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Scope of Innovations, Knowledge Spillovers and Growth

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Abstract

This paper exploits the formalization of a circular product differentiation model of Salop (1979) to propose an endogenous growth quality ladder model in which the knowledge inherent in a given sector can spread variously across the sectors of the economy, ranging from local to global influence. Accordingly, this affects the size of the pool of knowledge in which innovations draw themselves on in order to be produced. Therefore, the law of knowledge accumulation, and thus the growth rate of the economy, depend positively on the expected scope of diffusion of innovations, *i.e.* on the intensity of knowledge spillovers. This approach generalizes the endogenous growth theory as developed in the seminal models of Grossman & Helpman (1991) and Aghion & Howitt (1992), extending their analysis to the possibility of considering stochastic and partial knowledge spillovers.

This framework allows us to mitigate the positive externality of knowledge and thus to apprehend the issue of the funding of research with more parsimony. We characterize the set of steady-state Schumpeterian equilibria as a function of the public tools. We provide an explanation for the fact that research effort can either be suboptimal or over-optimal, depending on the expected scope of knowledge. Accordingly, we find that the optimal public tool dedicated to foster R&D activity depends positively on it.

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Keywords: Schumpeterian Growth, Scope of Diffusion of Innovations, Knowledge Spillovers, Pareto Optimality, Distortions, R&D Public Policy.

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1 Introduction

The innovation-based growth theory initialized by the seminal works of Romer (1990), Grossman & Helpman (1991), and Aghion & Howitt (1992), considers technological progress to be at the source of growth process. How should economists think about technological progress is still an ongoing question. The prevalent approach has been to view it as an incremental process that improves the efficiency of inputs deployment. Moreover, it is generally agreed that the source of technological progress is the accumulation of knowledge which consists in a succession of innovations which diffuse among the various R&D activities in the economy. The diffusion of innovations enables the creation of new knowledge, establishing a virtuous circle. All innovations, though, do not have the same scope. In any given economic era there are major technological innovations, such as electricity, the transistor, or Internet, that have a far-reaching impact, spreading their influence globally in the economy. One often refers to them as radical innovations. On the contrary, other innovations can be very particular to a given sector of activity or diffuse quite locally[1]. The scope of innovations hence appears to be a keystone in the complex mechanism of knowledge accumulation and hence in the growth process.

This work presents a Schumpeterian endogenous growth quality ladder model in which R&D activities produce innovations that can appear to have miscellaneous scopes of influence on the output of the other R&D activities. The diffusion of the knowledge inherent in a given intermediate sector is uncertain in the sense that it can spill variously across the sectors of the economy, ