re}\Big(\big(\boldsymbol{f}(\boldsymbol{z_1}) - \boldsymbol{f}(\boldsymbol{z_2})\big)(\overline{\boldsymbol{z_1} - \boldsymbol{z_2}})\Big) \geqslant 0,$$

for any $(\boldsymbol{z_1}, \boldsymbol{z_2}) \in \mathbb{C}^2$.

Proof. The result is clear if $|\boldsymbol{z_1}| + |\boldsymbol{z_2}| = 0$. Otherwise, apply Lemma 1.9.1. □

Remark 1.9.3. Corollary 1.9.2 still holds for any $m > 0$ and can be directly obtained as follows. The mapping f (considered as a function from $\mathbb{R}^2$ onto $\mathbb{R}^2$) is the derivative of the convex function

$$F: \begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & \frac{1}{m+1}(x^2+y^2)^{\frac{m+1}{2}}. \end{array}$$

It follows that f is a monotone function (Proposition 5.5 p.25 of Ekeland and Temam [83]).

Lemma 1.9.4. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $0 < m < 1$, let $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2$ satisfying (1.2.3) and let $\boldsymbol{F_1},\boldsymbol{F_2} \in \boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$ be such that $\boldsymbol{F_1}-\boldsymbol{F_2} \in \boldsymbol{L^2}(\Omega)$. Let $\boldsymbol{u_1},\boldsymbol{u_2} \in \boldsymbol{H^1_0}(\Omega)\cap\boldsymbol{L^{m+1}}(\Omega)$ be two solutions of (1.5.1) and (1.5.2), respectively. Then there exists a positive constant $C = C(N,m)$ satisfying the following property. If $\boldsymbol{a} \neq \boldsymbol{0}$ then*

$$\operatorname{Im}(\boldsymbol{a})\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}} + C|\boldsymbol{a}|^2\int\limits_\omega \frac{|\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)|^2}{(|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|)^{1-m}}\mathrm{d}x + \operatorname{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right)\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}}$$
$$\leqslant \operatorname{Re}\int\limits_\Omega \overline{\boldsymbol{a}}\big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\big(\overline{\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)}\big)\mathrm{d}x, \quad (1.9.1)$$

where $\omega = \Big\{x \in \Omega; |\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)| > 0\Big\}$. *If $\boldsymbol{a} = \boldsymbol{0}$ then*

$$\operatorname{Re}(\boldsymbol{b})\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}} = \operatorname{Re}\int\limits_\Omega \big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\big(\overline{\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)}\big)\mathrm{d}x, \quad (1.9.2)$$

$$\|\nabla\boldsymbol{u_1}-\nabla\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}} + \operatorname{Im}(\boldsymbol{b})\|\boldsymbol{u_1}-\boldsymbol{u_2}\|^2_{\boldsymbol{L^2}} = \operatorname{Im}\int\limits_\Omega \big(\boldsymbol{F_1}(x)-\boldsymbol{F_2}(x)\big)\big(\overline{\boldsymbol{u_1}(x)-\boldsymbol{u_2}(x)}\big)\mathrm{d}x. \quad (1.9.3)$$

Proof. Let $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ be two solutions of (1.1.1) and (1.2.1) and set $\boldsymbol{u} = \boldsymbol{u_1}-\boldsymbol{u_2}$ and $\boldsymbol{F} = \boldsymbol{F_1}-\boldsymbol{F_2}$. Then $\boldsymbol{u}$ satisfies

$$-\mathbf{i}\Delta\boldsymbol{u} + \boldsymbol{a}\big(\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2})\big) + \boldsymbol{b}\boldsymbol{u} = \boldsymbol{F}, \text{ in } \boldsymbol{H^{-1}}(\Omega)+\boldsymbol{L^{\frac{m+1}{m}}}(\Omega). \quad (1.9.4)$$

Assume $\boldsymbol{a} \neq \boldsymbol{0}$. We take the $\boldsymbol{H^{-1}}+\boldsymbol{L^{\frac{m+1}{m}}} - \boldsymbol{H^1_0}\cap\boldsymbol{L^{m+1}}$ duality product of (1.9.4) with $\boldsymbol{a}\boldsymbol{u}$. We obtain,

$$\operatorname{Im}(\boldsymbol{a})\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}} + |\boldsymbol{a}|^2\langle\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2}),\boldsymbol{u}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}},\boldsymbol{L^{m+1}}} + \operatorname{Re}\left(\boldsymbol{a}\overline{\boldsymbol{b}}\right)\|\boldsymbol{u}\|^2_{\boldsymbol{L^2}} = \langle\overline{\boldsymbol{a}}\boldsymbol{F},\boldsymbol{u}\rangle_{\boldsymbol{L^2},\boldsymbol{L^2}}. \quad (1.9.5)$$

Applying Lemma 1.9.1, there exists a positive constant $C = C(N,m)$ such that

$$\langle\boldsymbol{f}(\boldsymbol{u_1})-\boldsymbol{f}(\boldsymbol{u_2}),\boldsymbol{u}\rangle_{\boldsymbol{L^{\frac{m+1}{m}}},\boldsymbol{L^{m+1}}} \geqslant C\int\limits_\omega \frac{|\boldsymbol{u}(x)|^2}{(|\boldsymbol{u_1}(x)|+|\boldsymbol{u_2}(x)|)^{1-m}}\mathrm{d}x. \quad (1.9.6)$$

Then (1.9.1) follows from (1.9.5) and (1.9.6). We turn out the case $\boldsymbol{a} = \boldsymbol{0}$. Taking the $\boldsymbol{H^{-1}}+\boldsymbol{L^{\frac{m+1}{m}}} - \boldsymbol{H^1_0}\cap\boldsymbol{L^{m+1}}$ duality product of (1.9.4) with $\boldsymbol{u}$ and $\mathbf{i}\boldsymbol{u}$, one respectively obtains (1.9.2) and (1.9.3). □

Proof of Theorem 1.5.1. Note that since $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{C}^2\setminus\{(\boldsymbol{0},\boldsymbol{0})\}$ satisfies (1.2.3), if $\boldsymbol{a} = \boldsymbol{0}$ and $\operatorname{Re}(\boldsymbol{b}) = 0$ then one necessarily has $\operatorname{Im}(\boldsymbol{b}) > 0$. We apply estimates (1.9.1)–(1.9.3) of Lemma 1.9.4, according to the different cases, and Cauchy-Schwarz's inequality. Estimates (1.5.3) and (1.5.4) follow. □

Proof of Theorem 1.5.2. Let $\boldsymbol{F} \in \boldsymbol{L}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$ and let $\boldsymbol{u_1}, \boldsymbol{u_2} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega) \cap \boldsymbol{L}^{\boldsymbol{m}+\mathbf{1}}(\Omega)$ be two solutions of (1.1.1) and (1.2.1). By Lemma 1.9.4, (1.9.1)–(1.9.3) hold with $\boldsymbol{F_1} - \boldsymbol{F_2} = \mathbf{0}$. We first note that, since $\boldsymbol{u_1} - \boldsymbol{u_2} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$, if $\|\nabla \boldsymbol{u_1} - \nabla \boldsymbol{u_2}\|_{\boldsymbol{L^2}} = 0$ then $\boldsymbol{u_1} - \boldsymbol{u_2} = \mathbf{0}$, a.e. in Ω and uniqueness holds. It follows from hypotheses (1.2.3) and Lemma 1.9.4 that one necessarily has $\|\boldsymbol{u_1} - \boldsymbol{u_2}\|_{\boldsymbol{L^2}} = 0$, $\|\nabla \boldsymbol{u_1} - \nabla \boldsymbol{u_2}\|_{\boldsymbol{L^2}} = 0$ or $\int\limits_\omega \frac{|\boldsymbol{u_1} - \boldsymbol{u_2}|^2}{(|\boldsymbol{u_1}(x)| + |\boldsymbol{u_2}(x)|^{1-m})} \mathrm{d}x$, where $\omega = \big\{x \in \Omega; |\boldsymbol{u_1}(x)| + |\boldsymbol{u_2}(x)| > 0\big\}$. Those three cases imply that $\boldsymbol{u_1} = \boldsymbol{u_2}$, a.e. in Ω. This finishes the proof of the theorem. □

Proof of Corollary 1.5.3. Apply Theorem 1.4.1, Theorem 1.5.2 and Remark 1.6.1. □

Proof of Corollary 1.5.4. By uniqueness (Theorem 1.5.2), $\boldsymbol{u} \equiv \mathbf{0}$ is the unique solution. □

Proof of Corollary 1.5.5. Apply Theorem 1.3.6, Theorem 1.4.1, Proposition 1.4.5, Theorem 1.5.2 and Remark 1.6.1. □

Proof of Theorem 1.1.1. Apply Theorem 1.3.6 and Corollary 1.5.3. □

Proof of Theorem 1.1.2. Apply Theorem 1.3.5 and Corollary 1.5.3. □

Chapitre 2

Existence of weak solutions to some stationary Schrödinger equations with singular nonlinearity

with Jesús Ildefonso Díaz*

Abstract

We prove some existence (and sometimes also uniqueness) of solutions to some stationary equations associated to the complex Schrödinger operator under the presence of a singular nonlinear term. Among other new facts, with respect some previous results in the literature for such type of nonlinear potential terms, we include the case in which the spatial domain is possibly unbounded (something which is connected with some previous localization results by the authors), the presence of possible non-local terms at the equation, the case of boundary conditions different to the Dirichlet ones and, finally, the proof of the existence of solutions when the right-hand side term of the equation is beyond the usual L^2-space.

2.1 Introduction

This paper is concerned by existence of solutions for two kinds of equations related to the complex Schrödinger operator,

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega), \tag{2.1.1}$$

$$-\Delta u + a|u|^{-(1-m)}u + bu + cV^2u = F, \text{ in } L^2(\Omega), \tag{2.1.2}$$

with homogeneous Dirichlet boundary condition

$$u_{|\Gamma} = 0, \tag{2.1.3}$$

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or homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu}_{|\Gamma} = 0, \tag{2.1.4}$$

where Ω is a subset of $\mathbb{R}^N$ with boundary Γ, $0 < m < 1$, $(a,b,c) \in \mathbb{C}^3$ and $V \in L^\infty(\Omega;\mathbb{R})$ is a real potential. Here and in what follows, when Γ is of class C^1, ν denotes the outward unit normal vector to Γ. Moreover, $\Delta = \sum\limits_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian in Ω.

In Bégout and Díaz [25], the authors study the spatial localization property compactness of the support of solutions of equation (2.1.1) (see Theorems 1.3.1, 1.3.5, 1.3.6, 1.4.1, 1.4.4 and 1.5.2). Existence, uniqueness and *a priori* bound are also established with the homogeneous Dirichlet boundary condition, $F \in L^p(\Omega)$ $(2 < p < \infty)$ and $(a,b) \in \mathbb{C}^2$ satisfying assumptions (2.2.7) below. In this paper, we give such existence and *a priori* bound results but for the weaker assumption $F \in L^2(\Omega)$ (Theorems 2.2.8 and 2.2.9) and also for some different hypotheses on $(a,b) \in \mathbb{C}^2$ (Theorems 2.2.1 and 2.2.3). Additionally, we consider homogeneous Neumann boundary condition (Theorems 2.2.8 and 2.2.9).

In Bégout and Díaz [26], spatial localization property for the partial differential equation (2.1.2) associated to self-similar solutions of the nonlinear Schrödinger equation

$$\mathrm{i}u_t + \Delta u = a|u|^{-(1-m)}u + f(t,x),$$

is studied.

In this paper, we prove existence of solutions with homogeneous Dirichlet or Neumann boundary conditions (Theorems 2.2.4) and establish *a priori* bounds (Theorem 2.2.6), for both equations (2.1.1) and (2.1.2) with any of both boundary conditions (2.1.3) or (2.1.4). We also show uniqueness (Theorem 2.2.10) and regularity results (Theorem 2.2.12), under suitable additional conditions. We send the reader to the long introduction of Bégout and Díaz [26] for many comments on the frameworks in which the equation arises (Quantum Mechanics, Nonlinear Optics and Hydrodynamics) and their connections with some other papers in the literature.

This paper is organized as follows. In the next section, we give results about existence, uniqueness, regularity and *a priori* bounds for equations (2.1.1) and (2.1.2), with boundary conditions (2.1.3) or (2.1.4), and notations are given in Section 2.3. Section 2.4, is devoted to the establishment of *a priori* bounds for the different truncated nonlinearities of equations studied in this paper. In Section 2.5, we prove the results given in Section 2.2. In Bégout and Díaz [25], localization property is studied for equation (2.1.1). The results we give require, sometimes, the same assumptions on $(a,b) \in \mathbb{C}^2$ as in Bégout and Díaz [25] but with a change of notation. See Comments 2.2.7 below for the motivation of this change. In Section 2.6 we will show the existence of solutions to equation (2.1.2) for data in a weighted subspace. Finally, in the last section, we state the principal results obtained in this paper and give some applications. Existence of solutions for equation (2.1.2) is used in Bégout and Díaz [26] while existence of solutions for equation (2.1.1) is used in Bégout and Díaz [27].

2.2 Main results

Here, we state the main results of this paper.

Theorem 2.2.1 (Existence). *Let Ω an open subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and assume $0 < m < 1$, $(a, b) \in \mathbb{C}^2$ and $F \in L^2(\Omega)$. If $\mathrm{Re}(b) < 0$ then assume further that $\mathrm{Im}(b) \neq 0$ or $-\frac{1}{C_{\mathrm{P}}^2} < \mathrm{Re}(b)$, where C_{P} is the Poincaré's constant in (2.4.1) below. Then there exists at least a solution $u \in H_0^1(\Omega)$ of (2.1.1). In addition, Symmetry Property 2.2.2 below holds.*

Symmetry Property 2.2.2. *If furthermore, for any $\mathcal{R} \in SO_N(\mathbb{R})$, $\mathcal{R}\Omega = \Omega$ and if F is spherically symmetric then we may construct a solution which is additionally spherically symmetric. For $N = 1$, this means that if F is an even (respectively, an odd) function then u is also an even (respectively, an odd) function.*

Theorem 2.2.3 (*A priori* bound). *Let Ω an open subset of $\mathbb{R}^N$ be such that $|\Omega| < \infty$ and assume $0 < m < 1$, $(a, b) \in \mathbb{C}^2$ and $F \in L^2(\Omega)$. If $\mathrm{Re}(b) < 0$ then assume further that $\mathrm{Im}(b) \neq 0$ or $-\frac{1}{C_{\mathrm{P}}^2} < \mathrm{Re}(b)$, where C_{P} is the constant in (2.4.1) below. Let $u \in H_0^1(\Omega)$ be any solution to (2.1.1). Then we have the following estimate.*

$$\|u\|_{H_0^1(\Omega)} \leqslant C,$$

where $C = C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m)$.

Theorem 2.2.4 (Existence). *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset and assume $V \in L^\infty(\Omega; \mathbb{R})$, $0 < m < 1$, $(a, b, c) \in \mathbb{C}^3$ is such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then assume further that $\mathrm{Im}(a) < 0$. Then we have the following result.*

1) *For any $F \in L^2(\Omega)$, there exists at least a solution $u \in H_0^1(\Omega) \cap L^{m+1}(\Omega)$ to (2.1.2).*
2) *If we assume furthermore that Ω is bounded with a C^1 boundary then the conclusion 1) still holds true with $u \in H^1(\Omega)$ and the boundary condition (2.1.4) instead of $u \in H_0^1(\Omega)$.*

If, in addition, V is spherically symmetric then Symmetry Property 2.2.2 holds.

Remark 2.2.5. Here are some comments about boundary condition.

1) If $u \notin C(\overline{\Omega})$ and Ω has not a $C^{0,1}$ boundary, the condition $u_{|\Gamma} = 0$ does not make sense (in the sense of the trace) and, in this case, has to be understood as $u \in H_0^1(\Omega)$.
2) Assume that Ω is bounded and has a $C^{1,1}$ boundary. Let $u \in H^1(\Omega)$ be any solution to (2.1.2) with the boundary condition (2.1.4). Then $u \in H^2(\Omega)$ and boundary condition $\frac{\partial u}{\partial \nu}_{|\Gamma} = 0$ makes sense in the sense of the trace $\gamma(\nabla u.\nu) = 0$. If, in addition, $u \in C^1(\overline{\Omega})$ then obviously for any $x \in \Gamma$, $\frac{\partial u}{\partial \nu}(x) = 0$. Indeed, since $u \in H^1(\Omega)$, $\Delta u \in L^2(\Omega)$ and (2.1.2) makes sense almost everywhere in Ω, we have $\gamma\left(\frac{\partial u}{\partial \nu}\right) \in H^{-\frac{1}{2}}(\Gamma)$ and by Green's formula,

$$\mathrm{Re}\int_\Omega \nabla u(x).\overline{\nabla v(x)}\mathrm{d}x - \left\langle \gamma\left(\frac{\partial u}{\partial \nu}\right), \gamma(v)\right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} + \mathrm{Re}\int_\Omega f\big(u(x)\big)\overline{v(x)}\mathrm{d}x = \mathrm{Re}\int_\Omega F(x)\overline{v(x)}\mathrm{d}x, \quad (2.2.1)$$

for any $v \in H^1(\Omega)$, where $fu) = a|u|^{-(1-m)}u + bu + cV^2u$. (see Lemma 4.1, Theorem 4.2 and Corollary 4.1, p.155, in Lions and Magenes [129] and (1,5,3,10) in Grisvard [93], p.62).

This implies that

$$\left\langle \gamma\left(\frac{\partial u}{\partial \nu}\right), \gamma(v) \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0, \tag{2.2.2}$$

for any $v \in H^1(\Omega)$. Let $w \in H^{\frac{1}{2}}(\Gamma)$. Let $v \in H^1(\Omega)$ be such that $\gamma(v) = w$ (Theorem 1.5.1.3, p.38, in Grisvard [93]). We then deduce from (2.2.2) that,

$$\forall w \in H^{\frac{1}{2}}(\Gamma), \ \left\langle \gamma\left(\frac{\partial u}{\partial \nu}\right), w \right\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0,$$

and so $\gamma\left(\frac{\partial u}{\partial \nu}\right) = 0$. But also $u \in L^2(\Omega)$ and $\Delta u \in L^2(\Omega)$. It follows that $u \in H^2(\Omega)$ (Proposition 2.5.2.3, p.131, in Grisvard [93]). Hence the result.

Theorem 2.2.6 (***A priori* bound**)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $V \in L^\infty(\Omega;\mathbb{R})$, let $0 < m < 1$, let $(a,b,c) \in \mathbb{C}^3$ be such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then assume further that $\mathrm{Im}(a) < 0$. Let $F \in L^2(\Omega)$ and let $u \in H^1(\Omega)$ be any solution to (2.1.2) with boundary condition (2.1.3) or (2.1.4)* [1]*. Then we have the following estimate.*

$$\|u\|_{H^1(\Omega)}^2 + \|u\|_{L^{m+1}(\Omega)}^{m+1} \leqslant M\big(\|V\|_{L^\infty(\Omega)}^4 + 1\big)\|F\|_{L^2(\Omega)}^2,$$

where $M = M(|a|,|b|,|c|)$.

Comments 2.2.7. In the context of the paper of Bégout and Díaz [25], we can establish an existence result with the homogeneous Neumann boundary condition (instead of the homogeneous Dirichlet condition) and $F \in L^2(\Omega)$ $\big($instead of $F \in L^{\frac{m+1}{m}}(\Omega)\big)$. In Bégout and Díaz [25], we introduced the set,

$$\widetilde{\mathbb{A}} = \mathbb{C} \setminus \big\{z \in \mathbb{C}; \mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) \leqslant 0\big\},$$

and assumed that $(\widetilde{a},\widetilde{b}) \in \mathbb{C}^2$ satisfies,

$$(\widetilde{a},\widetilde{b}) \in \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \quad \text{and} \quad \begin{cases} \mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b}) \geqslant 0, \\ \text{or} \\ \mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b}) < 0 \ \text{ and } \ \mathrm{Im}(\widetilde{b}) > \dfrac{\mathrm{Re}(\widetilde{b})}{\mathrm{Re}(\widetilde{a})}\mathrm{Im}(\widetilde{a}), \end{cases} \tag{2.2.3}$$

with possibly $\widetilde{b} = 0$, and we worked with

$$-\mathrm{i}\Delta u + \widetilde{a}|u|^{-(1-m)}u + \widetilde{b}u = \widetilde{F}.$$

Nevertheless, to maintain a closer notation to many applied works in the literature (see, e.g., the introduction of Bégout and Díaz [26]), we do not work any more with this equation but with,

$$-\Delta u + a|u|^{-(1-m)}u + bu = F,$$

1. for which we additionally assume that Ω has a C^1 boundary.

and $b \neq 0$. This means that we chose, $\widetilde{a} = \mathrm{i}a$, $\widetilde{b} = \mathrm{i}b$ and $\widetilde{F} = \mathrm{i}F$. Then assumptions on (a,b) are changed by the fact that for $\widetilde{z} = \mathrm{i}z$,

$$\mathrm{Re}(z) = \mathrm{Re}(-\mathrm{i}\widetilde{z}) = \mathrm{Im}(\widetilde{z}), \tag{2.2.4}$$

$$\mathrm{Im}(z) = \mathrm{Im}(-\mathrm{i}\widetilde{z}) = -\mathrm{Re}(\widetilde{z}). \tag{2.2.5}$$

It follows that the set $\widetilde{\mathbb{A}}$ and (2.2.3) become,

$$\mathbb{A} = \mathbb{C} \setminus \big\{ z \in \mathbb{C}; \mathrm{Re}(z) \leqslant 0 \text{ and } \mathrm{Im}(z) = 0 \big\}, \tag{2.2.6}$$

$$(a,b) \in \mathbb{A} \times \mathbb{A} \quad \text{and} \quad \begin{cases} \mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0, \\ \text{or} \\ \mathrm{Im}(a)\mathrm{Im}(b) < 0 \ \text{ and } \ \mathrm{Re}(b) > \dfrac{\mathrm{Im}(b)}{\mathrm{Im}(a)}\mathrm{Re}(a). \end{cases} \tag{2.2.7}$$

Obviously,

$$\Big((\widetilde{a}, \widetilde{b}) \in \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \text{ satisfies (2.2.3)} \Big) \iff \Big((a,b) \in \mathbb{A} \times \mathbb{A} \text{ satisfies (2.2.7)} \Big).$$

Assumptions (2.2.7) are made to prove the existence and the localization property of solutions to equation (2.1.1). Now, we give some results about equation (2.1.1) when $(a,b) \in \mathbb{A} \times \mathbb{A}$ satisfies (2.2.7).

Theorem 2.2.8 (Existence). *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$, let $0 < m < 1$ and let $(a,b) \in \mathbb{A}^2$ satisfies (2.2.7).*

1) *For any $F \in L^2(\Omega)$, there exists at least a solution $u \in H^1_0(\Omega) \cap L^{m+1}(\Omega)$ to*

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega). \tag{2.2.8}$$

2) *If we assume furthermore that Ω is bounded with a C^1 boundary then the conclusion* 1) *still holds true with $u \in H^1(\Omega)$ and the boundary condition (2.1.4) instead of $u \in H^1_0(\Omega)$.*

In addition, Symmetry Property 2.2.2 holds.

Theorem 2.2.9 (*A priori* bound). *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$, let $0 < m < 1$ and let $(a,b) \in \mathbb{A}^2$ satisfies (2.2.7). Let $F \in L^2(\Omega)$ and let $u \in H^1(\Omega) \cap L^{m+1}(\Omega)$ be any solution to (2.2.8) with boundary condition (2.1.3) or (2.1.4)*[1]*. Then we have the following estimate.*

$$\|u\|^2_{H^1(\Omega)} + \|u\|^{m+1}_{L^{m+1}(\Omega)} \leqslant M\|F\|^2_{L^2(\Omega)},$$

where $M = M(|a|,|b|)$.

Theorem 2.2.10 (Uniqueness). *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $V \in L^\infty_{\mathrm{loc}}(\Omega;\mathbb{R})$, let $0 < m < 1$ and let $(a,b,c) \in \mathbb{C}^3$ satisfies one of the three following conditions.*

1) $a \neq 0$, $\mathrm{Re}(a) \geqslant 0$, $\mathrm{Re}(a\overline{b}) \geqslant 0$ *and* $\mathrm{Re}(a\overline{c}) \geqslant 0$.

2) $b \neq 0$, $\mathrm{Re}(b) \geqslant 0$, $a = kb$, *for some* $k \geqslant 0$ *and* $\mathrm{Re}(b\overline{c}) \geqslant 0$.

3) *$c \neq 0$, $\operatorname{Re}(c) \geqslant 0$, $a = kc$, for some $k > 0$ and $\operatorname{Re}(b\overline{c}) \geqslant 0$.*

Let $F \in L^1_{\mathrm{loc}}(\Omega)$. If there exist two solutions $u_1, u_2 \in H^1(\Omega) \cap L^{m+1}(\Omega)$ of (2.1.2) with the same boundary condition (2.1.3) or (2.1.4)[1] such that $Vu_1, Vu_2 \in L^2(\Omega)$ then $u_1 = u_2$.

Remark 2.2.11. Here are some comments about Theorems 2.2.1, 2.2.4, 2.2.8 and 2.2.10.

1) Assume F is spherically symmetric. Since we do not know, in general, if we have uniqueness of the solution, we are not able to show that any solution is radially symmetric.

2) In Theorem 1.5.2, uniqueness for equation

$$-\mathrm{i}\Delta u + \widetilde{a}|u|^{-(1-m)}u + \widetilde{b}u = \widetilde{F},$$

holds if $\widetilde{a} \neq 0$, $\operatorname{Im}(\widetilde{a}) \geqslant 0$ and $\operatorname{Re}(\widetilde{a}\overline{\widetilde{b}}) \geqslant 0$. By (2.2.4)–(2.2.5), those assumptions are equivalent to 1) of Theorem 2.2.10 above for equation (2.1.1) (of course, $c = 0$). It follows that Theorem 2.2.10 above extends Theorem 1.5.2.

3) In 2) of the above theorem, if we want to make an analogy with 1), assumption $a = kb$, for some $k \geqslant 0$ has to be replaced with $\operatorname{Re}(a\overline{b}) \geqslant 0$ and $\operatorname{Im}(a\overline{b}) = 0$. But,

$$\Big(\operatorname{Re}(a\overline{b}) \geqslant 0 \text{ and } \operatorname{Im}(a\overline{b}) = 0\Big) \iff \Big(\exists k \geqslant 0 / a = kb\Big).$$

In the same way,

$$\Big(\operatorname{Re}(a\overline{c}) > \text{ and } \operatorname{Im}(a\overline{c}) = 0\Big) \iff \Big(\exists k > 0 / a = kc\Big).$$

4) In the case of real solutions (with $F \equiv 0$ and $(a,b,c) \in \mathbb{R} \times \mathbb{R} \times \{0\}$), it is well-known that if $b < 0$ then it may appear multiplicity of solutions (once $m \in (0,1)$ and $a > 0$). For more details, see Theorem 1 in Díaz and Hernández [74].

Theorem 2.2.12 (Regularity). *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $V \in L^r_{\mathrm{loc}}(\Omega;\mathbb{C})$, for any $1 < r < \infty$, let $0 < m < 1$, let $(a,b) \in \mathbb{C}^2$, let $F \in L^1_{\mathrm{loc}}(\Omega)$, let $1 < q < \infty$ and let $u \in L^q_{\mathrm{loc}}(\Omega)$ be any local solution to*

$$-\Delta u + a|u|^{-(1-m)}u + Vu = F, \text{ in } \mathscr{D}'(\Omega). \tag{2.2.9}$$

Let $q \leqslant p < \infty$ and let $\alpha \in (0,m]$.

1) *If $F \in L^p_{\mathrm{loc}}(\Omega)$ then $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$. If $(F,V) \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \times C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ then $u \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$.*

2) *Assume further that Ω is bounded with a $C^{1,1}$ boundary, $F \in L^p(\Omega)$, $V \in L^r(\Omega;\mathbb{C})$, for any $1 < r < \infty$, $u \in L^q(\Omega)$ and $\gamma(u) = 0$. Then $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. If $(F,V) \in C^{0,\alpha}(\overline{\Omega}) \times C^{0,\alpha}(\overline{\Omega})$ then $u \in C^{2,\alpha}(\overline{\Omega}) \cap C_0(\Omega)$.*

3) *Assume further that Ω is bounded with a $C^{1,1}$ boundary, $F \in L^p(\Omega)$, $V \in L^r(\Omega;\mathbb{C})$, for any $1 < r < \infty$, $u \in L^q(\Omega)$ and $\gamma\left(\frac{\partial u}{\partial \nu}\right) = 0$. Then $u \in W^{2,p}(\Omega)$. If $(F,V) \in C^{0,\alpha}(\overline{\Omega}) \times C^{0,\alpha}(\overline{\Omega})$ then $u \in C^{2,\alpha}(\overline{\Omega})$ and for any $x \in \Gamma$, $\frac{\partial u}{\partial \nu}(x) = 0$.*

Remark 2.2.13. Assume Ω is bounded and has a $C^{1,1}$ boundary. Let $V \in \bigcap\limits_{1<r<\infty} L^r(\Omega;\mathbb{C})$, $0 < m < 1$, $(a,b) \in \mathbb{C}^2$, $1 < q \leqslant p < \infty$, $F \in L^p(\Omega)$ and let $u \in L^q(\Omega)$ be any solution to (2.2.9). Let

$T : u \longrightarrow \left\{\gamma(u), \gamma\left(\frac{\partial u}{\partial \nu}\right)\right\}$ be the trace function defined on $\mathscr{D}(\overline{\Omega})$. By density of $\mathscr{D}(\overline{\Omega})$ in $D_q(\Delta) \stackrel{\text{def}}{=} \left\{u \in L^q(\Omega); \Delta u \in L^q(\Omega)\right\}$, T has a linear and continuous extension from $D_q(\Delta)$ into $W^{-\frac{1}{q},q}(\Gamma) \times W^{-1-\frac{1}{q},q}(\Gamma)$ (Hörmander [107], Theorem 2 p.503 ; Lions and Magenes [129], Lemma 2.2 and Theorem 2.1 p.147 ; Lions and Magenes [130], Propositions 9.1, Proposition 9.2 and Theorem 9.1 p.82 ; Grisvard [93], p.54). Since $u \in L^q(\Omega)$, it follows from equation (2.2.9) and Hölder's inequality that $u \in D_q(\Delta)$, so that "$\gamma(u) = 0$" and "$\gamma\left(\frac{\partial u}{\partial \nu}\right) = 0$" make sense.

The main difficulty to apply Theorem 2.2.12 is to show that such a solution of (2.2.9) verifies some boundary condition. In the following result, we give a sufficient condition.

Proposition 2.2.14 (Regularity). *Let Ω be a bounded open subset of $\mathbb{R}^N$ with a $C^{1,1}$ boundary, let $V \in L^N(\Omega;\mathbb{C})$ ($V \in L^{2+\varepsilon}(\Omega;\mathbb{C})$, for some $\varepsilon > 0$, if $N = 2$ and $V \in L^2(\Omega;\mathbb{C})$ if $N = 1$), let $0 < m < 1$, let $a \in \mathbb{C}$ and let $F \in L^2(\Omega)$.*

1) *Let $u \in H_0^1(\Omega)$ be any solution to (2.2.9). Then $u \in H^2(\Omega)$ and $\gamma(u) = 0$.*
2) *Let $u \in H^1(\Omega)$ be any solution to (2.2.9) and (2.1.4). Then $u \in H^2(\Omega)$ and $\gamma\left(\frac{\partial u}{\partial \nu}\right) = 0$.*

Remark 2.2.15. Any solution given by Theorems 2.2.1, 2.2.4 or 2.2.8 belongs to $H^2_{\text{loc}}(\Omega)$ (Theorem 2.2.12).

2.3 Notations

We indicate here some of the notations used throughout this paper which have not been defined yet in the introduction (Section 2.1). We write $\mathrm{i}^2 = -1$. We denote by $\overline{z}$ the conjugate of the complex number z, $\mathrm{Re}(z)$ its real part and $\mathrm{Im}(z)$ its imaginary part. For $1 \leqslant p \leqslant \infty$, p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. The symbol Ω always indicates a nonempty open subset of $\mathbb{R}^N$ (bounded or not) ; its closure is denoted by $\overline{\Omega}$ and its boundary by Γ. For $A \in \{\Omega; \overline{\Omega}\}$, the space $C(A) = C^0(A)$ is the set of continuous functions from A to $\mathbb{C}$ and $C^k(A)$ ($k \in \mathbb{N}$) is the space of functions lying in $C(A)$ and having all derivatives of order lesser or equal than k belonging to $C(A)$. For $0 < \alpha \leqslant 1$ and $k \in \mathbb{N}_0 \stackrel{\text{def}}{=} \mathbb{N} \cup \{0\}$, $C^{k,\alpha}_{\text{loc}}(\Omega) = \left\{u \in C^k(\Omega); \forall \omega \Subset \Omega, \sum\limits_{|\beta|=k} H^\alpha_\omega(D^\beta u) < +\infty\right\}$, where $H^\alpha_\omega(u) = \sup\limits_{\left\{\substack{(x,y)\in\omega^2 \\ x\neq y}\right.} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$. The notation $\omega \Subset \Omega$ means that ω is a bounded open subset of $\mathbb{R}^N$ and $\overline{\omega} \subset \Omega$. In the same way, $C^{k,\alpha}(\overline{\Omega}) = \left\{u \in C^k(\overline{\Omega}); \sum\limits_{|\beta|=k} H^\alpha_\Omega(D^\beta u) < +\infty\right\}$. The space $C_0(\Omega)$ consists of functions belonging to $C(\overline{\Omega})$ and vanishing at the boundary Γ, $\mathscr{D}(\Omega)$ is the space of C^∞ functions with compact support and $\mathscr{D}(\overline{\Omega})$ is the restriction to $\overline{\Omega}$ of functions lying in $\mathscr{D}(\mathbb{R}^N)$. The trace function defined on $\mathscr{D}(\overline{\Omega})$ is denoted by γ. For $1 \leqslant p \leqslant \infty$ and $m \in \mathbb{N}$, the usual Lebesgue and Sobolev spaces are respectively denoted by $L^p(\Omega)$ and $W^{m,p}(\Omega)$, $W^{m,p}_0(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ under the $W^{m,p}$-norm, $H^m(\Omega) = W^{m,2}(\Omega)$ and $H^m_0(\Omega) = W^{m,2}_0(\Omega)$. For a Banach space E, its topological dual is denoted by $E^\star$ and $\langle\, .\, ,\, .\,\rangle_{E^\star,E} \in \mathbb{R}$ is the $E^\star - E$ duality product. In particular, for any $T \in L^{p'}(\Omega)$ and $\varphi \in L^p(\Omega)$ with $1 \leqslant p < \infty$, $\langle T, \varphi\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \mathrm{Re} \int\limits_\Omega T(x)\overline{\varphi(x)}\mathrm{d}x$. We write, $W^{-m,p'}(\Omega) = (W^{m,p}_0(\Omega))^\star$ ($p < \infty$) and $H^{-m}(\Omega) = (H^m_0(\Omega))^\star$. Unless if specified, any function belonging in a functional space $\big(W^{m,p}(\Omega)$,

$C^k(\Omega)$, etc$\big)$ is supposed to be a complex-valued function $\big(W^{m,p}(\Omega;\mathbb{C})$, $C^k(\Omega;\mathbb{C})$, etc$\big)$. We denote by $SO_N(\mathbb{R})$ the special orthogonal group of $\mathbb{R}^N$. Finally, we denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1,\ldots,a_n$, write $C(a_1,\ldots,a_n)$ to indicate that the constant C continuously depends only on $a_1,\ldots,a_n$ (this convention also holds for constants which are not denoted by "C").

2.4 A priori estimates

The proofs of the existence theorems relies on *a priori* bounds, in order to truncate the nonlinearity and pass to the limit. These bounds are formally obtained by multiplying the equation by $\overline{u}$ and $\mathrm{i}\overline{u}$, integrate by parts and by making some linear combinations with the obtained results. Now, we recall the well-known Poincaré's inequality. If $|\Omega|<\infty$ then,

$$\forall u\in H^1_0(\Omega),\ \|u\|_{L^2(\Omega)}\leqslant C_{\mathrm{P}}\|\nabla u\|_{L^2(\Omega)}. \tag{2.4.1}$$

where $C_{\mathrm{P}}=C_{\mathrm{P}}(|\Omega|,N)$. We will frequently use Hölder's inequality in the following form. If $|\Omega|<\infty$ and $0\leqslant m\leqslant 1$ then $L^2(\Omega)\hookrightarrow L^{m+1}(\Omega)$ and

$$\forall u\in L^2(\Omega),\ \|u\|^{m+1}_{L^{m+1}(\Omega)}\leqslant |\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\Omega)}. \tag{2.4.2}$$

Finally, we recall the well-known Young's inequality. For any real $x\geqslant 0$, $y\geqslant 0$ and $\mu>0$, one has

$$xy\leqslant \frac{\mu^2}{2}x^2+\frac{1}{2\mu^2}y^2. \tag{2.4.3}$$

Lemma 2.4.1. *Let Ω an open subset of $\mathbb{R}^N$ be such that $|\Omega|<\infty$, let ω an open subset of $\mathbb{R}^N$ be such that $\omega\subseteq\Omega$, let $0\leqslant m\leqslant 1$, let $(a,b)\in\mathbb{C}^2$, let $\alpha,\beta\geqslant 0$ and let $F\in L^2(\Omega)$. Let $u\in H^1_0(\Omega)$ satisfies*

$$\begin{multlined}\Big|\|\nabla u\|^2_{L^2(\Omega)}+\mathrm{Re}(a)\left(\|u\|^{m+1}_{L^{m+1}(\omega)}+\alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\right)\\ +\mathrm{Re}(b)\left(\|u\|^2_{L^2(\omega)}+\beta\|u\|_{L^1(\omega^{\mathrm{c}})}\right)\Big|\leqslant\int_\Omega |Fu|\mathrm{d}x,\end{multlined} \tag{2.4.4}$$

$$\left|\mathrm{Im}(a)\left(\|u\|^{m+1}_{L^{m+1}(\omega)}+\alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\right)+\mathrm{Im}(b)\left(\|u\|^2_{L^2(\omega)}+\beta\|u\|_{L^1(\omega^{\mathrm{c}})}\right)\right|\leqslant\int_\Omega |Fu|\mathrm{d}x. \tag{2.4.5}$$

Here, $\omega^{\mathrm{c}}=\Omega\setminus\omega$. Assume that one of the three following assertions holds.

1) $\mathrm{Re}(b)\geqslant 0$. *If* $\mathrm{Re}(a)<0$ *and* $|\omega|<|\Omega|$ *then assume further that* $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\leqslant\|u\|^{m+1}_{L^{m+1}(\omega^{\mathrm{c}})}$.
2) $\mathrm{Re}(b)<0$ *and* $\mathrm{Im}(b)\neq 0$. *If* $|\omega|<|\Omega|$ *then assume further that* $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\leqslant\|u\|^{m+1}_{L^{m+1}(\omega^{\mathrm{c}})}$, $F\in L^\infty(\Omega)$ *and* $-\alpha|\mathrm{Im}(a)|+\frac{\beta}{2}|\mathrm{Im}(b)|>\|F\|_{L^\infty(\Omega)}$.
3) $-C_{\mathrm{P}}^{-2}<\mathrm{Re}(b)<0$, *where* C_{P} *is the constant in* (2.4.1), $\alpha\|u\|_{L^1(\omega^{\mathrm{c}})}\leqslant\|u\|^{m+1}_{L^{m+1}(\omega^{\mathrm{c}})}$ *and* $\beta\|u\|_{L^1(\omega^{\mathrm{c}})}\leqslant\|u\|^2_{L^2(\omega^{\mathrm{c}})}$.

Then we have the following estimate.

$$\|u\|_{H^1_0(\Omega)}\leqslant C, \tag{2.4.6}$$

where $C=C(\|F\|_{L^2(\Omega)},|\Omega|,|a|,|b|,N,m)$.

Remark 2.4.2. Obviously, if $|\omega| = |\Omega|$ then $\alpha\|u\|_{L^1(\omega^c)} \leqslant \|u\|^{m+1}_{L^{m+1}(\omega^c)}$ and $\beta\|u\|_{L^1(\omega^c)} \leqslant \|u\|^2_{L^2(\omega^c)}$.

Proof of Lemma 2.4.1. By Poincaré's inequality (2.4.1), it is sufficient to establish

$$\|\nabla u\|_{L^2(\Omega)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, N, m). \tag{2.4.7}$$

Moreover, it follows from (2.4.3) and (2.4.1) that for any $\mu > 0$,

$$\int_\Omega |Fu| \mathrm{d}x \leqslant \frac{C_\mathrm{P}^2}{2}\|F\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}. \tag{2.4.8}$$

Finally, it follows from (2.4.2) and (2.4.1) that if $\alpha\|u\|_{L^1(\omega^c)} \leqslant \|u\|^{m+1}_{L^{m+1}(\omega^c)}$ then one has,

$$\|u\|^{m+1}_{L^{m+1}(\omega)} + \alpha\|u\|_{L^1(\omega^c)} \leqslant \|u\|^{m+1}_{L^{m+1}(\Omega)} \leqslant C_\mathrm{P}^{m+1}|\Omega|^{\frac{1-m}{2}}\|\nabla u\|^{m+1}_{L^2(\Omega)}. \tag{2.4.9}$$

We divide the proof in 3 steps.
Step 1. Proof of (2.4.7) with Assumption 1).
Assume hypothesis 1) holds true. If $\mathrm{Re}(a) \geqslant 0$ then (2.4.7) follows from (2.4.4) and (2.4.8), while if $\mathrm{Re}(a) < 0$ we then deduce from (2.4.4), (2.4.8) and (2.4.9) that,

$$\left(\|\nabla u\|^{1-m}_{L^2(\Omega)} - |\mathrm{Re}(a)|C_\mathrm{P}^{m+1}|\Omega|^{\frac{1-m}{2}}\right)\|\nabla u\|^{m+1}_{L^2(\Omega)} \leqslant C_\mathrm{P}^2\|F\|^2_{L^2(\Omega)}.$$

Hence (2.4.7).
Step 2. Proof of (2.4.7) with Assumption 2).
As for Step 1, it follows from (2.4.5), (2.4.2), (2.4.3) and Hölder's inequality that

$$\begin{aligned}\mathrm{Im}(b)|\left(\|u\|^2_{L^2(\omega)} + \beta\|u\|_{L^1(\omega^c)}\right) \leqslant{}& |\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\|u\|^{m+1}_{L^2(\omega)} + \alpha|\mathrm{Im}(a)|\|u\|_{L^1(\omega^c)} \\ &+ \frac{1}{2|\mathrm{Im}(b)|}\|F\|^2_{L^2(\omega)} + \frac{|\mathrm{Im}(b)|}{2}\|u\|^2_{L^2(\omega)} + \|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)}.\end{aligned}$$

Recalling that when $|\omega| < |\Omega|$, $-\alpha|\mathrm{Im}(a)| + \frac{\beta}{2}|\mathrm{Im}(b)| > \|F\|_{L^\infty(\Omega)}$, the above estimate yields

$$\left(|\mathrm{Im}(b)|\|u\|^{1-m}_{L^2(\omega)} - 2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}\right)\|u\|^{m+1}_{L^2(\omega)} + \beta|\mathrm{Im}(b)|\|u\|_{L^1(\omega^c)} \leqslant \frac{1}{|\mathrm{Im}(b)|}\|F\|^2_{L^2(\omega)}. \tag{2.4.10}$$

If $|\mathrm{Im}(b)|\|u\|^{1-m}_{L^2(\omega)} - 2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}} \leqslant 1$ then

$$\|u\|_{L^2(\omega)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, m) \overset{\text{not.}}{=} C_0, \tag{2.4.11}$$

and it follows from (2.4.5), (2.4.2), (2.4.11) and Hölder's inequality that,

$$\begin{aligned}\left(\beta|\mathrm{Im}(b)| - \alpha|\mathrm{Im}(a)|\right)\|u\|_{L^1(\omega^c)} &\leqslant C(C_0) + \|F\|_{L^\infty(\omega^c)}\|u\|_{L^1(\omega^c)} \\ &\leqslant C(C_0) + \left(\frac{\beta}{2}|\mathrm{Im}(b)| - \alpha|\mathrm{Im}(a)|\right)\|u\|_{L^1(\omega^c)},\end{aligned}$$

so that,

$$\beta\|u\|_{L^1(\omega^c)} \leqslant C(\|F\|_{L^2(\Omega)}, |\Omega|, |a|, |b|, m) \overset{\text{not.}}{=} C_1. \tag{2.4.12}$$

But if $|\mathrm{Im}(b)|\|u\|_{L^2(\omega)}^{1-m}-2|\mathrm{Im}(a)||\Omega|^{\frac{1-m}{2}}>1$ then (2.4.11) and (2.4.12) come from (2.4.10).
Finally, by (2.4.4), (2.4.8), (2.4.9), (2.4.11) and (2.4.12), one obtains

$$\|\nabla u\|_{L^2(\Omega)}^2\leqslant|\mathrm{Re}(a)|C_{\mathrm{P}}^{m+1}|\Omega|^{\frac{1-m}{2}}\|\nabla u\|_{L^2(\Omega)}^{m+1}+C(C_0,C_1)+\frac{C_{\mathrm{P}}^2}{2}\|F\|_{L^2(\Omega)}^2+\frac{1}{2}\|\nabla u\|_{L^2(\Omega)}^2.$$

It follows that $\big(\|\nabla u\|_{L^2(\Omega)}^{1-m}-C\big)\|\nabla u\|_{L^2(\Omega)}^{m+1}\leqslant C+C_{\mathrm{P}}^2\|F\|_{L^2(\Omega)}^2$, from which we easily deduce (2.4.7).
Step 3. Proof of (2.4.7) with Assumption 3).
By Assumption 3), (2.4.1), (2.4.3) and (2.4.9)

$$\|\nabla u\|_{L^2(\Omega)}^2\leqslant C\|\nabla u\|_{L^2(\Omega)}^{m+1}+\left(|\mathrm{Re}(b)|C_{\mathrm{P}}^2+\frac{C_{\mathrm{P}}^2}{2\mu^2}\right)\|\nabla u\|_{L^2(\Omega)}^2+\frac{\mu^2}{2}\|F\|_{L^2(\Omega)}^2,$$

where $C=C(|\Omega|,|a|,N,m)$. We then deduce,

$$\left(\left(1-|\mathrm{Re}(b)|C_{\mathrm{P}}^2-\frac{C_{\mathrm{P}}^2}{2\mu^2}\right)\|\nabla u\|_{L^2(\Omega)}^{1-m}-C\right)\|\nabla u\|_{L^2(\Omega)}^{m+1}\leqslant\frac{\mu^2}{2}\|F\|_{L^2(\Omega)}^2.$$

Since $|\mathrm{Re}(b)|<C_{\mathrm{P}}^{-2}$, there exists $\mu_0>0$ such that $C_2\overset{\text{def.}}{=}1-|\mathrm{Re}(b)|C_{\mathrm{P}}^2-\frac{C_{\mathrm{P}}^2}{2\mu_0^2}>0$. For such a μ_0, $\left(C_2\|\nabla u\|_{L^2(\Omega)}^{1-m}-C\right)\|\nabla u\|_{L^2(\Omega)}^{m+1}\leqslant\frac{\mu_0^2}{2}\|F\|_{L^2(\Omega)}^2$, from which (2.4.7) follows. □

Corollary 2.4.3. *Let $(\Omega_n)_{n\in\mathbb{N}}$ a sequence of open subsets of $\mathbb{R}^N$ be such that $\sup\limits_{n\in\mathbb{N}}|\Omega_n|<\infty$, let $0<m<1$, let $(a,b)\in\mathbb{C}^2$ and let $(F_n)_{n\in\mathbb{N}}\subset L^\infty(\Omega_n)$ be such that $\sup\limits_{n\in\mathbb{N}}\|F_n\|_{L^2(\Omega_n)}<\infty$. If $\mathrm{Re}(b)<0$ then assume further that $\mathrm{Im}(b)\neq0$ or $-\frac{1}{C_{\mathrm{P}}^2}<\mathrm{Re}(b)$, where C_{P} is the constant in (2.4.1). Let $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2}\subset H_0^1(\Omega_n)$ be a sequence satisfying*

$$\forall n\in\mathbb{N},\ \forall\ell\in\mathbb{N},\ -\Delta u_\ell^n+f_\ell\big(u_\ell^n\big)=F_n,\ \text{ in } L^2(\Omega_n),\tag{2.4.13}$$

where for any $\ell\in\mathbb{N}$,

$$\forall u\in L^2(\Omega_n),\ f_\ell(u)=\begin{cases}a|u|^{-(1-m)}u+bu, & \text{if } |u|\leqslant\ell,\\ a\ell^m\dfrac{u}{|u|}+b\ell\dfrac{u}{|u|}, & \text{if } |u|>\ell.\end{cases}\tag{2.4.14}$$

Then there exists a diagonal extraction $\left(u_{\varphi(n)}^n\right)_{n\in\mathbb{N}}$ of $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2}$ such that the following estimate holds.

$$\forall n\in\mathbb{N},\ \left\|u_{\varphi(n)}^n\right\|_{H_0^1(\Omega_n)}\leqslant C,$$

where $C=C\left(\sup\limits_{n\in\mathbb{N}}\|F_n\|_{L^2(\Omega_n)},\sup\limits_{n\in\mathbb{N}}|\Omega_n|,|a|,|b|,N,m\right)$.

Proof. Choosing u_ℓ^n and $\mathrm{i}u_\ell^n$ as test functions, we get

$$\|\nabla u_\ell^n\|_{L^2(\Omega_n)}^2+\mathrm{Re}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1}+\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)$$
$$+\mathrm{Re}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+\ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)=\mathrm{Re}\int_{\Omega_n}F_n\overline{u_\ell^n}\mathrm{d}x,$$

$$\operatorname{Im}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1}+\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)\\ +\operatorname{Im}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+\ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)=\operatorname{Im}\int_{\Omega_n}F_n\overline{u_\ell^n}\mathrm{d}x,$$

for any $(n,\ell)\in\mathbb{N}^2$. We first note that,

$$\forall(n,\ell)\in\mathbb{N}^2,\begin{cases}\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\leqslant\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|>\ell\})}^{m+1},\\ \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\leqslant\|u_\ell^n\|_{L^2(\{|u_\ell^n|>\ell\})}^2,\end{cases}\tag{2.4.15}$$

For each $n\in\mathbb{N}$, we choose $\varphi(n)\in\mathbb{N}$ large enough to have $\varphi(n)^{1-m}>2\frac{\|F_n\|_{L^\infty(\Omega_n)}+|\operatorname{Im}(a)|}{|\operatorname{Im}(b)|}$, when $\operatorname{Im}(b)\neq0$ and $\varphi(n)=n$, when $\operatorname{Im}(b)=0$. Thus for any $n\in\mathbb{N}$, as soon as $\operatorname{Im}(b)\neq0$, one has

$$\|F_n\|_{L^\infty(\Omega_n)}<-\varphi(n)^m|\operatorname{Im}(a)|+\frac{\varphi(n)}{2}|\operatorname{Im}(b)|.\tag{2.4.16}$$

With help of (2.4.15) and (2.4.16), we may apply Lemma 2.4.1 to $u_{\varphi(n)}^n$, for each $n\in\mathbb{N}$, with $\omega=\left\{x\in\Omega_n;\left|u_{\varphi(n)}^n(x)\right|\leqslant\varphi(n)\right\}$, $\alpha=\varphi(n)^m$ and $\beta=\varphi(n)$. □

Lemma 2.4.4. *Let $\Omega\subseteq\mathbb{R}^N$ be an open subset, let ω an open subset of $\mathbb{R}^N$ be such that $\omega\subseteq\Omega$, let $m\geqslant0$ and let $(a,b,c)\in\mathbb{C}^3$ be such that $\operatorname{Im}(b)\neq0$. If $\operatorname{Re}(a)\leqslant0$ then assume further that $\operatorname{Im}(a)\neq0$. Let $\alpha,\beta,R\geqslant0$, let $F\in L^2(\Omega)$ and let*

$$A=\begin{cases}\max\left\{1,\frac{1+|b|+R^2|c|}{|\operatorname{Im}(b)|},\frac{|\operatorname{Re}(a)|}{|\operatorname{Im}(a)|}\right\}, & \text{if } \operatorname{Re}(a)\leqslant0,\\ \max\left\{1,\frac{1+|b|+R^2|c|}{|\operatorname{Im}(b)|}\right\}, & \text{if } \operatorname{Re}(a)>0.\end{cases}$$

If $|\omega|<|\Omega|$ then assume further that $F\in L^\infty(\Omega)$ and $\beta\geqslant2A\|F\|_{L^\infty(\Omega)}+1$. Let $u\in H^1(\Omega)$ satisfies

$$\|\nabla u\|_{L^2(\Omega)}^2+\operatorname{Re}(a)\left(\|u\|_{L^{m+1}(\omega)}^{m+1}+\alpha\|u\|_{L^1(\omega^c)}\right)\\ -(|b|+R^2|c|)\left(\|u\|_{L^2(\omega)}^2+\beta\|u\|_{L^1(\omega^c)}\right)\leqslant\int_\Omega|Fu|\mathrm{d}x,\tag{2.4.17}$$

$$|\operatorname{Im}(a)|\left(\|u\|_{L^{m+1}(\omega)}^{m+1}+\alpha\|u\|_{L^1(\omega^c)}\right)+|\operatorname{Im}(b)|\left(\|u\|_{L^2(\omega)}^2+\beta\|u\|_{L^1(\omega^c)}\right)\leqslant\int_\Omega|Fu|\mathrm{d}x.\tag{2.4.18}$$

Then there exists a positive constant $M=M(|a|,|b|,|c|)$ such that,

$$\|\nabla u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\omega)}^2+\|u\|_{L^{m+1}(\omega)}^{m+1}+\|u\|_{L^1(\omega^c)}\leqslant M(R^4+1)\|F\|_{L^2(\Omega)}^2.\tag{2.4.19}$$

Proof. Let A be as in the lemma. We multiply (2.4.18) by A and sum the result to (2.4.17). This yields,

$$\|\nabla u\|_{L^2(\Omega)}^2+A_0\left(\|u\|_{L^{m+1}(\omega)}^{m+1}+\alpha\|u\|_{L^1(\omega^c)}\right)+\|u\|_{L^2(\omega)}^2+\beta\|u\|_{L^1(\omega^c)}\leqslant2A\int_\Omega|Fu|\mathrm{d}x,$$

where $A_0=A|\operatorname{Im}(a)|+\operatorname{Re}(a)$. Applying Hölder's inequality and (2.4.3), we get

$$\|\nabla u\|_{L^2(\Omega)}^2+\|u\|_{L^2(\omega)}^2+A_0\|u\|_{L^{m+1}(\omega)}^{m+1}+\beta\|u\|_{L^1(\omega^c)}\\ \leqslant2A\|F\|_{L^\infty(\Omega)}\|u\|_{L^1(\omega^c)}+2A^2\|F\|_{L^2(\Omega)}^2+\frac12\|u\|_{L^2(\omega)}^2,$$

from which we deduce the result if $|\omega| = |\Omega|$. Now, suppose $|\omega| < |\Omega|$. The above estimate leads to,

$$\|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\omega)}^2 + A_0\|u\|_{L^{m+1}(\omega)}^{m+1} + \big(\beta - 2A\|F\|_{L^\infty(\Omega)}\big)\,\|u\|_{L^1(\omega^c)} \leqslant 4A^2\|F\|_{L^2(\Omega)}^2,$$

from which we prove the lemma since $\beta - 2A\|F\|_{L^\infty(\Omega)} \geqslant 1$. □

Lemma 2.4.5. *Let $(a,b) \in \mathbb{A}^2$ satisfies (2.2.7). Then there exists $\delta_\star = \delta_\star(|a|,|b|) \in (0,1]$, $L = L(|a|,|b|)$ and $M = M(|a|,|b|)$ satisfying the following property. If $\delta \in [0,\delta_\star]$ and C_0, C_1, C_2, C_3, C_4 are six nonnegative real numbers satisfying*

$$\big|C_1 + \delta C_2 + \mathrm{Re}(a)C_3 + \big(\mathrm{Re}(b) - \delta\big)C_4\big| \leqslant C_0, \tag{2.4.20}$$

$$\big|\mathrm{Im}(a)C_3 + \mathrm{Im}(b)C_4\big| \leqslant C_0, \tag{2.4.21}$$

then

$$0 \leqslant C_1 + LC_3 + LC_4 \leqslant MC_0. \tag{2.4.22}$$

Proof. We split the proof in 4 cases. Let $\gamma > 0$ be small enough to be chosen later. Note that when $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$ then estimate (2.4.21) can be rewritten as

$$|\mathrm{Im}(a)|C_3 + |\mathrm{Im}(b)|C_4 \leqslant C_0. \tag{2.4.23}$$

Case 1. $\mathrm{Re}(a) \geqslant 0$, $\mathrm{Re}(b) \geqslant 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$. We add (2.4.23) with (2.4.20) and obtain,

$$C_1 + \big(\mathrm{Re}(a) + |\mathrm{Im}(a)|\big)C_3 + \big(\mathrm{Re}(b) - \delta_\star + |\mathrm{Im}(b)|\big)C_4 \leqslant 2C_0.$$

Case 2. $\Big(\mathrm{Re}(a) \geqslant 0,\ \mathrm{Re}(b) < 0 \text{ and } \mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0\Big)$ or $\Big(\mathrm{Im}(a)\mathrm{Im}(b) < 0\Big)$. Then,

$$C_1 + \frac{\mathrm{Re}(a)\mathrm{Im}(b) - \mathrm{Re}(b)\mathrm{Im}(a) + \gamma\mathrm{Im}(a)}{\mathrm{Im}(b)}C_3 + (\gamma - \delta_\star)C_4 \leqslant \frac{|\mathrm{Re}(b)| + |\mathrm{Im}(b)| + \gamma}{|\mathrm{Im}(b)|}C_0.$$

where we computed (2.4.20) $- \frac{\mathrm{Re}(b)-\gamma}{\mathrm{Im}(b)}$(2.4.21).

Case 3. $\mathrm{Re}(a) < 0$, $\mathrm{Re}(b) \geqslant 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$. By computing (2.4.20) $- \frac{\mathrm{Re}(a)-\gamma}{\mathrm{Im}(a)}$(2.4.21), we get,

$$C_1 + \gamma C_3 + \left(\frac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)} - \delta_\star\right)C_4 \leqslant \frac{|\mathrm{Re}(a)| + |\mathrm{Im}(a)| + \gamma}{|\mathrm{Im}(a)|}C_0.$$

Case 4. $\mathrm{Re}(a) < 0$, $\mathrm{Re}(b) < 0$ and $\mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0$. Note that since $(a,b) \in \mathbb{A}^2$ then necessarily $\mathrm{Im}(a)\mathrm{Im}(b) \neq 0$. Thus, we can compute (2.4.20) $+ \max\left\{\frac{|\mathrm{Re}(a)|+\gamma}{|\mathrm{Im}(a)|}, \frac{|\mathrm{Re}(b)|+\gamma}{|\mathrm{Im}(b)|}\right\}$(2.4.23) and obtain,

$$C_1 + \gamma C_3 + (\gamma - \delta_\star)C_4 \leqslant \left(\frac{|\mathrm{Re}(a)| + |\mathrm{Im}(a)| + \gamma}{|\mathrm{Im}(a)|} + \frac{|\mathrm{Re}(b)| + |\mathrm{Im}(b)| + \gamma}{|\mathrm{Im}(b)|}\right)C_0.$$

In both cases, we may choose $\gamma > 0$ small enough to have

$$\begin{cases} \dfrac{\mathrm{Re}(a)\mathrm{Im}(b) - \mathrm{Re}(b)\mathrm{Im}(a) + \gamma\mathrm{Im}(a)}{\mathrm{Im}(b)} > 0, & \text{in Case 2,} \\[2ex] \dfrac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)} > 0, & \text{in Case 3.} \end{cases}$$

Then we choose $0 < \delta_\star < \min\{1, \gamma, |\mathrm{Im}(b)| + |\mathrm{Re}(b)|\}$ such that

$$\delta_\star < \frac{\mathrm{Re}(b)\mathrm{Im}(a) - \mathrm{Re}(a)\mathrm{Im}(b) + \gamma\mathrm{Im}(b)}{\mathrm{Im}(a)}, \text{ in Case 3.}$$

This ends the proof. □

Corollary 2.4.6. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $V \in L^\infty(\Omega;\mathbb{R})$, let $0 < m < 1$ and let $(a,b,c) \in \mathbb{C}^3$ be such that $\mathrm{Im}(a) \leqslant 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. If $\mathrm{Re}(a) \leqslant 0$ then assume further that $\mathrm{Im}(a) < 0$. Let $\delta \geqslant 0$. Let $(F_n)_{n\in\mathbb{N}} \subset L^\infty(\Omega)\cap L^2(\Omega)$ be bounded in $L^2(\Omega)$ and let $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2} \subset H^1(\Omega)\cap L^{m+1}(\Omega)$ be a sequence satisfying*

$$\forall n \in \mathbb{N}, \tag{2.4.24}$$

with boundary condition (2.1.3) or (2.1.4), where for any $\ell \in \mathbb{N}$,

$$\forall u \in L^2(\Omega),\ f_\ell(u) = \begin{cases} a|u|^{-(1-m)}u + (b-\delta)u + cV^2u, & \text{if } |u| \leqslant \ell, \\ a\ell^m\dfrac{u}{|u|} + (b-\delta)\ell\dfrac{u}{|u|} + cV^2\ell\dfrac{u}{|u|}, & \text{if } |u| > \ell. \end{cases} \tag{2.4.25}$$

For (2.1.4), Ω is assumed to have a C^1 boundary. Then there exist $M = M\big(\|V\|_{L^\infty(\Omega)}, |a|, |b|, |c|\big)$ and a diagonal extraction $\Big(u^n_{\varphi(n)}\Big)_{n\in\mathbb{N}}$ of $(u_\ell^n)_{(n,\ell)\in\mathbb{N}^2}$ for which,

$$\begin{aligned}\big\|\nabla u^n_{\varphi(n)}\big\|^2_{L^2(\Omega)} + \big\|u^n_{\varphi(n)}\big\|^2_{L^2\left(\left\{\left|u^n_{\varphi(n)}\right|\leqslant\varphi(n)\right\}\right)} + \big\|u^n_{\varphi(n)}\big\|^{m+1}_{L^{m+1}\left(\left\{\left|u^n_{\varphi(n)}\right|\leqslant\varphi(n)\right\}\right)} \\ + \big\|u^n_{\varphi(n)}\big\|_{L^1\left(\left\{\left|u^n_{\varphi(n)}\right|>\varphi(n)\right\}\right)} \leqslant M\sup_{n\in\mathbb{N}}\|F_n\|^2_{L^2(\Omega)},\end{aligned}$$

for any $n \in \mathbb{N}$. The same is true if we replace the conditions on (a,b,c) by $(a,b,c) \in \mathbb{A}\times\mathbb{A}\times\{0\}$ satisfies (2.2.7) and $\delta \leqslant \delta_\star$, where $\delta_\star$ is given by Lemma 2.4.5. In this case, $M = M(|a|,|b|)$.

Proof. Choosing u_ℓ^n and $\mathrm{i}u_\ell^n$ as test functions, we obtain

$$\begin{aligned}\|\nabla u_\ell^n\|^2_{L^2(\Omega)} + \mathrm{Re}(a)\left(\|u_\ell^n\|^{m+1}_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \left(\mathrm{Re}(b) - \|V\|^2_{L^\infty(\Omega)}|\mathrm{Re}(c)|\right)\left(\|u_\ell^n\|^2_{L^2(\{|u_\ell^n|\leqslant\ell\})} + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \leqslant \mathrm{Re}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x,\end{aligned} \tag{2.4.26}$$

$$\begin{aligned}\mathrm{Im}(a)\left(\|u_\ell^n\|^{m+1}_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) + \mathrm{Im}(b)\left(\|u_\ell^n\|^2_{L^2(\{|u_\ell^n|\leqslant\ell\})} + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \mathrm{Im}(c)\left(\|Vu\|^2_{L^2(\{|u_\ell^n|\leqslant\ell\}))} + \ell\|V^2u\|_{L^1(\{|u_\ell^n|>\ell\}))}\right) = \mathrm{Im}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x,\end{aligned} \tag{2.4.27}$$

for any $(n,\ell) \in \mathbb{N}^2$. If $(a,b,c) \in \mathbb{A}\times\mathbb{A}\times\{0\}$ satisfies (2.2.7), then we obtain

$$\begin{aligned}\|\nabla u_\ell^n\|^2_{L^2(\Omega)} + \delta\|u_\ell^n\|^2_{L^2(\Omega)} + \mathrm{Re}(a)\left(\|u_\ell^n\|^{m+1}_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})} + \ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) \\ + \left(\mathrm{Re}(b) - \delta\right)\left(\|u_\ell^n\|^2_{L^2(\{|u_\ell^n|\leqslant\ell\})} + \ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right) = \mathrm{Re}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x,\end{aligned} \tag{2.4.28}$$

$$\mathrm{Im}(a)\left(\|u_\ell^n\|_{L^{m+1}(\{|u_\ell^n|\leqslant\ell\})}^{m+1}+\ell^m\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)$$
$$+\,\mathrm{Im}(b)\left(\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+\ell\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\right)=\mathrm{Im}\int_\Omega F_n\overline{u_\ell^n}\mathrm{d}x, \quad (2.4.29)$$

for any $(n,\ell)\in\mathbb{N}^2$. For this last case, it follows from Lemma 2.4.5, Hölder's inequality and (2.4.3) that

$$\|\nabla u_\ell^n\|_{L^2(\Omega)}^2+\frac{L}{2}\|u_\ell^n\|_{L^2(\{|u_\ell^n|\leqslant\ell\})}^2+L\big\|u_\ell^n\big\|_{L^{m+1}}^{m+1}\big(\big\{|u_\ell^n|\leqslant\ell\big\}\big)$$
$$+\left(L\ell-M\|F\|_{L^\infty(\Omega)}\right)\|u_\ell^n\|_{L^1(\{|u_\ell^n|>\ell\})}\leqslant\frac{M^2}{2L}\|F\|_{L^2(\Omega)}^2.$$

Then the result follows by choosing for each $n\in\mathbb{N}$, $\varphi(n)\in\mathbb{N}$ large enough to have $L\varphi(n)-M\|F\|_{L^\infty(\Omega)}\geqslant 1$. Now we turn out to the case (2.4.26)–(2.4.27). Let M and A be given by Lemma 2.4.4 with $R=\|V\|_{L^\infty(\Omega)}$. For each $n\in\mathbb{N}$, let $\varphi(n)\in\mathbb{N}$ be large enough to have $\varphi(n)\geqslant 2A\|F_n\|_{L^\infty(\Omega)}+1$, if $|\omega|<|\Omega|$ and $\varphi(n)=n$, if $|\omega|=|\Omega|$. For each $n\in\mathbb{N}$, with help of (2.4.26) and (2.4.27), we may apply Lemma 2.4.4 to $u_{\varphi(n)}^n$ with $\omega=\left\{x\in\Omega;\left|u_{\varphi(n)}^n(x)\right|\leqslant\varphi(n)\right\}$, $\alpha=\varphi(n)^m$, $\beta=\varphi(n)$ and $R=\|V\|_{L^\infty(\Omega)}$. Hence the result. □

2.5 Proofs of the main results

Proof of Theorem 2.2.12. Property 1) follows from Proposition 1.4.5 while Property 2) comes from Remark 1.4.7. It remains to establish Property 3). Assume first that $F\in L^p(\Omega)$ and $V\in\bigcap\limits_{1<r<\infty}L^r(\Omega)$. It follows from the equation that for any $\varepsilon\in(0,q-1)$, $\Delta u\in L^{q-\varepsilon}(\Omega)$. We now recall an elliptic regularity result. If for some $1<s<\infty$, $u\in L^s(\Omega)$ satisfies $\Delta u\in L^s(\Omega)$ and $\gamma(\nabla u.\nu)=0$ then $u\in W^{2,s}(\Omega)$ (Proposition 2.5.2.3, p.131, in Grisvard [93]). Since for any $\varepsilon\in(0,q-1)$, $u,\Delta u\in L^{q-\varepsilon}(\Omega)$ and $\gamma(\nabla u.\nu)=0$ (by assumption), by following the bootstrap method of the proof p.21–22 of Property 1) of Proposition 1.4.5, we obtain the result. Indeed, therein, it is sufficient to apply the global regularity result in Grisvard [93] (Proposition 2.5.2.3, p.131) in place of the local regularity result in Cazenave [58] (Proposition 4.1.2, p.101-102). Now, you turn out to the Hölder regularity. Assume $F\in C^{0,\alpha}(\overline{\Omega})$ and $V\in C^{0,\alpha}(\overline{\Omega})$. By global smoothness property in $W^{2,p}$ proved above, we know that $u\in W^{2,N+1}(\Omega)$ and $\gamma(\nabla u.\nu)=0$ in $L^{N+1}(\Gamma)$. It follows from the Sobolev's embedding, $W^{2,N+1}(\Omega)\hookrightarrow C^{1,\frac{1}{N+1}}(\overline{\Omega})\hookrightarrow C^{0,1}(\overline{\Omega})$, that for any $x\in\Gamma$, $\frac{\partial u}{\partial\nu}(x)=0$ and $u\in C^{0,1}(\overline{\Omega})$. A straightforward calculation yields,

$$\forall(x,y)\in\overline{\Omega}^2,\ \left||u(x)|^{-(1-m)}u(x)-|u(y)|^{-(1-m)}u(y)\right|\leqslant 5|u(x)-u(y)|^m\leqslant 5|x-y|^m.$$

Setting, $g=F-(a|u|^{-(1-m)}u+(b-1)u+cVu)$, we deduce that $g\in C^{0,\alpha}(\overline{\Omega})$. Let $v\in C^{2,\alpha}(\overline{\Omega})$ be the unique solution to

$$\begin{cases}-\Delta v+v=g, & \text{in }\Omega,\\ \frac{\partial v}{\partial\nu}=0, & \text{on }\Gamma,\end{cases}$$

(see, for instance, Theorem 3.2 p.137 in Ladyzhenskaya and Ural'tseva [125]). It follows that u and v are two H^1-solutions of the above equations and since uniqueness holds in $H^1(\Omega)$ (Lax-Milgram's

Theorem), we deduce that $u = v$. Hence $u \in C^{2,\alpha}(\overline{\Omega})$. This concludes the proof[2]. $\square$

Proof of Proposition 2.2.14. We first establish Property 1). Since Ω has $C^{0,1}$ boundary and $u \in H^1_0(\Omega)$, it follows that $\gamma(u) = 0$. Moreover, Sobolev's embedding and equation (2.2.9) imply that $\Delta u \in L^2(\Omega)$. We then obtain that $u \in H^2(\Omega)$ (Grisvard [93], Corollary 2.5.2.2, p.131). Hence Property 1). We turn out to Property 2). It follows from equation (2.2.9) that $\Delta u \in L^2(\Omega)$, so that (2.2.9) makes sense a.e. in Ω. Then Property 2) comes from the arguments of 2) of Remark 2.2.5. $\square$

Lemma 2.5.1. *Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded open subset, let $V \in L^\infty(\Omega;\mathbb{C})$, let $0 < m < 1$, let $(a,b,c) \in \mathbb{C}^3$ and let $F \in L^2(\mathcal{O})$. Let $\delta \in [0,1]$. Then for any $\ell \in \mathbb{N}$, there exist a solution $u^1_\ell \in H^1_0(\mathcal{O})$ to*

$$-\Delta u_\ell + \delta u_\ell + f_\ell(u_\ell) = F, \text{ in } L^2(\mathcal{O}), \tag{2.5.1}$$

with boundary condition (2.1.3) and a solution $u^2_\ell \in H^1(\mathcal{O})$ to (2.5.1) with boundary condition (2.1.4) (in this case, $\mathcal{O}$ is assumed to have a C^1 boundary and $\delta > 0$), where

$$\forall u \in L^2(\Omega),\ f_\ell(u) = \begin{cases} a|u|^{-(1-m)}u + (b-\delta)u + cV^2u, & \text{if } |u| \leqslant \ell, \\ a\ell^m \dfrac{u}{|u|} + (b-\delta)\ell\dfrac{u}{|u|} + cV^2\ell\dfrac{u}{|u|}, & \text{if } |u| > \ell. \end{cases} \tag{2.5.2}$$

If, in addition, V is spherically symmetric then Symmetry Property 2.2.2 holds.

Proof. We proceed with the proof in two steps. Let $H = H^1_0(\mathcal{O})$, in the homogeneous Dirichlet case, and $H = H^1(\mathcal{O})$, in the homogeneous Neumann case. Let $\delta \in [0,1]$ $\big($with additionally $\delta > 0$ and Γ of class C^1 if $H = H^1(\mathcal{O})\big)$. Step 1 below being obvious, we omit the proof.

Step 1. $\forall G \in L^2(\mathcal{O}), \exists! u \in H$ s.t. $-\Delta u + \delta u = G$. Moreover, $\exists \alpha > 0$ s.t. $\forall G \in L^2(\mathcal{O})$, $\big\|(-\Delta + \delta I)^{-1}G\big\|_{H^1(\mathcal{O})} \leqslant \alpha\|G\|_{L^2(\mathcal{O})}$. Finally, Symmetry Property 2.2.2 holds.

Step 2. Conclusion.

For each $\ell \in \mathbb{N}$, we define $g_\ell = -f_\ell + F \in C\big(L^2(\mathcal{O}); L^2(\mathcal{O})\big)$. With help of the continuous and compact embedding $i : H \hookrightarrow L^2(\mathcal{O})$ and Step 1, we may define a continuous and compact sequence of mappings $(T_\ell)_{\ell\in\mathbb{N}}$ of H as follows. For any $\ell \in \mathbb{N}$, set

$$\begin{array}{ccccccc} T_\ell : H & \overset{i}{\hookrightarrow} & L^2(\mathcal{O}) & \overset{g_\ell}{\longrightarrow} & L^2(\mathcal{O}) & \xrightarrow{(-\Delta+\delta I)^{-1}} & H \\ u & \longmapsto & i(u) = u & \longmapsto & g_\ell(u) & \longmapsto & (-\Delta + \delta u)^{-1}(g_\ell)(u) \end{array}$$

Set $\rho = 2\alpha(|a| + |b| + |c| + 1)\left(\left(\|V\|^2_{L^\infty(\Omega)} + 2\right)\ell|\mathcal{O}|^{\frac{1}{2}} + \|F\|_{L^2(\mathcal{O})}\right)$. Let $u \in H$. It follows that,

$$\|T_\ell(u)\|_{H^1(\mathcal{O})} = \big\|(-\Delta + \delta I)^{-1}(g_\ell)(u)\big\|_{H^1(\mathcal{O})} \leqslant \alpha\|g_\ell(u)\|_{L^2(\mathcal{O})} \leqslant \rho.$$

Existence comes from the Schauder's fixed point Theorem applied to T_ℓ. The Symmetry Property 2.2.2 is obtained by working in H_{rad} in place of H $\big($and in H_{even} and H_{odd} for $N = 1\big)$. $\square$

2. More directly, we could have said that since $u \in W^{2,N+1}(\Omega)$, $\gamma(\nabla u.\nu) = 0$ and $\Delta u \in C^{0,\alpha}(\overline{\Omega})$ (by the estimate of the nonlinearity) then by Theorem 6.3.2.1, p.287, in Grisvard [93], $u \in C^{2,\alpha}(\overline{\Omega})$. But this theorem requires Ω to have a $C^{2,1}$ boundary.

Proof of Theorem 2.2.1. Let for any $u\in L^2(\Omega)$, $f(u)=a|u|^{-(1-m)}u+bu$. Set $\Omega_n=\Omega\cap B(0,n)$. Let $(G_n)_{n\in\mathbb{N}}\subset\mathscr{D}(\Omega)$ be such that $G_n\xrightarrow[n\to\infty]{L^2(\Omega)}F$. Let $\left(u_\ell^n\right)_{(n,\ell)\in\mathbb{N}^2}\subset H_0^1(\Omega_n)$ a sequence of solutions of (2.5.1) be given by Lemma 2.5.1 with $\mathcal{O}=\Omega_n$, $c=\delta=0$ and $F_n=G_{n|\Omega_n}$. We define $\widetilde{u_\ell^n}\in H_0^1(\Omega)$ by extending u_n by 0 in $\Omega\cap\Omega_n^{\rm c}$. We also denote by $\widetilde{f_\ell}$ the extension by 0 of f_ℓ in $\Omega\cap\Omega_n^{\rm c}$. By Corollary 2.4.3, there exists a diagonal extraction $\left(\widetilde{u_{\varphi(n)}^n}\right)_{n\in\mathbb{N}}$ of $\left(\widetilde{u_\ell^n}\right)_{(n,\ell)\in\mathbb{N}^2}$ which is bounded in $H_0^1(\Omega)$. By reflexivity of $H_0^1(\Omega)$, Rellich-Kondrachov's Theorem and converse of the dominated convergence theorem, there exist $u\in H_0^1(\Omega)$ and $g\in L^2_{\rm loc}(\Omega;\mathbb{R})$ such that, up to a subsequence that we still denote by $\left(\widetilde{u_{\varphi(n)}^n}\right)_{n\in\mathbb{N}}$, $\widetilde{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{L^2_{\rm loc}(\Omega)}u$, $\widetilde{u_{\varphi(n)}^n}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}u$ and $\left|\widetilde{u_{\varphi(n)}^n}\right|\leqslant g$, a.e. in Ω, By these two last estimates, $\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}f(u)$ and $\left|\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\right|\leqslant C(g^m+g)\in L^2_{\rm loc}(\Omega)$, a.e. in Ω. From the dominated convergence Theorem, $\widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)\xrightarrow[n\to\infty]{L^2_{\rm loc}(\Omega)}f(u)$. Let $\varphi\in\mathscr{D}(\Omega)$. Let $n_\star\in\mathbb{N}$ be large enough to have $\operatorname{supp}\varphi\subset\Omega_{n_\star}$. We have by (2.5.1),

$$\forall n>n_\star,\ \left\langle -\mathrm{i}\Delta u_{\varphi(n)}^n+f_{\varphi(n)}\left(u_{\varphi(n)}^n\right)-F_n,\varphi_{|\Omega_n}\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)}=0.$$

The above convergencies lead to,

$$\begin{aligned}
&\langle -\Delta u+f(u)-F,\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}\\
&=\langle -u,\Delta\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}+\langle f(u)-F,\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}\\
&=\lim_{n\to\infty}\left\langle -\widetilde{u_{\varphi(n)}^n},\Delta\varphi\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}+\lim_{n\to\infty}\left\langle \widetilde{f_{\varphi(n)}}\left(\widetilde{u_{\varphi(n)}^n}\right)-G_n,\varphi\right\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}\\
&=\lim_{n\to\infty}\left\langle -\Delta u_{\varphi(n)}^n+f_{\varphi(n)}\left(u_{\varphi(n)}^n\right)-F_n,\varphi_{|\Omega_n}\right\rangle_{\mathscr{D}'(\Omega_n),\mathscr{D}(\Omega_n)}\\
&=0.
\end{aligned}$$

By density, we then obtain that $u\in H_0^1(\Omega)$ is a solution to $-\Delta u+f(u)=F$, in $L^2(\Omega)$. Finally, if F is spherically symmetric then u (obtained as a limit of solutions given by Lemma 2.5.1) is also spherically symmetric. For $N=1$, this includes the case where F is an even function. □

Proof of Theorems 2.2.3 and 2.2.9. Choosing u and $\mathrm{i}u$ as test functions, we obtain

$$\|\nabla u\|^2_{L^2(\Omega)}+\operatorname{Re}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)}+\operatorname{Re}(b)\|u\|^2_{L^2(\Omega)}=\operatorname{Re}\int_\Omega F\overline{u}\,\mathrm{d}x,$$
$$\operatorname{Im}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)}+\operatorname{Im}(b)\|u\|^2_{L^2(\Omega)}=\operatorname{Im}\int_\Omega F\overline{u}\,\mathrm{d}x.$$

Theorem 2.2.3 follows immediately from Lemma 2.4.1 applied with $\omega=\Omega$, while Theorem 2.2.9 is a consequence of Lemma 2.4.5 applied with $\delta=0$ and (2.4.3). This ends the proof. □

Proof of Theorem 2.2.6. Choosing u and $\mathrm{i}u$ as test functions, we obtain

$$\|\nabla u\|^2_{L^2(\Omega)}+\operatorname{Re}(a)\|u\|^{m+1}_{L^{m+1}(\Omega)}+\left(\operatorname{Re}(b)-|\operatorname{Re}(c)|\|V\|^2_{L^\infty(\Omega)}\right)\|u\|^2_{L^2(\Omega)}\leqslant\int_\Omega|Fu|\mathrm{d}x,$$
$$|\operatorname{Im}(a)|\|u\|^{m+1}_{L^{m+1}(\Omega)}+|\operatorname{Im}(b)|\|u\|^2_{L^2(\Omega)}+|\operatorname{Im}(c)|\|Vu\|^2_{L^2(\Omega)}\leqslant\int_\Omega|Fu|\mathrm{d}x.$$

The theorem follows Lemma 2.4.4 applied with $\omega=\Omega$, $R=\|V\|_{L^\infty(\Omega)}$ and $\alpha=\beta=0$. □

Proof of Theorems 2.2.4 and 2.2.8. We first assume that Ω is bounded. Let $H=H^1_0(\Omega)$, in the homogeneous Dirichlet case, and $H=H^1(\Omega)$, in the homogeneous Neumann case. Let $\delta_\star$ be given by Lemma 2.4.5 and let for any $u\in L^2(\Omega)$, $f(u)=a|u|^{-(1-m)}u+bu+cV^2u$ (with $c=0$ in the case of Theorem 2.2.8). Let $(F_n)_{n\in\mathbb{N}}\subset\mathscr{D}(\Omega)$ be such that $F_n\xrightarrow[n\to\infty]{L^2(\Omega)}F$. Let $\left(u^n_\ell\right)_{(n,\ell)\in\mathbb{N}^2}\subset H$ a sequence of solutions of (2.5.1) be given by Lemma 2.5.1 with $\mathcal{O}=\Omega$, $\delta=1$ for Theorem 2.2.4, $\delta=\delta_\star$ for Theorem 2.2.8 and such F_n. By Corollary 2.4.6, there exists a diagonal extraction $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ of $\left(u^n_\ell\right)_{(n,\ell)\in\mathbb{N}^2}$ which is bounded in $W^{1,1}(\Omega)\cap\dot H^1(\Omega)$. Let $1<p<2$ be such that $W^{1,1}(\Omega)\hookrightarrow L^p(\Omega)$. Then $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and there exist $u\in W^{1,p}(\Omega)\cap\dot H^1(\Omega)$ and $g\in L^p(\Omega;\mathbb{R})$ such that, up to a subsequence that we still denote by $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$, $u^n_{\varphi(n)}\xrightarrow[n\to\infty]{L^p(\Omega)}u$, $\nabla u^n_{\varphi(n)}\rightharpoonup\nabla u$ in $\left(L^2_{\mathrm{w}}(\Omega)\right)^N$, as $n\longrightarrow\infty$, $u^n_{\varphi(n)}\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}u$, $\left|u^n_{\varphi(n)}\right|\leqslant g$, a.e. in Ω and $\left(u^n_{\varphi(n)}\mathbb{1}_{\left\{\left|u^n_{\varphi(n)}\right|\leqslant\varphi(n)\right\}}\right)_{n\in\mathbb{N}}$ is bounded in $L^2(\Omega)$, where the last estimate comes from Corollary 2.4.6. By these three last estimates and Fatou's Lemma, $u\in L^2(\Omega)$, $f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\xrightarrow[n\to\infty]{\text{a.e. in }\Omega}f(u)-\delta u$ and $\left|f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\right|\leqslant C(g^m+g)\in L^p(\Omega)$, a.e. in Ω. It follows that $u\in H^1(\Omega)$. From the dominated convergence Theorem, $f_{\varphi(n)}\left(u^n_{\varphi(n)}\right)\xrightarrow[n\to\infty]{L^p(\Omega)}f(u)-\delta u$. Consider the Dirichlet boundary condition. We recall a Gagliardo-Nirenberg's inequality.

$$\forall w\in H^1_0(\Omega),\ \|w\|^{N+2}_{L^2(\Omega)}\leqslant C\|w\|^2_{L^1(\Omega)}\|\nabla w\|^N_{L^2(\Omega)},$$

where $C=C(N)$. In particular, C does not depend on Ω. Since $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}\subset H^1_0(\Omega)$ is bounded in $W^{1,1}(\Omega)\cap\dot H^1(\Omega)$, it follow from the above Gagliardo-Nirenberg's inequality that $\left(u^n_{\varphi(n)}\right)_{n\in\mathbb{N}}$ is bounded in $H^1_0(\Omega)$, so that $u\in H^1_0(\Omega)$. Now, we show that $u\in H$ is a solution. Let $m_0\in\mathbb{N}$ be large enough to have $H^{m_0}(\Omega)\hookrightarrow L^{p'}(\Omega)$. Let $v\in\mathscr{D}(\Omega)$, if $H=H^1_0(\Omega)$ and let $v\in H^{m_0}(\Omega)$, if $H=H^1(\Omega)$. By (2.5.1), we have for any $n\in\mathbb{N}$,

$$\left\langle\nabla u^n_{\varphi(n)},\nabla v\right\rangle_{L^2(\Omega),L^2(\Omega)}+\left\langle\delta u^n_{\varphi(n)}+f_{\varphi(n)}\left(u^n_{\varphi(n)}\right),v\right\rangle_{L^p(\Omega),L^{p'}(\Omega)}-\langle F_n,v\rangle_{L^2(\Omega),L^2(\Omega)}=0.\quad(2.5.3)$$

Above convergencies lead to allow us to pass in the limit in (2.5.3) and by density of $\mathscr{D}(\Omega)$ in $H^1_0(\Omega)$ and density of $H^{m_0}(\Omega)$ in $H^1(\Omega)$ (see, for instance, Corollary 9.8, p.277, in Brezis [44]), it follows that

$$\forall v\in H,\ \langle\nabla u,\nabla v\rangle_{L^2(\Omega),L^2(\Omega)}+\langle f(u),v\rangle_{L^2(\Omega),L^2(\Omega)}=\langle F,v\rangle_{L^2(\Omega),L^2(\Omega)}.$$

This finishes the proof of the existence for Ω bounded. Approximating Ω by an exhaustive sequence of bounded sets $(\Omega\cap B(0,n))_{n\in\mathbb{N}}$, the case Ω unbounded can be treated in the same way as in the proof of Theorem 2.2.1. The symmetry property also follows as in the proof of Theorem 2.2.1. □

Proof of Theorem 2.2.10. Let $u_1,u_2\in H^1(\Omega)\cap L^{m+1}(\Omega)$ be two solutions of (2.1.2) such that $Vu_1,Vu_2\in L^2(\Omega)$. We set $u=u_1-u_2$, $f(v)=|v|^{-(1-m)}v$ and $g(v)=af(v)+bv+cV^2v$. From Lemma 1.9.1, there exists a positive constant C such that,

$$C\int\limits_\omega\frac{|u_1(x)-u_2(x)|^2}{(|u_1(x)|+|u_2(x)|)^{1-m}}\mathrm{d}x\leqslant\langle f(u_1)-f(u_2),u_1-u_2\rangle_{L^{\frac{m+1}{m}}(\Omega),L^{m+1}(\Omega)},\quad(2.5.4)$$

where $\omega=\Big\{x\in\Omega;|u_1(x)|+|u_2(x)|>0\Big\}$. We have that u satisfies $-\Delta u+g(u_1)-g(u_2)=0$. Choosing $v=au$ as a test function, we get

$$\mathrm{Re}(a)\|\nabla u\|_{L^2}^2+|a|^2\langle f(u_1)-f(u_2),u_1-u_2\rangle_{L^{\frac{m+1}{m}},L^{m+1}}+\mathrm{Re}(a\bar{b})\|u\|_{L^2}^2+\mathrm{Re}\,(a\bar{c})\,\|Vu\|_{L^2}^2=0.$$

It follows from the above estimate and (2.5.4) that,

$$\mathrm{Re}(a)\|\nabla u\|_{L^2}^2+C|a|^2\int\limits_{\omega}\frac{|u_1(x)-u_2(x)|^2}{(|u_1(x)|+|u_2(x)|)^{1-m}}\mathrm{d}x+\mathrm{Re}(a\bar{b})\|u\|_{L^2}^2+\mathrm{Re}\,(a\bar{c})\,\|Vu\|_{L^2}^2\leqslant 0,$$

which yields Property 1). Properties 2) and 3) follow in the same way. □

Remark 2.5.2. It is not hard to adapt the above proof to find other criteria of uniqueness.

2.6 On the existence of solutions of the Dirichlet problem for data beyond $L^2(\Omega)$

In this section we shall indicate how some of the precedent results of this paper can be extended to some data F which are not in $L^2(\Omega)$ but in the more general Hilbert space $L^2(\Omega;\delta^\alpha)$, where $\delta(x)=\mathrm{dist}(x,\Gamma)$ and $\alpha\in(0,1)$.

In order to justify the associated notion of solution, we start by assuming that a function u solves equation

$$-\Delta u+f(u)=F,\quad\text{in}\quad\Omega,\tag{2.6.1}$$

with the Dirichlet boundary condition (2.1.3), $u_{|\Gamma}=0$, and we multiply (formally) by $\overline{v(x)}\delta(x)$, with $v\in H_0^1(\Omega;\delta^\alpha)$ $\big($the weighted Sobolev space associated to the weight $\delta^\alpha(x)\big)$, we integrate by parts (by Green's formula) and we take the real part. Then we get,

$$\mathrm{Re}\int\limits_{\Omega}\nabla u.\overline{\nabla v}\,\delta^\alpha\mathrm{d}x+\mathrm{Re}\int\limits_{\Omega}\overline{v}\,\nabla u.\nabla\delta^\alpha\mathrm{d}x+\mathrm{Re}\int\limits_{\Omega}f(u)\,\overline{v}\,\delta^\alpha\mathrm{d}x=\mathrm{Re}\int\limits_{\Omega}F\,\overline{v}\,\delta^\alpha\mathrm{d}x.\tag{2.6.2}$$

To give a meaning to the condition (2.6.2), we must assume that

$$F\in L^2(\Omega;\delta^\alpha),\tag{2.6.3}$$

where $\|F\|_{L^2(\Omega;\delta^\alpha)}^2=\int\limits_{\Omega}|F(x)|^2\delta^\alpha(x)\mathrm{d}x$, and to include in the definition of solution the conditions

$$u\in H_0^1(\Omega;\delta^\alpha)\quad\text{and}\quad f(u)\in L^2(\Omega;\delta^\alpha).\tag{2.6.4}$$

The justification of the second term in (2.6.2) is far to be trivial and requires the use of a version of the following Hardy type inequality,

$$\int\limits_{\Omega}|v(x)|^2\delta^{-(2-\alpha)}(x)\mathrm{d}x\leqslant C\int\limits_{\Omega}|\nabla v(x)|^2\delta^\alpha(x)\mathrm{d}x,\tag{2.6.5}$$

which holds for some constant C independent of v, for any $v \in H_0^1(\Omega;\delta^\alpha)$ once we assume that

$$\Omega \text{ is a bounded open subset of } \mathbb{R}^N \text{ with Lipschitz boundary} \tag{2.6.6}$$

(see, e.g., Kufner [121] and also Drábek, Kufner and Nicolosi [79], Kufner and Opic [122], Kufner and Sänding [123] and Nečas [141]). Notice that under (2.6.6), we know that $\delta \in W^{1,\infty}(\Omega)$ and so

$$\left|\int_\Omega \overline{v}\,\nabla u.\nabla\delta^\alpha \mathrm{d}x\right| = \left|\int_\Omega \left(\delta^{\frac{\alpha}{2}}\nabla u\right).\left(\frac{\overline{v}}{\delta^{\frac{\alpha}{2}}}\nabla\delta^\alpha\right)\mathrm{d}x\right| \leqslant \alpha\|\nabla\delta\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega;\delta^\alpha)}\|v\|_{L^2(\Omega;\delta^{-(2-\alpha)})} < \infty,$$

by Cauchy-Schwarz's inequality and (2.6.5).

Definition 2.6.1. Assumed (2.6.3), (2.6.6) and $\alpha \in (0,1)$, we say that $u \in H_0^1(\Omega;\delta^\alpha)$ is a *solution* of (2.6.1) and (2.1.3) in $H_0^1(\Omega;\delta^\alpha)$ if (2.6.4) holds and the integral condition (2.6.2) holds for any $v \in H_0^1(\Omega;\delta^\alpha)$.

Remark 2.6.2. Notice that $H_0^1(\Omega;\delta^\alpha) \hookrightarrow L^2(\Omega)$ (by the Hardy's inequality (2.6.5) and (2.6.6)). Moreover, since

$$\delta^{-s\alpha} \in L^1(\Omega), \text{ for any } s \in (0,1), \tag{2.6.7}$$

we know (Drábek, Kufner and Nicolosi [79], p.30) that

$$H_0^1(\Omega;\delta^\alpha) \hookrightarrow W^{1,p_s}(\Omega), \text{ with } p_s = \frac{2s}{s+1}.$$

Remark 2.6.3. Obviously, there are many functions F such that $F \in L^2(\Omega;\delta^\alpha)\setminus L^2(\Omega)$ (for instance, if $F(x) \sim \frac{1}{\delta(x)^\beta}$, for some $\beta > 0$, then $F \in L^2(\Omega;\delta^\alpha)$, if $\beta < \frac{\alpha+1}{2}$ but $F \notin L^2(\Omega)$, once $\beta \geqslant \frac{1}{2}$. This fact is crucial when the nonlinear term $f(u)$ involves a singular term of the form as in (2.1.2) but with $m \in (-1,0)$ (see Díaz, Hernández and Rakotoson [75] for the real case).

Remark 2.6.4. We point out that in most of the papers dealing with weighted solutions of semilinear equations, the notion of solution is not justified in this way but merely by replacing the Laplace operator by a bilinear form which becomes coercive on the space $H_0^1(\Omega;\delta^\alpha)$. The second integral term in (2.6.2) is not mentioned (since, formally, the multiplication of the equation is merely by $v \in H_0^1(\Omega;\delta^\alpha)$) but then it is quite complicated to justify that such alternative solutions satisfy the pde equation (2.1.2) when they are assumed, additionally, that $\Delta u \in L^2_{\mathrm{loc}}(\Omega)$. We also mention now (although it is a completely different approach) the notion of $L^1(\Omega;\delta)$-very weak solution developed recently for many scalars semilinear equations : see, e.g., Brezis, Cazenave, Martel and Ramiandrisoa [46], Díaz and Rakotoson [77] and the references therein).

By using exactly the same *a priori* estimates, but now adapted to the space $H_0^1(\Omega;\delta^\alpha)$, we get the following result.

Theorem 2.6.5. *Let Ω be a bounded open subset with Lipschitz boundary, $V \in L^\infty(\Omega;\mathbb{R})$, $0 < \alpha < 1$, $0 < m < 1$, $(a,b,c) \in \mathbb{C}^3$ as in Theorem 2.2.4 and let $F \in L^2(\Omega;\delta^\alpha)$. Then we have the following result.*

1) *There exists at least a solution* $u \in H_0^1(\Omega;\delta^\alpha)$ *to* (2.1.2)*. Furthermore, any such solution belongs to* $H^2_{\rm loc}(\Omega)$*.*
2) *If, in addition, we assume the conditions of Theorem* 2.2.10*, this solution is unique in the class of* $H_0^1(\Omega;\delta^\alpha)$*-solutions.*

Remark 2.6.6. In the proof of the *a priori* estimates, it is useful to replace the weighted function δ by a more smooth function having the same behavior near Γ. This is the case, for instance of the first eigenfunction φ_1 of the Laplace operator,

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1, & \text{in } \Omega,\\ \varphi_{1|\Gamma} = 0, & \text{on } \Gamma. \end{cases}$$

It is well-known that $\varphi_1 \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ and that $C_1\delta(x) \leqslant \varphi_1(x) \leqslant C_2\delta(x)$, for any $x\in\Omega$, for some positive constants C_1 and C_2, independent of x. Now, with this new weighted function, it is easy to see that the second term in (2.6.2) does not play any important role since, for instance, when taking $v=u$ as test function, we get that

$$\begin{aligned} &\mathrm{Re}\int_\Omega \overline{u}\,\nabla u.\nabla\varphi_1^\alpha \mathrm{d}x = \frac{1}{2}\int_\Omega \nabla|u|^2.\nabla\varphi_1^\alpha\mathrm{d}x = -\frac{1}{2}\int_\Omega |u|^2\Delta\varphi_1^\alpha\mathrm{d}x\\ &= \frac{\alpha\lambda_1}{2}\int_\Omega |u|^2\varphi_1^\alpha\mathrm{d}x + \frac{\alpha(1-\alpha)}{2}\int_\Omega |u|^2\varphi_1^{-(2-\alpha)}|\nabla\varphi_1|^2\mathrm{d}x \geqslant 0. \end{aligned}$$

2.7 Conclusions

In this section, we summarize the results obtained in Section 2.2 and give some applications.

The next result comes from Theorems 2.2.1, 2.2.3 and 2.2.10.

Theorem 2.7.1. *Let* Ω *an open subset of* $\mathbb{R}^N$ *be such that* $|\Omega|<\infty$ *and assume* $0<m<1$, $(a,b)\in\mathbb{C}^2$ *and* $F\in L^2(\Omega)$*. Assume that* $\mathrm{Re}(b) > -\frac{1}{C_{\rm P}^2}$ *or* $\mathrm{Im}(b)\neq 0$*, where* $C_{\rm P}$ *is the Poincaré's constant in* (2.4.1)*. Then there exists at least a solution* $u\in H_0^1(\Omega)$ *to*

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega). \tag{2.7.1}$$

Furthermore, $\|u\|_{H_0^1(\Omega)} \leqslant C(\|F\|_{L^2(\Omega)},|\Omega|,|a|,|b|,N,m)$*. Finally, if* $\overrightarrow{a}.\overrightarrow{b}>0$ *then the solution is unique.*

In the above theorem, the complex numbers a and b are seen as vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ of $\mathbb{R}^2$. Consequently, $\overrightarrow{a}.\overrightarrow{b}$ denotes the scalar product between these vectors of $\mathbb{R}^2$.

The novelty of Theorem 2.7.1 is about the range of (a,b) : we obtain existence of solution with, for instance, $(a,b)\in\mathbb{R}_-\times(-\varepsilon,0)$, with $\varepsilon>0$ small enough, or $(a,b)=(-1+\mathrm{i},-1-\mathrm{i})$. Recall that, up to today, existence was an open question when $(a,b)\in\mathbb{R}_-\times\mathbb{R}_-$ or $[a,b]\cap\mathbb{R}_-\times\mathrm{i}\{0\}\neq\emptyset$ (Bégout and Díaz [25]). Knowing that for such (a,b) equation (2.7.1) admits solutions, it would be interesting if, whether or not, solutions with compact support exist, as in Bégout and Díaz [25].

By Theorems 2.2.4, 2.2.6 and 2.2.10, we get the following result.

Theorem 2.7.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded open subset, let $0 < m < 1$ and let $(a,b,c) \in \mathbb{C}^3$ be such that $\mathrm{Im}(a) < 0$, $\mathrm{Im}(b) < 0$ and $\mathrm{Im}(c) \leqslant 0$. For any $F \in L^2(\Omega)$, there exists at least a solution $u \in H^1(\Omega)$ to*

$$-\Delta u + a|u|^{-(1-m)}u + bu + c|x|^2u = F, \text{ in } L^2(\Omega), \tag{2.7.2}$$

with boundary condition (2.1.3) or (2.1.4)[1]. Furthermore,

$$\|u\|_{H^1(\Omega)} \leqslant C(|a|,|b|,|c|)(R^2+1)\|F\|_{L^2(\Omega)},$$

where $B(0,R) \supset \Omega$. Finally, if $\overrightarrow{a}.\overrightarrow{b} > 0$ and $\overrightarrow{a}.\overrightarrow{c} > 0$ then the solution is unique.

Since, now, we are able to show that equation (2.7.2) admits solutions, we can study the propagation support phenomena. Indeed, we can show that, under some suitable conditions, there exists a self-similar solution u to

$$\mathrm{i}u_t + \Delta u = a|u|^{-(1-m)}u + f(t,x), \text{ in } \mathbb{R}^N,$$

such that for any $t > 0$, $\operatorname{supp} u(t)$ is compact (see Bégout and Díaz [27]).

Now, we turn out to equation (2.7.1) by extending some results found in Bégout and Díaz [25]. These results are due to Theorems 2.2.8, 2.2.9 and 2.2.10.

Theorem 2.7.3. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset of $\mathbb{R}^N$, let $0 < m < 1$ and let $(a,b) \in \mathbb{A}^2$ satisfies (2.2.7). For any $F \in L^2(\Omega)$, there exists at least a solution $u \in H^1(\Omega) \cap L^{m+1}(\Omega)$ to*

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega), \tag{2.7.3}$$

with boundary condition (2.1.3) or (2.1.4)[1] (in this last case, Ω is assumed bounded). Furthermore,

$$\|u\|^2_{H^1(\Omega)} + \|u\|^{m+1}_{L^{m+1}(\Omega)} \leqslant M(|a|,|b|)\|F\|^2_{L^2(\Omega)}.$$

Finally, if $\overrightarrow{a}.\overrightarrow{b} > 0$ then the solution is unique.

When $|\Omega| < \infty$, Theorem 2.7.3 is an improvement of Theorem 1.4.1, since we may choose $F \in L^2(\Omega)$, instead of $F \in L^{\frac{m+1}{m}}(\Omega)$ and that $L^{\frac{m+1}{m}}(\Omega) \subsetneq L^2(\Omega)$. In addition, this existence result extends to the homogeneous Neumann boundary condition. In this context, we may show three kinds of new results, under assumptions of Theorem 2.7.3.

- If $\Omega = \mathbb{R}^N$ and if $F \in L^2(\mathbb{R}^N)$ has compact support then equation (2.7.3) admits solutions and any solution is compactly supported.
- If $\|F\|_{L^2(\Omega)}$ is small enough and if F has compact support then equation (2.7.3) admits solutions with the homogeneous Dirichlet boundary condition and any solution is compactly supported in Ω.
- If $\|F\|_{L^2(\Omega)}$ is small enough, if $\overrightarrow{a}.\overrightarrow{b} > 0$ and if F has compact support then equation (2.7.3) admits a unique solution with the homogeneous Neumann boundary condition and, in fact, this solution is compactly supported in Ω.

For more details, see Bégout and Díaz [27]. Finally, in Section 2.6 we extended our techniques of proofs to the case in which the datum F is very singular near the boundary of Ω but still is in some weighted Lebesgue space (see Theorem 2.6.5).

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Chapitre 3

A sharper energy method for the localization of the support to some stationary Schrödinger equations with a singular nonlinearity

with Jesús Ildefonso Díaz*

Abstract

We prove the compactness of the support of the solution of some stationary Schrödinger equations with a singular nonlinear order term. We present here a sharper version of some energy methods previously used in the literature and, in particular, by the authors.

3.1 Introduction

Since the beginnings of the eighties of the last century, it is already well-known that the absence of the maximum principle for the case of systems and higher order nonlinear partial differential equations was one of the main motivations of the introduction of suitable energy methods allowing to conclude the compactness of the support of their solutions (see, e.g., the presentation made in the monograph Antontsev, Díaz and Shmarev [11]).

The application of such type of methods to the case of nonlinear Schrödinger equations with a singular zero order term required some important improvements of the method. That was the main object of the previous author's papers of Bégout and Díaz [24, 25].

The main goal of this new paper is to present a sharper version of the mentioned method potentially able to be applied to many other problems related to this type of Schrödinger equations such as the study of self-similar solutions, case of Neumann boundary conditions, presence of nonlocal terms

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(such as, for instance, in Hartree-Fock theory : Cazenave [57]), etc., which can not be treated with the mere technique presented in Bégout and Díaz [24, 25]. As a matter of fact, the concrete application of this sharper energy method to the concrete case of self-similar solutions of the evolution Schrödinger problem requires many additional arguments justifying the special structure of those solutions, reason why we decided to present it in a separated work (Bégout and Díaz [26]). We send the reader to Bégout and Díaz [26] for a long description of the important role of the compactness of the solution in this context and for many other references related to this qualitative property of the solution.

This paper is organized as follows. Below, we give some notations which will be used throughout this paper. In Section 3.2, we give the precise "localization" estimates which imply a solution of a partial differential equation to be compactly supported $\Big($see Theorems 3.2.1 and 3.2.2, and especially estimates (3.2.1) and (3.2.3)$\Big)$. In Section 3.3, we give a tool which permits, from a solution of some partial differential equation, to establish the "localization" estimate (Theorem 3.3.1). The results of these two sections are proved in Section 3.4. In Bégout and Díaz [25], localization property is studied for the complex-valued equation

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } \Omega. \tag{3.1.1}$$

We also study this property here, but with a change of notation (see Remark 3.5.1 below for the motivation of this change). Section 3.5 is devoted to the study of the localization property of the solutions of equation (3.1.1), in the same spirit as Bégout and Díaz [25], but with the homogeneous Neumann boundary condition instead of the homogeneous Dirichlet boundary condition (compare Theorem 3.5.6 below with Theorem 1.3.5). Finally, at the end of the paper, we treat equation (3.1.1) with the homogeneous Dirichlet boundary condition (Remark 3.5.8). We state the same results as in Bégout and Díaz [25], but with now the weaker assumption $F \in L^2(\Omega)$.

Before ending this section, we shall indicate here some of the notations used throughout. We write $\mathrm{i}^2 = -1$. We denote by $\overline{z}$ the conjugate of the complex number z. For $1 \leqslant p \leqslant \infty$, p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. For $j,k \in \mathbb{Z}$ with $j < k$, $[\![j,k]\!] = [j,k] \cap \mathbb{Z}$. We denote by Γ the boundary of a nonempty subset $\Omega \subseteq \mathbb{R}^N$ and $\Omega^{\mathrm{c}} = \mathbb{R}^N \setminus \Omega$ its complement. Unless if specified, any function lying in a functional space $\big(L^p(\Omega), W^{m,p}(\Omega)$, etc$\big)$ is supposed to be a complex-valued function $\big(L^p(\Omega;\mathbb{C}), W^{m,p}(\Omega;\mathbb{C})$, etc$\big)$. For a Banach space E, we denote by $E^\star$ its topological dual and by $\langle\ .\ ,\ .\ \rangle_{E^\star,E} \in \mathbb{R}$ the $E^\star - E$ duality product. In particular, for any $T \in L^{p'}(\Omega)$ and $\varphi \in L^p(\Omega)$ with $1 \leqslant p < \infty$, $\langle T,\varphi\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \mathrm{Re}\int\limits_\Omega T(x)\overline{\varphi(x)}\mathrm{d}x$. As usual, we denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1,\ldots,a_n$, write $C(a_1,\ldots,a_n)$ to indicate that the constant C continuously depends only on $a_1,\ldots,a_n$ (this convention also holds for constants which are not denoted by "C").

3.2 From suitable local inequalities to the vanishing of the involved complex functions on some small ball

In this section, we establish some results improving the presentation of some energy methods of Antontsev, Díaz and Shmarev [11] which allow to prove localization properties of solutions of a general class of nonlinear partial differential equations (Section 3.5, Remark 3.5.8 below and Bégout and Díaz [26]). In contrast to the presentation in Bégout and Díaz [25] (see e.g. Theorem 1.1.1), the following statement does not need any information on the second order equation but it will merely use a suitable balance between the total local energy (diffusion + absorption local energies) and the local boundary flux. This will be crucial for the applicability of the method to cases for which the techniques of Bégout and Díaz [24, 25] can not be applied.

Theorem 3.2.1. *Assume $0<m<1$ and let $N\in\mathbb{N}$. Then there exists $C=C(N,m)$ satisfying the following property: let $x_0\in\mathbb{R}^N$, $\rho_0>0$ and $u\in H^1_{\mathrm{loc}}\big(B(x_0,\rho_0)\big)$. If there exist $L>0$ and $M>0$ such that for almost every $\rho\in(0,\rho_0)$,*

$$\|\nabla u\|^2_{L^2(B(x_0,\rho))}+L\|u\|^{m+1}_{L^{m+1}(B(x_0,\rho))}\leqslant M\left|\int_{\mathbb{S}(x_0,\rho)}u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right|, \tag{3.2.1}$$

then $u_{|B(x_0,\rho_{\max})}\equiv0$, where

$$\rho^\nu_{\max}=\left(\rho_0^\nu-CM^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{\rho_0^{\nu-1},1\right\}\times\min_{\tau\in\left(\frac{m+1}{2},1\right]}\left\{\frac{E(\rho_0)^{\gamma(\tau)}\max\{b(\rho_0)^{\mu(\tau)},b(\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right\}\right)_+, \tag{3.2.2}$$

and where,

$$E(\rho_0)=\|\nabla u\|^2_{L^2(B(x_0,\rho_0))},\qquad b(\rho_0)=\|u\|^{m+1}_{L^{m+1}(B(x_0,\rho_0))},$$
$$k=2(1+m)+N(1-m),\qquad \nu=\tfrac{k}{m+1}>2,$$
$$\gamma(\tau)=\frac{2\tau-(1+m)}{k}\in(0,1),\quad \mu(\tau)=\frac{2(1-\tau)}{k},\quad \eta(\tau)=\frac{1-m}{1+m}-\gamma(\tau)>0.$$

for any $\tau\in\left(\frac{m+1}{2},1\right]$.

Here and in what follows, $r_+=\max\{0,r\}$ denotes the positive part of the real number r. For $x_0\in\mathbb{R}^N$ and $r>0$, $B(x_0,r)$ is the open ball of $\mathbb{R}^N$ of center x_0 and radius r, $\mathbb{S}(x_0,r)$ is its boundary and $\overline{B}(x_0,r)$ is its closure. Finally, σ is the surface measure on a sphere. A sharper estimate, in the same line of extension of the applicability of the techniques of Bégout and Díaz [24, 25] indicated before, can be obtained under some additional assumption on F.

Theorem 3.2.2. *Let $0<m<1$, $x_0\in\mathbb{R}^N$, $\rho_1>\rho_0>0$, $F\in L^2\big(B(x_0,\rho_1)\big)$ and $u\in H^1_{\mathrm{loc}}\big(B(x_0,\rho_1)\big)$. If there exist $L>0$ and $M>0$ such that for almost every $\rho\in(0,\rho_1)$,*

$$\|\nabla u\|^2_{L^2(B(x_0,\rho))}+L\|u\|^{m+1}_{L^{m+1}(B(x_0,\rho))}+L\|u\|^2_{L^2(B(x_0,\rho))}\leqslant M\left(\left|\int_{\mathbb{S}(x_0,\rho)}u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right|+\int_{B(x_0,\rho)}|F(x)u(x)|\mathrm{d}x\right), \tag{3.2.3}$$

then there exist $E_\star>0$ and $\varepsilon_\star>0$ satisfying the following property: if $\|\nabla u\|^2_{L^2(B(x_0,\rho_1))}<E_\star$ and

$$\|F\|^2_{L^2(B(x_0,\rho))}\leqslant\varepsilon_\star\big((\rho-\rho_0)_+\big)^p,\ \forall\rho\in(0,\rho_1), \tag{3.2.4}$$

where $p=\frac{2(1+m)+N(1-m)}{1-m}$, then $u_{|B(x_0,\rho_0)}\equiv0$. In other words, with the notation of Theorem 3.2.1, $\rho_{\max}=\rho_0$.

Remark 3.2.3. We may estimate $E_\star$ and $\varepsilon_\star$ as

$$E_\star = E_\star\left(\|u\|^{-1}_{L^{m+1}(B(x_0,\rho_1))}, \rho_1, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m\right),$$
$$\varepsilon_\star = \varepsilon_\star\left(\|u\|^{-1}_{L^{m+1}(B(x_0,\rho_1))}, \frac{\rho_0}{\rho_1}, \frac{L}{M}, N, m\right).$$

The dependence on $\frac{1}{\delta}$ means that if δ goes to 0 then $E_\star$ and $\varepsilon_\star$ may be very large. Note that $p = \frac{1}{\gamma(1)}$, where γ is the function defined in Theorem 3.2.1.

Remark 3.2.4. Note that by Cauchy-Schwarz's inequality, the right-hand side in (3.2.1) belongs to $L^1_{\text{loc}}([0,\rho_0);\mathbb{R})$ and so is defined almost everywhere in $(0,\rho_0)$. Consequently, by Hölder's inequality, the right-hand side in (3.2.3) is defined almost everywhere in $(0,\rho_1)$.

3.3 A general framework of applications related to the Schrödinger operator

The following result will be applied later to many concrete equations associated to the Schrödinger operator.

Theorem 3.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty open subset of $\mathbb{R}^N$, let $x_0 \in \Omega$, let $\rho_0 > 0$, let $1 \leqslant p_1, \dots, p_{n_1}, q_1, \dots, q_{n_2} < \infty$, let $F \in L^1_{\text{loc}}(\Omega)$ be such that $F_{|\Omega\cap B(x_0,\rho_0)} \in L^2\big(\Omega \cap B(x_0,\rho_0)\big)$ and let*

$$f \in C\left(\bigcap_{k=1}^{n_2} L^{q_k}_{\text{loc}}(\Omega); \sum_{j=1}^{n_1} L^{p'_j}_{\text{loc}}(\Omega)\right).$$

Let $u \in H^1_{\text{loc}}(\Omega) \cap L^{p_j}_{\text{loc}}(\Omega) \cap L^{q_k}_{\text{loc}}(\Omega)$, for any $(j,k) \in [\![1,n_1]\!] \times [\![1,n_2]\!]$, be any solution to the complex-valued equation

$$-\Delta u + f(u) = F, \text{ in } \mathscr{D}'(\Omega). \tag{3.3.1}$$

If $\rho_0 > \operatorname{dist}(x_0,\Gamma)$ then assume further that

$$f \in C\left(\bigcap_{k=1}^{n_2} L^{q_k}(\Omega); \sum_{j=1}^{n_1} L^{p'_j}(\Omega)\right), \ u \in H^1_0(\Omega),$$
$$u_{|\Omega\cap B(x_0,\rho_0)} \in L^{p_j}\big(\Omega\cap B(x_0,\rho_0)\big) \cap L^{q_k}\big(\Omega\cap B(x_0,\rho_0)\big),$$

for any $(j,k) \in [\![1,n_1]\!] \times [\![1,n_2]\!]$. Set for every $\rho \in [0,\rho_0)$,

$$I(\rho) = \left|\int_{\Omega\cap\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right|, \quad J(\rho) = \int_{\Omega\cap B(x_0,\rho)} |F(x)u(x)|\mathrm{d}x, \tag{3.3.2}$$

$$w(\rho) = \int_{\Omega\cap\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma, \quad I_{\text{Re}}(\rho) = \operatorname{Re}\big(w(\rho)\big), \quad I_{\text{Im}}(\rho) = \operatorname{Im}\big(w(\rho)\big). \tag{3.3.3}$$

Then we have,

$$I, J, I_{\mathrm{Re}}, I_{\mathrm{Im}} \in C([0,\rho_0);\mathbb{R}), \tag{3.3.4}$$

$$\|\nabla u\|^2_{L^2(\Omega\cap B(x_0,\rho))} + \mathrm{Re}\left(\int\limits_{\Omega\cap B(x_0,\rho)} f(u)\overline{u}\mathrm{d}x\right) = \mathrm{Re}\left(\int\limits_{\Omega\cap B(x_0,\rho)} F(x)\overline{u(x)}\mathrm{d}x\right) + I_{\mathrm{Re}}(\rho), \tag{3.3.5}$$

$$\mathrm{Im}\left(\int\limits_{\Omega\cap B(x_0,\rho)} f(u)\overline{u}\mathrm{d}x\right) = \mathrm{Im}\left(\int\limits_{\Omega\cap B(x_0,\rho)} F(x)\overline{u(x)}\mathrm{d}x\right) + I_{\mathrm{Im}}(\rho), \tag{3.3.6}$$

for any $\rho \in [0,\rho_0)$.

Remark 3.3.2. One easily sees that if $\rho_0 < \mathrm{dist}(x_0,\Gamma)$ then $I, J, I_{\mathrm{Re}}, I_{\mathrm{Im}} \in C([0,\rho_0];\mathbb{R})$.

Example 3.3.3. We give some functions f for which Theorem 3.3.1 applies.

1) Typically, we apply Theorem 3.3.1 to

$$f(u) = a|u|^{-(1-m)}u + bu + Vu,$$

with $(a,b) \in \mathbb{C}^2$, $V \in L^\infty_{\mathrm{loc}}(\Omega)$ and $0 < m < 1$. One easily checks that,

$$f \in C\left(L^2_{\mathrm{loc}}(\Omega) \cap L^{m+1}_{\mathrm{loc}}(\Omega); L^2_{\mathrm{loc}}(\Omega) + L^{\frac{m+1}{m}}_{\mathrm{loc}}(\Omega)\right).$$

If in addition, $V \in L^\infty(\Omega)$ then one also has,

$$f \in C\left(L^2(\Omega) \cap L^{m+1}(\Omega); L^2(\Omega) + L^{\frac{m+1}{m}}(\Omega)\right).$$

Let $z \in \mathbb{C}\setminus\{0\}$. Since $\left||z|^{-(1-m)}z\right| = |z|^m$, it is understood in the above example that $\left||z|^{-(1-m)}z\right| = 0$ when $z=0$.

2) **Hartree-Fock type equations.** Let $V \in L^p(\mathbb{R}^N;\mathbb{R}) + L^\infty(\mathbb{R}^N;\mathbb{R})$, with $\min\left\{1,\frac{N}{2}\right\} < p < \infty$ and let $W \in L^q(\mathbb{R}^N;\mathbb{R}) + L^\infty(\mathbb{R}^N;\mathbb{R})$, with $\min\left\{1,\frac{N}{4}\right\} < q < \infty$. Set $r = \frac{2p}{p-1}$, $s = \frac{4q}{q-1}$,

$$E = L^2(\mathbb{R}^N) \cap L^4(\mathbb{R}^N) \cap L^r(\mathbb{R}^N) \cap L^s(\mathbb{R}^N),$$
$$f(u) = Vu + (W \star |u|^2)u,$$

for any $u \in H^1(\mathbb{R}^N)$. Then $H^1(\mathbb{R}^N) \hookrightarrow E$ with dense embedding and, by density of $\mathscr{D}(\mathbb{R}^N)$ in spaces $L^m(\mathbb{R}^N)$, for any $m \in [1,\infty)$, we have

$$E^\star = L^2(\mathbb{R}^N) + L^{\frac{4}{3}}(\mathbb{R}^N) + L^{r'}(\mathbb{R}^N) + L^{s'}(\mathbb{R}^N),$$
$$f \in C\big(E;E^\star\big),$$
$$f \in C\big(H^1(\mathbb{R}^N); H^{-1}(\mathbb{R}^N)\big).$$

See Cazenave [57] (Proposition 1.1.3, Proposition 3.2.2, Remark 3.2.3, Proposition 3.2.9, Remark 3.2.10 and Example 3.2.11).

3.4 Proofs of the main results

Before proceeding to the proof of Theorems 3.2.1 and 3.2.2, we recall the well-known Young's inequality. For any real $x \geqslant 0$, $y \geqslant 0$, $\lambda > 1$ and $\varepsilon > 0$, one has

$$xy \leqslant \frac{1}{\lambda'}\varepsilon^{\lambda'}x^{\lambda'} + \frac{1}{\lambda}\varepsilon^{-\lambda}y^{\lambda}. \tag{3.4.1}$$

Proof of Theorems 3.2.1 and 3.2.2. We write $\rho_\star = \rho_0$, for the proof of Theorem 3.2.1 and $\rho_\star = \rho_1$, for the proof of Theorem 3.2.2. Let us introduce some notations. Let $\rho \in (0,\rho_\star)$. We set

$$E(\rho) = \|\nabla u\|^2_{L^2(B(x_0,\rho))}, \qquad b(\rho) = \|u\|^{m+1}_{L^{m+1}(B(x_0,\rho))}, \quad a(\rho) = \|u\|^2_{L^2(B(x_0,\rho))},$$
$$\theta = \tfrac{(1+m)+N(1-m)}{k} \in (0,1), \quad \ell = \tfrac{1}{\theta(1+m)}, \qquad \delta = \tfrac{k}{2(1+m)}.$$

We may assume that $u \in H^1\big(B(x_0,\rho_\star)\big)$. Indeed, the case $u \in H^1_{\mathrm{loc}}\big(B(x_0,\rho_\star)\big)$ can be treated by following the method in Bégout and Díaz [25] (see the end of Step 6, p.18, for Theorem 3.2.1 and the end of Step 7, p.19, for Theorem 3.2.2. We now proceed with the proof in 3 steps.

Step 1. $E \in W^{1,1}(0,\rho_\star)$, for a.e. $\rho \in (0,\rho_\star)$, $E'(\rho) = \|\nabla u\|^2_{L^2(\mathbb{S}(x_0,\rho))}$ and

$$E(\rho) + b(\rho) \leqslant \frac{1}{2}\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{2}}\left(E(\rho)+b(\rho)\right)^{\frac{\gamma(\tau)+1}{2}} + (L_1M)^2\|F\|^2_{L^2(B(x_0,\rho))}, \tag{3.4.2}$$

where $K_1(\tau) = C(N,m)L_1^2M^2\max\left\{\rho_\star^{\nu-1},1\right\}\max\{b(\rho_\star)^{\mu(\tau)}, b(\rho_\star)^{\eta(\tau)}\}$ and $L_1 = \max\left\{1,\frac{1}{L}\right\}$.
By the first lines of Step 2, p.16, we only have to show (3.4.2). Let $\rho \in (0,\rho_\star)$. We have to slightly modify the proof of Bégout and Díaz [25]. Indeed, since $F \in L^2$, we need of the term $\|u\|^2_{L^2}$. We have,

$$\left|\int_{\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right| \leqslant E'(\rho)^{\frac{1}{2}}\|u\|_{L^2(\mathbb{S}(x_0,\rho))}, \tag{3.4.3}$$

$$\|u\|_{L^2(\mathbb{S}(x_0,\rho))} \leqslant C(N,m)\left(\|\nabla u\|_{L^2(B(x_0,\rho))} + \rho^{-\delta}\|u\|_{L^{m+1}(B(x_0,\rho))}\right)^{\theta}\|u\|^{1-\theta}_{L^{m+1}(B(x_0,\rho))}. \tag{3.4.4}$$

See (1.7.11)–(1.7.12). Putting together (3.2.1) (for Theorem 3.2.1), (3.2.3) (for Theorem 3.2.2), (3.4.3) and (3.4.4), we obtain,

$$\begin{aligned} &E(\rho) + b(\rho) + \kappa a(\rho) \\ &\qquad\leqslant CL_1ME'(\rho)^{\frac{1}{2}}\left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta}b(\rho)^{\frac{1-\theta}{m+1}} + L_1M\int\limits_{B(x_0,\rho)}|F(x)u(x)|\mathrm{d}x, \end{aligned} \tag{3.4.5}$$

where $\kappa = 0$, in the case of Theorem 3.2.1 and where $\kappa = 1$, in the case of Theorem 3.2.2. In the case of Theorem 3.2.2, we apply (3.4.1) with $x = |F|$, $y = |u|$, $\lambda = 2$ and $\varepsilon = \sqrt{L_1M}$, and we get

$$\int\limits_{B(x_0,\rho)}|F(x)u(x)|\mathrm{d}x \leqslant \frac{L_1M}{2}\|F\|^2_{L^2(B(x_0,\rho))} + \frac{1}{2L_1M}a(\rho), \tag{3.4.6}$$

for any $\rho \in (0,\rho_\star)$. Putting together (3.4.5) and (3.4.6), we obtain for both theorems, for a.e. $\rho \in (0,\rho_\star)$,

$$E(\rho) + b(\rho) \leqslant C_0L_1ME'(\rho)^{\frac{1}{2}}\left(E(\rho)^{\frac{1}{2}} + \rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)^{\theta}b(\rho)^{\frac{1-\theta}{m+1}} + (L_1M)^2\|F\|^2_{L^2(B(x_0,\rho))}. \tag{3.4.7}$$

Let $\tau\in\left(\frac{m+1}{2},1\right]$ and let $\rho\in(0,\rho_\star)$. A straightforward calculation yields

$$\begin{aligned}
&\left(E(\rho)^{\frac12}+\rho^{-\delta}b(\rho)^{\frac{1}{m+1}}\right)b(\rho)^{\frac{1-\theta}{\theta(m+1)}}\\
&=E(\rho)^{\frac12}b(\rho)^{\frac{1-\theta}{\theta(m+1)}}+\rho^{-\delta}b(\rho)^{\frac{1}{\theta(m+1)}}\\
&=E(\rho)^{\frac12}b(\rho)^{\tau(1-\theta)\ell}b(\rho)^{(1-\tau)(1-\theta)\ell}+\rho^{-\delta}b(\rho)^{\frac12+\tau(1-\theta)\ell}b(\rho)^{\ell-\tau(1-\theta)\ell-\frac12}\\
&\leqslant 2\rho^{-\delta}\max\left\{\rho_\star^\delta,1\right\}K_2^2(\tau)^{\frac{1}{2\theta}}\left(E(\rho)+b(\rho)\right)^{\frac12+\tau(1-\theta)\ell},
\end{aligned}$$

where $K_2^2(\tau)=\max\{b(\rho_\star)^{\mu(\tau)},b(\rho_\star)^{\eta(\tau)}\}$, since $\frac{\mu(\tau)}{2\theta}=(1-\tau)(1-\theta)\ell$ and $\frac{\eta(\tau)}{2\theta}=\ell-\tau(1-\theta)\ell-\frac12$. Hence (3.4.2) follows from (3.4.7) and the above estimate with $K_1(\tau)=16C_0^2L_1^2M^2K_2^2(\tau)\max\left\{\rho_\star^{\nu-1},1\right\}$, since $2\delta\theta=\nu-1$ and $\theta\left(\frac12+\tau(1-\theta)\ell\right)=\frac{\gamma(\tau)+1}{2}$.

Step 2. For any $\tau\in\left(\frac{m+1}{2},1\right]$ and for a.e. $\rho\in(0,\rho_\star)$,

$$0\leqslant E(\rho)^{1-\gamma(\tau)}\leqslant K_1(\tau)\rho^{-(\nu-1)}E'(\rho)+(2L_1M)^{2(1-\gamma(\tau))}\|F\|_{L^2(B(x_0,\rho))}^{2(1-\gamma(\tau))}.$$

Following Step 4, p.17, but with Young's inequality (3.4.1) applied with $x=\frac12\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac12}$, $y=(E(\rho)+b(\rho))^{\frac{\gamma(\tau)+1}{2}}$, $\lambda=\lambda(\tau)=\frac{2}{\gamma(\tau)+1}$ and $\varepsilon=\varepsilon(\tau)=(\gamma(\tau)+1)^{\frac{1}{\lambda(\tau)}}$, Step 2 follows from the estimates

$$\begin{aligned}
&E(\rho)+b(\rho)\\
&\leqslant\frac12\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac12}(E(\rho)+b(\rho))^{\frac{\gamma(\tau)+1}{2}}+(L_1M)^2\|F\|_{L^2(B(x_0,\rho))}^2,\\
&\leqslant\frac{C(\tau)}{2^{\frac{\lambda(\tau)}{\lambda(\tau)-1}}}\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}}+\frac12(E(\rho)+b(\rho))+(L_1M)^2\|F\|_{L^2(B(x_0,\rho))}^2,\\
&\leqslant\frac12\left(K_1(\tau)\rho^{-(\nu-1)}E'(\rho)\right)^{\frac{1}{1-\gamma(\tau)}}+\frac12(E(\rho)+b(\rho))+(L_1M)^2\|F\|_{L^2(B(x_0,\rho))}^2,\\
C(\tau)&=\frac{\lambda(\tau)-1}{\lambda(\tau)}\varepsilon(\tau)^{\frac{\lambda(\tau)}{\lambda(\tau)-1}}<\frac{\lambda\left(\frac{m+1}{2}\right)-1}{\lambda\left(\frac{m+1}{2}\right)}(\gamma(\tau)+1)^{\frac{1}{\lambda(\tau)-1}}<\frac12 2^{\frac{1}{\lambda(\tau)-1}}<\frac12 2^{\frac{\lambda(\tau)}{\lambda(\tau)-1}}.
\end{aligned}$$

Step 3. Conclusion.

Now, following from Step 5 to Step 7, p.18–19, where estimate (1.7.16) therein has to be replaced with estimate of the above Step 2 and where the mapping $\rho\longmapsto F(\rho)$ has to be replaced with the new function $\rho\longmapsto(2L_1M)^{2(1-\gamma)}\|F\|_{L^2(B(x_0,\rho))}^{2(1-\gamma)}$, we prove Theorems 3.2.1 and 3.2.2. This achieves the proof. □

Proof of Theorem 3.3.1. If $\rho_0>\operatorname{dist}(x_0,\Gamma)$ then $u\in H_0^1(\Omega)$. So we may extend u by 0 on $\Omega^c\cap B(x_0,\rho_0)$. Denoting $\widetilde{u}$ this extension, we have $\widetilde{u}\in H_0^1\big(\Omega\cup B(x_0,\rho_0)\big)$. We first consider the case where $\rho_0\neq\operatorname{dist}(x_0,\Gamma)$. We deal with $\rho_0=\operatorname{dist}(x_0,\Gamma)$ at the end of the proof. It follows that $J\in C([0,\rho_0];\mathbb{R})$ and by Cauchy-Schwarz's inequality, $I\in L^1(0,\rho_0)$. Thus, $I,J,I_{\mathrm{Re}},I_{\mathrm{Im}}$ are defined almost everywhere on $(0,\rho_0)$. It follows from (3.3.1) that,

$$\langle\nabla u,\nabla\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}+\langle f(u),\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}=\langle F,\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},\tag{3.4.8}$$

for any $\varphi\in\mathscr{D}(\Omega)$. Let $\rho\in(0,\rho_0)$. For any $n\in\mathbb{N}$, $n>\frac{1}{\rho}$, we define $\psi_n\in W^{1,\infty}(\mathbb{R};\mathbb{R})$ by

$$\forall t\in\mathbb{R},\ \psi_n(t)=\begin{cases}1, & \text{if } |t|\in\left[0,\rho-\frac{1}{n}\right],\\ n(\rho-|t|), & \text{if } |t|\in\left(\rho-\frac{1}{n},\rho\right),\\ 0, & \text{if } |t|\in[\rho,\infty),\end{cases}$$

and we set $\widetilde{\varphi}_n(x)=\psi_n(|x-x_0|)\widetilde{u}(x)$ and $\varphi_n=\widetilde{\varphi}_{n|\Omega}$, for almost every $x\in\Omega\cup B(x_0,\rho_0)$. We easily check that for any $(j,k)\in[\![1,n_1]\!]\times[\![1,n_2]\!]$,

$$\begin{gathered}\varphi_{n|\Omega\cap B(x_0,\rho_0)}\in H^1_0\big(\Omega\cap B(x_0,\rho_0)\big)\cap L^{p_j}\big(\Omega\cap B(x_0,\rho_0)\big)\cap L^{q_k}\big(\Omega\cap B(x_0,\rho_0)\big),\\ \widetilde{\varphi}_n\in H^1_0\big(\Omega\cup B(x_0,\rho_0)\big)\cap L^{p_j}\big(\Omega\cup B(x_0,\rho_0)\big)\cap L^{q_k}\big(\Omega\cup B(x_0,\rho_0)\big),\\ \varphi_n\in H^1_0(\Omega)\cap L^{p_j}(\Omega)\cap L^{q_k}(\Omega).\end{gathered}$$

Then there exists $(\varphi_n^m)_{m\in\mathbb{N}}\subset\mathscr{D}(\Omega)$ such that for any $(n,m)\in\mathbb{N}^2$, $\operatorname{supp}\varphi_n^m\subset\Omega\cap B(x_0,\rho_0)$ and

$$\varphi_n^m\xrightarrow[m\longrightarrow\infty]{H^1_0(\Omega)\cap L^{p_j}(\Omega)\cap L^{q_k}(\Omega)}\varphi_n,$$

for any $(j,k)\in[\![1,n_1]\!]\times[\![1,n_2]\!]$. Consequently, $\varphi=\varphi_n$ are admissible test functions in (3.4.8). We have,

$$\begin{aligned}&\big\langle\nabla u,\nabla\varphi_n\big\rangle_{L^2(\Omega),L^2(\Omega)}=\langle\nabla\widetilde{u},\nabla\widetilde{\varphi}_n\rangle_{L^2(\Omega\cup B(0,\rho_0)),L^2(\Omega\cup B(0,\rho_0))}\\ &=\int\limits_{B(x_0,\rho)}\psi_n\big(|x-x_0|\big)|\nabla\widetilde{u}|^2\mathrm{d}x-n\operatorname{Re}\left(\int\limits_{\rho-\frac{1}{n}}^{\rho}\left(\int\limits_{\mathbb{S}(x_0,r)}\overline{\widetilde{u}}\nabla\widetilde{u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right)\mathrm{d}r\right),\end{aligned}$$

where we introduced the spherical coordinates (r,σ) at the last line. We now let $n\nearrow\infty$. Using the Lebesgue's dominated convergence Theorem and recalling that $I_{\mathrm{Re}}\in L^1((0,\rho_0);\mathbb{R})$, we obtain

$$\lim_{n\to\infty}\big\langle\nabla u,\nabla\varphi_n\big\rangle_{L^2(\Omega),L^2(\Omega)}=\|\nabla u\|^2_{L^2(\Omega\cap B(x_0,\rho))}-I_{\mathrm{Re}}(\rho).\tag{3.4.9}$$

Proceeding as above but also with $\varphi=\mathrm{i}\varphi_n$, we get $\lim\limits_{n\to\infty}\big\langle\nabla u,\mathrm{i}\nabla\varphi_n\big\rangle_{L^2,L^2}=-I_{\mathrm{Im}}(\rho)$ and

$$\begin{aligned}&\lim_{n\to\infty}\langle f(u),\varphi_n\rangle_{F^\star,E}=\operatorname{Re}\big(A(u)\big), &&\lim_{n\to\infty}\langle f(u),\mathrm{i}\varphi_n\rangle_{F^\star,E}=\operatorname{Im}\big(A(u)\big),\\ &\lim_{n\to\infty}\langle F,\varphi_n\rangle_{L^2,L^2}=\operatorname{Re}\big(B(u)\big), &&\lim_{n\to\infty}\langle F,\mathrm{i}\varphi_n\rangle_{L^2,L^2}=\operatorname{Im}\big(B(u)\big).\end{aligned}$$

where $E=\bigcap\limits_{j=1}^{n_2}L^{q_j}(\Omega)$, $F=\bigcap\limits_{j=1}^{n_1}L^{p_j}(\Omega)$, $A(u)=\displaystyle\int_{\Omega\cap B(x_0,\rho)}f(u)\,\overline{u}\,\mathrm{d}x$ and $B(u)=\displaystyle\int_{\Omega\cap B(x_0,\rho)}F(x)\overline{u(x)}\mathrm{d}x$. Estimates (3.3.5) and (3.3.6) then follow from (3.4.9) and these five last estimates. Since all terms in (3.3.5) and (3.3.6) are continuous on $[0,\rho_0]$, except eventually I_{Re} and I_{Im}, we deduce that I_{Re} and I_{Im} are continuous and (3.3.5) and (3.3.6) hold for any $\rho\in[0,\rho_0]$. The case $\rho_0=\operatorname{dist}(x_0,\Gamma)$ follows from the above proof applied with $\rho_0^n=\rho_0-\frac{1}{n}$ in place of ρ_0 and letting $n\nearrow\infty$. □

3.5 Application to the localization property to the case of Neumann boundary conditions

In Bégout and Díaz [25], the authors study the localization property for equation (3.5.6) below with the homogeneous Dirichlet boundary condition (see, for instance, Theorem 1.3.5). In Theorem 3.5.6 below, we show that the same property holds with the homogeneous Neumann boundary condition. Before, we need to prove that solutions exist. This can be found in Bégout and Díaz [28]. Note that from Bégout and Díaz [25] to this paper, there was a slight change of notation. See Remark 3.5.1 below.

Remark 3.5.1. In the context of the paper of Bégout and Díaz [25], we can establish an existence result with the homogeneous Neumann boundary condition (instead of the homogeneous Dirichlet condition) and $F \in L^2(\Omega)$ $\big($instead of $F \in L^{\frac{m+1}{m}}(\Omega)\big)$. In Bégout and Díaz [25], we introduced the set,

$$\widetilde{\mathbb{A}} = \mathbb{C} \setminus \big\{z \in \mathbb{C}; \mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) \leqslant 0\big\},$$

and assumed that $(\widetilde{a}, \widetilde{b}) \in \mathbb{C}^2$ satisfies,

$$(\widetilde{a}, \widetilde{b}) \in \widetilde{\mathbb{A}} \times \widetilde{\mathbb{A}} \quad \text{and} \quad \begin{cases} \mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b}) \geqslant 0, \\ \text{or} \\ \mathrm{Re}(\widetilde{a})\mathrm{Re}(\widetilde{b}) < 0 \text{ and } \mathrm{Im}(\widetilde{b}) > \dfrac{\mathrm{Re}(\widetilde{b})}{\mathrm{Re}(\widetilde{a})}\mathrm{Im}(\widetilde{a}), \end{cases} \tag{3.5.1}$$

with possibly $\widetilde{b} = 0$, and we worked with

$$-\mathrm{i}\Delta u + \widetilde{a}|u|^{-(1-m)}u + \widetilde{b}u = \widetilde{F}.$$

But here in order to follow a closer notation with most of the works dealing with Schrödinger equations, we do not work any more with this equation but with,

$$-\Delta u + a|u|^{-(1-m)}u + bu = F,$$

and $b \neq 0$. This means that we choose, $\widetilde{a} = \mathrm{i}a$, $\widetilde{b} = \mathrm{i}b$ and $\widetilde{F} = \mathrm{i}F$. Then assumptions on (a, b) are changed by the fact that for $\widetilde{z} = \mathrm{i}z$,

$$\mathrm{Re}(z) = \mathrm{Re}(-\mathrm{i}\widetilde{z}) = \mathrm{Im}(\widetilde{z}), \tag{3.5.2}$$

$$\mathrm{Im}(z) = \mathrm{Im}(-\mathrm{i}\widetilde{z}) = -\mathrm{Re}(\widetilde{z}). \tag{3.5.3}$$

It follows that the set $\widetilde{\mathbb{A}}$ and (3.5.1) become,

$$\mathbb{A} = \mathbb{C} \setminus \big\{z \in \mathbb{C}; \mathrm{Re}(z) \leqslant 0 \text{ and } \mathrm{Im}(z) = 0\big\}, \tag{3.5.4}$$

$$(a, b) \in \mathbb{A} \times \mathbb{A} \quad \text{and} \quad \begin{cases} \mathrm{Im}(a)\mathrm{Im}(b) \geqslant 0, \\ \text{or} \\ \mathrm{Im}(a)\mathrm{Im}(b) < 0 \text{ and } \mathrm{Re}(b) > \dfrac{\mathrm{Im}(b)}{\mathrm{Im}(a)}\mathrm{Re}(a). \end{cases} \tag{3.5.5}$$

Obviously,

$$\Big((\widetilde{a},\widetilde{b})\in\mathbb{C}^2 \text{ satisfies } (3.5.1)\Big) \iff \Big((a,b)\in\mathbb{C}^2 \text{ satisfies } (3.5.5)\Big).$$

Assumptions (3.5.5) are made to prove the existence and the localization property of solutions to

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega). \tag{3.5.6}$$

For uniqueness, the hypotheses are the following (Theorem 2.2.10).

Assumption 3.5.2 (Uniqueness). Assume that $(a,b)\in\mathbb{C}^2$ satisfies one of the two following conditions.

1) $a\neq 0$, $\mathrm{Re}(a)\geqslant 0$ and $\mathrm{Re}(a\overline{b})\geqslant 0$.

2) $b\neq 0$, $\mathrm{Re}(b)\geqslant 0$ and $a=kb$, for some $k\geqslant 0$.

A geometric interpretation of (3.5.5) and 1) of Assumption 3.5.2 is given in Section 1.6 of Chapter 1, modulus a rotation in the complex plane. Now, we give some results about equation (3.5.6) when $(a,b)\in\mathbb{C}^2$ satisfies (3.5.5).

Corollary 3.5.3 (Neumann boundary conditions). *Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ having a C^1 boundary, let ν be the outward unit normal vector to Γ, let $0<m<1$ and let $(a,b)\in\mathbb{C}^2$ satisfies (3.5.5). For any $F\in L^2(\Omega)$, there exists at least one solution $u\in H^1(\Omega)$ to*

$$\begin{cases} -\Delta u + a|u|^{-(1-m)}u + bu = F, \text{ in } L^2(\Omega), \\ \dfrac{\partial u}{\partial\nu}_{|\Gamma} = 0. \end{cases} \tag{3.5.7}$$

If furthermore (a,b) satisfies Assumption 3.5.2 then the solution of (3.5.7) is unique. Let $v\in H^1(\Omega)$ be any solution to (3.5.7). Then $v\in H^2_{\mathrm{loc}}(\Omega)$. In addition,

$$\|v\|_{H^1(\Omega)}\leqslant M\|F\|_{L^2(\Omega)}, \tag{3.5.8}$$

where $M=M(|a|,|b|)$. Finally, if for some $\alpha\in(0,m]$, $F\in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ then $u\in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$.

Symmetry Property 3.5.4. *If furthermore, for any $\mathcal{R}\in SO_N(\mathbb{R})$, $\mathcal{R}\Omega=\Omega$ and if F is spherically symmetric then we may construct a solution which is additionally spherically symmetric. For $N=1$, this means that if F is an even (respectively, an odd) function then u is also an even (respectively, an odd) function.*

Here and in what follows, $SO_N(\mathbb{R})$ denotes the special orthogonal group of $\mathbb{R}^N$.

Remark 3.5.5. One easily checks that if $(a,b)\in\mathbb{A}^2$ satisfies $\mathrm{Re}(a)\geqslant 0$ and $\mathrm{Re}(a\overline{b})\geqslant 0$ then $(a,b)\in\mathbb{C}^2$ verifies (3.5.5). In this case, uniqueness assumptions imply existence assumptions.

Proof of Corollary 3.5.3 and Symmetry Property 3.5.4. The result comes from Chapter 2 : Theorem 2.2.8 (existence and symmetry property), Theorem 2.2.10 (uniqueness), Theorem 2.2.9 (*a priori* estimate (3.5.8)) and Theorem 2.2.12 (local smoothness). □

Concerning the support of solution of (3.5.7) we have :

Theorem 3.5.6. *Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$ having a C^1 boundary, let $0 < m < 1$ and let $(a,b) \in \mathbb{C}^2$ satisfies (3.5.5). Then there exists $\varepsilon_\star > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_\star$, there exists $\delta_0 = \delta_0(\varepsilon, |a|, |b|, N, m) > 0$ satisfying the following property. Let $F \in L^2(\Omega)$ and let $u \in H^1(\Omega)$ be a solution to (3.5.7). If uniqueness holds for the problem (3.5.7)* [1]*, $\operatorname{supp} F$ is a compact set and $\|F\|_{L^2(\Omega)} \leqslant \delta_0$ then $\operatorname{supp} u \subset K(\varepsilon) \subset \Omega$, where*

$$K(\varepsilon) = \Big\{ x \in \mathbb{R}^N;\ \exists y \in \operatorname{supp} F \textit{ such that } |x - y| \leqslant \varepsilon \Big\},$$

which is compact.

The proof relies on the following lemma.

Lemma 3.5.7. *Let $\Omega \subset \mathbb{R}^N$ be a nonempty open subset of $\mathbb{R}^N$, let $0 < m < 1$ and let $(a,b) \in \mathbb{C}^2$ satisfies (3.5.5). Let $F \in L^1_{\mathrm{loc}}(\Omega)$ and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be any solution to*

$$-\Delta u + a|u|^{-(1-m)}u + bu = F, \ \textit{in } \mathscr{D}'(\Omega). \tag{3.5.9}$$

Then there exist two positive constants $L = L(|a|,|b|)$ and $M = M(|a|,|b|)$ satisfying the following property. Let $x_0 \in \Omega$ and $\rho_\star > 0$. If $F_{|\Omega \cap B(x_0,\rho_\star)} \in L^2\big(\Omega \cap B(x_0,\rho_\star)\big)$ then for any $\rho \in [0, \rho_\star)$,

$$\begin{aligned} &\|\nabla u\|^2_{L^2(\Omega\cap B(x_0,\rho))} + L\|u\|^{m+1}_{L^{m+1}(\Omega\cap B(x_0,\rho))} + L\|u\|^2_{L^2(\Omega\cap B(x_0,\rho))} \\ &\qquad \leqslant M \left(\left| \int_{\Omega\cap\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma \right| + \int_{\Omega\cap B(x_0,\rho)} |F(x)u(x)|\mathrm{d}x \right), \end{aligned} \tag{3.5.10}$$

where it is additionally assumed that $u \in H^1_0(\Omega)$ if $\rho_\star > \operatorname{dist}(x_0,\Gamma)$.

Proof. Let $x_0 \in \Omega$ and let $\rho_\star > 0$. We set for every $\rho \in [0,\rho_\star)$,

$$I(\rho) = \left| \int_{\Omega\cap\mathbb{S}(x_0,\rho)} u\overline{\nabla u}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma \right| \text{ and } J(\rho) = \int_{\Omega\cap B(x_0,\rho)} |F(x)u(x)|\mathrm{d}x.$$

It follows from Theorem 3.3.1 that $I, J \in C([0,\rho_\star);\mathbb{R})$ and

$$\Big| \|\nabla u\|^2_{L^2(\Omega\cap B(x_0,\rho))} + \operatorname{Re}(a)\|u\|^{m+1}_{L^{m+1}(\Omega\cap B(x_0,\rho))} + \operatorname{Re}(b)\|u\|^2_{L^2(\Omega\cap B(x_0,\rho))} \Big| \leqslant I(\rho) + J(\rho), \tag{3.5.11}$$

$$\Big| \operatorname{Im}(a)\|u\|^{m+1}_{L^{m+1}(\Omega\cap B(x_0,\rho))} + \operatorname{Im}(b)\|u\|^2_{L^2(\Omega\cap B(x_0,\rho))} \Big| \leqslant I(\rho) + J(\rho), \tag{3.5.12}$$

for any $\rho \in [0,\rho_\star)$. Estimate (3.5.10) then follows from (3.5.11), (3.5.12) and Lemma 2.4.5 with $\delta = 0$. Hence the result. □

Proof of Theorem 3.5.6. Let $F \in L^2(\Omega)$ with $\operatorname{supp} F \subset \Omega$ and let $u \in H^1(\Omega)$ a solution to (3.5.7) be given by Theorem 3.5.3. Set $K = \operatorname{supp} F$ and

$$\mathcal{O}(\varepsilon) = \Big\{ x \in \mathbb{R}^N;\ \exists y \in K \text{ such that } |x - y| < \varepsilon \Big\}.$$

Then $K(\varepsilon) = \overline{\mathcal{O}(\varepsilon)}$. Let $\varepsilon_\star > 0$ be small enough to have $K(5\varepsilon_\star) \subset \Omega$ and let $\varepsilon \in (0, \varepsilon_\star]$. Let L and M be given by Lemma 3.5.7 applied with $\rho_\star = 2\varepsilon$. By Theorem 3.2.1 and estimate (3.5.8) in Theorem 3.5.3

1. which is the case, for instance, if $(a,b) \in \mathbb{C}^2$ satisfies Assumption 3.5.2.

above, there exists $\delta_0=\delta_0(\varepsilon,|a|,|b|,N,m)>0$ such that if $\|F\|_{L^2(\Omega)}\leqslant\delta_0$ then $u_{|B(x_0,\varepsilon)}\equiv 0$, for any $x_0\in\Omega$ such that $B(x_0,2\varepsilon)\cap K=\emptyset$ and $B(x_0,2\varepsilon)\subset\Omega$. One easily sees that $B(x_0,2\varepsilon)\cap K=\emptyset$, for any $x_0\in\overline{K(2\varepsilon)^{\rm c}}\cap K(3\varepsilon)$. We deduce that for any $x_0\in\overline{K(2\varepsilon)^{\rm c}}\cap K(3\varepsilon)$, $u_{|B(x_0,\varepsilon)}\equiv 0$. By compactness, there exist $n\in\mathbb{N}$ and $x_1,\dots,x_n\in\overline{K(2\varepsilon)^{\rm c}}\cap K(3\varepsilon)$ such that,

$$\overline{K(\varepsilon)^{\rm c}}\cap\mathcal{O}(4\varepsilon)\subset\bigcup_{j=1}^n B(x_j,\varepsilon)\subset\bigcup_{j=1}^n B(x_j,2\varepsilon)\subset K(5\varepsilon)\subset\Omega.$$

It follows that $u_{|K(\varepsilon)^{\rm c}\cap\mathcal{O}(4\varepsilon)}\equiv 0$. Let us define $\widetilde{u}$ in Ω by,

$$\widetilde{u}=\begin{cases}u, & \text{in } \mathcal{O}(2\varepsilon),\\ 0, & \text{in } \Omega\setminus\mathcal{O}(2\varepsilon).\end{cases}$$

It follows that $\operatorname{supp}\widetilde{u}\subset K(\varepsilon)$ and $\widetilde{u}\in H^1_0(\Omega)$ is a solution to (3.5.7). By uniqueness assumption, $\widetilde{u}=u$ so that $\operatorname{supp}u\subset K(\varepsilon)\subset\Omega$, which is the desired result. □

Remark 3.5.8. In Bégout and Díaz [25], the authors study existence, uniqueness, smoothness and localization property for the equations (3.5.6) with an external source F belonging to $L^{\frac{m+1}{m}}(\Omega)$ with $0<m<1$ (see, for instance, Theorem 1.3.5). Below, we explain how the same results hold true with the weaker assumption $F\in L^2(\Omega)$. Indeed, when $|\Omega|<\infty$ and $0<m<1$, $L^{\frac{m+1}{m}}(\Omega)\hookrightarrow L^2(\Omega)$ and $L^{\frac{m+1}{m}}(\Omega)\neq L^2(\Omega)$. Results of existence can be found in Bégout and Díaz [28] jointly to some others additional results. Hypotheses on $(a,b)\in\mathbb{C}^2$ are the same as in Bégout and Díaz [25], except we have to require $b\neq 0$. Note that from Bégout and Díaz [25] to the present paper, there was a change of notation. See Remark 3.5.1 for precision. Throughout this remark, equation (3.5.6) with homogeneous Dirichlet boundary condition are considered and F is always assumed to belong in $L^2(\Omega)$ (instead of $L^{\frac{m+1}{m}}(\Omega)$ in Bégout and Díaz [25]) and assumptions on (a,b) are (3.5.5) and Assumption 3.5.2, instead of (1.2.2) and (1.2.3).

Analogous results to Theorems 1.4.1, 1.4.4 and Corollary 1.5.3 can be easily adapted. Indeed, by Theorems 2.2.8, 2.2.9, 2.2.10 and 2.2.12, these results hold but with $u\in H^2_{\rm loc}(\Omega)$ and

$$\|u\|^2_{H^1(\Omega)}+\|u\|^{m+1}_{L^{m+1}(\Omega)}\leqslant M\|F\|^2_{L^2(\Omega)},\tag{3.5.13}$$

instead of $u\in W^{2,\frac{m+1}{m}}_{\rm loc}(\Omega)$, (1.4.1) and (1.4.2). Concerning the localization property, Theorems 1.3.1 and 1.3.5 still hold true but with $F\in L^2(\Omega)$ and

$$\forall\rho\in(0,\rho_1),\ \|F\|^2_{L^2(\Omega\cap B(x_0,\rho))}\leqslant\varepsilon_\star(\rho-\rho_0)^p_+,\tag{3.5.14}$$

instead of (1.3.1). The proofs are essentially the same where we use Lemma 3.5.7 and (3.5.13) above instead of (1.4.1). It follows that Theorems 1.1.1 and 1.1.2 can be easily adapted with the obvious modifications.

Chapitre 4

Self-similar solutions with compactly supported profile of some nonlinear Schrödinger equations

with Jesús Ildefonso Díaz*

Abstract

"*Sharp localized*" solutions (*i.e.* with compact support for each given time t) of a singular nonlinear type Schrödinger equation in the whole space $\mathbb{R}^N$ are constructed here under the assumption that they have a self-similar structure. It requires the assumption that the external forcing term satisfies that $\boldsymbol{f}(t,x) = t^{-(\boldsymbol{p}-2)/2}\boldsymbol{F}(t^{-1/2}x)$ for some complex exponent $\boldsymbol{p}$ and for some profile function $\boldsymbol{F}$ which is assumed to be with compact support in $\mathbb{R}^N$. We show the existence of solutions of the form $\boldsymbol{u}(t,x) = t^{\boldsymbol{p}/2}\boldsymbol{U}(t^{-1/2}x)$, with a profile $\boldsymbol{U}$, which also has compact support in $\mathbb{R}^N$. The proof of the localization of the support of the profile $\boldsymbol{U}$ uses some suitable energy method applied to the stationary problem satisfied by $\boldsymbol{U}$ after some unknown transformation.

4.1 Introduction and main result

This paper deals with the study of "*sharp localized*" solutions of the nonlinear type Schrödinger equation in the whole space $\mathbb{R}^N$,

$$\mathbf{i}\frac{\partial \boldsymbol{u}}{\partial t} + \Delta \boldsymbol{u} = \boldsymbol{a}|\boldsymbol{u}|^{-(1-m)}\boldsymbol{u} + \boldsymbol{f}(t,x), \tag{4.1.1}$$

under the fundamental assumption $m \in (0,1)$ and for different choices of the complex coefficient $\boldsymbol{a}$. Here we use the notation of bold symbols for complex mathematics entities, $\mathbf{i}^2 = -1$ and $\Delta = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ for the Laplacian in the variables x.

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By the term "*sharp localized solutions*" we understand solutions which are more than merely the so called "*localized solutions*" considered earlier by many authors. For instance, most of the "*localized type solutions*" in the previous literature must vanish at infinity in an asymptotic way : $|\boldsymbol{u}(t,x)| \longrightarrow 0$ as $|x| \longrightarrow \infty$. They have been intensively studied mostly when some other structure property is added to the solution. It is the case of the special solutions which receive also other names such as *standing waves, travelling waves, solitons,* etc.

Here we are interested on solutions which have a sharper decay when $|x|$ goes to infinity in the sense that we will require the support of the function $\boldsymbol{u}(t,\,.\,)$ to be a compact set of $\mathbb{R}^N$, for any $t \geqslant 0$.

We recall that equations of the type (4.1.1) arise in many different contexts : Nonlinear Optics, Quantum Mechanics, Hydrodynamics, etc., and that, for instance, in Quantum Mechanics the main interest concerns the case in which $\mathrm{Re}(\boldsymbol{a}) > 0$, $\mathrm{Im}(\boldsymbol{a}) = 0$ (here and in which follows $\mathrm{Re}(\boldsymbol{a})$ is the real part of the complex number $\boldsymbol{a}$ and $\mathrm{Im}(\boldsymbol{a})$ is its imaginary part) and that in Nonlinear Optics the t does not represent time but the main scalar variable which appears in the propagation of the wave guide direction (see Agrawal and Kivshar [3], p.7 ; Temam and Miranville [169], p.517). Sometimes equations of the type (4.1.1) are named as Gross-Pitaevskiĭ type of equations in honor of two famous papers by those authors in 1961 (Gross [94] and Pitaevskiĭ [149]). For some physical details and many references, we send the reader to the general presentations made in the books Ablowitz, Prinari and Trubatch [1], Cazenave [57] and Sulem and Sulem [165].

In most of the papers on equations of the type (4.1.1), it is assumed that $m = 3$ (the so called *cubic case*). Nevertheless there are applications in which the general case $m > 0$ is of interest. For instance, it is the case of the so called "*non-Kerr type equations*" arising in the study of *optical solitons* (see, e.g., Agrawal and Kivshar [3], p.14 and following).

The case $m \in (0,1)$ has been studied before by other authors but under different points of view : some explicit self-similar solutions (the so called *algebraic solitons*) can be found in Polyanin and Zaitsev [151] (see also Agrawal and Kivshar [3], p.33). We also mention here the series of interesting papers by Rosenau and co-authors (Kashdan and Rosenau [116], Rosenau and Schuss [156]) in which "*sharp localized*" solutions are also considered with other type of statements and methods.

We also mention that the case $\mathrm{Re}(\boldsymbol{a}) > 0$ (which corresponds to the dissipative case, also called defocusing or repulsive case, when $\mathrm{Im}(\boldsymbol{a}) = 0$) must be well distinguished of the so called attractive problem (or also focusing case) in which it is assumed that $\mathrm{Re}(\boldsymbol{a}) < 0$ (and $\mathrm{Im}(\boldsymbol{a}) = 0$). See, e.g., Ablowitz, Prinari and Trubatch [1], Cazenave [57], Sulem and Sulem [165] and their references).

The case of complex potentials with certain types of singularities, i.e. corresponding to the choice $\mathrm{Im}(\boldsymbol{a}) \neq 0$, has been previously considered by several authors, and arises in many different situations (see, for instance, Brezis and Kato [47], Carles and Gallo [53], LeMesurier [127], Liskevich and Stollmann [131] and the references therein).

Here we assume that the datum $\boldsymbol{f}$ is not zero and represents some other physical magnitude which may arise in the possible coupling with some different phenomenon : see the different chapters of Part IV of the book Sulem and Sulem [165], the interaction phenomena between long waves and short waves (Benney [34], Dias and Figueira [72], Urrea [177] and their references), etc.

Obviously, the property of the compactness of the support of $\boldsymbol{u}(t,\ .\ )$ requires the assumption that "the support" of the datum function $\boldsymbol{f}(t,\ .\ )$ is a compact set of $\mathbb{R}^N$, for a.e. $t>0$. Because of that, the qualitative property we consider in this paper can be understood as a "finite speed of propagation property" typical of linear wave equations. We point out that our treatment is very different than other "propagation properties" studied previously in the literature for Schrödinger equations which are formulated in terms of the spectrum of the solutions. See, e.g., the so called Anderson localization (Anderson [9]), Jensen [115], etc.

One of the main reasons of the study of "*sharp localized*" solutions arises from the fact that, if we assume for the moment $\boldsymbol{f}\equiv\boldsymbol{0}$, then

$$\frac{\partial}{\partial t}|\boldsymbol{u}|^2+\operatorname{div}\boldsymbol{J}=2\operatorname{Im}(\boldsymbol{a})|\boldsymbol{u}|^{m+1},$$

where

$$\boldsymbol{J}\stackrel{\text{def}}{=}\big(\boldsymbol{u}\overline{\nabla\boldsymbol{u}}-\overline{\boldsymbol{u}}\nabla\boldsymbol{u}\big)=-2\operatorname{Re}(\mathbf{i}\,\overline{\boldsymbol{u}}\,\nabla\boldsymbol{u}),$$

($\overline{\boldsymbol{u}}$ denotes the conjugate of the complex function $\boldsymbol{u}$) and so we get (at least formally) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}|\boldsymbol{u}(t,x)|^2\mathrm{d}x=\operatorname{Im}(\boldsymbol{a})\int_{\mathbb{R}^N}|\boldsymbol{u}(t,x)|^{m+1}\mathrm{d}x.$$

Notice that if $\operatorname{Im}(\boldsymbol{a})\neq0$ then there is no mass conservation. For instance, this is the case studied by Carles and Gallo [53] where they prove that actually the solution vanishes after a finite time, once that $m\in(0,1)$. More generally, it is easy to see that the two following conservation laws hold, once $a\in\mathbb{R}$ and $\boldsymbol{f}\equiv\boldsymbol{0}$: if $\boldsymbol{u}(t)\in\boldsymbol{H^1}(\mathbb{R}^N)\cap\boldsymbol{L^{m+1}}(\mathbb{R}^N)$ then we have the mass conservation $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L^2}(\mathbb{R}^N)}=0$, moreover, if $\boldsymbol{u}(t)\in\boldsymbol{H^2}(\mathbb{R}^N)\cap\boldsymbol{L^{2m}}(\mathbb{R}^N)$ then $\boldsymbol{u}(t)\in\boldsymbol{L^{m+1}}(\mathbb{R}^N)$ and we have conservation of energy $\frac{\mathrm{d}}{\mathrm{d}t}E\big(\boldsymbol{u}(t)\big)=0$, where

$$E\big(\boldsymbol{u}(t)\big)=\frac{1}{2}\|\nabla\boldsymbol{u}(t)\|^2_{\boldsymbol{L^2}(\mathbb{R}^N)}+\frac{a}{m+1}\|\boldsymbol{u}(t)\|^{m+1}_{\boldsymbol{L^{m+1}}(\mathbb{R}^N)}.$$

Indeed, in the first case, $\Delta\boldsymbol{u}(t)\in\boldsymbol{H^{-1}}(\mathbb{R}^N)$ and $|\boldsymbol{u}(t)|^{-(1-m)}\boldsymbol{u}(t)\in\boldsymbol{L}^{\frac{m+1}{m}}(\mathbb{R}^N)$. It follows from the equation (4.1.1) that $\frac{\partial\boldsymbol{u}(t)}{\partial t}\in\boldsymbol{H^{-1}}(\mathbb{R}^N)+\boldsymbol{L}^{\frac{m+1}{m}}(\mathbb{R}^N)$ and since $\big(\boldsymbol{H^1}(\mathbb{R}^N)\cap\boldsymbol{L^{m+1}}(\mathbb{R}^N)\big)^\star=\boldsymbol{H^{-1}}(\mathbb{R}^N)+\boldsymbol{L}^{\frac{m+1}{m}}(\mathbb{R}^N)$, it follows that we may take the duality product of equation (4.1.1) with $\mathbf{i}\boldsymbol{u}(t)$, from which the mass conservation follows. In the same way, since $\boldsymbol{u}(t)\in\boldsymbol{L^2}(\mathbb{R}^N)\cap\boldsymbol{L^{2m}}(\mathbb{R}^N)$ and $0<m<1$, we get that $\boldsymbol{u}(t)\in\boldsymbol{L^{m+1}}(\mathbb{R}^N)$. We also easily have that $\Delta\boldsymbol{u}(t)\in\boldsymbol{L^2}(\mathbb{R}^N)$ and $|\boldsymbol{u}(t)|^{-(1-m)}\boldsymbol{u}(t)\in\boldsymbol{L^2}(\mathbb{R}^N)$. It follows from the equation (4.1.1) that $\frac{\partial\boldsymbol{u}(t)}{\partial t}\in\boldsymbol{L^2}(\mathbb{R}^N)$ and so we may take the duality product of equation (4.1.1) with $\frac{\partial\boldsymbol{u}(t)}{\partial t}$, from which the conservation of energy follows.

Like in the pioneering study by Schrödinger, the condition $\operatorname{Im}(\boldsymbol{a})=0$ implies that $|\boldsymbol{u}|^2$ represents a probability density, and so the study of "*sharp localized solutions*" becomes very relevant (recall the Heisenberg Uncertainty Principle). As we will show here (sequel of previous papers by the authors, Bégout and Díaz [24, 25]), if $m\in(0,1)$, under suitable conditions on the coefficient $\boldsymbol{a}$ (for instance for $\operatorname{Re}(\boldsymbol{a})>0$ and $\operatorname{Im}(\boldsymbol{a})=0$), it is possible to get some estimates on the support of solutions $\boldsymbol{u}(t,x)$ showing that the probability $|\boldsymbol{u}(t,x)|^2$ to localize a particle is zero outside of a compact set of $\mathbb{R}^N$.

The natural structure for searching self-similar solutions is based on the transformation $\lambda \longmapsto \boldsymbol{u}_{\boldsymbol{\lambda}}$, where for $\lambda > 0$, $\boldsymbol{p} \in \mathbb{C}$ and $\boldsymbol{u} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^1_{\mathrm{loc}}}(\mathbb{R}^N)\big)$, we define

$$\boldsymbol{u}_{\boldsymbol{\lambda}}(t,x) = \lambda^{-\boldsymbol{p}}\boldsymbol{u}(\lambda^2 t, \lambda x), \ \forall t > 0, \text{ for a.e. } x \in \mathbb{R}^N. \tag{4.1.2}$$

Recall that since $\boldsymbol{p} \in \mathbb{C}$ then $\lambda^{\boldsymbol{p}} \overset{\text{def}}{=} \boldsymbol{e}^{\boldsymbol{p}\ln\lambda} = e^{\mathrm{Re}(\boldsymbol{p})\ln\lambda}\boldsymbol{e}^{\mathbf{i}\mathrm{Im}(\boldsymbol{p})\ln\lambda} = \lambda^{\mathrm{Re}(\boldsymbol{p})}\boldsymbol{e}^{\mathbf{i}\mathrm{Im}(\boldsymbol{p})\ln\lambda}$ and that $|\lambda^{\boldsymbol{p}}| = \lambda^{\mathrm{Re}(\boldsymbol{p})}$. Our main assumption on the datum $\boldsymbol{f}$ is that

$$\boldsymbol{f}(t,x) = \lambda^{-(\boldsymbol{p}-2)}\boldsymbol{f}(\lambda^2 t, \lambda x), \ \forall \lambda > 0, \tag{4.1.3}$$

for some $\boldsymbol{p} \in \mathbb{C}$, for any $t > 0$ and almost every $x \in \mathbb{R}^N$, or equivalently, that

$$\boldsymbol{f}(t,x) = t^{\frac{\boldsymbol{p}-2}{2}}\boldsymbol{F}\left(\frac{x}{\sqrt{t}}\right), \tag{4.1.4}$$

for any $t > 0$ and almost every $x \in \mathbb{R}^N$, where $\boldsymbol{F} = \boldsymbol{f}(1)$. It is easy to build functions $\boldsymbol{f}$ satisfying (4.1.3). Indeed, for any given function $\boldsymbol{F}$, we define $\boldsymbol{f}$ by (4.1.4). Then $\boldsymbol{f}(1) = \boldsymbol{F}$ and $\boldsymbol{f}$ satisfies (4.1.3). Finally, if we assume $\mathrm{Re}(\boldsymbol{p}) = \frac{2}{1-m}$ then a direct calculation show that if $\boldsymbol{u}$ is a solution to (4.1.1) then for any $\lambda > 0$, $\boldsymbol{u}_{\boldsymbol{\lambda}}$ is also a solution to (4.1.1), and conversely.

We easily check that if $\boldsymbol{u}$ satisfies the invariance property $\boldsymbol{u} = \boldsymbol{u}_{\boldsymbol{\lambda}}$, for any $\lambda > 0$, then

$$\boldsymbol{u}(t,x) = t^{\frac{\boldsymbol{p}}{2}}\boldsymbol{U}\left(\frac{x}{\sqrt{t}}\right), \tag{4.1.5}$$

for any $t > 0$ and almost every $x \in \mathbb{R}^N$, where $\boldsymbol{U} = \boldsymbol{u}(1)$. Thus, we arrive to the following notion :

Definition 4.1.1. Let $0 < m < 1$, let $\boldsymbol{f} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)\big)$ satisfies (4.1.3) and let $\boldsymbol{p} \in \mathbb{C}$ be such that $\mathrm{Re}(\boldsymbol{p}) = \frac{2}{1-m}$. A solution $\boldsymbol{u}$ of (4.1.1) is said to be *self-similar* if $\boldsymbol{u} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)\big)$ and if for any $\lambda > 0$, $\boldsymbol{u}_{\boldsymbol{\lambda}} = \boldsymbol{u}$, where $\boldsymbol{u}_{\boldsymbol{\lambda}}$ is defined by (4.1.2). In this cases, $\boldsymbol{u}(1)$ is called the *profile* of $\boldsymbol{u}$ and is denoted by $\boldsymbol{U}$.

It follows from equation (4.1.1) and (4.1.5) that $\boldsymbol{U}$ satisfies

$$-\Delta\boldsymbol{U} + \boldsymbol{a}|\boldsymbol{U}|^{-(1-m)}\boldsymbol{U} - \frac{\mathbf{i}\boldsymbol{p}}{2}\boldsymbol{U} + \frac{\mathbf{i}}{2}x.\nabla\boldsymbol{U} = -\boldsymbol{F}, \tag{4.1.6}$$

in $\mathscr{D}'(\mathbb{R}^N)$, where $\boldsymbol{F} = \boldsymbol{f}(1)$. Conversely, if $\boldsymbol{U} \in \boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)$ verifies (4.1.6), in $\mathscr{D}'(\mathbb{R}^N)$, then the function $\boldsymbol{u}$ defined by (4.1.5) belongs to $\boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)\big)$ and is a self-similar solution to (4.1.1), where $\boldsymbol{f}$ is defined by (4.1.4) and satisfies (4.1.3). It is useful to introduce the unknown transformation

$$\boldsymbol{g}(x) = \boldsymbol{U}(x)\boldsymbol{e}^{-\mathbf{i}\frac{|x|^2}{8}}. \tag{4.1.7}$$

Then for any $m \in \mathbb{R}$, $\boldsymbol{p} \in \mathbb{C}$ and $\boldsymbol{U} \in \boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)$, $\boldsymbol{U}$ is a solution to (4.1.6) in $\mathscr{D}'(\mathbb{R}^N)$ if and only if $\boldsymbol{g} \in \boldsymbol{L^2_{\mathrm{loc}}}(\mathbb{R}^N)$ is a solution to

$$-\Delta\boldsymbol{g} + \boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g} - \mathbf{i}\frac{N+2\boldsymbol{p}}{4}\boldsymbol{g} - \frac{1}{16}|x|^2\boldsymbol{g} = -\boldsymbol{F}\boldsymbol{e}^{-\mathbf{i}\frac{|\,.\,|^2}{8}}, \tag{4.1.8}$$

in $\mathscr{D}'(\mathbb{R}^N)$. It will be convenient to study (4.1.8) instead of (4.1.6). Indeed, formally, if we multiply (4.1.8) by $\pm\overline{\boldsymbol{g}}$ or $\pm\mathbf{i}\overline{\boldsymbol{g}}$, integrate by parts and take the real part, one obtains some positive or negative quantities. But the same method applied to (4.1.6) gives (at least directly) nothing because of the term $\mathbf{i}x.\nabla \boldsymbol{U}$.

Notice that if $\boldsymbol{p} \in \mathbb{C}$ is such that $\mathrm{Re}(\boldsymbol{p}) = \frac{2}{1-m}$ and if $\boldsymbol{f} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2}(\mathbb{R}^N)\big)$ and satisfies (4.1.3) with $\boldsymbol{f}(t_0)$ compactly supported for some $t_0 > 0$, then it follows from (4.1.3) that for any $t > 0$, $\operatorname{supp}\boldsymbol{f}(t)$ is compact. Moreover, from (4.1.5), if $\boldsymbol{u}$ is a self-similar solution of (4.1.1) and if $\operatorname{supp}\boldsymbol{U}$ is compact then for any $t > 0$, $\operatorname{supp}\boldsymbol{u}(t)$ is compact. As a matter of fact, it is enough to have that $\boldsymbol{u}(t_0)$ is compactly supported for some $t_0 > 0$ to have that $\boldsymbol{u}$ satisfies (4.1.9) below and $\operatorname{supp}\boldsymbol{u}(t)$ is compact, for any $t > 0$. Indeed, $\boldsymbol{U} = \boldsymbol{u}(1)$ satisfies (4.1.6) and by (4.1.5), $\operatorname{supp}\boldsymbol{U}$ and $\operatorname{supp}\boldsymbol{u}(t)$ are compact for any $t > 0$. Let $\boldsymbol{g}$ be defined by (4.1.7). Then $\boldsymbol{g}$ is a solution compactly supported to (4.1.8) and it follows the results of Section 4.3 below that $\boldsymbol{g} \in \boldsymbol{H^2_c}(\mathbb{R}^N)$. By (4.1.7), we obtain that $\boldsymbol{U} \in \boldsymbol{H^2_c}(\mathbb{R}^N)$ and we deduce easily from (4.1.5) that $\boldsymbol{u}$ satisfies (4.1.9).

The main result of this paper is the following.

Theorem 4.1.2. *Let $0 < m < 1$, let $\boldsymbol{a} \in \mathbb{C}$ be such that $\mathrm{Im}(\boldsymbol{a}) \leqslant 0$. If $\mathrm{Re}(\boldsymbol{a}) \leqslant 0$ then assume further that $\mathrm{Im}(\boldsymbol{a}) < 0$. Let $\boldsymbol{p} \in \mathbb{C}$ be such that $\mathrm{Re}(\boldsymbol{p}) = \frac{2}{1-m}$ and let $\boldsymbol{f} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2}(\mathbb{R}^N)\big)$ satisfying (4.1.3). Assume also that $\operatorname{supp}\boldsymbol{f}(1)$ is compact.*

1. *If $\|\boldsymbol{f}(1)\|_{\boldsymbol{L^2}(\mathbb{R}^N)}$ is small enough then there exists a self-similar solution*
$$\boldsymbol{u} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{H^2}(\mathbb{R}^N)\big) \cap \boldsymbol{C^1}\big((0,\infty);\boldsymbol{H^1}(\mathbb{R}^N)\big) \cap \boldsymbol{C^2}\big((0,\infty);\boldsymbol{L^2}(\mathbb{R}^N)\big) \tag{4.1.9}$$
to (4.1.1) such that for any $t > 0$, $\operatorname{supp}\boldsymbol{u}(t)$ is compact. In particular, $\boldsymbol{u}$ is a strong solution and verifies (4.1.1) for any $t > 0$ in $\boldsymbol{L^2}(\mathbb{R}^N)$, and so almost everywhere in $\mathbb{R}^N$.
2. *Let $R > 0$. For any $\varepsilon > 0$, there exists $\delta_0 = \delta_0(R,\varepsilon,|\boldsymbol{a}|,|\boldsymbol{p}|,N,m) > 0$ satisfying the following property: if $\operatorname{supp}\boldsymbol{f}(1) \subset \overline{B}(0,R)$ and if $\|\boldsymbol{f}(1)\|_{\boldsymbol{L^2}(\mathbb{R}^N)} \leqslant \delta_0$ then the profile $\boldsymbol{U}$ of the solution obtained above verifies $\operatorname{supp}\boldsymbol{U} \subset K(\varepsilon) \subset \overline{B}(0,R+\varepsilon)$, where*
$$K(\varepsilon) = \Big\{x \in \mathbb{R}^N;\ \exists y \in \operatorname{supp}\boldsymbol{f}(1) \text{ such that } |x-y| \leqslant \varepsilon\Big\},$$
which is compact.
3. *Let $R_0 > 0$. Assume now further that $\mathrm{Re}(\boldsymbol{a}) > 0$, $\mathrm{Im}(\boldsymbol{a}) = 0$ and*
$$4\mathrm{Im}(\boldsymbol{p}) + 2\sqrt{4\mathrm{Im}^2(\boldsymbol{p}) + 2} \geqslant R_0^2.$$
Then the solution is unique in the set of functions $\boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_c}(\mathbb{R}^N)\big)$ whose profile $\boldsymbol{V}$ satisfies $\operatorname{supp}\boldsymbol{V} \subset \overline{B}(0,R_0)$.

In contrast with many other papers on self-similar solutions of equations dealing with exponents $m > 1$ (see Cazenave and Weissler [61, 62, 63] and their references), in this paper we do not prescribe any initial data $\boldsymbol{u}(0)$ to (4.1.1) since we are only interested on any solution $\boldsymbol{u}(t)$ by an external source $\boldsymbol{f}(t)$ compactly supported. Moreover, we point out that if $\boldsymbol{u} \in \boldsymbol{C}\big([0,\infty);\boldsymbol{L^q}(\mathbb{R}^N)\big)$ is a self-similar solution to (4.1.1), for some $0 < q \leqslant \infty$, then necessarily $\boldsymbol{u}(0) = \boldsymbol{0}$. Indeed, with help of (4.1.5), we easily show that $\boldsymbol{U} \in \boldsymbol{L^q}(\mathbb{R}^N)$ and that for any $t > 0$, $\|\boldsymbol{u}(t)\|_{\boldsymbol{L^q}(\mathbb{R}^N)} = t^{\frac{1}{1-m}+\frac{N}{2q}}\|\boldsymbol{U}\|_{\boldsymbol{L^q}(\mathbb{R}^N)}$, implying necessarily that $\boldsymbol{u}(0) = \boldsymbol{0}$. On the other hand, notice that if $\boldsymbol{u} \in \boldsymbol{C}\big([0,\infty);\mathscr{D}'(\mathbb{R}^N)\big)$ is a self-similar solution to (4.1.1) then one cannot expect to have $\boldsymbol{u}(0) \in \boldsymbol{L^q}(\mathbb{R}^N)$, unless $\boldsymbol{u}(0) = \boldsymbol{0}$. Indeed, we would have $\boldsymbol{u_\lambda}(0)=\boldsymbol{u}(0)$ in $\boldsymbol{L^q}(\mathbb{R}^N)$ and for any $\lambda>0$, $\|\boldsymbol{u}(0)\|_{\boldsymbol{L^q}(\mathbb{R}^N)}=\lambda^{\frac{2}{1-m}+\frac{N}{q}}\|\boldsymbol{u}(0)\|_{\boldsymbol{L^q}(\mathbb{R}^N)}$ and again we deduce that necessarily $\boldsymbol{u}(0)=\boldsymbol{0}$. More generally, the set of functions $\boldsymbol{u}$ satisfying the invariance property,

$$\forall\lambda>0,\ \text{for a.e. } x\in\mathbb{R}^N,\ \boldsymbol{u_\lambda}(x)\stackrel{\text{def}}{=}\lambda^{-\boldsymbol{p}}\boldsymbol{u}(\lambda x)=\boldsymbol{u}(x),$$

and lying in $\boldsymbol{L^q}(\mathbb{R}^N)$ is reduced to $\boldsymbol{0}$.

In the special case of self-similar solution, the above arguments show that if $\boldsymbol{f}\equiv\boldsymbol{0}$, $a\in\mathbb{R}$ and $\boldsymbol{u}\in\boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_c}(\mathbb{R}^N)\big)$ then necessarily $\boldsymbol{u}(t)=0$, for any $t>0$. Indeed, if $\boldsymbol{u}\in\boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_c}(\mathbb{R}^N)\big)$ is a self-similar solution to (4.1.1) then its profile $\boldsymbol{U}$ belongs to $\boldsymbol{L^2}(\mathbb{R}^N)$ and $\boldsymbol{u}\in\boldsymbol{C}^2((0;\infty)\times\mathbb{R}^N)$ (see Section 4.3 below). So for any $t>0$, we can multiply the above equation by $-\mathrm{i}\overline{\boldsymbol{u}}(t)$, integrate by parts over $\mathbb{R}^N$ and take the real part. We then deduce the mass conservation, $\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}(t)\|^2_{\boldsymbol{L^2}(\mathbb{R}^N)}=0$, which yields with the above identity,

$$\|\boldsymbol{U}\|_{\boldsymbol{L^2}(\mathbb{R}^N)}=\|\boldsymbol{u}(t)\|_{\boldsymbol{L^2}(\mathbb{R}^N)}=t^{\frac{1}{1-m}+\frac{N}{4}}\|\boldsymbol{U}\|_{\boldsymbol{L^2}(\mathbb{R}^N)},$$

for any $t>0$. Hence the result. As a matter of fact, if $\ell\in\{0,1,2\}$ and if $\boldsymbol{u}\in\boldsymbol{C}\big((0,\infty);\boldsymbol{H^\ell}(\mathbb{R}^N)\big)$ is a self-similar solution to (4.1.1) then one easily deduces from (4.1.5) that actually $\lim\limits_{t\searrow0}\|\boldsymbol{u}(t)\|_{\boldsymbol{H^\ell}(\mathbb{R}^N)}=0$.

We also mention here that our treatment of sharp localized solutions has some indirect connections with the study of the "unique continuation property". Indeed, we are showing that this property does not hold when $m\in(0,1)$, in contrast to the case of linear and other type of nonlinear Schrödinger equations (see, e.g., Kenig, Ponce and Vega [118], Urrea [177]).

The paper is organized as follows. In the next section, we introduce some notations and give general versions of the main results (Theorems 4.2.3 and 4.2.5). In Section 4.3, we recall some existence, uniqueness, *a priori* bound and smoothness results of solutions to equation (4.1.8) associated to the evolution equation (4.1.1). Finally, Section 4.4 is devoted to the proofs of the mentioned results, which we carry out by improving some energy methods presented in Antontsev, Díaz and Shmarev [11].

4.2 Notations and general versions of the main result

Before stating our main results, we will indicate here some of the notations used throughout. For $1\leqslant p\leqslant\infty$, p' is the conjugate of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. We denote by $\overline{\Omega}$ the closure of a nonempty subset $\Omega\subseteq\mathbb{R}^N$ and by $\Omega^{\mathrm{c}}=\mathbb{R}^N\setminus\Omega$ its complement. We note $\omega\Subset\Omega$ to mean that $\overline{\omega}\subset\Omega$ and that $\overline{\omega}$ is a compact subset of $\mathbb{R}^N$. Unless if specified, any function lying in a functional space $\big(\boldsymbol{L^p}(\Omega)$, $\boldsymbol{W^{m,p}}(\Omega)$, etc$\big)$ is supposed to be a complex-valued function $\big(\boldsymbol{L^p}(\Omega;\mathbb{C})$, $\boldsymbol{W^{m,p}}(\Omega;\mathbb{C})$, etc$\big)$. For a functional space $\boldsymbol{E}\subset\boldsymbol{L^1_{loc}}(\Omega;\mathbb{C})$, we denote by $\boldsymbol{E_c}=\big\{\boldsymbol{f}\in\boldsymbol{E};\operatorname{supp}\boldsymbol{f}\Subset\Omega\big\}$. For a Banach space $\boldsymbol{E}$, we denote by $\boldsymbol{E}^\star$ its topological dual and by $\langle\,.\,,\,.\,\rangle_{\boldsymbol{E}^\star,\boldsymbol{E}}\in\mathbb{R}$ the $\boldsymbol{E}^\star-\boldsymbol{E}$ duality product. In particular, for any $\boldsymbol{T}\in\boldsymbol{L^{p'}}(\Omega)$ and $\boldsymbol{\varphi}\in\boldsymbol{L^p}(\Omega)$ with $1\leqslant p<\infty$, $\langle\boldsymbol{T},\boldsymbol{\varphi}\rangle_{\boldsymbol{L^{p'}}(\Omega),\boldsymbol{L^p}(\Omega)}=\operatorname{Re}\int\limits_\Omega\boldsymbol{T}(x)\overline{\boldsymbol{\varphi}(x)}\mathrm{d}x$. For $x_0\in\mathbb{R}^N$ and $r>0$, we denote by $B(x_0,r)$ the open ball of $\mathbb{R}^N$ of center x_0 and radius r, by $\mathbb{S}(x_0,r)$ its boundary and by $\overline{B}(x_0,r)$ its closure. As usual, we denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1,\ldots,a_n$, write $C(a_1,\ldots,a_n)$ to indicate that the constant

C continuously depends only on $a_1, \dots, a_n$ (this convention also holds for constants which are not denoted by "C").

Now, we state the precise notion of solution.

Definition 4.2.1. Let Ω be a nonempty bounded open subset of $\mathbb{R}^N$, let $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{C}^{\mathbf{3}}$, let $0 < m \leqslant 1$ and let $\boldsymbol{G} \in \boldsymbol{L}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$.

1. We say that $\boldsymbol{g}$ is a *local very weak solution* to
$$-\Delta \boldsymbol{g} + \boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g} + \boldsymbol{b}\boldsymbol{g} + \boldsymbol{c}x.\nabla \boldsymbol{g} = \boldsymbol{G}, \tag{4.2.1}$$
in $\mathscr{D}'(\Omega)$, if $\boldsymbol{g} \in \boldsymbol{L}^{\mathbf{2}}_{\mathbf{loc}}(\Omega)$ and if
$$\langle \boldsymbol{g}, -\Delta\boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} + \langle \boldsymbol{H}(\boldsymbol{g}), \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \boldsymbol{G}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{4.2.2}$$
for any $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$, where
$$\boldsymbol{H}(\boldsymbol{h}) = \boldsymbol{a}|\boldsymbol{h}|^{-(1-m)}\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h} + \boldsymbol{c}x.\nabla \boldsymbol{h}, \tag{4.2.3}$$
for any $\boldsymbol{h} \in \boldsymbol{L}^{\mathbf{2}}_{\mathbf{loc}}(\Omega)$. If, in addition, $\boldsymbol{g} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$ then we say that $\boldsymbol{g}$ is a *global very weak solution* to (4.2.1).
2. We say that $\boldsymbol{g}$ is a *local weak solution* to (4.2.1) in $\mathscr{D}'(\Omega)$, if $\boldsymbol{g} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$ and if
$$\langle \nabla\boldsymbol{g}, \nabla\boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} + \langle \boldsymbol{H}(\boldsymbol{g}), \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \boldsymbol{G}, \boldsymbol{\varphi}\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{4.2.4}$$
for any $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$, where $\boldsymbol{H} \in \boldsymbol{C}\big(\boldsymbol{L}^{\mathbf{2}}_{\mathbf{loc}}(\Omega); \mathscr{D}'(\Omega)\big)$ is defined by (4.2.3).
3. We say that $\boldsymbol{g}$ is a *local weak solution* to
$$-\Delta \boldsymbol{g} + \boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g} + \boldsymbol{b}\boldsymbol{g} + \boldsymbol{c}|x|^2\boldsymbol{g} = \boldsymbol{G}, \tag{4.2.5}$$
in $\mathscr{D}'(\Omega)$, if $\boldsymbol{g} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$ and if $\boldsymbol{g}$ satisfies (4.2.4), for any $\boldsymbol{\varphi} \in \mathscr{D}(\Omega)$, where
$$\boldsymbol{H}(\boldsymbol{h}) = \boldsymbol{a}|\boldsymbol{h}|^{-(1-m)}\boldsymbol{h} + \boldsymbol{b}\boldsymbol{h} + \boldsymbol{c}|x|^2\boldsymbol{h}, \tag{4.2.6}$$
for any $\boldsymbol{h} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{loc}}(\Omega)$.
4. Assume further that $\boldsymbol{G} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$. We say that $\boldsymbol{g}$ is a *global weak solution* to (4.2.1) and
$$\boldsymbol{g}_{|\Gamma} = \boldsymbol{0}, \tag{4.2.7}$$
in $\boldsymbol{L}^{\mathbf{2}}(\Omega)$, if $\boldsymbol{g} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ and if
$$\langle \nabla\boldsymbol{g}, \nabla\boldsymbol{v}\rangle_{\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega)} + \langle \boldsymbol{H}(\boldsymbol{g}), \boldsymbol{v}\rangle_{\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega)} = \langle \boldsymbol{G}, \boldsymbol{v}\rangle_{\boldsymbol{L}^2(\Omega),\boldsymbol{L}^2(\Omega)}, \tag{4.2.8}$$
for any $\boldsymbol{v} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$, where $\boldsymbol{H} \in \boldsymbol{C}\big(\boldsymbol{H}^{\mathbf{1}}(\Omega); \boldsymbol{L}^{\mathbf{2}}(\Omega)\big)$ is defined by (4.2.3). Note that $\Delta\boldsymbol{g} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$, so that equation (4.2.1) makes sense in $\boldsymbol{L}^{\mathbf{2}}(\Omega)$ and almost everywhere in Ω.
5. Assume further that $\boldsymbol{G} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$. We say that $\boldsymbol{g}$ is a *global weak solution* to (4.2.5) and (4.2.7), in $\boldsymbol{L}^{\mathbf{2}}(\Omega)$, if $\boldsymbol{g} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$ and if $\boldsymbol{g}$ satisfies (4.2.8), for any $\boldsymbol{v} \in \boldsymbol{H}^{\mathbf{1}}_{\mathbf{0}}(\Omega)$, where $\boldsymbol{H} \in \boldsymbol{C}\big(\boldsymbol{L}^{\mathbf{2}}(\Omega); \boldsymbol{L}^{\mathbf{2}}(\Omega)\big)$ is defined by (4.2.6). Note that $\Delta\boldsymbol{g} \in \boldsymbol{L}^{\mathbf{2}}(\Omega)$, so that equation (4.2.5) makes sense in $\boldsymbol{L}^{\mathbf{2}}(\Omega)$ and almost everywhere in Ω.

In the above definition, Γ denotes the boundary of Ω and $\boldsymbol{C}(\Omega)=\boldsymbol{C^0}(\Omega)$ is the space of complex-valued functions which are defined and continuous over Ω. Obviously, for $k\in\mathbb{N}$, $\boldsymbol{C^k}(\Omega)$ denotes the space of complex-valued functions lying in $\boldsymbol{C}(\Omega)$ and having all derivatives of order lesser or equal than k belonging to $\boldsymbol{C}(\Omega)$.

Remark 4.2.2. Here are some comments about Definition 4.2.1.

1. Note that in Definition 4.2.1, any global weak solution is a local weak and a global very weak solution, and any local weak or global very weak solution is a local very weak solution.
2. Assume that Ω has a $C^{0,1}$ boundary. Let $\boldsymbol{g}\in\boldsymbol{H^1}(\Omega)$. Then boundary condition $\boldsymbol{g}_{|\Gamma}=0$ makes sense in the sense of the trace $\boldsymbol{\gamma}(\boldsymbol{g})=\boldsymbol{0}$. Thus, it is well-known that $\boldsymbol{g}\in\boldsymbol{H^1_0}(\Omega)$ if and only if $\boldsymbol{\gamma}(\boldsymbol{g})=\boldsymbol{0}$. If furthermore Ω has a C^1 boundary and if $\boldsymbol{g}\in\boldsymbol{C}(\overline{\Omega})\cap\boldsymbol{H^1_0}(\Omega)$ then for any $x\in\Gamma$, $\boldsymbol{g}(x)=\boldsymbol{0}$ (Theorem 9.17, p.288, in Brezis [44]). Finally, if $\boldsymbol{g}\notin\boldsymbol{C}(\overline{\Omega})$ and Ω has not a $C^{0,1}$ boundary, the condition $\boldsymbol{g}_{|\Gamma}=\boldsymbol{0}$ does not make sense and, in this case, has to be understood as $\boldsymbol{g}\in\boldsymbol{H^1_0}(\Omega)$.
3. Let $0<m\leqslant 1$ and let $\boldsymbol{z}\in\mathbb{C}\setminus\{\boldsymbol{0}\}$. Since $\left||\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}\right|=|\boldsymbol{z}|^m$, it is understood in Definition 4.2.1 that $\left||\boldsymbol{z}|^{-(1-m)}\boldsymbol{z}\right|=0$ when $\boldsymbol{z}=\boldsymbol{0}$.

The main results of this section are the two following theorems implying, as a special case, the statement of Theorem 4.1.2.

Theorem 4.2.3. *Let $\Omega\subset B(0,R)$ be a nonempty bounded open subset of $\mathbb{R}^N$, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\in\mathbb{C}^3$ be such that $\operatorname{Im}(\boldsymbol{a})\leqslant 0$, $\operatorname{Im}(\boldsymbol{b})<0$ and $\operatorname{Im}(\boldsymbol{c})\leqslant 0$. If $\operatorname{Re}(\boldsymbol{a})\leqslant 0$ then assume further that $\operatorname{Im}(\boldsymbol{a})<0$. Then there exist three positive constants $C=C(N,m)$, $L=L(R,|\boldsymbol{a}|,|\boldsymbol{p}|,N,m)$ and $M=M(R,|\boldsymbol{a}|,|\boldsymbol{p}|,N,m)$ satisfying the following property: let $\boldsymbol{G}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, let $\boldsymbol{g}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be any local weak solution to (4.2.5), let $x_0\in\Omega$ and let $\rho_0>0$. If $\rho_0>\operatorname{dist}(x_0,\Gamma)$ then assume further that $\boldsymbol{g}\in\boldsymbol{H^1_0}(\Omega)$. Assume now that $\boldsymbol{G}_{|\Omega\cap B(x_0,\rho_0)}\equiv\boldsymbol{0}$. Then $\boldsymbol{g}_{|\Omega\cap B(x_0,\rho_{\max})}\equiv\boldsymbol{0}$, where*

$$\rho_{\max}^\nu=\left(\rho_0^\nu-CM^2\max\left\{1,\frac{1}{L^2}\right\}\max\left\{\rho_0^{\nu-1},1\right\}\right.\\ \left.\times\min_{\tau\in\left(\frac{m+1}{2},1\right]}\left\{\frac{E(\rho_0)^{\gamma(\tau)}\max\{b(\rho_0)^{\mu(\tau)},b(\rho_0)^{\eta(\tau)}\}}{2\tau-(1+m)}\right\}\right)_+,\qquad(4.2.9)$$

where

$$E(\rho_0)=\|\nabla\boldsymbol{g}\|^2_{\boldsymbol{L^2}(\Omega\cap B(x_0,\rho_0))},\qquad b(\rho_0)=\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L^{m+1}}(\Omega\cap B(x_0,\rho_0))},$$
$$k=2(1+m)+N(1-m),\qquad \nu=\tfrac{k}{m+1}>2,$$

and where

$$\gamma(\tau)=\frac{2\tau-(1+m)}{k}\in(0,1),\quad \mu(\tau)=\frac{2(1-\tau)}{k},\quad \eta(\tau)=\frac{1-m}{1+m}-\gamma(\tau)>0.$$

for any $\tau\in\left(\frac{m+1}{2},1\right]$.

Here and in what follows, $r_+=\max\{0,r\}$ denotes the positive part of the real number r.

Remark 4.2.4. If the solution is too "large", it may happen that $\rho_{\max}=0$ and so the above result is not consistent. A sufficient condition to observe a localizing effect is that the solution is small enough, in a suitable sense. We give below a sufficient condition on the data $\boldsymbol{a}\in\mathbb{C}$, $\boldsymbol{p}\in\mathbb{C}$ and $\boldsymbol{G}$ to have $\rho_{\max}>0$.

Theorem 4.2.5. *Let $\Omega\subset B(0,R)$ be a nonempty bounded open subset of $\mathbb{R}^N$, let $0<m<1$, let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\in\mathbb{C}^3$ be such that $\mathrm{Im}(\boldsymbol{a})\leqslant 0$, $\mathrm{Im}(\boldsymbol{b})<0$ and $\mathrm{Im}(\boldsymbol{c})\leqslant 0$. If $\mathrm{Re}(\boldsymbol{a})\leqslant 0$ then assume further that $\mathrm{Im}(\boldsymbol{a})<0$. Let $\boldsymbol{G}\in\boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$, let $\boldsymbol{g}\in\boldsymbol{H}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$ be any local weak solution to (4.2.5), let $x_0\in\Omega$ and let $\rho_1>0$. If $\rho_1>\mathrm{dist}(x_0,\Gamma)$ then assume further that $\boldsymbol{g}\in\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$. Then there exist two positive constants $E_\star>0$ and $\varepsilon_\star>0$ satisfying the following property: let $\rho_0\in(0,\rho_1)$ and assume that $\|\nabla\boldsymbol{g}\|^2_{\boldsymbol{L^2}(\Omega\cap B(x_0,\rho_1))}<E_\star$ and*

$$\forall\rho\in(0,\rho_1),\ \|\boldsymbol{G}\|^2_{\boldsymbol{L^2}(\Omega\cap B(x_0,\rho))}\leqslant\varepsilon_\star(\rho-\rho_0)^p_+, \tag{4.2.10}$$

where $p=\frac{2(1+m)+N(1-m)}{1-m}$. Then $\boldsymbol{g}_{|\Omega\cap B(x_0,\rho_0)}\equiv\boldsymbol{0}$. In other words (with the notation of Theorem 4.2.3), $\rho_{\max}=\rho_0$.

Remark 4.2.6. We may estimate $E_\star$ and $\varepsilon_\star$ as

$$E_\star=E_\star\left(\|\boldsymbol{g}\|^{-1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho_1))},\rho_1,\frac{\rho_0}{\rho_1},\frac{L}{M},N,m\right),$$
$$\varepsilon_\star=\varepsilon_\star\left(\|\boldsymbol{g}\|^{-1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho_1))},\frac{\rho_0}{\rho_1},\frac{L}{M},N,m\right),$$

where $L>0$ and $M>0$ are given by Theorem 4.2.3. The dependence on $\frac{1}{\delta}$ means that if δ goes to 0 then $E_\star$ and $\varepsilon_\star$ may be very large. Note that $p=\frac{1}{\gamma(1)}$, where γ is the function defined in Theorem 4.2.3.

4.3 Existence, uniqueness and smoothness

We recall the following results which are taken from other works by the authors (Bégout and Díaz [28], Theorems 2.2.4, 2.2.6 and 2.2.12). Let $\Omega\subset B(0,R)$ be a nonempty bounded open subset of $\mathbb{R}^N$, let $0<m<1$ and let $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})\in\mathbb{C}^3$ be such that $\mathrm{Im}(\boldsymbol{a})\leqslant 0$, $\mathrm{Im}(\boldsymbol{b})<0$ and $\mathrm{Im}(\boldsymbol{c})\leqslant 0$. If $\mathrm{Re}(\boldsymbol{a})\leqslant 0$ then assume further that $\mathrm{Im}(\boldsymbol{a})<0$. For any $\boldsymbol{G}\in\boldsymbol{L^2}(\Omega)$, there exists at least one global weak solution $\boldsymbol{g}\in\boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)\cap\boldsymbol{H}^{\boldsymbol{2}}_{\mathbf{loc}}(\Omega)$ to (4.2.5) and (4.2.7). Moreover, if Ω has a $C^{1,1}$ boundary then $\boldsymbol{g}\in\boldsymbol{H^2}(\Omega)$. Finally,

$$\|\boldsymbol{g}\|_{\boldsymbol{H^1}(\Omega)}\leqslant M_0(R^2+1)\|\boldsymbol{G}\|_{\boldsymbol{L^2}(\Omega)}, \tag{4.3.1}$$

where $M_0=M_0(|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$. Finally, if $\boldsymbol{U}$ belongs to $\boldsymbol{L}^{\boldsymbol{2}}_{\mathbf{loc}}(\Omega)$ with $\boldsymbol{U}$ a local very weak solution to

$$-\Delta\boldsymbol{U}+\boldsymbol{a}|\boldsymbol{U}|^{-(1-m)}\boldsymbol{U}+\boldsymbol{b}\boldsymbol{U}+\mathbf{i}cx.\nabla\boldsymbol{U}=\boldsymbol{F},\ \text{in } \mathscr{D}'(\Omega),$$

(with any $(\boldsymbol{a},\boldsymbol{b},c)\in\mathbb{C}\times\mathbb{C}\times\mathbb{R}$) then $\boldsymbol{U}\in\boldsymbol{H}^{\boldsymbol{2}}_{\mathbf{loc}}(\Omega)$. Indeed, by the unknown transformation described at the beginning of Section 4.4 below, we are brought back to the study of the smoothness of solutions to equation,

$$-\Delta\boldsymbol{g}+\boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g}+\left(\boldsymbol{b}-\mathbf{i}\frac{cN}{2}\right)\boldsymbol{g}-\frac{c^2}{4}|x|^2\boldsymbol{g}=\boldsymbol{F}(x)\boldsymbol{e}^{-\mathbf{i}c\frac{|x|^2}{4}},\ \text{in } \mathscr{D}'(\Omega),$$

for which the above smoothness result applies. Concerning the uniqueness of solutions, we have the following result.

Theorem 4.3.1 (Uniqueness). *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset let $0<m<1$, let $(a,\boldsymbol{b},c) \in \mathbb{R}\times\mathbb{C}\times\mathbb{R}$ be such that $a>0$, $\mathrm{Re}(\boldsymbol{b}) \geqslant 0$ and $c \geqslant 0$. Then for any $\boldsymbol{F} \in \boldsymbol{L^2}(\Omega)$, equation*

$$-\Delta \boldsymbol{U} - \mathbf{i}a|\boldsymbol{U}|^{-(1-m)}\boldsymbol{U} - \mathbf{i}\boldsymbol{b}\boldsymbol{U} + \mathbf{i}cx.\nabla \boldsymbol{U} = \boldsymbol{F}, \text{ in } \mathscr{D}'(\Omega),$$

admits at most one global very weak solution compact with support $\boldsymbol{U} \in \boldsymbol{L^2_c}(\Omega)$.

Proof. Let $\boldsymbol{U_1},\boldsymbol{U_2} \in \boldsymbol{L^2_c}(\Omega)$ be two global very weak solutions both compactly supported to the above equation. By the results above, one has $\boldsymbol{U_1},\boldsymbol{U_2} \in \boldsymbol{H^2_c}(\Omega)$. Setting $\boldsymbol{g_1} = \boldsymbol{U_1}\boldsymbol{e}^{-\mathbf{i}c\frac{|\,.\,|^2}{4}}$ and $\boldsymbol{g_2} = \boldsymbol{U_2}\boldsymbol{e}^{-\mathbf{i}c\frac{|\,.\,|^2}{4}}$, a straightforward calculation shows that (see also the beginning of Section 4.4 below) $\boldsymbol{g_1},\boldsymbol{g_2} \in \boldsymbol{H^2_c}(\Omega)$ satisfy

$$-\Delta \boldsymbol{g} + \widetilde{\boldsymbol{a}}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g} + \widetilde{\boldsymbol{b}}\boldsymbol{g} + \widetilde{c}V^2\boldsymbol{g} = \widetilde{\boldsymbol{F}}, \text{ in } \boldsymbol{L^2}(\Omega),$$

where $\widetilde{\boldsymbol{a}} = -\mathbf{i}a$, $\widetilde{\boldsymbol{b}} = -\mathbf{i}\left(\boldsymbol{b} + \frac{cN}{2}\right)$, $\widetilde{c} = -\frac{c^2}{4}$, $V(x) = |x|$ and $\widetilde{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{e}^{-\mathbf{i}c\frac{|\,.\,|^2}{4}}$. Note that,

$$\begin{aligned}
&\widetilde{\boldsymbol{a}} \neq 0, \quad \mathrm{Re}(\widetilde{\boldsymbol{a}}) = 0,\\
&\mathrm{Re}\left(\widetilde{\boldsymbol{a}}\,\overline{\widetilde{\boldsymbol{b}}}\right) = \mathrm{Re}\left(a\left(\overline{\boldsymbol{b} + \frac{cN}{2}}\right)\right) = a\mathrm{Re}(\boldsymbol{b}) + \frac{1}{2}acN \geqslant 0,\\
&\mathrm{Re}\left(\widetilde{\boldsymbol{a}}\,\overline{\widetilde{c}}\right) = \frac{ac^2}{4}\mathrm{Re}(\mathbf{i}) = 0.
\end{aligned}$$

It follows from 1) of Theorem 2.2.10 that $\boldsymbol{g_1} = \boldsymbol{g_2}$ and hence, $\boldsymbol{U_1} = \boldsymbol{U_2}$. □

Remark 4.3.2. Notice that uniqueness for self-similar solution is relied to uniqueness for (4.1.8). Using Theorem 2.2.10, we can show that the uniqueness of self-similar solutions to equation (4.1.1) holds in the class of functions $\boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_c}(\mathbb{R}^N)\big)$ when, for instance, $\mathrm{Re}(\boldsymbol{a}) = 0$ and $\mathrm{Im}(\boldsymbol{a}) < 0$ (Theorem 4.3.1). These hypotheses are the same as in Carles and Gallo [53]. We point out that it seems possible to adapt the uniqueness method of Theorem 2.2.10 to obtain other criteria of uniqueness.

Remark 4.3.3. In the proof of uniqueness of Theorem 4.1.2, we will use the Poincaré's inequality (4.4.9). This estimate can be improved in several ways. For instance, for any $x_0 \in \mathbb{R}^N$ and any $R>0$, we have

$$\|\boldsymbol{u}\|_{\boldsymbol{L^2}(B(x_0,R))} \leqslant \frac{2R}{\pi}\|\nabla \boldsymbol{u}\|_{\boldsymbol{L^2}(B(x_0,R))}, \tag{4.3.2}$$

which is substantially better than (4.4.9), since $\frac{2}{\pi} < 1 < \sqrt{2}$. Actually, (4.3.2) holds for any $\boldsymbol{u} \in \boldsymbol{H^1}\big(B(x_0,R)\big)$ such that

$$\int\limits_{B(x_0,R)} \boldsymbol{u}(x)\mathrm{d}x = \boldsymbol{0},$$

and $\dfrac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} \in \boldsymbol{L^\infty}\big(B(x_0,R)\big)$, for any $(j,k) \in [\![1,N]\!]\times[\![1,N]\!]$. See Payne and Weinberger [148] for more details.

4.4 Proofs of the localization properties

We start by pointing out that if $\Omega\subseteq\mathbb{R}^N$ is a nonempty open subset and if $0<m\leqslant 1$, we have the following property : let $\boldsymbol{U}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ be a local weak solution to

$$-\Delta\boldsymbol{U}+\boldsymbol{a}|\boldsymbol{U}|^{-(1-m)}\boldsymbol{U}+\boldsymbol{b}\boldsymbol{U}+\mathbf{i}cx.\nabla\boldsymbol{U}=\boldsymbol{F}(x),\ \text{in } \mathscr{D}'(\Omega),$$

for some $(\boldsymbol{a},\boldsymbol{b},c)\in\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ and $\boldsymbol{F}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$. Setting $\boldsymbol{g}(x)=\boldsymbol{U}(x)\boldsymbol{e}^{-\mathbf{i}c\frac{|x|^2}{4}}$, for almost every $x\in\Omega$, it follows that $\boldsymbol{g}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ is a local weak solution to

$$-\Delta\boldsymbol{g}+\boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g}+\left(\boldsymbol{b}-\mathbf{i}\frac{cN}{2}\right)\boldsymbol{g}-\frac{c^2}{4}|x|^2\boldsymbol{g}=\boldsymbol{F}(x)\boldsymbol{e}^{-\mathbf{i}c\frac{|x|^2}{4}},\ \text{in } \mathscr{D}'(\Omega).$$

Conversely, if $\boldsymbol{g}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ is a local weak solution to

$$-\Delta\boldsymbol{g}+\boldsymbol{a}|\boldsymbol{g}|^{-(1-m)}\boldsymbol{g}+\boldsymbol{b}\boldsymbol{g}-c^2|x|^2\boldsymbol{g}=\boldsymbol{G}(x),\ \text{in } \mathscr{D}'(\Omega),$$

for some $(\boldsymbol{a},\boldsymbol{b},c)\in\mathbb{C}\times\mathbb{C}\times\mathbb{R}$ and $\boldsymbol{G}\in\boldsymbol{L^1_{\mathrm{loc}}}(\Omega)$, then setting $\boldsymbol{U}(x)=\boldsymbol{g}(x)\boldsymbol{e}^{\mathbf{i}c\frac{|x|^2}{2}}$, for almost every $x\in\Omega$, it follows that $\boldsymbol{U}\in\boldsymbol{H^1_{\mathrm{loc}}}(\Omega)$ is a local weak solution to

$$-\Delta\boldsymbol{U}+\boldsymbol{a}|\boldsymbol{U}|^{-(1-m)}\boldsymbol{U}+(\boldsymbol{b}+\mathbf{i}cN)\boldsymbol{U}+2\mathbf{i}cx.\nabla\boldsymbol{U}=\boldsymbol{G}(x)\boldsymbol{e}^{\mathbf{i}c\frac{|x|^2}{2}},\ \text{in } \mathscr{D}'(\Omega).$$

The proof of Theorems 4.2.3 and 4.2.5 follows the main structure of application of the energy methods introduced to the study of free boundary (see, e.g., the general presentation made in the monograph Antontsev, Díaz and Shmarev [11]). In both cases, the conclusions follow quite easily once it is obtained a general differential inequality for the local energy $E(\rho)$ of the type

$$E(\rho)^\alpha\leqslant C\rho^{-\beta}E'(\rho)+K(\rho-\rho_0)^\omega_+,\tag{4.4.1}$$

for some positive constants C, β and ω with $K=0$, in case of Theorem 4.2.3 and $K>0$ small enough, in case of Theorem 4.2.5. The key estimate which leads to desired local behaviour is that the exponent α arising in (4.4.1) satisfies that $\alpha\in(0,1)$.

Although the main steps to prove (4.4.1) follow the same steps already indicated in the monograph Antontsev, Díaz and Shmarev [11], it turns out that the concrete case of the systems of scalar equations generated by the Schrödinger operator does not fulfill the assumptions imposed in Antontsev, Díaz and Shmarev [11] for the case of systems of nonlinear equations. The extension of the method which applied to the system associated to the complex Schrödinger operator is far to be trivial and it was the main object of Bégout and Díaz [25]. Unfortunately, the extension of the method presented in Bégout and Díaz [25] is not enough to be applied to the fundamental equation of the present paper $\big($*i.e.* (4.1.8) or (4.2.5)$\big)$ mainly due to the presence of the source term $-c^2|x|^2g$. A sharper version of the energy method, also applicable to a different type of nonlinear complex Schrödinger type equations (for instance containing a Hartree-Fock type nonlocal term), was developed in Bégout and Díaz [27], where the applicability of the energy method was reduced to prove a certain local energy balance. Such a local balance will be proved here in the following lemma. Thanks to that, the proofs of Theorems 4.2.3 and 4.2.5 are then a corollary of Theorems 3.2.1 and 3.2.2.

Lemma 4.4.1. *Let* $\Omega \subset B(0,R)$ *be a nonempty bounded open subset of* $\mathbb{R}^N$, *let* $0 < m < 1$, *let* $(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}) \in \mathbb{C}^3$ *be such that* $\mathrm{Im}(\boldsymbol{a}) \leqslant 0$, $\mathrm{Im}(\boldsymbol{b}) < 0$ *and* $\mathrm{Im}(\boldsymbol{c}) \leqslant 0$. *If* $\mathrm{Re}(\boldsymbol{a}) \leqslant 0$ *then assume further that* $\mathrm{Im}(\boldsymbol{a}) < 0$. *Let* $\boldsymbol{G} \in \boldsymbol{L}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$ *and let* $\boldsymbol{g} \in \boldsymbol{H}^{\boldsymbol{1}}_{\mathbf{loc}}(\Omega)$ *be any local weak solution to* (4.2.5). *Then there exist two positive constants* $L = L(R,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$ *and* $M = M(R,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$ *such that for any* $x_0 \in \Omega$ *and any* $\rho_\star > 0$, *if* $\boldsymbol{G}_{|\Omega\cap B(x_0,\rho_\star)} \in \boldsymbol{L}^{\boldsymbol{2}}\big(\Omega \cap B(x_0,\rho_\star)\big)$ *then we have*

$$\|\nabla \boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} + L\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega\cap B(x_0,\rho))} + L\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} \\ \leqslant M\left(\left|\int_{\Omega\cap\mathbb{S}(x_0,\rho)} \boldsymbol{g}\overline{\nabla \boldsymbol{g}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right| + \int_{\Omega\cap B(x_0,\rho)} |\boldsymbol{G}(x)\boldsymbol{g}(x)|\mathrm{d}x\right), \quad (4.4.2)$$

for every $\rho \in [0,\rho_\star)$, *where it is additionally assumed that* $\boldsymbol{g} \in \boldsymbol{H}^{\boldsymbol{1}}_{\boldsymbol{0}}(\Omega)$ *if* $\rho_\star > \mathrm{dist}(x_0,\Gamma)$.

Proof. Let $x_0 \in \Omega$ and let $\rho_\star > 0$. Let σ be the surface measure on a sphere and set for every $\rho \in [0,\rho_*)$,

$$I(\rho) = \left|\int_{\Omega\cap\mathbb{S}(x_0,\rho)} \boldsymbol{g}\overline{\nabla \boldsymbol{g}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma\right|, \quad J(\rho) = \int_{\Omega\cap B(x_0,\rho)} |\boldsymbol{G}(x)\boldsymbol{g}(x)|\mathrm{d}x,$$

$$w(\rho) = \int_{\Omega\cap\mathbb{S}(x_0,\rho)} \boldsymbol{g}\overline{\nabla \boldsymbol{g}}.\frac{x-x_0}{|x-x_0|}\mathrm{d}\sigma, \quad I_{\mathrm{Re}}(\rho) = \mathrm{Re}\big(w(\rho)\big), \quad I_{\mathrm{Im}}(\rho) = \mathrm{Im}\big(w(\rho)\big).$$

It follows from Theorem 3.3.1 that $I, J, I_{\mathrm{Re}}, I_{\mathrm{Im}} \in C([0,\rho_*);\mathbb{R})$ and that,

$$\|\nabla \boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} + \mathrm{Re}(\boldsymbol{a})\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega\cap B(x_0,\rho))} + \mathrm{Re}(\boldsymbol{b})\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} \\ + \mathrm{Re}(\boldsymbol{c})\||x|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} = I_{\mathrm{Re}}(\rho) + \mathrm{Re}\left(\int\limits_{\Omega\cap B(x_0,\rho)} \boldsymbol{G}(x)\overline{\boldsymbol{g}(x)}\mathrm{d}x\right), \quad (4.4.3)$$

$$\mathrm{Im}(\boldsymbol{a})\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(\Omega\cap B(x_0,\rho))} + \mathrm{Im}(\boldsymbol{b})\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} + \mathrm{Im}(\boldsymbol{c})\||x|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(\Omega\cap B(x_0,\rho))} \\ = I_{\mathrm{Im}}(\rho) + \mathrm{Im}\left(\int\limits_{\Omega\cap B(x_0,\rho)} \boldsymbol{G}(x)\overline{\boldsymbol{g}(x)}\mathrm{d}x\right), \quad (4.4.4)$$

for any $\rho \in [0,\rho_\star)$. From these estimates, we obtain

$$\Big|\|\nabla \boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} + \mathrm{Re}(\boldsymbol{a})\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))} + \mathrm{Re}(\boldsymbol{b})\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} \\ + \mathrm{Re}(\boldsymbol{c})\||x|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))}\Big| \leqslant I(\rho) + J(\rho), \quad (4.4.5)$$

$$|\mathrm{Im}(\boldsymbol{a})|\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))} + |\mathrm{Im}(\boldsymbol{b})|\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} + |\mathrm{Im}(\boldsymbol{c})|\||x|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} \leqslant I(\rho) + J(\rho), \quad (4.4.6)$$

for any $\rho \in [0,\rho_\star)$. Let $A > 1$ to be chosen later. We multiply (4.4.6) by A and sum the result with (4.4.5). This leads to,

$$\|\nabla \boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} + A_1\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L}^{\boldsymbol{m+1}}(B(x_0,\rho))} + A_2\|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} \\ + \mathrm{Re}(\boldsymbol{c})\||x|\boldsymbol{g}\|^2_{\boldsymbol{L}^{\boldsymbol{2}}(B(x_0,\rho))} \leqslant 2A\big(I(\rho) + J(\rho)\big), \quad (4.4.7)$$

where

$$A_1 = \begin{cases} \mathrm{Re}(\boldsymbol{a}), & \text{if } \mathrm{Re}(\boldsymbol{a}) > 0, \\ A|\mathrm{Im}(\boldsymbol{a})| - |\mathrm{Re}(\boldsymbol{a})|, & \text{if } \mathrm{Re}(\boldsymbol{a}) \leqslant 0, \end{cases}$$
$$A_2 = A|\mathrm{Im}(\boldsymbol{b})| - |\mathrm{Re}(\boldsymbol{b})|.$$

But (4.4.7) yields,

$$\|\nabla \boldsymbol{g}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))} + A_1\|\boldsymbol{g}\|^{m+1}_{\boldsymbol{L^{m+1}}(B(x_0,\rho))} + \big(A_2 - R^2|\mathrm{Re}(\boldsymbol{c})|\big)\|\boldsymbol{g}\|^2_{\boldsymbol{L^2}(B(x_0,\rho))} \leqslant 2A\big(I(\rho) + J(\rho)\big) \quad (4.4.8)$$

We choose $A = A(R,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$ large enough to have $A|\mathrm{Im}(\boldsymbol{a})| - |\mathrm{Re}(\boldsymbol{a})| \geqslant 1$ (when $\mathrm{Re}(\boldsymbol{a}) \leqslant 0$) and $A_2 - R^2|\mathrm{Re}(\boldsymbol{c})| \geqslant 1$. Then (4.4.2) comes from (4.4.8) with $L = \min\{A_1, 1\}$ and $M = 2A$. Note that $L = L(R,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$ and $M = M(R,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|)$. This concludes the proof. □

Remark 4.4.2. When $\rho_\star \leqslant \mathrm{dist}(x_0,\Gamma)$ and $\boldsymbol{G} \in \boldsymbol{L^2_{\mathrm{loc}}}(\Omega)$, one may easily obtain (4.4.3)–(4.4.4) without the technical Theorem 3.3.1. Indeed, it follows from Proposition 1.4.5 that $\boldsymbol{g} \in \boldsymbol{H^2_{\mathrm{loc}}}(\Omega)$, so that equation (4.2.5) makes sense in $\boldsymbol{L^2_{\mathrm{loc}}}(\Omega)$ and almost everywhere in Ω. Thus, if $\rho_\star \leqslant \mathrm{dist}(x_0,\Gamma)$ then $\boldsymbol{g}_{|B(x_0,\rho)} \in \boldsymbol{H^2}\big(B(x_0,\rho)\big)$ and (4.4.3) $\big($respectively, (4.4.4)$\big)$ is obtained by multiplying (4.2.5) by $\overline{\boldsymbol{g}}$ (respectively, by $\overline{\mathbf{i}\boldsymbol{g}}$), integrating by parts over $B(x_0,\rho)$ and taking the real part.

Proof of Theorem 4.2.3. By Lemma 4.4.1, u satisfies (3.2.1). The result then comes from Theorem 3.2.1. □

Proof of Theorem 4.2.5. By Lemma 4.4.1, u satisfies (3.2.3). The result then comes from Theorem 3.2.2. □

Proof of Theorem 4.1.2. Let $R > 0$. Let $\varepsilon > 0$ and let $\boldsymbol{f} \in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2}(\mathbb{R}^N)\big)$ satisfying (4.1.3) and $\mathrm{supp}\,\boldsymbol{f}(1) \subset \overline{B}(0,R)$. Let M_0 be the constant in (4.3.1). Let $\boldsymbol{b} = -\mathbf{i}\frac{N+2\boldsymbol{p}}{4}$, $\boldsymbol{c} = -\frac{1}{16}$ and $\boldsymbol{G} = -\boldsymbol{f}(1)\boldsymbol{e}^{-\mathbf{i}\frac{|\,.\,|^2}{8}}$. Note that $\mathrm{Im}(\boldsymbol{a}) \leqslant 0$, $\mathrm{Im}(\boldsymbol{b}) = -\frac{N(1-m)+4}{4(1-m)} < 0$ and $\mathrm{Im}(\boldsymbol{c}) = 0$. In addition, if $\mathrm{Re}(\boldsymbol{a}) \leqslant 0$ then $\mathrm{Im}(\boldsymbol{a}) < 0$. It follows that the existence result of Section 4.3 applies to equation (4.1.8) : let $\boldsymbol{g} \in \boldsymbol{H^1_0}(B(0,2R+2\varepsilon)) \cap \boldsymbol{H^2}(B(0,2R+2\varepsilon))$ be such a solution to (4.1.8) and (4.2.7). We apply Theorem 4.2.3 with $\rho_0 = 2\varepsilon$. By (4.3.1), there exists $\delta_0 = \delta_0(R,\varepsilon,|\boldsymbol{a}|,|\boldsymbol{b}|,|\boldsymbol{c}|,N,m) > 0$ such that if $\|\boldsymbol{f}(1)\|_{\boldsymbol{L^2}(\mathbb{R}^N)} \leqslant \delta_0$ then $\rho_{\max} \geqslant \varepsilon$. Set $K = \mathrm{supp}\,\boldsymbol{f}(1) = \mathrm{supp}\,\boldsymbol{G}$. Let $x_0 \in \overline{K(2\varepsilon)^{\mathrm{c}}} \cap B(0,2R+2\varepsilon)$. Let $y \in B(x_0,2\varepsilon)$ and let $z \in K$. By definition of $K(2\varepsilon)$, $\mathrm{dist}(\overline{K(2\varepsilon)^{\mathrm{c}}},K) = 2\varepsilon$. We then have

$$2\varepsilon = \mathrm{dist}(\overline{K(2\varepsilon)^{\mathrm{c}}},K) \leqslant |x_0 - z| \leqslant |x_0 - y| + |y - z| < 2\varepsilon + |y - z|.$$

It follows that for any $z \in K$, $|y - z| > 0$, so that $y \notin K$. This means that $B(x_0,2\varepsilon) \cap K = \emptyset$, for any $x_0 \in \overline{K(2\varepsilon)^{\mathrm{c}}} \cap B(0,2R+2\varepsilon)$. By Theorem 4.2.3 we deduce that for any $x_0 \in \overline{K(2\varepsilon)^{\mathrm{c}}} \cap B(0,2R+2\varepsilon)$, $\boldsymbol{g}_{|B(x_0,\varepsilon)} \equiv \boldsymbol{0}$. By compactness, $\overline{K(\varepsilon)^{\mathrm{c}}} \cap B(0,2R+2\varepsilon)$ may be covered by a finite number of sets $B(x_0,\varepsilon) \cap B(0,2R+2\varepsilon)$ with $x_0 \in \overline{K(2\varepsilon)^{\mathrm{c}}}$. It follows that $\boldsymbol{g}_{|K(\varepsilon)^{\mathrm{c}} \cap B(0,2R+2\varepsilon)} \equiv \boldsymbol{0}$. This means that $\mathrm{supp}\,\boldsymbol{g} \subset K(\varepsilon) \subset B(0,2R+2\varepsilon)$. We then extend $\boldsymbol{g}$ by $\boldsymbol{0}$ outside of $B(0,2R+2\varepsilon)$. Thus, $\boldsymbol{g} \in \boldsymbol{H^2_{\mathrm{c}}}(\mathbb{R}^N)$ is a solution to (4.1.8) in $\mathbb{R}^N$. Now, let $\boldsymbol{U} = \boldsymbol{g}\boldsymbol{e}^{\mathbf{i}\frac{|\,.\,|^2}{8}}$ and let for any $t > 0$, $\boldsymbol{u}(t) = t^{\frac{\boldsymbol{p}}{2}}\boldsymbol{U}\left(\frac{\cdot}{\sqrt{t}}\right)$. It follows that $\mathrm{supp}\,\boldsymbol{U} = \mathrm{supp}\,\boldsymbol{g} \subset K(\varepsilon)$, $\boldsymbol{U} \in \boldsymbol{H^2_{\mathrm{c}}}(\mathbb{R}^N)$ and $\boldsymbol{U}$ is a solution to (4.1.6) in $\mathbb{R}^N$. By (4.1.5), $\boldsymbol{u}$ verifies (4.1.9) and is a solution to (4.1.1) in $(0,\infty) \times \mathbb{R}^N$ with $\boldsymbol{u}(1) = \boldsymbol{U}$ compactly supported in

$K(\varepsilon)$. By Definition 4.1.1, $\boldsymbol{u}$ is self-similar and still by (4.1.5), $\operatorname{supp}\boldsymbol{u}(t)$ is compact for any $t>0$. Hence Properties 1 and 2. It remains to show Property 3. Let $R_0>0$ and assume further that $\operatorname{Re}(\boldsymbol{a})>0$, $\operatorname{Im}(\boldsymbol{a})=0$ and $0<R_0^2\leqslant 4\operatorname{Im}(\boldsymbol{p})+2\sqrt{4\operatorname{Im}^2(\boldsymbol{p})+2}$. Let $\boldsymbol{u_1},\boldsymbol{u_2}\in \boldsymbol{C}\big((0,\infty);\boldsymbol{L^2_c}(\mathbb{R}^N)\big)$ be two solutions to (4.1.1) whose profile $\boldsymbol{U_1},\boldsymbol{U_2}$ satisfy $\operatorname{supp}\boldsymbol{U},\operatorname{supp}\boldsymbol{V}\subset\overline{B}(0,R_0)$. By Section 4.3, $\boldsymbol{U_1},\boldsymbol{U_2}\in\boldsymbol{H^2_c}(\mathbb{R}^N)$. For $j\in\{1,2\}$, let $\boldsymbol{g_j}=\boldsymbol{U_j}\boldsymbol{e}^{-\mathbf{i}\frac{|\,.\,|^2}{8}}$. It follows that $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ belong to $\boldsymbol{H^2_c}(\mathbb{R}^N)$, are compactly supported in $\overline{B}(0,R_0)$ and satisfy the same equation (4.1.8). Let $\boldsymbol{g}=\boldsymbol{g_1}-\boldsymbol{g_2}$ and set for any $\boldsymbol{h}\in\boldsymbol{L^2_c}(\mathbb{R}^N)$, $\boldsymbol{H}(\boldsymbol{h})=|\boldsymbol{h}|^{-(1-m)}\boldsymbol{h}$. It follows that,

$$-\Delta\boldsymbol{g}+a\big(\boldsymbol{H}(\boldsymbol{g_1})-\boldsymbol{H}(\boldsymbol{g_2})\big)-\mathbf{i}\frac{N+2\boldsymbol{p}}{4}\boldsymbol{g}-\frac{1}{16}|x|^2\boldsymbol{g}=\boldsymbol{0},\ \text{a.e. in } \mathbb{R}^N.$$

Multiplying this equation by $\overline{\boldsymbol{g}}$, integrating by parts over $\mathbb{R}^N$ and taking the real part, we get

$$\begin{aligned}&\|\nabla\boldsymbol{g}\|^2_{\boldsymbol{L^2}}+a\langle\boldsymbol{H}(\boldsymbol{g_1})-\boldsymbol{H}(\boldsymbol{g_2}),\boldsymbol{g_1}-\boldsymbol{g_2}\rangle_{\boldsymbol{L^2},\boldsymbol{L^2}}-\operatorname{Re}\left(\mathbf{i}\frac{N+2\boldsymbol{p}}{4}\right)\|\boldsymbol{g}\|^2_{\boldsymbol{L^2}}-\frac{1}{16}\||\,.\,|\boldsymbol{g}\|^2_{\boldsymbol{L^2}}\\ &=\|\nabla\boldsymbol{g}\|^2_{\boldsymbol{L^2}}+a\langle\boldsymbol{H}(\boldsymbol{g_1})-\boldsymbol{H}(\boldsymbol{g_2}),\boldsymbol{g_1}-\boldsymbol{g_2}\rangle_{\boldsymbol{L^2},\boldsymbol{L^2}}+\frac{1}{2}\operatorname{Im}(\boldsymbol{p})\|\boldsymbol{g}\|^2_{\boldsymbol{L^2}}-\frac{1}{16}\||\,.\,|\boldsymbol{g}\|^2_{\boldsymbol{L^2}}\\ &=0,\end{aligned}$$

We recall the following refined Poincaré's inequality (Bégout and Torri [31]).

$$\forall\boldsymbol{u}\in\boldsymbol{H^1_0}\big(B(0,R_0)\big),\ \|\boldsymbol{u}\|^2_{\boldsymbol{L^2}(B(0,R_0))}\leqslant 2R_0^2\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L^2}(B(0,R_0))},\tag{4.4.9}$$

If follows from (4.4.9) and Lemma 1.9.1 that there exists a positive constant C such that,

$$\left(\frac{1}{2R_0^2}+\frac{1}{2}\operatorname{Im}(\boldsymbol{p})-\frac{R_0^2}{16}\right)\|\boldsymbol{g}\|^2_{\boldsymbol{L^2}}+Ca\int\limits_\omega\frac{|\boldsymbol{g_1}(x)-\boldsymbol{g_2}(x)|^2}{(|\boldsymbol{g_1}(x)|+|\boldsymbol{g_2}(x)|)^{1-m}}\mathrm{d}x\leqslant 0,$$

where $\omega=\Big\{x\in\Omega;|\boldsymbol{g_1}(x)|+|\boldsymbol{g_2}(x)|>0\Big\}$. But,

$$\frac{1}{2R_0^2}+\frac{1}{2}\operatorname{Im}(\boldsymbol{p})-\frac{R_0^2}{16}=\frac{1}{16R_0^2}\left(-R_0^4+8\operatorname{Im}(\boldsymbol{p})R_0^2+8\right)\geqslant 0,$$

when

$$0\leqslant R_0^2\leqslant 4\operatorname{Im}(\boldsymbol{p})+2\sqrt{4\operatorname{Im}^2(\boldsymbol{p})+2}.$$

It follows that $\boldsymbol{g_1}=\boldsymbol{g_2}$ which implies that $\boldsymbol{U_1}=\boldsymbol{U_2}$ and for any $t>0$, $\boldsymbol{u_1}(t)=\boldsymbol{u_2}(t)$. This ends the proof. □

Chapitre 5

Finite time extinction for the strongly damped nonlinear Schrödinger equation in bounded domains

with Jesús Ildefonso Díaz*

Abstract

We prove the *finite time extinction property* ($u(t) \equiv 0$ on Ω for any $t \geqslant T_\star$, for some $T_\star > 0$) for solutions of the nonlinear Schrödinger problem $\mathrm{i}u_t + \Delta u + a|u|^{-(1-m)}u = f(t,x)$, on a bounded domain Ω of $\mathbb{R}^N$, $N \leqslant 3$, $a \in \mathbb{C}$ with $\mathrm{Im}(a) > 0$ (the damping case) and under the crucial assumptions $0 < m < 1$ and the dominating condition $2\sqrt{m}\,\mathrm{Im}(a) \geqslant (1-m)|\mathrm{Re}(a)|$. We use an energy method as well as several a priori estimates to prove the main conclusion. The presence of the non-Lipschitz nonlinear term in the equation introduces a lack of regularity of the solution requiring a study of the existence and uniqueness of solutions satisfying the equation in some different senses according to the regularity assumed on the data.

5.1 Introduction

This paper deals with the *finite time extinction property* of solutions of the nonlinear Schrödinger problem

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} + \Delta u + a|u|^{-(1-m)}u = f(t,x), & \text{in } (0,\infty)\times\Omega, \\ u(t)_{|\Gamma} = 0, & \text{on } (0,\infty)\times\Gamma, \\ u(0) = u_0, & \text{in } \Omega, \end{cases} \tag{5.1.1}$$

when, roughly speaking, we assume that $N \leqslant 3$,

$$a \in \mathbb{C} \text{ with } \mathrm{Im}(a) > 0, \tag{5.1.2}$$

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and

$$0 < m < 1. \tag{5.1.3}$$

We start by pointing out that this *finite time extinction property* ($u(t) \equiv 0$ on Ω for any $t \geqslant T_\star$, for some $T_\star > 0$) represents, clearly, the most opposite property to the famous Max Born result on the *conservation of the mass*

$$\|u(t)\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}, \text{ for any } t \geqslant 0,$$

which arises (when $f = 0$) in the linear case (and more generally if $\mathrm{Im}(a) = 0$: see Proposition 5.2.3 below) and which allows the probabilistic understanding of the complex wave solution $u(t,x)$ in the context of the applications of the linear Schrödinger equation in Quantum Mechanics. It is well known that the presence of a damping term (5.1.2) makes the equation irreversible with respect the time.

We also recall that the Schrödinger equation in presence of a nonlinear term in the equation (as, e.g., problem (5.1.1) when $a \in \mathbb{C}$ and $a \neq 0$) arises in many other different contexts as, e.g., Nonlinear Optics, Hydrodynamics, etc., and that those other contexts, for instance in Nonlinear Optics, the variable t does not represent time but the main scalar spacial variable which appears in the propagation of the waveguide direction (see e.g. Agrawal and Kivshar [3], Sulem and Sulem [165], Shi, Xu, Yang, Yang and Yin [158] and its many references).

As a matter of fact, the nonlinear Schrödinger equation under condition (5.1.2) is referred in the literature as the *damped* case and it was intensively studied since the middle of the past century under different additional conditions (but most of them for $m > 1$) (see, e.g., Nelson [142], Pozzi [152], Bardos and Brezis [17], Lions [128], Kato [117], Brezis and Kato [47], Vladimirov [179], Tsutsumi [171], Temam and Miranville [169], Kita and Shimomura [120], Carles and Gallo [53], Carles and Ozawa [55] and Hayashi, Li and Naumkin [104], among others).

In our above formulation we assume that $a \in \mathbb{C}$ and thus a possible, non-dominant non-dissipative nonlinear term may coexists with the damping term (i.e., we allow $\mathrm{Re}(a) \neq 0$). Nevertheless, our main result on the finite time extinction for $|\Omega| < \infty$ requires the dominating condition

$$2\sqrt{m}\,\mathrm{Im}(a) \geqslant (1-m)|\mathrm{Re}(a)|,$$

as well as the assumption (5.1.3) on a strong damping.

We also recall that in most of the papers on the nonlinear equation (5.1.1) it is assumed that $m = 3$ (the so called cubic case). Nevertheless there are several applications in which the general case $m > 0$ is of interest. For instance, it is the case of the so called *non-Kerr type equations* arising in the study of optical solitons (see, e.g., [3]). For some other physical details and many references, we refer the reader to the general presentations made in the books [3] and [165]. Some other references concerning the case $m \in (0,1)$ are quoted in our previous paper Bégout and Díaz [26]. We also mention that the spacial localization phenomenon (solutions with support $u(t,\,.\,)$ being a compact, when Ω is unbounded) requires a different balance between the damping and non-damping components (mainly with $\mathrm{Im}(a) > 0$) of the nonlinear term $a|u|^{m-1}u$ (see [25, 26, 27]).

In spite of the large amount of papers devoted to the existence and uniqueness results of nonlinear

Schrödinger equations with a damping term only very few of them allowed the consideration of a strong damping term (i.e. condition (5.1.3)). This is the reason why we presented here some new results on the general theory of the existence, uniqueness and regularity of solutions of the strongly damped Schrödinger equation improving several previous papers in the literature (see, e.g. Carles and Gallo [53], Lions [128], Brezis and Cazenave [45] and Vrabie [181]) which are needed for the study of the finite time extinction property.

Since the comparison principle does not apply to our problem, the main tool to prove the finite time extinction property is a suitable *energy method* in the spirit of the collection of energy methods quoted in the monograph Antontsev, Díaz and Shmarev [11]. Nevertheless, the adaptation to the nonlinear Schrödinger equation requires some new estimates and also a sharper study of the ordinary differential inequality satisfied by the mass. We start by giving, in Section 5.2, a *semi-abstract* result (which is proved in Section 5.5) in which the finite time extinction property is derived under a general regularity condition on the solution. The presence of the non-Lipschitz nonlinear term in the equation introduces a lack of regularity of the solution (in contrast to the case in which $m \geqslant 1$) and so we shall devote Section 5.4 to present a separated study of the existence and uniqueness of solutions satisfying the equation in some different sense according to the regularity assumed on the data. To this purpose, we use mainly some monotonicity methods, jointly with suitable regularizations and passing to the limit, improving previous results in the literature. Section 5.3 concerns the finite time extinction and the asymptotic behavior of the solution. The proofs of the results of Sections 5.3 and 5.4 are presented in Sections 5.7 and 5.6, respectively. An Appendix (p.201), collecting some technical auxiliary results, is also presented for the convenience of the reader.

We point out that in our formulation it may arise a non-homogeneous term (on which we assume a finite time extinction T_0) and that, surprisingly enough, under some critical decay to zero of $f(t,\,.\,)$ at $t=T_0$, we can conclude that the corresponding solution u also vanishes after the same time $t=T_0$ (see Theorem 5.2.1 part 2). Our energy method allows us also to get some large time decay estimates in some cases, always under the presence of a damping term, in which the conditions on the finite time extinction property fails (see Theorems 5.3.5 and 5.3.6 below). See Shimomura [159] for a related result with $m=1+\frac{2}{N}$.

We mention that it seems possible to apply the techniques of this paper to the consideration of some other complex-valued nonlinear equations such as the Gross-Pitaevskii equations, the Hartree-Fock equations, and the Ginzburg-Landau equations (see, e.g., Bégout and Díaz [28], Antontsev, Dias and Figueira [10], Okazawa and Yokota [147] and its many references).

Finally, we collect here some notations which will be used along with this paper. We let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Let $t\in\mathbb{R}$. Then $t_+=\max\{t,0\}$ is the positive part of t. We denote by $\overline{z}$ the conjugate of the complex number z, by $\mathrm{Re}(z)$ its real part and by $\mathrm{Im}(z)$ its imaginary part. For $1\leqslant p\leqslant\infty$, p' is the conjugate of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. We write Γ the boundary of a subset $\Omega\subset\mathbb{R}^N$. Unless if specified, all functions are complex-valued ($H^1(\Omega)=H^1(\Omega;\mathbb{C})$, etc). The notations $L^p(\Omega)$ ($p\in(0,\infty]$), $W^{k,p}(\Omega)$, $W^{k,p}_0(\Omega)$, $H^k(\Omega)$, $H^k_0(\Omega)$ ($p\in[1,\infty]$, $k\in\mathbb{N}$), $W^{-k,p'}(\Omega)$ and $H^{-k}(\Omega)$ ($p\in[1,\infty)$, $k\in\mathbb{N}$) refer as the usual well known different Lebesgue, Sobolev and Hilbert spaces and their topological dual. By convention of notation, $W^{0,p}(\Omega)=W^{0,p}_0(\Omega)=L^p(\Omega)$. For a Banach space X, we denote by $X^\star$ its topological dual and by $\langle\,.\,,\,.\,\rangle_{X^\star,X}\in\mathbb{R}$ the $X^\star-X$ duality product. In particular, for any $T\in L^{p'}(\Omega)$

and $\varphi \in L^p(\Omega)$ with $1 \leqslant p < \infty$, $\langle T, \varphi \rangle_{L^{p'}(\Omega),L^p(\Omega)} = \operatorname{Re} \int_\Omega T(x)\overline{\varphi(x)}\mathrm{d}x$. The scalar product in $L^2(\Omega)$ between two functions u, v is, $(u,v)_{L^2(\Omega)} = \operatorname{Re} \int_\Omega u(x)\overline{v(x)}\mathrm{d}x$. For a Banach space X and $p \in [1,\infty]$, $u \in L^p_{\mathrm{loc}}\big([0,\infty); X\big)$ means that $u \in L^p_{\mathrm{loc}}\big((0,\infty); X\big)$ and for any $T > 0$, $u_{|(0,T)} \in L^p\big((0,T); X\big)$. In the same way, $u \in W^{1,p}_{\mathrm{loc}}\big([0,\infty); X\big)$ means that $u \in L^p_{\mathrm{loc}}\big([0,\infty); X\big)$, u is absolutely continuous over $[0,\infty)$ (so it has a derivative u' almost everywhere on $(0,\infty)$) and $u' \in L^p_{\mathrm{loc}}\big([0,\infty); X\big)$. For a real x, $[x]$ denotes its integer part. As usual, we denote by C auxiliary positive constants, and sometimes, for positive parameters $a_1, \ldots, a_n$, write as $C(a_1, \ldots, a_n)$ to indicate that the constant C depends only on $a_1, \ldots, a_n$ and that this dependence is continuous (we will use this convention for constants which are not denoted merely by "C").

5.2 A semi-abstract result for finite time extinction

We consider the following nonlinear Schrödinger equation.

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} + \Delta u + a|u|^{-(1-m)}u = f(t,x), \text{ in } (0,\infty)\times\Omega, & (5.2.1)\\ u(t)_{|\Gamma} = 0, \text{ on } (0,\infty)\times\Gamma, & (5.2.2)\\ u(0) = u_0, \text{ in } \Omega, & (5.2.3)\end{cases}$$

The next result proves the finite time extinction of solutions (in some cases even in the same time in which the source $f(t,x)$ vanishes) under suitable "regularity" conditions on the solution (this is the reason why we denote as "semi-abstract" such a framework). In the following sections we shall obtain sufficient conditions implying that such a framework holds.

Theorem 5.2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $0 < m \leqslant 1$, $a \in \mathbb{C}$, $f \in L^1_{\mathrm{loc}}\big([0,\infty); L^2(\Omega)\big)$ and $u_0 \in L^2(\Omega)$. Assume that u is any strong solution to (5.2.1)–(5.2.3) (see Definition 5.4.1 below) and that,*

$$u \in L^\infty\big((0,\infty); H^\ell_0(\Omega)\big), \tag{5.2.4}$$

where $\ell = \left[\frac{N}{2}\right] + 1$ (or $H^\ell(\Omega)$ instead of $H^\ell_0(\Omega)$, if Ω is a half-space or if Ω has a bounded $C^{0,1}$-boundary). Then the following conclusions hold.

1) *If there exists $T_0 \geqslant 0$ such that,*

$$\textit{for almost every } t > T_0,\ f(t) = 0, \tag{5.2.5}$$

then there exists a finite time $T_\star \geqslant T_0$ such that,

$$\forall t \geqslant T_\star,\ \|u(t)\|_{L^2(\Omega)} = 0. \tag{5.2.6}$$

Furthermore,

$$T_\star \leqslant \frac{2\,\ell\, C_{\mathrm{GN}}\, \|u\|^{\frac{N(1-m)}{2\ell}}_{L^\infty((0,\infty);H^\ell(\Omega))}}{\operatorname{Im}(a)(1-m)(2\ell-N)} \|u(T_0)\|^{\frac{(1-m)(2\ell-N)}{2\ell}}_{L^2(\Omega)} + T_0, \tag{5.2.7}$$

where $C_{\mathrm{GN}} = C_{\mathrm{GN}}(N,m)$ is the constant in the inequality (5.5.6) below.

2) *There exist* $\varepsilon_\star = \varepsilon_\star(\operatorname{Im}(a), N, m)$ *satisfying the following property. Let* $T_0 > 0$ *and let* $C_{\rm GN}$ *be the constant in* (5.5.6). *If,*

$$\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{1-m} \leqslant \operatorname{Im}(a)\, C_{\rm GN}^{-1}\, \delta\,(1-\delta)\, T_0, \tag{5.2.8}$$

and if for almost every $t > 0$,

$$\|f(t)\|_{L^2(\Omega)}^2 \leqslant \varepsilon_\star \|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{-\frac{2N}{2\ell-N}} \big(T_0 - t\big)_+^{\frac{2\delta-1}{1-\delta}}, \tag{5.2.9}$$

where $\delta = \frac{(2\ell+N)+m(2\ell-N)}{4\ell} \in \left(\frac12, 1\right)$, *then* (5.2.6) *holds true with* $T_\star = T_0$.

Remark 5.2.2. Notice that $\delta\,(1-\delta) = \frac{(2\ell-N)(1-m)((2\ell+N)+m(2\ell-N))}{16\ell^2}$ and $\frac{2\delta-1}{1-\delta} = 2\frac{N(1-m)+2\ell m}{(2\ell-N)(1-m)}$.

The following result collects several very useful *a priori* estimates and some time differentiability conditions.

Proposition 5.2.3. *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open subset,* $0 < m \leqslant 1$, $a \in \mathbb{C}$, $f \in L^1_{\rm loc}\big([0,\infty); L^2(\Omega)\big)$ *and* $u_0 \in L^2(\Omega)$. *Assume that* u *is any weak solution to* (5.2.1)–(5.2.3) (*see Definition* 5.4.1 *below*). *Then we have the following results.*

$$u \in L^{m+1}_{\rm loc}\big([0,\infty); L^{m+1}(\Omega)\big), \tag{5.2.10}$$

$$\begin{cases} \dfrac12\|u(t)\|_{L^2(\Omega)}^2 + \operatorname{Im}(a)\displaystyle\int_s^t \|u(\sigma)\|_{L^{m+1}(\Omega)}^{m+1}\mathrm{d}\sigma \geqslant \dfrac12\|u(s)\|_{L^2(\Omega)}^2 \\ \qquad\qquad + \operatorname{Im}\displaystyle\int_s^t\!\!\int_\Omega f(\sigma,x)\,\overline{u(\sigma,x)}\,\mathrm{d}x\,\mathrm{d}\sigma, \quad \textit{if } \operatorname{Im}(a) \leqslant 0, \\ \dfrac12\|u(t)\|_{L^2(\Omega)}^2 + \operatorname{Im}(a)\displaystyle\int_s^t \|u(\sigma)\|_{L^{m+1}(\Omega)}^{m+1}\mathrm{d}\sigma \leqslant \dfrac12\|u(s)\|_{L^2(\Omega)}^2 \\ \qquad\qquad + \operatorname{Im}\displaystyle\int_s^t\!\!\int_\Omega f(\sigma,x)\,\overline{u(\sigma,x)}\,\mathrm{d}x\,\mathrm{d}\sigma, \quad \textit{if } \operatorname{Im}(a) \geqslant 0, \end{cases} \tag{5.2.11}$$

for any $t \geqslant s \geqslant 0$. *Finally, if* u *satisfies one of the conditions below then the map* $t \longmapsto \|u(t)\|_{L^2(\Omega)}^2$ *belongs to* $W^{1,1}_{\rm loc}\big([0,\infty);\mathbb{R}\big)$ *and we have equality in* (5.2.11).

a) u *is a strong solution* (*see Definition* 5.4.1 *below*),

b) $|\Omega| < \infty$,

c) $m = 1$,

d) $\operatorname{Im}(a) = 0$.

Remark 5.2.4. Here are some comments about Theorem 5.2.1.

1) Let f satisfies (5.2.5) and let u be a weak solution (see Definition 5.4.1 below). By (5.2.11) we obtain that for any $t \geqslant T_0$,

$$\begin{cases} \|u(t)\|_{L^2(\Omega)} = \|u(T_0)\|_{L^2(\Omega)}, & \text{if } \operatorname{Im}(a) = 0, \\ \|u(t)\|_{L^2(\Omega)} \geqslant \|u(T_0)\|_{L^2(\Omega)}, & \text{if } \operatorname{Im}(a) < 0. \end{cases}$$

It follows that in those cases the finite time extinction is not reachable. If $m = 1$ then we have, thanks to Proposition 5.2.3,

$$\forall t \geqslant T_0,\ \|u(t)\|_{L^2(\Omega)} = \|u(T_0)\|_{L^2(\Omega)} e^{-\mathrm{Im}(a)(t-T_0)}.$$

And again, there is no finite time extinction.

2) Let u be a weak solution of (5.2.1) (see Definition 5.4.1). It is obvious from the equation and 1) of this remark that if u vanishes at a finite time $T_\star \geqslant 0$ then necessarily f must satisfy (5.2.5) (but not necessarily the decay condition (5.2.9)) and that necessarily $\mathrm{Im}(a) > 0$ and $m < 1$. If, in addition, $|\Omega| < \infty$ then we have,

$$T_\star \geqslant \frac{\|u(T_0\|_{L^2(\Omega)}^{1-m}}{(1-m)\mathrm{Im}(a)|\Omega|^{\frac{1-m}{2}}} + T_0. \tag{5.2.12}$$

Indeed, it follows from (5.2.5), Proposition 5.2.3 and Hölder's inequality that for almost every $t > T_0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2(\Omega)}^2 = -\mathrm{Im}(a)\|u(t)\|_{L^{m+1}(\Omega)}^{m+1} \geqslant -\mathrm{Im}(a)|\Omega|^{\frac{1-m}{2}}\|u(t)\|_{L^2(\Omega)}^{m+1},$$

that is, $y' \geqslant -2\mathrm{Im}(a)|\Omega|^{\frac{1-m}{2}} y^{\frac{m+1}{2}}$, where $y(\,.\,) = \|u(\,.\,)\|_{L^2(\Omega)}^2$. After integration we get,

$$y(t)^{\frac{1-m}{2}} \geqslant \Big(y(T_0)^{\frac{1-m}{2}} - (1-m)\mathrm{Im}(a)|\Omega|^{\frac{1-m}{2}}(t-T_0)\Big)_+,$$

for any $t \geqslant T_0$, since $y \geqslant 0$. Hence the result.

3) The proof of the finite time extinction of u strongly relies on Gagliardo-Nirenberg's inequality (Lemma 5.5.4 below), that is : for any $v \in H_0^\ell(\Omega) \cap L^{m+1}(\Omega)$ (or $H^\ell(\Omega)$ instead of $H_0^\ell(\Omega)$, if Ω is a half-space or if Ω has a bounded $C^{0,1}$-boundary),

$$\|v\|_{L^2(\Omega)}^{\frac{(2\ell+N)+m(2\ell-N)}{2\ell}} \leqslant C_{\mathrm{GN}}\|v\|_{L^{m+1}(\Omega)}^{m+1}\|v\|_{H^\ell(\Omega)}^{\frac{N(1-m)}{2\ell}}, \tag{5.2.13}$$

to get the ordinary differential inequality (5.5.11) below :

$$y'(t) + 2\,\mathrm{Im}(a)\,C_{\mathrm{GN}}^{-1}\,\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{-\frac{N(1-m)}{2\ell}}\,y(t)^\delta \leqslant 0, \quad t > T_0, \tag{5.2.14}$$

where $\delta = \frac{(2\ell+N)+m(2\ell-N)}{4\ell}$, $y = \|u(\,.\,)\|_{L^2(\Omega)}^2$ and $C_{\mathrm{GN}} = C_{\mathrm{GN}}(N,m,\ell)$. This holds thanks to the non increasing property (5.2.11) of the mass (we recall that $\mathrm{Im}(a) > 0$ is necessary to have finite time extinction, by 1) of this remark). But this method fails if $N \geqslant 2\ell$. Indeed, first of all, Gagliardo-Nirenberg's inequality imposes that $0 \leqslant m \leqslant 1$. And as seen in 1) of this remark, finite time extinction is not reachable for $m = 1$. So, assume that $0 \leqslant m < 1$, (5.2.5) is fulfilled and u satisfies (5.2.4), where the integer ℓ has to be chosen later. Then for any $\ell \geqslant 1$, we may apply Lemma 5.5.4 below, which is (5.2.13) with $v = u(t)$, and we finally get (5.2.14). But if N is even and $\ell = \frac{N}{2}$ then $\delta = 1$ and Lemma 5.5.1 below yield,

$$\|u(t)\|_{L^2(\Omega)} \leqslant \|u(T_0)\|_{L^2(\Omega)} e^{-\mathrm{Im}(a)\,C^{-1}\,(t-T_0)}, \tag{5.2.15}$$

for any $t \geqslant T_0$, where $C = C(\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}, N, m)$. In the same way, if $1 \leqslant \ell < \frac{N}{2}$ then $\delta > 1$ and Lemma 5.5.1 below yield,

$$\|u(t)\|_{L^2(\Omega)} \leqslant \frac{\|u(T_0)\|_{L^2(\Omega)}}{\left(1 + \operatorname{Im}(a)\, C^{-1}(1-m)(N-2\ell)\|u(T_0)\|_{L^2(\Omega)}^{\frac{(1-m)(N-2\ell)}{2\ell}}(t-T_0)\right)^{\frac{2\ell}{(1-m)(N-2\ell)}}}, \tag{5.2.16}$$

for any $t \geqslant T_0$, where $C = C(\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}, N, m)$, and again this estimate does not give necessarily any finite time extinction result.

5.3 Finite time extinction and asymptotic behavior of solutions

Most of the results in this paper hold under the structural assumptions below.

Assumption 5.3.1. We assume that $\Omega \subseteq \mathbb{R}^N$ is a nonempty subset, $0 < m \leqslant 1$ and $a \in \mathbb{C}$ with $\operatorname{Im}(a) > 0$. If $m < 1$ then we assume further that,

$$2\sqrt{m}\operatorname{Im}(a) \geqslant (1-m)|\operatorname{Re}(a)|, \tag{5.3.1}$$

$$|\Omega| < \infty. \tag{5.3.2}$$

Theorem 5.3.2. *Let Assumption 5.3.1 be fulfilled with $N \in \{1,2,3\}$ and $m < 1$. Let $f \in W^{1,1}_{\text{loc}}\big([0,\infty); L^2(\Omega)\big)$, $u_0 \in H^1_0(\Omega)$ and assume that one of the following hypotheses holds.*

1) *$N = 1$ and $f \in W^{1,1}_{\text{loc}}\big([0,\infty); H^1_0(\Omega)\big)$.*
2) *$N \in \{1,2,3\}$, Ω is bounded with a $C^{1,1}$-boundary and $u_0 \in H^2(\Omega) \cap H^1_0(\Omega)$.*

Let u be the unique strong solution of (5.2.1)–(5.2.3) (see Definition 5.4.1, Theorems 5.4.4 and 5.4.5 and Remark 5.4.6 below). Finally, assume that there exists $T_0 \geqslant 0$ such that,

$$\textit{for almost every } t > T_0,\ f(t) = 0.$$

Then we have the following results.

a) *There exists a finite time $T_\star \geqslant T_0$ such that,*

$$\forall t \geqslant T_\star,\ \|u(t)\|_{L^2(\Omega)} = 0. \tag{5.3.3}$$

Furthermore, $T_\star$ satisfies the estimates (5.2.7) and (5.2.12).

b) *There exists $\varepsilon_\star = \varepsilon_\star(|a|, |\Omega|, N, m)$ satisfying the following property. Let δ be given in Property 2) of Theorem 5.2.1. If $f \in W^{1,1}\big((0,\infty); H^1_0(\Omega)\big)$,*

$$\begin{cases} \left(\|u_0\|_{H^1_0(\Omega)} + \|f\|_{L^1((0,\infty);H^1_0(\Omega))}\right)^{1-m} \leqslant \varepsilon_\star \min\left\{1, T_0\right\}, & \textit{if } N = 1, \\ \left(\|u_0\|^m_{H^2(\Omega)} + \|f\|^m_{W^{1,1}((0,\infty);H^1_0(\Omega))}\right)^{1-m} \leqslant \varepsilon_\star \min\left\{1, T_0\right\}, & \textit{if } N \in \{2,3\}, \end{cases}$$

and if for almost every $t > 0$,

$$\|f(t)\|^2_{L^2(\Omega)} \leqslant \varepsilon_\star (T_0 - t)_+^{\frac{2\delta-1}{1-\delta}},$$

then (5.3.3) holds with $T_\star = T_0$.

Remark 5.3.3. Notice that $\frac{2\delta-1}{1-\delta}=2\frac{1+m}{1-m}$, if $N\in\{1,2\}$ and $\frac{2\delta-1}{1-\delta}=2\frac{3+m}{1-m}$, if $N=3$.

Remark 5.3.4. Theorem 5.3.2 is an extension of the main result of Carles and Gallo [53] in the sense that they obtain the same conclusion as in *a*) but under the additional conditions $\mathrm{Re}(a)=0$, $f=0$ and without the lower bound for $T_\star$. As far as we know, the result in *b*) is new.

The following result gives some asymptotic decay estimates, for large time, for the case of higher dimensions $N\geqslant 4$.

Theorem 5.3.5. *Let Assumption 5.3.1 be fulfilled with $N\geqslant 4$ and $m<1$. Let $f\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$ and let $u_0\in H^1_0(\Omega)$. Assume further that $f\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);H^1_0(\Omega)\big)$ or $u_0\in H^2(\Omega)$ and that Ω is bounded with a $C^{1,1}$-boundary. Let u be the unique strong solution of (5.2.1)–(5.2.3) (see Definition 5.4.1, Theorems 5.4.4 and 5.4.5 and Remark 5.4.6 below). Finally, assume that there exists $T_0\geqslant 0$ such that*

$$\text{for almost every } t>T_0,\ f(t)=0.$$

Then we have for any $t\geqslant T_0$,

$$\|u(t)\|_{L^2(\Omega)}\leqslant\|u(T_0)\|_{L^2(\Omega)}e^{-\mathrm{Im}(a)\,C^{-1}\,(t-T_0)},$$

if $N=4$ and $u_0\in H^2(\Omega)$, and,

$$\|u(t)\|_{L^2(\Omega)}\leqslant\frac{\|u(T_0)\|_{L^2(\Omega)}}{\left(1+\mathrm{Im}(a)\,C^{-1}(1-m)(N-2\ell)\|u(T_0)\|_{L^2(\Omega)}^{\frac{(1-m)(N-2\ell)}{2\ell}}(t-T_0)\right)^{\frac{2\ell}{(1-m)(N-2\ell)}}},$$

if $N\geqslant 5$ or $u_0\in H^1_0(\Omega)$, where $C=C(\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))},N,m)$.

Theorem 5.3.6. *Let Assumption 5.3.1 be fulfilled, let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$, let $u_0\in L^2(\Omega)$ and let u be the unique weak solution of (5.2.1)–(5.2.3) (see Definition 5.4.1 and Theorem 5.4.3 below). If*

$$f\in L^1\big((0,\infty);L^2(\Omega)\big),$$

then,

$$\lim_{t\nearrow\infty}\|u(t)\|_{L^p(\Omega)}=0,$$

for any $p\in(0,2]$ (with $p=2$, if $m=1$ and $|\Omega|=\infty$).

Remark 5.3.7. Note that for $m=1$ in Theorem 5.3.6, if the stronger assumption (5.2.5) holds then we have,

$$\forall t\geqslant T_0,\ \|u(t)\|_{L^2(\Omega)}=\|u(T_0)\|_{L^2(\Omega)}e^{-\mathrm{Im}(a)(t-T_0)}.$$

See 1) of Remark 5.2.4.

5.4 Existence and uniqueness of solutions

Here and after, we shall always identify $L^2(\Omega)$ with its topological dual. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $0 < m \leqslant 1$ and let $X = H \cap L^{m+1}(\Omega)$, where $H = L^2(\Omega)$ or $H = H^1_0(\Omega)$. It follows from Lemma B.2 and 2) of Lemma B.4 below that,

$$X^\star = H^\star + L^{\frac{m+1}{m}}(\Omega),$$
$$L^{m+1}_{\mathrm{loc}}\big([0,\infty);X\big) \cap W^{1,\frac{m+1}{m}}_{\mathrm{loc}}\big([0,\infty);X^\star\big) \hookrightarrow C\big([0,\infty);L^2(\Omega)\big).$$

This justifies the notion of solution below (and it explains the sense in which the initial condition is satisfied).

Definition 5.4.1. Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $0 < m \leqslant 1$, $a \in \mathbb{C}$, $f \in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$ and $u_0 \in L^2(\Omega)$. Let us consider the following assertions.

1) $u \in L^{m+1}_{\mathrm{loc}}\big([0,\infty);H^1_0(\Omega) \cap L^{m+1}(\Omega)\big) \cap W^{1,\frac{m+1}{m}}_{\mathrm{loc}}\big([0,\infty);H^\star + L^{\frac{m+1}{m}}(\Omega)\big)$,
2) For almost every $t > 0$, $\Delta u(t) \in H^\star$.
3) u satisfies (5.2.1) in $\mathscr{D}'\big((0,\infty)\times\Omega\big)$.
4) $u(0) = u_0$.

We shall say that u is a *strong solution* if u is a H^2-solution or a H^1_0-solution. We shall say that u is a *H^2-solution of* (5.2.1)–(5.2.3) $\big($respectively, a *H^1_0-solution of* (5.2.1)–(5.2.3)$\big)$, if u satisfies the Assertions 1)–4) with $H = L^2(\Omega)$ $\big($respectively, with $H = H^1_0(\Omega)\big)$.
We shall say that u is a L^2-*solution* or simply a *weak solution* of (5.2.1)–(5.2.3) is there exists a pair,

$$(f_n,u_n)_{n\in\mathbb{N}} \subset L^1_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big) \times C\big([0,\infty);L^2(\Omega)\big), \tag{5.4.1}$$

such that for any $n \in \mathbb{N}$, u_n is a H^2-solution of (5.2.1)–(5.2.2) where the right-hand side member of (5.2.1) is f_n, and if

$$f_n \xrightarrow[n\to\infty]{L^1((0,T);L^2(\Omega))} f \quad \text{and} \quad u_n \xrightarrow[n\to\infty]{C([0,T];L^2(\Omega))} u, \tag{5.4.2}$$

for any $T > 0$.

Remark 5.4.2. Before making some comments on the above definition, it is useful to analyze some peculiar properties which arise when Ω is unbounded. Let $0 < m \leqslant 1$. Set for any $z \in \mathbb{C}$, $g(z) = |z|^{-(1-m)}z$ $(g(0) = 0)$ and let us define the mapping for any measurable function $u : \Omega \longrightarrow \mathbb{C}$, which we still denote by g, by $g(u)(x) = g(u(x))$. Let $H = L^2(\Omega)$ or $H = H^1_0(\Omega)$. It follows from (5.6.4) below that,

$$g \in C\big(L^{m+1}(\Omega);L^{\frac{m+1}{m}}(\Omega)\big) \text{ and } g \text{ is bounded on bounded sets.} \tag{5.4.3}$$

In particular, if $|\Omega| < \infty$ or if $m = 1$ then $H^1_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{m+1}(\Omega)$ with dense embedding and thus, $L^{\frac{m+1}{m}}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. We then obtain,

$$g \in C\big(L^2(\Omega);L^2(\Omega)\big) \cap C\big(H^1_0(\Omega);H^{-1}(\Omega)\big) \text{ and } g \text{ is bounded on bounded sets,} \tag{5.4.4}$$

and Assertion 1) becomes,

$$u\in L^{m+1}_{\mathrm{loc}}\big([0,\infty);H^1_0(\Omega)\big)\cap W^{1,\frac{m+1}{m}}_{\mathrm{loc}}\big([0,\infty);H^\star\big). \tag{5.4.5}$$

But if $|\Omega|=\infty$ and $m<1$ then the regularity (5.4.4) is not anymore valid. By Lemma B.2 below, we have,

$$\mathscr{D}(\Omega)\hookrightarrow X\hookrightarrow L^{m+1}(\Omega) \text{ with both dense embeddings,} \tag{5.4.6}$$

where $X=H\cap L^{m+1}(\Omega)$. It follows that,

$$L^{\frac{m+1}{m}}(\Omega)\hookrightarrow X^\star\hookrightarrow \mathscr{D}'(\Omega). \tag{5.4.7}$$

This gives with (5.4.3),

$$g\in C(X;X^\star) \text{ and } g \text{ is bounded on bounded sets.} \tag{5.4.8}$$

It follows from (5.4.3) and (5.4.6)–(5.4.8) that,

$$\langle g(u),v\rangle_{X^\star,X}=\langle g(u),v\rangle_{L^{\frac{m+1}{m}}(\Omega),L^{m+1}(\Omega)}=\mathrm{Re}\int\limits_\Omega g(u)\overline{v}\mathrm{d}x, \tag{5.4.9}$$

for any $u,v\in X$. Now, let us make some comments about Definition 5.4.1.

1) As seen at the beginning of this section, any strong or weak solution belongs to $C\big([0,\infty);L^2(\Omega)\big)$ and Assertion 4) makes sense in $L^2(\Omega)$.

2) It is obvious that a H^2-solution is also a H^1_0-solution and a weak solution. But it is not clear that a H^1_0-solution is a weak solution, without assuming a continuous dependence of the solution with respect to the initial data. Such a result will be established with the additional assumption (5.3.1) on a (see Lemma 5.6.5 below).

3) If $|\Omega|<\infty$ or if $m=1$ then it follows from (5.4.4), (5.4.5) and Assertion 2) that any H^2-solution (respectively, any H^1_0-solution) satisfies (5.2.1) in $L^2(\Omega)$ $\big($respectively, in $H^{-1}(\Omega)\big)$, for almost every $t>0$. Note also that Assertion 2) of Definition 5.4.1 is not an additional assumption for the H^1_0-solutions.

4) If $|\Omega|=\infty$ and if $m<1$ then it follows from (5.4.8) and Assertions 1) and 2) that any H^2-solution (respectively, any H^1_0-solution) satisfies (5.2.1) in $L^2(\Omega)+L^{\frac{m+1}{m}}(\Omega)$ $\big($respectively, in $H^{-1}(\Omega)+L^{\frac{m+1}{m}}(\Omega)\big)$, for almost every $t>0$.

5) Assume that u is a weak solution. By Definition 5.4.1, there exists $(f_n,u_n)_{n\in\mathbb{N}}$ satisfying (5.4.1)–(5.4.2) such that for any $n\in\mathbb{N}$, u_n is a H^2-solution of (5.2.1)–(5.2.2) where the right-hand side of (5.2.1) is f_n. Applying (5.6.4)–(5.6.5) below, we deduce that for any $T>0$,

$$\begin{gathered}\Delta u_n\xrightarrow[n\to\infty]{C([0,T];H^{-2}(\Omega))}\Delta u,\\ g(u_n)\xrightarrow[n\to\infty]{C([0,T];L^2(\Omega))}g(u),\quad\text{if } |\Omega|<\infty,\\ g(u_n)\xrightarrow[n\to\infty]{C([0,T];L^{\frac{2}{m}}(\Omega))}g(u).\end{gathered}$$

Now, we set : $Y=H^2_0(\Omega)\cap L^{\frac{2}{2-m}}(\Omega)$. By Lemma B.2 below, we have,

$$\begin{gathered}
Y^\star=H^{-2}(\Omega)+L^{\frac{2}{m}}(\Omega),\\
\mathscr{D}(\Omega)\hookrightarrow Y\hookrightarrow H^2_0(\Omega),L^2(\Omega),L^{\frac{2}{2-m}}(\Omega)\ \text{ with dense embedding,}\\
H^{-2}(\Omega),L^2(\Omega),L^{\frac{2}{m}}(\Omega)\hookrightarrow Y^\star\hookrightarrow\mathscr{D}'(\Omega).
\end{gathered}$$

Using the above uniform convergences and (5.4.2), we deduce that,

$$\int\limits_0^\infty\left\langle \mathrm{i}\frac{\partial u}{\partial t}+\Delta u+ag(u),\varphi\right\rangle_{Y^\star,Y}\psi(t)\,\mathrm{d}t=\int\limits_0^\infty\big\langle f(t),\varphi\big\rangle_{Y^\star,Y}\psi(t)\mathrm{d}t.$$

for any $\varphi\in Y$ and $\psi\in\mathscr{D}\big((0,\infty);\mathbb{R}\big)$.
As a conclusion, if u is a weak solution then $u\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);Y^\star\big)$ and it solves (5.2.1) in $Y^\star$, for almost every $t>0$. In particular, u satisfies (5.2.1) in $\mathscr{D}'\big((0,\infty)\times\Omega\big)$. If, in addition, $|\Omega|<\infty$ or if $m=1$ then we deduce from the above that $u\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);H^{-2}(\Omega)\big)$ and u solves (5.2.1) in $H^{-2}(\Omega)$, for almost every $t>0$.

6) When $m<1$ then except for Theorem 5.2.1 and Proposition 5.2.3, all the results of the following Sections 5.2–5.4 will be stated with $|\Omega|<\infty$.
7) Notice that the boundary condition $u(t)_{|\Gamma}=0$ is included in the assumption $u(t)\in H^1_0(\Omega)$.

Theorem 5.4.3 (**Existence and uniqueness of L^2-solutions**). *Let Assumption 5.3.1 be fulfilled and let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$. Then for any $u_0\in L^2(\Omega)$, there exists a unique weak solution u to (5.2.1)–(5.2.3). In addition, we have the following properties.*

1) *The map $t\longmapsto\|u(t)\|^2_{L^2(\Omega)}$ belongs to $W^{1,1}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ and we have,*

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}+\mathrm{Im}(a)\|u(t)\|^{m+1}_{L^{m+1}(\Omega)}=\mathrm{Im}\int\limits_\Omega f(t,x)\,\overline{u(t,x)}\,\mathrm{d}x,\tag{5.4.10}$$

for almost every $t>0$.

2) *If v is another weak solution of (5.2.1)–(5.2.2) with $v(0)=v_0\in L^2(\Omega)$ and $h\in L^1_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, instead of f in (5.2.1) then,*

$$\|u(t)-v(t)\|_{L^2(\Omega)}\leqslant\|u(s)-v(s)\|_{L^2(\Omega)}+\int\limits_s^t\|f(\sigma)-h(\sigma)\|_{L^2(\Omega)}\mathrm{d}\sigma,\tag{5.4.11}$$

for any $t\geqslant s\geqslant0$.

Theorem 5.4.4 (**Existence and uniqueness of H^1_0-solutions**). *Let Assumption 5.3.1 be fulfilled and let $f\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);H^1_0(\Omega)\big)$. Then for any $u_0\in H^1_0(\Omega)$, there exists a unique H^1_0-solution u to (5.2.1)–(5.2.3). Furthermore, u is also a weak solution and satisfies the following properties.*

1) *$u\in C\big([0,\infty);L^2(\Omega)\big)\cap C^1\big([0,\infty);H^{-2}(\Omega)\big)$ and u satisfies (5.2.1) in $H^{-2}(\Omega)$, for any $t\geqslant0$.*

2) $u\in C_{\rm w}\big([0,\infty);H^1_0(\Omega)\big)\cap W^{1,\infty}_{\rm loc}\big([0,\infty);H^{-1}(\Omega)\big)$ *and,*

$$\begin{cases}\|u(t)-u(s)\|_{L^2(\Omega)}\leqslant M|t-s|^{\frac12}, & (5.4.12)\\ \|\nabla u(t)\|_{L^2(\Omega)}\leqslant\|\nabla u_0\|_{L^2(\Omega)}+\displaystyle\int_0^t\|\nabla f(s)\|_{L^2(\Omega)}\mathrm{d}s, & (5.4.13)\end{cases}$$

for any $t\geqslant s\geqslant0$, *where* $M^2=2\|u\|_{L^\infty((s,t);H^1_0(\Omega))}\|u_t\|_{L^\infty((s,t);H^{-1}(\Omega))}$.

3) *The map* $t\longmapsto\|u(t)\|^2_{L^2(\Omega)}$ *belongs to* $C^1\big([0,\infty);\mathbb{R}\big)$ *and* (5.4.10) *holds for any* $t\geqslant0$.

4) *If* $f\in W^{1,1}\big((0,\infty);H^1_0(\Omega)\big)$ *then we have,*

$$u\in L^\infty\big((0,\infty);H^1_0(\Omega)\big)\cap W^{1,\infty}\big((0,\infty);H^{-1}(\Omega)\big)\cap C^1_{\rm b}\big([0,\infty);H^{-2}(\Omega)\big).$$

Theorem 5.4.5 (Existence and uniqueness of H^2-solutions). *Let Assumption 5.3.1 be fulfilled and let* $f\in W^{1,1}_{\rm loc}\big([0,\infty);L^2(\Omega)\big)$. *Then for any* $u_0\in H^1_0(\Omega)$ *with* $\Delta u_0\in L^2(\Omega)$, *there exists a unique* H^2*-solution* u *to* (5.2.1)–(5.2.3). *Furthermore,* u *satisfies the following properties.*

1) $u\in C\big([0,\infty);H^1_0(\Omega)\big)\cap C^1\big([0,\infty);H^{-1}(\Omega)\big)$, u *satisfies* (5.2.1) *in* $H^{-1}(\Omega)$, *for any* $t\geqslant0$.

2) $u\in W^{1,\infty}_{\rm loc}\big([0,\infty);L^2(\Omega)\big)$, $\Delta u\in L^\infty_{\rm loc}\big([0,\infty);L^2(\Omega)\big)$ *and,*

$$\begin{cases}\|u(t)-u(s)\|_{L^2(\Omega)}\leqslant\|u_t\|_{L^\infty((s,t);L^2(\Omega))}|t-s|, & (5.4.14)\\ \|\nabla u(t)-\nabla u(s)\|_{L^2(\Omega)}\leqslant M|t-s|^{\frac12}, & (5.4.15)\\ \|u_t\|_{L^\infty((0,t);L^2(\Omega))}\leqslant\|\Delta u_0+a|u_0|^{m-1}u_0-f(0)\|_{L^2(\Omega)}+\displaystyle\int_0^t\|f'(\sigma)\|_{L^2(\Omega)}\mathrm{d}\sigma, & (5.4.16)\end{cases}$$

for any $t\geqslant s\geqslant0$, *where* $M^2=2\|u_t\|_{L^\infty((s,t);L^2(\Omega))}\|\Delta u\|_{L^\infty((s,t);L^2(\Omega))}$.

3) *The map* $t\longmapsto\|u(t)\|^2_{L^2(\Omega)}$ *belongs to* $C^1\big([0,\infty);\mathbb{R}\big)$ *and* (5.4.10) *holds for any* $t\geqslant0$.

4) *If* $f\in W^{1,1}\big((0,\infty);L^2(\Omega)\big)$ *then we have,*

$$u\in C_{\rm b}\big([0,\infty);H^1_0(\Omega)\big)\cap C^1_{\rm b}\big([0,\infty);H^{-1}(\Omega)\big)\cap W^{1,\infty}\big((0,\infty);L^2(\Omega)\big),$$
$$\Delta u\in L^\infty\big((0,\infty);L^2(\Omega)\big).$$

Remark 5.4.6. Let $E=\big\{u\in H^1_0(\Omega);\Delta u\in L^2(\Omega)\big\}$ with $\|u\|^2_E=\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)}$. We recall that $E\subset H^2_{\rm loc}(\Omega)$ (Theorem 8.8, in Gilbarg and Trudinger [90]). If $\Omega=\mathbb{R}^N$ then $E=H^2(\mathbb{R}^N)$ with equivalent norms (by the Fourier transform and Plancherel's formula), while if Ω is bounded and Γ is of class $C^{1,1}$ then $E=H^2(\Omega)\cap H^1_0(\Omega)$ with equivalent norms (Theorem 8.12, in Gilbarg and Trudinger [90] and Corollary 2.5.2.2, in Grisvard [93]). In order to get the equivalence of norms, we may use the inequalities,

$$\|\nabla u\|^2_{L^2(\Omega)}\leqslant\|u\|_{L^2(\Omega)}\|\Delta u\|_{L^2(\Omega)}\leqslant\|u\|^2_{L^2(\Omega)}+\|\Delta u\|^2_{L^2(\Omega)},\tag{5.4.17}$$

which hold for any subset $\Omega\subseteq\mathbb{R}^N$ and any $u\in H^2(\Omega)\cap H^1_0(\Omega)$.

Remark 5.4.7. Since $f\in C\big([0,\infty);L^2(\Omega)\big)$ (by 1) of Lemma B.4), estimate (5.4.16) with $f(0)$ makes sense.

Remark 5.4.8. It follows from (5.4.11) and (5.4.13) that if $N=1$ then the decay assumptions (5.2.8) and (5.2.9) may be replaced with,

$$\Big(\|u_0\|_{H^1_0(\Omega)}+\|f\|_{L^1((0,\infty);H^1_0(\Omega))}\Big)^{1-m}\leqslant\varepsilon_\star\min\big\{1,T_\star\big\},$$
$$\|f(t)\|^2_{L^2(\Omega)}\leqslant\varepsilon_\star\big(T_\star-t\big)_+^{\frac{2\delta-1}{1-\delta}},\tag{5.4.18}$$

for almost every $t>0$, where $\nu_\star=\varepsilon_\star(\mathrm{Im}(a),N,m)$. In the same way, it follows from (5.4.11), (5.4.13), (5.4.16), Remark 5.4.6 and (5.2.1) that if $N\leqslant3$ and Ω is bounded with a $C^{1,1}$-boundary then (5.2.8) may be replaced with,

$$\Big(\|u_0\|^m_{H^2(\Omega)}+\|f\|^m_{W^{1,1}((0,\infty);H^1_0(\Omega))}\Big)^{1-m}\leqslant\varepsilon_\star\min\big\{1,T_\star\big\},$$

and (5.2.9) with (5.4.18), where $\varepsilon_\star=\varepsilon_\star(|a|,|\Omega|,N,m)$.

5.5 Proof of the semi-abstract result on the finite time extinction

The proof of Theorem 5.2.1 relies on the three following lemmas.

Lemma 5.5.1. *Let $y\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ with $y\geqslant0$ over $(0,\infty)$, $\delta\in\mathbb{R}$, $\alpha>0$ and $T_0\geqslant0$. If*

$$y'+2\alpha y^\delta\leqslant0,$$

almost everywhere on (T_0,∞), then we have,

$$y(t)\leqslant\begin{cases}\Big(y(T_0)^{1-\delta}+2\alpha(1-\delta)(T_0-t)\Big)_+^{\frac{1}{1-\delta}}, & \text{if } \delta<1,\\ y(T_0)e^{-2\alpha(t-T_0)}, & \text{if } \delta=1,\\ \dfrac{y(T_0)}{\big(1+2\alpha(\delta-1)y(T_0)^{\delta-1}(t-T_0)\big)^{\frac{1}{\delta-1}}} & \text{if } \delta>1,\end{cases}$$

for any $t\geqslant T_0$. In particular, if $\delta<1$ then for any $t\geqslant T_\star$, $y(t)=0$ where,

$$T_\star\leqslant\frac{1}{2\alpha(1-\delta)}y(T_0)^{1-\delta}+T_0.$$

Proof. The result follows by integration of the ordinary differential inequality over (T_0,t). □

The following lemma improves a similar result contained in Antontsev, Díaz and Shmarev [11] (Proposition 1.1).

Lemma 5.5.2. *Let $y\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ with $y\geqslant0$ over $[0,\infty)$, $\delta\in(0,1)$, $\alpha,T_0>0$ and,*

$$y_\star=\big(\alpha\,\delta^\delta(1-\delta)\big)^{\frac{1}{1-\delta}},\tag{5.5.1}$$
$$x_\star=\left(\alpha\,\delta\,(1-\delta)\,T_0\right)^{\frac{1}{1-\delta}}.\tag{5.5.2}$$

If,

$$y(0) \leqslant x_\star, \tag{5.5.3}$$

and if for almost every $t > 0$,

$$y'(t) + \alpha y(t)^\delta \leqslant y_\star \left(T_0 - t\right)_+^{\frac{\delta}{1-\delta}}, \tag{5.5.4}$$

then for any $t \geqslant T_0$, $y(t) = 0$.

Proof. Set for any $t \in [0, T_0]$, $z(t) = x_\star T_0^{-\frac{1}{1-\delta}} \left(T_0 - t\right)^{\frac{1}{1-\delta}}$. We have for almost every $t \in (0, T_0)$,

$$z'(t) + \alpha z(t)^\delta = y_\star \left(T_0 - t\right)^{\frac{\delta}{1-\delta}} \geqslant y'(t) + \alpha y(t)^\delta. \tag{5.5.5}$$

We claim that for any $t \in [0, T_0]$, $y(t) \leqslant z(t)$. If not, since by (5.5.3) $z(0) \geqslant y(0)$ and y and z are continuous over $[0, T_0]$ (by 1) of Lemma B.4), there exist $t_\star \in [0, T_0)$ and $\varepsilon \in (0, T_0 - t_\star)$ such that $y(t_\star) = z(t_\star)$ and $y(t) > z(t)$, for any $t \in (t_\star, t_\star + \varepsilon)$. This leads with (5.5.5) to, $y' \leqslant z'$, almost everywhere on $(t_\star, t_\star + \varepsilon)$. Integrating over $(t_\star, t)$ for $t \in (t_\star, t_\star + \varepsilon)$, we obtain that $y(t) \leqslant z(t)$, for any $t \in [t_\star, t_\star + \varepsilon]$. A contradiction. Hence the claim. In particular, $y(T_0) \leqslant z(T_0) = 0$. But from (5.5.4), y is non increasing over (T_0, ∞). Hence the result, since $y \geqslant 0$ everywhere. □

Remark 5.5.3. Let us explain how we found $y_\star$ and $x_\star$ in Lemma 5.5.2. We look for a solution of the ordinary differential inequality (5.5.4). Set for any $x \geqslant 0$,

$$\forall x \geqslant 0,\ f(x) = (1-\delta)^{-1} T_0^{-\frac{1}{1-\delta}} x^\delta \left(\alpha(1-\delta)T_0 - x^{1-\delta}\right),$$
$$\forall t \in [0, T_0],\ z(t) = x T_0^{-\frac{1}{1-\delta}} \left(T_0 - t\right)_+^{\frac{1}{1-\delta}}.$$

We want $z(0) = x \geqslant y(0)$ to apply our proof. A straightforward calculation yields,

$$z'(t) + \alpha z(t)^\delta = f(x) \left(T_0 - t\right)^{\frac{\delta}{1-\delta}}.$$

We compute, $\underset{x \geqslant 0}{\operatorname{argmax}} f(x) = x_\star$, where $x_\star$ is given by (5.5.2), and $f(x_\star) = y_\star$, where $y_\star$ is given by (5.5.1). We then choose $x = x_\star$ in the definition of z and we obtain the condition (5.5.3).

Lemma 5.5.4 (Gagliardo-Nirenberg's inequality). *Let* $N \in \mathbb{N}$, *let* $\Omega \subseteq \mathbb{R}^N$ *be an open subset, let* $0 \leqslant m \leqslant 1$ *and let* $\ell \in \mathbb{N}$. *Then for any* $v \in H_0^\ell(\Omega) \cap L^{m+1}(\Omega)$,

$$\|v\|_{L^2(\Omega)}^{\frac{(2\ell+N)+m(2\ell-N)}{2\ell}} \leqslant C \|v\|_{L^{m+1}(\Omega)}^{m+1} \|v\|_{H^\ell(\Omega)}^{\frac{N(1-m)}{2\ell}}, \tag{5.5.6}$$

where $C = C(m, \ell, N)$. *If* Ω *is a half-space or if* Ω *has a bounded* $C^{0,1}$*-boundary then* (5.5.6) *holds for any* $v \in H^\ell(\Omega)$.

Proof. See, for instance, Friedman [86], Theorem 9.3, for $v \in \mathscr{D}(\mathbb{R}^N)$ and so, by extension and density, for $v \in H_0^\ell(\Omega) \cap L^{m+1}(\Omega)$. If Ω is a half-space or if Ω has a bounded $C^{0,1}$-boundary then there exists a linear extension operator E such that for any $k \in \mathbb{N}_0$ and $p \in [1, \infty]$,

$$E \in \mathscr{L}\big(W^{k,p}(\Omega); W^{k,p}(\mathbb{R}^N)\big),$$

and $Eu=u$, almost everywhere in Ω (Stein [161], Theorem 5, §3.2, §3.3 ; Adams [2], Theorem 4.26 ; see also Grisvard [93], Theorem 1.4.3.1). □

Proof of Proposition 5.2.3. Let the assumptions of the theorem be fulfilled. We first assume that u is a strong solution. Let H be as in Definition 5.4.1 and let $X=H\cap L^{m+1}(\Omega)$. By Definition 5.4.1, we have (5.2.10) and by 3) and 4) of Remark 5.4.2, we can take the $X^\star-X$ duality product with $\mathrm{i}u$. Estimate (5.2.11) with equality then follows from (5.4.9) and 1) of Lemma B.5. Now, assume that u is a weak solution. Let $(f_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be as in Definition 5.4.1. According to the above, it follows from Hölder's inequality that $f\overline{u}\in L^1_{\mathrm{loc}}\big([0,\infty);L^1(\Omega)\big)$ and,

$$f_n\overline{u_n}\xrightarrow[n\to\infty]{L^1_{\mathrm{loc}}([0,\infty);L^1(\Omega))}f\overline{u}, \tag{5.5.7}$$

$$\frac12\|u_n(t)\|^2_{L^2(\Omega)}+\mathrm{Im}(a)\int_s^t\|u_n(\sigma)\|^{m+1}_{L^{m+1}(\Omega)}\mathrm{d}\sigma = \frac12\|u_n(s)\|^2_{L^2(\Omega)}+\mathrm{Im}\int_s^t\!\!\int_\Omega f_n(\sigma,x)\,\overline{u_n(\sigma,x)}\,\mathrm{d}x\,\mathrm{d}\sigma, \tag{5.5.8}$$

for any $n\in\mathbb{N}$ and $t\geqslant s\geqslant0$. If $|\Omega|<\infty$ or if $m=1$ then for any $T>0$, $C([0,T];L^2(\Omega))\hookrightarrow C([0,T];L^{m+1}(\Omega))$ and then we are allowed to pass to the limit in (5.5.8) under the integral symbol. We then get with (5.5.7) the desired result under the hypotheses b), c) or d). If $|\Omega|=\infty$, $m<1$ and $\mathrm{Im}(a)\geqslant0$ then for any $T>0$, $C([0,T];L^2(\Omega))\hookrightarrow C([0,T];L^{m+1}_{\mathrm{loc}}(\Omega))$. By (5.5.8),

$$\frac12\|u_n(t)\|^2_{L^2(\Omega)}+\mathrm{Im}(a)\int_s^t\|u_n(\sigma)\|^{m+1}_{L^{m+1}(\Omega\cap B(0,R))}\mathrm{d}\sigma \leqslant \frac12\|u_n(s)\|^2_{L^2(\Omega)}+\mathrm{Im}\int_s^t\!\!\int_\Omega f_n(\sigma,x)\,\overline{u_n(\sigma,x)}\,\mathrm{d}x\,\mathrm{d}\sigma,$$

for any $t>s>0$, $R>0$ and $n\in\mathbb{N}$. Passing to the limit in n first and then in R then, we obtain (5.2.10) and (5.2.11) with the help of the monotone convergence Theorem and (5.5.7). We proceed in the same way if $|\Omega|=\infty$, $m<1$ and $\mathrm{Im}(a)\leqslant0$. □

Proof of Theorem 5.2.1. By (5.5.6) and Proposition 5.2.3, we have for almost every $t>0$,

$$\|u(t)\|^{\frac{(2\ell+N)+m(2\ell-N)}{2\ell}}_{L^2(\Omega)}\leqslant C_{\mathrm{GN}}\|u\|^{\frac{N(1-m)}{2\ell}}_{L^\infty((0,\infty);H^\ell(\Omega))}\|u(t)\|^{m+1}_{L^{m+1}(\Omega)},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}+2\mathrm{Im}(a)\|u(t)\|^{m+1}_{L^{m+1}(\Omega)}=2\mathrm{Im}\int_\Omega f(t,x)\overline{u(t,x)}\mathrm{d}x.$$

It follows that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}+2\alpha\|u(t)\|^{2\delta}_{L^2(\Omega)}\leqslant2\int_\Omega|f(t,x)||u(t,x)|\mathrm{d}x, \tag{5.5.9}$$

for almost every $t>0$, where $\alpha=\operatorname{Im}(a)C_{\rm GN}^{-1}\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{-\frac{N(1-m)}{2\ell}}$ and $\delta=\frac{(2\ell+N)+m(2\ell-N)}{4\ell}$. Since $0<m<1$ and $\ell=\left[\frac{N}{2}\right]+1$, we have $\frac12<\delta<1$. Using the Young inequality,

$$xy\leqslant\frac{\varepsilon^{-p'}}{p'}x^{p'}+\frac{\varepsilon^p}{p}y^p,$$

with $x=\|f(t)\|_{L^2(\Omega)}$, $y=\|u(t)\|_{L^2(\Omega)}$, $p=2\delta$ and $\varepsilon=(\alpha\delta)^{\frac{1}{2\delta}}$, one obtains with Cauchy-Schwarz's inequality,

$$2\int\limits_\Omega|f(t,x)||u(t,x)|{\rm d}x\leqslant\frac{2\delta-1}{\delta}(\alpha\delta)^{-\frac{1}{2\delta-1}}\|f(t)\|_{L^2(\Omega)}^{\frac{2\delta}{2\delta-1}}+\alpha\|u(t)\|_{L^2(\Omega)}^{2\delta}. \tag{5.5.10}$$

Finally, set for any $t\geqslant0$, $y(t)=\|u(t)\|_{L^2(\Omega)}^2$ and let us prove Property 1). If f satisfies (5.2.5) then (5.5.9) may be rewritten as,

$$y'(t)+2\alpha y(t)^\delta\leqslant0, \tag{5.5.11}$$

for almost every $t>T_0$. We then conclude with the help of Lemma 5.5.1. Now assume that (5.2.8)–(5.2.9) hold where the constant $\varepsilon_\star$ has to be determined later. We then have,

$$y(0)^{1-\delta}\leqslant\alpha\,\delta\,(1-\delta)\,T_0, \tag{5.5.12}$$

$$\|f(t)\|_{L^2(\Omega)}^2\leqslant\varepsilon_\star\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{-\frac{N(1-m)}{2\ell}\frac{1}{1-\delta}}\big(T_0-t\big)_+^{\frac{2\delta-1}{1-\delta}}, \tag{5.5.13}$$

where (5.5.12) is a consequence of (5.2.8) and (5.5.13) is nothing else but (5.2.9). Gathering together (5.5.9), (5.5.10) and (5.5.13), one gets

$$y'(t)+\alpha y(t)^\delta\leqslant\frac{2\delta-1}{\delta}(\operatorname{Im}(a)C_{\rm GN}^{-1}\delta)^{-\frac{1}{2\delta-1}}\varepsilon_\star^{\frac{\delta}{2\delta-1}}\|u\|_{L^\infty((0,\infty);H^\ell(\Omega))}^{-\frac{N(1-m)}{2\ell}\frac{1}{1-\delta}}\big(T_0-t\big)_+^{\frac{\delta}{1-\delta}}.$$

Choosing $\varepsilon_\star=(2\delta-1)^{-\frac{2\delta-1}{\delta}}(\operatorname{Im}(a)C_{\rm GN}^{-1}\delta)^{\frac{1}{1-\delta}}(1-\delta)^{\frac{2\delta-1}{\delta(1-\delta)}}$, one obtains,

$$y'(t)+\alpha y(t)^\delta\leqslant y_\star\big(T_0-t\big)_+^{\frac{\delta}{1-\delta}}.$$

for almost every $t>0$, where $y_\star$ is given by (5.5.1). Notice that (5.5.12) is nothing else but (5.5.3). We infer by Lemma 5.5.2 that $y(t)=0$, for any $t\geqslant T_0$. □

5.6 Proofs of the existence and uniqueness theorems

Lemma 5.6.1. *Let Assumption 5.3.1 be fulfilled. Let us define the following (nonlinear) operator on $L^2(\Omega)$.*

$$\begin{cases}D(A)=\left\{u\in H_0^1(\Omega);\ \Delta u\in L^2(\Omega)\right\},\\ \forall u\in D(A),\ Au=-{\rm i}\Delta u-{\rm i}a|u|^{-(1-m)}u,\end{cases} \tag{5.6.1}$$

Then A is a maximal monotone operator on $L^2(\Omega)$ (and so m-accretive) with domain dense.

The proof relies on the following lemmas.

Lemma 5.6.2 ([132]). *Let $0<m\leqslant 1$. Set for any $z\in\mathbb{C}$, $g(z)=|z|^{-(1-m)}z$ $(g(0)=0)$. Then for any $(z_1,z_2)\in\mathbb{C}\times\mathbb{C}$,*

$$2\sqrt{m}\left|\operatorname{Im}\Big(\big(g(z_1)-g(z_2)\big)\big(\overline{z_1-z_2}\big)\Big)\right|\leqslant(1-m)\operatorname{Re}\Big(\big(g(z_1)-g(z_2)\big)\big(\overline{z_1-z_2}\big)\Big), \tag{5.6.2}$$

$$|g(z_1)-g(z_2)|\leqslant 3|z_1-z_2|^m. \tag{5.6.3}$$

Let $\Omega\subseteq\mathbb{R}^N$ be an open subset. We define the mapping for any measurable function $u:\Omega\longrightarrow\mathbb{C}$, which we still denote by g, by $g(u)(x)=g(u(x))$. Then for any $p\in[1,\infty)$,

$$g\in C\big(L^p(\Omega);L^{\frac{p}{m}}(\Omega)\big) \text{ and } g \text{ is bounded on bounded sets,} \tag{5.6.4}$$

$$g\in C\big(L^2(\Omega);L^2(\Omega)\big) \text{ and } g \text{ is bounded on bounded sets, if } |\Omega|<\infty. \tag{5.6.5}$$

Finally, let $a\in\mathbb{C}$ with $\operatorname{Im}(a)>0$ satisfying (5.3.1). If $\big(g(u)-g(v)\big)(\overline{u-v})\in L^1(\Omega)$ then,

$$\operatorname{Re}\left(-\mathrm{i}\,a\int_\Omega(g(u)-g(v))(\overline{u-v})\mathrm{d}x\right)\geqslant 0. \tag{5.6.6}$$

We may choose, for instance, $u,v\in L^2(\Omega)$, if $|\Omega|<\infty$, or $u,v\in L^{m+1}(\Omega)$, in the general case.

Proof. Estimate (5.6.2) is Lemma 2.2 of Liskevich and Perel′muter [132] while (1.2.7) comes from Lemma B.1, implying (5.6.4) and (5.6.5). Finally, by (5.6.4), (5.6.5) and Hölder's inequality, we have $\big(g(u)-g(v)\big)(\overline{u-v})\in L^1(\Omega)$, for any u,v as in the statement of the lemma and by (5.6.2),

$$\begin{aligned}
&\operatorname{Re}\left(-\mathrm{i}\,a\int_\Omega\big(g(u)-g(v)\big)(\overline{u-v})\mathrm{d}x\right)\\
&=\operatorname{Im}(a)\operatorname{Re}\int_\Omega\big(g(u)-g(v)\big)\big(\overline{u-v}\big)\mathrm{d}x+\operatorname{Re}(a)\operatorname{Im}\int_\Omega\big(g(u)-g(v)\big)\big(\overline{u-v}\big)\mathrm{d}x\\
&\geqslant\left(\operatorname{Im}(a)-|\operatorname{Re}(a)|\frac{1-m}{2\sqrt{m}}\right)\operatorname{Re}\int_\Omega\big(g(u)-g(v)\big)\big(\overline{u-v}\big)\mathrm{d}x\\
&\geqslant 0.
\end{aligned}$$

This ends the proof. □

Proof of Lemma 5.6.1. The density of the domain of the operator is obvious. Let g be as in Lemma 5.6.2. It is well known that $(-\mathrm{i}\Delta,D(A))$ is a maximal monotone operator on $L^2(\Omega)$ (Proposition 2.6.12 in Cazenave and Haraux [59]). In addition, if we define B on $L^2(\Omega)$ by $Bu=-\mathrm{i}ag(u)$, it follows from (5.6.4)–(5.6.6) that $B\in C(L^2(\Omega);L^2(\Omega))$ and

$$(Bu-Bv,u-v)_{L^2(\Omega)}=\operatorname{Re}\left(-\mathrm{i}\,a\int_\Omega(g(u)-g(v))(\overline{u-v})\mathrm{d}x\right)\geqslant 0,$$

for any $u,v\in L^2(\Omega)$. We then infer that $A=-\mathrm{i}\Delta+B$ is a maximal monotone operator (Brezis [43], Corollaries 2.5 and 2.7). □

To obtain (5.4.13), we need to regularize the nonlinearity in order to apply the ∇ operator. We then establish the next lemma.

Lemma 5.6.3. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let $0 < m < 1$, let $a \in \mathbb{C}$ with $\operatorname{Im}(a) > 0$ satisfying (5.3.1) and let $\varepsilon \in (0,1)$. Let for any $u \in L^2(\Omega)$, $g_\varepsilon(u) = (|u|^2+\varepsilon)^{-\frac{1-m}{2}}u$. Finally, let g be as in Lemma 5.6.2 and let $D(A)$ be defined by (5.6.1). Then,*

$$g_\varepsilon \in C\big(L^2(\Omega);L^2(\Omega)\big) \cap C\big(H_0^1(\Omega);H_0^1(\Omega)\big), \tag{5.6.7}$$

$$\forall u \in D(A),\ \operatorname{Re}\left(\mathrm{i}a\int_\Omega g_\varepsilon(u)\overline{\Delta u}\mathrm{d}x\right) \geqslant 0, \tag{5.6.8}$$

$$\forall u \in D(A) \text{ such that } u^m\Delta u \in L^1(\Omega),\ \operatorname{Re}\left(\mathrm{i}a\int_\Omega g(u)\overline{\Delta u}\mathrm{d}x\right) \geqslant 0. \tag{5.6.9}$$

Remark 5.6.4. If $\Omega \subseteq \mathbb{R}^N$ is arbitrary, $m=1$ and $\operatorname{Im}(a) > 0$ then for any $u \in D(A)$,

$$\operatorname{Re}\left(\mathrm{i}a\int_\Omega g(u)\overline{\Delta u}\mathrm{d}x\right) = \operatorname{Im}(a)\|\nabla u\|_{L^2(\Omega)}^2 \geqslant 0.$$

In other words, one directly obtains (5.6.9).

Proof of Lemma 5.6.3. A straightforward calculation shows that for any $\varepsilon \in (0,1)$,

$$|g_\varepsilon(u) - g_\varepsilon(v)| \leqslant C\varepsilon^{-1}|u-v|,$$
$$|\nabla g_\varepsilon(u)| \leqslant C\varepsilon^{-1}|\nabla u|.$$

It follows that if $u \in H_0^1(\Omega)$ then $g_\varepsilon(u) \in H_0^1(\Omega)$ and (5.6.7) comes from the above estimates and the partial converse of the dominated convergence Theorem (see, for instance, Brezis [44], Theorem 4.9). Let us turn out to the proof of (5.6.8). Let $u \in D(A)$. It follows from (5.6.7) that we can take the scalar product in L^2 between $\mathrm{i}ag_\varepsilon(u)$ and Δu. We then obtain,

$$\begin{aligned}
&\operatorname{Re}\left(\mathrm{i}a\int_\Omega g_\varepsilon(u)\overline{\Delta u}\mathrm{d}x\right) = (\mathrm{i}ag_\varepsilon(u),\Delta u)_{L^2(\Omega)} = -(\mathrm{i}a\nabla g_\varepsilon(u),\nabla u)_{L^2(\Omega)}\\
&= \operatorname{Re}\left(-\mathrm{i}a\int_\Omega \frac{|\nabla u|^2(|u|^2+\varepsilon) - (1-m)\operatorname{Re}(u\overline{\nabla u}).u\overline{\nabla u}}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x\right)\\
&= \operatorname{Im}(a)\int_\Omega \frac{|\nabla u|^2(|u|^2+\varepsilon) - (1-m)|\operatorname{Re}(u\overline{\nabla u})|^2}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x - \operatorname{Re}(a)\int_\Omega \frac{(1-m)\operatorname{Re}(u\overline{\nabla u}).\operatorname{Im}(u\overline{\nabla u})}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x\\
&= \varepsilon\operatorname{Im}(a)\int_\Omega \frac{|\nabla u|^2}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x\\
&\quad + \operatorname{Im}(a)\int_\Omega \frac{m|\operatorname{Re}(u\overline{\nabla u})|^2 + |\operatorname{Im}(u\overline{\nabla u})|^2}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x - \operatorname{Re}(a)\int_\Omega \frac{(1-m)\operatorname{Re}(u\overline{\nabla u}).\operatorname{Im}(u\overline{\nabla u})}{(|u|^2+\varepsilon)^{\frac{3-m}{2}}}\mathrm{d}x
\end{aligned}$$

where we used in the last equality the fact that, $|\nabla u|^2|u|^2 = |\operatorname{Re}(u\overline{\nabla u})|^2 + |\operatorname{Im}(u\overline{\nabla u})|^2$. To conclude, it remains to show that,

$$(1-m)|\operatorname{Re}(a)|\,|\operatorname{Re}(u\overline{\nabla u})|\,|\operatorname{Im}(u\overline{\nabla u})| \leqslant \operatorname{Im}(a)\left(m|\operatorname{Re}(u\overline{\nabla u})|^2 + |\operatorname{Im}(u\overline{\nabla u})|^2\right). \tag{5.6.10}$$

Using our assumption on a and the following Young inequality,

$$2|xy| \leqslant \delta x^2 + \frac{y^2}{\delta},$$

with $x = |\mathrm{Re}(u\overline{\nabla u})|$, $y = |\mathrm{Im}(u\overline{\nabla u})|$ and $\delta = \sqrt{m}$, we obtain,

$$\begin{aligned} &(1-m)|\mathrm{Re}(a)|\,|\mathrm{Re}(u\overline{\nabla u})|\,|\mathrm{Im}(u\overline{\nabla u})| \\ &\leqslant 2\sqrt{m}\,\mathrm{Im}(a)|\mathrm{Re}(u\overline{\nabla u})|\,|\mathrm{Im}(u\overline{\nabla u})| \\ &\leqslant \sqrt{m}\,\mathrm{Im}(a)\left(\sqrt{m}|\mathrm{Re}(u\overline{\nabla u})|^2 + \frac{|\mathrm{Im}(u\overline{\nabla u})|^2}{\sqrt{m}}\right) \\ &\leqslant \mathrm{Im}(a)\left(m|\mathrm{Re}(u\overline{\nabla u})|^2 + |\mathrm{Im}(u\overline{\nabla u})|^2\right), \end{aligned}$$

which is (5.6.10). Finally, since we have $g_\varepsilon(u) \xrightarrow[\varepsilon\searrow 0]{\text{a.e. on }\Omega} g(u)$ and $|g_\varepsilon(u)| \overset{\text{a.e.}}{\leqslant} |g(u)|$, for any $\varepsilon > 0$, (5.6.9) is a consequence of (5.6.8) and the dominated convergence Theorem. □

Concerning the continuous dependence with respect to the data we have :

Lemma 5.6.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, $0 < m \leqslant 1$ and $a \in \mathbb{C}$ with $\mathrm{Im}(a) > 0$ satisfying (5.3.1). Let $X = L^2(\Omega) \cap L^{m+1}(\Omega)$ or $X = H_0^1(\Omega) \cap L^{m+1}(\Omega)$. Finally, let $f_1, f_2 \in L^1_{\mathrm{loc}}([0,\infty); L^2(\Omega))$ and let*

$$u, v \in L^p_{\mathrm{loc}}\big([0,\infty); X\big) \cap W^{1,p'}_{\mathrm{loc}}\big([0,\infty); X^\star\big),$$

for some $1 < p < \infty$. If,

$$\begin{aligned} \mathrm{i}u_t + \Delta u + a|u|^{-(1-m)}u &= f_1, \\ \mathrm{i}v_t + \Delta v + a|v|^{-(1-m)}v &= f_2, \end{aligned}$$

in $\mathscr{D}'\big((0,\infty)\times\Omega\big)$, then $u, v \in C\big([0,\infty); L^2(\Omega)\big)$ and

$$\|u(t) - v(t)\|_{L^2(\Omega)} \leqslant \|u(s) - v(s)\|_{L^2(\Omega)} + \int_s^t \|f_1(\sigma) - f_2(\sigma)\|_{L^2(\Omega)} \mathrm{d}\sigma, \tag{5.6.11}$$

for any $t \geqslant s \geqslant 0$.

Proof. By Lemma B.2 and the dense embedding $X \hookrightarrow L^2(\Omega)$, we have $L^2(\Omega) \hookrightarrow X^\star \hookrightarrow \mathscr{D}'\Omega)$ and for any $(x,y) \in L^2(\Omega) \times X$,

$$(x,y)_{L^2(\Omega)} = \langle x, y\rangle_{L^2(\Omega),L^2(\Omega)} = \langle x, y\rangle_{X^\star,X}. \tag{5.6.12}$$

It follows from above and (5.4.8) that the equations in the lemma make sense in $X^\star$ and we then have,

$$\mathrm{i}(u-v)_t + \Delta(u-v) + \big(ag(u) - ag(v)\big) = f_1 - f_2, \ \text{ in } X^\star,$$

almost everywhere on $(0,\infty)$, where g is as in Lemma 5.6.2. Taking the $X^\star - X$ duality product of the above equation with $\mathrm{i}(u-v)$, it follows from 2) of Lemma B.4, 1) of Lemma B.5 and (5.6.12) that $u, v \in C\big([0,\infty); L^2(\Omega)\big)$, the mapping $t \longmapsto \|u(t) - v(t)\|^2_{L^2(\Omega)}$ belongs to $W^{1,1}_{\mathrm{loc}}\big([0,\infty); \mathbb{R}\big)$ and,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\,.\,) - v(\,.\,)\|^2_{L^2(\Omega)} + \big\langle ag(u) - ag(v), \mathrm{i}(u-v)\big\rangle_{X^\star,X} = \big(f_1 - f_2, \mathrm{i}(u-v)\big)_{L^2(\Omega)},$$

almost everywhere on $(0,\infty)$. Applying (5.4.9), (5.6.6) and Cauchy-Schwarz's inequality to the above, one infers

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\,.\,)-v(\,.\,)\|_{L^2(\Omega)}^2 \leqslant \|f_1-f_2\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)},$$

almost everywhere on $(0,\infty)$. Integrating over (s,t), one obtains (5.6.11). □

Proof of Theorem 5.4.5. By Lemma 5.6.1 and Vrabie [181] (Theorem 1.7.1), there exists a unique $u\in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)$ satisfying $u(t)\in H^1_0(\Omega)$, $\Delta u(t)\in L^2(\Omega)$ and (5.2.1) in $L^2(\Omega)$, for almost every $t>0$, $u(0)=u_0$ and (5.4.16). Then (5.4.14) comes from (5.4.16). It follows from 1) of Lemma B.4, (5.6.4)–(5.6.5), (5.4.16), (5.4.17) and (5.2.1) that,

$$f\in C\big([0,\infty);L^2(\Omega)\big), \tag{5.6.13}$$
$$|u|^{-(1-m)}u\in C\big([0,\infty);L^2(\Omega)\big), \tag{5.6.14}$$
$$\Delta u\in L^\infty_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big), \tag{5.6.15}$$
$$u\in L^\infty_{\mathrm{loc}}\big([0,\infty);H^1_0(\Omega)\big),$$

so that u is a H^2-solution and $u\in C\big([0,\infty);H^1_0(\Omega)\big)$ (by 3) of Lemma B.4). So,

$$\Delta u\in C\big([0,\infty);H^{-1}(\Omega)\big). \tag{5.6.16}$$

It then follows from (5.6.13), (5.6.14), (5.6.16) and (5.2.1) that,

$$u_t\in C\big([0,\infty);H^{-1}(\Omega)\big).$$

By (5.4.17), (5.4.14) and (5.6.15), one obtains (5.4.15) and Properties 1) and 2) are proved. Property 3) follows easily from Property 1), (B.3) and Proposition 5.2.3. Finally, Property 4) comes from (5.6.11), (5.4.16), (5.4.17), (5.6.4), (5.6.5), the embedding 1) of Lemma B.4 and (5.2.1). This concludes the proof of the theorem. □

Proof of Theorem 5.4.3. Existence comes from density of $H^2_0(\Omega)\times W^{1,1}_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ in $L^2(\Omega)\times L^1_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, Theorem 5.4.5, (5.6.11) and completeness of $C\big([0,T];L^2(\Omega)\big)$, for any $T>0$. Property 1) comes from Proposition 5.2.3. Estimate (5.4.11) being stable by passing to the limit in $C\big([0,T];L^2(\Omega)\big)\times L^1\big((0,T);L^2(\Omega)\big)$, for any $T>0$, it is sufficient to establish it for the H^2-solutions. This then comes from Lemma 5.6.5 and the uniqueness conclusion of the theorem follows. Finally, Property 1) comes from Proposition 5.2.3. □

Proof of Theorem 5.4.4. The uniqueness of solutions comes from Lemma 5.6.5. Let $f\in W^{1,1}_{\mathrm{loc}}([0,\infty);H^1_0(\Omega))$ and let $u_0\in H^1_0(\Omega)$. Let $(\varphi_n)_{n\in\mathbb{N}}\subset H^2_0(\Omega)$ be such that $\varphi_n\xrightarrow[n\to\infty]{H^1_0(\Omega)}u_0$. Finally, let g be defined as in Lemma 5.6.2 and for each $n\in\mathbb{N}$, let u_n be the unique H^2-solution of (5.2.1)–(5.2.2) such that $u_n(0)=\varphi_n$, given by Theorem 5.4.5. By Lemma 5.6.5, we have for any $T>0$ and $n,p\in\mathbb{N}$,

$$\|u_n\|_{C([0,T];L^2(\Omega))}\leqslant\|\varphi_n\|_{L^2(\Omega)}+\int_0^T\|f(t)\|_{L^2(\Omega)}\mathrm{d}t, \tag{5.6.17}$$
$$\|u_n-u_p\|_{L^\infty((0,\infty);L^2(\Omega))}\leqslant\|\varphi_n-\varphi_p\|_{L^2(\Omega)},$$

It follows that for any $T>0$, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,T];L^2(\Omega)\big)$. As a consequence, and with (5.6.4)–(5.6.5), there exists $u\in C\big([0,\infty);L^2(\Omega)\big)$ such that for any $T>0$,

$$u_n \xrightarrow[n\to\infty]{C([0,T];L^2(\Omega))} u, \tag{5.6.18}$$

$$g(u)\in C\big([0,T];L^2(\Omega)\big), \tag{5.6.19}$$

$$g(u_n) \xrightarrow[n\to\infty]{C([0,T];L^2(\Omega))} g(u). \tag{5.6.20}$$

By definition, it follows from (5.6.18) that u is a weak solution of (5.2.1)–(5.2.3) (take $f_n=f$, for any $n\in\mathbb{N}$). By 3) of Remark 5.4.2, we can take the L^2-scalar product of (5.2.1) with $-\mathrm{i}\Delta u_n$ and it follows from (B.4) that for any $n\in\mathbb{N}$ and almost every $s>0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n(s)\|_{L^2(\Omega)}^2+\mathrm{Re}\left(\mathrm{i}a\int\limits_\Omega g(u_n(s))\overline{\Delta u_n(s)}\mathrm{d}x\right)=\big(\nabla f(s),\mathrm{i}\nabla u_n(s)\big)_{L^2(\Omega)},$$

which gives with (5.6.9), Remark 5.6.4 and Cauchy-Schwarz's inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n(s)\|_{L^2(\Omega)}^2\leqslant\|\nabla f_n(s)\|_{L^2(\Omega)}\|\nabla u_n(s)\|_{L^2(\Omega)}.$$

By integration, we obtain for any $t>0$ and any $n\in\mathbb{N}$,

$$\|\nabla u_n(t)\|_{L^2(\Omega)}\leqslant\|\nabla\varphi_n\|_{L^2(\Omega)}+\int_0^t\|\nabla f(s)\|_{L^2(\Omega)}\mathrm{d}s. \tag{5.6.21}$$

By the Sobolev embedding 1) of Lemma B.4,

$$W^{1,1}_{\mathrm{loc}}\big([0,\infty);L^2(\Omega)\big)\hookrightarrow C\big([0,\infty);L^2(\Omega)\big), \tag{5.6.22}$$

(5.6.17), (5.6.20), (5.6.21) and (5.2.1), we infer that,

$$(u_n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\big((0,T);H^1_0(\Omega)\big)\cap W^{1,\infty}\big((0,T);H^{-1}(\Omega)\big), \tag{5.6.23}$$

for any $T>0$. Applying Propositions 1.3.14 and 1.1.2 in Cazenave [57], it follows from (5.6.18) and (5.6.23) that,

$$u\in C_{\mathrm{w}}\big([0,\infty);H^1_0(\Omega)\big)\cap W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);H^{-1}(\Omega)\big), \tag{5.6.24}$$

$$\Delta u\in C\left([0,\infty);H^{-2}(\Omega)\right), \tag{5.6.25}$$

$$u_n(t)\rightharpoonup u(t),\quad\text{in }\ H^1_w(\Omega),\quad\text{as }\ n\to\infty, \tag{5.6.26}$$

for any $t\geqslant 0$. Since u is a weak solution, u solves (5.2.1) in $H^{-2}(\Omega)$, for almost every $t>0$ (Property 5) of Remark 5.4.2). As a consequence, and with help of (5.6.19), (5.6.22) and (5.6.25), we have that $u_t\in C\big([0,\infty);H^{-2}(\Omega)\big)$ and u satisfies (5.2.1) in $H^{-2}(\Omega)$, for any $t\geqslant 0$. We then infer with (5.6.24) that u is a H^1_0-solution and Property 1) holds. Still by (5.6.24), we have for any $t\geqslant s\geqslant 0$,

$$\begin{aligned}\|u(t)-u(s)\|_{L^2(\Omega)}^2&\leqslant 2\|u\|_{L^\infty((s,t);H^1_0(\Omega))}\|u(t)-u(s)\|_{H^{-1}(\Omega)}\\&\leqslant 2\|u\|_{L^\infty((s,t);H^1_0(\Omega))}\|u_t\|_{L^\infty((s,t);H^{-1}(\Omega))}|t-s|,\end{aligned}$$

which is (5.4.12). By (5.6.26), the weak lower semicontinuity of the norm and (5.6.21), one obtains (5.4.13) and Property 2) is proved. Property 3) follows easily from Proposition 5.2.3 and the fact that $u,f\in C\big([0,\infty);L^2(\Omega)\big)$ and $L^2(\Omega)\hookrightarrow L^{m+1}(\Omega)$. Finally, Property 4) comes from (5.4.11), (5.4.13), (5.6.4), (5.6.5), 1) of Lemma B.4 and (5.2.1). This concludes the proof of the theorem. □

5.7 Proofs of the finite time extinction property and asymptotic behavior theorems

Proof of Theorem 5.3.2. For the Property *a*), apply Theorems 5.4.4, 5.4.5, Remark 5.4.6 and Theorem 5.2.1 (with $\ell=1$, if $u_0\in H^1_0(\Omega)$ and $\ell=2$, if $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$). We then obtain the finite time extinction result and the upper bound on $T_\star$. The lower bound on $T_\star$ comes from 2) of Remark 5.2.4. Property *b*) comes from Remark 5.4.8. ☐

Proof of Theorem 5.3.5. By Theorems 5.4.4, 5.4.5 and Remark 5.4.6, $u\in L^\infty\big((0,\infty);H^\ell(\Omega)\big)$, where $\ell=1$, if $u_0\in H^1_0(\Omega)$ and $\ell=2$, if $u_0\in H^2(\Omega)\cap H^1_0(\Omega)$. The result then comes from 3) of Remark 5.2.4. ☐

Proof of Theorem 5.3.6. Let the assumptions of the theorem be fulfilled. We proceed to the proof in two steps.

Step 1. Assume further that $f\in\mathscr{D}\big([0,\infty);L^2(\Omega)\big)$ and $u_0\in H^2_0(\Omega)$. Then, $\lim\limits_{t\nearrow\infty}\|u(t)\|_{L^2(\Omega)}=0$.

It follows from uniqueness and Theorem 5.4.5 that u is a H^2-solution and $u\in L^\infty\big((0,\infty);H^1_0(\Omega)\big)$. Let $[0,T_0]\supset\operatorname{supp}f$. By (5.4.10), $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}\leqslant 0$, for any $t>T_0$. It follows that $\lim\limits_{t\nearrow\infty}\|u(t)\|_{L^2(\Omega)}=\ell_0$, for some $\ell_0\in[0,\infty)$. If $m=1$ then we have, one more time by (5.4.10), $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}\leqslant-2\mathrm{Im}(a)\ell_0^2$, for any $t>T_0$. It follows that $\ell_0=0$. Now, assume that $m<1$ and suppose, by contradiction, that $\ell_0\neq0$. Let $q\in(2,\infty)$ with $(N-2)q<2N$. By Hölder's inequality and Sobolev's embedding $H^1_0(\Omega)\hookrightarrow L^q(\Omega)$, there exists $\theta\in(0,1)$ such that,

$$0<\ell_0\leqslant\|u(t)\|_{L^2(\Omega)}\leqslant\|u(t)\|^\theta_{L^{m+1}(\Omega)}\|u(t)\|^{1-\theta}_{L^q(\Omega)}\leqslant C\|u(t)\|^\theta_{L^{m+1}(\Omega)}\|u\|^{1-\theta}_{L^\infty((0,\infty);H^1_0(\Omega))},$$

for any $t>T_0$. We infer that, $\inf\limits_{t>T_0}\|u(t)\|_{L^{m+1}(\Omega)}>0$, which implies with (5.4.10),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\Omega)}\leqslant-2\mathrm{Im}(a)\inf_{t>T_0}\|u(t)\|^{m+1}_{L^{m+1}(\Omega)}<0,$$

for any $t>T_0$. As a consequence, $\lim\limits_{t\nearrow\infty}\|u(t)\|_{L^2(\Omega)}=-\infty$, a contradiction.

Step 2. Conclusion.

Let $(\varphi_n)_{n\in\mathbb{N}}\subset H^2_0(\Omega)$ and $(f_n)_{n\in\mathbb{N}}\subset\mathscr{D}\big([0,\infty);L^2(\Omega)\big)$ be such that,

$$\varphi_n\xrightarrow[n\to\infty]{L^2(\Omega)}u_0\quad\text{and}\quad f_n\xrightarrow[n\to\infty]{L^1((0,\infty);L^2(\Omega))}f.$$

For each $n\in\mathbb{N}$, let u_n the H^2-solution to (5.2.1)–(5.2.2), with f_n instead of f, be such that $u_n(0)=\varphi_n$, given by Theorem 5.4.5. Let $n\in\mathbb{N}$. It follows from (5.4.11) that,

$$\begin{aligned}\|u(t)\|_{L^2(\Omega)}&\leqslant\|u-u_n\|_{L^\infty((0,\infty);L^2(\Omega))}+\|u_n(t)\|_{L^2(\Omega)}\\&\leqslant\|u_0-\varphi_n\|_{L^2(\Omega)}+\|f-f_n\|_{L^1((0,\infty);L^2(\Omega))}+\|u_n(t)\|_{L^2(\Omega)},\end{aligned}$$

for any $t>0$. We get from Step 1,

$$\limsup_{t\nearrow\infty}\|u(t)\|_{L^2(\Omega)}\leqslant\|u_0-\varphi_n\|_{L^2(\Omega)}+\|f-f_n\|_{L^1((0,\infty);L^2(\Omega))}.$$

Letting $n \nearrow \infty$, we obtain $\lim\limits_{t\nearrow\infty} \|u(t)\|_{L^2(\Omega)} = 0$. Finally, the general case comes from the embedding $L^2(\Omega) \hookrightarrow L^p(\Omega)$, which holds for any $p \in (0, 2]$, as soon as $|\Omega| < \infty$. This concludes the proof. □

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Chapitre 6

Finite time extinction for a damped nonlinear Schrödinger equation in the whole space

Abstract

We consider a nonlinear Schrödinger equation set in the whole space with a single power of interaction and an external source. We first establish existence and uniqueness of the solutions and then show, in low space dimension, that the solutions vanish at a finite time. Under a smallness hypothesis of the initial data and some suitable additional assumptions on the external source, we also show that we can choose the upper bound on which time the solutions vanish.

6.1 Introduction and explanation of the method

Let us consider the following Schrödinger equation with a nonlinear damping term,

$$\mathrm{i}u_t + \Delta u + a|u|^{m-1}u = f(t,x), \quad \text{in } (0,\infty)\times\Omega, \tag{6.1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open subset, $a \in \mathbb{C}$, $0 < m < 1$ and $f : (0,\infty)\times\Omega \longrightarrow \mathbb{C}$ measurable is an external source. When $a \in \mathbb{R}$, $m \geqslant 1$ and $f = 0$, equation (6.1.1) has been intensively studied, especially with $\Omega = \mathbb{R}^N$ (among which existence, uniqueness, blow-up, scattering theory, time decay). The literature is too extensive to give an exhaustive list. See, for instance, the monographs of Cazenave [57], Sulem and Sulem [165], Tao [167] and the references therein. The case $a \in \mathbb{C}$ is more anecdotic. See, for instance, Bardos and Brezis [17], Lions [128], Tsutsumi [171] and Shimomura [159]. Note that except in [128], it is always assumed $m > 1$.

In this paper, we are looking for solutions which vanishes at a finite time. For many reasons, we have to consider $0 < m < 1$. When $m = 1$, existence is not hard to obtain, since the equation is linear, while the finite time property is not possible (which is a direct consequence of (6.1.4)). To our knowledge the first paper in this direction is due to Carles and Gallo [53] with $a = \mathrm{i}$, $f = 0$ and Ω is a compact manifold without boundary. To construct solutions, they regularize the nonlinearity and use

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a compactness method to pass in the limit. They prove the finite time extinction property for $N \leqslant 3$ including the case $m = 0$. More recently, Carles and Ozawa [55] obtain the existence, uniqueness and finite time extinction for $\Omega = \mathbb{R}^N$, $a \in \mathrm{i}\mathbb{R}_+$ and $f = 0$. Due to the lack of compactness, they restrict their study to $N \leqslant 2$ and add an harmonic confinement in (6.1.1) for some technical reasons. For the finite time property with $N = 2$ they also restrict the range of m to $\left[\frac{1}{2}, 1\right)$ and make a smallness assumption of the initial data. In this paper, we work in the whole space and we remove of all these restrictions and extend the previous results to a large class of values of a (see, for instance, Theorems 6.2.7 and 6.3.1). Indeed, we shall assume that the complex number a is in a cone of the complex plane. More precisely,

$$a \in C(m) \overset{\text{def}}{=} \Big\{ z \in \mathbb{C};\ \mathrm{Im}(z) > 0 \ \text{ and } \ 2\sqrt{m}\,\mathrm{Im}(z) \geqslant (1-m)|\mathrm{Re}(z)| \Big\}. \tag{6.1.2}$$

The assumption that a belongs to the cone $C(m)$ was considered in a series of papers by Okazawa and Yokota [145, 146, 147]. They studied the asymptotic behavior of the solutions to the complex Ginzburg-Landau equation in a bounded domain with the assumption (6.1.2) and, sometimes, with $m > 1$. See also Kita and Shimomura [120] and Hou, Jiang, Li and You [108] where (6.1.2) is assumed but with (among others restrictive assumptions) $m > 1$. In all these papers, there is no finite time extinction result. We would also like mention the (very complete) work of Antontsev, Dias and Figueira [10] where they consider the complex Ginzburg-Landau equation,

$$e^{-\mathrm{i}\gamma} u_t - \Delta u + |u|^{m-1} u = f(t,x), \ \text{ in } (0,\infty) \times \Omega, \tag{6.1.3}$$

where Ω is bounded, $0 < m < 1$ and $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$. In particular, $e^{-\mathrm{i}\gamma} \neq \pm\mathrm{i}$. They show spatial localization, waiting time and finite time extinction properties. The case of equation (6.1.3) with a delayed nonlocal perturbation is studied in the recent paper of Díaz, Padial, Tello and Tello [76]. Finally, Hayashi, Li and Naumkin [104] study time decay for a more classical Schrödinger equation (6.1.1) (a satisfying (6.1.2), $m > 1$ and $\Omega = \mathbb{R}^N$).

In this paper, we are interested in the finite time extinction of the solution. Formally, this result is not too hard to obtain (the method we explain below for the finite time extinction property is that used in [53, 55, 29]). Suppose $f = 0$. It is well known that solutions that vanish in finite time do not exist when $m \geqslant 1$ (at least when $a \in \mathbb{R}$). Indeed, multiplying (6.1.1) by $\overline{\mathrm{i}u}$, integrating by parts and taking the real part, we obtain,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{L^2}^2 + \mathrm{Im}(a)\|u(t)\|_{L^{m+1}}^{m+1} = 0. \tag{6.1.4}$$

To expect a finite time extinction, the mass has to be non increasing and so $\mathrm{Im}(a) > 0$. Now, since $m+1 < 2$, we may interpolate L^2 between L^{m+1} and L^p, for some $p > 2$, and control the L^p-norm by a Sobolev norm. Using a Gagliardo-Nirenberg's inequality,

$$\|u(t)\|_{L^2}^{2\frac{m+1}{2\theta_\ell}} \leqslant \|u(t)\|_{L^{m+1}}^{m+1} \|u(t)\|_{H^\ell}^{\frac{(m+1)(1-\theta_\ell)}{\theta_\ell}}, \tag{6.1.5}$$

for some an explicit constant $\theta_\ell \in (0,1)$, if u is bounded in H^ℓ then putting together (6.1.4)–(6.1.5), we arrive at the ordinary differential equation,

$$y' + Cy^\delta \leqslant 0, \tag{6.1.6}$$

with $\delta=\frac{m+1}{2\theta_\ell}$, where $y(t)=\|u(t)\|_{L^2}^2$. By integration, we then obtain the asymptotic behavior of u with respect to the value of δ.

- If $\delta<1$ then $y(t)^{1-\delta}\leqslant(y(0)^{1-\delta}-Ct)_+$ and so u vanishes before time $T_\star=C^{-1}y(0)^{1-\delta}$.
- If $\delta=1$ then $y(t)\leqslant y(0)e^{-Ct}$.
- If $\delta>1$ then $y(t)^{\delta-1}\leqslant y(0)^{\delta-1}(1+Ct)^{-1}$.

As a consequence, a sufficient condition to have extinction in finite time is $\delta<1$ which turns out to be equivalent to $N=1$ when $\ell=1$. To increase the space dimension, we assume that u is bounded in H^2 and we deduce that $\delta<1$ when $N\leqslant 3$. Theoretically, we can reach any space dimension if u is bounded in H^ℓ for ℓ large enough (actually, if $\ell=\left[\frac{N}{2}\right]+1$, where $\left[\frac{N}{2}\right]$ denotes the integer part of $\frac{N}{2}$; see Theorem 5.2.1). But this is not reasonable due to the lack of regularity of the nonlinearity, which is merely Hölder continuous. A reachable goal is to obtain existence and boundedness of the solutions in H^2.

Now, we focus on the construction of a solution to (6.1.1) in $\mathbb{R}^N$ with $f=0$ (to fix ideas). First of all, we would like to uniformly control $\|u(t)\|_{H^1}^2$. Estimate (6.1.4) partially answers this question. For $\|\nabla u(t)\|_{L^2}^2$, we multiply (6.1.1) by $\mathrm{i}\overline{\Delta u}$ and take the real part. We get,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u(t)\|_{L^2}^2+\operatorname{Re}\left(\mathrm{i}a\int\limits_{\mathbb{R}^N}|u(t)|^{m-1}u(t)\overline{\Delta u(t)}\mathrm{d}x\right)=0.$$

We then expect to have,

$$\operatorname{Re}\left(\mathrm{i}a\int\limits_{\mathbb{R}^N}|u(t)|^{m-1}u(t)\overline{\Delta u(t)}\mathrm{d}x\right)\geqslant 0. \tag{6.1.7}$$

Regularizing the nonlinearity, integrating by parts and passing to the limit, (6.1.7) can be proved under assumption (6.1.2) (Lemma 6.4.4). Actually, we extended the method found in Carles and Gallo [53], where the situation is simpler since $a=\mathrm{i}$. Assume $\Omega\subseteq\mathbb{R}^N$. To construct a solution to (6.1.1), we use theory of the maximal monotone operators in the Hilbert space L^2. We then consider the operator,

$$Au=-\mathrm{i}\Delta u-\mathrm{i}a|u|^{m-1}u, \tag{6.1.8}$$

with the natural domain [1] $D(A)=\left\{u\in H_0^1(\Omega);u^m\in L^2(\Omega)\text{ and }\Delta u\in L^2(\Omega)\right\}$. Monotonicity relies on the inequality,

$$\operatorname{Re}\left(-\mathrm{i}\,a\int\limits_{\Omega}\left(|u|^{m-1}u-|v|^{m-1}v\right)(\overline{u-v})\mathrm{d}x\right)\geqslant 0. \tag{6.1.9}$$

Once (6.1.9) is proved, it remains to show that $R(I+A)=L^2$ (Theorem 6.4.1 and Corollary 6.4.5). This means that for any $F\in L^2$, the equation

$$-\mathrm{i}\Delta u-\mathrm{i}a|u|^{m-1}u+u=F, \tag{6.1.10}$$

admits a solution belonging to $D(A)$. Existence, uniqueness, *a priori* estimates and smoothness of the solutions of (6.1.10) for a large class of values of a (including (6.1.2)) have been intensively studied

1. It is natural in the sense that it is the smallest domain, in the sense of the inclusion, for which $D(A)\subset L^2$.

in the papers by Bégout and Díaz [25, 28]. The natural [2] space to look for a solution is $H^1_0 \cap L^{m+1}$. When Ω is bounded with a smooth boundary, a bootstrap method yields $u \in H^2(\Omega)$. Note that in this case, the condition $u^m \in L^2(\Omega)$ is automatically verified since $u^m \in L^{\frac{2}{m}}(\Omega) \hookrightarrow L^2(\Omega)$ and then $u \in D(A)$. Although this method works very well, we proposed another one in Bégout and Díaz [29] : we make the sum of two monotone operators, where one of them is maximal monotone $(-\mathrm{i}\Delta)$ and the other one is continuous over $L^2(\Omega)$ $(-\mathrm{i}a|u|^{m-1}u)$. A difficulty appears when Ω is unbounded, say $\Omega = \mathbb{R}^N$. In this case, we have $D(A) = H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$ and we have to show that a solution $u \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ belongs to $L^{2m}(\mathbb{R}^N)$, or equivalently $\Delta u \in L^2(\mathbb{R}^N)$. Having (6.1.7) in mind, a natural method would be to multiply (6.1.10) by $-\overline{\Delta u}$ and take the real part. But then we lose the term $\|\Delta u\|^2_{L^2(\mathbb{R}^N)}$. The original idea is to rotate a in the complex plane and stay in the cone $C(m)$ to still have (6.1.7) (see Lemma 6.4.2 and the picture p.111). If we can find $b \in \mathbb{C}$ such that $ab \in C(m)$ then multiplying (6.1.10) by $-b\overline{\Delta u}$, integrating by parts and taking the real part, we arrive at,

$$-\mathrm{Im}(b)\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \mathrm{Re}\left(\mathrm{i}ab\int\limits_{\mathbb{R}^N}|u|^{m-1}u\overline{\Delta u}\mathrm{d}x\right) + \mathrm{Re}(b)\|\nabla u\|^2_{L^2(\mathbb{R}^N)} = -\mathrm{Re}\left(b\int\limits_{\mathbb{R}^N}F\overline{\Delta u}\mathrm{d}x\right).$$

We see that we must have $\mathrm{Im}(b) < 0$ and so the rotation has to be made in the negative sense. So we exclude the boundary of $C(m)$ located in the first quarter complex plane.Hence Assumption 6.2.1. Note that the sign of $\mathrm{Re}(b)$ has no importance since we already have an estimate in $H^1(\mathbb{R}^N)$. Having *a priori* estimates, we may construct a solution $u \in H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$ of (6.1.10) as a limit of solutions with compact support. The existence of such solutions is provided in Bégout and Díaz [25] (see also Bégout and Díaz [27]). To conclude the explanation of our method, we go back to the proof of (6.1.9). When $a = \mathrm{i}$, this is very simple since this estimate is equivalent to the monotonicity of the derivative of the convex function defined on $\mathbb{R}^2$ by, $(x,y) \longmapsto \frac{1}{m+1}(x^2+y^2)^{\frac{m+1}{2}}$ (see Remark 1.9.3). But when $\mathrm{Re}(a) \neq 0$ then the imaginary part of the integral in (6.1.9) is still there. Fortunately, this can be controlled by its real part under assumption (6.1.2) and a consequence of Liskevich and Perel′muter [132] (Lemma 2.2).

Finally, we consider the limit cases $m = 0$ and $m = 1$ for the values of a. Since $\lim\limits_{m\searrow 0} C(m) = \{0\} \times \mathrm{i}(0,\infty)$, it seems that no extension of [53, 55] is possible. The other limit case $\lim\limits_{m\nearrow 1} C(m) = \mathbb{R} \times \mathrm{i}(0,\infty)$ is entirely treated in Bégout and Díaz [29] : existence, uniqueness and boundedness for any subset $\Omega \subseteq \mathbb{R}^N$.

We will use the following notations throughout this paper. We denote by $\overline{z}$ the conjugate of the complex number z, by $\mathrm{Re}(z)$ its real part and by $\mathrm{Im}(z)$ its imaginary part. Unless if specified, all functions are complex-valued ($H^1(\Omega) = H^1(\Omega;\mathbb{C})$, etc). For $1 \leqslant p \leqslant \infty$, p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$. For a Banach space X, we denote by $X^\star$ its topological dual and by $\langle\, .\, ,\, .\,\rangle_{X^\star,X} \in \mathbb{R}$ the $X^\star - X$ duality product. In particular, for any $T \in L^{p'}(\Omega)$ and $\varphi \in L^p(\Omega)$ with $1 \leqslant p < \infty$, $\langle T,\varphi\rangle_{L^{p'}(\Omega),L^p(\Omega)} = \mathrm{Re}\int_\Omega T(x)\overline{\varphi(x)}\mathrm{d}x$. The scalar product in $L^2(\Omega)$ between two functions u, v is, $(u,v)_{L^2(\Omega)} = \mathrm{Re}\int_\Omega u(x)\overline{v(x)}\mathrm{d}x$. For a Banach space X and $p \in [1,\infty]$, $u \in L^p_{\mathrm{loc}}\big([0,\infty);X\big)$ means that for any $T > 0$, $u_{|(0,T)} \in L^p\big((0,T);X\big)$. In the same way, we will use the notation $u \in W^{1,p}_{\mathrm{loc}}\big([0,\infty);X\big)$. As usual, we denote by C auxiliary positive constants, and sometimes, for

2. Multiply (6.1.10) by $\overline{\mathrm{i}u}$ and $\overline{u}$, integrate by parts and take the real part.

positive parameters $a_1,\dots,a_n$, write as $C(a_1,\dots,a_n)$ to indicate that the constant C depends only on $a_1,\dots,a_n$ and that dependence is continuous (we will use this convention for constants which are not denoted by "C").

This paper is organized as follows. In Section 6.2, we state the mains results about existence, uniqueness and boundness for (6.1.1) (Theorem 6.2.4, 6.2.6 and 6.2.7). In Section 6.3, we give the results about the finite time extinction property and the asymptotic behavior (Theorems 6.3.1, 6.3.4 and 6.3.5). The proofs of the existence, uniqueness and boundness are made in Section 6.4 while those of the finite time extinction property and the asymptotic behavior are given in Section 6.5.

6.2 Existence and uniqueness of the solutions

Let $0<m<1$, let $a\in\mathbb{C}$, let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$ and let $u_0\in L^2(\mathbb{R}^N)$. We consider the following nonlinear Schrödinger equation.

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t}+\Delta u+a|u|^{-(1-m)}u=f(t,x), \text{ in } (0,\infty)\times\mathbb{R}^N, & (6.2.1)\\ u(0)=u_0, \text{ in } \mathbb{R}^N, & (6.2.2)\end{cases}$$

The main results in this paper hold with the assumptions below.

Assumption 6.2.1. We assume that $0<m<1$ and $a\in\mathbb{C}$ satisfy,

$$2\sqrt{m}\,\mathrm{Im}(a)\geqslant(1-m)|\mathrm{Re}(a)|. \tag{6.2.3}$$

If $\mathrm{Re}(a)\geqslant0$ then we assume further that,

$$2\sqrt{m}\,\mathrm{Im}(a)>(1-m)\mathrm{Re}(a). \tag{6.2.4}$$

Here and after, we shall always identify $L^2(\mathbb{R}^N)$ with its topological dual. Let $0<m<1$ and let $X=H\cap L^{m+1}(\mathbb{R}^N)$, where $H=L^2(\mathbb{R}^N)$ or $H=H^1(\mathbb{R}^N)$. We recall that (Lemmas B.2 and B.4),

$$X^\star=H^\star+L^{\frac{m+1}{m}}(\mathbb{R}^N), \tag{6.2.5}$$

$$\mathscr{D}(\mathbb{R}^N)\hookrightarrow X\hookrightarrow L^{m+1}(\mathbb{R}^N) \text{ with both dense embeddings}, \tag{6.2.6}$$

$$L^{\frac{m+1}{m}}(\mathbb{R}^N)\hookrightarrow X^\star\hookrightarrow\mathscr{D}'(\mathbb{R}^N) \text{ with both dense embeddings}, \tag{6.2.7}$$

$$L^{m+1}_{\mathrm{loc}}\big([0,\infty);X\big)\cap W^{1,\frac{m+1}{m}}_{\mathrm{loc}}\big([0,\infty);X^\star\big)\hookrightarrow C\big([0,\infty);L^2(\mathbb{R}^N)\big). \tag{6.2.8}$$

This justifies the notion of solution below (and especially 4)).

Definition 6.2.2. Let $0<m<1$, let $a\in\mathbb{C}$, let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$ and let $u_0\in L^2(\mathbb{R}^N)$. Let us consider the following assertions.

1) $u\in L^{m+1}_{\mathrm{loc}}\big([0,\infty);H^1(\mathbb{R}^N)\cap L^{m+1}(\mathbb{R}^N)\big)\cap W^{1,\frac{m+1}{m}}_{\mathrm{loc}}\big([0,\infty);H^\star+L^{\frac{m+1}{m}}(\mathbb{R}^N)\big)$,
2) For almost every $t>0$, $\Delta u(t)\in H^\star$.
3) u satisfies (6.2.1) in $\mathscr{D}'\big((0,\infty)\times\mathbb{R}^N\big)$.

4) $u(0)=u_0$.

We shall say that u is a *strong solution* if u is an H^2-solution or an H^1-solution. We shall say that u is an H^2-*solution of* (6.2.1)–(6.2.2) $\big($respectively, an H^1-*solution of* (6.2.1)–(6.2.2)$\big)$, if u satisfies the Assertions 1)–4) with $H=L^2(\mathbb{R}^N)$ $\big($respectively, with $H=H^1(\mathbb{R}^N)\big)$.
We shall say that u is a L^2-*solution* or a *weak solution* of (6.2.1)–(6.2.2) is there exists a pair,

$$(f_n,u_n)_{n\in\mathbb{N}}\subset L^1_{\text{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)\times C\big([0,\infty);L^2(\mathbb{R}^N)\big), \tag{6.2.9}$$

such that for any $n\in\mathbb{N}$, u_n is an H^2-solution of (6.2.1) where the right-hand side of (6.2.1) is f_n, and if

$$f_n \xrightarrow[n\to\infty]{L^1((0,T);L^2(\mathbb{R}^N))} f \text{ and } u_n \xrightarrow[n\to\infty]{C([0,T];L^2(\mathbb{R}^N))} u, \tag{6.2.10}$$

for any $T>0$, and if u satisfies (6.2.2).

Remark 6.2.3. Let $0<m<1$. Set for any $z\in\mathbb{C}$, $g(z)=|z|^{-(1-m)}z$ $(g(0)=0)$. We define the mapping for any measurable function $u:\mathbb{R}^N\longrightarrow\mathbb{C}$, which we still denote by g, by $g(u)(x)=g(u(x))$. Let X be as in the beginning of this section (see (6.2.5)–(6.2.8)). From (6.2.6), (6.2.7) and the basic estimate,

$$\forall(z_1,z_2)\in\mathbb{C}^2,\ |g(z_1)-g(z_2)|\leqslant C|z_1-z_2|^m, \tag{6.2.11}$$

we deduce easily that,

$$g\in C\big(L^{m+1}(\mathbb{R}^N);L^{\frac{m+1}{m}}(\mathbb{R}^N)\big) \text{ and } g \text{ is bounded on bounded sets}, \tag{6.2.12}$$

$$g\in C(X;X^\star) \text{ and } g \text{ is bounded on bounded sets}. \tag{6.2.13}$$

By (6.2.6)–(6.2.7) and (6.2.12)–(6.2.13), it follows that,

$$\langle g(u),v\rangle_{X^\star,X}=\langle g(u),v\rangle_{L^{\frac{m+1}{m}}(\mathbb{R}^N),L^{m+1}(\mathbb{R}^N)}=\operatorname{Re}\int\limits_{\mathbb{R}^N} g(u)\overline{v}\mathrm{d}x, \tag{6.2.14}$$

for any $u,v\in X$. Now, let us collect some basic informations about the solutions.

1) Any strong or weak solution belongs to $C\big([0,\infty);L^2(\mathbb{R}^N)\big)$ and Assertion 4) makes sense in $L^2(\mathbb{R}^N)$ (by (6.2.8)).
2) It is obvious that an H^2-solution is also an H^1-solution and a weak solution. But it is not clear that an H^1-solution is a weak solution, without a continuous dependence of the solution with respect to the initial data. Such a result will be established with the additional assumptions (6.2.3)–(6.2.4) on a (see Lemma 6.4.6 below). Note also that Assertion 2) of Definition 6.2.2 is not an additional assumption for the H^1-solutions.
3) Any H^2-solution (respectively, any H^1-solution) satisfies (6.2.1) in $L^2(\mathbb{R}^N)+L^{\frac{m+1}{m}}(\mathbb{R}^N)$ $\big($respectively, in $H^{-1}(\mathbb{R}^N)+L^{\frac{m+1}{m}}(\mathbb{R}^N)\big)$, for almost every $t>0$. Indeed, this is a direct consequence of Definition 6.2.2 and (6.2.13).

4) If u is a weak solution then $u\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);Y^\star\big)$ and it solves (6.2.1) in $Y^\star$, for almost every $t>0$, where $Y=H^2(\mathbb{R}^N)\cap L^{\frac{2}{2-m}}(\mathbb{R}^N)$ and $Y^\star=H^{-2}(\mathbb{R}^N)+L^{\frac{2}{m}}(\mathbb{R}^N)\hookrightarrow \mathscr{D}'(\mathbb{R}^N)$ (by Lemma B.2). Indeed, using the notation of Definition 6.2.2 and (6.2.11), this comes from (6.2.10) and the uniform convergences,

$$\Delta u_n \xrightarrow[n\to\infty]{C([0,T];H^{-2}(\mathbb{R}^N))} \Delta u, \tag{6.2.15}$$

$$g(u_n) \xrightarrow[n\to\infty]{C([0,T];L^{\frac{2}{m}}(\mathbb{R}^N))} g(u), \tag{6.2.16}$$

for any $T>0$. In particular, u solves (6.2.1) in $\mathscr{D}'\big((0,\infty)\times\mathbb{R}^N\big)$.

Theorem 6.2.4 (**Existence and uniqueness of $\boldsymbol{L^2}$-solutions**). *Let Assumption 6.2.1 be fulfilled and let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$. Then for any $u_0\in L^2(\mathbb{R}^N)$, there exists a unique weak solution u to (6.2.1)–(6.2.2). In addition,*

$$u\in L^{m+1}_{\mathrm{loc}}\big([0,\infty);L^{m+1}(\mathbb{R}^N)\big), \tag{6.2.17}$$

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)}+\mathrm{Im}(a)\int\limits_s^t\|u(\sigma)\|^{m+1}_{L^{m+1}(\mathbb{R}^N)}\mathrm{d}\sigma\leqslant\frac{1}{2}\|u(s)\|^2_{L^2(\mathbb{R}^N)}+\mathrm{Im}\int\limits_s^t\int\limits_{\mathbb{R}^N} f(\sigma,x)\,\overline{u(\sigma,x)}\,\mathrm{d}x\,\mathrm{d}\sigma, \tag{6.2.18}$$

for any $t\geqslant s\geqslant 0$. Finally, if v is a weak solution of (6.2.1) with $v(0)=v_0\in L^2(\mathbb{R}^N)$ and $g\in L^1_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^N))$ instead of f in (6.2.1) then,

$$\|u(t)-v(t)\|_{L^2(\mathbb{R}^N)}\leqslant\|u(s)-v(s)\|_{L^2(\mathbb{R}^N)}+\int\limits_s^t\|f(\sigma)-g(\sigma)\|_{L^2(\mathbb{R}^N)}\mathrm{d}\sigma, \tag{6.2.19}$$

for any $t\geqslant s\geqslant 0$.

Remark 6.2.5. Let Assumption 6.2.1 be fulfilled. Let $p\in[m+1,2)$. It follows from (6.2.18) and Hölder's and Young's inequalities that if $f\in L^1\big((0,\infty);L^2(\mathbb{R}^N)\big)$ then,

$$u\in L^\infty\big((0,\infty);L^2(\mathbb{R}^N)\big)\cap L^{m+1}\big((0,\infty);L^{m+1}(\mathbb{R}^N)\big).$$

By interpolation, we infer that for any $p\in[m+1,2)$,

$$u\in C_{\mathrm{b}}\big([0,\infty);L^2(\mathbb{R}^N)\big)\cap L^{\frac{p(1-m)}{2-p}}\big((0,\infty);L^p(\mathbb{R}^N)\big). \tag{6.2.20}$$

If, in addition, $(\varphi_n)_{n\in\mathbb{N}}\subset L^2(\mathbb{R}^N)$, $(f_n)_{n\in\mathbb{N}}\subset L^1\big((0,\infty);L^2(\mathbb{R}^N)\big)$ and,

$$\varphi_n\xrightarrow[n\to\infty]{L^2(\mathbb{R}^N)}u_0\quad\text{and}\quad f_n\xrightarrow[n\to\infty]{L^1((0,\infty);L^2(\mathbb{R}^N))}f,$$

then by (6.2.19), (6.2.20) and again by interpolation, we have for any $p\in(m+1,2)$,

$$u_n\xrightarrow[n\to\infty]{C_{\mathrm{b}}([0,\infty);L^2(\mathbb{R}^N))\cap L^{\frac{p(1-m)}{2-p}}((0,\infty);L^p(\mathbb{R}^N))}u,$$

where for each $n\in\mathbb{N}$, u_n is the weak solution of (6.2.1) with $u_n(0)=\varphi_n$ and f_n instead of f.

Theorem 6.2.6 (Existence and uniqueness of H^1-solutions). *Let Assumption 6.2.1 be fulfilled and let $f \in W^{1,1}_{\mathrm{loc}}\big([0,\infty);H^1(\mathbb{R}^N)\big)$. Then for any $u_0 \in H^1(\mathbb{R}^N)$, there exists a unique H^1-solution u to (6.2.1)–(6.2.2). Furthermore, u is also a weak solution and satisfies the following properties.*

1) *$u \in C\big([0,\infty);L^2(\mathbb{R}^N)\big) \cap C^1\big([0,\infty);Y^\star\big)$ and u satisfies (6.2.1) in $Y^\star$, for any $t \geqslant 0$, where $Y^\star = H^{-2}(\mathbb{R}^N) + L^{\frac{2}{m}}(\mathbb{R}^N)$.*
2) *$u \in C_{\mathrm{w}}\big([0,\infty);H^1(\mathbb{R}^N)\big) \cap W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);H^{-1}(\mathbb{R}^N)+L^{\frac{2}{m}}(\mathbb{R}^N)\big)$ and,*
$$\|\nabla u(t)\|_{L^2(\mathbb{R}^N)} \leqslant \|\nabla u_0\|_{L^2(\mathbb{R}^N)} + \int_0^t \|\nabla f(s)\|_{L^2(\mathbb{R}^N)}\mathrm{d}s, \tag{6.2.21}$$
for any $t \geqslant 0$.
3) *The map $t \longmapsto \|u(t)\|^2_{L^2(\mathbb{R}^N)}$ belongs to $W^{1,1}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ and we have,*
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^N)} + \mathrm{Im}(a)\|u(t)\|^{m+1}_{L^{m+1}(\mathbb{R}^N)} = \mathrm{Im}\int\limits_{\mathbb{R}^N} f(t,x)\,\overline{u(t,x)}\,\mathrm{d}x, \tag{6.2.22}$$
for almost every $t > 0$.

Theorem 6.2.7 (Existence and uniqueness of H^2-solutions). *Let Assumption 6.2.1 be fulfilled and let $f \in W^{1,1}_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$. Then for any $u_0 \in H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$, there exists a unique H^2-solution u to (6.2.1)–(6.2.2). Furthermore, u satisfies (6.2.1) in $L^2(\mathbb{R}^N)$, for almost every $t > 0$, and the following properties.*

1) *$u \in C\big([0,\infty);H^1(\mathbb{R}^N)\cap L^{m+1}(\mathbb{R}^N)\big) \cap C^1\big([0,\infty);H^{-1}(\mathbb{R}^N)+L^{\frac{m+1}{m}}(\mathbb{R}^N)\big)$ and u satisfies (6.2.1) in $H^{-1}(\mathbb{R}^N)+L^{\frac{m+1}{m}}(\mathbb{R}^N)$, for any $t \geqslant 0$.*
2) *$u \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big) \cap L^\infty_{\mathrm{loc}}\big([0,\infty);H^2(\mathbb{R}^N)\cap L^{2m}(\mathbb{R}^N)\big)$ and,*
$$\begin{cases} \|u(t)-u(s)\|_{L^2(\mathbb{R}^N)} \leqslant \|u_t\|_{L^\infty((s,t);L^2(\mathbb{R}^N))}|t-s|, & (6.2.23)\\ \|\nabla u(t)-\nabla u(s)\|_{L^2(\mathbb{R}^N)} \leqslant M|t-s|^{\frac{1}{2}}, & (6.2.24)\\ \|u_t\|_{L^\infty((0,t);L^2(\mathbb{R}^N))} \leqslant \|\Delta u_0 + a|u_0|^{m-1}u_0 - f(0)\|_{L^2(\mathbb{R}^N)} + \displaystyle\int_0^t \|f'(\sigma)\|_{L^2(\mathbb{R}^N)}\mathrm{d}\sigma, & (6.2.25) \end{cases}$$
for any $t \geqslant s \geqslant 0$, where $M^2 = 2\|u_t\|_{L^\infty((s,t);L^2(\mathbb{R}^N))}\|\Delta u\|_{L^\infty((s,t);L^2(\mathbb{R}^N))}$.
3) *The map $t \longmapsto \|u(t)\|^2_{L^2(\mathbb{R}^N)}$ belongs to $C^1\big([0,\infty);\mathbb{R}\big)$ and (6.2.22) holds for any $t \geqslant 0$.*
4) *If $f \in W^{1,1}\big((0,\infty);L^2(\mathbb{R}^N)\big)$ then we have,*
$$u \in C_{\mathrm{b}}\big([0,\infty);H^1(\mathbb{R}^N)\big) \cap L^\infty\big((0,\infty);H^2(\mathbb{R}^N)\cap L^{2m}(\mathbb{R}^N)\big) \cap W^{1,\infty}\big((0,\infty);L^2(\mathbb{R}^N)\big).$$

Remark 6.2.8. It follows from Lemma B.4 below that $f \in C\big([0,\infty);L^2(\mathbb{R}^N)\big)$ and so, estimate (6.2.25) with $f(0)$ makes sense.

Remark 6.2.9. We recall that if $u \in L^2(\mathbb{R}^N)$ with $\Delta u \in L^2(\mathbb{R}^N)$ then $u \in H^2(\mathbb{R}^N)$. Furthermore, if $\|u\|^2_{H^{2,2}(\mathbb{R}^N)} = \|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}$ then $\|\,.\,\|_{H^{2,2}(\mathbb{R}^N)}$ and $\|\,.\,\|_{H^2(\mathbb{R}^N)}$ are equivalent norms. Indeed, this us due to the Fourier transform and Plancherel's formula. Finally, note that,
$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \leqslant \|u\|_{L^2(\mathbb{R}^N)}\|\Delta u\|_{L^2(\mathbb{R}^N)} \leqslant \|u\|^2_{L^2(\mathbb{R}^N)} + \|\Delta u\|^2_{L^2(\mathbb{R}^N)}, \tag{6.2.26}$$
for any $u \in H^2(\mathbb{R}^N)$.

Remark 6.2.10. Using a radically different method than the one we propose here, we may show that all the results of this section remain valid if we replace $\mathbb{R}^N$ with an unbounded domain $\Omega \neq \mathbb{R}^N$. This will be the subject of a future work.

6.3 Finite time extinction and asymptotic behavior

Following the method by Carles and Gallo [53] (also used by Carles and Ozawa [55]) and Bégout and Díaz [29], we are able to prove the finite time extinction and asymptotic behavior results.

Theorem 6.3.1. *Let Assumption 6.2.1 be fulfilled with* $N \in \{1,2,3\}$, *let* $f \in W^{1,1}\big((0,\infty);L^2(\mathbb{R}^N)\big)$, *let* $u_0 \in H^1(\mathbb{R}^N)$ *and assume that one of the following hypotheses holds.*

1) $N=1$ *and* $f \in W^{1,1}\big((0,\infty);H^1(\mathbb{R})\big)$.
2) $N \in \{1,2,3\}$ *and* $u_0 \in H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$.

Let u *be the unique strong solution of* (6.2.1)–(6.2.2). *Finally, assume that there exists* $T_0 \geqslant 0$ *such that,*

$$\textit{for almost every } \ t > T_0, \ f(t) = 0.$$

Let ℓ *be the exponent in* $u_0 \in H^\ell(\mathbb{R}^N)$. *We have the following results.*

a) *There exists a finite time* $T_\star \geqslant T_0$ *such that,*

$$\forall t \geqslant T_\star, \ \|u(t)\|_{L^2(\mathbb{R}^N)} = 0. \tag{6.3.1}$$

Furthermore,

$$T_\star \leqslant C\|u\|_{L^\infty((0,\infty);H^\ell(\mathbb{R}^N))}^{\frac{N(1-m)}{2\ell}} \|u(T_0)\|_{L^2(\mathbb{R}^N)}^{\frac{(1-m)(2\ell-N)}{2\ell}} + T_0, \tag{6.3.2}$$

where $C = C(\mathrm{Im}(a),N,m,\ell)$.

b) *There exists* $\varepsilon_\star = \varepsilon_\star(|a|,N,m)$ *satisfying the following property. Let* $\delta = \frac{(2\ell+N)+m(2\ell-N)}{4\ell} \in \big(\frac{1}{2},1\big)$. *If* $f \in W^{1,1}\big((0,\infty);H^1(\mathbb{R}^N)\big)$,

$$\begin{cases} \big(\|u_0\|_{H^1(\mathbb{R}^N)} + \|f\|_{L^1((0,\infty);H^1(\mathbb{R}^N))}\big)^{1-m} \leqslant \varepsilon_\star \min\big\{1,T_0\big\}, & \textit{if } N=1, \\ \Big(\|u_0\|_{H^2(\mathbb{R}^N)}^m + \|f\|_{W^{1,1}((0,\infty);H^1(\mathbb{R}^N))}^m\Big)^{1-m} \leqslant \varepsilon_\star \min\big\{1,T_0\big\}, & \textit{if } N \in \{2,3\}, \end{cases}$$

and if for almost every $t>0$,

$$\|f(t)\|_{L^2(\mathbb{R}^N)}^2 \leqslant \varepsilon_\star \big(T_0 - t\big)_+^{\frac{2\delta-1}{1-\delta}}, \tag{6.3.3}$$

then (6.3.1) *holds with* $T_\star = T_0$.

Remark 6.3.2. If $(N,\ell) \in \{(1,1),(2,2)\}$ then $\frac{2\delta-1}{1-\delta} = 2\frac{1+m}{1-m}$, if $(N,\ell)=(1,2)$ then $\frac{2\delta-1}{1-\delta} = 2\frac{1+3m}{3(1-m)}$ and if $(N,\ell)=(3,2)$ then $\frac{2\delta-1}{1-\delta} = 2\frac{3+m}{1-m}$. Note that if $N=1$ and $u_0 \in H^2(\mathbb{R}^N)$ then there are two possible choices for $\frac{2\delta-1}{1-\delta}$ in (6.3.3) : $2\frac{1+m}{1-m}$ or $2\frac{1+3m}{3(1-m)}$. Since for t near T_0, $T_0 - t < 1$ then the choice the less restrictive is that for which $\frac{2\delta-1}{1-\delta}$ is the smallest as possible, that is $2\frac{1+3m}{3(1-m)}$.

Remark 6.3.3. In the case of our nonlinearity, Theorem 6.3.1 is an improvement of the result of Carles and Ozawa [55] in the sense they obtain the same conclusion as in *a*) but with a presence harmonic confinement in (6.2.1), $\mathrm{Re}(a)=0$, $f=0$, $N\in\{1,2\}$ and $\big(u_0\in H^1(\mathbb{R})\cap\mathscr{F}(H^1(\mathbb{R}))$ [3]$\big)$, if $N=1$ and $\big(u_0\in H^2(\mathbb{R}^2)\cap\mathscr{F}(H^2(\mathbb{R}^2))$[3], $\|u_0\|_{L^2(\mathbb{R}^2)}$ small enough and $\frac{1}{2}\leqslant m<1\big)$, if $N=2$. Additional nonlinearities are also considered in [55].

Theorem 6.3.4. *Let Assumption 6.2.1 be fulfilled with $N\geqslant 4$, let $f\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$ and let $u_0\in H^1(\mathbb{R}^N)$. Suppose further that $f\in W^{1,1}_{\mathrm{loc}}\big([0,\infty);H^1(\mathbb{R}^N)\big)$ or $u_0\in H^2(\mathbb{R}^N)$. Let u be the unique strong solution of (6.2.1)–(6.2.2). Finally, assume that there exists $T_0\geqslant 0$ such that,*

$$\textit{for almost every } t>T_0,\ f(t)=0.$$

Then we have for any $t\geqslant T_0$,

$$\|u(t)\|_{L^2(\mathbb{R}^N)}\leqslant\|u(T_0)\|_{L^2(\mathbb{R}^N)}e^{-C(t-T_0)},$$

if $N=4$ and $u_0\in H^2(\mathbb{R}^N)$,

$$\|u(t)\|_{L^2(\mathbb{R}^N)}\leqslant\frac{\|u(T_0)\|_{L^2(\mathbb{R}^N)}}{\left(1+C\|u(T_0)\|_{L^2(\mathbb{R}^N)}^{\frac{(1-m)(N-2\ell)}{2\ell}}(t-T_0)\right)^{\frac{2\ell}{(1-m)(N-2\ell)}}},$$

if $N\geqslant 5$ or $u_0\in H^1(\mathbb{R}^N)$, where $C=C(\|u\|_{L^\infty((0,\infty);H^\ell(\mathbb{R}^N))},\mathrm{Im}(a),N,m,\ell)$.

Theorem 6.3.5. *Let Assumption 6.2.1 be fulfilled, let $f\in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$, let $u_0\in L^2(\mathbb{R}^N)$ and let u be the unique weak solution of (6.2.1)–(6.2.2). If*

$$f\in L^1\big((0,\infty);L^2(\mathbb{R}^N)\big),$$

then,

$$\lim_{t\nearrow\infty}\|u(t)\|_{L^2(\mathbb{R}^N)}=0.$$

6.4 Proofs of the existence and uniqueness theorems

Since we have to prove existence in the whole space, the method is radically different than that used in Bégout and Díaz [29].

Theorem 6.4.1. *Let Assumption 6.2.1 be fulfilled and let $\lambda,b_0>0$. Then for any $F\in L^2(\mathbb{R}^N)$, there exists a unique solution u to,*

$$\begin{cases}u\in H^2(\mathbb{R}^N)\cap L^{2m}(\mathbb{R}^N),\\ -\lambda\Delta u-a\lambda|u|^{-(1-m)}u-\mathrm{i}b_0u=F,\quad\textit{in}\ \ L^2(\mathbb{R}^N).\end{cases}\tag{6.4.1}$$

In addition,

$$\|u\|_{H^2(\mathbb{R}^N)}^2+\|u\|_{L^{m+1}(\mathbb{R}^N)}^{m+1}+\|u\|_{L^{2m}(\mathbb{R}^N)}^{2m}\leqslant M\|F\|_{L^2(\mathbb{R}^N)}^2,\tag{6.4.2}$$

3. $\mathscr{F}(H^1(\mathbb{R}))\hookrightarrow L^{2m}(\mathbb{R})$ and $\mathscr{F}(H^2(\mathbb{R}^2))\hookrightarrow L^{2m}(\mathbb{R}^2)$, for any $\frac{1}{3}<m\leqslant 1$.

where $M = M(|a|, \mathrm{Arg}(a), b_0, \lambda)$*. Furthermore, if* F *is compactly supported then so is* u*. Finally, let* $G \in L^2(\mathbb{R}^N)$*. If* v *is a solution to* (6.4.1) *with* G *instead of* F *then,*

$$\|u - v\|_{L^2(\mathbb{R}^N)} \leqslant \frac{1}{b_0}\|F - G\|_{L^2(\mathbb{R}^N)}. \tag{6.4.3}$$

Here and after, $\mathrm{Arg}(a) \in (0, \pi)$ *denotes the principal value of the argument of* a*.*

The proof of the theorem relies on the following lemmas.

Lemma 6.4.2. *Let Assumption* 6.2.1 *be fulfilled. Then there exists* $b \in \mathbb{C}$*, with* $|b| = 1$*, satisfying the following property.*

$$\mathrm{Re}(b) > 0 \ \ \textit{and} \ \ \mathrm{Im}(b) < 0, \tag{6.4.4}$$

$$2\sqrt{m}\,\mathrm{Im}(ab) > (1-m)\mathrm{Re}(ab) \geqslant 0. \tag{6.4.5}$$

In addition, $b = b(\mathrm{Arg}(a))$*. In particular,* ab *satisfies* (6.2.3)*–*(6.2.4) *of Assumption* 6.2.1*.*

Proof. Let $\theta_a = \mathrm{Arg}(a) \in (0, \pi)$, since $\mathrm{Im}(a) > 0$. We look for $b = e^{-\mathrm{i}\theta_b}$, where $0 < \theta_b < \frac{\pi}{2}$.
Case 1 : $\mathrm{Re}(a) < 0$.
If follows that, $\frac{\pi}{2} < \theta_a < \pi$. We choose $\theta_b = \theta_a - \frac{\pi}{2}$. We then have $ab = \mathrm{i}|a|$ and the conclusion is clear.
Case 2 : $\mathrm{Re}(a) \geqslant 0$.
If follows that, $0 < \theta_a \leqslant \frac{\pi}{2}$ and by (6.2.4), one has

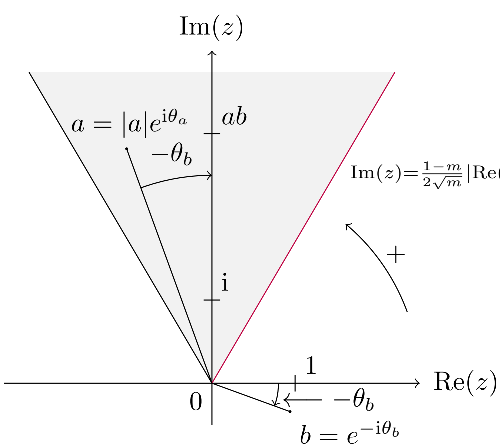
$$2\sqrt{m}\sin(\theta_a) > (1-m)\cos(\theta_a) \geqslant 0. \tag{6.4.6}$$

By continuity and (6.4.6), there exists $\theta_b \in (0, \theta_a)$ such that,

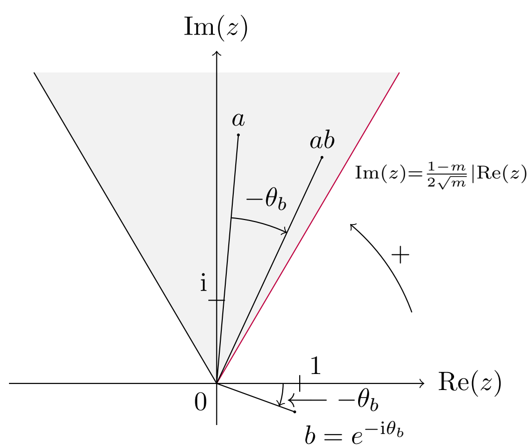
$$2\sqrt{m}\sin(\theta_a - \theta_b) > (1-m)\cos(\theta_a - \theta_b) > 0. \tag{6.4.7}$$

Then, $0 < \theta_a - \theta_b < \frac{\pi}{2}$, $ab = |a|e^{\mathrm{i}(\theta_a - \theta_b)}$ and again the conclusion is clear. □

We may summarize the proof of Lemma 6.4.2 with the picture below.



$\theta_b = \theta_a - \frac{\pi}{2}$
Case 1 : $\mathrm{Re}(a) < 0$



$0 < \theta_b \ll 1$
Case 2 : $\mathrm{Re}(a) \geqslant 0$

Lemma 6.4.3. *Let $0 < m < 1$. Set for any $z \in \mathbb{C}$, $g(z) = |z|^{-(1-m)}z$ ($g(0) = 0$). We define the mapping for any measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{C}$, which we still denote by g, by $g(u)(x) = g(u(x))$. Then for any $p \in [1, \infty)$,*

$$g \in C\big(L^p(\mathbb{R}^N); L^{\frac{p}{m}}(\mathbb{R}^N)\big) \text{ and } g \text{ is bounded on bounded sets.} \tag{6.4.8}$$

Let $a \in \mathbb{C}$ with $\mathrm{Im}(a) > 0$ satisfying (6.2.3). Then $\big(g(u) - g(v)\big)\big(\overline{u - v}\big) \in L^1(\mathbb{R}^N)$ and,

$$\mathrm{Re}\left(-\mathrm{i}\, a \int\limits_{\mathbb{R}^N} \big(g(u) - g(v)\big)\big(\overline{u - v}\big)\mathrm{d}x\right) \geqslant 0, \tag{6.4.9}$$

for any $u, v \in L^{m+1}(\mathbb{R}^N)$.

Proof. Property (6.4.8) is an obvious consequence of (6.2.11) which implies the integrability property in the lemma. By Lemma 2.2 of Liskevich and Perel′muter [132], we have

$$2\sqrt{m}\left|\mathrm{Im}\Big(\big(g(z_1) - g(z_2)\big)\big(\overline{z_1 - z_2}\big)\Big)\right| \leqslant (1 - m)\mathrm{Re}\Big(\big(g(z_1) - g(z_2)\big)\big(\overline{z_1 - z_2}\big)\Big), \tag{6.4.10}$$

for any $(z_1, z_2) \in \mathbb{C}^2$. Let $u, v \in L^{m+1}(\mathbb{R}^N)$. We have by (6.4.10),

$$\begin{aligned}
&\mathrm{Re}\left(-\mathrm{i}\, a \int\limits_{\mathbb{R}^N} \big(g(u) - g(v)\big)\big(\overline{u - v}\big)\mathrm{d}x\right)\\
&= \mathrm{Im}(a)\mathrm{Re}\int\limits_{\mathbb{R}^N} \big(g(u) - g(v)\big)\big(\overline{u - v}\big)\mathrm{d}x + \mathrm{Re}(a)\mathrm{Im}\int\limits_{\mathbb{R}^N} \big(g(u) - g(v)\big)\big(\overline{u - v}\big)\mathrm{d}x\\
&\geqslant \left(\mathrm{Im}(a) - |\mathrm{Re}(a)|\frac{1 - m}{2\sqrt{m}}\right)\mathrm{Re}\int\limits_{\mathbb{R}^N} \big(g(u) - g(v)\big)\big(\overline{u - v}\big)\mathrm{d}x\\
&\geqslant 0.
\end{aligned}$$

The lemma is proved. □

Lemma 6.4.4 ([29])**.** *Let $0 < m < 1$ and let $a \in \mathbb{C}$ with $\mathrm{Im}(a) > 0$ satisfying (6.2.3). Let g be as in Lemma 6.4.3. Then $g(u)\overline{\Delta u} \in L^1(\mathbb{R}^N)$ and,*

$$\mathrm{Re}\left(\mathrm{i}a \int\limits_{\mathbb{R}^N} g(u)\overline{\Delta u}\mathrm{d}x\right) \geqslant 0, \tag{6.4.11}$$

for any $u, v \in H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$.

Proof. See Lemma 5.6.3. □

Proof of Theorem 6.4.1. Let Assumption 6.2.1 be fulfilled, $\lambda, b_0 > 0$ and $F \in L^2(\mathbb{R}^N)$. Let g be as in Lemma 6.4.3. We want to solve,

$$-\lambda\Delta u - a\lambda g(u) - \mathrm{i}b_0 u = F, \quad \text{in} \;\; H^{-1}(\mathbb{R}^N) + L^{\frac{m+1}{m}}(\mathbb{R}^N). \tag{u_F}$$

We proceed with the proof in five steps.
Step 1 : A first estimate. Let $G \in L^2(\mathbb{R}^N)$. If $u, v \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ are solutions of (u_F) and (v_G), respectively, then estimate (6.4.3) holds true.
We multiply by $\mathrm{i}\overline{\varphi}$, for $\varphi \in \mathscr{D}(\mathbb{R}^N)$, the equation satisfied by $u-v$, we integrate by parts and we take the real part. By density of $\mathscr{D}(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ and (6.4.8), $\big(g(u)-g(v)\big)(\overline{u-v}) \in L^1(\mathbb{R}^N)$ and we may choose $\varphi = u - v$. It follows that,

$$\lambda \mathrm{Re}\left(-\mathrm{i}a \int\limits_{\mathbb{R}^N} \big(g(u)-g(v)\big)(\overline{u-v})\mathrm{d}x\right) + b_0\|u-v\|^2_{L^2(\mathbb{R}^N)} = -\mathrm{Im}\left(\int\limits_{\mathbb{R}^N} (F-G)\overline{(u-v)}\mathrm{d}x\right). \tag{6.4.12}$$

Estimate (6.4.3) then comes from (6.4.12), (6.4.9) and Cauchy-Schwarz's inequality.
Step 2 : A second estimate. If u is a solution to (6.4.1) then $u \in L^{m+1}(\mathbb{R}^N)$ and satisfies (6.4.2). Since $2m < m+1 < 2$, then $L^{2m}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \subset L^{m+1}(\mathbb{R}^N)$. By Theorem 2.2.9,

$$\|u\|^2_{H^1(\mathbb{R}^N)} + \|u\|^{m+1}_{L^{m+1}(\mathbb{R}^N)} \leqslant M(|a|, b_0, \lambda)\|F\|^2_{L^2(\mathbb{R}^N)}. \tag{6.4.13}$$

Let $b \in \mathbb{C}$ be given by Lemma 6.4.2. We multiply the equation in (6.4.1) by $-\mathrm{i}b\overline{\Delta u}$, integrate by parts and take the real part. We obtain,

$$\begin{aligned} &-\lambda \mathrm{Im}(b)\|\Delta u\|^2_{L^2(\mathbb{R}^N)} + \lambda \mathrm{Re}\left(\mathrm{i}ab \int\limits_{\mathbb{R}^N} g(u)\overline{\Delta u}\mathrm{d}x\right) + b_0 \mathrm{Re}(b)\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \\ &\qquad = \mathrm{Im}\left(b \int_{\mathbb{R}^N} F\overline{\Delta u}\mathrm{d}x\right). \end{aligned} \tag{6.4.14}$$

By (6.4.5), we may apply Lemma 6.4.4. Using (6.4.4), (6.4.11) and applying Cauchy-Schwarz's inequality in (6.4.14), one obtains,

$$\|\Delta u\|_{L^2(\mathbb{R}^N)} \leqslant \frac{|b|}{\lambda|\mathrm{Im}(b)|}\|F\|_{L^2(\mathbb{R}^N)}. \tag{6.4.15}$$

Now, since by Plancherel's formula, $\|u\|_{\dot{H}^2(\mathbb{R}^N)} \leqslant C\||\xi|^2\widehat{u}\|_{L^2(\mathbb{R}^N)} \leqslant C\|\Delta u\|_{L^2(\mathbb{R}^N)}$, putting together (6.4.13) and (6.4.15), one obtains (6.4.2).
Step 3 : Compactness of the solution. If $\operatorname{supp} F$ is compact and if $u \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ is a solution to (u_F) then $\operatorname{supp} u$ is compact.
This comes from Theorem 1.3.6.
Step 4 : Existence and uniqueness. There exists a unique solution $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ to (u_F).
By Theorem 2.2.8, equation (u_F) admits a solution $u \in H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$. By Proposition 1.4.5, $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, by Step 1 this solution is unique.
Step 5 : Conclusion.
Estimates (6.4.2)–(6.4.3), uniqueness and compactness property come from Steps 1–3, once the existence of a solution to (6.4.1) is proved. Let $u \in H^2_{\mathrm{loc}}(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \cap L^{m+1}(\mathbb{R}^N)$ the solution of (u_F) be given by Step 4. Let $(F_n)_{n\in\mathbb{N}} \subset \mathscr{D}(\mathbb{R}^N)$ be such that $F_n \xrightarrow[n\to\infty]{L^2(\mathbb{R}^N)} F$. Finally, for each $n \in \mathbb{N}$, denote

by u_n the unique solution to (6.4.1), where the right-hand side is F_n instead of F (Steps 4 and 3). By Steps 1 and 2, $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^2(\mathbb{R}^N)$ and $u_n \xrightarrow[n\to\infty]{L^2(\mathbb{R}^N)} u$. It follows that $u \in H^2(\mathbb{R}^N)$ and, from the equation in (6.4.1), $g(u) \in L^2(\mathbb{R}^N)$. Hence u is a solution to (6.4.1). This concludes the proof. □

Corollary 6.4.5. *Let Assumption 6.2.1 be fulfilled. Let us define the following (nonlinear) operator on* $L^2(\mathbb{R}^N)$.

$$\begin{cases} D(A) = H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N), \\ \forall u \in D(A),\ Au = -\mathrm{i}\Delta u - \mathrm{i}a|u|^{-(1-m)}u, \end{cases}$$

Then A is maximal monotone on $L^2(\mathbb{R}^N)$ (and so m-accretive) with dense domain.

Proof. The density is obvious. For any $\lambda > 0$, $I + \lambda A$ is bijective from $D(A)$ onto $L^2(\mathbb{R}^N)$ and $(I + \lambda A)^{-1}$ is a contraction (Theorem 6.4.1). It follows that A is maximal monotone (Brezis [43], Proposition 2.2, p.23). □

Proof of Theorem 6.2.7. Let g be as in Lemma 6.4.3. We first recall that by 1) of Lemma B.4,

$$f \in C\big([0,\infty); L^2(\mathbb{R}^N)\big). \tag{6.4.16}$$

By Corollary 6.4.5 and Barbu [16] (Theorem 2.2, p.131), there exists a unique $u \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty); L^2(\mathbb{R}^N)\big)$ satisfying $u(t) \in H^2(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N)$ and (6.2.1) in $L^2(\mathbb{R}^N)$, for almost every $t > 0$, $u(0) = u_0$ and (6.2.25). This last estimate yields (6.2.23). Since $u \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty); L^2(\mathbb{R}^N)\big)$, it follows from Lemma B.5 that the map $M : t \longmapsto \frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^N)}$ belongs to $W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);\mathbb{R}\big)$ and $M'(t) = \big(u(t), u_t(t)\big)_{L^2(\mathbb{R}^N)}$, for almost every $t > 0$. Multiplying (6.2.1) by $\overline{\mathrm{i}u}$, integrating by parts over $\mathbb{R}^N$ and taking the real part, we obtain (6.2.22), for almost every $t > 0$. We deduce easily from (6.2.22), (6.4.16) and Hölder's inequality that $u \in L^\infty_{\mathrm{loc}}\big([0,\infty); L^{m+1}(\mathbb{R}^N)\big)$. Multiplying again (6.2.1) by $\overline{u}$, integrating by parts and taking the real part, we get

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)} \leqslant |\mathrm{Re}(a)|\|u(t)\|^{m+1}_{L^{m+1}(\mathbb{R}^N)} + \big(\|u_t(t)\|_{L^2(\mathbb{R}^N)} + \|f(t)\|_{L^2(\mathbb{R}^N)}\big)\,\|u(t)\|_{L^2(\mathbb{R}^N)},$$

for almost every $t > 0$. It follows that $u \in L^\infty_{\mathrm{loc}}\big([0,\infty); H^1(\mathbb{R}^N)\big)$. We infer that u is an H^2-solution. Let $b \in \mathbb{C}$ be given by Lemma 6.4.2. We multiply (6.2.1) by $\overline{\mathrm{i}abg(u)}$, integrate and take the real part. We get,

$$\mathrm{Re}\left(\overline{ab}\int_{\mathbb{R}^N} u_t\overline{g(u)}\mathrm{d}x\right) + \mathrm{Re}\left(\overline{\mathrm{i}ab}\int_{\mathbb{R}^N}\overline{g(u)}\Delta u\mathrm{d}x\right) + |a|^2\mathrm{Re}(\overline{\mathrm{i}b})\|g(u)\|^2_{L^2(\mathbb{R}^N)} = \mathrm{Re}\left(\overline{\mathrm{i}ab}\int_{\mathbb{R}^N} f\overline{g(u)}\mathrm{d}x\right). \tag{6.4.17}$$

By Lemma 6.4.2, we have (6.4.11). This implies,

$$\mathrm{Re}\left(\overline{\mathrm{i}ab}\int_{\mathbb{R}^N}\overline{g(u)}\Delta u\mathrm{d}x\right) = \mathrm{Re}\left(\mathrm{i}ab\int_{\mathbb{R}^N} g(u)\overline{\Delta u}\mathrm{d}x\right) \geqslant 0, \tag{6.4.18}$$

and (6.4.17) becomes,

$$|a||\mathrm{Im}(b)|\,\|u\|_{L^{2m}(\mathbb{R}^N)}^{2m} \leqslant \int\limits_{\mathbb{R}^N} |(u_t+\mathrm{i}f)\overline{g(u)}|\mathrm{d}x, \tag{6.4.19}$$

since $\mathrm{Re}(\mathrm{i}\overline{b}) = -\mathrm{Im}(b) > 0$, by (6.4.4). By Cauchy-Schwarz's and Young's inequalities, we get

$$\int\limits_{\mathbb{R}^N} |(u_t+\mathrm{i}f)\overline{g(u)}|\mathrm{d}x \leqslant \frac{1}{2|a||\mathrm{Im}(b)|}\|u_t+\mathrm{i}f\|_{L^2(\mathbb{R}^N)}^2 + \frac{|a||\mathrm{Im}(b)|}{2}\|u\|_{L^{2m}(\mathbb{R}^N)}^{2m}. \tag{6.4.20}$$

Putting together (6.4.19) and (6.4.20), we arrive at,

$$\|u(t)\|_{L^{2m}(\mathbb{R}^N)}^{2m} \leqslant \frac{1}{|a|^2|\mathrm{Im}(b)|^2}\left(\|u_t(t)\|_{L^2(\mathbb{R}^N)} + \|f(t)\|_{L^2(\mathbb{R}^N)}\right)^2, \tag{6.4.21}$$

for almost every $t>0$. Multiplying again (6.2.1) by $\mathrm{i}b\overline{\Delta u}$, using (6.4.18) and proceeding as above, we arrive at,

$$\|\Delta u(t)\|_{L^2(\mathbb{R}^N)} \leqslant \frac{1}{|\mathrm{Im}(b)|}\left(\|u_t(t)\|_{L^2(\mathbb{R}^N)} + \|f(t)\|_{L^2(\mathbb{R}^N)}\right), \tag{6.4.22}$$

for almost every $t>0$. By (6.4.16), (6.4.21), (6.4.22), Remark 6.2.9 and Hölder's inequality (recalling that $2m < m+1 < 2$), we obtain,

$$u \in L^\infty_{\mathrm{loc}}\big([0,\infty);H^2(\mathbb{R}^N)\big) \cap L^\infty_{\mathrm{loc}}\big([0,\infty);L^{2m}(\mathbb{R}^N)\big), \tag{6.4.23}$$

$$u \in C\big([0,\infty);L^2(\mathbb{R}^N)\big) \cap L^\infty_{\mathrm{loc}}\big([0,\infty);L^{2m}(\mathbb{R}^N)\big) \hookrightarrow C\big([0,\infty);L^{m+1}(\mathbb{R}^N)\big). \tag{6.4.24}$$

Recalling that $u \in W^{1,\infty}_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$, by (6.4.23) and the embedding 3) of Lemma B.4, we have $u \in C\big([0,\infty);H^1(\mathbb{R}^N)\big)$. We then deduce Property 1), with help of (6.2.13), (6.4.16) and (6.2.1). With (6.2.26), (6.2.23) and (6.4.23), we get (6.2.24) and Property 2) is proved. Property 3) comes from (6.2.22), (6.4.16) and (6.4.24). Finally, Property 4) follows easily from the embedding 1) of Lemma B.4, Remarks 6.2.5 and 6.2.9, (6.2.25), (6.4.21) and (6.4.22). This concludes the proof of the theorem. □

Lemma 6.4.6. *Let Assumption 6.2.1 be fulfilled and* $f,g \in L^1_{\mathrm{loc}}\big([0,\infty);L^2(\mathbb{R}^N)\big)$. *If u and v are strong solutions or weak solutions of*

$$\mathrm{i}u_t + \Delta u + a|u|^{-(1-m)}u = f_1,$$
$$\mathrm{i}v_t + \Delta v + a|v|^{-(1-m)}v = f_2,$$

respectively, then $u,v \in C\big([0,\infty);L^2(\Omega)\big)$ *and*

$$\|u(t)-v(t)\|_{L^2(\Omega)} \leqslant \|u(s)-v(s)\|_{L^2(\Omega)} + \int\limits_s^t \|f_1(\sigma)-f_2(\sigma)\|_{L^2(\Omega)}\mathrm{d}\sigma, \tag{6.4.25}$$

for any $t \geqslant s \geqslant 0$.

Proof. Let $X=H^1(\mathbb{R}^N)\cap L^{m+1}(\mathbb{R}^N)$ and let u,v be as in the lemma. Continuity comes from (6.2.8) and Definition 6.2.2. Estimate (6.4.25) being stable by passing to the limit in $C\big([0,T];L^2(\mathbb{R}^N)\big)\times L^1\big((0,T);L^2(\mathbb{R}^N)\big)$, for any $T>0$, it is sufficient to establish it for the H^2-solutions. And since an H^2-solution is an H^1 solution, we may assume that u,v are H^1 solution. Making the difference between the two equations, it follows from 3) of Remark 6.2.3 that we can take the $X^\star-X$ duality product of the result with $\mathrm{i}(u-v)$. With help of (B.3) below, (6.2.14), (6.4.9) and Cauchy-Schwarz's inequality, we then arrive at,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\,.\,)-v(\,.\,)\|_{L^2(\Omega)}^2\leqslant\|f_1-f_2\|_{L^2(\Omega)}\|u-v\|_{L^2(\Omega)},$$

almost everywhere on $(0,\infty)$. Integrating over (s,t), one obtains (6.4.25). □

Proof of Theorem 6.2.4. Existence, estimate (6.2.19) and uniqueness comes from density of $H^2(\mathbb{R}^N)\times W^{1,1}_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^N))$ in $L^2(\mathbb{R}^N)\times L^1_{\mathrm{loc}}([0,\infty);L^2(\mathbb{R}^N))$, Theorem 6.2.7, Lemma 6.4.6 and completeness of $C\big([0,T];L^2(\mathbb{R}^N)\big)$, for any $T>0$. Finally, estimates (6.2.17)–(6.2.18) comes from Proposition 5.2.3. This ends the proof of the theorem. □

Proof of Theorem 6.2.6. Uniqueness comes from Lemma 6.4.6. Let $f\in W^{1,1}_{\mathrm{loc}}([0,\infty);H^1(\mathbb{R}^N))$ and let $u_0\in H^1(\mathbb{R}^N)$. Let $(\varphi_n)_{n\in\mathbb{N}}\subset\mathscr{D}(\mathbb{R}^N)$ be such that $\varphi_n\xrightarrow[n\to\infty]{H^1(\mathbb{R}^N)}u_0$. Finally, let g be defined as in Lemma 6.4.3 and for each $n\in\mathbb{N}$, let u_n the unique H^2-solution of (6.2.1) such that $u_n(0)=\varphi_n$, be given by Theorem 6.2.7. By Lemma 6.4.6, we have for any $T>0$ and $n,p\in\mathbb{N}$,

$$\begin{gathered}\|u_n\|_{C([0,T];L^2(\mathbb{R}^N))}\leqslant\|\varphi_n\|_{L^2(\mathbb{R}^N)}+\int_0^T\|f(t)\|_{L^2(\mathbb{R}^N)}\mathrm{d}t,\\ \|u_n-u_p\|_{L^\infty((0,\infty);L^2(\mathbb{R}^N))}\leqslant\|\varphi_n-\varphi_p\|_{L^2(\mathbb{R}^N)},\end{gathered}\tag{6.4.26}$$

It follows that for any $T>0$, $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C\big([0,T];L^2(\mathbb{R}^N)\big)$. As a consequence, there exists $u\in C\big([0,\infty);L^2(\mathbb{R}^N)\big)$ such that for any $T>0$,

$$u_n\xrightarrow[n\to\infty]{C([0,T];L^2(\mathbb{R}^N))}u.\tag{6.4.27}$$

By definition, it follows from (6.4.27) that u is a weak solution of (6.2.1)–(6.2.2). By Theorem 6.2.7, we can take the L^2-scalar product of (6.2.1) with $-\mathrm{i}\Delta u_n$ and it follows from (B.4) that for any $n\in\mathbb{N}$ and almost every $s>0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n(s)\|_{L^2(\mathbb{R}^N)}^2+\mathrm{Re}\left(\mathrm{i}a\int\limits_{\mathbb{R}^N}g(u_n(s))\overline{\Delta u_n(s)}\mathrm{d}x\right)=\big(\nabla f(s),\mathrm{i}\nabla u_n(s)\big)_{L^2(\mathbb{R}^N)}.$$

which gives with (6.4.11) and Cauchy-Schwarz's inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n(s)\|_{L^2(\mathbb{R}^N)}^2\leqslant\|\nabla f(s)\|_{L^2(\mathbb{R}^N)}\|\nabla u_n(s)\|_{L^2(\mathbb{R}^N)}.$$

By integration, we obtain for any $t>0$ and any $n\in\mathbb{N}$,

$$\|\nabla u_n(t)\|_{L^2(\mathbb{R}^N)}\leqslant\|\nabla\varphi_n\|_{L^2(\mathbb{R}^N)}+\int_0^t\|\nabla f(s)\|_{L^2(\mathbb{R}^N)}\mathrm{d}s.\tag{6.4.28}$$

By the Sobolev embedding 1) of Lemma B.4,

$$W^{1,1}_{\rm loc}\big([0,\infty);L^2(\mathbb{R}^N)\big)\hookrightarrow C\big([0,\infty);L^2(\mathbb{R}^N)\big),\tag{6.4.29}$$

(6.4.26), (6.4.28), (6.4.8) and (6.2.1), we infer that,

$$(u_n)_{n\in\mathbb{N}} \text{ is bounded in } L^\infty\big((0,T);H^1(\mathbb{R}^N)\big)\cap W^{1,\infty}\big((0,T);Z^\star\big),\tag{6.4.30}$$

for any $T>0$, where $Z^\star=H^{-1}(\mathbb{R}^N)+L^{\frac{2}{m}}(\mathbb{R}^N)$ is the topological dual space of $Z=H^1(\mathbb{R}^N)\cap L^{\frac{2}{2-m}}(\mathbb{R}^N)$. Note that $Z^\star$ is reflexive (Lemma B.2) and since $H^1(\mathbb{R}^N)\hookrightarrow Z^\star$, it follows from (6.4.27), (6.4.30), (6.2.15) and Proposition 1.1.2, p.2, and (ii) of Remark 1.3.13, p.12, in Cazenave [57] that,

$$u\in C_{\rm w}\left([0,\infty);H^1(\mathbb{R}^N)\right)\cap W^{1,\infty}_{\rm loc}\big([0,\infty);Z^\star\big),\tag{6.4.31}$$

$$\Delta u\in C\left([0,\infty);H^{-2}(\mathbb{R}^N)\right),\tag{6.4.32}$$

$$u_n(t)\rightharpoonup u(t),\ \text{ in }\ H^1_{\rm w}(\mathbb{R}^N),\ \text{ as }\ n\to\infty,\tag{6.4.33}$$

for any $t\geqslant 0$. After integration of (6.2.22), we see with help of (6.4.26) that for any $T>0$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^{m+1}\big((0,T);L^{m+1}(\mathbb{R}^N)\big)\cong L^{m+1}\big((0,T)\times\mathbb{R}^N\big)$, which is reflexive. We infer with (6.4.27),

$$u\in L^{m+1}_{\rm loc}\big([0,\infty);L^{m+1}(\mathbb{R}^N)\big).\tag{6.4.34}$$

By (6.4.29), (6.4.31), (6.4.34) and (6.2.1), it follows that u satisfies 1) of Definition 6.2.2 and then u is an H^1-solution. By 3) of Remark 6.2.3, we can take the $X-X^\star$ duality product with iu, where $X=H^1(\mathbb{R}^N)\cap L^{m+1}(\mathbb{R}^N)$. Applying Lemma B.5 and (6.2.14), Property 3) follows. Estimate (6.2.21) comes from (6.4.33), (6.4.28) and the weak lower semicontinuity of the norm. Finally, smoothness of the solution in Properties 1) and 2) follows easily from (6.4.29), (6.4.31), (6.4.32), (6.4.8) and the equation (6.2.1). This concludes the proof of the theorem. □

6.5 Proofs of the finite time extinction and asymptotic behavior theorems

Proof of Theorem 6.3.1. Apply Theorems 6.2.6, 6.2.7 and use the general theorem of finite time extinction (Theorem 5.2.1 and Remark 5.4.8). Nevertheless, to make the proof more understandable, we briefly explain how to obtain (6.3.1)–(6.3.2). Let $\ell=1$, if $u_0\in H^1(\mathbb{R}^N)$ and $\ell=2$, if $u_0\in H^2(\mathbb{R}^N)$. Assume that for some $T_0\geqslant 0$, $f(t)=0$, for almost every $t>T_0$. It follows from Theorems 6.2.6, 6.2.7 and Remark 6.2.5 that $u\in L^\infty\big((0,\infty);H^\ell(\mathbb{R}^N)\big)$. We have by Gagliardo-Nirenberg's inequality and (6.2.22),

$$\|u(t)\|_{L^2(\mathbb{R}^N)}^{\frac{(2\ell+N)+m(2\ell-N)}{2\ell}}\leqslant C\|u\|_{L^\infty((0,\infty);H^\ell(\mathbb{R}^N))}^{\frac{N(1-m)}{2\ell}}\|u(t)\|_{L^{m+1}(\mathbb{R}^N)}^{m+1},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^N)}+2\mathrm{Im}(a)\|u(t)\|^{m+1}_{L^{m+1}(\mathbb{R}^N)}=0,$$

for almost every $t>T_0$. It follows that,

$$y'(t)+Cy(t)^\delta\leqslant 0,\tag{6.5.1}$$

for almost every $t > T_0$, where $y(t) = \|u(t)\|^2_{L^2(\mathbb{R}^N)}$ and $\delta = \frac{(2\ell+N)+m(2\ell-N)}{4\ell}$. By our assumption on ℓ, we have $\delta \in (0,1)$ if $N \leqslant 3$. Hence (6.3.1)–(6.3.2) by integration. □

Proof of Theorem 6.3.4. Let $\ell = 1$, if $u_0 \in H^1(\mathbb{R}^N)$ and $\ell = 2$, if $u_0 \in H^2(\mathbb{R}^N)$. By Theorems 6.2.6, 6.2.7 and Remark 6.2.5, $u \in L^\infty\big((0,\infty); H^\ell(\mathbb{R}^N)\big)$. Repeating the proof of Theorem 6.3.1, we obtain obtain (6.5.1). According to the different cases as in the theorem, we have $\delta = 1$ or $\delta > 1$. The results then follow by integration (see also (6.1.6) and the lines below). For more details, see 3) of Remark 5.2.4. □

Proof of Theorem 6.3.5. By Remark 6.2.5, we may assume that $f \in \mathscr{D}\big([0,\infty); L^2(\mathbb{R}^N)\big)$ and $u_0 \in H^2(\mathbb{R}^N)$. Let $[0,T_0] \supset \operatorname{supp} f$. By (6.2.22), $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leqslant 0$, for any $t > T_0$. It follows that $\lim\limits_{t\nearrow\infty} \|u(t)\|_{L^2(\mathbb{R}^N)} = \ell_0$, for some $\ell_0 \in [0,\infty)$. Let $q \in (2,\infty)$ with $(N-2)q < 2N$. By Hölder's inequality and Sobolev's embedding $H^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$, there exists $\theta \in (0,1)$ such that,

$$\ell_0 \leqslant \|u(t)\|_{L^2(\mathbb{R}^N)} \leqslant \|u(t)\|^\theta_{L^{m+1}(\mathbb{R}^N)}\|u(t)\|^{1-\theta}_{L^q(\mathbb{R}^N)} \leqslant C\|u(t)\|^\theta_{L^{m+1}(\mathbb{R}^N)}\|u\|^{1-\theta}_{L^\infty((0,\infty);H^1(\mathbb{R}^N))},$$

for any $t > T_0$. We get, still by (6.2.22),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^N)} \leqslant -C\ell_0^{\frac{m+1}{\theta}} \leqslant 0,$$

for any $t > T_0$. Hence $\ell_0 = 0$. □

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Chapitre 7

A Generalized Interpolation Inequality and its Application to the Stabilization of Damped Equations

with Fernando Soria*

Abstract

In this paper, we establish a generalized Hölder's or interpolation inequality for weighted spaces in which the weights are non-necessarily homogeneous. We apply it to the stabilization of some damped wave-like evolution equations. This allows obtaining explicit decay rates for smooth solutions for more general classes of damping operators. In particular, for $1-d$ models, we can give an explicit decay estimate for pointwise damping mechanisms supported on any strategic point.

7.1 Introduction

We are interested on a generalized Hölder's or interpolation inequality, in order to establish explicit decay rates for smooth solutions of damped wave-like equations with weak damping.

Let $(\Omega, \mathscr{T}, \mu)$ be a measure space and let ω_1 and ω_2 be two μ-measurable weights on Ω. The problem we address consists in finding suitable functions Φ and Ψ such that

$$1 \leqslant \Phi\left(\frac{\displaystyle\int_\Omega |f(x)|\omega_1(x)\mathrm{d}\mu(x)}{\|f\|_{L^1(\Omega,\mathscr{T},\mu)}}\right)\Psi\left(\frac{\displaystyle\int_\Omega |f(x)|\omega_2(x)\mathrm{d}\mu(x)}{\|f\|_{L^1(\Omega,\mathscr{T},\mu)}}\right), \tag{7.1.1}$$

for any $f \in L^1(\Omega, \mathscr{T}, \mu) \cap L^1(\Omega, \mathscr{T}, \omega_1\mathrm{d}\mu) \cap L^1(\Omega, \mathscr{T}, \omega_2\mathrm{d}\mu)$.

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The case where the weights functions are homogeneous is well-known. Indeed, if $\omega_1(x) = |x|^\alpha$ and $\omega_2(x) = |x|^{-\beta}$ ($\alpha, \beta > 0$), the classical Hölder's inequality gives

$$\int_\Omega |f(x)|\mathrm{d}x \leqslant \left(\int_\Omega |f(x)|\,|x|^\alpha \mathrm{d}x\right)^{\frac{\beta}{\alpha+\beta}} \left(\int_\Omega |f(x)|\,|x|^{-\beta}\mathrm{d}x\right)^{\frac{\alpha}{\alpha+\beta}}, \tag{7.1.2}$$

where dx denotes the Lebesgue's measure or, equivalently,

$$1 \leqslant \left(\frac{\displaystyle\int_\Omega |f(x)|\,|x|^\alpha \mathrm{d}x}{\displaystyle\int_\Omega |f(x)|\mathrm{d}x}\right)^{\frac{\beta}{\alpha+\beta}} \left(\frac{\displaystyle\int_\Omega |f(x)|\,|x|^{-\beta} \mathrm{d}x}{\displaystyle\int_\Omega |f(x)|\mathrm{d}x}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Obviously, (7.1.2) is a particular case of (7.1.1), in which the functions Φ and Ψ are respectively $\Phi(t) = t^{\frac{\beta}{\alpha+\beta}}$ and $\Psi(t) = t^{\frac{\alpha}{\alpha+\beta}}$.

This paper is devoted to obtain a generalization of (7.1.2) for non-homogeneous weights. We are typically interested in situations in which, for instance, $\omega_1(x) = e^{-|x|}$ and $\omega_2(x) = |x|^2$. As we shall see, if we are able to get an interpolation inequality of the form (7.1.1) in this case, we will be able to give new explicit decay rates for damped $1-d$ wave equations with pointwise damping.

Let us briefly illustrate the connection between these two issues.

Let $a \in L^\infty(0,1)$ be a nonnegative and bounded damping potential and consider the damped wave equation in one space dimension,

$$\begin{cases} u_{tt}(t,x) - u_{xx}(t,x) + a(x)u_t(t,x) = 0, & \text{for} \quad (t,x) \in (0,\infty)\times(0,1), \\ u(t,0) = u(t,1) = 0, & \text{for} \quad t \in [0,\infty), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), & \text{for} \quad x \in (0,1). \end{cases} \tag{7.1.3}$$

This system is well-posed. More precisely, for any initial data $u^0 \in H_0^1(0,1)$ and $u^1 \in L^2(0,1)$, there exists a unique solution in the class $\mathcal{C}([0,\infty); H_0^1(0,1)) \cap \mathcal{C}^1([0,\infty); L^2(0,1))$. The energy of solutions

$$E(t) = \frac{1}{2}\left(\|u_t(t)\|^2_{L^2(0,1)} + \|u_x(t)\|^2_{L^2(0,1)}\right),$$

decreases along trajectories according to the dissipation law

$$\frac{d}{\mathrm{d}t}E(t) = -\int_0^1 a(x)|u_t(t,x)|^2\mathrm{d}x. \tag{7.1.4}$$

The decay rate of the energy depends on the efficiency of the damping term when absorbing the energy of the system according to (7.1.4).

Using LaSalle's invariance principle, it is easy to see that the energy of every solution tends to zero as $t \longrightarrow \infty$ whenever the damping potential a satisfies for almost every $x \in I$, $a(x) \geqslant a_0 > 0$, for some constant $a_0 > 0$, where $I \subset (0,1)$ is a set of positive measure (Haraux [95]). In the $1-d$ case under consideration, in fact, one can even show that the energy of solutions tends to zero exponentially. To prove this fact, it is sufficient to show that for some $T > 0$ and $C > 0$ the following inequality holds

$$E(0) \leqslant C \int_0^T \int_0^1 a(x)|u_t(t,x)|^2 \mathrm{d}x\mathrm{d}t, \tag{7.1.5}$$

for every solution.

This inequality, which is often referred to as *observability inequality*, asserts that the damping mechanism during a time interval $(0, T)$ suffices to capture a fraction of the total energy of all solutions.

Combining (7.1.4), (7.1.5) and the semigroup property, it is easy to see that the exponential decay property holds, i.e. there exist $C > 0$ and $\omega > 0$ such that

$$\forall t \geqslant 0,\ E(t) \leqslant CE(0)e^{-\omega t}, \tag{7.1.6}$$

for every solution.

In fact, to prove that (7.1.5) is fulfilled, one can use the fact that it is sufficient to prove it for the solutions of the corresponding conservative systems (7.1.3) with $a = 0$. In that case, the inequality is easy to get for $T = 2$ using the Fourier decomposition of solutions.

Let us now consider a case where the control is supported simply on a point $a \in (0, 1)$ through a Dirac mass,

$$u_{tt} - u_{xx} + \delta_a u_t(t,a) = 0, \quad (t,x) \in (0,\infty) \times (0,1), \tag{7.1.7}$$

with the same boundary conditions, initial data and energy as before. Here, δ_a denotes the Dirac mass concentrated in a.

When the point $a \in \mathbb{Q}$, there are solutions of (7.1.7) that do not decay and for which the energy is constant in time. This is due to the fact that rational points are nodal ones for the corresponding Sturm-Liouville problem.

When $a \notin \mathbb{Q}$, LaSalle's invariance principle allows proving that the energy of each solution tends to zero as $t \longrightarrow \infty$. However, in this case the exponential decay rate does not hold. This is due to the fact that, even if $a \notin \mathbb{Q}$, the damping term does not dissipative uniformly all the Fourier components of the solutions. This can be easily seen when analyzing the analogue of (7.1.5). Indeed, there exists a sequence of separate variable solutions of the conservative problem (7.1.3) with $a = 0$ for which the energy $E(0)$ is of order one and the dissipated quantity, $\int_0^T |u_t(t,a)|^2 \mathrm{d}t$, tends to zero. This sequence can be built in separated variables, based on the sequence of eigenfunction $\sin(nx)$ such that $\sin(na)$ tends to zero as n tends to infinity. The main difference with the case where the damping potential $a \geqslant 0$ is positive on a set of positive measure is that, in that case, $\inf_{n \geqslant 1} \int_0^1 a(x)\sin^2(nx)\mathrm{d}x > 0$.

In view of this, one may only expect a weaker observability inequality to hold. A natural way of proceeding in this case is to obtain a weakened version of (7.1.5) in which the energy $E(0)$ in the left hand side is replaced by a weaker energy $E_-(0)$ which, roughly speaking, is the Fourier norm of solutions with weights $\sin^2(na)$. More precisely,

$$E_-(0) \leqslant C \int_0^T |u_t(t,a)|^2 \mathrm{d}t = -C(E(T) - E(0)). \tag{7.1.8}$$

The problem is then how to derive an explicit decay rate for the energy E out of (7.1.8). First, we need to assume some more regularity on the initial data, say, $(u^0, u^1) \in [H^2(0,1) \cap H_0^1(0,1)] \times H_0^1(0,1)$. We denote by E_+ the corresponding energy, $E_+(0) = \frac{1}{2}\|(u^0,u^1)\|^2_{H^2(0,1)\times H_0^1(0,1)}$.

In this way, we have three different energies with different degrees of strength : E, which is the reference energy in which we are interested, E_+, which is finite because the initial data have been taken to be smooth, and E_- which is the weaker energy the damping really damps out according to (7.1.8).

Applying (7.1.1), one can deduce an interpolation inequality of the form

$$1 \leqslant \Phi\left(\frac{E_-(0)}{E(0)}\right)\Psi\left(\frac{E_+(0)}{E(0)}\right), \tag{7.1.9}$$

where Φ and Ψ depend on the energies E_+ and E_- under consideration, $E_+(0)$ being the strong norm $E_+(0) = \frac{1}{2}\|(u^0,u^1)\|^2_{H^2(0,1)\times H_0^1(0,1)}$. This clearly implies

$$E(0)\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{E_+(0)}{E(0)}\right)}\right) \leqslant E_-(0), \tag{7.1.10}$$

which, together with the weak observability inequality (7.1.8) yields,

$$E(0)\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{E_+(0)}{E(0)}\right)}\right) \leqslant C(E(0) - E(T)), \tag{7.1.11}$$

which, together with the semigroup property yield (see Ammari and Tucsnak [8]),

$$\forall t \geqslant 0,\ E(t) \leqslant \frac{C}{\Psi^{-1}\left(\frac{1}{\Phi\left(\frac{1}{t+1}\right)}\right)}\|(u^0,u^1)\|^2_{H^2(0,1)\times H_0^1(0,1)}. \tag{7.1.12}$$

Our method is closely of that one developed by Nicaise [143], in which the decay estimate of the energy looks like (7.1.12) (see Section 5 in [143]). But unfortunately, his method cannot apply in this paper because the damping term has to be more regular, in some sense, that one we consider (see [143]).

Obviously, the decay rate in (7.1.12) depends on the behavior of the functions Ψ and Φ. More precisely, it depends on the behavior of $\Phi(t)$ near $t = 0$ and then of that of Ψ^{-1} at infinity. Therefore, in order to determine the decay of solutions it is necessary to have a sharp description of the functions Φ and Ψ entering in the interpolation inequality.

The behavior of Φ and Ψ depends on the energies E, E_+ and E_- under consideration. We recall that E_- is given by the weak observability inequality (7.1.8). This is intimately related to the weakness of the damping mechanism and no choice can be done at that level. By the contrary, there is some liberty at the level of choosing E_+ since the initial data can be chosen to be as smooth as we like. Obviously, one expects a faster decay rate for solutions when they are smoother. This is indeed the case as our analysis shows. All this can be precisely quantified by the analysis of the functions Φ and Ψ in the interpolation inequality.

How Φ and Ψ depend on the energies E_+ and E_-, in the general context of the interpolation inequality (7.1.1), corresponds to analyzing how the functions Φ and Ψ depend on the weight functions ω_1 and ω_2. This article is precisely devoted to prove a rather general version of (7.1.1) with a careful analysis of the behavior of Φ and Ψ. This will allow us to get explicit decay rates not only for the model problem above of the $1-d$ wave equation with pointwise damping but also for some other models that we shall discuss below. In particular, we will be able to give explicit decay rates for the stabilization of a beam by means of a piezoelectric actuators, a problem that was discussed by Tucsnak [173, 174] in the context of control.

There is an extensive literature concerning the stabilization of damped wave-like equations. But most of it refers to the case where the damping term (linear or nonlinear one) is able to capture the whole energy of the system (see, for instance, Haraux and Zuazua [103], Nicaise [143] and Zuazua [184]). In these works, the multiplier method is implied, as a tool to quantify the amount of energy that the dissipative mechanism is able to observe. But to apply this method, the damping term has to be active in a large subset of the domain or of the boundary where the equation holds. Much less is known when the damping term is located in a narrow set, like, for instance, pointwise dampers in one space dimension. But, as we have shown above, the results one may expect in that setting need to be necessarily of a weaker nature since in those situations the damping term is only able to absorb a lower order energy. In particular, in this context, multiplier methods do not apply.

We focus mainly on the wave equation with a damping control concentrated on an interior point. Some partial results of explicit decay rates already exist and can be found in Ammari, Henrot and Tucsnak [5, 6], Jaffard, Tucsnak and Zuazua [110] and Tucsnak [175]. As explained above, our generalized interpolation inequality allows answering to this in much more generality. We will also address the stabilization of Bernoulli–Euler beams with force and moment damping. For partial results of explicit decay rates, see Ammari and Tucsnak [7].

This paper is organized as follows. In Section 7.2, we establish our generalized Hölder's inequality or interpolation inequality (Theorems 7.2.1 and 7.2.2). In Section 7.3, we give a criterion of optimality for Theorem 7.2.1 (Definition 7.3.3) and a sufficient condition to have optimality in our interpolation inequality (Proposition 7.3.5). In Section 7.4, we apply these results to get explicit decay rates for the damped wave (see (7.4.2.1)) with Dirichlet boundary condition and in Section 7.5 we briefly explain how these results can be applied to the wave equation with mixed boundary condition (Subsection 7.5.1, equation (7.5.1.1)) and to some beam equations (Subsection 7.5.2, equation (7.5.2.1)). The explicit decay rates are given. These results extend the previous ones by Ammari, Henrot and Tucsnak [6], Ammari and Tucsnak [7] and Jaffard, Tucsnak and Zuazua [110].

We end this section by introducing some notations. For a real valued function f defined on an open interval I (respectively, (m,∞) for some $m\in\mathbb{R}$) and for $a\in\partial I$ (respectively, $a\in\{m,\infty\}$), the notation $f(a)$ means $\lim\limits_{\substack{t\to a\\ t\in I}} f(t)$. For $a\in\mathbb{R}$, we denote by δ_a the Dirac mass concentrated in a.

7.2 An interpolation inequality

Our analysis requires some elementary notions and results on convex functions.

Recall that if $f:I\longrightarrow\mathbb{R}$ is a convex function on an open interval I, then it is continuous, locally absolutely continuous on I and it is of class $\mathcal{C}^1$ almost everywhere. More precisely, there exists a finite or countable set $\mathcal{N}\subset I$ such that f is of class $\mathcal{C}^1$ relatively to $I\setminus\mathcal{N}$. In particular, for any $t,s\in I$, $f(t)-f(s)=\displaystyle\int_s^t f'(\sigma)\mathrm{d}\sigma$. In addition, f' is nondecreasing relatively to $I\setminus\mathcal{N}$. Furthermore, f has a left derivative f'_ℓ and a right derivative f'_{r} at each point of I and for any $t,s\in I$ such that $s<t$, $f'_\ell(s)\leqslant f'_{\mathrm{r}}(s)\leqslant f'_\ell(t)\leqslant f'_{\mathrm{r}}(t)$. For more details, see Niculescu and Persson [144] (Theorems 1.3.1 and 1.3.3, p.12, Proposition 3.4.2, p.87 and Theorem 3.7.3, p.96) and Rockafellar [153] (Corollary 10.1.1, p.83, Theorem 10.4, p.86 and Theorem 25.3, p.244). Finally, we recall that f is a concave function if $-f$ is a convex function.

Let $(\Omega,\mathscr{T},\mu)$ be a measure space and let $\omega_1,\omega_2:\Omega\longrightarrow[0,\infty)$ be two μ–measurable weights. In order to establish our generalized Hölder's inequality, we need the following hypotheses.

$$\begin{cases}\Phi:I_1\longrightarrow[0,\infty)\text{ is a concave function, }I_1\text{ is an}\\ \text{open interval and for a.e. }x\in\Omega,\ \omega_1(x)\in I_1,\end{cases}\tag{7.2.1}$$

$$\begin{cases}\Psi:I_2\longrightarrow[0,\infty)\text{ is a concave function, }I_2\text{ is an}\\ \text{open interval and for a.e. }x\in\Omega,\ \omega_2(x)\in I_2,\end{cases}\tag{7.2.2}$$

$$\text{for a.e. }x\in\Omega,\ 1\leqslant\Phi(\omega_1(x))\Psi(\omega_2(x)).\tag{7.2.3}$$

Theorem 7.2.1. *Let $(\Omega,\mathscr{T},\mu)$ be a measure space, $\omega_1,\omega_2:\Omega\longrightarrow[0,\infty)$ be two μ–measurable weights and $0<p<\infty$. If there exist two functions Φ et Ψ satisfying (7.2.1) – (7.2.3) then for any $f\in L^p(\Omega,\mathscr{T},\mu)$, $f\not\equiv0$, we have*

$$1\leqslant\Phi\left(\frac{\displaystyle\int_\Omega|f|^p\omega_1\mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right)\Psi\left(\frac{\displaystyle\int_\Omega|f|^p\omega_2\mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right),\tag{7.2.4}$$

as soon as $L^p(\Omega,\mathscr{T},\omega_1\mathrm{d}\mu)\cap L^p(\Omega,\mathscr{T},\omega_2\mathrm{d}\mu)$.

Obviously, one of the main issues to be clarified is whether there exist functions Φ and Ψ satisfying the requirements (7.2.1), (7.2.2) and (7.2.3). This, of course, depends on the properties that the weight functions ω_1 and ω_2 satisfy. Below we shall give sufficient conditions on the weights ω_1 and ω_2 guaranteeing that Φ and Ψ as above exist. This can be done by imposing some stronger conditions on the weight functions. More precisely, assume that $\Omega = (m, \infty)$ (for some $m \in \mathbb{R}$), $\mathrm{d}\mu = \mathrm{d}x$ is the Lebesgue's measure and

$$\omega_1 : (m, \infty) \longrightarrow (0, \omega_1(m)) \text{ is a convex and decreasing function and } \omega_1(\infty) = 0, \tag{7.2.5}$$

$$\omega_2 : (m, \infty) \longrightarrow (0, \infty) \text{ is a convex and increasing function and } \omega_2(\infty) = \infty, \tag{7.2.6}$$

$$\Phi : (0, \omega_1(m)) \longrightarrow (0, \infty) \text{ is a concave and increasing function and } \Phi(0) = 0, \tag{7.2.7}$$

$$\Psi : (\omega_2(m), \infty) \longrightarrow (0, \infty) \text{ is a concave and increasing function and } \Psi(\infty) = \infty, \tag{7.2.8}$$

$$\forall t \in (m, \infty),\ 1 \leqslant \Phi(\omega_1(t))\Psi(\omega_2(t)). \tag{7.2.9}$$

Note that in (7.2.7), hypothesis $\Phi(0) = 0$ means that Φ can be extended by continuity in 0 by 0.

The following result asserts that functions satisfying (7.2.7)–(7.2.9) (and so (7.2.1)–(7.2.3)) exist, if the weights ω_1 and ω_2 verify the additional assumptions (7.2.5)–(7.2.6).

Theorem 7.2.2. *Let $m > 0$ and let ω_1, ω_2, be two weights satisfying (7.2.5) – (7.2.6). We define the function φ by*

$$\forall t > m,\ \varphi(t) = m\frac{\omega_1(t)}{t}. \tag{7.2.10}$$

Then the following assertions hold.

1. *The function Φ defined on $[0, \omega_1(m))$ by $\Phi(0) = 0$ and $\Phi(t) = \dfrac{1}{\varphi^{-1}(t)}$, for $t \neq 0$, satisfies (7.2.7).*
2. *The function Ψ defined on $(\omega_2(m), \infty)$ by $\Psi(t) = \omega_2^{-1}(t)$ satisfies (7.2.8).*
3. *For Φ and Ψ defined as above, estimate (7.2.9) holds.*

Before proving Theorems 7.2.1–7.2.2, let us establish some preliminaries lemmas. The following result being a direct consequence of the definition of convex functions, we omit the proof.

Lemma 7.2.3. *Let $I \subset \mathbb{R}$ be an interval and let $\varphi : I \longrightarrow \mathbb{R}$ be a function. Then φ is increasing and concave on I if and only if φ^{-1} is increasing and convex on $\varphi(I)$.*

The next lemma is the inverse version of the classical Jensen's inequality (W. Rudin [157]).

Lemma 7.2.4 (Inverse Jensen's inequality)**.** *Let $(\Omega, \mathscr{T}, \nu)$ be a measure space such that $\nu(\Omega) = 1$ and let $-\infty \leqslant a < b \leqslant +\infty$. Assume that*
1) *$\varphi : (a, b) \longrightarrow \mathbb{R}$ is a concave function,*
2) *$f \in L^1(\Omega, \mathscr{T}, \nu)$ is such that for almost every $x \in \Omega$, $f(x) \in (a, b)$.*
Then $\varphi(f)_+ \in L^1(\Omega, \mathscr{T}, \nu)$ and

$$\int_\Omega \varphi(f)\mathrm{d}\nu \leqslant \varphi\left(\int_\Omega f\mathrm{d}\nu\right). \tag{7.2.11}$$

Remark 7.2.5. Since φ is concave on (a, b), it is continuous and $\varphi \circ f$ is a $\mathscr{T}$-measurable function. Furthermore, $\varphi(f)_+ \in L^1(\Omega, \mathscr{T}, \nu)$ so the left-hand side of (7.2.11) makes sense and $\displaystyle\int_\Omega \varphi(f)\mathrm{d}\nu \in$

$[-\infty,+\infty)$. Indeed, since φ is a concave function, it follows from the discussion at the beginning of this section that for any $t,s\in(a,b)$, $\varphi(t)\leqslant\varphi(s)+\varphi_\ell'(s)(t-s)$. In particular,

$$\begin{aligned}\varphi(f)&\leqslant\varphi(t_0)+\varphi_\ell'(t_0)(f-t_0),\ \text{a.e. in }\Omega,\\ \varphi(f)_+&\leqslant|\varphi(t_0)|+|\varphi_\ell'(t_0)|(|f|+|t_0|)\in L^1(\Omega,\mathscr{T},\nu),\end{aligned}\tag{7.2.12}$$

where $t_0=\int_\Omega f\mathrm{d}\nu$. Integrating (7.2.12) over Ω, we obtain (7.2.11). For more details, see Theorem 3.3 p.62 in W. Rudin [157].

Now, we are in the conditions to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. Let $0<p<\infty$, let $f\in L^p(\Omega,\mathscr{T},\mu)\cap L^p(\Omega,\mathscr{T},\omega_1\mathrm{d}\mu)\cap L^p(\Omega,\mathscr{T},\omega_2\mathrm{d}\mu)$, $f\not\equiv0$, and let ν be the measure defined by $\nu=\frac{|f|^p}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\mu$. Then $\nu(\Omega)=1$. We apply twice Lemma 7.2.4 with $\varphi_1=\Phi$, $f_1=\omega_1$, $\varphi_2=\Psi$ and $f_2=\omega_2$. Then $\Phi\circ\omega_1\in L^1(\Omega,\mathscr{T},\nu)$, $\Psi\circ\omega_2\in L^1(\Omega,\mathscr{T},\nu)$ and it follows from (7.2.3), Cauchy-Schwarz's inequality and (7.2.11) that

$$\begin{aligned}1=\left(\int_\Omega 1^{\frac12}\mathrm{d}\nu\right)^2&\leqslant\left(\int_\Omega\Phi^{\frac12}(\omega_1(x))\Psi^{\frac12}(\omega_2(x))\mathrm{d}\nu(x)\right)^2\\ &\leqslant\int_\Omega\Phi(\omega_1(x))\mathrm{d}\nu(x)\int_\Omega\Psi(\omega_2(x))\mathrm{d}\nu(x)\\ &\leqslant\Phi\left(\int_\Omega\omega_1(x)\mathrm{d}\nu(x)\right)\Psi\left(\int_\Omega\omega_2(x)\mathrm{d}\nu(x)\right)\\ &=\Phi\left(\frac{\int_\Omega|f|^p\omega_1\mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right)\Psi\left(\frac{\int_\Omega|f|^p\omega_2\mathrm{d}\mu}{\|f\|^p_{L^p(\Omega,\mathscr{T},\mu)}}\right).\end{aligned}$$

Hence (7.2.4). □

The proof of Theorem 7.2.2 relies on the following lemma.

Lemma 7.2.6. *Let $m\in[0,\infty)$, $0<M\leqslant\infty$ and $p\in[1,\infty)$. Let $f:(m,\infty)\longrightarrow(0,M)$ be a nonincreasing function such that $f(m)=M$. Define the function φ_p on (m,∞) by*

$$\forall t>m,\ \varphi_p(t)=\frac{f(t)}{t^p}.\tag{7.2.13}$$

If f is convex on (m,∞) then φ_p is convex on (m,∞) and $\frac{1}{\varphi_p^{-1}}$ is concave and increasing on $\left(0,\frac{M}{m^p}\right)$, where we have used the notation $\frac{M}{m^p}=+\infty$ if $m=0$ and/or $M=+\infty$. Furthermore, $\lim\limits_{t\searrow0}\frac{1}{\varphi^{-1}(t)}=0$.

Remark 7.2.7. If $0<p<1$ then the conclusion of Lemma 7.2.6 may be false. Indeed, let $q_0\in(p,1)$ and set $q=\frac{1}{q_0}>1$. We then choose $f(t)=\frac{1}{t^{q_0-p}}$, $t>0$. Then f and φ_p are obviously convex and decreasing on $(0,\infty)$. But for any $t>0$, $\frac{1}{\varphi_p^{-1}(t)}=t^q$. So that φ_p is not concave on $(0,\infty)$ since $q>1$.

Remark 7.2.8. Let $f:(m,\infty)\longrightarrow(0,\infty)$ be an application, where $m\in\mathbb{R}$. Assume that f is convex on (m,∞) and that $\lim\limits_{t\to\infty}f(t)=0$. If f is nonincreasing on (m,∞) then it is in fact decreasing on (m,∞). Indeed, if f is not decreasing on (m,∞) then $f(t)=f(a)>0$ for any $t\in(a,b)$, for some interval $(a,b)\subset(m,\infty)$. Since $\lim\limits_{t\to\infty}f(t)=0$, we necessarily have $b<\infty$. Then $f'\equiv0$ on (a,b) and, by hypothesis $\lim\limits_{t\to\infty}f(t)=0$, this implies that $f'(t_0)<0$, for some $t_0\in(b,\infty)$. This contradicts hypothesis f is convex.

Proof of Lemma 7.2.6. Let φ_p be defined by (7.2.13). Note that $\varphi_p:(m,\infty)\longrightarrow\left(0,\frac{M}{m^p}\right)$ being bijective, continuous and decreasing, it follows that $\varphi_p^{-1}:\left(0,\frac{M}{m^p}\right)\longrightarrow(m,\infty)$ is well-defined, continuous and decreasing. So $\frac{1}{\varphi_p^{-1}}:\left(0,\frac{M}{m^p}\right)\longrightarrow\left(0,\frac{1}{m}\right)$ is continuous and increasing, where we have used the notation $\frac{1}{m}=+\infty$ if $m=0$. The product of two positive and convex functions with the same monotonicity being convex, it follows that the function $t\longmapsto\frac{f(t)}{t^p}$ is convex and so φ_p is convex. Moreover, hypothesis $\lim\limits_{t\nearrow\infty}\varphi(t)=0$ implies that $\lim\limits_{t\searrow0}\frac{1}{\varphi^{-1}(t)}=0$. Since f is convex, according to the basic properties on convex functions we recalled in the beginning of this section, there exists a sequence $(a_n)_{n\in\mathbb{N}}\subset(m,\infty)$ such that f is $\mathcal{C}^1$ and f' is nondecreasing relatively to $(m,\infty)\setminus\mathcal{N}$, with $\mathcal{N}=\bigcup\limits_{n=1}^{\infty}\{a_n\}$. Now, we proceed to the proof in 3 steps.

Step 1. Set for every $t\in(m,\infty)\setminus\mathcal{N}$,

$$h(t)=-(f'(t)t-pf(t))\quad\text{ and }\quad g(t)=\frac{h(t)}{t^{p-1}}. \tag{7.2.14}$$

Then g is nonincreasing and nonnegative on $(m,\infty)\setminus\mathcal{N}$.

Indeed, let $s,t\in(m,\infty)\setminus\mathcal{N}$ be such that $s<t$. Since f is convex, it follows from the discussion at the beginning of this section that $f(t)-f(s)\leqslant f'(t)(t-s)$. Using this estimate, $p\geqslant1$ and again the fact that f is nonincreasing and f' is nondecreasing relatively to $(m,\infty)\setminus\mathcal{N}$, we obtain that

$$\begin{aligned}h(t)-h(s)&=p(f(t)-f(s))-(t-s)f'(t)-s(f'(t)-f'(s))\\&\leqslant f(t)-f(s)-f'(t)(t-s)\leqslant0.\end{aligned}$$

Consequently, h is is nonincreasing. Since it is nonnegative (because f is nonnegative and nonincreasing), it follows that g is also nonincreasing and nonnegative relatively to $(m,\infty)\setminus\mathcal{N}$.

Step 2. We claim that, for any $t>m$, $\varphi_p(t)=\displaystyle\int_0^{1/t}g\left(\frac{1}{s}\right)ds$.

Indeed, by (7.2.13)–(7.2.14), we have for every $\sigma\in(m,\infty)\setminus\mathcal{N}$,

$$-\varphi_p'(\sigma)=-\frac{f'(\sigma)\sigma^p-pf(\sigma)\sigma^{p-1}}{\sigma^{2p}}=-\frac{f'(\sigma)\sigma-pf(\sigma)}{\sigma^{p+1}}=\frac{h(\sigma)}{\sigma^{p-1}}\frac{1}{\sigma^2}=\frac{g(\sigma)}{\sigma^2}.$$

Then for any $\varepsilon>0$, $\varphi_p'\in L^1(m+\varepsilon,\infty)$ and so $\varphi_p(t)=\displaystyle\int_t^\infty\frac{g(\sigma)}{\sigma^2}\mathrm{d}\sigma$, which yields the desired result, by using the change of variables $\sigma=\frac{1}{s}$.

Step 3. Conclusion.

Let ψ be defined on $\left(0,\frac{M}{m^p}\right)$ by $\psi(t)=\frac{1}{\varphi_p^{-1}(t)}$. Thus by Step 2, we have for any $t\in\left(0,\frac{1}{m}\right)$,

$$\psi^{-1}(t)=\varphi_p\left(\frac{1}{t}\right)=\int\limits_0^t g\left(\frac{1}{s}\right)ds.$$

Then ψ^{-1} is absolutely continuous and for almost every $t \in \left(0, \frac{1}{m}\right)$, $\left(\psi^{-1}\right)'(t) = g\left(\frac{1}{t}\right) \geqslant 0$. Since g is nonincreasing relatively to $(m, \infty) \setminus \mathcal{N}$ (Step 1), it follows that ψ^{-1} is increasing and convex on $\left(0, \frac{1}{m}\right)$. By Lemma 7.2.3, $\psi \overset{\text{def}}{=} \frac{1}{\varphi_p^{-1}}$ is increasing and concave on $\left(0, \frac{M}{m^p}\right)$. Hence the result. □

Proof of Theorem 7.2.2. Let φ be defined on (m, ∞) by (7.2.10). By (7.2.5)–(7.2.6), ω_2 is invertible on $(\omega_2(m), \infty)$ and $\varphi : (m, \infty) \longrightarrow (0, \omega_1(m))$ is a bijective and decreasing function. Then definition of Φ and Ψ makes sense.

Proof of 1–2. Assertion 1 is a direct consequence of Lemma 7.2.6 applied to $f = m\omega_1$ and assertion 2 comes from (7.2.6) and Lemma 7.2.3.

Proof of 3. By (7.2.10) and definition of Φ, $\Phi^{-1}\left(\frac{1}{t}\right) = \varphi(t) \leqslant \omega_1(t)$, for any $t > m$. Since φ and ω_1 are both decreasing, this implies that

$$\forall t \in (0, \omega_1(m)),\ \Phi(t) = \frac{1}{\varphi^{-1}(t)} \geqslant \frac{1}{\omega_1^{-1}(t)}.$$

With the above estimate, we obtain that

$$\forall t > m,\ \Phi(\omega_1(t))\Psi(\omega_2(t)) = \Phi(\omega_1(t))t \geqslant \frac{t}{\omega_1^{-1}(\omega_1(t))} = 1.$$

Hence (7.2.9). This concludes the proof. □

We now give an example where the assumptions of Theorem 7.2.1 are satisfied. The weight functions ω_1, ω_2 are of a particular form that arises naturally in applications : While ω_1 tends to zero exponentially at ∞, ω_2 grows as a polynomial function. This is a case that may not be covered by Hölder's inequality. In the sequel, we compute explicitly the functions Φ and Ψ for which the generalized interpolation inequality holds.

Example 7.2.9. Let $\Omega = \mathbb{R}^N \setminus \overline{B}(0,1)$ and $A \geqslant 1$. We consider the weights defined on Ω by $\omega_1(x) = e^{-A|x|}$ and $\omega_2(x) = |x|^2$. We define the interpolating functions $\Psi(t) = \sqrt{t}$ $(t \geqslant 0)$ and

$$\forall t \in [0, e^{A-2}], \quad \Phi(t) = \begin{cases} 0, & \text{if } t = 0, \\ \dfrac{2A}{A - \ln t}, & \text{if } 0 < t \leqslant e^{A-2}. \end{cases}$$

The hypotheses of Theorem 2.1 are satisfied since the weights ω_1 and ω_2 and the interpolation functions Φ and Ψ defined as above, satisfy the pointwise inequality (7.2.3) as it is immediate to check. Indeed, for any $x \in \Omega$,

$$\Phi(\omega_1(x))\Psi(\omega_2(x)) = \frac{2A|x|}{A + A|x|} = \frac{2|x|}{1 + |x|} \geqslant 1,$$

since $|x| > 1$. Moreover, a straightforward calculation shows that Φ is concave on $[0, e^{A-2}]$. As a consequence of Theorem 2.1 we obtain the following functional generalized interpolation inequality. Let $f \in L^2(\Omega;\mathbb{C}) \setminus \{0\}$ be such that $|\,.\,|f(\,.\,) \in L^2(\Omega;\mathbb{C})$. Then,

$$\|f\|_{L^2(\Omega)} \leqslant 2\sqrt{\int\limits_\Omega |f(x)|^2 |x|^2 \mathrm{d}x \frac{A}{A + \ln\left(\dfrac{1}{\|f\|^2_{L^2(\Omega)}} \displaystyle\int\limits_\Omega |f(x)|^2 e^{-A|x|} \mathrm{d}x\right)}}. \tag{7.2.15}$$

In the same way, we have

$$\|u\|_{\ell^2(\mathbb{N})} \leqslant 2\sqrt{\sum_{n=1}^{\infty} n^2|u_n|^2}\frac{A}{A-\ln\left(\frac{1}{\|u\|^2_{\ell^2(\mathbb{N})}}\sum_{n=1}^{\infty} e^{-An}|u_n|^2\right)}, \tag{7.2.16}$$

for any $u=(u_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N};\mathbb{C})\setminus\{0\}$ such that $(nu_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{N};\mathbb{C})$. Note that one always has for any $A\geqslant 1$,

$$0<\frac{1}{\|f\|^2_{L^2(\Omega)}}\int_\Omega |f(x)|^2e^{-A|x|}\mathrm{d}x \leqslant e^{-A}\leqslant e^{A-2}$$

and

$$0<\frac{1}{\|u\|^2_{\ell^2(\mathbb{N})}}\sum_{n=1}^{\infty} e^{-An}|u_n|^2\leqslant e^{-A}\leqslant e^{A-2},$$

(since $e^{-A}\leqslant e^{A-2}\iff A\geqslant 1$) so the above quantities takes their values in the domain of concavity of Φ. It follows that estimates (7.2.15) and (7.2.16) always make sense.

7.3 Optimality

It this section, we discuss the notion of optimality for the pairs of functions (Φ,Ψ) satisfying the interpolation inequalities above. We will also give sufficient conditions guaranteeing the pair is optimal. Throughout this section, for simplicity, we assume that $\Omega=(m,\infty)$ (for some $m\in\mathbb{R}$) and that $\mathrm{d}\mu=\mathrm{d}x$ is the Lebesgue's measure. Before introducing the definition of optimality, we need the following lemma.

Lemma 7.3.1. *Let $m\in\mathbb{R}$ and let ω_1, ω_2, Φ and Ψ satisfy (7.2.5) – (7.2.9). Let $\delta\in(0,\omega_1(m)]$ be such that $\Phi(\delta)=\frac{1}{\Psi(\omega_2(m))}$, if $\Psi(\omega_2(m))>0$ and let $\delta=+\infty$, if $\Psi(\omega_2(m))=0$. We define*

$$\forall t\in(0,\delta),\ \mathcal{H}_{\Phi,\Psi}(t)=\frac{1}{\Psi^{-1}\left(\frac{1}{\Phi(t)}\right)}. \tag{7.3.1}$$

Then $\mathcal{H}_{\Phi,\Psi}$ is a positive, increasing and continuous function on $(0,\delta)$ and $\lim\limits_{t\searrow 0}\mathcal{H}_{\Phi,\Psi}(t)=0$. Furthermore,

$$\forall t\in(0,\delta),\ 0<\frac{1}{\omega_2\circ\omega_1^{-1}(t)}\leqslant\mathcal{H}_{\Phi,\Psi}(t). \tag{7.3.2}$$

Finally,

$$\mathcal{H}^{-1}_{\Phi,\Psi}(t)=\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{1}{t}\right)}\right), \tag{7.3.3}$$

for any $t\in(0,\mathcal{H}_{\Phi,\Psi}(\delta))$.

Remark 7.3.2. Note that such a $\delta\in(0,\omega_1(m)]$ exists because of the continuity of Φ.

Assuming for the moment that Lemma 7.3.1 holds (we shall return to its proof later), the following definition makes sense.

Definition 7.3.3. Let $m \in \mathbb{R}$ and ω_1, ω_2, Φ and Ψ satisfy (7.2.5)–(7.2.9). We say that $(\Phi,\, \Psi)$ is an *optimal pair for the weights* (ω_1, ω_2) if the function $\mathcal{H}_{\Phi,\Psi}$ defined by (7.3.1) satisfies

$$\mathcal{H}_{\Phi,\Psi} \overset{0}{\approx} \frac{1}{\omega_2 \circ \omega_1^{-1}}. \tag{7.3.4}$$

Here and in the sequel, by $\mathcal{H}_{\Phi,\Psi} \overset{0}{\approx} \dfrac{1}{\omega_2 \circ \omega_1^{-1}}$ we mean that there exist two constants $C > 0$ and $\varepsilon \in (0, \delta)$ such that

$$\forall t \in (0, \varepsilon),\ \frac{1}{\omega_2 \circ \omega_1^{-1}(t)} \leqslant \mathcal{H}_{\Phi,\Psi}(t) \leqslant \frac{C}{\omega_2 \circ \omega_1^{-1}(t)}, \tag{7.3.5}$$

where $\delta > 0$ is given in Lemma 7.3.1.

In view of (7.3.2) when (7.3.4) holds, the function $\mathcal{H}_{\Phi,\Psi}(t)$ goes to 0 as $t \searrow 0$ as rapidly as possible. The pair (Φ, Ψ) is then optimal in that sense. As we shall see in applications, this will yield the optimal decay rate for the energy of solutions of damped wave-like equations.

Remark 7.3.4. It is important to note that the notion of optimal pair (Φ, Ψ) depends on the weights (ω_1, ω_2). On the other hand, given two weights ω_1 and ω_2 satisfying (7.2.5)–(7.2.6) and a pair (Φ, Ψ) satisfying (7.2.8)–(7.2.9), if $\Phi^{-1}\left(\frac{1}{\Psi \circ \omega_2}\right)$ is convex then the pair (Φ, Ψ) is necessarily optimal with respect to the weights $\widetilde{\omega_1}$ and ω_2, where we have chosen $\widetilde{\omega_1}(t) = \Phi^{-1}\left(\frac{1}{\Psi(\omega_2(t))}\right)$. Indeed, (7.2.5)–(7.2.8) hold for $(\widetilde{\omega_1}, \omega_2, \Phi, \Psi)$. Moreover,

$$\Phi(\widetilde{\omega_1}(t))\Psi(\omega_2(t)) = \frac{1}{\Psi(\omega_2(t))}\Psi(\omega_2(t)) = 1,$$

and (7.2.9) is fulfilled. Finally, a straightforward calculation gives

$$\mathcal{H}_{\Phi,\Psi}(t) \overset{\text{def}}{=} \frac{1}{\Psi^{-1}\left(\dfrac{1}{\Phi(t)}\right)} = \frac{1}{\omega_2 \circ \widetilde{\omega_1}^{-1}(t)}.$$

Hence (7.3.4).

Now we give a sufficient condition for the pair (Φ, Ψ) to be optimal.

Proposition 7.3.5. *Let $m \in \mathbb{R}$ and let ω_1 and ω_2 be satisfying (7.2.5) − (7.2.6). Let $1 \leqslant p < \infty$, and set*

$$\forall t > \omega_2(m), \quad \Psi_p(t) = \left(\omega_2^{-1}(t)\right)^{\frac{1}{p}}, \tag{7.3.6}$$

and

$$\forall t \in (0, \omega_1(m)), \quad \Phi_p(t) = \frac{1}{\left(\omega_1^{-1}(t)\right)^{\frac{1}{p}}}, \tag{7.3.7}$$

together with $\Phi_p(0) = 0$.
If $\dfrac{1}{\left(\omega_1^{-1}\right)^{\frac{1}{p}}}$ is concave on $(0, \omega_1(m))$ then (Φ_p, Ψ_p) constitutes an optimal pair for the weights (ω_1, ω_2).

On the other hand, the following Proposition guarantees that, once we have an optimal pair (Φ,Ψ) it is easy to build other optimal pairs. Of course, in practice, when applying the interpolation inequalities to obtain decay rates for evolution equations, it is irrelevant whether one uses an optimal pair or another since all of them, by definition, yield the same decay rates.

Proposition 7.3.6. *Let $m\in\mathbb{R}$ and let ω_1, ω_2, Φ and Ψ be satisfying* (7.2.5) – (7.2.7)*. Let $0<p<\infty$, let $(0,\delta)$ be the interval of definition of $\mathcal{H}_{\Phi,\Psi}$ and let $(0,\delta_p)$ be the interval of definition of $\mathcal{H}_{\Phi^p,\Psi^p}$* (*see Lemma* 7.3.1)*. Then*

$$\forall t\in(0,\inf\{\delta,\delta_p\}),\ \mathcal{H}_{\Phi,\Psi}(t)=\mathcal{H}_{\Phi^p,\Psi^p}(t).$$

In particular, if (Φ,Ψ) is an optimal pair for the weights (ω_1,ω_2), then the same holds for (Φ^p,Ψ^p).

Remark 7.3.7. In other words, Proposition 7.3.6 means that, from the point of view of the decay of $\mathcal{H}_{\Phi,\Psi}$, the inequalities $1\leqslant\Phi(\omega_1)\Psi(\omega_2)$ and $1\leqslant\Phi^p(\omega_1)\Psi^p(\omega_2)$, yield the same result.

Proof of Lemma 7.3.1. Let Φ and Ψ be any functions satisfying (7.2.8)–(7.2.9) and $\delta>0$ be defined as in Lemma 7.3.1. It follows from (7.2.5)–(7.2.9) and definition of δ that

$$\forall t\in(0,\omega_1(m)),\ 1\leqslant\Phi(t)\Psi\left(\omega_2\circ\omega_1^{-1}(t)\right)\quad\text{and}\quad\forall t\in(0,\delta),\ 0<\Phi(t)<\frac{1}{\Psi(\omega_2(m))}\leqslant+\infty.$$

We then have

$$\forall t\in(0,\delta),\ 0\leqslant\Psi(\omega_2(m))<\frac{1}{\Phi(t)}\leqslant\Psi\left(\omega_2\circ\omega_1^{-1}(t)\right).$$

Since Ψ^{-1} is increasing on $(\Psi(\omega_2(m)),\infty)$, this gives

$$\forall t\in(0,\delta),\ 0<\Psi^{-1}\left(\frac{1}{\Phi(t)}\right)\overset{\text{def}}{=}\frac{1}{\mathcal{H}_{\Phi,\Psi}(t)}\leqslant\omega_2\circ\omega_1^{-1}(t),$$

which yields (7.3.2). Properties of $\mathcal{H}_{\Phi,\Psi}$ follows easily from (7.2.7)–(7.2.8). □

Proof of Proposition 7.3.6. Let $s\in\mathcal{H}_{\Phi,\Psi}((0,\delta))\cap\mathcal{H}_{\Phi^p,\Psi^p}((0,\delta_p))$. Then we have,

$$\mathcal{H}_{\Phi^p,\Psi^p}(t)=s\iff\frac{1}{(\Psi^p)^{-1}\left(\frac{1}{\Phi^p(t)}\right)}=s\iff(\Psi^p)^{-1}\left(\frac{1}{\Phi^p(t)}\right)=\frac{1}{s}$$
$$\iff\frac{1}{\Phi^p(t)}=\Psi^p\left(\frac{1}{s}\right)\iff\frac{1}{\Phi(t)}=\Psi\left(\frac{1}{s}\right)\iff\mathcal{H}_{\Phi,\Psi}(t)=s.$$

Hence the result. □

Proof of Proposition 7.3.5. Assume that hypotheses of Proposition 7.3.5 are satisfied. It follows from Lemma 7.2.3 and (7.2.6) that Ψ_p satisfies (7.2.8). By (7.2.5) and the fact that $\frac{1}{(\omega_1^{-1})^{\frac{1}{p}}}$ is concave on $(0,\omega_1(m))$, the function Φ_p defined as in (7.3.7) satisfies (7.2.7). By (7.3.6) and (7.3.7), (7.2.9) and (7.3.4) are verified. Indeed, by Proposition 7.3.6,

$$\mathcal{H}_{\Phi_p,\Psi_p}(t)=\mathcal{H}_{\Phi_p^p,\Psi_p^p}(t)=\frac{1}{(\Psi_p^p)^{-1}\left(\frac{1}{\Phi_p^p(t)}\right)}$$
$$=\frac{1}{(\Psi_p^p)^{-1}(\omega_1^{-1}(t))}=\frac{1}{(\omega_2^{-1})^{-1}(\omega_1^{-1}(t))}=\frac{1}{\omega_2\circ\omega_1^{-1}(t)}.$$

This concludes the proof. □

Remark 7.3.8. Note that the hypothesis $p \geqslant 1$ in Proposition 7.3.5 is made to ensure that $\left(\omega_2^{-1}\right)^{\frac{1}{p}}$ is a concave function. So it follows from the above proof that, if $0 < p < 1$ is such that $\left(\omega_2^{-1}\right)^{\frac{1}{p}}$ is concave, then the conclusion of Proposition 7.3.5 still holds.

Remark 7.3.9. Proposition 7.3.6 shows the non uniqueness of the optimal pairs (Φ, Ψ). One may give other examples. Let $m \in \mathbb{R}$ and let ω_1 and ω_2 be satisfying (7.2.5)–(7.2.6). Following the proof of Proposition 7.3.5, we can show that if $\frac{1}{\omega_2 \circ \omega_1^{-1}}$ is concave then the functions $\Psi = \mathrm{Id}$ and $\Phi = \frac{1}{\omega_2 \circ \omega_1^{-1}}$ are an optimal pair of functions.

7.4 Application to the stabilization on the wave equation with Dirichlet boundary condition

In this section, we give some applications of Section 7.2. We recover and extend the results of Ammari, Henrot and Tucsnak [6], Ammari and Tucsnak [7] and Jaffard, Tucsnak and Zuazua [110]. We will detail the first example (Subsection 7.4.2) and we will indicate how we proceed for the others equations (for conciseness of the paper, we will not detail the proof, the method being very technical). We apply our interpolation inequality to the stabilization of a wave equation with a damping control concentrated on an interior point (Subsection 7.4.2) and to the stabilization of a Bernoulli–Euler beam with a damping control concentrated in an interior point (Subsection 7.5.2).

7.4.1 Explanation of the method

To set the context, we introduce some notations and refer to Ammari and Tucsnak [8] for more details. We consider u the solution of the following equation.

$$\begin{cases} u_{tt} + Au + BB^\star u_t = 0, & (t,x) \in (0,\infty) \times I, \\ u(0,x) = u^0(x), & x \in I, \\ u_t(0,x) = u^1(x), & x \in I, \end{cases} \tag{7.4.1.1}$$

where A is a linear unbounded self-adjoint operator, $B \in \mathcal{L}(U; D(A^{\frac{1}{2}})^\star)$, $(U, \| \,.\, \|_U)$ is a complex Hilbert space, $D(A^{\frac{1}{2}}) = \overline{D(A)}^{\| \,\cdot\, \|_{\frac{1}{2}}}$, $\|u\|_{\frac{1}{2}} = \sqrt{\langle Au, u\rangle}$, $D(A^{\frac{1}{2}})^\star$ is the topological dual of the space $D(A^{\frac{1}{2}})$, $I = (0, L)$ is an interval of $\mathbb{R}$ and where the initial data (u^0, u^1) are chosen in a Banach space $V \times L^2(I)$, in which equation (7.4.1.1) is well set. The associated energy E of u is given by

$$\forall t \geqslant 0,\ E(u(t)) = \frac{1}{2}\left(\|u_t(t)\|^2_{L^2(I)} + \|A^{\frac{1}{2}}u(t)\|^2_{L^2(I)}\right), \tag{7.4.1.2}$$

and satisfies

$$\forall t \geqslant s \geqslant 0,\ E(u(t)) - E(u(s)) = -\int_s^t \|(B^\star u)_t(\sigma)\|^2_U \mathrm{d}\sigma \leqslant 0. \tag{7.4.1.3}$$

Typically, $V \times L^2(I) = D(A^{\frac{1}{2}}) \times L^2(I)$ is the space for which the energy is well-defined and $U = \mathbb{R}$. But we need more regularity and we choose $(u^0, u^1) \in \mathcal{D}(\mathcal{A})$, where

$$\mathcal{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -A & -BB^\star \end{pmatrix}.$$

Denote by $(a_n)_{n\geqslant 0}$ the sequence of the Fourier's coefficient of u^0 and by $(b_n)_{n\geqslant 0}$ the u^1 one. We also consider v the solution of

$$\begin{cases} v_{tt} + Av = 0, & (t,x) \in (0,\infty) \times I, \\ v(0,x) = u^0(x), & x \in I, \\ v_t(0,x) = u^1(x), & x \in I. \end{cases} \tag{7.4.1.4}$$

Depending of the spaces V and $\mathcal{D}(\mathcal{A})$ we have chosen, we obtain for $(u^0, u^1) \in D(A) \times V$,

$$\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A})} = \sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2)\omega_2(n),$$

$$E(u(0)) \overset{\text{def}}{=} \frac{1}{2}\|(u^0,u^1)\|^2_{V\times L^2(I)} = \frac{1}{2}\sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2),$$

for some weight ω_2 satisfying (7.2.6) and some $p \in [0,\infty)$. Roughly speaking, in our examples, this comes from the expansion of u^0 and u^1 in Fourier's series and Parseval's identity.

First, we show that there exist a time $T > 0$, two constants $C > 0$ and $C_1 > 0$ and a weight ω_1 satisfying (7.2.5), such that for any initial data $(u^0, u^1) \in V \times L^2(I)$,

$$\int_0^T \|(B^\star u)_t(t)\|^2_U \mathrm{d}t \geqslant C \int_0^T \|(B^\star v)_t(t)\|^2_U \mathrm{d}t \geqslant C_1 \sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2)\omega_1(n), \tag{7.4.1.5}$$

where the last estimate comes from Ingham's inequality (Ingham [109]). For a complete example, see Lemmas 7.4.3.10 and 7.4.3.11.

Second, we define the weak energy E_- and the strong energy E_+ as follow.

$$E_+(0) = \sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2)\omega_2(n), \tag{7.4.1.6}$$

$$E(0) = \sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2), \tag{7.4.1.7}$$

$$E_-(0) = \sum_{n=0}^{\infty} n^p (a_n^2 + b_n^2)\omega_1(n). \tag{7.4.1.8}$$

Third, we show that there exist two functions Φ and Ψ satisfying (7.2.7) and (7.2.8). From Theorem 7.2.1, we have (7.2.4). Typically, we choose $\Phi(t) = \frac{1}{\varphi^{-1}(t)}$ and $\Psi(t) = \omega_2^{-1}(t)$, where $\varphi(t) = \frac{\omega_1(t)}{t^p}$ with $p \in \{0,2,4\}$. From (7.2.4) and (7.4.1.6)–(7.4.1.8), we deduce that

$$E_-(0) \geqslant E(0)\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{E_+(0)}{E(0)}\right)}\right) = E(0)\mathcal{H}^{-1}_{\Phi,\Psi}\left(\frac{E(0)}{E_+(0)}\right), \tag{7.4.1.9}$$

where $\mathcal{H}_{\Phi,\Psi}^{-1}$ is defined by (7.3.3). Putting together (7.4.1.3), (7.4.1.5) and (7.4.1.9), we obtain

$$E(T) \leqslant E(0) - C_1 E(0)\mathcal{H}_{\Phi,\Psi}^{-1}\left(\frac{E(0)}{E_+(0)}\right). \tag{7.4.1.10}$$

See Lemma 7.4.3.12 for a complete example.

Fourth, we use (7.4.1.10), the semigroup property and the method of Ammari and Tucsnak [8] to show that

$$\forall t \geqslant 0,\ E(t) \leqslant C\mathcal{H}_{\Phi,\Psi}\left(\frac{1}{t+1}\right)\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A})}^2. \tag{7.4.1.11}$$

Their proof is based on an interpolation method. See Theorem 7.4.3.5 for a complete example.

7.4.2 Notations for the wave equation (7.4.2.1) with Dirichlet boundary condition and known results

We consider a wave equation with a damping control concentrated on an interior point $a \in (0,1)$ with homogenous Dirichlet boundary condition,

$$\begin{cases} u_{tt} - u_{xx} + \delta_a u_t(t,a) = 0, & (t,x) \in (0,\infty)\times(0,1), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), & x \in (0,1), \\ u(t,0) = u(t,1) = 0 & t \in [0,\infty). \end{cases} \tag{7.4.2.1}$$

Let $V_1 = H_0^1(0,1)$. A direct calculation gives that for any $u \in V_1$, $\|u\|_{L^2(0,1)} \leqslant \|u_x\|_{L^2(0,1)}$, so we may endow V_1 of the norm $\|u\|_{V_1} = \|u_x\|_{L^2(0,1)}$, for any $u \in V_1$. Let $X_1 = V_1 \times L^2(0,1)$,

$$Y_1 = \big(H_0^1(0,1)\cap H^2(0,a)\cap H^2(a,1)\big)\times H_0^1(0,1), \quad D(A_1) = H_0^1(0,1)\cap H^2(0,1), \quad A_1 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

$$\mathcal{D}(\mathcal{A}_1) = \left\{(u,v)\in Y_1;\ \frac{\mathrm{d}u}{\mathrm{d}x}(a_+) - \frac{\mathrm{d}u}{\mathrm{d}x}(a_-) = v(a)\right\},$$

with

$$\|(u,v)\|_{\mathcal{D}(\mathcal{A}_1)}^2 = \|(u,v)\|_{Y_1}^2 = \|u\|_{H^2(0,a)}^2 + \|u\|_{H^2(a,1)}^2 + \|v\|_{H_0^1(0,1)}^2,$$

and let $\mathcal{A}_1 = \begin{pmatrix} 0 & \mathrm{Id} \\ -A_1 & -\delta_a \end{pmatrix}$. We define the energy E_1 for u solution of equation (7.4.2.1) by

$$\forall t \geqslant 0,\ E_1(u(t)) = \frac{1}{2}\left(\|u_t(t)\|_{L^2(0,1)}^2 + \|u_x(t)\|_{L^2(0,1)}^2\right) = \frac{1}{2}\|(u(t),u_t(t))\|_{X_1}^2. \tag{7.4.2.2}$$

Well-posedness and regularity results

Let $a \in (0,1)$. We recall that for any $(u^0,u^1) \in X_1$, there exists a unique solution $(u,u_t) \in \mathcal{C}([0,\infty);X_1)$ of (7.4.2.1). Moreover, $u(\ .\ ,a) \in H_{\mathrm{loc}}^1([0,\infty))$. Thus equation (7.4.2.1) makes sense in $L_{\mathrm{loc}}^2([0,\infty);H^{-1}(0,1))$. In addition, u satisfies the following energy estimate.

$$\forall t \geqslant s \geqslant 0,\ E_1(u(t)) - E_1(u(s)) = -\int_s^t |u_t(\sigma,a)|^2 \mathrm{d}\sigma \leqslant 0. \tag{7.4.2.3}$$

If furthermore $(u^0,u^1)\in\mathcal{D}(\mathcal{A}_1)$ then $(u,u_t)\in\mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A}_1))$. Finally, $\mathcal{A}_1$ is m–dissipative with domain dense in X_1 so that $\mathcal{A}_1$ generates a semigroup of contractions $(\mathcal{S}_1(t))_{t\geqslant 0}$ on X_1 and on $\mathcal{D}(\mathcal{A}_1)$, which means that

$$\begin{gathered}\forall(u^0,u^1)\in X_1,\ \|(u(t),u_t(t))\|_{X_1}\leqslant\|(u^0,u^1)\|_{X_1},\\ \forall(u^0,u^1)\in\mathcal{D}(\mathcal{A}_1),\ \|(u(t),u_t(t))\|_{\mathcal{D}(\mathcal{A}_1)}\leqslant\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_1)},\end{gathered}\tag{7.4.2.4}$$

for any $t\geqslant 0$. For more details, see for example Theorem 1.1 and Lemma 2.1 of Tucsnak [175] and Proposition 2.1 of Ammari and Tucsnak [8]. We also recall that $E_1(u(t))\xrightarrow{t\longrightarrow\infty}0$, or equivalently

$$\begin{gathered}\lim_{t\to\infty}\left(\|u(t)\|_{V_1}+\|u_t(t)\|_{L^2(0,1)}\right)=0,\\ \text{if and only if}\\ a\notin\mathbb{Q}.\end{gathered}\tag{7.4.2.5}$$

And if furthermore a satisfies (7.4.2.5) and if $(u^0,u^1)\in\mathcal{D}(\mathcal{A}_1)$ then we have the estimate

$$\forall t\geqslant 0,\ \|(u(t),u_t(t))\|_{X_1}\leqslant\|\mathcal{S}_1(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_1);X_1)}\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_1)},$$

with $\lim\limits_{t\to\infty}\|\mathcal{S}_1(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_1);X_1)}=0$ (Proposition 1.1 of Tucsnak [175]). Finally, it follows from (7.4.2.2)–(7.4.2.3) that

$$\forall t\geqslant s\geqslant 0,\ \|(u(t),u_t(t))\|_{X_1}\leqslant\|(u(s),u_t(s))\|_{X_1}.\tag{7.4.2.6}$$

Our goal is to describe the decay rate of $E_1(u(t))$ as $t\longrightarrow\infty$, for any $a\in(0,1)$ as soon as $E_1(u(t))\xrightarrow{t\longrightarrow\infty}0$, when the lack of observability occurs. By (7.4.2.5), this means that $a\notin\mathbb{Q}$.

Known decay

Now, we show that our method allows us to recover the known results (Jaffard, Tucsnak and Zuazua [110]). We recall the definition of an irrational algebraic number.

Definition 7.4.2.1. Let $d\in\mathbb{N}$, $d\geqslant 2$. An irrational number a is said to be *algebraic of degree* d if there exists a minimal polynomial function P of degree d with rational coefficients such that $P(a)=0$. P is *minimal* in the sense that if Q is a polynomial function with rational coefficients such that $Q(a)=0$ then $\deg Q\geqslant\deg P$.

If a is an irrational algebraic number of degree d then it follows from Liouville's Theorem that there exists a positive constant $C=C(d)$ such that for any $(m,n)\in\mathbb{Z}\times\mathbb{N}$, $\left|a-\frac{m}{n}\right|\geqslant\frac{C}{n^d}$. This implies that there exists a positive constant $c_1=c_1(d)$ such that

$$\forall n\in\mathbb{N},\ |\sin(n\pi a)|\geqslant\frac{c_1}{n^{d-1}}\quad\text{and}\quad\left|\sin\left(\left(n+\frac{1}{2}\right)\pi a\right)\right|\geqslant\frac{c_1}{(2n+1)^{d-1}}.\tag{7.4.2.7}$$

Notation 7.4.2.2. We denote by $\mathcal{S}$ the set of all irrational numbers $a\in(0,1)$ such that if $[0,a_1,\ldots,a_n,\ldots]$ is the expansion of a as a continued fraction, then $(a_n)_{n\in\mathbb{N}}$ is bounded.

Let us notice that $\mathcal{S}$ is obviously infinite and not countable and by classical results on Diophantine approximation (see Cassals [56], p.120), $\lambda(\mathcal{S})=0$, where λ is the Lebesgue's measure. Moreover, by

Euler–Lagrange's Theorem (see Lang [126], p.57), $\mathcal{S}$ contains the set of algebraic irrational numbers $a \in (0,1)$ of degree 2. According to a classical result (see Tucsnak [175] and the references therein), if $a \in \mathcal{S}$ then estimates (7.4.2.7) hold with $d = 2$. Finally, for any $\varepsilon > 0$, there exist two λ–measurable sets $I_\varepsilon \subset (0,1)$ and $J_\varepsilon \subset (0,1)$ and a constant $c_2 = c_2(\varepsilon) > 0$ such that $\lambda(I_\varepsilon) = \lambda(J_\varepsilon) = 1$ and such that for any $a \in I_\varepsilon$ and any $b \in J_\varepsilon$,

$$\forall n \in \mathbb{N},\ |\sin(n\pi a)| \geqslant \frac{c_2}{n^{1+\varepsilon}} \quad \text{and} \quad \left|\sin\left(\left(n + \frac{1}{2}\right)\pi b\right)\right| \geqslant \frac{c_2}{(2n+1)^{1+\varepsilon}}. \tag{7.4.2.8}$$

Let us notice that by Roth's Theorem (see Cassals [56], p.104), I_ε and J_ε contain all algebraic irrational numbers of $(0,1)$. The following result is due to Jaffard, Tucsnak and Zuazua [110] (Theorem 3.3).

Proposition 7.4.2.3 ([110])**.** *Let $\mathcal{S}$ be defined in Notation 7.4.2.2 and let for any $t \geqslant 0$, $\omega_2(t) = t^2$. We have the following result.*

1. *Let $a \in \mathcal{S}$ and set for any $t > 0$, $\omega_1(t) = \frac{c_1}{t}$, where c_1 is given by (7.4.2.7) with $d = 2$. Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$, the corresponding solution u of (7.4.2.1) verifies*

$$E_1(u(t)) \leqslant \frac{C}{(t+1)} \|(u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}, \tag{7.4.2.9}$$

   for any $t \geqslant 0$. Furthermore, time decay in (7.4.2.9) is optimal in the sense of Definition 7.3.3.

2. *Let $\varepsilon > 0$ and set for any $t > 0$, $\omega_1(t) = \frac{c_2}{t^{1+\varepsilon}}$, where c_2 is given by (7.4.2.8). For almost every $a \in (0,1) \cap \mathbb{Q}^c$, there exists a constant $C = C(a, \varepsilon) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$, the corresponding solution u of (7.4.2.1) verifies*

$$E_1(u(t)) \leqslant \frac{C}{(t+1)^{\frac{1}{1+\varepsilon}}} \|(u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}, \tag{7.4.2.10}$$

   for any $t \geqslant 0$. Furthermore, time decay in (7.4.2.10) is optimal in the sense of Definition 7.3.3.

7.4.3 New results

Before stating the main results, let us make the following definition.

Definition 7.4.3.1. We say that the functions $(\omega_1, \omega_2, \Phi, \Psi)$ are an *admissible quadruplet* if the following assertions hold.

1. The quadruplet $(\omega_1, \omega_2, \Phi, \Psi)$ satisfies (7.2.5)–(7.2.8) on $(0, \infty)$ and (7.2.9) holds on $(1, \infty)$.
2. One of the two following conditions is satisfied.
   (a) The function $t \longmapsto \dfrac{1}{t}\mathcal{H}^{-1}_{\Phi,\Psi}(t)$ is nondecreasing on $(0,1)$, where $\mathcal{H}^{-1}_{\Phi,\Psi}$ defined by (7.3.3) has to verify $\mathcal{H}_{\Phi,\Psi}((0,\delta)) \supset (0,1)$.
   (b) For any $t > 0$, $\Phi(t) = C_1 t^{\frac{1}{p}}$ and $\Psi(t) = C_2 t^{\frac{1}{q}}$ for some $p \geqslant 1$, $q \geqslant 1$ and constants $C_1, C_2 > 0$. In particular, we have for any $t > 0$, $\mathcal{H}_{\Phi,\Psi}(t) = \left(C_1 C_2^{-1}\right)^q t^{\frac{q}{p}}$.

In our applications, the weight ω_1 comes from an oscillating function and it is not clear that it satisfies (7.2.5). So we precise how we obtain such a weight.

Lemma 7.4.3.2. *Let $-\infty < a < b \leqslant \infty$ and let $\varepsilon : [a,b) \longrightarrow (0,\infty)$ be a continuous function such that $\liminf\limits_{t \nearrow b} \varepsilon(t) = 0$. Then there exists a convex function $\varphi \in \mathcal{C}^1_{\mathrm{b}}([a,b);\mathbb{R})$ such that $0 < \varphi \leqslant \varepsilon$ and $\varphi' < 0$ on $[a,b)$.*

Proof. Firstly, we note that we can find a positive function $\widetilde{\varepsilon} \in \mathcal{C}^1([a,b);\mathbb{R})$ such that $0 < \widetilde{\varepsilon} \leqslant \varepsilon$ and $\widetilde{\varepsilon}\,' < 0$ on $[a,b)$. So it is enough to consider ε to be such a function. Secondly, up to a bijective transformation conserving the convexity, we may assume that $[a,b) = [0,1)$. Set

$$\forall t \in [0,1),\ f(t) = \max\{\varepsilon'(s);\ 0 \leqslant s \leqslant t\}.$$

Define φ by

$$\forall t \in [0,1),\ \varphi(t) = -\int_t^1 f(s)ds \ \text{ and } \ \varphi(1) = 0.$$

Since f is monotone and ε' is continuous, it follows that $f \in \mathcal{C}_{\mathrm{b}}([0,1);\mathbb{R})$. Then φ is well-defined, $\varphi \in \mathcal{C}_{\mathrm{b}}([0,1];\mathbb{R}) \cap \mathcal{C}^1_{\mathrm{b}}([0,1);\mathbb{R})$ and $\varphi' = f$ on $[0,1)$. Clearly, $\varphi > 0$ and $\varphi' < 0$ on $[0,1)$. In addition, φ' is nondecreasing so that φ is convex. Finally, for any $\sigma \in [0,1)$, $\varphi'(\sigma) \geqslant \varepsilon'(\sigma)$. Integrating this expression on $(t,1)$, for any $t \in [0,1)$, and using that $\varphi(1) = \varepsilon(1) = 0$, we get $\varphi(t) \leqslant \varepsilon(t)$. This concludes the proof. □

Let $(u_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be such that $\liminf\limits_{n\to\infty} u_n = 0$. Let $\varepsilon \in \mathcal{C}([0,\infty);\mathbb{R})$ be such that $0 < \varepsilon(n) \leqslant u_n$, for any $n \in \mathbb{N}$. Let $\varphi \in \mathcal{C}([0,\infty);\mathbb{R})$ be a decreasing and convex function such that for any $t \geqslant 0$, $0 < \varphi(t) \leqslant \varepsilon(t)$ (which exists by Lemma 7.4.3.2) and consider $\mathcal{C} \subset [1,\infty) \times [0,\infty)$ the closure of the convex envelope of the set $\{(n,u_n);\ n \in \mathbb{N}\}$. Finally, fix arbitrarily $t \geqslant 1$. Then the set $\mathcal{C}_t \overset{\text{def}}{=} \mathcal{C} \cap (\{t\} \times \mathbb{R})$ is nonempty, closed and Lemma 7.4.3.2 ensures that for any $s_t \in \mathbb{R}$ such that $(t,s_t) \in \mathcal{C}_t$,

$$0 < \varphi(t) \leqslant s_t.$$

So by compactness, we may define the function ω_1 as

$$\forall t \geqslant 1,\ \omega_1(t) = \min\{s_t;\ (t,s_t) \in \mathcal{C}_t\} \tag{7.4.3.1}$$

and extend ω_1 as a decreasing, continuous and convex way on $[0,1]$. From the above discussion, Lemma 7.4.3.2 and Remark 7.2.8, ω_1 satisfies (7.2.5) with $m = 0$. This justifies the following definition.

Definition 7.4.3.3. Let $(u_n)_{n\in\mathbb{N}} \subset (0,\infty)$ such that $\liminf\limits_{n\to\infty} u_n = 0$. The function ω_1 defined on $[0,\infty)$ by (7.4.3.1) is called the *lower convex envelope of the sequence* $(u_n)_{n\in\mathbb{N}}$.

In some sense, ω_1 is the "nearest" convex and decreasing function of $(u_n)_{n\in\mathbb{N}}$ satisfying $0 < \omega_1(n) \leqslant u_n$, for any $n \in \mathbb{N}$. It will be useful to consider the weights ω_1 and ω_2 defined as following. Let $a \in (0,1) \cap \mathbb{Q}^{\mathrm{c}}$.

$$\omega_1 \text{ is the lower convex envelope of the sequence } (\sin^2(n\pi a))_{n\in\mathbb{N}}, \tag{7.4.3.2}$$

$$\forall t \geqslant 0,\ \omega_2(t) = t^2. \tag{7.4.3.3}$$

The following lemma shows that such definition for weights is consistent with the notion of admissible quadruplet.

Proposition 7.4.3.4. *Let $(u_n)_{n\in\mathbb{N}} \subset (0,\infty)$ be such that $\liminf_{n\to\infty} u_n = 0$, let ω_1 be its lower convex envelope (Definition 7.4.3.2), let $p \geqslant 1$, let $\alpha \in [0,1]$ and set for any $t \geqslant 0$, $\omega_2(t) = (t+\alpha)^p$. Define for any $t \geqslant \alpha^p$, $\Psi(t) = t^{\frac{1}{p}} - \alpha$ and for any $t > 0$,*

$$\varphi(t) = \frac{\omega_1(t)}{t^p} \quad \text{and} \quad \Phi(t) = \frac{1}{\varphi^{-1}(t)}.$$

Then the quadruplet $(\omega_1, \omega_2, \Phi, \Psi)$ is admissible and for any $t > 0$,

$$\mathcal{H}_{\Phi,\Psi}(t) = \frac{1}{(\varphi^{-1}(t) + \alpha)^p}.$$

Proof. By definition of ω_1, ω_2 and Ψ, (7.2.5), (7.2.6) and (7.2.8) are satisfied. By Lemma 7.2.6 applied to $f = \omega_1$ and with $m = 0$ and $M = \omega(0)$, it follows that Φ satisfies (7.2.7). Moreover, we easily check that $\Phi \geqslant \frac{1}{\omega^{-1}}$ on $(0, \omega_1(1)]$. As a consequence, (7.2.9) holds on $[1,\infty)$, so that condition 1 of Definition 7.4.3.1 is fulfilled. Finally, by Lemma 7.3.1 we have

$$\forall t \in \left(0, \alpha^{-p}\right), \ \widetilde{\mathcal{H}}(t) \overset{\text{def}}{=} \frac{1}{t}\mathcal{H}_{\Phi,\Psi}^{-1}(t) = \left(1 - \alpha t^{\frac{1}{p}}\right)^{-p} \omega_1\left(t^{-\frac{1}{p}} - \alpha\right),$$
$$\forall t > 0, \ \mathcal{H}_{\Phi,\Psi}(t) = \frac{1}{(\varphi^{-1}(t) + \alpha)^p},$$

where we used the notation $\alpha^{-p} = +\infty$ if $\alpha = 0$. It is clear that $\widetilde{\mathcal{H}}$ is increasing on $(0, \alpha^{-p}) \supset (0,1)$, so that (2a) of Definition 7.4.3.1 holds and $(\omega_1, \omega_2, \Phi, \Psi)$ is an admissible quadruplet. □

The main results are the following.

Theorem 7.4.3.5. *Let $a \in (0,1) \cap \mathbb{Q}^c$ and let ω_1 and ω_2 be defined by (7.4.3.2) − (7.4.3.3). Let Φ and Ψ be two functions such that the quadruplet $(\omega_1, \omega_2, \Phi, \Psi)$ is admissible (Definition 7.4.3.1). Let $\mathcal{H}_{\Phi,\Psi}$ be defined by (7.3.1). Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$, the corresponding solution u of (7.4.2.1) verifies*

$$\forall t \geqslant 0, \ E_1(u(t)) \leqslant C\mathcal{H}_{\Phi,\Psi}\left(\frac{1}{t+1}\right) \|(u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}, \tag{7.4.3.4}$$

if Φ and Ψ satisfy the hypothesis (2a) of Definition 7.4.3.1 and

$$\forall t \geqslant 0, \ E_1(u(t)) \leqslant \frac{C}{(t+1)^{\frac{q}{p}}} \|(u^0, u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}, \tag{7.4.3.5}$$

if for any $t > 0$, $\Phi(t) = C_1 t^{\frac{1}{p}}$ and $\Psi(t) = C_2 t^{\frac{1}{q}}$ for some $p \in [1,\infty)$, $q \in [1,\infty)$ and constants $C_1, C_2 > 0$ (case (2b) of Definition 7.4.3.1).

Remark 7.4.3.6. At the light of estimate (7.4.3.4), it is clear that we would like to find some functions Φ and Ψ such that $\mathcal{H}_{\Phi,\Psi}(t)$ goes to 0 as $t \searrow 0$ as rapidly as possible. This justifies Definition 7.3.3. Moreover, Proposition 7.4.3.4 ensures that there exists a quadruplet of functions $(\omega_1, \omega_2, \Phi, \Psi)$ which is admissible.

Concerning the explicit decay, the results are the following.

Theorem 7.4.3.7. *Let $a \in (0,1) \cap \mathbb{Q}^c$ and let ω_1 be defined by (7.4.3.2). We set*

$$\forall t > 0,\ \varphi(t) = \frac{\omega_1(t)}{t^2}.$$

Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$, the corresponding solution u of (7.4.2.1) satisfies

$$\forall t \geqslant 0,\ \|(u(t), u_t(t))\|_{V_1 \times L^2(0,1)} \leqslant \frac{C}{\varphi^{-1}\left(\frac{1}{t+1}\right)} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_1)}.$$

Remark 7.4.3.8. By Theorem 7.4.3.7, we are able to give the explicit decay of the energy for any $a \in (0,1) \cap \mathbb{Q}^c$. This completes the lack, since the decay was known for almost every $a \in (0,1)$ (Jaffard, Tucsnak and Zuazua [110], Theorem 3.3).

Remark 7.4.3.9. It follows from Theorem 7.4.3.7 and Proposition 7.4.3.4 that for any $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$, the corresponding solution u of (7.4.2.1) satisfies

$$\|(u(t), u_t(t))\|_{V_1 \times L^2(0,1)} \leqslant C\Phi\left(\frac{1}{t+1}\right) \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_1)}.$$

for any $t \geqslant 0$. In other words, decay of the energy directly depends on the behavior of the interpolation function Φ near 0.

Proof of Theorem 7.4.3.7. The result comes from Proposition 7.4.3.4 (applied with $(u_n)_{n\in\mathbb{N}} = (\sin^2(n\pi a))_{n\in\mathbb{N}}$, $p = 2$ and $\alpha = 0$) and from (7.4.3.4) of Theorem 7.4.3.5. □

Proof of Proposition 7.4.2.3. Let $\mathcal{S}$ be defined in Notation 7.4.2.2.
Case of 1. Let $a \in \mathcal{S}$ and let c_1 be the constant in (7.4.2.7) with $d = 2$.
Case of 2. Let $\varepsilon > 0$, let $I_\varepsilon \subset (0,1)$ be the set introduced after the Notation 7.4.2.2, let c_2 be the constant in (7.4.2.8) and let $a \in I_\varepsilon$.
Preliminary. Let $\nu \geqslant 0$ and $\ell \in \{1, 2\}$. We define on $(0, \infty)$ the following functions.

$$\omega_1(t) = \frac{c_\ell^2}{t^{2(1+\nu)}}, \quad \Psi(t) = t^{\frac{1}{2}}, \quad \Phi(t) = 2\left(\frac{t}{c_\ell^2}\right)^{\frac{1}{2(1+\nu)}}.$$

Let ω_2 be defined by (7.4.3.3) and let $\mathcal{H}_{\Phi,\Psi}$ be the corresponding functions given by (7.3.1). Then

$$\forall t > 0,\ \mathcal{H}_{\Phi,\Psi}(t) = 4\left(\frac{t}{c_\ell^2}\right)^{\frac{1}{1+\nu}}.$$

Furthermore for any $t > 0$, $\Phi(\omega_1(t))\Psi(\omega_2(t)) \geqslant 1$ and $\mathcal{H}_{\Phi,\Psi}(t) = \frac{C}{\omega_2 \circ \omega_1^{-1}(t)}$.
Proof of 1. Let $\nu = 0$ and $\ell = 1$. The result follows by applying (7.4.3.5) of Theorem 7.4.3.5.
Proof of 2. Let $\nu = \varepsilon$ and $\ell = 2$. The result follows by applying (7.4.3.5) of Theorem 7.4.3.5. This concludes the proof. □

Before proving Theorem 7.4.3.5, we need several results. Let us decompose the solution u as following. For u solution of (7.4.2.1) with initial data $(u^0, u^1) \in X_1$, we write

$$u(t,x) = v(t,x) + w(t,x), \tag{7.4.3.6}$$

for $(t,x) \in [0,\infty) \times (0,1)$, where v is the unique solution of

$$\begin{cases} v_{tt} - v_{xx} = 0, & (t,x) \in (0,\infty) \times (0,1), \\ v(0,x) = u^0(x), & x \in (0,1), \\ v_t(0,x) = u^1(x), & x \in (0,1), \\ v(t,0) = v(t,1) = 0, & t \in [0,\infty). \end{cases} \tag{7.4.3.7}$$

Then we have the well-known result (see for example Lemmas 4.1 and 5.3 of Ammari and Tucsnak [8] for the proof).

Lemma 7.4.3.10. *Let $a \in (0,1)$ and let $T = 10$. Then there exists a constant $C_1 = C_1(a) > 0$ satisfying the following property. For any initial data $(u^0,u^1) \in X_1$, the corresponding solutions u and v of (7.4.2.1) and (7.4.3.7) satisfy*

$$C_1 \int_0^T v_t^2(t,a)\mathrm{d}t \leqslant \int_0^T u_t^2(t,a)\mathrm{d}t \leqslant 4 \int_0^T v_t^2(t,a)\mathrm{d}t.$$

Now, we decompose $u^0 \in V_1$ and $u^1 \in L^2(0,1)$ as

$$u^0(x) = \sum_{n=0}^{\infty} a_n \sin(n\pi x), \quad u^1(x) = \pi \sum_{n=0}^{\infty} n b_n \sin(n\pi x). \tag{7.4.3.8}$$

We then have

$$\|u^0\|^2_{L^2(0,1)} = \frac{1}{2} \sum_{n=0}^{\infty} a_n^2, \quad \|u_x^0\|^2_{L^2(0,1)} = \frac{\pi^2}{2} \sum_{n=0}^{\infty} n^2 a_n^2, \quad \|u^1\|^2_{L^2(0,1)} = \frac{\pi^2}{2} \sum_{n=0}^{\infty} n^2 b_n^2. \tag{7.4.3.9}$$

It follows that the solution v of (7.4.3.7) is defined by

$$\forall (t,x) \in \mathbb{R} \times (0,1),\ v(t,x) = \sum_{n=0}^{\infty} \left\{ (a_n \cos(n\pi t) + b_n \sin(n\pi t)) \sin(n\pi x) \right\}. \tag{7.4.3.10}$$

If furthermore $(u^0,u^1) \in D(A_1) \times V_1$ then

$$\|u_{xx}^0\|^2_{L^2(0,a)} + \|u_{xx}^0\|^2_{L^2(a,1)} = \frac{\pi^4}{2} \sum_{n=0}^{\infty} n^4 a_n^2, \quad \|u_x^1\|^2_{L^2(0,1)} = \frac{\pi^4}{2} \sum_{n=0}^{\infty} n^4 b_n^2. \tag{7.4.3.11}$$

We have the following simple result.

Lemma 7.4.3.11. *Let $a \in (0,1)$, let $T = 10$, let $(u^0,u^1) \in X_1$ and let $(a_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ and $(b_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$ be given by (7.4.3.8). Then*

$$\int_0^T v_t^2(t,a)\mathrm{d}t \geqslant \pi^2 \sum_{n=0}^{\infty} n^2(a_n^2 + b_n^2)\sin^2(n\pi a), \tag{7.4.3.12}$$

where v is the solution of (7.4.3.7) given by (7.4.3.10).

Proof. Using (7.4.3.10), we have

$$\int_0^T v_t^2(t,a)\mathrm{d}t \geqslant \pi^2 \int_0^2 \left(\sum_{n=0}^{\infty} \sin(n\pi a)(-na_n \sin(n\pi t) + nb_n \cos(n\pi t)) \right)^2 \mathrm{d}t$$
$$= \pi^2 \sum_{n=0}^{\infty} \sin^2(n\pi a)(n^2 a_n^2 + n^2 b_n^2),$$

where the last line comes from Parseval's identity. Hence (7.4.3.12). □

Lemma 7.4.3.12. *Let $a \in (0,1) \cap \mathbb{Q}^c$, let $T = 10$, let ω_1 be given by (7.4.3.2) and let ω_2 be given by (7.4.3.3). Let Φ and Ψ be two functions such that the quadruplet $(\omega_1, \omega_2, \Phi, \Psi)$ satisfies hypothesis 1 of Definition 7.4.3.1 and such that $\mathcal{H}_{\Phi,\Psi}((0,\delta)) \supset (0,1)$. Then there exists a constant $C_2 = C_2(a) > 0$ such that for any initial data $(u^0,u^1) \in \mathcal{D}(\mathcal{A}_1)$,*

$$\|(u^0,u^1)\|_{X_1}^2 - \|(u(T),u_t(T))\|_{X_1}^2 \geqslant C_2 \|(u^0,u^1)\|_{X_1}^2 \mathcal{H}_{\Phi,\Psi}^{-1}\left(\frac{\|(u^0,u^1)\|_{X_1}^2}{\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_1)}^2} \right), \tag{7.4.3.13}$$

where u is the solution of (7.4.2.1), and where $\mathcal{H}_{\Phi,\Psi}^{-1}$ is defined by (7.3.3).

Proof. By Proposition 7.4.3.4, Φ and Ψ exist. We decompose u^0 and u^1 as in (7.4.3.8). We write

$$E_-(0) = \frac{\pi^2}{2} \sum_{n=0}^{\infty} n^2(a_n^2 + b_n^2)\omega_1(n), \tag{7.4.3.14}$$

where ω_1 verifies 1 of Definition 7.4.3.1. By (7.4.2.3) and Lemmas 7.4.3.10 and 7.4.3.11, there exists a constant $C_2 = C_2(a) > 0$ such that

$$\|(u^0,u^1)\|_{X_1}^2 - \|(u(T),u_t(T))\|_{X_1}^2 \geqslant C_2 E_-(0). \tag{7.4.3.15}$$

Assume further that $(u^0,u^1) \in D(A_1) \times V_1$. We define

$$E_+(0) = \frac{\pi^4}{4} \sum_{n=0}^{\infty} n^4(a_n^2 + b_n^2). \tag{7.4.3.16}$$

Putting together (7.4.3.16) and (7.4.3.11), we have that for any initial data $(u^0,u^1) \in D(A_1) \times V_1$,

$$E_+(0) = \frac{1}{2}\left(\|u_{xx}^0\|_{L^2(0,a)}^2 + \|u_{xx}^0\|_{L^2(a,1)}^2 + \|u_x^1\|_{L^2(0,1)}^2 \right).$$

These estimates imply that

$$E_+(0) \leqslant \|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_1)}^2. \tag{7.4.3.17}$$

Recall that by (7.4.2.2) and (7.4.3.9),

$$E(0) = \frac{\pi^2}{4} \sum_{n=0}^{\infty} n^2(a_n^2 + b_n^2) \stackrel{\text{def}}{=} \frac{1}{2}\|(u^0,u^1)\|_{X_1}^2, \tag{7.4.3.18}$$

where we have set $E(0) = E_1(u(0))$. Let $u = (u_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N};\mathbb{R})$ be defined by

$$\forall n \in \mathbb{N},\ u_n = n^2(a_n^2 + b_n^2).$$

Then it follows from Theorem 7.2.1 (applied to the function $f = u$, with $p = 1$, the discrete measure on $\mathcal{P}(\mathbb{N})$ and the weights ω_1 and ω_2), (7.4.3.14) and (7.4.3.16)–(7.4.3.18) that

$$1 \leqslant \Phi\left(\frac{E_-(0)}{\|(u^0,u^1)\|^2_{X_1}}\right)\Psi\left(\frac{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}{\|(u^0,u^1)\|^2_{X_1}}\right),$$

which yields

$$E_-(0) \geqslant \|(u^0,u^1)\|^2_{X_1}\Phi^{-1}\left(\frac{1}{\Psi\left(\frac{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}{\|(u^0,u^1)\|^2_{X_1}}\right)}\right).$$

Then for any $(u^0, u^1) \in D(A_1) \times V_1$,

$$E_-(0) \geqslant \|(u^0,u^1)\|^2_{X_1}\mathcal{H}^{-1}_{\Phi,\Psi}\left(\frac{\|(u^0,u^1)\|^2_{X_1}}{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}\right). \tag{7.4.3.19}$$

From (7.4.3.15) and (7.4.3.19), it follows that (7.4.3.13) holds for any $(u^0, u^1) \in D(A_1) \times V_1$. By continuity of $\mathcal{H}^{-1}_{\Phi,\Psi}$ and by density of $D(A_1)\times V_1$ in Y_1 (which contains $\mathcal{D}(\mathcal{A}_1)$ and has the same norm of $\mathcal{D}(\mathcal{A}_1)$), it follows that (7.4.3.13) holds for any $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_1)$. Hence the result. □

Proof of Theorem 7.4.3.5. We follow the proof of Theorem 2.4 of Ammari and Tucsnak [8]. Let $T = 10$. By Lemma 7.4.3.12, we have that

$$\|(u(T), u_t(T))\|^2_{X_1} \leqslant \|(u^0,u^1)\|^2_{X_1} - C_2\|(u^0,u^1)\|^2_{X_1}\mathcal{H}^{-1}_{\Phi,\Psi}\left(\frac{\|(u^0,u^1)\|^2_{X_1}}{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}\right).$$

This estimate remains valid in successive intervals $[\ell T, (\ell+1)T]$. So with (7.4.2.4), (7.4.2.6) and the fact that $\mathcal{H}^{-1}_{\Phi,\Psi}$ is increasing (Lemma 7.3.1), we obtain that

$$\begin{aligned}\|(u((\ell+1)T), u_t((\ell+1)T))\|^2_{X_1} \leqslant& \|(u(\ell T), u_t(\ell T))\|^2_{X_1}\\ &-C_2\|(u(\ell T), u_t(\ell T))\|^2_{X_1}\mathcal{H}^{-1}_{\Phi,\Psi}\left(\frac{\|(u((\ell+1)T), u_t((\ell+1)T))\|^2_{X_1}}{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}\right),\end{aligned} \tag{7.4.3.20}$$

for every $\ell \in \mathbb{N}\cup\{0\}$.

Case 1. The functions Φ and Ψ satisfy hypothesis (2a) of Definition 7.4.3.1.
Our expression (7.4.3.20) is the same that (4.16) in Ammari and Tucsnak [8] (with $X \times V = X_1$, $\|\,.\,\|_{Y_1\times Y_2} = \|\,.\,\|_{\mathcal{D}(\mathcal{A}_1)}$, $\mathcal{G} = \mathcal{H}^{-1}_{\Phi,\Psi}$ and $\theta = \frac{1}{2}$). The rest of the proof follows as in [8] (where (2a) of Definition 7.4.3.1 is used). Then (7.4.3.4) follows.

Case 2. The functions Φ and Ψ satisfy hypothesis (2b) of Definition 7.4.3.1.

It follows that for any $t > 0$, $\mathcal{H}^{-1}_{\Phi,\Psi}(t) = C_3 t^{\frac{p}{q}}$. Using again (7.4.2.6) and the definition of $\mathcal{H}^{-1}_{\Phi,\Psi}$, (7.4.3.20) becomes

$$\|(u((\ell+1)T), u_t((\ell+1)T))\|^2_{X_1} \leqslant \|(u(\ell T), u_t(\ell T))\|^2_{X_1} - C_4 \frac{\|(u((\ell+1)T), u_t((\ell+1)T))\|^{2\frac{p+q}{q}}_{X_1}}{\|(u^0, u^1\|^{2\frac{p}{q}}_{\mathcal{D}(\mathcal{A}_1)}}, \tag{7.4.3.21}$$

for every $\ell \in \mathbb{N} \cup \{0\}$. Our expression (7.4.3.21) is the same that (4.23) in Ammari and Tucsnak [8] (with $X \times V = X_1$, $\| \, . \, \|_{Y_1 \times Y_2} = \| \, . \, \|_{\mathcal{D}(\mathcal{A}_1)}$ and $\theta = \frac{q}{p+q}$). The rest of the proof follows as in [8]. □

Remark 7.4.3.13. We are not able to apply directly Theorem 2.4 of Ammari and Tucsnak [8]. Indeed, in their theorem, the assumption (2.8) is

$$\int_0^2 v_t^2(t,a)\mathrm{d}t \geqslant C\|(u^0,u^1)\|^2_{V_1 \times L^2(0,1)} \mathcal{G}\left(\frac{\|(u^0,u^1)\|^2_{L^2(0,1)\times H^{-1}(0,1)}}{\|(u^0,u^1)\|^2_{V_1\times L^2(0,1)}}\right),$$

(where $\mathcal{G} = \mathcal{H}^{-1}_{\Phi,\Psi}$) and we can only show the weaker estimate (by the inequalities of interpolation)

$$\int_0^2 v_t^2(t,a)\mathrm{d}t \geqslant C\|(u^0,u^1)\|^2_{V_1 \times L^2(0,1)} \mathcal{G}\left(\frac{\|(u^0,u^1)\|^2_{V_1\times L^2(0,1)}}{\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_1)}}\right).$$

7.5 Others applications

7.5.1 Wave equation with mixed boundary condition

We consider a wave equation with a damping control concentrated on an interior point $a \in (0,1)$ with a homogenous Dirichlet boundary condition at the left end and a homogenous Neumann boundary condition at the right end,

$$\begin{cases} u_{tt} - u_{xx} + \delta_a u_t(t,a) = 0, & (t,x) \in (0,\infty)\times(0,1), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x), & x \in (0,1), \\ u(t,0) = u_x(t,1) = 0, & t \in [0,\infty). \end{cases} \tag{7.5.1.1}$$

Notations for the wave equation (7.5.1.1) with homogenous mixed Dirichlet and Neumann boundary condition

Let $V_2 = \{u \in H^1(0,1);\ u(0) = 0\}$. A direct calculation gives that for any $u \in V_2$, $\|u\|_{L^2(0,1)} \leqslant \|u_x\|_{L^2(0,1)}$, so we may endow V_2 of the norm $\|u\|_{V_2} = \|u_x\|_{L^2(0,1)}$, for any $u \in V_2$. Let $X_2 = V_2 \times L^2(0,1)$,

$$Y_2 = \left\{u \in V_2 \cap H^2(0,a) \cap H^2(a,1);\ \frac{\mathrm{d}u}{\mathrm{d}x}(1) = 0\right\} \times V_2,$$

$$D(A_2) = \left\{u \in V_2 \cap H^2(0,1);\ \frac{\mathrm{d}u}{\mathrm{d}x}(1) = 0\right\}, \quad A_2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2},$$

$$\mathcal{D}(\mathcal{A}_2) = \left\{(u,v) \in Y_2;\ \frac{\mathrm{d}u}{\mathrm{d}x}(a_+) - \frac{\mathrm{d}u}{\mathrm{d}x}(a_-) = v(a)\right\},$$

with

$$\|(u,v)\|^2_{\mathcal{D}(\mathcal{A}_2)} = \|(u,v)\|^2_{Y_2} = \|u\|^2_{H^2(0,a)} + \|u\|^2_{H^2(a,1)} + \|v\|^2_{H^1(0,1)},$$

and let $\mathcal{A}_2 = \begin{pmatrix} 0 & \mathrm{Id} \\ -A_2 & -\delta_a \end{pmatrix}$. We define the energy E_2 for u solution of equation (7.5.1.1) by (7.4.2.2).

Well-posedness and regularity results

Let $a \in (0,1)$. We recall that for any $(u^0,u^1) \in X_2$, there exists a unique solution $(u,u_t) \in \mathcal{C}([0,\infty);X_2)$ of (7.5.1.1). Moreover, $u(\ .\ ,a) \in H^1_{\mathrm{loc}}([0,\infty))$. Thus equation (7.5.1.1) makes sense in $L^2_{\mathrm{loc}}([0,\infty);H^{-1}(0,1))$. In addition, u satisfies the following energy estimate.

$$\forall t \geqslant s \geqslant 0,\ E_2(u(t)) - E_2(u(s)) = -\int_s^t |u_t(\sigma,a)|^2 \mathrm{d}\sigma \leqslant 0. \tag{7.5.1.2}$$

If furthermore $(u^0,u^1) \in \mathcal{D}(\mathcal{A}_2)$ then $(u,u_t) \in \mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A}_2))$. Finally, $\mathcal{A}_2$ is m–dissipative with domain dense in X_2 so that $\mathcal{A}_2$ generates a semigroup of contractions $(\mathcal{S}_2(t))_{t\geqslant 0}$ on X_2 and on $\mathcal{D}(\mathcal{A}_2)$, which means that

$$\forall (u^0,u^1) \in X_2,\ \|(u(t),u_t(t))\|_{X_2} \leqslant \|(u^0,u^1)\|_{X_2},$$
$$\forall (u^0,u^1) \in \mathcal{D}(\mathcal{A}_2),\ \|(u(t),u_t(t))\|_{\mathcal{D}(\mathcal{A}_2)} \leqslant \|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_2)},$$

for any $t \geqslant 0$. For more details, see Proposition 1.1 and Section 3 p.223 of Ammari, Henrot and Tucsnak [6]. We also recall that $E_2(u(t)) \xrightarrow{t\longrightarrow\infty} 0$, or equivalently

$$\lim_{t\to\infty} \left(\|u(t)\|_{V_2} + \|u_t(t)\|_{L^2(0,1)}\right) = 0$$

if and only if

$$\forall (p,q) \in \mathbb{N}\times\mathbb{N},\ a \neq \frac{2p}{2q-1}, \tag{7.5.1.3}$$

And if furthermore a satisfies (7.5.1.3) and if $(u^0,u^1) \in \mathcal{D}(\mathcal{A}_2)$ then we have the estimate

$$\forall t \geqslant 0,\ \|(u(t),u_t(t))\|_{X_2} \leqslant \|\mathcal{S}_2(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_2);X_2)} \|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_2)},$$

with $\lim\limits_{t\to\infty} \|\mathcal{S}_2(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_2);X_2)} = 0$ (Proposition 3.1 of Ammari, Henrot and Tucsnak [6]). Finally,

$$\begin{cases} \exists \omega > 0,\ \exists C = C(\omega) > 0 \ \text{ such that } \ \forall (u^0,u^1) \in X_2, \\ \forall t \geqslant 0,\ E_2(u(t)) \leqslant Ce^{-\omega t} E_2(u(0)) \end{cases}$$

if and only if

$$a = \frac{2p-1}{q}, \quad \text{for some } \ (p,q) \in \mathbb{N}\times\mathbb{N}. \tag{7.5.1.4}$$

See Theorem 1.2 of Ammari, Henrot and Tucsnak [6]. It follows from (7.4.2.2) and (7.5.1.2) that

$$\forall t \geqslant s \geqslant 0,\ \|(u(t),u_t(t))\|_{X_2} \leqslant \|(u(s),u_t(s))\|_{X_2}.$$

We are concerned by the decay rate of the energy $E_2(u(t))$ when it is not exponentially stable. In particular, by (7.5.1.3) and (7.5.1.4) this implies that $a \notin \mathbb{Q}$.

The main results are the following.

Theorem 7.5.1.1. *Let $a \in (0,1) \cap \mathbb{Q}^c$ and let ω_1 be the lower convex envelope of the sequence*

$$\left(\sin^2\left(\left(n+\frac{1}{2}\right)\pi a\right)\right)_{n\in\mathbb{N}}$$

(Definition 7.4.3.3). Let ω_2 be defined on $[0,\infty)$ by $\omega_2(t) = \left(t+\frac{1}{2}\right)^2$. Let Φ and Ψ be two functions such that the quadruplet $(\omega_1, \omega_2, \Phi, \Psi)$ is admissible (Definition 7.4.3.1). Let $\mathcal{H}_{\Phi,\Psi}$ be defined by (7.3.1). Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_2)$, the corresponding solution u of (7.5.1.1) verifies

$$\forall t \geqslant 0,\ E_2(u(t)) \leqslant C\mathcal{H}_{\Phi,\Psi}\left(\frac{1}{t+1}\right)\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_2)},$$

if Φ and Ψ satisfy the hypothesis (2a) of Definition 7.4.3.1 and

$$\forall t \geqslant 0,\ E_2(u(t)) \leqslant \frac{C}{(t+1)^{\frac{q}{p}}}\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_2)}, \tag{7.5.1.5}$$

if for any $t > 0$, $\Phi(t) = C_1 t^{\frac{1}{p}}$ and $\Psi(t) = C_2 t^{\frac{1}{q}}$ for some $p \in [1,\infty)$, $q \in [1,\infty)$ and constants $C_1, C_2 > 0$ (case (2b) of Definition 7.4.3.1).

Proof. We write $u^0(x) = \sum\limits_{n=0}^{\infty} a_n \sin\left(\left(n+\frac{1}{2}\right)\pi x\right)$ and $u^1(x) = \pi \sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right) b_n \sin\left(\left(n+\frac{1}{2}\right)\pi x\right)$ and we consider the solution v of (7.4.3.7) satisfying the same boundary condition as u. We follow the method as for (7.4.2.1). Then from Ingham's inequality (Ingham [109]) and the results of Ammari, Henrot and Tucsnak [6] (Lemma 4.2 of [6]; see also Lemma 2.5 of [6] and Lemma 4.1 of [8]), we obtain for $T = 10$,

$$\int_0^T u_t^2(t,a)\mathrm{d}t \geqslant C(a)\int_0^T v_t^2(t,a)\mathrm{d}t \geqslant C(a)\pi^2 \sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^2 (a_n^2+b_n^2)\sin^2\left(\left((n+\frac{1}{2}\right)\pi a\right).$$

Then we define

$$E_+(0) = \frac{\pi^4}{4}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^4(a_n^2+b_n^2) = \frac{\pi^4}{4}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^2(a_n^2+b_n^2)\omega_2(n),$$

$$E(0) = \frac{\pi^2}{4}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^2(a_n^2+b_n^2) \stackrel{\text{def}}{=} \frac{1}{2}\|(u^0,u^1)\|^2_{X_2},$$

$$E_-(0) = \frac{\pi^2}{2}\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^2(a_n^2+b_n^2)\omega_1(n).$$

The result follows from the discussion at the beginning of Section 7.4. □

Using Theorem 7.5.1.1 and Proposition 7.4.3.4 (applied with $(u_n)_{n\in\mathbb{N}} = \left(\sin^2\left(\left(n+\frac{1}{2}\right)\pi a\right)\right)_{n\in\mathbb{N}}$, $p = 2$ and $\alpha = \frac{1}{2}$), we obtain the following result.

Theorem 7.5.1.2. *Let $a \in (0,1) \cap \mathbb{Q}^c$ and let ω_1 and ω_2 be defined as in Theorem 7.5.1.1. We set*

$$\forall t > 0,\ \varphi(t) = \frac{\omega_1(t)}{t^2}.$$

Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_2)$, the corresponding solution u of (7.5.1.1) satisfies

$$\forall t \geqslant 0,\ \|(u(t), u_t(t))\|_{V_2\times L^2(0,1)} \leqslant \frac{C}{\varphi^{-1}\left(\frac{1}{t+1}\right)} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_2)}.$$

Remark 7.5.1.3. By Theorem 7.5.1.2, we are able to give the explicit decay of the energy for any $a \in (0,1) \cap \mathbb{Q}^c$. This completes the lack, since the decay was known for almost every $a \in (0,1)$, as stated in Theorem 1.4 of Ammari, Henrot and Tucsnak [6]. In addition, with help of (7.5.1.5) of Theorem 7.5.1.1, our method allows us to recover the results of that Theorem 1.4.

7.5.2 Bernoulli–Euler beam with a pointwise interior damping control

We consider a Bernoulli–Euler beam with a damping control concentrated in an interior point $a \in (0,1)$,

$$\begin{cases} u_{tt} + u_{xxxx} + \delta_a u_t(t,a) = 0, & (t,x) \in (0,\infty)\times(0,1), \\ u(0,x) = u^0(x), \quad u_t(0,x) = u^1(x), & x \in (0,1), \\ u(t,0) = u(t,1) = u_{xx}(t,0) = u_{xx}(t,1) = 0, & t \in [0,\infty). \end{cases} \tag{7.5.2.1}$$

We also could have chosen the boundary condition

$$\forall t \geqslant 0,\ u(t,0) = u_x(t,1) = u_{xx}(t,0) = u_{xxx}(t,1) = 0,$$

as in Ammari and Tucsnak [7]. But for conciseness of the paper, we do not consider this case.

Notations for the Bernoulli–Euler beam equation (7.5.2.1)
Let $V_3 = H_0^1(0,1) \cap H^2(0,1)$. By Cauchy–Schwarz's inequality, we have $\|u\|_{L^2(0,1)} \leqslant \|u_x\|_{L^2(0,1)} \leqslant \|u_{xx}\|_{L^2(0,1)}$, for any $u \in V_3$. So we may endow V_3 of the norm $\|u\|_{V_3} = \|u_{xx}\|_{L^2(0,1)}$, for any $u \in V_3$. Let $X_3 = V_3 \times L^2(0,1)$,

$$Y_3 = \left\{ u \in H_0^1(0,1) \cap H^2(0,1) \cap H^4(0,a) \cap H^4(a,1);\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(1) = 0 \right\} \times V_3,$$

$$D(A_3) = \left\{ u \in H_0^1(0,1) \cap H^4(0,1);\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(0) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(1) = 0 \right\}, \quad A_3 = \frac{\mathrm{d}^4}{\mathrm{d}x^4},$$

$$\mathcal{D}(\mathcal{A}_3) = \left\{ (u,v) \in Y_3;\ \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(a_+) = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(a_-) \text{ and } \frac{\mathrm{d}^3 u}{\mathrm{d}x^3}(a_+) - \frac{\mathrm{d}^3 u}{\mathrm{d}x^3}(a_-) = -v(a) \right\},$$

with

$$\|(u,v)\|^2_{\mathcal{D}(\mathcal{A}_3)} = \|(u,v)\|^2_{Y_3} = \|u\|^2_{H^4(0,a)} + \|u\|^2_{H^4(a,1)} + \|v\|^2_{H^2(0,1)},$$

and let $\mathcal{A}_3 = \begin{pmatrix} 0 & \mathrm{Id} \\ -A_3 & -\delta_a \end{pmatrix}$. We define the energy E_3 for u solution of equation (7.5.2.1) by

$$\forall t \geqslant 0,\ E_3(u(t)) = \frac{1}{2}\left(\|u_t(t)\|^2_{L^2(0,1)} + \|u_{xx}(t)\|^2_{L^2(0,1)} \right) = \frac{1}{2}\|(u(t), u_t(t))\|^2_{X_3}. \tag{7.5.2.2}$$

Well-posedness and regularity results
We recall that for any $(u^0, u^1) \in X_3$, there exists a unique solution $(u, u_t) \in \mathcal{C}([0,\infty); X_3)$ of (7.5.2.1).

Moreover, $u(\,.\,,a)\in H^1_{\text{loc}}([0,\infty))$ and thus equation (7.5.2.1) makes sense in $L^2_{\text{loc}}([0,\infty);H^{-2})$. In addition, u satisfies the following energy estimate.

$$\forall t\geqslant s\geqslant 0,\; E_3(u(t))-E_3(u(s))=-\int_s^t|u_t(\sigma,a)|^2\mathrm{d}\sigma\leqslant 0. \tag{7.5.2.3}$$

If furthermore $(u^0,u^1)\in\mathcal{D}(\mathcal{A}_3)$ then $(u,u_t)\in\mathcal{C}([0,\infty);\mathcal{D}(\mathcal{A}_3))$. Finally, $\mathcal{A}_3$ is m–dissipative with domain dense in X_3 so that $\mathcal{A}_3$ generates a semigroup of contractions $(\mathcal{S}_3(t))_{t\geqslant 0}$ on X_3 and on $\mathcal{D}(\mathcal{A}_3)$, which means that

$$\forall(u^0,u^1)\in X_3,\;\|(u(t),u_t(t))\|_{X_3}\leqslant\|(u^0,u^1)\|_{X_3},$$
$$\forall(u^0,u^1)\in\mathcal{D}(\mathcal{A}_3),\;\|(u(t),u_t(t))\|_{\mathcal{D}(\mathcal{A}_3)}\leqslant\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_3)},$$

for any $t\geqslant 0$. For more details, see for example Proposition 2.1 of Ammari and Tucsnak [8] ; Section 2 p.1161, Proposition 2.1 and Section 5 p.1173–1174 of Ammari and Tucsnak [7]. We also recall that $E_3(u(t))\xrightarrow{t\longrightarrow\infty}0$, or equivalently

$$\lim_{t\to\infty}\left(\|u(t)\|_{V_3}+\|u_t(t)\|_{L^2(0,1)}\right)=0$$
$$\text{if and only if}$$
$$a\notin\mathbb{Q}. \tag{7.5.2.4}$$

And if furthermore a satisfies (7.5.2.4) and if $(u^0,u^1)\in\mathcal{D}(\mathcal{A}_3)$ then we have the estimate

$$\forall t\geqslant 0,\;\|(u(t),u_t(t))\|_{X_3}\leqslant\|\mathcal{S}_3(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_3);X_3)}\|(u^0,u^1)\|_{\mathcal{D}(\mathcal{A}_3)},$$

with $\lim\limits_{t\to\infty}\|\mathcal{S}_3(t)\|_{\mathcal{L}(\mathcal{D}(\mathcal{A}_3);X_3)}=0$ (Proposition 2.1 and Section 5 p.1174 of Ammari and Tucsnak [7]). Finally, it follows from (7.5.2.2)–(7.5.2.3) that the following holds.

$$\forall t\geqslant s\geqslant 0,\;\|(u(t),u_t(t))\|_{X_3}\leqslant\|(u(s),u_t(s))\|_{X_3}.$$

The goal is to establish the decay rate of $E_3(u(t))$ as $t\longrightarrow\infty$, for any $a\in(0,1)$ as soon as $E_3(u(t))\xrightarrow{t\longrightarrow\infty}0$, when the lack of observability occurs. In particular, by (7.5.2.4), this implies that $a\notin\mathbb{Q}$.

Theorem 7.5.2.1. *Let $a\in(0,1)\cap\mathbb{Q}^c$, let ω_1 be the lower convex envelope of the sequence $(\sin^2(n\pi a))_{n\in\mathbb{N}}$ (Definition 7.4.3.3) and let ω_2 be defined on $[0,\infty)$ by $\omega_2(t)=t^4$. Let Φ and Ψ be two functions such that the quadruplet $(\omega_1,\omega_2,\Phi,\Psi)$ is admissible (see Definition 7.4.3.1) and let $\mathcal{H}_{\Phi,\Psi}$ be defined by (7.3.1). Then there exists a constant $C=C(a)>0$ such that for any initial data $(u^0,u^1)\in\mathcal{D}(\mathcal{A}_3)$, the corresponding solution u of (7.5.2.1) verifies*

$$\forall t\geqslant 0,\;E_3(u(t))\leqslant C\mathcal{H}_{\Phi,\Psi}\left(\frac{1}{t+1}\right)\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_3)},$$

if Φ and Ψ satisfy the hypothesis (2a) of Definition 7.4.3.1 and

$$\forall t\geqslant 0,\;E_3(u(t))\leqslant\frac{C}{(t+1)^{\frac{q}{p}}}\|(u^0,u^1)\|^2_{\mathcal{D}(\mathcal{A}_3)}, \tag{7.5.2.5}$$

if for any $t>0$, $\Phi(t)=C_1t^{\frac{1}{p}}$ and $\Psi(t)=C_2t^{\frac{1}{q}}$ for some $p\in[1,\infty)$, $q\in[1,\infty)$ and constants $C_1,C_2>0$ (case (2b) of Definition 7.4.3.1).

Proof. We write $u^0(x) = \sum_{n=0}^{\infty} a_n \sin(n\pi x)$ and $u^1(x) = \pi^2 \sum_{n=0}^{\infty} n^2 b_n \sin(n\pi x)$ and we consider the solution v of $v_{tt} + v_{xxxx} = 0$, satisfying the same boundary condition and having the same initial data as u. We follow the method as for (7.4.2.1). From Ingham's inequality (Ingham [109]) and Lemmas 3.3 and 5.1 of Ammari and Tucsnak [7] (see also Lemmas 4.1 and 5.7 of Ammari and Tucsnak [8]), we obtain for $T = 10$,

$$\int_0^T u_t^2(t,a)\mathrm{d}t \geqslant C(a) \int_0^T v_t^2(t,a)\mathrm{d}t \geqslant C(a) \sum_{n=0}^{\infty} n^4(a_n^2 + b_n^2)\sin^2(n\pi a).$$

Then we define

$$E_+(0) = \frac{\pi^8}{4} \sum_{n=0}^{\infty} n^8(a_n^2 + b_n^2) = \frac{\pi^8}{4} \sum_{n=0}^{\infty} n^4(a_n^2 + b_n^2)\omega_2(n),$$

$$E(0) = \frac{\pi^4}{4} \sum_{n=0}^{\infty} n^4(a_n^2 + b_n^2) \stackrel{\text{def}}{=} \frac{1}{2}\|(u^0, u^1)\|_{X_3}^2,$$

$$E_-(0) = \frac{\pi^4}{2} \sum_{n=0}^{\infty} n^4(a_n^2 + b_n^2)\omega_1(n).$$

The result follows from the discussion at the beginning of Section 7.4. □

Using Theorem 7.5.2.1 and Proposition 7.4.3.4 (applied with $(u_n)_{n\in\mathbb{N}} = (\sin^2(n\pi a))_{n\in\mathbb{N}}$, $p = 4$ and $\alpha = 0$), we obtain the following result.

Theorem 7.5.2.2. *Let $a \in (0,1) \cap \mathbb{Q}^c$ and let ω_1 and ω_2 be defined as in Theorem 7.5.2.1. We set*

$$\forall t > 0,\ \varphi(t) = \frac{\omega_1(t)}{t^4}.$$

Then there exists a constant $C = C(a) > 0$ such that for any initial data $(u^0, u^1) \in \mathcal{D}(\mathcal{A}_3)$, the solution u of (7.5.2.1) satisfies

$$\forall t \geqslant 0,\ \|(u(t), u_t(t))\|_{V_3 \times L^2(0,1)} \leqslant \frac{C}{\left(\varphi^{-1}\left(\frac{1}{t+1}\right)\right)^2} \|(u^0, u^1)\|_{\mathcal{D}(\mathcal{A}_3)}.$$

Remark 7.5.2.3. By Theorem 7.5.2.2, we are able to give the explicit decay of the energy for any $a \in (0,1) \cap \mathbb{Q}^c$. This completes the lack, since the decay was known for almost every $a \in (0,1)$ (Ammari and Tucsnak [7], Theorem 2.2). In addition, with help of (7.5.2.5) of Theorem 7.5.2.1, our method allows us to recover the decay of Theorem 2.2 in Ammari and Tucsnak [7].

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Chapitre 8

On damped second-order gradient systems

with Jérôme Bolte* and Mohamed Ali Jendoubi†

Abstract

Using small deformations of the total energy, as introduced in [97], we establish that damped second order gradient systems

$$u''(t) + \gamma u'(t) + \nabla G(u(t)) = 0,$$

may be viewed as quasi-gradient systems. In order to study the asymptotic behavior of these systems, we prove that any (nontrivial) desingularizing function appearing in KL inequality satisfies $\varphi(s) \geqslant c\sqrt{s}$ whenever the original function is definable and C^2. Variants to this result are given. These facts are used in turn to prove that a desingularizing function of the potential G also desingularizes the total energy and its deformed versions. Our approach brings forward several results interesting for their own sake : we provide an asymptotic alternative for quasi-gradient systems, either a trajectory converges, or its norm tends to infinity. The convergence rates are also analyzed by an original method based on a one-dimensional worst-case gradient system.

We conclude by establishing the convergence of solutions of damped second order systems in various cases including the definable case. The real-analytic case is recovered and some results concerning convex functions are also derived.

8.1 Introduction

8.1.1 A global view on previous results

In this paper, we develop some new tools for the asymptotic behavior as t goes to infinity of solutions $u : \mathbb{R}_+ \longrightarrow \mathbb{R}^N$ of the following second order system

$$u''(t) + \gamma u'(t) + \nabla G(u(t)) = 0, \quad t \in \mathbb{R}_+. \tag{8.1.1.1}$$

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Here, $\gamma > 0$ is a positive real number which can be seen as a *damping coefficient*, $N \geqslant 1$ is an integer and $G \in C^2(\mathbb{R}^N)$ is a real-valued function. In Mechanics, (8.1.1.1) models, among other problems, the motion of an object subject to a force deriving from a potential G (e.g. gravity) and to a viscous friction force $-\gamma u'$. In particular, the above may be seen as a qualitative model for the motion of a material point subject to gravity, constrained to evolve on the graph of G and subject to a damping force, further insights and results on this view may be found in [13, 50]. This type of dynamical system has been the subject of several works in various fields and along different perspectives, one can quote for instance [14] for Nonsmooth Mechanics, [48, 37] for recent advances in Optimization and [150] for pioneer works on the topic, partial differential equations and related aspects [100, 114, 18].

The aim of this work is to provide a deeper understanding of the asymptotic behavior of such a system and of the mechanisms behind the stabilization of trajectories at infinity (making each bounded orbit approach some specific critical point). Such behaviors have been widely investigated for gradient systems,

$$u'(t) + \nabla G(u(t)) = 0,$$

for a long time now. The first decisive steps were made by Łojasiewicz for analytic functions through the introduction of the so-called gradient inequality [134, 133]. Many other works followed among which two important contributions : [49] for convex functions and [124] for definable functions. Surprisingly the asymptotic behavior of the companion dynamics (8.1.1.1) has only been "recently" analyzed. The motivation for studying (8.1.1.1) seems to come from three distinct fields PDEs, Mechanics and Optimization. Out of the convex realm [135, 4], the seminal paper is probably [97]. Like many of the works on gradient systems the main assumption, borrowed from Łojasiewicz original contributions, is the analyticity of the function – or more precisely the fact that the function satisfies the Łojasiewicz inequality. This work paved the way for many developments : convergence rates studies [99], extension to partial differential equations [160, 112, 111, 98, 106, 100, 66, 85, 84, 101, 18], use of various kind of dampings [64, 65] (see also [52, 102, 88, 113]). Despite the huge amount of subsequent works, some deep questions remained somehow unanswered ; in particular it is not clear to see :

- *What are the exact connections between gradient systems and damped second-order gradient systems ?*
- *Within these relationships, how central is the role of the properties/geometry of the potential function G ?*

Before trying to provide some answers, we recall some fundamental notions related to these questions ; they will also constitute the main ingredients in our analysis of (8.1.1.1).

Quasi-gradient fields. The notion is natural and simple : a vector field V is called *quasi-gradient* for a function L if it has the same singular point (as ∇L) and if the angle α between the field V and the gradient ∇L remains acute and bounded away from $\pi/2$. Proper definitions are recalled in Section 8.3.1. Of course, such systems have a behavior which is very similar to those of gradient systems (see Theorem 8.3.1.2). We refer to [19] and the references therein for further geometrical insights on the topic.

Liapunov functions for damped second order gradient systems. The most striking common point between (8.1.1.1) and gradient systems is that of a "natural" Liapunov function. In our case, it is given by the *total energy*, sum of the potential energy and the kinetic energy,

$$E_T(u,v) = G(u) + \frac{1}{2}\|v\|^2.$$

The above is a Liapunov function in the phase space, more concretely

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_T(u(t),u'(t)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}|u'(t)|^2 + G(u(t))\Big) \\ &= -\gamma\|u'(t)\|^2.\end{aligned}$$

Contrary to what happens for classical gradient systems the vector field associated with (8.1.1.1) is not strictly Lyapunov for E_T : it obviously degenerates on the subspace $[v=0]$ (or $[u'=0]$). The use of E_T is however at the heart of most results attached to this dynamical system.

KL functions. A KL function is a function whose values can be reparametrized in the neighborhood of each of its critical point so that the resulting functions become *sharp*(1). More formally, G is called KL on the slice of level lines $[0 < G < r_0] \overset{\text{def}}{=} \big\{u \in \mathbb{R}^N; 0 < G(u) < r_0\big\}$, if there exists $\varphi \in C^0\big([0,r_0)\big) \cap C^1(0,r_0)$ concave such that $\varphi(0)=0$, $\varphi' > 0$ and

$$\|\nabla(\varphi \circ G)(u)\| \geqslant 1, \quad \forall u \in [0 < G < r_0].$$

Proper definitions and local versions can be found in the next section. The above definition originates in [40] and is based on the fundamental work of Kurdyka [124], where it was introduced in the framework of o-minimal structure(2) as a generalization of the famous Łojasiewicz inequality.

KL functions are central in the analysis of gradient systems, the readers are referred to [40] and the references therein.

Desingularizing functions. The function appearing above, namely φ, is called a *desingularizing function* : the faster φ' tends to infinity at 0, the flatter is G around critical points. As opposed to the Łojasiewicz gradient inequality, this behavior, in the o-minimal world, is not necessarily of a "power-type". Highly degenerate functions can be met, like for instance $G(u) = \exp\big(-1/p^2(u)\big)$ where $p : \mathbb{R}^N \longrightarrow \mathbb{R}$ is any real polynomial function. This class of functions belongs to the log-exp structure, an o-minimal class that contains semi-algebraic sets and the graph of the exponential function [182]. Finally, observe that if it is obvious that φ might have an arbitrarily brutal behavior at 0, it is also pretty clear that the smoothness of G is related to a lower-control of the behavior of φ, for instance we must have $\varphi'(0) = \infty$ – which is not the case in general in the nonsmooth world (see e.g. [39]).

8.1.2 Main results

Several auxiliary theorems were necessary to establish our main result, we believe they are interesting for their own sake. Here they are :

– An asymptotic alternative for quasi-gradient systems : either a trajectory converges or it escapes to infinity,

– A general convergence rate result for the solutions of the gradient systems that brings forward a worst-case gradient dynamical system in dimension one,

1. That is, the norms of its gradient remain bounded away from zero.
2. A far reaching concept that generalizes semi-algebraic or (globally) subanalytic classes of sets and functions.

– Lower bounds for desingularizing functions of C^2 KL functions.

We are now in position to describe the strategy we followed in that paper for the asymptotic study of the damped second order gradient system (8.1.1.1). Our method was naturally inspired by the Liapunov function provided in [97].

1. First we show that E_T can be slightly and "semi-algebraically" (respectively, definably) deformed into a smooth function E_T^{def}, so that the gradient of the new energy $\nabla E_T^{\mathrm{def}}$ makes an uniformly acute angle with the vector field associated with (8.1.1.1) – this property only holds on bounded sets of the phase space. The system (8.1.1.1) appears therefore as a quasi-gradient system for E_T^{def}.
2. In a second step we establish/verify that the solutions of the quasi-gradient systems converge whenever they originate from a KL function.

   We also provide rates of convergence and we explain how they may be naturally and systematically derived from a one-dimensional worst-case gradient dynamics.

   At this stage it is possible to proceed abstractly to the proof of the convergence of solutions to (8.1.1.1) in several cases. For instance the definable case : we simply have to use the fact that E_T^{def} is definable whenever G is, so it is a KL function and the conclusion follows.

   Although direct and fast, this approach has an important drawback from a conceptual viewpoint since it relies on a desingularizing function attached to an auxiliary function E_T^{def} whose meaning is unclear. Whatever perspectives we may adopt (Mechanics, Optimization, PDEs), an important question is indeed to *understand what happens when G is KL and how the desingularizing function of G actually impacts the convergence of solutions to* (8.1.1.1).
3. We answer to this question in the following way.
   (a) We prove that desingularizing functions of C^2 definable functions have a lower bound. Roughly speaking, we prove that for nontrivial critical points the desingularizing function has the property $\varphi(s) \geqslant c\sqrt{s}$ $\left(\text{or equivalently}^{(3)}\ \varphi'(s) \geqslant \frac{c'}{\sqrt{s}}\right)$.
   (b) We establish that if φ is definable and desingularizing for G at $\overline{u}$ then it is desingularizing for both E_T and E_T^{def} at $(\overline{u}, 0)$.
4. We conclude by combining previous results to obtain in particular the convergence of solutions to (8.1.1.1) under definability assumptions. We also provide convergence rates that depend on the desingularizing function of G, *i.e. on the geometry of the potential.*

We would like to point out and emphasize two facts that we think are of interest. First the property $\varphi(s) \geqslant c\sqrt{s}$ (see Lemma 8.2.2.1 below) is a new result and despite its "intuitive" aspect the proof is nontrivial. We believe it has an interest in its own sake.

More related to our work is the fact that (in the definable case and in many other relevant cases) our results show that the desingularizing function of G is conditioning the asymptotic behavior of solutions of the system. Within an Optimization perspective this means that the "complexity", or at least the convergence rate, of the dynamical system is entirely embodied in G when G is smooth. From a mechanical viewpoint, stabilization at infinity is determined by the conditioning of G provided the

3. Recall that φ is definable.

latter is smooth enough ; in other words the intuition that for large time behaviors, the potential has a predominant effect on the system is correct – a fact which is of course related to the dissipation of the kinetic energy at a "constant rate".

Notation. The finite-dimensional space $\mathbb{R}^N$ ($N \geqslant 1$) is endowed with the canonical scalar product $\langle\, .\,,\, .\,\rangle$ whose norm is denoted by $\|\,.\,\|$. The product space $\mathbb{R}^N \times \mathbb{R}^N$ is endowed with the natural product metric which we still denote by $\langle\, .\,,\, .\,\rangle$. We also define for any $\overline{u} \in \mathbb{R}^N$ and $r > 0$, $B(\overline{u}, r) = \{u \in \mathbb{R}^N; \|u - \overline{u}\| < r\}$. When S is a subset of $\mathbb{R}^N$ its interior is denoted by $\operatorname{int} S$ and its closure by $\overline{S}$. If $F : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a differentiable function, its gradient is denoted by ∇F. When F is a twice differentiable function, its Hessian is denoted by $\nabla^2 F$. The set of critical points of F is defined by

$$\operatorname{crit} F = \Big\{u \in \mathbb{R}^N; \nabla F(u) = 0\Big\}.$$

This paper is organized as follows. In Section 8.2, we provide a lower bound for desingularizing function of C^2 functions under various assumptions, like definability (Proposition 8.2.1.3 and Lemma 8.2.2.1). In Section 8.3, we recall the behavior of a first order system having a quasi-gradient structure for some KL function and we provide an asymptotic alternative (Theorem 8.3.1.2). In Theorem 8.3.2.4, the convergence rate of any solution to a first order system having a quasi-gradient structure is proved to be better than that of a one-dimensional worst-case gradient dynamics (various known results are recovered in a transparent way). Finally, we establish that any function which desingularizes G in (8.1.1.1) also desingularizes the total energy and various relevant deformation of the latter (Proposition 8.3.3.3). In Section 8.4, we study the asymptotic behavior of solutions to (8.1.1.1) (Theorem 8.4.1) while in Section 8.5, we describe several consequences of our main results. Appendix (p.199) provides, for the comfort of the reader, some elementary facts on o-minimal structures.

8.2 Structural results : lower bounds for desingularizing functions of C^2 functions

To keep the reading smooth and easy, we will not formally define here o-minimal structure. The definition is postponed in Appendix (p.199). Let us however recall, at this stage, that the simplest o-minimal structure (containing the graph of the real product) is given by the class of real semi-algebraic sets and functions. A semi-algebraic set is the finite union of sets of the form

$$\Big\{u \in \mathbb{R}^N;\ p(u) = 0,\ p_i(u) < 0, \forall i \in I\Big\}, \tag{8.2.1}$$

where I is a finite set and $p, \{p_i\}_{i \in I}$ are real polynomial functions.

Let us recall a fundamental concept for dissipative dynamical systems of gradient type.

Definition 8.2.1 (**Kurdyka-Łojasiewicz property and desingularizing function**).
Let $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ be a differentiable function.

(i) We shall say that G has the *KL property* at $\overline{u} \in \mathbb{R}^N$ if there exist $r_0 > 0$, $\eta > 0$ and $\varphi \in C([0, r_0); \mathbb{R}_+)$ such that

1. $\varphi(0) = 0$, $\varphi \in C^1((0, r_0); \mathbb{R}_+)$ concave and φ' positive on $(0, r_0)$,

2. $u \in B(\overline{u},\eta) \implies |G(u)-G(\overline{u})| < r_0$; and for each $u \in B\big(\overline{u},\eta\big)$, such that $G(u) \neq G(\overline{u})$,

$$\big\|\nabla(\varphi \circ |G(\,.\,)-G(\overline{u})|)(u)\big\| \geqslant 1. \tag{8.2.2}$$

Such a function φ is called a *desingularizing function* of G at $\overline{u}$ on $B(\overline{u},\eta)$.

(ii) The function G is called a *KL function* if it has the KL property at each of its points.

The following result is due to Łojasiewicz in its real-analytic version (see e.g. [133, 134]), it was generalized to o-minimal structures and considerably simplified by Kurdyka in [124] (see Appendix p.199).

Theorem 8.2.2 (**Kurdyka-Łojasiewicz inequality** [124][4]). *Let $\mathcal{O}$ be an o-minimal structure and let $G \in C^1(\mathbb{R}^N;\mathbb{R})$ be a definable function. Then G is a KL function.*

Remark 8.2.3. (a) Theorem 8.2.2 is of course trivial when $\overline{u} \notin \operatorname{crit} G$ – take indeed, $\varphi(s) = cs$ where $c = \frac{1+\varepsilon}{\|\nabla G(\overline{u})\|}$ and $\varepsilon > 0$.
(b) Restrictions of real-analytic functions to compact sets included in their (open) domain belong to the o-minimal structure of globally analytic sets [82]. They are therefore KL functions (see indeed Example A.2). In some o-minimal structures there are nontrivial functions for which all derivatives vanish on some nonempty set, like $G(u) = \exp(-1/f^2(u))$ where $f \neq 0$ is any smooth semi-algebraic function achieving the value 0[5] (see also Example A.2). For these cases, φ is not of power-type – as it is the case when G is semi-algebraic or real-analytic. Other types of functions satisfying the KL property in various contexts are provided in [12] (see also Corollary 8.5.5).
(c) Desingularizing functions of definable functions can be chosen to be definable, strictly concave and C^k (where k is arbitrary).

The following trivial notion is quite convenient.

Definition 8.2.4 (**Trivial critical points**). A critical point u of a differentiable function $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ is called *trivial* if $u \in \operatorname{int}\operatorname{crit} G$. It is *nontrivial* otherwise. Observe that u is nontrivial if, and only if, there exists $u_n \xrightarrow{n\to\infty} u$ such that $G(u_n) \neq G(u)$, for any $n \in \mathbb{N}$.

When $\overline{u}$ is a trivial critical point of G, any concave function $\varphi \in C^0\big([0,r_0)\big) \cap C^1(0,r_0)$ such that $\varphi' > 0$ and $\varphi(0) = 0$ is desingularizing at $\overline{u}$.

An immediate consequence of the KL inequality is a local and strong version of Sard's theorem.

Remark 8.2.5 (**Local finiteness of critical values**). Let $G \in C^1(\mathbb{R}^N;\mathbb{R})$ and $\overline{u} \in \mathbb{R}^N$. Assume that G satisfies the KL property at $\overline{u}$ on $B(\overline{u},\eta)$. Then

$$u \in B(\overline{u},\eta) \text{ and } \nabla G(u) = 0 \implies G(u) = G(\overline{u}).$$

The simplest functions we can think of with respect to the behavior of the solutions to (8.1.1.1) are given by functions with linear gradients, that is quadratic forms

$$G(u) = \frac{1}{2}\langle Au, u\rangle,\ u \in \mathbb{R}^N,\ \text{where } A \in \mathscr{M}_N(\mathbb{R}),\ A^T = A.$$

4. See comments in Appendix A.
5. This function is definable in the log-exp structure of Wilkie [182].

When $A \neq 0$, it is easy to establish directly that $\varphi(s) = \sqrt{\frac{1}{|\lambda|}s}$ (where λ is a nonzero eigenvalue with smallest absolute value) provides a desingularizing function. In the subsections to come, we show that the best we can hope in general for a desingularizing function φ attached to a C^2 function G is precisely a quantitative behavior of square-root type.

8.2.1 Lower bounds for desingularizing functions of potentials having a simple critical point structure

Our first assumption, formally stated below, asserts that points having critical value must be critical points. The assumption is rather strong in general but it will be complemented in the next section by a far more general result for definable functions.

$$\begin{cases} \text{Let } \overline{u} \in \operatorname{crit} G. \\ \text{There exists } \eta > 0 \text{ such that for any } u \in B(\overline{u}, \eta), \\ \big(G(u) = G(\overline{u}) \implies u \in \operatorname{crit} G\big). \end{cases} \tag{8.2.1.1}$$

Example 8.2.1.1. (a) When $N = 1$ and $G \in C^1$ is KL then assumption (8.2.1.1) holds.
[If the result does not hold then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{n\longrightarrow\infty} \overline{u}$ and

$$G(x_n) = G(\overline{u}), \tag{8.2.1.2}$$

$$G'(x_n) \neq 0, \tag{8.2.1.3}$$

for any $n \in \mathbb{N}$. Without loss of generality, we may assume that $(x_n)_{n\in\mathbb{N}}$ is monotone, say decreasing. From (8.2.1.2)–(8.2.1.3) and Rolle's Theorem, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that $x_{n+1} < u_n < x_n$, $G'(u_n) = 0$, $G(u_n) \neq G(\overline{u})$, for any $n \in \mathbb{N}$. Thus $G(u_n)$ are critical values distinct from $G(\overline{u})$ such that $G(u_n) \longrightarrow G(\overline{u})$; this contradicts the local finiteness of critical values – see Remark 8.2.5.]
(b) Of course, the result in (a) cannot be extended to higher dimensions. Consider for instance

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad G(u_1, u_2) = u_1^2 - u_2^2,$$

which is obviously KL. One has $\nabla G(u) = 0$ if, and only if, $u = 0$, yet $G(t, -t) = 0$ for any t in $\mathbb{R}$.
(c) If G is convex, (8.2.1.1) holds globally, *i.e.*, with $\eta = \infty$. [This follows directly from the well-known fact that $G(u) = \min G$ if, and only if, $\nabla G(u) = 0$.]

Lemma 8.2.1.2 (**Comparing values growth with gradients growth**).
Let $G \in C^{1,1}_{\text{loc}}(\mathbb{R}^N; \mathbb{R})$ and $\overline{u} \in \operatorname{crit} G$. Assume there exists $\varepsilon > 0$ such that

$$u \in B(\overline{u}, 2\varepsilon) \text{ and } G(u) = G(\overline{u}) \implies u \in \operatorname{crit} G,$$

in other words (8.2.1.1) holds (with $\eta = 2\varepsilon$). Then there exists $c > 0$ such that

$$|G(u) - G(\overline{u})| \geqslant c\|\nabla G(u)\|^2, \tag{8.2.1.4}$$

for any $u \in B(\overline{u}, \varepsilon)$.

Proof. Working if necessary with $\widetilde{G}(u) = G(u) - G(\overline{u})$, we may assume, without loss of generality, that $G(\overline{u}) = 0$. Let us proceed in two steps.

Step 1. Let $H \in C^{1,1}\big(\overline{B}(\overline{u}, 2\varepsilon); \mathbb{R}\big)$ with $\overline{u} \in \operatorname{crit} H$ and assume further that $H \geqslant 0$. We claim that there exists $c > 0$ such that

$$\forall u \in B(\overline{u}, \varepsilon), \ H(u) \geqslant c\|\nabla H(u)\|^2. \tag{8.2.1.5}$$

Denote by L_2 the Lipschitz constant of ∇H on $\overline{B}(\overline{u}, 2\varepsilon)$, let $L_1 = \max\limits_{u \in \overline{B}(\overline{u}, 2\varepsilon)} \|\nabla H(u)\|$ and set $L = L_1 + L_2$. Since,

$$\big(L_1 = 0 \ \text{ or } \ L_2 = 0\big) \implies \nabla H_{|B(\overline{u}, \varepsilon)} \equiv 0 \implies (8.2.1.5),$$

we may assume that $L_2 > 0$ and $L_1 > 0$. Let $u \in B(\overline{u}, \varepsilon)$. We have for any $v \in B(0, 2\varepsilon)$,

$$\begin{aligned} H(v) - H(u) &= \int_0^1 \langle \nabla H\big((1-t)u + tv\big), v - u\rangle \mathrm{d}t \\ &= \int_0^1 \langle \nabla H\big((1-t)u + tv\big) - \nabla H(u), v - u\rangle \mathrm{d}t + \langle \nabla H(u), v - u\rangle, \end{aligned}$$

so that for any $v \in B(0, 2\varepsilon)$,

$$\Big| H(v) - H(u) - \langle \nabla H(u), v - u\rangle \Big| \leqslant \frac{L_2}{2}\|v - u\|^2. \tag{8.2.1.6}$$

Note that $\big\|\big(u - \frac{\varepsilon}{L}\nabla H(u)\big) - \overline{u}\big\| \leqslant \|u - \overline{u}\| + \frac{\varepsilon}{L}\|\nabla H(u)\| < \varepsilon + \varepsilon\frac{L_1}{L} < 2\varepsilon$. By convexity, we infer that $\big[u, u - \frac{\varepsilon}{L}\nabla H(u)\big] \subset B(\overline{u}, 2\varepsilon)$. It follows that $v = u - \frac{\varepsilon}{L}\nabla H(u)$ is an admissible choice in (8.2.1.6). Without loss of generality, we may assume that $\varepsilon \leqslant 1$. This leads to

$$0 \leqslant H(v) \leqslant H(u) - \frac{\varepsilon}{2L}\|\nabla H(u)\|^2.$$

Whence the claim.

Step 2. Define for any $u \in \overline{B}(\overline{u}, 2\varepsilon)$, $H(u) = |G(u)|$. Since $\big(G(u) = 0 \implies \nabla G(u) = 0\big)$, we easily deduce that $H \in C^1_{\mathrm{b}}\big(B(\overline{u}, 2\varepsilon); \mathbb{R}\big)$ and for any $u \in B(\overline{u}, 2\varepsilon)$, $\nabla H(u) = \operatorname{sign}\big(G(u)\big)\nabla G(u)$. Denote by L_2 the Lipschitz constant of ∇G on $\overline{B}(\overline{u}, 2\varepsilon)$. We claim that,

$$\|\nabla H(u) - \nabla H(v)\| \leqslant L_2\|u - v\|, \tag{8.2.1.7}$$

for any $(u, v) \in B(\overline{u}, 2\varepsilon) \times B(\overline{u}, 2\varepsilon)$. Let $(u, v) \in B(\overline{u}, 2\varepsilon) \times B(\overline{u}, 2\varepsilon)$. Estimate (8.2.1.7) being clear if $G(u)G(v) \geqslant 0$, we may assume that $G(u)G(v) < 0$. By the Mean Value Theorem and the assumptions on G, it follows that there exists $t \in (0, 1)$ such that for $w = (1-t)u + tv$, $G(w) = 0$ and $\nabla G(w) = 0$. We then infer,

$$\begin{aligned} \|\nabla H(u) - \nabla H(v)\| &= \|\nabla G(u) + \nabla G(v)\| \leqslant \|\nabla G(u)\| + \|\nabla G(v)\| \\ &= \|\nabla G(u) - \nabla G(w)\| + \|\nabla G(w) - \nabla G(v)\| \\ &\leqslant L_2\|u - w\| + L_2\|w - v\| = L_2\|u - v\|. \end{aligned}$$

Hence (8.2.1.7). It follows that $H \in C^{1,1}\big(\overline{B}(\overline{u}, 2\varepsilon); \mathbb{R}\big)$ and H satisfies the assumptions of Step 1. Applying (8.2.1.5) to H, we get (8.2.1.4). This concludes the proof. □

Proposition 8.2.1.3 (Lower bound for desingularizing functions). *Let $G \in C^{1,1}_{\text{loc}}(\mathbb{R}^N;\mathbb{R})$ and let $\overline{u}$ be a nontrivial critical point, i.e. $\overline{u} \in \operatorname{crit} G \setminus \operatorname{int} \operatorname{crit} G$. Assume that G satisfies the KL property at $\overline{u}$ and that assumption (8.2.1.1) holds at $\overline{u}$.*
Then there exists $\beta > 0$ such that for any desingularizing function φ of G at $\overline{u}$,

$$\varphi'(s) \geqslant \frac{\beta}{\sqrt{s}}, \tag{8.2.1.8}$$

for any small positive s.

Proof. We may assume $G(\overline{u}) = 0$. Combining (8.2.2) and (8.2.1.4), we deduce that $\varphi'(|G(u)|) \geqslant \frac{1}{\|\nabla G(u)\|} \geqslant \frac{\beta}{\sqrt{|G(u)|}}$, for any $u \in B(\overline{u},\varepsilon)$ such that $G(u) \neq G(\overline{u})$ (Remark 8.2.5). Changing G into $-G$ if necessary, there is no loss of generality to assume that there exists u_n such that $u_n \longrightarrow \overline{u}$ with $G(u_n) > 0$ (recall $\overline{u}$ is a nontrivial critical point). Since G is continuous, this implies by a connectedness argument that for some ρ there exists $r > 0$ such that $\big|G\big(B(\overline{u},\rho)\big)\big| \supset (0,r)$. Using the parametrization $s \in (0,r)$ we conclude that $\varphi'(s) \geqslant \frac{\beta}{\sqrt{s}}$, for any s sufficiently small. □

8.2.2 Lower bounds for desingularizing functions of definable C^2 functions

This part makes a strong use of definability arguments (these are recalled in the last section).

Lemma 8.2.2.1 (Lower bounds for desingularizing functions of C^2 definable functions). *Let $G : \Omega \longrightarrow \mathbb{R}$ be a C^2 definable function on an open subset $\Omega \ni 0$ of $\mathbb{R}^N$. We assume that 0 is a nontrivial critical point*[6] *and that $G(0) = 0$.*
Since G is definable it has the KL property[7] *that is, there exist $\eta, r_0 > 0$ and $\varphi : [0,r_0) \longrightarrow \mathbb{R}$ as in Definition 8.2.1 such that*

$$\|\nabla\big(\varphi \circ |G|\big)(u)\| \geqslant 1, \tag{8.2.2.1}$$

for any u in $B(0,\eta)$ such that $G(u) \neq 0$.
Then there exists $c > 0$ such that

$$\varphi'(s) \geqslant \frac{c}{\sqrt{s}}, \tag{8.2.2.2}$$

so that $\varphi(s) \geqslant 2c\sqrt{s}$, for any small $s > 0$.

Proof. Let us outline the ideas of the proof : after a simple reduction step, we show that the squared norm of a/the smallest gradient on a level line increases at most linearly with the function values. In the second step, we show that this estimate is naturally linked to the increasing rate of φ itself and to property (8.2.2.2). Let $\varphi : [0,r_0) \longrightarrow \mathbb{R}$ be any desingularizing function of G at 0 on $B(0,\eta)$, as in Definition 8.2.1.

Changing G in $-G$ if necessary, we may assume by Definition 8.2.4, without loss of generality, that there exists a sequence $(u_n)_n$ such that $u_n \xrightarrow{n\to\infty} 0$ and $G(u_n) > 0$, for any $n \in \mathbb{N}$. Let us proceed

6. Equivalently, we assume that there exists $u_n \xrightarrow{n\to\infty} 0$ such that $G(u_n) \neq 0$.
7. See Theorem 8.2.2.

with the proof in three steps.

Step 1. We first modify the function G as follows. Let $\rho \in C^2(\mathbb{R}^N;[0,1])$ be a semi-algebraic function such that

$$\begin{cases} \operatorname{supp} \rho \subset B(0,\eta) \subset \Omega, \\ \rho(x) = 1, \text{ if } x \in B\left(0, \frac{\eta}{2}\right). \end{cases}$$

Let us define $\widehat{G}$ on $\mathbb{R}^N$ by

$$\widehat{G}(u) = \begin{cases} \rho(u)G(u) + \operatorname{dist}\left(u, B\left(0, \frac{\eta}{2}\right)\right)^3, & \text{if } u \in \Omega, \\ 0, & \text{if } u \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

It follows that $\widehat{G} \in C^2(\mathbb{R}^N;\mathbb{R})$, leaves the set of desingularizing functions at 0 unchanged, has compact lower level sets and is definable in the same structure (recall Definition A.1 (iii)). Finally, we obviously have,

$$u_n \xrightarrow{n\to\infty} 0 \text{ with } \widehat{G}(u_n) > 0, \ \forall n \in \mathbb{N}. \tag{8.2.2.3}$$

Without loss of generality, we may assume that $\eta \leqslant 1$ and $r_0 \leqslant \frac{\eta^3}{8}$. Let $u \in \mathbb{R}^N \setminus B(0,\eta)$. One has,

$$\widehat{G}(u) = \operatorname{dist}\left(u, B\left(0, \frac{\eta}{2}\right)\right)^3 = \left(\|u\| - \frac{\eta}{2}\right)^3 \geqslant \frac{\eta^3}{8} \geqslant r_0.$$

It follows that,

$$\inf_{u \in B(0,\eta)\cap[\widehat{G}=r]} \|\nabla \widehat{G}(u)\| = \min_{u\in[\widehat{G}=r]} \|\nabla \widehat{G}(u)\|, \ \ \forall r \in (0, r_0). \tag{8.2.2.4}$$

Step 2. For $r > 0$, we introduce

$$(P_r) \qquad \psi(r) = \min\left\{\frac{1}{2}\|\nabla \widehat{G}(u)\|^2;\ u \in \mathbb{R}^N,\ \widehat{G}(u) = r\right\}.$$

Since the set of critical values of a definable function is finite and since the level sets are compact, we may choose, if necessary, r_0 so that $\psi > 0$ on $(0, r_0)$ (the fact that 0 is a nontrivial critical point excludes the case when ψ vanishes around 0). If we denote by $S(r)$ the nonempty compact set of solutions to (P_r), one easily sees that

$$S : (0, r_0) \rightrightarrows \mathbb{R}^N,$$

is a definable point-to-set mapping – this follows by a straightforward use of quantifier elimination (*i.e.*, by the use of Definition A.1). Using the Definable Selection Lemma (Lemma A.4), one obtains a definable curve $u : (0, r_0) \longrightarrow \mathbb{R}^N$ such that $u(r) \in S(r)$, for any $r \in (0, r_0)$. Finally, using the Monotonicity Lemma (Lemma A.3) repeatedly on the coordinates u_i of u, one can shrink r_0 so that u is actually in $C^1((0, r_0);\mathbb{R}^N)$.

Fix now r in $(0, r_0)$. Since r is noncritical the problem (P_r) is qualified and we can apply Lagrange's Theorem for constrained problems. This yields the existence of a real multiplier $\lambda(r)$ such that

$$\nabla^2 \widehat{G}(u(r))\nabla \widehat{G}(u(r)) - \lambda(r)\nabla \widehat{G}(u(r)) = 0, \tag{8.2.2.5}$$

with of course $\widehat{G}(u(r)) = r$.

Note that for any $r \in (0, r_0)$, $\nabla\widehat{G}(u(r)) \neq 0$ (as seen at the beginning of this step) so that $\lambda(r)$ is an actual eigenvalue of $\nabla^2\widehat{G}(u(r))$. Since $\widehat{G}$ is C^2, the curve $\nabla^2\widehat{G}(u(r))$ is bounded in the space of matrices $\mathscr{M}_N(\mathbb{R})$. Since eigenvalues depend continuously on operators, one deduces from the previous remarks that there exists $\overline{\lambda} \geqslant 0$ such that

$$|\lambda(r)| \leqslant \overline{\lambda},\ \forall r \in (0, r_0).$$

Multiplying (8.2.2.5) by $u'(r)$ gives $\langle \nabla^2\widehat{G}(u(r))\nabla\widehat{G}(u(r)), u'(r)\rangle = \lambda(r)\langle \nabla\widehat{G}(u(r)), u'(r)\rangle$, which is nothing else than

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\|\nabla\widehat{G}(u(r))\|^2 = \lambda(r)\frac{\mathrm{d}}{\mathrm{d}r}\widehat{G}(u(r)).$$

Since $\widehat{G}(u(r)) = r$, one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\|\nabla\widehat{G}(u(r))\|^2 = \lambda(r),$$

so after integration on $[s, r] \subset (0, r_0)$, one obtains

$$\left|\|\nabla\widehat{G}(u(r))\|^2 - \|\nabla\widehat{G}(u(s))\|^2\right| = 2\left|\int_s^r \lambda(\tau)d\tau\right| \leqslant 2\overline{\lambda}|r - s| \xrightarrow{r,s\to 0} 0. \tag{8.2.2.6}$$

It follows that $\left(\|\nabla\widehat{G}(u(r))\|^2\right)_{s>0}$ is a Cauchy's family, so that the limit ℓ of $\|\nabla\widehat{G}(u(s))\|^2$ as s goes to zero exists in $[0, \infty)$. We recall that by assumption (8.2.2.3), $u_n \xrightarrow{n\to\infty} 0$, $\widehat{G}(u_n) > 0$ and $\nabla\widehat{G}(u_n) \xrightarrow{n\to\infty} 0$. Now, setting $r_n = \widehat{G}(u_n)$, one has by definition of $u(r_n)$, $\|\nabla\widehat{G}(u_n)\| \geqslant \|\nabla\widehat{G}(u(r_n))\|$. This implies that $\ell = 0$ and as a consequence (8.2.2.6) yields

$$\frac{1}{2}\|\nabla\widehat{G}(u(r))\|^2 = \int_0^r \lambda(\tau)d\tau \leqslant \overline{\lambda}r, \tag{8.2.2.7}$$

in other words

$$\psi(r) \leqslant \overline{\lambda}r,\quad \forall r \in (0, r_0). \tag{8.2.2.8}$$

Step 3. Let us now conclude. By KL inequality one has for any $r \in (0, r_0)$,

$$\varphi'(r) \geqslant \frac{1}{\|\nabla\widehat{G}(u)\|},\quad \forall u \in B(0,\eta) \cap [G = r]. \tag{8.2.2.9}$$

As a consequence, we can use (8.2.2.4) in (8.2.2.9) and the linear estimate (8.2.2.8) above to conclude as follows :

$$\begin{aligned}
\varphi'(r) &\geqslant \frac{1}{\inf\left\{\|\nabla\widehat{G}(u)\|;\ u \in B(0,\eta) \cap [\widehat{G} = r]\right\}} \\
&= \frac{1}{\min\left\{\|\nabla\widehat{G}(u)\|;\ u \in [\widehat{G} = r]\right\}} \\
&\geqslant \frac{1}{\sqrt{2\psi(r)}} \\
&\geqslant \frac{c}{\sqrt{r}},
\end{aligned}$$

for any $r \in (0, r_0)$, with $c = \left(\sqrt{2\lambda}\right)^{-1}$. Hence (8.2.2.2). □

Remark 8.2.6. (a) Note that if $G \notin C^2$ then (8.2.2.2) does not hold. Indeed, take $G(u) = u^{\frac{3}{2}}$ and $\varphi(s) = s^{\frac{2}{3}}$ as a (semi-algebraic) counter-example.
(b) When we omit the assumption that 0 is a nontrivial critical point, *i.e.* $0 \in \operatorname{int} \operatorname{crit} G$, then G vanishes in a neighborhood of 0. In that case, the result is not true in general since any concave increasing function adequately regular is desingularizing for G. However a function $\varphi(s) = c\sqrt{s}$ can still be chosen as a desingularizing function.
Hence, *for an arbitrary C^2 definable function, we can always assume that for any critical point, the corresponding desingularizing function satisfies $\varphi'(s) \geqslant c\frac{1}{\sqrt{s}}$ (locally for some positive constant c).*

8.3 Damped second order gradient systems

8.3.1 Quasi-gradient structure and KL inequalities

Definition 8.3.1.1. Let Γ be a nonempty closed subset of $\mathbb{R}^N$ and let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a locally Lipschitz continuous mapping.

(i) We say that the first order system

$$u'(t) + F\big(u(t)\big) = 0, \ t \in \mathbb{R}_+, \tag{8.3.1.1}$$

has a *quasi-gradient structure for E on Γ*, if there exist a differentiable function $E : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $\alpha_\Gamma = \alpha > 0$ such that

(angle condition) $$\big\langle \nabla E(u), F(u) \big\rangle \geqslant \alpha \, \|\nabla E(u)\| \, \|F(u)\|, \text{ for any } u \in \Gamma, \tag{8.3.1.2}$$

(rest-points equivalence) $$\operatorname{crit} E \cap \Gamma = F^{-1}(\{0\}) \cap \Gamma. \tag{8.3.1.3}$$

(ii) Equivalently a vector field F having the above properties is said to be *quasi-gradient for E on Γ*.

The following result involves classical material and ideas, yet, the fact that an asymptotic alternative can be derived in this setting does not seem to be well-known (see however [12] in a discrete context).

Theorem 8.3.1.2 (Asymptotic alternative for quasi-gradient fields). *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a locally Lipschitz mapping that defines a quasi-gradient vector field for E on $\mathbb{R}^N$, for some differentiable function $E : \mathbb{R}^N \longrightarrow \mathbb{R}$. Assume further that the function E is KL. Let u be any solution to (8.3.1.1). Then,*

(i) either $\|u(t)\| \xrightarrow{t\to\infty} \infty$,

(ii) or u converges to a singular point u_∞ of F as $t \longrightarrow \infty$.

When (ii) holds then $u' \in L^1\big((0,\infty); \mathbb{R}^N\big)$ and $u'(t) \xrightarrow{t\to\infty} 0$. Moreover, we have the following estimate,

$$\|u(t) - u_\infty\| \leqslant \frac{1}{\alpha}\varphi\big(E(u(t)) - E(u_\infty)\big), \tag{8.3.1.4}$$

where φ is a desingularizing function of E at u_∞ and α is the constant in (8.3.1.2).

Proof. We assume that (i) does not hold, so there exist $u_\infty \in \mathbb{R}^N$ and a sequence $s_n \nearrow \infty$ such that $u(s_n) \xrightarrow{n\longrightarrow\infty} u_\infty$. Note that by continuity of E, one has $E\big(u(s_n)\big) \xrightarrow{n\longrightarrow\infty} E(u_\infty)$. Observe also that from equation (8.3.1.1) and the angle condition (8.3.1.2), one has for any $t \geqslant 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(E\circ u\big)(t) &= \big\langle \nabla E\big(u(t)\big), u'(t)\big\rangle \\ &= -\big\langle \nabla E\big(u(t)\big), F\big(u(t)\big)\big\rangle \\ &\leqslant -\alpha\|\nabla E(u(t))\|\,\|F(u(t))\|, \end{aligned} \tag{8.3.1.5}$$

and thus the mapping $t \longmapsto E(u(t))$ is nonincreasing, which implies

$$\lim_{t\to\infty} E(u(t)) = E(u_\infty).$$

Note that if $E(u(\bar{t})) = E(u_\infty)$ for some $\bar{t}$, one would have $\frac{\mathrm{d}}{\mathrm{d}t}\big(E\circ u\big)(t) = 0$ for any $t > \bar{t}$, which would in turn imply, by (8.3.1.5), that $\|\nabla E(u(t))\|\,\|F(u(t))\| = 0$, for any such t. In view of the rest point equivalence (8.3.1.3), this would mean that $F(u(t)) = 0$, hence by uniqueness of solution curves, that $u(t) = u_\infty$ for any $t \geqslant 0$. We can thus assume without loss of generality that

$$E(u(t)) > E(u_\infty),\ \forall t \geqslant 0. \tag{8.3.1.6}$$

Let $t_0 > 0$ be such that $u(t_0) \in B\left(u_\infty, \frac{\eta}{2}\right)$ and $\varphi\big(E\big(u(t_0)\big) - E(u_\infty)\big) \in \left(0, \frac{\eta\alpha}{2}\right)$, where $\alpha > 0$ is the constant in (8.3.1.2) [in view of our preliminary comments and of the continuity of E such a t_0 exists]. By continuity of u, there exists $\tau > 0$ such that for any $t \in [t_0, t_0+\tau)$, $u(t) \in B(u_\infty, \eta)$. So we may define $T \in (t_0, \infty]$ as

$$T = \sup\Big\{t > t_0\,;\, \forall s \in [t_0, t),\ u(s) \in B(u_\infty, \eta)\Big\}.$$

By (8.3.1.5), the Kurdyka-Łojasiewicz inequality (8.2.2) and equation (8.3.1.1), we have for any $t \in (t_0, T)$,

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}\Big(\varphi\circ\Big(E\big(u(\,.\,)\big) - E\big(u_\infty\big)\Big)\Big)(t) \\ &= -\,\varphi'\big(E\big(u(t)\big) - E(u_\infty)\big)\frac{\mathrm{d}}{\mathrm{d}t}\big(E\circ u\big)(t) \\ &\geqslant \alpha\,\varphi'\big(E\big(u(t)\big) - E(u_\infty)\big)\,\big\|\nabla E\big(u(t)\big)\big\|\,\big\|F\big(u(t)\big)\big\| \\ &= \alpha\,\big\|F\big(u(t)\big)\big\|\,\big\|\nabla\big(\varphi\circ\big(E(\,.\,) - E(u_\infty)\big)\big(u(t)\big)\big\| \\ &\geqslant \alpha\|u'(t)\|. \end{aligned} \tag{8.3.1.7}$$

It follows from the above estimate that

$$\|u(t) - u(t_0)\| \leqslant \int_{t_0}^{t} \|u'(s)\|\mathrm{d}s \leqslant \frac{\varphi\big(E\big(u(t_0)\big) - E(u_\infty)\big)}{\alpha} < \frac{\eta}{2}, \tag{8.3.1.8}$$

for any $t \in (t_0, T)$. We claim that $T = \infty$. Indeed, otherwise $T < \infty$ and (8.3.1.8) applies with $t = T$. Hence,

$$\|u(T) - u_\infty\| \leqslant \|u(T) - u(t_0)\| + \|u(t_0) - u_\infty\| < \eta.$$

Then $u(T) \in B(u_\infty, \eta)$, which contradicts the definition of T. As a consequence the curve u' belongs to $L^1\big((t_0,\infty);\mathbb{R}^N\big)$ by (8.3.1.8) and the curve u converges to u_∞ by Cauchy's criterion. Finally since 0 must be a cluster point of u' $\big($recall indeed $\int_0^\infty \|u'(t)\| \mathrm{d}t < \infty$ and u' is uniformly continuous by (8.3.1.1)$\big)$, one must have $F(u_\infty) = 0$. The announced estimate follows readily from (8.3.1.8) and the fact that $T = \infty$. □

Corollary 8.3.1.3. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be locally Lipschitz continuous and assume that for any $R > 0$ the mapping F defines a quasi-gradient vector field for some differentiable function $E_R : \mathbb{R}^N \longrightarrow \mathbb{R}$ on $\overline{B}(0,R)$. Assume further that each of the functions E_R is KL.*
Let u be any bounded solution to (8.3.1.1). Then u converges to a singular point u_∞ of F, u' is integrable and converges to 0. In particular, if we take $R \geqslant \sup\big\{\|u(t)\|;\ t \in [0,\infty)\big\}$, we have the following estimate,

$$\|u(t) - u_\infty\| \leqslant \frac{1}{\alpha_R}\varphi\Big(E_R(u(t)) - E_R(u_\infty)\Big), \tag{8.3.1.9}$$

where φ is a desingularizing function of E_R at u_∞ and α_R is the constant in (8.3.1.2), for the ball $\overline{B}(0,R)$.

Proof. Take $R \geqslant \sup\big\{\|u(t)\|; t \in [0,\infty)\big\}$ and observe that the previous proof may be reproduced as it is : just replace E by E_R. □

8.3.2 Convergence rate of quasi-gradient systems and worst-case dynamics

To simplify our presentation we consider first a proper gradient system :

$$u'(t) + \nabla E(u(t)) = 0, \tag{8.3.2.1}$$

where $E : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a twice continuously differentiable KL function. We assume that u is bounded so, by virtue of our previous considerations, the curve converges to some critical point u_∞ of E. Observe that if u_∞ is a trivial critical point, one actually has $u(0) = u_\infty$ and the asymptotic study is trivial.

We thus assume u_∞ to be nontrivial, and we denote by φ a desingularizing function of E at u_∞. We set

$$\psi = \varphi^{-1},$$

whose domain is denoted by $[0,a)$, (with $a \in (0,\infty]$) and we consider the *one-dimensional worst-case gradient dynamics* (see [38]) :

$$\nu'(t) + \psi'(\nu(t)) = 0, \quad \nu(0) = \nu_0 \in (0,a). \tag{8.3.2.2}$$

We shall assume that

$$\varphi'(s) \geqslant \frac{c}{\sqrt{s}}, \text{ on } (0,r_0), \tag{8.3.2.3}$$

which implies that solutions ν to (8.3.2.2) are globally defined on $[0,\infty)$ and satisfy $\lim_{t\nearrow\infty} \nu(t) = 0$ with $\nu(t) \geqslant \nu_0 e^{-c_0 t}$, for any $t \geqslant 0$ (and for some $c_0 > 0$). Uniqueness holds by concavity of φ. Finally, note that if E is a C^2 definable function then φ can be chosen to be C^2, strictly concave and satisfying (8.3.2.3) (Remark 8.2.3 (c) and Lemma 8.2.2.1).

Radial functions and worst-case dynamics. A full justification of the terminology "worst-case dynamics" is to be given further, but at this stage one can observe that E could be taken of the form

$$E_{\text{rad}}(u) = \varphi^{-1}(\|u - u_\infty\|), \text{ with } u \in B(u_\infty, \eta)\ (\eta > 0),$$

provided that φ^{-1} is smooth enough. In that case φ is clearly desingularizing and the solutions of the gradient system (8.3.2.1) are radial in the sense that they are of the form[8]

$$u(t) = u_\infty + \nu(t)\frac{u_0 - u_\infty}{\|u_0 - u_\infty\|}, \tag{8.3.2.4}$$

where ν is a solution to (8.3.2.2). In this case, the dynamics (8.3.2.2) exactly measures the convergence rates for (8.3.2.1), since one has for any $t \geqslant 0$ and any u_0 such that $\nu(0) = \|u_0 - u_\infty\|$,

$$E_{\text{rad}}(u(t)) = \psi(\nu(t)), \tag{8.3.2.5}$$

$$\|u(t) - u_\infty\| = \nu(t). \tag{8.3.2.6}$$

We are about to see that this behavior in terms of convergence rate is actually the worst we can expect.

Remark 8.3.2.1. (a) As can be seen below, the worst-case gradient system is introduced to measure the rate of convergence of solutions for large t. Since nontrivial solutions to (8.3.2.2) have the same asymptotic behavior (they are, indeed, all of the form $\nu_1(t) = \nu(t+t_0)$ where t_0 is some real number), the choice of the initial condition $\nu(0)$ in $(0,a)$ can be made arbitrarily.
(b) The above rewrites $\nu'(t)\varphi'\big(\varphi^{-1}(\nu(t))\big) = -1$. Thus if μ denotes an antiderivative of $\varphi' \circ \varphi^{-1}$, one has $\nu(t) = \mu^{-1}(-t + a_0)$ (where a_0 is a constant), for any $t > 0$ large enough.
(c) In general, the explicit integration of such a system depends on the integrability properties of ψ and on the fact that $\varphi' \circ \varphi^{-1}$ admits an antiderivative in a closed form.
For instance if $\varphi(s) = (\frac{s}{c})^\theta$, with $c > 0$ and $\theta \in \left(0, \frac{1}{2}\right)$, then $\psi(s) = cs^{\frac{1}{\theta}}$ and

$$\nu'(t) + \frac{c}{\theta}\,\nu(t)^{\frac{1-\theta}{\theta}} = 0, \quad \nu(0) \in (0,a).$$

Thus by integration

$$\frac{\mathrm{d}}{\mathrm{d}t}\nu^{1-\frac{1-\theta}{\theta}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\nu^{\frac{-1+2\theta}{\theta}}(t) = c_1,$$

with $c_1 > 0$. As a consequence,

$$\nu(t) = \big(c_2 + c_1 t\big)^{-\frac{\theta}{1-2\theta}},$$

with $c_2 > 0$. When $\theta = \frac{1}{2}$ one easily sees that $\nu(t) = \nu(0)\exp\left(-2ct\right).$

8. Just use the formula in (8.3.2.1).

Theorem 8.3.2.2 (**The worst-case rate and worst-case one-dimensional gradient dynamics**).
Let $E \in C^2(\mathbb{R}^N;\mathbb{R})$ be a KL function, let u be a bounded solution to (8.3.2.1) and let $u_\infty \in \operatorname{crit} E$ satisfying $u(t) \xrightarrow{t\to\infty} u_\infty$ (such a u_∞ exists by Theorem 8.3.1.2). Then for any t large enough,

$$E(u(t)) - E(u_\infty) \leqslant \psi(\nu(t)), \tag{8.3.2.7}$$

and

$$\|u(t) - u_\infty\| \leqslant \nu(t), \tag{8.3.2.8}$$

where ν is a solution to (8.3.2.2).

Proof. Without loss of generality, we may assume that $E(u_\infty) = 0$. From the previous results, we know that for any $t \geqslant t_0$, we have $u(t) \in B(u_\infty, \eta)$ and $E(u(t)) \in (0, r_0)$, so that the KL inequality gives (see Theorem 8.3.1.2 and (8.3.1.7)) :

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi \circ E(u)\big)(t) \geqslant \|u'(t)\|.$$

Set $z(t) = E(u(t))$. Since $\frac{\mathrm{d}}{\mathrm{d}t}(E \circ u)(t) = -\|u'(t)\|^2$, one has $-\frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ z)(t) \geqslant \sqrt{-z'(t)}$, or equivalently

$$\varphi'\big(z(t)\big)^2 z'(t) \leqslant -1.$$

Consider now the worst-case gradient system with initial condition $\nu(t_0) = \varphi\big(E(u(t_0))\big)$ and set $z_a(t) = \psi(\nu(t)) = \varphi^{-1}(\nu(t))$, for $t \geqslant t_0$. The system (8.3.2.2) becomes $\varphi'(z_a(t))z_a'(t) + \frac{1}{\varphi'(z_a(t))} = 0$, *i.e.*, $\varphi'(z_a(t))^2 z_a'(t) = -1$. If μ is an antiderivative of ${\varphi'}^2$ on $(0, r_0)$, it is an increasing function and one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu \circ z)(t) = \varphi'\big(z(t)\big)^2 z'(t) \leqslant -1 = \varphi'\big(z_a(t)\big)^2 z_a'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\mu \circ z_a)(t),$$

and $\mu(z(t_0)) = \mu(z_a(t_0))$. As a consequence, $\mu(z(t)) \leqslant \mu(z_a(t))$, hence $z(t) \leqslant z_a(t)$ for any $t \geqslant t_0$, which is exactly (8.3.2.7). Using (8.3.1.4), we conclude by observing that

$$\|u(t) - u_\infty\| \leqslant \varphi(E(u(t))) \leqslant \varphi(z_a(t)) = \nu(t).$$

The theorem is proved. □

Remark 8.3.2.3. Observe that in the case of a desingularizing function of power type (see Remark 8.3.2.1 (c)), we recover well-known estimates [99].

Theorem 8.3.2.4 (**The worst-case one-dimensional gradient dynamics for quasi-gradient systems**).
Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a locally Lipschitz continuous mapping that defines a quasi-gradient vector field for some function $E \in C^2(\mathbb{R}^N;\mathbb{R})$ on $\overline{B}(0,R)$, for any $R > 0$. Assume further that the function E is KL and that for any $R > 0$, there exists a positive constant $b > 0$ such that

$$\|\nabla E(u)\| \leqslant b\|F(u)\|, \tag{8.3.2.9}$$

for any $u \in \overline{B}(0,R)$*. Assume further that for a given initial data* $u_0 \in \mathbb{R}^N$ *the solution* u *to* (8.3.1.1) *converges to some rest point* u_∞*. Denote by* φ *some desingularizing function for* E *at* u_∞*.*
Then there exist some constants $c, d > 0, t_0 \in \mathbb{R}$ *such that*

$$\|u(t) - u_\infty\| \leqslant d\nu\,(ct + t_0)\,, \tag{8.3.2.10}$$

where ν *is a solution to* (8.3.2.2)*.*

Proof. Combining the techniques used in Theorems 8.3.1.2 and 8.3.2.2, the proof is almost identical to that of Theorem 8.3.2.2. Without loss of generality, we may assume that $E(u_\infty) = 0$. We simply need to check the following inequality which is itself a consequence of the assumption (8.3.2.9) applied with $R = \sup\limits_{t>0} \|u(t)\|$.

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}(E \circ u)(t) &= -\langle u'(t), \nabla E(u(t))\rangle \\ &\leqslant \|F(u(t))\|\,\|\nabla E(u(t))\| \\ &\leqslant b\,\|F(u(t))\|^2 \\ &\leqslant b\,\|u'(t)\|^2. \end{aligned}$$

From (8.3.1.7) one has $-\frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ E)(u(t)) \geqslant \alpha\|u'(t)\|$, for any t sufficiently large. Setting $z(t) = E(u(t))$, one obtains $-\frac{\mathrm{d}}{\mathrm{d}t}(\varphi \circ z)(t) \geqslant \frac{\alpha}{\sqrt{b}}\sqrt{-z'(t)}$. The conclusion follows as before by using a reparametrization of (8.3.2.2). □

Remark 8.3.2.5. Assumption (8.3.2.9) is of course necessary and simply means that the vector field F drives solutions to their rest points at least "as fast as ∇E" (see also [67]).

8.3.3 Damped second order systems are quasi-gradient systems

As announced earlier our approach to the asymptotic behavior of damped second order gradient system is based on the observation that (8.1.1.1) can be written as a system having a quasi-gradient structure. For $G \in C^2(\mathbb{R}^N; \mathbb{R})$, let us define $\mathcal{F} : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by

$$\mathcal{F}(u,v) = \big(-v,\, \gamma v + \nabla G(u)\big).$$

Then (8.1.1.1) is equivalent to

$$U'(t) + \mathcal{F}\big(U(t)\big) = 0, \quad t \in \mathbb{R}_+, \text{ with } U = (u,v). \tag{8.3.3.1}$$

As explained in the introduction the total energy function $E_T(u,v) = G(u) + \frac{1}{2}\|v\|^2$ (sum of the potential energy and the kinetic energy) is a Liapunov function for our dynamical system (8.1.1.1). Formally

$$\langle \nabla E_T(u,v), \mathcal{F}(u,v)\rangle = \gamma\|v\|^2.$$

From the above we see, that the damped system (8.1.1.1) is not quasi-gradient for E_T since one obviously has a degeneracy phenomenon

$$\big\langle \nabla E_T(u,v), \mathcal{F}(u,v)\big\rangle = 0 \text{ whenever } v = 0, \tag{8.3.3.2}$$

where in general $\nabla E_T(u,v) \neq 0$ and $\mathcal{F}(u,v) \neq 0$.
The idea that follows consists in continuously deforming the level sets of E_T, through a family of functions :

$$\mathcal{E}_\lambda : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R} \ \text{ with } \ \mathcal{E}_0 = E_T \ \ (\lambda \text{ denotes here a positive parameter}),$$

so that the angle formed between each of the gradients of the resulting functions $\mathcal{E}_\lambda, \lambda > 0$ and the vector $\mathcal{F}$ remains far away from $\pi/2$. In other words we seek for functions making $\mathcal{F}$ a quasi-gradient vector field.

Proposition 8.3.3.1 (**The second order gradient systems are quasi-gradient systems**). *Let $G \in C^2(\mathbb{R}^N;\mathbb{R})$ and let $\gamma > 0$. For $\lambda > 0$, define $\mathcal{E}_\lambda \in C^1(\mathbb{R}^N \times \mathbb{R}^N;\mathbb{R})$ by*

$$\mathcal{E}_\lambda(u,v) = \left(\frac{1}{2}\|v\|^2 + G(u)\right) + \lambda\langle\nabla G(u), v\rangle.$$

For any $R > 0$, there exists $\lambda_0 > 0$ satisfying the following property. For any $\lambda \in (0, \lambda_0]$, there exists $\alpha > 0$ such that

$$\big\langle \nabla\mathcal{E}_\lambda(u,v), \mathcal{F}(u,v)\big\rangle \geqslant \alpha\,\|\nabla\mathcal{E}_\lambda(u,v)\|\,\|\mathcal{F}(u,v)\|, \tag{8.3.3.3}$$

for any $(u,v) \in \overline{B}(0,R) \times \mathbb{R}^N$. Furthermore,

$$\operatorname{crit}\mathcal{E}_\lambda \cap \left(\overline{B}(0,R) \times \mathbb{R}^N\right) = \mathcal{F}^{-1}(\{0\}) \cap \left(\overline{B}(0,R) \times \mathbb{R}^N\right), \tag{8.3.3.4}$$

for any $\lambda \in [0, \lambda_0]$.

Proof. For each $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$, we have $\nabla\mathcal{E}_\lambda(u,v) = \big(\nabla G(u) + \lambda\,\nabla^2 G(u)v,\ v + \lambda\nabla G(u)\big)$. Let $R > 0$ be given and let $M = \max\big\{\|\nabla^2 G(u)\|;\ u \in \overline{B}(0,R)\big\}$. Choose $\lambda_0 > 0$ small enough to have

$$\gamma - \left(M + \frac{\gamma^2}{2}\right)\lambda_0 > 0.$$

Let $\lambda \in (0, \lambda_0]$. Then for any $(u,v) \in \overline{B}(0,R) \times \mathbb{R}^N$, we obtain by Young's inequality,

$$\begin{aligned}
\big\langle \nabla\mathcal{E}_\lambda(u,v), \mathcal{F}(u,v)\big\rangle &= \gamma\|v\|^2 - \lambda\,\langle\nabla^2 G(u)v, v\rangle + \lambda\,\langle\nabla G(u), \gamma v\rangle + \lambda\,\|\nabla G(u)\|^2 \\
&\geqslant \left(\gamma - M\lambda_0 - \frac{\lambda_0}{2}\gamma^2\right)\|v\|^2 + \frac{\lambda}{2}\,\|\nabla G(u)\|^2 \\
&\geqslant \alpha_0\,(\|v\|^2 + \|\nabla G(u)\|^2),
\end{aligned} \tag{8.3.3.5}$$

where $\alpha_0 = \min\left\{\gamma - \left(M + \frac{\gamma^2}{2}\right)\lambda_0,\ \frac{\lambda}{2}\right\} > 0$. Moreover,

$$\|\nabla\mathcal{E}_\lambda(u,v)\|\,\|\mathcal{F}(u,v)\| \leqslant \frac{1}{2}\|\nabla\mathcal{E}_\lambda(u,v)\|^2 + \frac{1}{2}\|\mathcal{F}(u,v)\|^2 \leqslant C(\|v\|^2 + \|\nabla G(u)\|^2). \tag{8.3.3.6}$$

Combining (8.3.3.6) with (8.3.3.5), we deduce that the angle condition (8.3.3.3) is satisfied with $\alpha = \frac{\alpha_0}{C}$. Finally, the rest point equivalence (8.3.3.4) follows from (8.3.3.5). □

Remark 8.3.3.2. Note that for $\lambda = 0$, we recover the *total energy* $E_T(u,v) = \mathcal{E}_0(u,v) = \frac{1}{2}\|v\|^2 + G(u)$.

The following result is of primary importance : roughly speaking it shows that functions which desingularize the potential G at some critical point $\overline{u}$, also desingularize the energy function E_T and more generally the family of deformed functions $\mathcal{E}_\lambda$ at the corresponding critical point $(\overline{u}, 0)$. This result implies in turn that the decay rate of the energy is essentially conditioned by the geometry of G as one might expect from a mechanical or an intuitive perspective.

In the proposition below one needs the kinetic energy to be desingularized by φ. This explains our main assumption.

Proposition 8.3.3.3 (**Desingularizing functions of the energy**). *Let $G \in C^2(\mathbb{R}^N;\mathbb{R})$, $\overline{u} \in \operatorname{crit} G$ and assume that there exists a desingularizing function $\varphi \in C^1\big((0,r_0);\mathbb{R}_+\big)$ of G at $\overline{u}$ on $B(\overline{u},\eta)$ such that $\varphi'(s) \geqslant \frac{c}{\sqrt{s}}$, for any $s \in (0,r_0)$.*
Then there exist $\lambda_1 > 0$, $\eta_1 > 0$ and $c > 0$ such that

$$\left\|\nabla\left(\varphi \circ \frac{1}{2}\left|\mathcal{E}_\lambda(\,.\,,\,.\,) - \mathcal{E}_\lambda(\overline{u},0)\right|\right)(u,v)\right\| \geqslant c, \tag{8.3.3.7}$$

for any $\lambda \in [0,\lambda_1]$ and any $(u,v) \in B(\overline{u},\eta_1) \times B(0,\eta_1)$ such that $\mathcal{E}_\lambda(u,v) \neq \mathcal{E}_\lambda(\overline{u},0)$.

Proof. By standard translation arguments, we may assume without loss of generality that $G(\overline{u}) = 0$ and $\overline{u} = 0$. Then $\mathcal{E}_\lambda(0,0) = 0$ and (8.3.3.7) consists in showing that for some constant $c > 0$,

$$\varphi'\left(\frac{1}{2}\left|\mathcal{E}_\lambda(u,v)\right|\right) \geqslant \frac{c}{\|\nabla\mathcal{E}_\lambda(u,v)\|},$$

for any $\lambda \in [0,\lambda_1]$ and any $(u,v) \in B(0,\eta_1) \times B(0,\eta_1)$ such that $\mathcal{E}_\lambda(u,v) \neq 0$. Recall that $0 \in \operatorname{crit} G$. Let $M = \max\Big\{\|\nabla^2 G(u)\|;\ u \in \overline{B}(0,\eta)\Big\}$ and define $\lambda_1 = \min\Big\{\frac{1}{4}\,,\,\frac{1}{2(M^2+1)}\Big\}$. We have,

$$\begin{aligned}
\|\nabla\mathcal{E}_\lambda(u,v)\|^2 &= \|\nabla G(u) + \lambda\nabla^2 G(u)v\|^2 + \|v + \lambda\nabla G(u)\|^2 \\
&\geqslant \|\nabla G(u)\|^2 + \|v\|^2 - \lambda_1(M^2+1)\|v\|^2 - 2\lambda_1\|\nabla G(u)\|^2 \\
&\geqslant \frac{1}{2}\big(\|v\|^2 + \|\nabla G(u)\|^2\big),
\end{aligned} \tag{8.3.3.8}$$

and in particular,

$$\|\nabla G(u)\| \leqslant 2\|\nabla\mathcal{E}_\lambda(u,v)\|, \tag{8.3.3.9}$$

for any $\lambda \in [0,\lambda_1]$ and any $(u,v) \in \mathbb{R}^N \times \mathbb{R}^N$. Let now $(\lambda,u,v) \in [0,\lambda_1] \times B(0,\eta) \times \mathbb{R}^N$ be such that $\mathcal{E}_\lambda(u,v) \neq 0$. Since φ' is nonincreasing, we have

$$\begin{aligned}
\varphi'\left(\frac{1}{2}\left|\mathcal{E}_\lambda(u,v)\right|\right) &\geqslant \varphi'\left(\frac{1}{2}\left|\mathcal{E}_\lambda(u,v) - \mathcal{E}_\lambda(u,0)\right| + \frac{1}{2}\left|\mathcal{E}_\lambda(u,0)\right|\right) \\
&\geqslant \varphi'\left(\max\left\{\left|\mathcal{E}_\lambda(u,v) - \mathcal{E}_\lambda(u,0)\right|, \left|\mathcal{E}_\lambda(u,0)\right|\right\}\right).
\end{aligned} \tag{8.3.3.10}$$

Let us first find a lower bound on $\varphi'(|\mathcal{E}_\lambda(u,0)|)$. Observe that necessarily $\mathcal{E}_\lambda(u,0) = G(u) \neq 0$. In particular, $\nabla G(u) \neq 0$ (Remark 8.2.5). We then have by (8.2.2) and (8.3.3.9), $\nabla\mathcal{E}_\lambda(u,v) \neq 0$ and

$$\varphi'\big(|\mathcal{E}_\lambda(u,0)|\big) = \varphi'\big(|G(u)|\big) \geqslant \frac{1}{\|\nabla G(u)\|} \geqslant \frac{1}{2\|\nabla\mathcal{E}_\lambda(u,v)\|}, \tag{8.3.3.11}$$

for any $\lambda \in [0, \lambda_1]$ and any $(u, v) \in B(0, \eta) \times \mathbb{R}^N$ such that $\mathcal{E}_\lambda(u, 0) \neq 0$.

Let us now estimate $\varphi'(|\mathcal{E}_\lambda(u, v) - \mathcal{E}_\lambda(u, 0)|)$ in (8.3.3.10) under the assumption $\mathcal{E}_\lambda(u, v) \neq \mathcal{E}_\lambda(u, 0)$. Cauchy-Schwarz' inequality implies that for any $\lambda \in [0, \lambda_1]$,

$$|\mathcal{E}_\lambda(u, v) - \mathcal{E}_\lambda(u, 0)| \leqslant \frac{1}{2}\big(\|v\|^2 + \lambda_1\|v\|^2 + \lambda_1\|\nabla G(u)\|^2\big). \tag{8.3.3.12}$$

Combining (8.3.3.12) with (8.3.3.8), we deduce that for any $\lambda \in [0, \lambda_1]$ and any $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$|\mathcal{E}_\lambda(u, v) - \mathcal{E}_\lambda(u, 0)| \leqslant (1 + \lambda_1)\|\nabla \mathcal{E}_\lambda(u, v)\|^2. \tag{8.3.3.13}$$

By continuity of ∇G, there exists $\eta_1 \in (0, \eta)$ such that

$$\sup\Big\{(1 + \lambda_1)\|\nabla \mathcal{E}_\lambda(u, v)\|^2;\ (\lambda, u, v) \in [0, \lambda_1] \times B(0, \eta_1) \times B(0, \eta_1)\Big\} < r_0.$$

Using successively the fact that φ' is nonincreasing and $\varphi'(s) \geqslant \frac{c}{\sqrt{s}}$, it follows from (8.3.3.13) that if $(u, v) \in B(0, \eta_1) \times B(0, \eta_1)$ with $\mathcal{E}_\lambda(u, v) \neq \mathcal{E}_\lambda(u, 0)$ then $\nabla \mathcal{E}_\lambda(u, v) \neq 0$ and

$$\varphi'\big(|\mathcal{E}_\lambda(u, v) - \mathcal{E}_\lambda(u, 0)|\big) \geqslant \varphi'\big((1 + \lambda_1)\|\nabla \mathcal{E}_\lambda(u, v)\|^2\big) \geqslant \frac{c_1}{\|\nabla \mathcal{E}_\lambda(u, v)\|}, \tag{8.3.3.14}$$

where $c_1 > 0$ is a constant. Finally, inequalities (8.3.3.11) and (8.3.3.14) together with (8.3.3.10) yield the existence of a constant $c > 0$ such that for any $\lambda \in [0, \lambda_1]$ and any $(u, v) \in B(0, \eta_1) \times B(0, \eta_1)$ such that $\mathcal{E}_\lambda(u, v) \neq 0$, there holds $\nabla \mathcal{E}_\lambda(u, v) \neq 0$ and $\varphi'\left(\frac{1}{2}|\mathcal{E}_\lambda(u, v)|\right)\|\nabla \mathcal{E}_\lambda(u, v)\| \geqslant c$, which is the desired result. □

8.4 Convergence results

Before providing our last results, we would like to recall to the reader that a bounded trajectory of (8.1.1.1) may not converge to a single critical point; finite-dimensional counterexamples for $N = 2$ are provided in [14, 114], in each case the trajectory of (8.1.1.1) ends up circling indefinitely around a disk.

We now proceed to establish a central result whose specialization to various settings will provide us with several extensions of Haraux-Jendoubi's initial work [97].

Theorem 8.4.1. *Let $G \in C^2(\mathbb{R}^N; \mathbb{R})$ and $(u_0, u_0') \in \mathbb{R}^N \times \mathbb{R}^N$ be a set of initial conditions for (8.1.1.1). Denote by $u \in C^2\big([0, \infty); \mathbb{R}^N\big)$ the unique regular solution to (8.1.1.1) with initial data (u_0, u_0'). Assume that the following holds.*

1. **(*The trajectory is bounded*)** $\sup_{t>0}\|u(t)\| < \infty$.
2. **(*Convergence to a critical point*)** *G is a KL function. Each desingularizing function φ of G satisfies*

$$\varphi'(s) \geqslant \frac{\beta}{\sqrt{s}}, \tag{8.4.1}$$

for any $s \in (0, \eta_0)$, where β and η_0 are positive constants (see Definition 8.2.1).

Then,

(i) *u' and u'' belong to $L^1\big((0,\infty);\mathbb{R}^N\big)$ and in particular u converges to a single limit u_∞ in* $\operatorname{crit} G$.

(ii) *When u converges to u_∞, we denote by φ the desingularizing function of G at u_∞. One has the following estimate*

$$\|u(t) - u_\infty\| \leqslant c\nu(t),$$

where ν is the solution of the worst-case gradient system

$$\nu'(t) + (\varphi^{-1})'(\nu(t)) = 0,\ \nu(0) > 0.$$

Proof of Theorem 8.4.1. Let $G \in C^2(\mathbb{R}^N;\mathbb{R})$, let $(u_0, u_0') \in \mathbb{R}^N \times \mathbb{R}^N$, let $u \in C^2\big([0,\infty);\mathbb{R}^N\big)$ and let $\overline{u} \in \mathbb{R}^N$. Set $U(t) = \big(u(t), u'(t)\big)$, $U_0 = (u_0, u_0')$ and $\overline{U} = (\overline{u}, 0)$. Let $\mathcal{F}$ and let $\mathcal{E}_\lambda$ be defined as in Subsection 8.3.1 and Proposition 8.3.3.1, respectively. Note that if $\overline{u} \notin \operatorname{crit} G$ then $\overline{U} \notin \operatorname{crit} \mathcal{E}_\lambda$ and $\varphi(t) = ct$ desingularizes $\mathcal{E}_\lambda$ at $\overline{U}$, for any $\lambda \geqslant 0$ $\big($Remark 8.2.3 (a) and (8.3.3.4)$\big)$. Otherwise, $\overline{u} \in \operatorname{crit} G$ and we shall apply Proposition 8.3.3.3. Since $\sup_{t>0} \|u(t)\| < \infty$, $u''(t) + \gamma u'(t) = A(t)$ where A is bounded. Thus, $u'(t) = u'(0)e^{-\gamma t} + \int_0^t \exp(-\gamma(t-s))A(s)\mathrm{d}s$, and by a straightforward calculation, $\sup_{t>0} \|u'(t)\| < \infty$. It follows that $\sup_{t>0} \|U(t)\| < \infty$. Let $R = \sup_{t>0} \|U(t)\|$. Let $\lambda_0 > 0$ and $0 < \lambda_1 < \lambda_0$ be given by Propositions 8.3.3.1 and 8.3.3.3, respectively. Let us fix $0 < \lambda_\star < \lambda_1$ and let $\alpha > 0$ be given by Proposition 8.3.3.1 for such $\mathcal{E}_{\lambda_\star}$ and R. By Proposition 8.3.3.1, the first order system

$$U'(t) + \mathcal{F}\big(U(t)\big) = 0, \quad t \in \mathbb{R}_+, \tag{8.4.2}$$

has a quasi-gradient structure for $\mathcal{E}_{\lambda_\star}$ on $\overline{B}(0,R)$ (Definition 8.3.1.1). Finally, since G has the KL property at $\overline{u}$, $\mathcal{E}_{\lambda_\star}$ also has the KL property at $\overline{U}$ (Proposition 8.3.3.3). It follows that Theorem 8.3.1.2 applies to U, from which (i) follows.

The estimate part of the proof of (ii) will follow from Theorem 8.3.2.4, if we establish that for any $R > 0$, there exists $b > 0$ such that for any $(u,v) \in \overline{B}(0,R) \times \overline{B}(0,R)$,

$$\|\nabla \mathcal{E}_{\lambda_\star}(u,v)\| \leqslant b\|\mathcal{F}(u,v)\|.$$

First we observe that for each $R > 0$ and for any $(u,v) \in \overline{B}(0,R) \times \overline{B}(0,R)$, there exists $k_1 \geqslant 0$ such that

$$\|\nabla \mathcal{E}_{\lambda_\star}(u,v)\|^2 \leqslant k_1\big(\|\nabla G(u)\|^2 + \|v\|^2\big). \tag{8.4.3}$$

This follows trivially by Cauchy-Schwarz' inequality and the fact that $\nabla^2 G$ is continuous hence bounded on bounded sets. Fix $\sigma > 0$ and recall the inequality $2ab \leqslant \sigma^2 a^2 + \frac{b^2}{\sigma^2}$ for all real numbers a, b. By Cauchy-Schwarz' inequality and the previous inequality

$$\begin{aligned}
\|\mathcal{F}(u,v)\|^2 &= \|v\|^2 + \|\gamma v + \nabla G(u)\|^2 \\
&\geqslant (1+\gamma^2)\|v\|^2 + \|\nabla G(u)\|^2 - 2\|\gamma v\|\|\nabla G(u)\| \\
&\geqslant (1+\gamma^2)\|v\|^2 + \|\nabla G(u)\|^2 - \sigma^2\|\gamma v\|^2 - \frac{1}{\sigma^2}\|\nabla G(u)\|^2 \\
&= (1-(\sigma^2-1)\gamma^2)\|v\|^2 + \left(1 - \frac{1}{\sigma^2}\right)\|\nabla G(u)\|^2.
\end{aligned}$$

Choosing $\sigma > 1$ so that $1-(\sigma^2-1)\gamma^2 > 0$ yields $k_2 > 0$ such that $\|\mathcal{F}(u,v)\|^2 \geqslant k_2\big(\|\nabla G(u)\|^2 + \|v\|^2\big)$, for any u, v in $\mathbb{R}^N$. Combining this last inequality with (8.4.3), we obtain $\|\nabla\mathcal{E}_{\lambda_\star}(u,v)\|^2 \leqslant \frac{k_1}{k_2}\|\mathcal{F}(u,v)\|^2$, for any $(u,v)\in \overline{B}(0,R)\times\overline{B}(0,R)$. Hence the result. □

Remark 8.4.2. (a) As announced previously convergence rates depend directly on the geometry of G through φ.
(b) The fact that the length of the velocity curve u' is finite suggests that highly oscillatory phenomena are unlikely.

8.5 Consequences

In the following corollaries, the mapping $\mathbb{R}_+ \ni t \longmapsto u(t)$ is a solution curve of (8.1.1.1).

Corollary 8.5.1 (**Convergence theorem for real-analytic functions** [97]). *Assume that $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ is real-analytic and let u be a bounded solution to (8.1.1.1). Then we have the following result.*

- (i) *(u, u') has a finite length. In particular u converges to a critical point u_∞.*
- (ii) *When u converges to u_∞, we denote by $\varphi(s) = cs^\theta$ $\big($with $c > 0$ and $\theta \in \big(0, \frac{1}{2}\big]\big)$ the desingularizing function of G at u_∞ – the quantity θ is the Łojasiewicz exponent associated with u_∞. One has the following estimates.*
  - (a) *$\|u(t)-u_\infty\| \leqslant ct^{-\frac{\theta}{1-2\theta}}$, with $c > 0$, when $\theta \in \big(0, \frac{1}{2}\big)$.*
  - (b) *$\|u(t)-u_\infty\| \leqslant c''\exp(-c't)$, with $c', c'' > 0$, when $\theta = \frac{1}{2}$.*

Proof. The proof follows directly from the original Łojasiewicz inequality [134, 133] and the fact that desingularizing functions for real-analytic functions are indeed of the form $\varphi(s) = cs^\theta$ with $\theta \in (0, \frac{1}{2}]$. Hence (8.2.1.8) holds and Theorem 8.4.1 applies, see also Remark 8.3.2.1 (c). □

Corollary 8.5.2 (**Convergence theorem for definable functions**). *Let $\mathcal{O}$ be an o-minimal structure that contains the collection of semi-algebraic sets. Assume $G : \mathbb{R}^N \longrightarrow \mathbb{R}$ is C^2 and definable in $\mathcal{O}$. Let u be a bounded solution to (8.1.1.1). Then we have the following result.*

- (i) *u' and u'' belong to $L^1\big((0,\infty);\mathbb{R}^N\big)$ and in particular u converges to a single limit u_∞ in $\operatorname{crit} G$.*
- (ii) *When u converges to u_∞ we denote by φ the desingularizing function of G at u_∞. One has the following estimate*

$$\|u(t)-u_\infty\| \leqslant c\nu(t),$$

where ν is a solution of the worst-case gradient system

$$\nu'(t) + (\varphi^{-1})'(\nu(t)) = 0,\ \nu(0) > 0.$$

Proof. G is a KL function by Kurdyka's version of the Łojasiewicz inequality. The fact that $\varphi'(s) \geqslant \frac{c}{\sqrt{s}}$ comes from Lemma 8.2.2.1. So, Theorem 8.4.1 applies. □

Corollary 8.5.3 (**Convergence theorem for the one-dimensional case** [96]). *Let $G \in C^2(\mathbb{R};\mathbb{R})$ and let u be a bounded solution to (8.1.1.1). Then u converges to a single point and we have the same type of rate of convergence as in the previous corollary.*

Proof. We proceed as in [183]. Argue by contradiction and assume that $\omega(u_0, u_0')$, the ω-limit set of (u_0, u_0'), is not a singleton. Since $\omega(u_0, u_0')$ is connected in $\mathbb{R}$, it is an interval and has a nonempty interior. Take $\overline{u}$ in the interior of $\omega(u_0, u_0')$ The Łojasiewicz inequality trivially holds at $\overline{u}$ for $G \equiv 0$ with $\varphi(s) = \sqrt{s}$ (recall $\overline{u}$ is interior). Apply then Theorem 8.4.1. □

Remark 8.5.4. In the one-dimensional case, convergence can be obtained with much more general forms of damping, see [51].

Corollary 8.5.5 (Convergence theorem for convex functions satisfying growth conditions).

Let $G \in C^2(\mathbb{R}^N; \mathbb{R})$ be a convex function such that

$$\operatorname{argmin} G \stackrel{\text{def}}{=} \big\{u \in \mathbb{R}^N; G(u) = \min G\big\},$$

is nonempty (note that $\operatorname{argmin} G = \operatorname{crit} G$*). Assume further that, for each minimizer x^*, there exists $\eta > 0$, such that G satisfies*

$$\forall u \in B(x^*, \eta),\ G(u) \geqslant \min G + c \operatorname{dist}(u, \operatorname{argmin} G)^r, \tag{8.5.1}$$

with $r \geqslant 1$ and $c > 0$. Then the solution curve $t \longmapsto (u(t), u'(t))$ has a finite length. In particular u converges to a minimizer u_∞ of G as t goes to ∞.

Proof. A general result of Alvarez [4] ensures that u is bounded (and even converges). On the other hand it has been shown in [40] that functions satisfying the growth assumption (8.5.1), also satisfy the Lojasiewicz inequality with desingularizing functions of the form $s \longmapsto c' s^{1-1/r}$ with $c' > 0$. Combining the previous arguments, the conclusion follows readily. □

Remark 8.5.6. An alternative and more general approach to establish that trajectories have a finite length has been developed for convex functions in [135, 71].

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Chapitre 9

Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation

with Ana Vargas*

Abstract

In this paper, we show that any solution of the nonlinear Schrödinger equation $iu_t + \Delta u \pm |u|^{\frac{4}{N}} u = 0$, which blows up in finite time, satisfies a mass concentration phenomena near the blow-up time. Our proof is essentially based on the Bourgain's one [42], which has established this result in the bidimensional spatial case, and on a generalization of Strichartz's inequality, where the bidimensional spatial case was proved by Moyua, Vargas and Vega [139]. We also generalize to higher dimensions the results in Keraani [119] and Merle and Vega [137].

9.1 Introduction and main results

Let $\gamma \in \mathbb{R} \setminus \{0\}$ and let $0 \leqslant \alpha \leqslant \frac{4}{N}$. It is well-known that for any $u_0 \in L^2(\mathbb{R}^N)$, there exists a unique maximal solution

$$u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^N)) \cap L^{\frac{4(\alpha+2)}{N\alpha}}_{\text{loc}}((-T_{\min}, T_{\max}); L^{\alpha+2}(\mathbb{R}^N)),$$

of

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} + \Delta u + \gamma |u|^\alpha u = 0, \; (t,x) \in (-T_{\min}, T_{\max}) \times \mathbb{R}^N, \\ u(0) = u_0, \text{ in } \mathbb{R}^N, \end{cases} \tag{9.1.1}$$

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satisfying the conservation of charge, that is for any $t \in (-T_{\min}, T_{\max})$, $\|u(t)\|_{L^2(\mathbb{R}^N)} = \|u_0\|_{L^2(\mathbb{R}^N)}$. The solution u also satisfies the following Duhamel's formula

$$\forall t \in (-T_{\min}, T_{\max}),\ u(t) = \mathcal{T}(t)u_0 + \mathrm{i}\gamma \int_0^t (\mathcal{T}(t-s)\{|u|^\alpha u\})(s)\mathrm{d}s, \tag{9.1.2}$$

where we design by $(\mathcal{T}(t))_{t\in\mathbb{R}}$ the group of isometries $(e^{it\Delta})_{t\in\mathbb{R}}$ generated by $i\Delta$ on $L^2(\mathbb{R}^N;\mathbb{C})$. Moreover u is maximal in the following sense. If $\alpha < \frac{4}{N}$ then $T_{\max} = T_{\min} = \infty$, if $\alpha = \frac{4}{N}$ and if $T_{\max} < \infty$ then

$$\|u\|_{L^{\frac{2(N+2)}{N}}((0,T_{\max});L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))} = \infty,$$

and if $\alpha = \frac{4}{N}$ and $T_{\min} < \infty$ then $\|u\|_{L^{\frac{2(N+2)}{N}}((-T_{\min},0);L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))} = \infty$ (see Cazenave and Weissler [60] and Tsutsumi [172], also Cazenave [57], Corollary 4.6.5 and Section 4.7). Now, assume that $\alpha = \frac{4}{N}$. It is well-known that if $\|u_0\|_{L^2}$ is small enough then $T_{\max} = T_{\min} = \infty$, whereas if $\gamma > 0$ then there exists some $u_0 \in L^2(\mathbb{R}^N)$ such that $T_{\max} < \infty$ and $T_{\min} < \infty$. For example, it is sufficient to choose $u_0 = \lambda\varphi$, where $\varphi \in H^1(\mathbb{R}^N) \cap L^2(|x|^2;dx)$, $\varphi \not\equiv 0$, and where $\lambda > 0$ is large enough (Glassey [92], Vlasov, Petrischev and Talanov [180], Cazenave and Weissler [60]).

In the case $\gamma > 0$, when blow-up in finite time occurs, a mass concentration phenomena was observed near the blow-up time (see Theorem 2 in Merle and Tsutsumi [136] and Theorem 6.6.7 in Cazenave [57]), under the conditions that $u_0 \in H^1(\mathbb{R}^N)$ is spherically symmetric, $N \geqslant 2$ and $\gamma > 0$. Theorem 6.6.7 in Cazenave [57] asserts that if $T_{\max} < \infty$ for a solution u of equation (9.1.4) below, then for any $\varepsilon \in \left(0, \frac{1}{2}\right)$,

$$\liminf_{t \nearrow T_{\max}} \int_{B(0,(T_{\max}-t)^{\frac{1}{2}-\varepsilon})} |u(t,x)|^2 dx \geqslant \|Q\|_{L^2(\mathbb{R}^N)}^2, \tag{9.1.3}$$

where Q is the ground state, *i.e.* the unique positive solution of $-\Delta Q + Q = |Q|^{\frac{4}{N}}Q$ (see Merle and Tsutsumi [136], Tsutsumi [172]). The proof uses the conservation of energy and the compactness property of radially symmetric functions lying in $H^1(\mathbb{R}^N)$. The spherical symmetry assumption was relaxed by Nawa [140]; see also Hmidi and Keraani [105]. Later, it was proved that for data in H^s, for some $s < 1$, (9.1.3) holds. This was proved by Colliander, Raynor, Sulem and Wright [68] for dimension 2, and extended by Tzirakis [176] to dimension 1 and by Visan and Zhang [178] to general dimension.

In Bourgain [42], a mass concentration phenomena, estimate (9.1.5) below, is obtained for any $u_0 \in L^2(\mathbb{R}^2)$, $\gamma \neq 0$, but in spatial dimension $N = 2$. Consider solutions of the following critical nonlinear Schrödinger equation,

$$\begin{cases} \mathrm{i}\dfrac{\partial u}{\partial t} + \Delta u + \gamma |u|^{\frac{4}{N}} u = 0,\ (t,x) \in (-T_{\min}, T_{\max}) \times \mathbb{R}^N, \\ u(0) = u_0, \text{ in } \mathbb{R}^N, \end{cases} \tag{9.1.4}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$ is a given parameter. Bourgain showed, in the case $N = 2$ (see Theorem 1 in [42]), that if $u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^2))$ is a solution of (9.1.4) with initial data $u_0 \in L^2(\mathbb{R}^2)$ which blows-up in finite time $T_{\max} < \infty$, then

$$\limsup_{t\nearrow T_{\max}} \sup_{c\in\mathbb{R}^N} \int_{B(c,C(T_{\max}-t)^{\frac{1}{2}})} |u(t,x)|^2 \mathrm{d}x \geqslant \varepsilon, \tag{9.1.5}$$

where the constants C and ε depend continuously and only on $\|u_0\|_{L^2}$ and $|\gamma|$. The proof is based on a refinement of Strichartz's inequality for $N = 2$, due to Moyua, Vargas and Vega (see Theorem 4.2 and Lemma 4.4 in [139]).

Very recently, Keraani [119] showed for $N \in \{1,2\}$ that there is some $\delta_0 > 0$, such that, under the same assumptions, if in addition $\|u_0\|_{L^2} < \sqrt{2}\delta_0$ then for any $\lambda(t) > 0$ such that $\lambda(t) \xrightarrow{t\nearrow T_{\max}} \infty$,

$$\liminf_{t\nearrow T_{\max}} \sup_{c\in\mathbb{R}^N} \int_{B(c,\lambda(t)(T_{\max}-t)^{\frac{1}{2}})} |u(t,x)|^2 \mathrm{d}x \geqslant \delta_0^2. \tag{9.1.6}$$

Keraani's proof uses a linear profile decomposition that was shown in dimension $N = 2$ by Merle and Vega [137] and in dimension $N = 1$ by Carles and Keraani [54] (see Theorem 9.5.4 below for the precise statement). The proofs of the decompositions are based on the above mentioned refinement of Strichartz's inequality by Moyua, Vargas and Vega and another one for the case $N = 1$ observed by Carles and Keraani [54]. In this paper, we generalize the refinement of Strichartz's inequality (see Theorem 9.1.4 below) in order to establish the higher dimensional versions of all these results. Our proofs (namely, those of Theorem 9.1.2 and Lemma 9.3.3) rely on the restriction theorems for paraboloids proved by Tao [166]. There is another minor technical point, because the Strichartz's exponent $\frac{2N+4}{N}$, is not a natural number when the dimension $N \geqslant 3$, except $N = 4$. We have to deal with this little inconvenience which did not appeared in $N \in \{1,2\}$.

This paper is organized as follows. At the end of this section, we state the main results (Theorems 9.1.1 and 9.1.4) and give some notations which will be used throughout this paper. Section 9.2 is devoted to the proof of the refinement of Strichartz's inequality (Theorems 9.1.2–9.1.4). In Section 9.3, we establish some preliminary results in order to prove a mass concentration result in Section 9.4 (Proposition 9.4.1). We prove Theorem 9.1.1 in Section 9.4. Finally, Section 9.5 is devoted to the generalization to higher dimensions of the results by Keraani [119] and Merle and Vega [137].

Throughout this paper, we use the following notation. For $1 \leqslant p \leqslant \infty$, p' denotes the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$; $L^p(\mathbb{R}^N) = L^p(\mathbb{R}^N;\mathbb{C})$ is the usual Lebesgue space. The Laplacian in $\mathbb{R}^N$ is written $\Delta = \sum\limits_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}$ and $\frac{\partial u}{\partial t} = u_t$ is the time derivative of the complex-valued function u. For $c \in \mathbb{R}^N$ and $R \in (0,\infty)$, we denote by $B(c,R) = \{x \in \mathbb{R}^N;\ |x-c| < R\}$ the open ball of $\mathbb{R}^N$ of center c and radius R. We design by $\mathcal{C}$ the set of half–closed cubes in $\mathbb{R}^N$. So $\tau \in \mathcal{C}$ if and only if there exist $(a_1,\ldots,a_N) \in \mathbb{R}^N$ and $R > 0$ such that $\tau = \prod\limits_{j=1}^{N} [a_j, a_j + R)$. The length of a side of $\tau \in \mathcal{C}$ is written $\ell(\tau) = R$. Given $A \subset \mathbb{R}^N$, we denote by $|A|$ its Lebesgue measure. Let $j,k \in \mathbb{N}$ with $j < k$. Then we denote $[\![j,k]\!] = [j,k] \cap \mathbb{N}$. We denote by $\mathcal{F}$ the Fourier transform in $\mathbb{R}^N$ defined by [1]

1. with this definition of the Fourier transform, $\|\mathcal{F}u\|_{L^2} = \|\mathcal{F}^{-1}u\|_{L^2} = \|u\|_{L^2}$, $\mathcal{F}^{-1}\mathcal{F} = \mathcal{F}\mathcal{F}^{-1} = \mathrm{Id}_{L^2}$, $\mathcal{F}(u*v) = \mathcal{F}u\mathcal{F}v$ and $\mathcal{F}^{-1}(u*v) = \mathcal{F}^{-1}u\mathcal{F}^{-1}v$.

$\widehat{u}(\xi) = \mathcal{F}u(\xi) = \displaystyle\int_{\mathbb{R}^N} e^{-2i\pi x.\xi}u(x)\mathrm{d}x$, and by $\mathcal{F}^{-1}$ its inverse given by $\mathcal{F}^{-1}u(x) = \displaystyle\int_{\mathbb{R}^N} e^{2i\pi\xi.x}u(\xi)\mathrm{d}\xi$. C are auxiliary positive constants and $C(a_1, a_2, \ldots, a_n)$ indicates that the constant C depends only on positive parameters $a_1, a_2, \ldots, a_n$ and that the dependence is continuous.

Finally, we recall the Strichartz's estimates (Stein–Tomas Theorem) (see Stein [162], Strichartz [164] and Tomas [170]). Let $I \subseteq \mathbb{R}$ be an interval, let $t_0 \in \overline{I}$ and let $\gamma \in \mathbb{C}$. Set for any $t \in I$, $\Phi_u(t) = i\gamma\displaystyle\int_{t_0}^{t}(\mathcal{T}(t-s)\{|u|^{\frac{4}{N}}u\})(s)\mathrm{d}s$. Then we have

$$\|\mathcal{T}(\,.\,)u_0\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \leqslant C_0\|u_0\|_{L^2(\mathbb{R}^N)}, \tag{9.1.7}$$

$$\|\Phi_u\|_{L^{\frac{2(N+2)}{N}}(I\times\mathbb{R}^N)} \leqslant C_1\|u\|^{\frac{N+4}{N}}_{L^{\frac{2(N+2)}{N}}(I\times\mathbb{R}^N)}, \tag{9.1.8}$$

where $C_0 = C_0(N) > 0$ and $C_1 = C_1(N, |\gamma|) > 0$. For more details, see Ginibre and Velo [91] (Lemma 3.1) and Cazenave and Weissler [60] (Lemma 3.1), also Cazenave [57] (Theorem 2.3.3). The main results of this paper are the following.

Theorem 9.1.1. *Let $\gamma \in \mathbb{R}\setminus\{0\}$, let $u_0 \in L^2(\mathbb{R}^N)\setminus\{0\}$ and let*

$$u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^N)) \cap L^{\frac{2(N+2)}{N}}_{\mathrm{loc}}((-T_{\min}, T_{\max}); L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))$$

be the maximal solution of (9.1.4) such that $u(0) = u_0$. There exists $\varepsilon = \varepsilon(\|u_0\|_{L^2}, N, |\gamma|) > 0$ satisfying the following property. If $T_{\max} < \infty$ then

$$\limsup_{t\nearrow T_{\max}} \sup_{c\in\mathbb{R}^N} \int_{B(c,(T_{\max}-t)^{\frac{1}{2}})} |u(t,x)|^2\mathrm{d}x \geqslant \varepsilon,$$

and if $T_{\min} < \infty$ then

$$\limsup_{t\searrow -T_{\min}} \sup_{c\in\mathbb{R}^N} \int_{B(c,(T_{\min}+t)^{\frac{1}{2}})} |u(t,x)|^2\mathrm{d}x \geqslant \varepsilon.$$

By keeping track of the constants through the proofs, it can be shown that $\varepsilon = C(N, |\gamma|)\|u_0\|_{L^2}^{-m}$ for some $m > 0$ (this was pointed out by Colliander). Notice that no hypothesis on the attractivity on the nonlinearity (that is on the γ's sign), on the spatial dimension N and on the smoothness on the initial data u_0 are made.

For each $j \in \mathbb{Z}$, we break up $\mathbb{R}^N$ into dyadic cubes $\tau_k^j = \prod\limits_{m=1}^{N}[k_m 2^{-j}, (k_m+1)2^{-j})$, where $k = (k_1, \ldots, k_N) \in \mathbb{Z}^N$ with $\ell(\tau_k^j) = 2^{-j}$. Define $f_k^j(x) = f\mathbb{1}_{\tau_k^j}(x)$. Let $1 \leqslant p < \infty$ and let $1 \leqslant q < \infty$. We define the space

$$X_{p,q} = \Big\{f \in L^p_{\mathrm{loc}}(\mathbb{R}^N);\ \|f\|_{X_{p,q}} < \infty\Big\},$$

where

$$\|f\|_{X_{p,q}} = \left[\sum_{j\in\mathbb{Z}} 2^{j\frac{N}{2}\frac{2-p}{p}q} \sum_{k\in\mathbb{Z}^N} \|f_k^j\|^q_{L^p(\mathbb{R}^N)}\right]^{\frac{1}{q}}.$$

Then $(X_{p,q}, \| \,.\, \|_{X_{p,q}})$ is a Banach space and the set of functions $f \in L^\infty(\mathbb{R}^N)$ with compact support is dense in $X_{p,q}$ for the norm $\| \,.\, \|_{X_{p,q}}$.

We prove the following improvement of Strichartz's (Stein–Tomas's) inequality.

Theorem 9.1.2. *Let $q = \frac{2(N+2)}{N}$ and $1 < p < 2$ be such that $\frac{1}{p'} > \frac{N+3}{N+1}\frac{1}{q}$. For every function g such that $g \in X_{p,q}$ or $\widehat{g} \in X_{p,q}$, we have*

$$\|\mathcal{T}(\,.\,)g\|_{L^q(\mathbb{R}^{N+1})} \leqslant C \min\left\{\|g\|_{X_{p,q}}, \|\widehat{g}\|_{X_{p,q}}\right\}, \tag{9.1.9}$$

where $C = C(N,p)$.

Theorem 9.1.3. *Let $q > 2$ and let $1 < p < 2$. Then there exists $\mu \in \left(0, \frac{1}{p}\right)$ such that for every function $f \in L^2(\mathbb{R}^N)$, we have*

$$\|f\|_{X_{p,q}} \leqslant C \left[\sup_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^N} 2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |f(x)|^p dx\right]^\mu \|f\|_{L^2(\mathbb{R}^N)}^{1-\mu p} \leqslant C\|f\|_{L^2(\mathbb{R}^N)}, \tag{9.1.10}$$

where $C = C(p,q)$ and $\mu = \mu(p,q)$. In particular, $L^2(\mathbb{R}^N) \hookrightarrow X_{p,q}$. Moreover, $L^2(\mathbb{R}^N) \neq X_{p,q}$.

As a corollary we obtain the following improvement of Strichartz's (Stein–Tomas's) inequality.

Theorem 9.1.4. *Let $q = \frac{2(N+2)}{N}$ and let $p < 2$ be such that $\frac{1}{p'} > \frac{N+3}{N+1}\frac{1}{q}$. Then, there exists $\mu \in \left(0, \frac{1}{p}\right)$ such that for every function $g \in L^2(\mathbb{R}^N)$, we have*

$$\|\mathcal{T}(\,.\,)g\|_{L^q(\mathbb{R}^{N+1})} \leqslant C \left[\sup_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^N} 2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |\widehat{g}(\xi)|^p \mathrm{d}\xi\right]^\mu \|g\|_{L^2(\mathbb{R}^N)}^{1-\mu p} \leqslant C\|g\|_{L^2(\mathbb{R}^N)}, \tag{9.1.11}$$

where $C = C(N,p)$ and $\mu = \mu(N,p)$.

Remark 9.1.5 (See Bourgain [42], p.262–263)**.** By Hölder's inequality, if $1 < p < 2$ then for any $(j,k) \in \mathbb{Z}\times\mathbb{Z}^N$,

$$\left[2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |\widehat{g}(\xi)|^p \mathrm{d}\xi\right]^{1/p} \leqslant \left[2^{j\frac{N}{2}} \int_{\tau_k^j} |\widehat{g}(\xi)| \mathrm{d}\xi\right]^\theta \|\widehat{g}\|_{L^2(\mathbb{R}^N)}^{1-\theta} \leqslant \|g\|_{B^0_{2,\infty}}^\theta \|\widehat{g}\|_{L^2(\mathbb{R}^N)}^{1-\theta},$$

for some $0 < \theta < 1$. Therefore, it follows from our Strichartz's refinement, Theorem 9.1.4, that the following holds.

$$\forall M > 0,\ \exists \eta > 0 \text{ such that if } \|u_0\|_{L^2} \leqslant M \text{ and } \|u_0\|_{B^0_{2,\infty}} < \eta \text{ then } T_{\max} = T_{\min} = \infty,$$

where u is the corresponding solution of (9.1.4). Furthermore, $u \in L^{\frac{2(N+2)}{N}}(\mathbb{R}; L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))$ and there exists a scattering state in $L^2(\mathbb{R}^N)$. The same result holds if the condition $\|u_0\|_{B^0_{2,\infty}} < \eta$ is replaced by

$$\sup_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^N} 2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |u_0(x)|^p \mathrm{d}x < \eta',$$

for a suitable η'.

Very recently, Rogers and Vargas [154] have proved, for the non–elliptic cubic Schrödinger equation $i\partial_t u + \partial_{x_1}^2 u - \partial_{x_2}^2 u + \gamma|u|^2 u = 0$ in dimension 2, some results analogous to Theorems 9.1.1, 9.1.2, 9.1.3 and 9.1.4.

9.2 Strichartz's refinement

We recall that $\mathcal{T}(t)g = K_t * g$, where $K_t(x) = (4\pi i t)^{-\frac{N}{2}} e^{i\frac{|x|^2}{4t}}$ and that $\widehat{K_t}(\xi) = e^{-i4\pi^2|\xi|^2 t}$. Using that for any $g \in L^2(\mathbb{R}^N)$, $\mathcal{T}(t)g = \mathcal{F}^{-1}(\widehat{K_t}\widehat{g})$ we have,

$$(\mathcal{T}(t)g)(x) = \int\limits_{\mathbb{R}^N} e^{2i\pi\left(x.\xi - 2\pi t|\xi|^2\right)}\widehat{g}(\xi)\mathrm{d}\xi. \tag{9.2.1}$$

Let $S = \left\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^N;\ \tau = -2\pi|\xi|^2\right\}$, let $\mathrm{d}\sigma(|\xi|^2,\xi) = \mathrm{d}\xi$ and let f be defined on S by $f(\tau,\xi) = f(-2\pi|\xi|^2,\xi) = \widehat{g}(\xi)$. Then,

$$\begin{aligned}(\mathcal{T}(t)g)(x) &= \int\limits_{\mathbb{R}^N} f(-2\pi|\xi|^2,\xi)e^{2i\pi(x.\xi-2\pi t|\xi|^2)}\mathrm{d}\xi \\ &= \iint\limits_S f(\tau,\xi)e^{2i\pi(t\tau+x.\xi)}\mathrm{d}\sigma(\tau,\xi) = \mathcal{F}^{-1}(f\mathrm{d}\sigma)(t,x).\end{aligned} \tag{9.2.2}$$

Our main tool will be the following bilinear restriction estimate proved by Tao [166]. We adapt the statements to our notation using the equivalence (9.2.2).

Theorem 9.2.1 (Theorem 1.1 in [166])**.** *Let Q, Q' be cubes of sidelength 1 in $\mathbb{R}^N$ such that*

$$\min\{d(x,y);\ x\in Q,\ y\in Q'\} \sim 1$$

and let $\widehat{f}$, $\widehat{g}$ functions respectively supported in Q and Q'. Then for any $r > \frac{N+3}{N+1}$ and $p \geqslant 2$, we have

$$\|\mathcal{T}(\,.\,)f\mathcal{T}(\,.\,)g\|_{L^r(\mathbb{R}^{N+1})} \leqslant C\|\widehat{f}\|_{L^p(Q)}\|\widehat{g}\|_{L^p(Q')},$$

with a constant C independent of f, g, Q and Q'.

By interpolation with the trivial estimate

$$\|\mathcal{T}(\,.\,)f\mathcal{T}(\,.\,)g\|_{L^\infty(\mathbb{R}^{N+1})} \leqslant C\|\widehat{f}\|_{L^1(Q)}\|\widehat{g}\|_{L^1(Q')} \leqslant C\|\widehat{f}\|_{L^p(Q)}\|\widehat{g}\|_{L^p(Q')},$$

for any $p \geqslant 1$, one obtains the following result.

Theorem 9.2.2 ([166])**.** *Let Q, Q' be cubes of sidelength 1 in $\mathbb{R}^N$ such that*

$$\min\{d(x,y);\ x\in Q,\ y\in Q'\} \sim 1$$

and $\widehat{f}$, $\widehat{g}$ functions respectively supported in Q and Q'. Then for any $r > \frac{N+3}{N+1}$ and for all p such that $\frac{2}{p'} > \frac{N+3}{N+1}\frac{1}{r}$, we have

$$\|\mathcal{T}(\,.\,)f\mathcal{T}(\,.\,)g\|_{L^r(\mathbb{R}^{N+1})} \leqslant C\|\widehat{f}\|_{L^p(\mathbb{R}^N)}\|\widehat{g}\|_{L^p(\mathbb{R}^N)},$$

with a constant C independent of f, g, Q and Q'.

By rescaling and taking $r = \frac{N+2}{N}$, we obtain the following.

Corollary 9.2.3. *Let τ, τ' be cubes of sidelength 2^{-j} such that*

$$\min\{d(x,y);\ x\in\tau,\ y\in\tau'\}\sim 2^{-j}$$

and $\widehat{f}$, $\widehat{g}$ functions respectively supported in τ and τ'. Then for $r=\frac{N+2}{N}$ and for any p such that $\frac{2}{p'}>\frac{N+3}{N+1}\frac{1}{r}$, we have

$$\|\mathcal{T}(\,.\,)f\mathcal{T}(\,.\,)g\|_{L^r(\mathbb{R}^{N+1})}\leqslant C2^{jN\frac{2-p}{p}}\|\widehat{f}\|_{L^p(\mathbb{R}^N)}\|\widehat{g}\|_{L^p(\mathbb{R}^N)},$$

with a constant C independent of f, g, τ and τ'.

We will need to use the orthogonality of functions with disjoint support. More precisely, the following lemma, a proof of which can be found, for instance, in Tao, Vargas, Vega [168], Lemma 6.1.

Lemma 9.2.4. *Let $(R_k)_{k\in\mathbb{Z}}$ be a collection of rectangles in frequency space and $c>0$, such that the dilates $(1+c)R_k$ are almost disjoint (i.e. $\sum_k \mathbb{1}_{(1+c)R_k}\leqslant C$), and suppose that $(f_k)_{k\in\mathbb{Z}}$ is a collection of functions whose Fourier transforms are supported on R_k. Then for all $1\leqslant p\leqslant\infty$, we have*

$$\|\sum_{k\in\mathbb{Z}}f_k\|_{L^p(\mathbb{R}^N)}\leqslant C(N,c)\left(\sum_{k\in\mathbb{Z}}\|f_k\|^{p^*}_{L^p(\mathbb{R}^N)}\right)^{\frac{1}{p^*}},$$

where $p^*=\min(p,p')$.

Proof of Theorem 9.1.2. We set $r=\frac{q}{2}=\frac{N+2}{N}$. We first consider the case where $\widehat{g}\in X_{p,q}$. We can assume that the support of $\widehat{g}$ is contained in the unit square. The general result follows by scaling and density. For each $j\in\mathbb{Z}$, we decompose $\mathbb{R}^N$ into dyadic cubes τ_k^j of sidelength 2^{-j}. Given a dyadic cube τ_k^j we will say that it is the "parent" of the 2^N dyadic cubes of sidelength 2^{-j-1} contained in it. We write $\tau_k^j\sim\tau_{k'}^j$ if τ_k^j, $\tau_{k'}^j$ are not adjacent but have adjacent parents. For each $j\geqslant 0$, write $g=\sum g_k^j$ where $\widehat{g}_k^j(\xi)=\widehat{g}\mathbb{1}_{\tau_k^j}(\xi)$. Denote by Γ the diagonal of $\mathbb{R}^N\times\mathbb{R}^N$, $\Gamma=\{(x,x);\ x\in\mathbb{R}^N\}$. We have the following decomposition (of Whitney type) of $\mathbb{R}^N\times\mathbb{R}^N\setminus\Gamma$ (see Figure 9.1),

$$(\mathbb{R}^N\times\mathbb{R}^N)\setminus\Gamma=\bigcup_j\ \bigcup_{k,k';\,\tau_k^j\sim\tau_{k'}^j}\tau_k^j\times\tau_{k'}^j.$$

Thus,

$$\begin{aligned}\mathcal{T}(t)g(x)\,\mathcal{T}(t)g(x)&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}e^{2i\pi\left(x.\xi-2\pi t|\xi|^2\right)}\widehat{g}(\xi)e^{2i\pi\left(x.\eta-2\pi t|\eta|^2\right)}\widehat{g}(\eta)\mathrm{d}\xi\mathrm{d}\eta\\&=\sum_j\sum_k\sum_{k';\tau_k^j\sim\tau_{k'}^j}\int\int_{\tau_k^j\times\tau_{k'}^j}e^{2i\pi\left(x.\xi-2\pi t|\xi|^2\right)}\widehat{g}(\xi)e^{2i\pi\left(x.\eta-2\pi t|\eta|^2\right)}\widehat{g}(\eta)\mathrm{d}\xi\mathrm{d}\eta\\&=\sum_j\sum_k\sum_{k';\tau_k^j\sim\tau_{k'}^j}\mathcal{T}(t)g_k^j\,\mathcal{T}(t)g_{k'}^j\end{aligned}$$

(see also Tao, Vargas and Vega [168]). Thus,

$$\|\mathcal{T}(\,.\,)g\|^2_{L^{2r}(\mathbb{R}^{N+1})}=\|\mathcal{T}(\,.\,)g\mathcal{T}(\,.\,)g\|_{L^r(\mathbb{R}^{N+1})}=\|\sum_j\sum_{\substack{k,k':\\ \tau_k^j\sim\tau_{k'}^j}}\mathcal{T}(\,.\,)g_k^j\mathcal{T}(\,.\,)g_{k'}^j\|_{L^r(\mathbb{R}^{N+1})}.$$

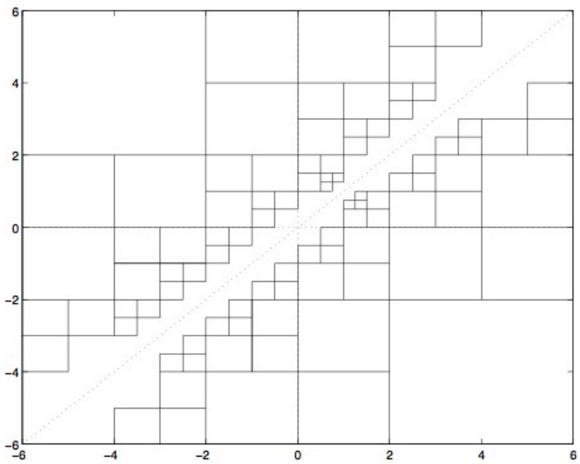

FIGURE 9.1 – $\mathbb{R}^N \times \mathbb{R}^N$

For each $k = (k_1, k_2, \ldots, k_N)$, the support of the $(N+1)$-dimensional Fourier transform of $\mathcal{T}(\,.\,)g_k^j$ is contained in the set $\tilde{\tau}_k^j = \{(-2\pi|\xi|^2, \xi);\ \xi \in \tau_k^j\}$. Hence the support of the Fourier transform of $\mathcal{T}(\,.\,)g_k^j\mathcal{T}(\,.\,)g_{k'}^j$ is contained in $\tilde{\tau}_k^j + \tilde{\tau}_{k'}^j = \{(-2\pi(|\xi|^2 + |\xi'|^2), \xi + \xi');\ \xi \in \tau_k^j,\ \xi' \in \tau_{k'}^j\}$. Using the identity $|\xi|^2 + |\xi'|^2 = \frac{1}{2}|\xi + \xi'|^2 + \frac{1}{2}|\xi - \xi'|^2$ we see that $\tilde{\tau}_k^j + \tilde{\tau}_{k'}^j$ is contained in the set $H_{j,k} = \{(a,b) \in \mathbb{R}^N \times \mathbb{R} :\ |a - 2^{-j+1}k| \leqslant C2^{-j},\ 2^{-2j} \leqslant -|a|^2 - \frac{b}{\pi} \leqslant 3N2^{-2j}\}$. Note that,

$$\sum_j \sum_k \sum_{k';\ \tau_k^j \sim \tau_{k'}^j} \mathbb{1}_{H_{j,k}} \leqslant C(N).$$

Hence, the functions $\mathcal{T}(\,.\,)g_k^j\mathcal{T}(\,.\,)g_{k'}^j$ are almost orthogonal in $L^2(\mathbb{R}^{N+1})$. A similar orthogonality condition was the key in the proof of the L^4–boundedness of the Bochner–Riesz multipliers given by Córdoba [69], see also Tao, Vargas and Vega [168], and implicitly appears in Bourgain [41], Moyua, Vargas and Vega [138, 139]. But we need something more, since we are not working in L^2 and we want to apply Lemma 9.2.4. For $M = 2[\ln(N+1)]$, we decompose each τ_j^k into dyadic subcubes of sidelength 2^{-j-M}. Consequently, we have a corresponding decomposition of $\tau_k^j \times \tau_{k'}^j$ and of $\mathbb{R}^N \times \mathbb{R}^N$, as follows : set $\mathcal{D}$ the family of multi-indices $(m, m', \ell) \in \mathbb{Z}^N \times \mathbb{Z}^N \times \mathbb{Z}$, so that, there exists some $\tau_k^{\ell-M}$ and $\tau_{k'}^{\ell-M}$ with $\tau_m^\ell \subset \tau_k^{\ell-M}$, $\tau_{m'}^\ell \subset \tau_{k'}^{\ell-M}$ and $\tau_k^{\ell-M} \sim \tau_{k'}^{\ell-M}$ $(j = \ell - M)$. Then,

$$(\mathbb{R}^N \times \mathbb{R}^N) \setminus \Gamma = \bigcup_{\mathcal{D}} \tau_m^\ell \times \tau_{m'}^\ell.$$

Hence,

$$\|\mathcal{T}(\,.\,)g\|_{L^{2r}(\mathbb{R}^{N+1})}^2 = \|\mathcal{T}(\,.\,)g\mathcal{T}(\,.\,)g\|_{L^r(\mathbb{R}^{N+1})} = \|\sum_{\mathcal{D}} \mathcal{T}(\,.\,)g_m^\ell \mathcal{T}(\,.\,)g_{m'}^\ell\|_{L^r(\mathbb{R}^{N+1})}.$$

Notice that if $(m, m', \ell) \in \mathcal{D}$, then the distance between τ_m^ℓ and $\tau_{m'}^\ell$ is bigger than $2^{-\ell+M} \geqslant N2^{-\ell}$, and smaller than $\sqrt{N}2^{-\ell+M}$. We **claim** that there are rectangles $R_{m,m',\ell}$, and $c = c(N)$, so that $\tilde{\tau}_m^\ell \times \tilde{\tau}_{m'}^\ell \subset R_{m,m',\ell}$ and $\sum_{\mathcal{D}} \mathbb{1}_{(1+c)R_{m,m',\ell}} \leqslant C(N)$. We postpone the proof of this claim to the end of the proof. Assuming that it holds, and by Lemma 9.2.4, since $r < 2$, we have

$$\|\sum_{\mathcal{D}} \mathcal{T}(\,.\,)g^{\ell}_{m}\mathcal{T}(\,.\,)g^{\ell}_{m'}\|_{L^r(\mathbb{R}^{N+1})} \leqslant C(N)\left[\sum_{\mathcal{D}} \|\mathcal{T}(\,.\,)\,g^{\ell}_{m}\mathcal{T}(\,.\,)g^{\ell}_{m'}\|^{r}_{L^r(\mathbb{R}^{N+1})}\right]^{\frac{1}{r}}.$$

Now use Corollary 9.2.3 to estimate

$$\left[\sum_{\mathcal{D}} \|\mathcal{T}(\,.\,)\,g^{\ell}_{m}\mathcal{T}(\,.\,)g^{\ell}_{m'}\|^{r}_{L^r(\mathbb{R}^{N+1})}\right]^{\frac{1}{r}}$$

$$\leqslant C(N,p)\left[\sum_{\ell}\sum_{m}\sum_{m';(m,m',\ell)\in\mathcal{D}} 2^{\ell N r\frac{2-p}{p}}\|\widehat{g}^{\ell}_{m}\|^{r}_{L^p(\mathbb{R}^N)}\|\widehat{g}^{\ell}_{m'}\|^{r}_{L^p(\mathbb{R}^N)}\right]^{\frac{1}{r}}.$$

Now, for each (m,ℓ) there are at most $4^N 2^{MN}$ indices m' such that $(m,m',\ell)\in\mathcal{D}$. Hence,

$$\left[\sum_{\ell}\sum_{m}\sum_{m';(m,m',\ell)\in\mathcal{D}} 2^{\ell N r\frac{2-p}{p}}\|\widehat{g}^{\ell}_{m}\|^{r}_{L^p(\mathbb{R}^N)}\|\widehat{g}^{\ell}_{m'}\|^{r}_{L^p(\mathbb{R}^N)}\right]^{\frac{1}{r}} \leqslant C(N)\left[\sum_{\ell}\sum_{m} 2^{\ell N r\frac{2-p}{p}}\|\widehat{g}^{\ell}_{m}\|^{2r}_{L^p(\mathbb{R}^N)}\right]^{\frac{1}{r}}.$$

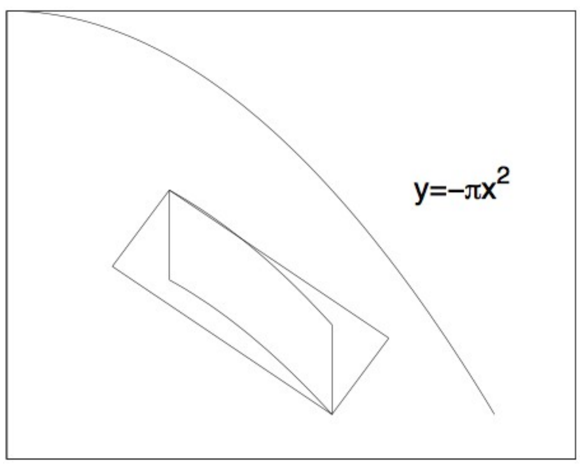

FIGURE 9.2 – $H_{m,m',\ell} \subset R_{m,m',\ell}$

We still have to justify the claim. Assume, for the sake of simplicity that

$$\tau^{\ell}_{m}\times\tau^{\ell}_{m'} \subset \{(x_1,x_2,\dots,x_N)\in\mathbb{R}^N;\ \forall j\in[\![1,N]\!],\ x_j\geqslant 0\}.$$

Then $\tilde{\tau}^{\ell}_{m}\times\tilde{\tau}^{\ell}_{m'}$ is contained on a set $H_{m,m',\ell} = \{(a,b)\in\mathbb{R}^N\times\mathbb{R};\ a = (m+m')2^{-\ell}+v,\ v = (v_1,v_2,\cdots,v_N),\ 0\leqslant v_i\leqslant 2^{-\ell+1},\ 2^{-2\ell+2M}\leqslant -|a|^2-\frac{b}{\pi}\leqslant 3N2^{-2\ell+2M}\}$. Consider the paraboloid defined by $-|a|^2-\frac{b}{\pi} = 2^{-2\ell+2M}$. Take $\Pi_{m,m',\ell}$ to be the tangent hyperplane to this paraboloid at the point of coordinates (a_0,b_0), with $a_0 = (m+m')2^{-\ell}$, $b_0 = -\pi|a_0|^2 - 2^{-2\ell+2M}$ (and passing through that point). Consider also the point (a_1,b_1) with $a_1 = a_0 + (2^{-\ell+1},2^{-\ell+1},\dots,2^{-\ell+1})$ and $b_1 = -\pi|a_1|^2 - 3N2^{-2\ell+2M}$. Then, the rectangle $R_{m,m',\ell}$ is defined as the only rectangle having a face contained in that hyperplane and the points (a_0,b_0), and (a_1,b_1) as opposite vertices. Due to the convexity of paraboloids, it follows that $H_{m,m',\ell} \subset R_{m,m',\ell}$ (see Figure 9.2). Moreover, one can also see that, for small $c = c(N)$, $(1+c)R_{m,m',\ell} \subset \{(a,b);\ a = (m+m')2^{-\ell}+v,\ v = (v_1, v_2, \dots, v_N),\ |v_i| \leqslant C(N)2^{-\ell+1},\ C'(N)2^{-2\ell+2M} \leqslant -|a|^2 - \frac{b}{\pi} \leqslant C''(N)2^{-2\ell+2M}\}$. Therefore, we have $\sum_{\mathcal{D}} \mathbb{1}_{(1+c)R_{m,m',\ell}} \leqslant C(N)$. Hence (9.1.9) in the case $\widehat{g} \in X_{p,q}$. Now, assume $g \in X_{p,q}$. By density, it is sufficient to prove (9.1.9) for $g \in L^2(\mathbb{R}^N)$. By a straightforward calculation and the above result, we obtain that $\|\mathcal{T}(\,.\,)g\|_{L^q(\mathbb{R}^{N+1})} = \|\mathcal{T}(\,.\,)\left(\mathcal{F}^{-1}\overline{g}\right)\|_{L^q(\mathbb{R}^{N+1})} \leqslant C(N,p)\|g\|_{X_{p,q}}$. Hence (9.1.9). □

Proof of Theorem 9.1.3. Notice first, that the second inequality follows from Hölder's. By homogeneity, we can assume that $\|f\|_{L^2(\mathbb{R}^N)} = 1$. Then, it suffices to show that for any function $f \in L^2(\mathbb{R}^N)$ such that $\|f\|_{L^2(\mathbb{R}^N)} = 1$,

$$\sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}q} \left(\int_{\tau_k^j} |f|^p\right)^{\frac{q}{p}} \leqslant C(p,q) \left[\sup_{j,k} \left\{2^{j\frac{N}{2}\frac{2-p}{p}} \left(\int_{\tau_k^j} |f|^p\right)^{\frac{1}{p}}\right\}\right]^{\alpha},$$

where $\alpha = \mu pq$ and where μ has to be determined. Take α and β such that $\frac{2}{q} < \beta < 1$, $\beta > \frac{p}{2}$ and $\alpha + q\beta = q$. Then,

$$\sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}q} \left(\int_{\tau_k^j} |f|^p\right)^{\frac{q}{p}} \leqslant \left\{\sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j} |f|^p\right)^{\beta\frac{q}{p}}\right\} \sup_{j,k} \left[2^{j\frac{N}{2}\frac{2-p}{p}} \left(\int_{\tau_k^j} |f|^p\right)^{\frac{1}{p}}\right]^{\alpha}.$$

We set $\mu = \frac{\alpha}{pq} = \frac{1-\beta}{p} \in \left(0, \frac{1}{p}\right)$. Hence, it is enough to show

$$\sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j} |f|^p\right)^{\beta\frac{q}{p}} \leqslant C(p,q).$$

We split the sum,

$$\begin{aligned}
&\sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j} |f|^p\right)^{\beta\frac{q}{p}} \\
&\leqslant C \sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j \cap \{|f| > 2^{jN/2}\}} |f|^p\right)^{\beta\frac{q}{p}} \\
&+ C \sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j \cap \{|f| \leqslant 2^{jN/2}\}} |f|^p\right)^{\beta\frac{q}{p}} \overset{\text{not}}{=} C(A+B),
\end{aligned}$$

where $C = C(p,q)$. We study the first term. Set for each $j \in \mathbb{Z}$, $f^j = f\mathbb{1}_{\{|f| > 2^{jN/2}\}}$. Then,

$$A = \sum_j \sum_k \left(2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |f^j|^p\right)^{\beta\frac{q}{p}}.$$

Since $\beta q > 2$, we also have $\beta \frac{q}{p} > 1$. Then,

$$A \leqslant \left(\sum_j \sum_k 2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |f^j|^p \right)^{\beta \frac{q}{p}} = \left(\sum_j 2^{j\frac{N}{2}(2-p)} \int_{\mathbb{R}^N} |f^j|^p \right)^{\beta \frac{q}{p}}$$
$$\leqslant \left(\int_{\mathbb{R}^N} |f|^p \sum_{\{j;\, |f| > 2^{jN/2}\}} 2^{j\frac{N}{2}(2-p)} \right)^{\beta \frac{q}{p}}.$$

Since $2 - p > 0$, we can sum the series and obtain

$$A \leqslant C \left(\int_{\mathbb{R}^N} |f|^p |f|^{(2-p)} \right)^{\beta \frac{q}{p}} \leqslant C \left(\int_{\mathbb{R}^N} |f|^2 \right)^{\beta \frac{q}{p}} \leqslant C,$$

by our assumption that $\|f\|_{L^2} = 1$. We now estimate B. Set for any $j \in \mathbb{Z}$, $f_j = f \mathbb{1}_{\{|f| \leqslant 2^{jN/2}\}}$. Then,

$$B = \sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \left(\int_{\tau_k^j} |f_j|^p \right)^{\beta \frac{q}{p}}$$

We use Hölder's inequality with exponents $\frac{\beta q}{p}$ and $\frac{\beta q}{\beta q - p}$. We obtain,

$$B \leqslant \sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \int_{\tau_k^j} |f_j|^{\beta q} \left(|\tau_k^j|^{\frac{\beta q - p}{\beta q}} \right)^{\beta \frac{q}{p}}$$
$$= \sum_j \sum_k 2^{j\frac{N}{2}\frac{2-p}{p}\beta q} \int_{\tau_k^j} |f_j|^{\beta q} \left(2^{-jN\frac{\beta q - p}{\beta q}} \right)^{\beta \frac{q}{p}}$$
$$= \sum_j \sum_k 2^{jN(1-\beta\frac{q}{2})} \int_{\tau_k^j} |f_j|^{\beta q} = \sum_j 2^{jN(1-\beta\frac{q}{2})} \int_{\mathbb{R}^N} |f_j|^{\beta q}$$
$$\leqslant \int_{\mathbb{R}^N} |f|^{\beta q} \sum_{\{j;\, |f| \leqslant 2^{jN/2}\}} 2^{jN(1-\beta\frac{q}{2})}.$$

Since $1 - \beta \frac{q}{2} < 0$, we sum the series to obtain

$$B \leqslant C \int_{\mathbb{R}^N} |f|^{\beta q} |f|^{(2-\beta q)} \leqslant C \int_{\mathbb{R}^N} |f|^2 \leqslant C,$$

since $\|f\|_{L^2} = 1$.
We give an example to show that $L^2(\mathbb{R}^N) \neq X_{p,q}$. Let

$$f(x) = \frac{1}{|x|^{\frac{N}{2}} |\ln |x||^{\frac{1}{2}}} \mathbb{1}_{(0,\frac{1}{2})^N}.$$

Then for any $1 \leqslant p < 2$ and any $q > 2$, $f \in X_{p,q}$ but $f \notin L^2(\mathbb{R}^N)$. □

9.3 Preliminary results

In this and next section, we follow Bourgain's arguments ([42]). We have to modify them in the proof of Lemma 9.3.3, because the Strichartz's exponent is not, in general, a natural number.

Lemma 9.3.1. *Let $f \in L^2(\mathbb{R}^N)\setminus\{0\}$. Then for any $\varepsilon > 0$, such that $\|\mathcal{T}(\cdot)f\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \varepsilon$, there exist $N_0 \in \mathbb{N}$ with $N_0 \leqslant C(\|f\|_{L^2}, N, \varepsilon)$, $(A_n)_{1\leqslant n\leqslant N_0} \subset (0,\infty)$ and $(f_n)_{1\leqslant n\leqslant N_0} \subset L^2(\mathbb{R}^N)$ satisfying the following properties.*

1. $\forall n \in [\![1, N_0]\!]$, $\operatorname{supp}\widehat{f_n} \subset \tau_n$, *where* $\tau_n \in \mathcal{C}$ *with* $\ell(\tau_n) \leqslant C\|f\|^c_{L^2(\mathbb{R}^N)}\varepsilon^{-\nu}A_n$, *and where the constants* C, c *and* ν *are positive and depend only on* N.
2. $\forall n \in [\![1, N_0]\!]$, $|\widehat{f_n}| < A_n^{-\frac{N}{2}}$.
3. $\|\mathcal{T}(\,.\,)f - \sum_{n=1}^{N_0} \mathcal{T}(\,.\,)f_n\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} < \varepsilon$.
4. $\|f\|^2_{L^2(\mathbb{R}^N)} = \sum_{n=1}^{N_0} \|f_n\|^2_{L^2(\mathbb{R}^N)} + \|f - \sum_{n=1}^{N_0} f_n\|^2_{L^2(\mathbb{R}^N)}$.

The proof relies on the following lemma.

Lemma 9.3.2. *Let $g \in L^2(\mathbb{R}^N)$ and let $\varepsilon > 0$ be such that $\|\mathcal{T}(\,.\,)g\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \varepsilon$. Then there exist $h \in L^2(\mathbb{R}^N)$ and $A > 0$ satisfying the following properties.*

1. $\operatorname{supp}\widehat{h} \subset \tau$, *where* $\tau \in \mathcal{C}$ *with* $\ell(\tau) \leqslant C\|g\|^c_{L^2(\mathbb{R}^N)}\varepsilon^{-\nu}A$, *and where the constants* C, c *and* ν *depend only on* N.
2. $|\widehat{h}| \leqslant A^{-\frac{N}{2}}$ *and* $\|h\|^2_{L^2(\mathbb{R}^N)} \geqslant C\|g\|^{-a}_{L^2(\mathbb{R}^N)}\varepsilon^b$, *where the constants* C, a *and* b *depend only on* N.
3. $\|g - h\|^2_{L^2(\mathbb{R}^N)} = \|g\|^2_{L^2(\mathbb{R}^N)} - \|h\|^2_{L^2(\mathbb{R}^N)}$.

Proof. We distinguish 3 cases.

Case 1. $\operatorname{supp}\widehat{g} \subset [-1,1]^N$. Then the function h will also satisfy $\operatorname{supp}\widehat{h} \subset \tau \subset [-1,1]^N$.
Let $\varepsilon > 0$ and let g be as in Lemma 9.3.2 such that $\operatorname{supp}\widehat{g} \subset [-1,1]^N$. It follows from Theorem 9.1.4 that

$$\varepsilon \leqslant \|\mathcal{T}(\,.\,)g\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \leqslant C\|g\|^{1-\mu p}_{L^2(\mathbb{R}^N)} \left[\sup_{(j,k)\in\mathbb{Z}\times\mathbb{Z}^N} 2^{j\frac{N}{2}(2-p)} \int_{\tau_k^j} |\widehat{g}(\xi)|^p \mathrm{d}\xi\right]^{\mu}.$$

So there exist $j \in \mathbb{Z}$ and $\tau \in \mathcal{C}$, with $\tau \subset [-1,1]^N$ and $\ell(\tau) = 2^{-j}$, such that

$$\int_\tau |\widehat{g}(\xi)|^p \mathrm{d}\xi \geqslant C(\|g\|^{\mu p-1}_{L^2(\mathbb{R}^N)}\varepsilon)^{\frac{1}{\mu}} 2^{-j\frac{N}{2}(2-p)}. \tag{9.3.1}$$

Let $M = \left((C\|g\|^{\mu(p-2)-1}_{L^2(\mathbb{R}^N)}\varepsilon)^{\frac{1}{\mu}} 2^{-j\frac{N}{2}(2-p)-1}\right)^{\frac{1}{p-2}}$, where C is the constant in (9.3.1). Then by Plancherel's Theorem,

$$\int_{\tau\cap\{|\widehat{g}|\geqslant M\}} |\widehat{g}(\xi)|^p \mathrm{d}\xi = M^{p-2} \int_{\tau\cap\{|\widehat{g}|\geqslant M\}} |\widehat{g}(\xi)|^p M^{2-p} \mathrm{d}\xi$$

$$\leqslant M^{p-2} \int |\widehat{g}|^p |\widehat{g}|^{2-p} \mathrm{d}\xi = M^{p-2}\|g\|^2_{L^2(\mathbb{R}^N)}. \tag{9.3.2}$$

It follows from (9.3.1)–(9.3.2) that

$$\begin{aligned}\int\limits_{\tau\cap\{|\widehat{g}|<M\}}|\widehat{g}(\xi)|^p\mathrm{d}\xi &= \int\limits_{\tau}|\widehat{g}(\xi)|^p\mathrm{d}\xi - \int\limits_{\tau\cap\{|\widehat{g}|\geqslant M\}}|\widehat{g}(\xi)|^p\mathrm{d}\xi \\ &\geqslant (C\|g\|_{L^2(\mathbb{R}^N)}^{\mu p-1}\varepsilon)^{\frac{1}{\mu}}2^{-j\frac{N}{2}(2-p)} - M^{p-2}\|g\|_{L^2(\mathbb{R}^N)}^2 \\ &\geqslant C\varepsilon^{\frac{1}{\mu}}2^{-j\frac{N}{2}(2-p)}\|g\|_{L^2(\mathbb{R}^N)}^{-\frac{1-\mu p}{\mu}}.\end{aligned}$$

By Hölder's inequality and the above estimate, we get

$$C\varepsilon^{\frac{1}{\mu}}2^{-j\frac{N}{2}(2-p)}\|g\|_{L^2(\mathbb{R}^N)}^{-\frac{1-\mu p}{\mu}} \leqslant \int\limits_{\tau\cap\{|\widehat{g}|<M\}}|\widehat{g}(\xi)|^p\mathrm{d}\xi \leqslant \left(\int\limits_{\tau\cap\{|\widehat{g}|<M\}}|\widehat{g}(\xi)|^2\mathrm{d}\xi\right)^{\frac{p}{2}}|\tau|^{\frac{2-p}{2}}.$$

Since $|\tau| = 2^{-jN}$, we then obtain,

$$\int\limits_{\tau\cap\{|\widehat{g}|<M\}}|\widehat{g}(\xi)|^2\mathrm{d}\xi \geqslant C\|g\|_{L^2(\mathbb{R}^N)}^{-\frac{2(1-\mu p)}{\mu p}}\varepsilon^{\frac{2}{\mu p}}. \tag{9.3.3}$$

Let $h \in L^2(\mathbb{R}^N)$ be such that $\widehat{h} = \widehat{g}\mathbb{1}_{\tau\cap\{|\widehat{g}|<M\}}$ and let $A = M^{-\frac{2}{N}}$. Then $\operatorname{supp}\widehat{h} \subset \tau \subset [-1,1]^N$ with $\ell(\tau) = 2^{-j} = C\|g\|_{L^2(\mathbb{R}^N)}^{\frac{2\mu(2-p)+2}{N\mu(2-p)}}\varepsilon^{-\frac{2}{N\mu(2-p)}}A$. So we have 1, and 2 follows from (9.3.3). Since $\widehat{h}$ and $\widehat{g}-\widehat{h}$ have disjoint supports, 3 follows.

Case 2. $\operatorname{supp}\widehat{g} \subset [-M,M]^N$ for some $M > 0$. Then h will also satisfy $\operatorname{supp}\widehat{h} \subset \tau \subset [-M,M]^N$. Let $\varepsilon > 0$ and let g be as in the Lemma 9.3.2 such that $\operatorname{supp}\widehat{g} \subset [-M,M]^N$ for some $M > 0$. Let $g' \in L^2(\mathbb{R}^N)$ be such that $\widehat{g}'(\xi) = M^{\frac{N}{2}}\widehat{g}(M\xi)$. Then $\operatorname{supp}\widehat{g}' \subset [-1,1]^N$ and so we may apply the Case 1 to g'. Thus there exist $h' \in L^2(\mathbb{R}^N)$, $\tau' \in \mathcal{C}$ and $A' > 0$ satisfying 1–3. We define $h \in L^2(\mathbb{R}^N)$ by $\widehat{h}(\xi) = M^{-\frac{N}{2}}\widehat{h}'\left(\frac{\xi}{M}\right)$. Then $\|g\|_{L^2(\mathbb{R}^N)} = \|g'\|_{L^2(\mathbb{R}^N)}$ and $\|h\|_{L^2(\mathbb{R}^N)} = \|h'\|_{L^2(\mathbb{R}^N)}$. In particular, second part of 2 holds for g and h. Setting $\tau = M\tau'$, it follows that $\operatorname{supp}\widehat{h} \subset \tau \subset [-M,M]^N$ and $\ell(\tau) = M\ell(\tau') \leqslant C\|g\|_{L^2(\mathbb{R}^N)}^q\varepsilon^\nu MA'$. So h satisfies 1 with $A = MA'$. Finally, $|\widehat{h}| < M^{-\frac{N}{2}}A'^{-\frac{N}{2}} = A^{-\frac{N}{2}}$, which implies 2. Finally, 3 follows from the similar identity for $\widehat{g}'$ and $\widehat{h}'$.

Case 3. General case.
Let $\varepsilon > 0$ and let g be as in the Lemma 9.3.2. For $M > 0$, we define $u_M \in L^2(\mathbb{R}^N)$ by $\widehat{u_M} = \widehat{g}\mathbb{1}_{[-M,M]^N}$. It follows from Strichartz's estimate (9.1.7) and Plancherel's Theorem that

$$\|\mathcal{T}(\,.\,)(u_M - g)\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \leqslant C\|u_M - g\|_{L^2(\mathbb{R}^N)} = C\|\widehat{u_M} - \widehat{g}\|_{L^2(\mathbb{R}^N)} \xrightarrow{M\longrightarrow\infty} 0.$$

Then there exists $M_0 > 0$ such that

$$\|\mathcal{T}(\,.\,)u_{M_0}\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \frac{\varepsilon}{2}.$$

Setting $g_0 = u_{M_0}$, we apply the Case 2 to g_0, obtaining h. Since $\|g_0\|_{L^2(\mathbb{R}^N)} \leqslant \|g\|_{L^2(\mathbb{R}^N)}$, Properties 1 and 2 are clear for g and h. Also, Property 3 holds for g and h, again because the disjointness of supports. This achieves the proof of the lemma. □

Proof of Lemma 9.3.1. Let $f \in L^2(\mathbb{R}^N) \setminus \{0\}$ and let $\varepsilon > 0$ be such that

$$\|\mathcal{T}(\,.\,)f\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \varepsilon.$$

We apply Lemma 9.3.2 to f. Let $h \in L^2(\mathbb{R}^N)$, $\tau \in \mathcal{C}$, $A > 0$, $a = a(N) > 0$, $b = b(N) > 0$, $c = c(N) > 0$ and $\nu = \nu(N) > 0$ be given by Lemma 9.3.2. We set $f_1 = h$, $\tau_1 = \tau$ and $A_1 = A$. By Lemma 9.3.2, we have

$$\ell(\tau_1) \leqslant C\|f\|_{L^2}^c \varepsilon^{-\nu} A_1, \tag{9.3.4}$$

$$\|f - f_1\|_{L^2}^2 = \|f\|_{L^2}^2 - \|f_1\|_{L^2}^2, \quad \|f - f_1\|_{L^2}^{-a} \geqslant \|f\|_{L^2}^{-a} \quad \text{and} \quad \|f_1\|_{L^2}^2 \geqslant C\|f\|_{L^2}^{-a}\varepsilon^b. \tag{9.3.5}$$

Now, we may assume that

$$\|\mathcal{T}(\,.\,)f - \mathcal{T}(\,.\,)f_1\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \varepsilon,$$

otherwise we set $N_0 = 1$ and the proof is finished. So we may apply Lemma 9.3.2 to $g = f - f_1$. Let $h \in L^2(\mathbb{R}^N)$, let $\tau \in \mathcal{C}$ and let $A > 0$ be given by Lemma 9.3.2. We set $f_2 = h$, $\tau_2 = \tau$ and $A_2 = A$. By Lemma 9.3.2 and (9.3.5), we have

$$\ell(\tau_2) \leqslant C\|f - f_1\|_{L^2}^c \varepsilon^{-\nu} A_2 \leqslant C\|f\|_{L^2}^c \varepsilon^{-\nu} A_2, \tag{9.3.6}$$

$$\|f - (f_1 + f_2)\|_{L^2}^2 = \|f - f_1\|_{L^2}^2 - \|f_2\|_{L^2}^2 = \|f\|_{L^2}^2 - (\|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2), \tag{9.3.7}$$

$$\|f_2\|_{L^2}^2 \geqslant C\|f - f_1\|_{L^2}^{-a}\varepsilon^b \geqslant C\|f\|_{L^2}^{-a}\varepsilon^b. \tag{9.3.8}$$

We repeat the process as long as

$$\|\mathcal{T}(\,.\,)f - \sum_{j=1}^{k-1} \mathcal{T}(\,.\,)f_j\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \varepsilon,$$

applying Lemma 9.3.2 to $g = f - \sum\limits_{j=1}^{k-1} f_j$. Then, by (9.3.4)–(9.3.8), we obtain functions $f_1, \ldots, f_n$ satisfying Properties 1 and 2 of Lemma 9.3.1 and

$$\|f - \sum_{j=1}^{k} f_j\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{j=1}^{k} \|f_j\|_{L^2}^2, \tag{9.3.9}$$

$$\|f_k\|_{L^2}^2 \geqslant C\|f\|_{L^2}^{-a}\varepsilon^b, \tag{9.3.10}$$

for any $k \in [\![1, n]\!]$, for some $n \geqslant 2$. From Strichartz's estimate (9.1.7) and (9.3.9)–(9.3.10), we obtain

$$\begin{aligned}
&\|\mathcal{T}(\,.\,)f - \sum_{j=1}^{n} \mathcal{T}(\,.\,)f_j\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)}^2 \\
&\leqslant C\|f - \sum_{j=1}^{n} f_j\|_{L^2}^2 \leqslant C(\|f\|_{L^2}^2 - Cn\|f\|_{L^2}^{-a}\varepsilon^b) \xrightarrow{n\longrightarrow\infty} -\infty.
\end{aligned}$$

So the process stops for some $n \leqslant C(\|f\|_{L^2}, N, \varepsilon)$. We set $N_0 = n$ and the proof is achieved. □

Lemma 9.3.3. *Let $g \in L^2(\mathbb{R}^N)$, let $\tau \in \mathcal{C}$, let $A > 0$ and let $C_0 > 0$ be such that $\operatorname{supp}\widehat{g} \subset \tau$, $\ell(\tau) \leqslant C_0 A$ and $|\widehat{g}| < A^{-\frac{N}{2}}$. Let ξ_0 be the center of τ. Then for any $\varepsilon > 0$, there exist $N_1 \in \mathbb{N}$ with $N_1 \leqslant C(N, C_0, \varepsilon)$ and $(Q_n)_{1 \leqslant n \leqslant N_1} \subset \mathbb{R} \times \mathbb{R}^N$ with*

$$Q_n = \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^N;\ t \in I_n \text{ and } (x - 4\pi t\xi_0) \in C_n\right\}, \tag{9.3.11}$$

where $I_n \subset \mathbb{R}$ is an interval with $|I_n| = \dfrac{1}{A^2}$ and $C_n \in \mathcal{C}$ with $\ell(C_n) = \dfrac{1}{A}$ such that

$$\left(\int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} Q_n} |(\mathcal{T}(t))g(x)|^{\frac{2(N+2)}{N}} \mathrm{d}t\mathrm{d}x \right)^{\frac{N}{2(N+2)}} < \varepsilon.$$

Notice that the functions f_n obtained in Lemma 9.3.1 satisfy the hypothesis of Lemma 9.3.3.

Proof of Lemma 9.3.3. We define $g' \in L^2(\mathbb{R}^N)$ by $\widehat{g'}(\xi') = A^{\frac{N}{2}} \widehat{g}(\xi_0 + A\xi')$. Then $\|g'\|_{L^2} = \|g\|_{L^2}$, $|\widehat{g'}| < 1$ and $\operatorname{supp}\widehat{g'} \subset \left[-\frac{C_0}{2}, \frac{C_0}{2}\right]^N$. It follows from (9.2.1) applied to g' that

$$
\begin{aligned}
|(\mathcal{T}(A^2 t)g')(A(x - 4\pi t\xi_0))| &= \left| \int\limits_{(-\frac{C_0}{2}, \frac{C_0}{2})^N} e^{2i\pi\left(A(x - 4\pi t\xi_0).\xi - 2\pi A^2 t|\xi|^2\right)} \widehat{g'}(\xi)\mathrm{d}\xi \right| \\
&= A^{\frac{N}{2}} \left| \int\limits_{(-\frac{C_0}{2}, \frac{C_0}{2})^N} e^{2i\pi\left(A(x - 4\pi t\xi_0).\xi - 2\pi A^2 t|\xi|^2\right)} \widehat{g}(\xi_0 + A\xi)\mathrm{d}\xi \right| \\
&= A^{-\frac{N}{2}} |(\mathcal{T}(t)g)(x)|,
\end{aligned}
$$

where the last identity follows from the change of variables $\zeta = \xi_0 + A\xi$. Setting

$$\begin{cases} t' = A^2 t, \\ x' = A(x - 4\pi t\xi_0), \end{cases} \tag{9.3.12}$$

we then have

$$|(\mathcal{T}(t)g)(x)| = A^{\frac{N}{2}} |(\mathcal{T}(t')g')(x')|. \tag{9.3.13}$$

By (9.2.1),

$$|(\mathcal{T}(t)g')(x)| = \left| \int\limits_{(-\frac{C_0}{2}, \frac{C_0}{2})^N} \widehat{g'}(\zeta) e^{2i\pi(x.\zeta - 2\pi t|\zeta|^2)} \mathrm{d}\zeta \right|. \tag{9.3.14}$$

By (9.2.2) (with g' in the place of g) and Corollary 1.2 of Tao [166], we obtain

$$\|\mathcal{T}(\,.\,)g'\|_{L^q(\mathbb{R} \times \mathbb{R}^N)} \leqslant C(N,q) \|\widehat{g'}\|_{L^p(\mathbb{R}^N)} = C(N,q) \|\widehat{g'}\|_{L^p\left((-\frac{C_0}{2}, \frac{C_0}{2})^N\right)}, \tag{9.3.15}$$

for any $q > \frac{2(N+3)}{(N+1)}$ and any $p \geqslant 1$ such that $q = \frac{N+2}{N}p'$. Let $p' = p'(N) \in (1,2)$ be such that

$$\frac{2(N+3)}{(N+1)} < \frac{N+2}{N}p' < \frac{2(N+2)}{N}.$$

Thus $q = q(N) = \dfrac{N+2}{N}p' < \dfrac{2(N+2)}{N}$ and it follows from (9.3.15) that and Hölder's inequality that

$$\|\mathcal{T}(\,.\,)g'\|_{L^q(\mathbb{R}\times\mathbb{R}^N)} \leqslant C(N)\|\widehat{g'}\|_{L^p\left(\left(-\frac{C_0}{2},\frac{C_0}{2}\right)^N\right)} \leqslant C(N)\left|\left(-\frac{C_0}{2},\frac{C_0}{2}\right)^N\right|^{\frac{1}{p}}\|\widehat{g'}\|_{L^\infty\left(\left(-\frac{C_0}{2},\frac{C_0}{2}\right)^N\right)},$$

so that

$$\|\mathcal{T}(\,.\,)g'\|_{L^q(\mathbb{R}\times\mathbb{R}^N)} \leqslant C(C_0,N).$$

This estimate implies that for any $\lambda > 0$,

$$\begin{aligned}&\int\limits_{\{|\mathcal{T}(\,.\,)g'|<\lambda\}} |\mathcal{T}(t')g'(x')|^{\frac{2(N+2)}{N}}\mathrm{d}t'\mathrm{d}x' \\ &= \int\limits_{\{|\mathcal{T}(\,.\,)g'|<\lambda\}} |\mathcal{T}(t')g'(x')|^{\left(\frac{2(N+2)}{N}-q\right)+q}\mathrm{d}t'\mathrm{d}x' \leqslant C(C_0,N)\lambda^{\frac{2(N+2)}{N}-q}.\end{aligned}$$

So there exists $\lambda_0 = \lambda_0(N,C_0,\varepsilon) \in (0,1)$ small enough such that

$$\int\limits_{\{|\mathcal{T}(\,.\,)g'|<2\lambda_0\}} |\mathcal{T}(t')g'(x')|^{\frac{2(N+2)}{N}}\mathrm{d}t'\mathrm{d}x' < \varepsilon^{\frac{2(N+2)}{N}}, \tag{9.3.16}$$

Since $\operatorname{supp}\widehat{g'} \subset \left[-\frac{C_0}{2},\frac{C_0}{2}\right]^N$ and $\|\widehat{g'}\|_{L^\infty} \leqslant 1$, it follows from formula (9.2.1) that for any $(t',x') \in \mathbb{R}\times\mathbb{R}^N$ and any $(t'',x'') \in \mathbb{R}\times\mathbb{R}^N$,

$$|\mathcal{T}(t')g'(x') - \mathcal{T}(t'')g'(x'')| \leqslant C(|t'-t''| + |x'-x''|),$$

where $C = C(C_0,N) \geqslant 1$. So for such a constant, if $(t',x') \in \{|\mathcal{T}(.)g'| \geqslant 2\lambda_0\}$ and if $(t'',x'') \in \mathbb{R}\times\mathbb{R}^N$ is such that $|t'-t''| \leqslant \frac{\lambda_0}{2C} < \frac{1}{2}$ and $|x'-x''| \leqslant \frac{\lambda_0}{2C} < \frac{1}{2}$ then $|\mathcal{T}(t'')g(x'')| \geqslant \lambda_0$, that is $(t'',x'') \in \{|\mathcal{T}(\,.\,)g'| \geqslant \lambda_0\}$. So there exist a set R and a family $(P_r)_{r\in R} = (J_r,K_r)_{r\in R} \subset \mathbb{R}\times\mathbb{R}^N$, where $J_r \subset \mathbb{R}$ is a closed interval of center $t' \in \mathbb{R}$ with $|J_r| = \frac{\lambda_0}{C}$ and $K_r \in \mathcal{C}$ of center $x' \in \mathbb{R}^N$ with $\ell(K_r) = \frac{\lambda_0}{C}$ and $(t',x') \in \{|\mathcal{T}(\,.\,)g'| \geqslant 2\lambda_0\}$, such that

$$\forall (r,s) \in R\times R \text{ with } r \neq s,\ \operatorname{Int}(P_r)\cap\operatorname{Int}(P_s) = \emptyset, \tag{9.3.17}$$

$$\{|\mathcal{T}(\,.\,)g'| \geqslant 2\lambda_0\} \subset \bigcup_{r\in R} P_r \subset \{|\mathcal{T}(\,.\,)g'| \geqslant \lambda_0\}, \tag{9.3.18}$$

where $\operatorname{Int}(P_r)$ denotes the interior of the set P_r. We set $N_1 = \#R$. It follows from (9.3.17)–(9.3.18) and Strichartz's estimate (9.1.7) that,

$$\begin{aligned}N_1\left(\frac{\lambda_0}{C}\right)^{N+1} &= \left|\bigcup_{r\in R} P_r\right| \leqslant |\{|(\mathcal{T}(\,.\,)g')| \geqslant \lambda_0\}| \\ &\leqslant \lambda_0^{-\frac{2(N+2)}{N}}\|\mathcal{T}(\,.\,)g'\|^{\frac{2(N+2)}{N}}_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \leqslant C\lambda_0^{-\frac{2(N+2)}{N}}\|g\|^{\frac{2(N+2)}{N}}_{L^2},\end{aligned}$$

from which we deduce that $N_1 < \infty$ and $N_1 \leqslant C(\|g\|_{L^2}, N, C_0, \varepsilon)$. Actually, since our hypothesis implies that $\|g\|_{L^2} \leqslant C_0^{N/2}$, we can write also $N_1 \leqslant C(N, C_0, \varepsilon)$. For any $n \in [\![1, N_1]\!]$, let (t_n, x_n) be the center of P_n, let $I_n \subset \mathbb{R}$ be the interval of center $\frac{t_n}{A^2}$ with $|I_n| = \frac{1}{A^2}$, let $I'_n = A^2 I_n$, let $C_n \in \mathcal{C}$ of center $\frac{1}{A}x_n$ with $\ell(C_n) = \frac{1}{A}$, let $C'_n = AC_n$ and let Q_n be defined by (9.3.11). Then $\bigcup\limits_{n=1}^{N_1} P_n \subset \bigcup\limits_{n=1}^{N_1} (I'_n \times C'_n)$, which yields with (9.3.16) and (9.3.18),

$$\int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} (I'_n \times C'_n)} |\mathcal{T}(t')g'(x')|^{\frac{2(N+2)}{N}} \mathrm{d}t'\mathrm{d}x' < \varepsilon^{\frac{2(N+2)}{N}}. \tag{9.3.19}$$

By (9.3.13),

$$\int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} Q_n} |\mathcal{T}(t)g(x)|^{\frac{2(N+2)}{N}} \mathrm{d}t\mathrm{d}x = A^{N+2} \int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} Q_n} |\mathcal{T}(t')g'(x')|^{\frac{2(N+2)}{N}} \mathrm{d}t'\mathrm{d}x'$$

But $(t, x) \in Q_n \iff (t', x') \in I'_n \times C'_n$, and so we deduce from the above estimate and (9.3.12) that

$$\int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} Q_n} |\mathcal{T}(t)g(x)|^{\frac{2(N+2)}{N}} \mathrm{d}t\mathrm{d}x = \int\limits_{\mathbb{R}^{N+1} \setminus \bigcup\limits_{n=1}^{N_1} (I'_n \times C'_n)} |\mathcal{T}(t')g(x')|^{\frac{2(N+2)}{N}} \mathrm{d}t'\mathrm{d}x'. \tag{9.3.20}$$

Putting together (9.3.19) and (9.3.20), we obtain the desired result. □

9.4 Mass concentration

Proposition 9.4.1. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, let $u_0 \in L^2(\mathbb{R}^N) \setminus \{0\}$ and let*

$$u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^N)) \cap L^{\frac{2(N+2)}{N}}_{\mathrm{loc}}((-T_{\min}, T_{\max}); L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))$$

be the maximal solution of (9.1.4) such that $u(0) = u_0$. Then there exists $\eta_0 = \eta_0(N, |\gamma|) > 0$ satisfying the following properties. Let $(T_0, T_1) \subset (-T_{\min}, T_{\max})$ be an interval and let

$$\eta = \|u\|_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)}. \tag{9.4.1}$$

If $\eta \in (0, \eta_0]$ then there exist $t_0 \in (T_0, T_1)$ and $c \in \mathbb{R}^N$ such that

$$\|u(t_0)\|_{L^2(B(c,R))} \geqslant \varepsilon, \tag{9.4.2}$$

where $R = \min\left\{(T_1 - t_0)^{\frac{1}{2}}, (t_0 - T_0)^{\frac{1}{2}}\right\}$ and $\varepsilon = \varepsilon(\|u_0\|_{L^2}, N, \eta) > 0$.

Proof. Let γ, u_0, u and (T_0, T_1) be as in the Proposition 9.4.1. Let $\eta > 0$ be as in (9.4.1). By (9.1.2), we have

$$\forall t \in (-T_{\min}, T_{\max}),\ u(t) = \mathcal{T}(t - T_0)u(T_0) + i\gamma \int_{T_0}^t (\mathcal{T}(t - s)\{|u|^{\frac{4}{N}} u\})(s)\mathrm{d}s. \tag{9.4.3}$$

Setting for any $t \in (-T_{\min}, T_{\max})$, $\Phi_u(t) = i\gamma \int_{T_0}^t (\mathcal{T}(t-s)\{|u|^{\frac{4}{N}} u\})(s)ds$ and applying Strichartz's estimate (9.1.8), we get with (9.4.1)

$$\|\Phi_u\|_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \leqslant C_1 \|u\|^{\frac{N+4}{N}}_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} = C_1 \eta^{\frac{N+4}{N}}, \tag{9.4.4}$$

where $C_1 = C_1(N, |\gamma|) \geqslant 1$. For every $a, b \geqslant 0$, $(a+b)^\alpha \leqslant C(\alpha)(a^\alpha + b^\alpha)$, where $C(\alpha) = 1$ if $0 < \alpha \leqslant 1$ and $C(\alpha) = 2^{\alpha-1}$ if $\alpha \geqslant 1$. Let C_2 be such a constant for $\alpha = \frac{4}{N}$. We choose $\eta_0 = \eta_0(N, |\gamma|) > 0$ small enough to have

$$2(2C_1)^{\frac{4}{N}} C_2 \eta_0^{\frac{16}{N^2}} \leqslant 1. \tag{9.4.5}$$

Assume that $\eta \leqslant \eta_0$. We proceed in 3 steps.

Step 1. We show that, there exist $f_0 \in L^2(\mathbb{R}^N)$, $A > 0$ and $\tau \in \mathcal{C}$ of center $\xi_0 \in \mathbb{R}^N$ satisfying supp $\widehat{f_0} \subset \tau$, $\ell(\tau) \leqslant C(\|u_0\|_{L^2}, N, \eta)A$ and $|\widehat{f_0}| < A^{-\frac{N}{2}}$, and there exist an interval $I \subset \mathbb{R}$ and $K \in \mathcal{C}$, with $|I| = \dfrac{1}{A^2}$ and $\ell(K) = \dfrac{1}{A}$, such that for $Q \subset \mathbb{R} \times \mathbb{R}^N$ defined by

$$Q = \left\{(t,x) \in \mathbb{R} \times \mathbb{R}^N;\ t \in I \text{ and } (x - 4\pi t\xi_0) \in K\right\},$$

we have

$$\iint\limits_{((T_0,T_1)\times\mathbb{R}^N)\cap Q} |u(t,x)|^2 |\mathcal{T}(t-T_0)f_0(x)|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \geqslant C\eta^{\frac{2(N+2)}{N}}, \tag{9.4.6}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$.
To prove this claim, we apply Lemma 9.3.1 to $f = u(T_0)$ with $\varepsilon_0 = \eta^{\frac{N+4}{N}}$. Note that, by (9.4.1), (9.4.3), (9.4.4), (9.4.5) and time translation, we have that

$$\|\mathcal{T}(\cdot)u(T_0)\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} = \|\mathcal{T}(\cdot - T_0)u(T_0)\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} \geqslant \eta/2 \geqslant \varepsilon_0.$$

It follows from Hölder's inequality (with $p = \frac{N+2}{N}$ and $p' = \frac{N+2}{2}$), (9.4.3)–(9.4.4) and Lemma 9.3.1 that

$$\begin{aligned}
&\int_{T_0}^{T_1}\!\!\int_{\mathbb{R}^N} |u(t,x)|^2 \left| u(t,x) - \sum_{n=1}^{N_0} \mathcal{T}(t-T_0)f_n(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \\
&\leqslant \|u\|^2_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \|u - \sum_{n=1}^{N_0} \mathcal{T}(\,\cdot - T_0\,)f_n\|^{\frac{4}{N}}_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \\
&\leqslant \eta^2 \left(\|\mathcal{T}(\,.\,)u(T_0) - \sum_{n=1}^{N_0} \mathcal{T}(\,.\,)f_n\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R}\times\mathbb{R}^N)} + C_1 \|u\|^{\frac{N+4}{N}}_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \right)^{\frac{4}{N}} \\
&\leqslant C_1^{\frac{4}{N}} \eta^2 (\varepsilon_0 + \eta^{\frac{N+4}{N}})^{\frac{4}{N}} \leqslant (2C_1)^{\frac{4}{N}} \eta_0^{\frac{16}{N^2}} \eta^{\frac{2(N+2)}{N}} \leqslant \frac{1}{2C_2} \eta^{\frac{2(N+2)}{N}}.
\end{aligned}$$

The above estimate and (9.4.1) yield

$$\eta^{\frac{2(N+2)}{N}} = \int_{T_0}^{T_1}\int_{\mathbb{R}^N} |u(t,x)|^2 \left| \left(u(t,x) - \sum_{n=1}^{N_0} \mathcal{T}(t-T_0) f_n(x) \right) + \sum_{n=1}^{N_0} \mathcal{T}(t-T_0) f_n(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x$$

$$\leqslant C_2 \left(\frac{1}{2C_2} \eta^{\frac{2(N+2)}{N}} + \int_{T_0}^{T_1}\int_{\mathbb{R}^N} |u(t,x)|^2 \left| \sum_{n=1}^{N_0} \mathcal{T}(t-T_0) f_n(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \right),$$

which gives

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^N} |u(t,x)|^2 \left| \sum_{n=1}^{N_0} \mathcal{T}(t-T_0) f_n(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \geqslant \frac{1}{2C_2} \eta^{\frac{2(N+2)}{N}}. \tag{9.4.7}$$

By Lemma 9.3.1 and conservation of charge, $N_0 \leqslant C(\|u_0\|_{L^2}, N, \eta)$. It follows from (9.4.7) that there exists $n_0 \in [\![1, N_0]\!]$ such that

$$\int_{T_0}^{T_1}\int_{\mathbb{R}^N} |u(t,x)|^2 \left| \mathcal{T}(t-T_0) f_{n_0}(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \geqslant C \eta^{\frac{2(N+2)}{N}}, \tag{9.4.8}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. Set $A = A_{n_0}$, $\tau = \tau_{n_0}$ and $C_0 = C(N) \|u_0\|_{L^2}^{c(N)} \varepsilon_0^{-\nu(N)}$, where we have used the notations of Lemma 9.3.1. Let $\xi_0 \in \mathbb{R}^N$ be the center of τ_{n_0}. We apply Lemma 9.3.3 to $g = f_{n_0}$ and $\varepsilon_1 = \left(\frac{C}{2}\right)^{\frac{N}{4}} \eta$, where C is the constant in (9.4.8). It follows from Hölder's inequality (with $p = \frac{N+2}{N}$ and $p' = \frac{N+2}{2}$), (9.4.1) and Lemma 9.3.3 that

$$\iint_{((T_0,T_1)\times\mathbb{R}^N) \setminus \bigcup_{n=1}^{N_1} Q_n} |u(t,x)|^2 \left| \mathcal{T}(t-T_0) f_{n_0}(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x$$

$$\leqslant \|u\|^2_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \|\mathcal{T}(\,.\,) f_{n_0}\|^{\frac{4}{N}}_{L^{\frac{2(N+2)}{N}}(\mathbb{R}^{N+1} \setminus \bigcup_{n=1}^{N_1} Q_n)}$$

$$\leqslant \eta^2 \varepsilon_1^{\frac{4}{N}} = \frac{C}{2} \eta^{\frac{2(N+2)}{N}}.$$

The above estimate with (9.4.8) yield

$$\iint_{((T_0,T_1)\times\mathbb{R}^N) \cap (\bigcup_{n=1}^{N_1} Q_n)} |u(t,x)|^2 \left| \mathcal{T}(t-T_0) f_{n_0}(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \geqslant C \eta^{\frac{2(N+2)}{N}}, \tag{9.4.9}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. By Lemma 9.3.3, $N_1 \leqslant C(\|u_0\|_{L^2}, N, \eta)$. With (9.4.9), this implies that there exists $n_1 \in [\![1, N_1]\!]$ such that

$$\iint_{((T_0,T_1)\times\mathbb{R}^N) \cap Q_{n_1}} |u(t,x)|^2 \left| \mathcal{T}(t-T_0) f_{n_0}(x) \right|^{\frac{4}{N}} \mathrm{d}t\mathrm{d}x \geqslant C \eta^{\frac{2(N+2)}{N}}, \tag{9.4.10}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. Hence we obtain the Step 1 claim with $f_0 = f_{n_0}$, $I = I_{n_1}$, $K = C_{n_1}$ and $Q = Q_{n_1}$.

Step 2. We show that $\dfrac{1}{A} \leqslant C(T_1 - T_0)^{\frac{1}{2}}$ and $\sup\limits_{t\in\mathbb{R}} \|\mathcal{T}(t-T_0)f_0\|_{L^\infty(\mathbb{R}^N)} \leqslant CA^{\frac{N}{2}}$, where $C = C(\|u_0\|_{L^2}, N, \eta)$.
By (9.2.1) and Step 1, $|\mathcal{T}(t-T_0)f_0| \leqslant \int_\tau |\widehat{f_0}(\xi)| d\xi \leqslant A^{-\frac{N}{2}} \int_\tau 1 d\xi \leqslant CA^{\frac{N}{2}}$, which yields second part of Step 2. Using this estimate, Step 1 and conservation of charge, we deduce

$$\begin{aligned} C\eta^{\frac{2(N+2)}{N}} &\leqslant \iint\limits_{((T_0,T_1)\times\mathbb{R}^N)\cap Q} |u(t,x)|^2 |\mathcal{T}(t-T_0)f_0(x)|^{\frac{4}{N}} \mathrm{d}x\mathrm{d}t \\ &\leqslant CA^2 \iint\limits_{((T_0,T_1)\times\mathbb{R}^N)\cap Q} |u(t,x)|^2 \mathrm{d}x\mathrm{d}t \leqslant CA^2 \int_{T_0}^{T_1}\int_{\mathbb{R}^N} |u(t,x)|^2 \mathrm{d}x\mathrm{d}t \\ &\leqslant CA^2 \|u_0\|_{L^2}^2 (T_1 - T_0). \end{aligned}$$

Hence we obtain the Step 2 claim.

Step 3. Conclusion.
Let $K \in \mathcal{C}$, I and Q be as in Step 1, and let $\eta' = C\eta^{\frac{2(N+2)}{N}}$, where C is the constant of (9.4.10). Let $K(t) = K + 4\pi t\xi_0$ and let $\kappa > 0$ be small enough to be chosen later. It follows from Step 1, Step 2 and Hölder's inequality (with $p = \frac{N+2}{N}$ and $p' = \frac{N+2}{2}$), that

$$\begin{aligned} \eta' &\leqslant \iint\limits_{((T_0,T_1)\times\mathbb{R}^N)\cap Q} |u(t,x)|^2 \, |\mathcal{T}(t-T_0)f_0(x)|^{\frac{4}{N}} \, \mathrm{d}x\mathrm{d}t \\ &\leqslant \|\mathcal{T}(\cdot - T_0)f_0\|_{L^\infty}^{\frac{4}{N}} \int\limits_{I\cap(T_0,T_1)} \left(\int_{K(t)} |u(t,x)|^2 \mathrm{d}x \right) \mathrm{d}t \\ &\leqslant CA^2 \int\limits_{I\cap(T_0,T_1)} \left(\int_{K(t)} |u(t,x)|^2 \mathrm{d}x \right) \mathrm{d}t \\ &\leqslant CA^2 \int\limits_{I\cap\left(T_0+\frac{\kappa\eta'}{A^2}, T_1-\frac{\kappa\eta'}{A^2}\right)} \left(\int_{K(t)} |u(t,x)|^2 \mathrm{d}x \right) \mathrm{d}t \\ &\quad + CA^2 \|u\|^2_{L^{\frac{2(N+2)}{N}}((T_0,T_1)\times\mathbb{R}^N)} \left(\int_{I\cap\left[\left(T_0, T_0+\frac{\kappa\eta'}{A^2}\right)\cup\left(T_1-\frac{\kappa\eta'}{A^2}, T_1\right)\right]} \left(\int_{K(t)} 1 \, \mathrm{d}x \right) \mathrm{d}t \right)^{\frac{2}{N+2}} \\ &\leqslant CA^2 |I| \sup_{t\in I\cap\left(T_0+\frac{\kappa\eta'}{A^2}, T_1 \frac{\kappa\eta'}{A^2}\right)} \int_{K(t)} |u(t,x)|^2 \mathrm{d}x + CA^2 \eta'^{\frac{N}{N+2}} \left(\frac{\kappa\eta'}{A^2}\right)^{\frac{2}{N+2}} \left(\frac{1}{A^2}\right)^{\frac{N}{N+2}} \\ &\leqslant C \sup_{t\in I\cap\left(T_0+\frac{\kappa\eta'}{A^2}, T_1-\frac{\kappa\eta'}{A^2}\right)} \int_{K(t)} |u(t,x)|^2 \mathrm{d}x + C\kappa^{\frac{2}{N+2}} \eta', \end{aligned}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. For such a C, let $\kappa > 0$ be small enough to have $C\kappa^{\frac{2}{N+2}} \leqslant \frac{1}{2}$. Then

$\kappa = \kappa(\|u_0\|_{L^2}, N, \eta)$ and

$$\sup_{t\in I\cap\left(T_0+\frac{\kappa\eta'}{A^2},T_1-\frac{\kappa\eta'}{A^2}\right)} \int_{K(t)} |u(t,x)|^2 \mathrm{d}x \geqslant C\eta^{\frac{2(N+2)}{N}},$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. So there exists $t_0 \in I \cap \left(T_0 + \frac{\kappa\eta'}{A^2}, T_1 - \frac{\kappa\eta'}{A^2}\right)$ such that

$$\int_{K(t_0)} |u(t_0,x)|^2 \mathrm{d}x \geqslant C\eta^{\frac{2(N+2)}{N}}, \tag{9.4.11}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. Since $\ell(K(t_0)) = \dfrac{1}{A}$, then $K(t_0)$ is contained in a ball of radius $\dfrac{\sqrt{N}}{A}$. Furthermore, $T_0 + \dfrac{\kappa\eta'}{A^2} < t_0 < T_1 - \dfrac{\kappa\eta'}{A^2}$, which yields

$$\frac{1}{A} \leqslant C \min\{(T_1 - t_0)^{\frac{1}{2}}, (t_0 - T_0)^{\frac{1}{2}}\}, \tag{9.4.12}$$

where $C = C(\|u_0\|_{L^2}, N, \eta)$. Using this and Step 2, it follows that $K(t_0)$ can be covered by a finite number (which depends only on $\|u_0\|_{L^2}$, N and η) of balls of radius $R = \min\left\{(T_1 - t_0)^{\frac{1}{2}}, (t_0 - T_0)^{\frac{1}{2}}\right\}$. Then, by (9.4.11), there is some $c \in \mathbb{R}^N$ such that

$$\int_{B(c,R)} |u(t_0,x)|^2 \mathrm{d}x \geqslant \varepsilon(\|u_0\|_{L^2}, N, \eta). \tag{9.4.13}$$

This concludes the proof. □

Proof of Theorem 9.1.1. Let γ, u_0 and u be as in Theorem 9.1.1. Let $\eta_0 = \eta_0(N, |\gamma|) > 0$ be given by Proposition 9.4.1. We apply Proposition 9.4.1 with $\eta = \eta_0$. Let $\varepsilon = \varepsilon(\|u_0\|_{L^2}, N, |\gamma|) > 0$ be given by Proposition 9.4.1. Assume that $T_{\max} < \infty$. Then $\|u\|_{L^{\frac{2(N+2)}{N}}((0,T_{\max});L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))} = \infty$ and so there exist

$$0 = T_1 < T_2 < \cdots < T_n < T_{n+1} < \cdots < T_{\max}$$

such that

$$\forall n \in \mathbb{N},\ \|u\|_{L^{\frac{2(N+2)}{N}}((T_n,T_{n+1})\times\mathbb{R}^N)} = \eta_0.$$

It follows from Proposition 9.4.1 that for each $n \in \mathbb{N}$, there exist $c_n \in \mathbb{R}^N$, $R_n > 0$ and $t_n \in (T_n, T_{n+1})$ such that

$$R_n \leqslant \min\{(T_{\max} - t_n)^{\frac{1}{2}}, (T_{\min} + t_n)^{\frac{1}{2}}\} \quad \text{and} \quad \|u(t_n)\|_{L^2(B(c_n,R_n))} \geqslant \varepsilon,$$

for every $n \in \mathbb{N}$. The case $T_{\min} < \infty$ follows in the same way. Hence we have proved the result. □

9.5 Further Results

As a corollary of the previous results, we can generalize to higher dimensions the 2–dimensional results proved by Merle and Vega [137] and the results proved by Keraani in [119] dimensions 1 and 2. We state here the most interesting of them. We need first some notation.

Definition 9.5.1. Let $\gamma \in \mathbb{R} \setminus \{0\}$. We define δ_0 as the supremum of δ such that if

$$\|u_0\|_{L^2} < \delta,$$

then (9.1.4) has a global solution $u \in C(\mathbb{R}; L^2(\mathbb{R}^N)) \cap L^{\frac{2(N+2)}{N}}(\mathbb{R}; L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))$.

We can prove the following result.

Theorem 9.5.2. *Let $\gamma \in \mathbb{R} \setminus \{0\}$, let $u_0 \in L^2(\mathbb{R}^N) \setminus \{0\}$, such that $\|u_0\|_{L^2(\mathbb{R}^N)} < \sqrt{2}\delta_0$, and let*

$$u \in C((-T_{\min}, T_{\max}); L^2(\mathbb{R}^N)) \cap L^{\frac{2(N+2)}{N}}_{\mathrm{loc}}((-T_{\min}, T_{\max}); L^{\frac{2(N+2)}{N}}(\mathbb{R}^N))$$

be the maximal solution of (9.1.4) such that $u(0) = u_0$. Assume that $T_{\max} < \infty$, and let $\lambda(t) > 0$, such that $\lambda(t) \longrightarrow \infty$ as $t \longrightarrow T_{\max}$. Then there exists $x(t) \in \mathbb{R}^N$ such that,

$$\liminf_{t \nearrow T_{\max}} \int_{B(x(t), \lambda(t)(T_{\max}-t)^{\frac{1}{2}})} |u(t,x)|^2 \mathrm{d}x \geqslant \delta_0^2.$$

If $T_{\min} < \infty$ and $\lambda(t) \longrightarrow \infty$ as $t \longrightarrow -T_{\min}$ then there exists $x(t) \in \mathbb{R}^N$ such that,

$$\liminf_{t \searrow -T_{\min}} \int_{B(x(t), \lambda(t)(T_{\min}+t)^{\frac{1}{2}})} |u(t,x)|^2 \mathrm{d}x \geqslant \delta_0^2.$$

The main ingredient in the proof of that theorem is a profile decomposition of the solutions of the free Schrödinger equation. This decomposition was shown in the case $N = 2$ by Merle and Vega [137] (see also Theorem 1.4 in [54]) and by Carles and Keraani [54] when $N = 1$. We generalize it to higher dimensions thanks to the improved Strichartz estimate, Theorem 9.1.4. To describe it we need a definition. We follow the notation of Carles and Keraani [54].

Definition 9.5.3. If $\Gamma^j = (\rho_n^j, t_n^j, \xi_n^j, x_n^j)_{n \in \mathbb{N}}$, $j = 1, 2, \dots$ is a family of sequences in $(0, \infty) \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, we say that it is an *orthogonal family* if for all $j \neq k$,

$$\limsup_{n \to \infty} \left(\frac{\rho_n^j}{\rho_n^k} + \frac{\rho_n^k}{\rho_n^j} + \frac{|t_n^j - t_n^k|}{(\rho_n^j)^2} + \left| \frac{x_n^j - x_n^k}{\rho_n^j} + \frac{t_n^j \xi_n^j - t_n^k \xi_n^k}{\rho_n^j} \right| \right) = \infty.$$

Now, we can state the theorem about the linear profiles.

Theorem 9.5.4. *Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $L^2(\mathbb{R}^N)$. Then, there exists a subsequence (that we name (u_n) for the sake of simplicity) that satisfies the following: there exists a family $(\phi^j)_{j \in \mathbb{N}}$ of functions in $L^2(\mathbb{R}^N)$ and a family of pairwise orthogonal sequences $\Gamma^j = (\rho_n^j, t_n^j, \xi_n^j, x_n^j)_{n \in \mathbb{N}}$, $j = 1, 2, \dots$ such that*

$$\mathcal{T}(t) u_n(x) = \sum_{j=1}^{\ell} H_n^j(\phi^j)(t,x) + w_n^\ell(t,x),$$

where

$$H_n^j(\phi)(t,x) = \mathcal{T}(t) \left(e^{i(\cdot)\frac{\xi_n^j}{2}} \mathcal{T}(-t_n^j) \frac{1}{(\rho_n^j)^{N/2}} \phi\left(\frac{\cdot - x_n^j}{\rho_n^j} \right) \right)(x),$$

with

$$\limsup_{n \to \infty} \|w_n^\ell\|_{L^{\frac{2(N+2)}{N}}(\mathbb{R} \times \mathbb{R}^N)} \longrightarrow 0 \quad \text{as} \quad \ell \longrightarrow \infty.$$

Moreover, for every $\ell \geqslant 1$,

$$\|u_n\|^2_{L^2(\mathbb{R}^N)} = \sum_{j=1}^{\ell} \|\phi^j\|^2_{L^2(\mathbb{R}^N)} + \|w_n^\ell(0)\|^2_{L^2(\mathbb{R}^N)} + \mathrm{o}(1),$$

as $n \longrightarrow \infty$.

A similar result has been proved for wave equations by Bahouri and Gérard [15]. To prove Theorem 9.5.4 one can follow Carles and Keraani (proof of Theorem 1.4) in [54]. It is observed in that paper (Remark 3.5) that the result follows from the refined Strichartz's estimate, our Theorem 9.1.4, once we overcome a technical issue, due to the fact that the Strichartz exponent $\frac{2(N+2)}{N}$ is an even natural number when $N \in \{1, 2\}$ (which covers the cases that the previous authors considered) but not in higher dimensions (except $N = 4$). Thus, to complete the proof we only need the following orthogonality result.

Lemma 9.5.5. *For any $M \geqslant 1$,*

$$\Big\|\sum_{j=1}^{M} H_n^j(\phi^j)\Big\|^{\frac{2(N+2)}{N}}_{L^{\frac{2(N+2)}{N}}(\mathbb{R}^{N+1})} \leqslant \sum_{j=1}^{M} \|H_n^j(\phi^j)\|^{\frac{2(N+2)}{N}}_{L^{\frac{2(N+2)}{N}}(\mathbb{R}^{N+1})} + \mathrm{o}(1) \quad as \quad n \longrightarrow \infty.$$

Proof. The proof if based on a well-known orthogonality property (see Gérard [89] and (3.47) in Merle and Vega [137]) : if we have two orthogonal families Γ^1 and Γ^2, and two functions in $L^2(\mathbb{R}^N)$, ϕ^1 and ϕ^2, then

$$\|H_n^1(\phi^1)H_n^2(\phi^2)\|_{L^{\frac{N+2}{N}}(\mathbb{R}^{N+1})} = \mathrm{o}(1) \quad as \quad n \longrightarrow \infty. \tag{9.5.1}$$

When $N = 1$ or $N = 2$, $\frac{2(N+2)}{N}$ is a natural number, so we can decompose the $L^{\frac{2(N+2)}{N}}$ norm as a product and, using (9.5.1), we obtain directly the lemma. In the higher dimensional case, write

$$\begin{aligned}
&\Big\|\sum_{j=1}^{M} H_n^j(\phi^j)\Big\|^{\frac{2(N+2)}{N}}_{L^{\frac{2(N+2)}{N}}} = \int \Big|\sum_j H_n^j(\phi^j)\Big|^2 \Big|\sum_j H_n^j(\phi^j)\Big|^{\frac{4}{N}}\\
&= \int \sum_j \sum_k |H_n^j(\phi^j)H_n^k(\phi^k)| \Big|\sum_\ell H_n^\ell(\phi^\ell)\Big|^{\frac{4}{N}}\\
&= \sum_j \int |H_n^j(\phi^j)|^2 \Big|\sum_\ell H_n^\ell(\phi^\ell)\Big|^{\frac{4}{N}} + \sum_j \sum_{k\neq j} \int |H_n^j(\phi^j)H_n^k(\phi^k)| \Big|\sum_\ell H_n^\ell(\phi^\ell)\Big|^{\frac{4}{N}}\\
&\overset{\text{not}}{=} A + B.
\end{aligned}$$

We estimate B using Hölder's inequality with exponents $\frac{N+2}{N}$ and $\frac{N+2}{2}$,

$$\begin{aligned}
&\int |H_n^j(\phi_j)H_n^k(\phi^k)| \Big|\sum_\ell H_n^\ell(\phi^j)\Big|^{\frac{4}{N}}\\
&\leqslant \|H_n^j(\phi^j)H_n^k(\phi^k)\|_{L^{\frac{N+2}{N}}(\mathbb{R}^{N+1})} \Big\|\sum_{\ell=1}^{M} H_n^\ell(\phi^\ell)\Big\|^{\frac{4}{N}}_{L^{\frac{2(N+2)}{N}}}.
\end{aligned}$$

Then, we use the orthogonality (9.5.1) and obtain $B = \mathrm{o}(1)$.

About A, when $N \geqslant 4$ then $\frac{4}{N} \leqslant 1$ and therefore,

$$\begin{aligned} A &\leqslant \sum_j \sum_\ell \int |H_n^j(\phi^j)|^2 |H_n^\ell(\phi^\ell)|^{\frac{4}{N}} \\ &= \sum_j \int |H_n^j(\phi^j)|^2 |H_n^j(\phi^j)|^{\frac{4}{N}} + \sum_j \sum_{\ell \neq j} \int |H_n^j(\phi^j)|^2 |H_n^\ell(\phi^\ell)|^{\frac{4}{N}}. \end{aligned}$$

The first term of the sum is

$$\sum_j \|H_n^j(\phi^j)\|_{L^{\frac{2(N+2)}{N}}}^{\frac{2(N+2)}{N}}.$$

The second one is

$$\sum_j \sum_{\ell \neq j} \int |H_n^j(\phi^j)|^{2-\frac{4}{N}} |H_n^j(\phi^j) H_n^\ell(\phi^\ell)|^{\frac{4}{N}}.$$

We apply Hölder's with exponents $\frac{N+2}{N-2}$ and $\frac{N+2}{4}$ and bound the last sum by

$$\sum_j \sum_{j \neq \ell} \|H_n^j(\phi_n^j)|\|_{L^{\frac{2N+4}{N}}}^{2-\frac{4}{N}} \|H_n^j(\phi^j) H_n^\ell(\phi^\ell)\|_{L^{\frac{N+2}{N}}}^{\frac{4}{N}}$$

which is o(1) by (9.5.1). This finishes the proof of the Lemma for $N \geqslant 4$.

When $N = 3$, then $\frac{4}{N} = \frac{4}{3} > 1$, which complicates a bit the argument. We write

$$A = \sum_j \int |H_n^j(\phi_j)|^2 |\sum_\ell H_n^\ell(\phi^\ell)| |\sum_m H_n^m(\phi^m)|^{\frac{1}{3}} \leqslant \sum_\ell \sum_j \sum_m \int |H_n^j(\phi^j)|^2 |H_n^\ell(\phi^\ell)| |H_n^m(\phi^m)|^{\frac{1}{3}}.$$

Using a similar argument as in the previous case, we show that the above integrals are o(1) except in the case $j = \ell = m$. This ends the proof of the lemma for $N = 3$. □

Proof of Theorem 9.5.2. To prove Theorem 9.5.2, one can follow the arguments given by Keraani in [119]. Again one has to deal with the fact that $\frac{4}{N}$ is not in general a natural number. Apart from Lemma 9.5.5, we just need an elementary inequality (see (1.10) in Gérard [89]) for the function $F(x) = |x|^{\frac{4}{N}} x$:

$$|F(\sum_{j=1}^{\ell} U^j) - \sum_{j=1}^{\ell} F(U^j)| \leqslant \sum_j \sum_{k \neq j} |U^j| |U^k|^{\frac{4}{N}}.$$

Then, the arguments given by Keraani generalize to higher dimensions without difficulty, and prove Theorem 9.5.2. □

Remark 9.5.6. As said in the beginning of this section, we generalize all the results of Keraani [119] to higher dimension N. In particular, we display two of them.

1. There exists an initial data $u_0 \in L^2(\mathbb{R}^N)$ with $\|u_0\|_{L^2} = \delta_0$, for which the solution u of (9.1.4) blows-up in finite time $T_{\max}$.
2. Let u be a blow-up solution of (9.1.4) at finite time $T_{\max}$ with initial data u_0, such that $\|u_0\|_{L^2} < \sqrt{2}\,\delta_0$. Let $(t_n)_{n\in\mathbb{N}}$ be any time sequence such that $t_n \xrightarrow{n\to\infty} T_{\max}$. Then there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$ (still denoted by $(t_n)_{n\in\mathbb{N}}$), which satisfies the following properties. There

exist $\psi \in L^2(\mathbb{R}^N)$ with $\|\psi\|_{L^2} \geqslant \delta_0$, and a sequence $(\rho_n, \xi_n, x_n)_{n\in\mathbb{N}} \in (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \frac{\rho_n}{\sqrt{T_{\max} - t_n}} \leqslant A,$$

for some $A \geqslant 0$, and

$$\rho_n^{\frac{N}{2}} e^{ix\xi_n} u(t_n, \rho_n x + x_n) \rightharpoonup \psi \quad \text{in} \quad L^2_w(\mathbb{R}^N),$$

as $n \longrightarrow \infty$.

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Appendix

A Some elements on o-minimal structures

Some references for o-minimal structures are [70, 82, 124, 80]. We only collect in this appendix the elements that are necessary to follow our main developments.

Definition A.1 (o-minimal structure [70, Definition 1.5]). An *o-minimal* structure on $(\mathbb{R}, +, .)$ is a sequence of Boolean algebras[2] $\mathcal{O} = \{\mathcal{O}_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ such that for each $n \in \mathbb{N}$,

(i) if A belongs to $\mathcal{O}_n$ then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathcal{O}_{n+1}$;

(ii) if $\Pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ is the canonical projection onto $\mathbb{R}^n$ then for any $A \in \mathcal{O}_{n+1}$, the set $\Pi(A)$ belongs to $\mathcal{O}_n$;

(iii) $\mathcal{O}_n$ contains the family of real algebraic subsets of $\mathbb{R}^n$, that is, every set of the form

$$\Big\{x \in \mathbb{R}^n;\, p(x) = 0\Big\},$$

where $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a real polynomial function;

(iv) the elements of $\mathcal{O}_1$ are exactly the finite unions of intervals and points.

Being given an o-minimal structure $\mathcal{O}$, a set $A \subset \mathbb{R}^n$ is called *definable* (in $\mathcal{O}$) if $A \in \mathcal{O}_n$. A mapping $F : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *definable in* $\mathcal{O}$ if its graph is definable in Ω as a subset of $\mathbb{R}^n \times \mathbb{R}^m$. A point-to-set mapping

$$S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m,$$

maps each point x in $\mathbb{R}^n$ to a subset $S(x)$ of $\mathbb{R}^m$. The *domain of* S, denoted by $\operatorname{dom} S$, is given by the set of elements x in $\mathbb{R}^n$ such that $S(x)$ is nonempty. The graph of S is defined by

$$\operatorname{graph} S = \Big\{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m;\, y \in S(x)\Big\}.$$

As previously a point-to-set mapping is called *definable* (*in* $\mathcal{O}$) if its graph is definable in $\mathbb{R}^n \times \mathbb{R}^m$.

Example A.2. (a) **Semi-algebraic sets.** The first and simplest example of o-minimal structure is given by the class of semi-algebraic objects (see (8.2.1)). Tarski-Seidenberg principle (see [36]) asserts that linear projections of semi-algebraic sets are semi-algebraic sets, in other words item (ii)

2. Recall that a Boolean algebra is stable by finite union, finite intersection and contains the empty set and the total space; here $\emptyset \in \mathcal{O}_n$ and $\mathbb{R}^n \in \mathcal{O}_n$.

of Definition A.1 holds for the class of semi-algebraic sets. The other items of the definition are easy to establish.

(b) **Globally subanalytic sets.** There exists an o-minimal structure that contains semi-algebraic sets and sets of the form $\{(x,t) \in [-1,1]^n \times \mathbb{R};\ f(x) = t\}$, where $f : [-1,1]^n \longrightarrow \mathbb{R}$ ($n \in \mathbb{N}$) is a real analytic function that can be extended analytically on a neighborhood of the square $[-1,1]^n$ – these are sometimes called restricted analytic functions. This result is essentially due to Gabrielov [87] ; sets belonging to this structure are called *globally subanalytic sets* (see [81] and the references therein).

(c) **Log-exp structure.** There exists an o-minimal structure containing the globally subanalytic sets and the graph of $\exp : \mathbb{R} \longrightarrow \mathbb{R}$, see [81].

There are other results on o-minimal structures and the field is still very active, but the above examples give a good idea of the power of the concept.

We now describe some stability/regularity results that we used in this paper.

Let $\mathcal{O}$ be an o-minimal structure on $(\mathbb{R}, +, \,.\,)$.

Lemma A.3 (**Monotonicity Lemma** [**82, Theorem 4.1**])**.** *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a definable function and $k \in \mathbb{N}$. Then there exists a finite partition of I into p intervals $I_1, \dots, I_p$, such that f restricted to each nontrivial interval I_j, $j \in \{1, \dots, p\}$, is C^k and either strictly monotone or constant. Observe that some I_j can be reduced to a singleton.*

Lemma A.4 (**Definable Selection Lemma** [**70**])**.** *Let $S : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a definable point-to-set mapping. Then there exists a definable mapping $F : \operatorname{dom} S \longrightarrow \mathbb{R}^m$ such that*

$$F(x) \in S(x),\ \forall x \in \operatorname{dom} S.$$

We recall the following theorem as stated in Kurdyka's original work [124].

Theorem A.5. *Let Ω be a nonempty open bounded subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ a differentiable definable function with $f > 0$ on Ω. Then there exist $r_0 > 0$ and a continuous definable function $\varphi : [0, r_0) \to \mathbb{R}_+$ such that $\varphi(0) = 0$, $\varphi \in C^1(0, r_0)$ and $\varphi' > 0$ such that*

$$\|\nabla(\varphi \circ f)(x)\| \geqslant 1,\ \forall x \in \Omega.$$

Remark A.6. Let us show how to recover the form of KL inequality given in Theorem 8.2.2.
We adopt the notation of Theorem 8.2.2. Fix $\mu > 0$. Apply first, the above result to $G - G(\overline{u})$ (respectively, to $G(\overline{u}) - G$) on $\Omega_1 = B(\overline{u}, \mu) \cap [G - G(\overline{u}) > 0]$ (respectively, on $\Omega_2 = B(\overline{u}, \mu) \cap [G(\overline{u}) - G > 0]$). This gives $\varphi_1 : [0, r_1) \longrightarrow \mathbb{R}_+$ and $\varphi_2 : [0, r_2) \longrightarrow \mathbb{R}_+$, as in Kurdyka's Theorem. Let us now build a "global" φ as in Theorem 8.2.2. First recall that the derivative of a differentiable definable function is definable in the same structure, see [70]. Set $p(s) = (\varphi_1' - \varphi_2')(s)$. By definability, p is positive, negative or null on an interval of the form $(0, \varepsilon)$. This yields the existence of r in $(0, \min\{r_1, r_2\})$ such that, for instance, $\varphi_1' > \varphi_2'$ on $(0, r)$. Set then $\varphi = \varphi_1$ and observe that

$$\|\nabla(\varphi \circ |G(\,\cdot\,) - G(\overline{u})|)(u)\| \geqslant 1,\ \forall u \in B(0, \eta) \setminus [G \neq G(\overline{u})],$$

when η is sufficiently small.

B Some useful estimates and results about Sobolev spaces

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and we use the convention, $W^{0,p}(\mathbb{R}^N) = L^p(\mathbb{R}^N)$.

Lemma B.1. *Let $0 < m \leqslant 1$. Then we have for any $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$,*

$$\Big||z_1|^{-(1-m)}z_1 - |z_2|^{-(1-m)}z_2\Big| \leqslant 3|z_1 - z_2|^m, \tag{B.1}$$

where $|z|^{-(1-m)}z = 0$, if $z = 0$.

Proof. Let $0 < m < 1$ (the case $m = 1$ being obvious). We proceed to the proof in four steps.
Step 1 : $\forall t, s \geqslant 0$, $|t^m - s^m| \leqslant |t - s|^m$.
Let for $x \geqslant 1$, $f(x) = (x-1)^m - (x^m - 1)$. Then $f' > 0$ on $(1, \infty)$ and so $f\left(\frac{t}{s}\right) \geqslant f(1) = 0$, for any $t \geqslant s > 0$. Hence Step 1.
Step 2 : $\forall a \geqslant 0$, $\forall \theta \in \mathbb{R}$, $\left|a^m - a^m e^{\mathrm{i}\theta}\right| \leqslant 2^{1-m}\left|a - ae^{\mathrm{i}\theta}\right|^m$.
We have for any $\theta \in \mathbb{R}$, $\left|1 - e^{\mathrm{i}\theta}\right|^{1-m} \leqslant 2^{1-m}$, implying $\left|1 - e^{\mathrm{i}\theta}\right| \leqslant 2^{1-m}\left|1 - e^{\mathrm{i}\theta}\right|^m$, therefore Step 2.
Step 3 : $\forall (z_1, z_2) \in \mathbb{C} \setminus \{0\} \times \mathbb{C}$, $\left||z_2| - \frac{\overline{z_1}}{|z_1|}z_2\right|^m \leqslant 2^m|z_1 - z_2|^m$.
We have,

$$\begin{aligned}
&\left||z_2| - \frac{\overline{z_1}}{|z_1|}z_2\right| = \left|\left(|z_2| - \frac{\overline{z_1}}{|z_1|}z_1\right) + \left(\frac{\overline{z_1}}{|z_1|}z_1 - \frac{\overline{z_1}}{|z_1|}z_2\right)\right| \\
&= \left|\big(|z_2| - |z_1|\big) + \left(\frac{\overline{z_1}}{|z_1|}z_1 - \frac{\overline{z_1}}{|z_1|}z_2\right)\right| \leqslant \big||z_2| - |z_1|\big| + |z_1 - z_2| \leqslant 2|z_1 - z_2|.
\end{aligned}$$

Hence Step 3.
Step 4 : Conclusion.
Let $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ with $z_1 z_2 \neq 0$, otherwise there is nothing to prove.

$$\begin{aligned}
&\Big||z_1|^{-(1-m)}z_1 - |z_2|^{-(1-m)}z_2\Big| = \left||z_1|^{-(1-m)}z_1\frac{\overline{z_1}}{|z_1|} - |z_2|^{-(1-m)}z_2\frac{\overline{z_1}}{|z_1|}\right| \\
&= \left|\big(|z_1|^m - |z_2|^m\big) + \left(|z_2|^m - |z_2|^m\frac{\overline{z_1}}{|z_1|}\frac{z_2}{|z_2|}\right)\right| \overset{\text{Steps 1 and 2}}{\leqslant} |z_1 - z_2|^m + 2^{1-m}\left||z_2| - |z_2|\frac{\overline{z_1}}{|z_1|}\frac{z_2}{|z_2|}\right|^m \\
&= |z_1 - z_2|^m + 2^{1-m}\left||z_2| - \frac{\overline{z_1}}{|z_1|}z_2\right|^m \overset{\text{Steps 3}}{\leqslant} 3|z_1 - z_2|^m.
\end{aligned}$$

The lemma is proved. □

The next lemmas are, more or less, a repetition of the unpublished book of Brezis and Cazenave [45].

Lemma B.2. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty open subset, let $k, m \in \mathbb{N}_0$ and let $1 \leqslant p, q < \infty$. Then $\mathscr{D}(\Omega) \hookrightarrow W_0^{k,p}(\Omega) \cap W_0^{m,q}(\Omega)$ with dense embedding. In addition, $W_0^{k,p}(\Omega) \cap W_0^{m,q}(\Omega)$ is separable and,*

$$\big(W_0^{k,p}(\Omega) \cap W_0^{m,q}(\Omega)\big)^\star = W^{-k,p'}(\Omega) + W^{-m,q'}(\Omega) \hookrightarrow \mathscr{D}'(\Omega). \tag{B.2}$$

Finally, if $p, q > 1$ then $W_0^{k,p}(\Omega) \cap W_0^{m,q}(\Omega)$ and $W^{-k,p'}(\Omega) + W^{-m,q'}(\Omega)$ are reflexive and separable.

Proof. Set $X = W_0^{k,p}(\Omega) \cap W_0^{m,q}(\Omega)$. Without loss of generality, we may assume that $p \leqslant q$. It is clear that $\mathscr{D}(\Omega) \hookrightarrow X$. The equality in (B.2) comes from the density of $\mathscr{D}(\Omega)$ in the spaces $W_0^{j,r}(\Omega)$

and Bergh and Löfström [35] (Lemma 2.3.1 and Theorem 2.7.1). Since for any $j \in \mathbb{N}_0$ and $r \in [1, \infty)$, $W^{-j,r'}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$, we have by the equality in (B.2),

$$X^\star = \left\{ T \in \mathscr{D}'(\Omega); T = T_1 + T_2, (T_1, T_2) \in W^{-k,p'}(\Omega) \times W^{-m,q'}(\Omega) \right\}.$$

Let $T \in X^\star$ be such that $\langle T, \varphi \rangle_{X^\star, X} = 0$, for any $\varphi \in \mathscr{D}(\Omega)$. It follows from above that for any $\varphi \in \mathscr{D}(\Omega)$, $\langle T, \varphi \rangle_{\mathscr{D}'(\Omega), \mathscr{D}(\Omega)} = \langle T, \varphi \rangle_{X^\star, X} = 0$. Then $T = 0$ in $\mathscr{D}'(\Omega)$, hence in $X^\star$. We deduce that $\mathscr{D}(\Omega) \hookrightarrow X$ is dense (Brezis [44], Corollary 1.8) and so $X^\star \hookrightarrow \mathscr{D}'(\Omega)$. Now, let $n > k + m$ be large enough to have $W_0^{n,p}(\Omega) \hookrightarrow X$. Since this embedding is dense and $W_0^{n,p}(\Omega)$ is separable, we infer that X is separable. Finally, separability and reflexivity of the last part of the lemma present no difficulty and follow easily from reflexivity and separability of the spaces $W_0^{j,r}(\Omega)$, (B.2) and Eberlein–Šmulian's Theorem (Brezis [44], Theorem 3.19 and Corollary 3.27). □

Lemma B.3 ([45])**.** *Let $I \subseteq \mathbb{R}$ be an open interval, let $1 \leqslant p, q < \infty$ and let $X \hookrightarrow Y$ be two Banach spaces. Then $\mathscr{D}(\overline{I}; X)$ is dense in $L^p(I; X) \cap W^{1,q}(I; Y)$. Moreover, if Z is a Banach space such that $Z \hookrightarrow X$ with dense embedding then $\mathscr{D}(\overline{I}; Z)$ is dense in $L^p(I; X) \cap W^{1,q}(I; Y)$.*

Proof. We first construct a linear extension operator to bring back to the case $I = \mathbb{R}$. The first statement then follows from the standard procedure of truncation and regularization, while the second statement comes from the density of $\mathscr{D}(\mathbb{R}; Z)$ in $C_\mathrm{c}^1(\mathbb{R}; X)$, for the norm of $C_\mathrm{b}^1(\mathbb{R}; X)$. □

Lemma B.4. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset. We consider below the following Hilbert space $D(A)$.*

$$D(A) = \left\{ u \in H_0^1(\Omega);\ \Delta u \in L^2(\Omega) \right\},$$
$$\|u\|_{D(A)}^2 = \|u\|_{H_0^1(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2,$$

for any $u \in D(A)$. Let X be a Banach space, let I be an open interval and let $1 < p < \infty$. We have the following results.

1) $W^{1,1}\big(I; X\big) \hookrightarrow C_{\mathrm{b,u}}\big(\overline{I}; X\big)$.
2) $L^p(I; X) \cap W^{1,p'}(I; X^\star) \hookrightarrow C_\mathrm{b}\big(\overline{I}; L^2(\Omega)\big)$, *if $X \hookrightarrow L^2(\Omega)$ with dense embedding.*
3) $L^p\big(I; D(A)\big) \cap W^{1,p'}\big(I; L^2(\Omega)\big) \hookrightarrow C_\mathrm{b}\big(\overline{I}; H_0^1(\Omega)\big)$.

Lemma B.5. *Let $\Omega \subseteq \mathbb{R}^N$ be an open subset, let I be an open interval and let $1 < p < \infty$. For $t \in I$ and $u = u(t, x) \in \mathbb{C}$, let us define (formally),*

$$M(t) = \frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 \ \ \text{and} \ \ E(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2(\Omega)}^2.$$

Let $D(A)$ be the Hilbert space be defined in Lemma B.4 and let $X \hookrightarrow L^2(\Omega)$ be a Banach space with dense embedding. We then have the following results.

1) *If $u \in L^p(I; X) \cap W^{1,p'}(I; X^\star)$ or if $u \in W^{1,1}(I; L^2(\Omega))$ then $M \in W^{1,1}(I; \mathbb{R})$ and,*

$$M'(t) = \begin{cases} \big\langle u(t), u'(t) \big\rangle_{X, X^\star}, & \text{if } u \in L^p(I; X) \cap W^{1,p'}(I; X^\star), \\ \big(u(t), u'(t)\big)_{L^2(\Omega)}, & \text{if } u \in W^{1,1}(I; L^2(\Omega)), \end{cases} \tag{B.3}$$

for almost every $t \in I$.

2) *If* $u \in L^p(I;D(A)) \cap W^{1,p'}(I;L^2(\Omega))$ *then* $E \in W^{1,1}(I;\mathbb{R})$ *and,*

$$E'(t) = \big(-\Delta u(t), u'(t) \big)_{L^2(\Omega)}, \tag{B.4}$$

for almost every $t \in I$.

Proof of Lemmas B.4 and B.5. The proof of the embedding $W^{1,1}\big(I;X\big) \hookrightarrow C_{\mathrm{b},\boldsymbol{u}}\big(\overline{I};X\big)$ is very standard and we omit its proof. Now, assume that $X \hookrightarrow L^2(\Omega)$ with dense embedding. For any $v \in L^2(\Omega)$ and $\varphi \in X$, let $\Phi_v(\varphi) = (v,\varphi)_{L^2(\Omega)}$. Since $X \hookrightarrow L^2(\Omega)$, it is clear that $\Phi \in \mathscr{L}\big(L^2(\Omega);X^\star\big)$. The embedding $X \hookrightarrow L^2(\Omega)$ being dense, we easily show that Φ is injective. Identifying Φ_v with v, it follows that $L^2(\Omega) \hookrightarrow X^\star$ and for any $v \in L^2(\Omega)$ and $\varphi \in X$,

$$\langle v,\varphi\rangle_{X^\star,X} = \langle v,\varphi\rangle_{L^2(\Omega),L^2(\Omega)}.$$

In particular, if $v \in L^2(\Omega)$ then $\langle v,v\rangle_{X^\star,X} = \|v\|^2_{L^2(\Omega)}$. We then note that $M \in C^1(\overline{I};\mathbb{R})$, $E \in C^1(\overline{I};\mathbb{R})$ and,

$$M(t) = M(s) + \int_s^t \big\langle u(\sigma), u'(\sigma) \big\rangle_{X,X^\star} \mathrm{d}\sigma, \tag{B.5}$$

$$E(t) = E(s) + \int_s^t \big(-\Delta u(\sigma), u'(\sigma) \big)_{L^2(\Omega)} \mathrm{d}\sigma, \tag{B.6}$$

for any $t,s \in \overline{I}$, as soon as $u \in \mathscr{D}(\overline{I};X)$, for (B.5) and $u \in \mathscr{D}(\overline{I};D(A))$, for (B.6). Applying Hölder's inequality in time and Young's inequality, one obtains,

$$\begin{aligned}
\|u(t)\|^2_{L^2(\Omega)} &\leqslant \|u(s)\|_X \|u(s)\|_{X^\star} + \|u\|^2_{L^p(I;X)} + \|u'\|^2_{L^{p'}(I;X^\star)}, \\
\|\nabla u(t)\|^2_{L^2(\Omega)} &\leqslant \|u(s)\|_{L^2(\Omega)} \|\Delta u(s)\|_{L^2(\Omega)} + \|\Delta u\|^2_{L^p(I;L^2)} + \|u'\|^2_{L^{p'}(I;L^2)},
\end{aligned} \tag{B.7}$$

for any $t,s \in \overline{I}$. Let $(I_n)_{n\in\mathbb{N}} \subset I$ be a increasing[3] sequence of open bounded intervals such that $\bigcup_{n\in\mathbb{N}} I_n = I$. Integrating in s and applying, one more time, Hölder's and Young's inequalities, we have,

$$|I_n|\, \|u\|^2_{C_\mathrm{b}(\overline{I_n};L^2)} \leqslant (1+|I_n|) \Big(\|u\|_{L^p(I;X)} + \|u\|_{W^{1,p'}(I;X^\star)} \Big)^2,$$

for any $n \in \mathbb{N}$. Dividing by $|I_n|$, letting $n \nearrow \infty$ and proceeding in the same way in (B.7), we arrive at,

$$\|u\|_{C_\mathrm{b}(\overline{I};L^2)} \leqslant (1+|I|^{-\frac12}) \Big(\|u\|_{L^p(I;X)} + \|u\|_{W^{1,p'}(I;X^\star)} \Big), \tag{B.8}$$

$$\|\nabla u\|_{C_\mathrm{b}(\overline{I};L^2)} \leqslant (1+|I|^{-\frac12}) \Big(\|u\|_{L^p(I;D(A))} + \|u\|_{W^{1,p'}(I;L^2)} \Big), \tag{B.9}$$

with the convention $|I|^{-\frac12} = 0$, if $|I| = \infty$. Since $X \hookrightarrow X^\star$ and $D(A) \hookrightarrow L^2(\Omega)$, we prove Lemma B.4 by density with (B.8)–(B.9) (Lemma B.3). Finally, Lemma B.5 is a consequence of (B.5)–(B.6) and Lemmas B.3–B.4. □

3. in the sense of the inclusion.

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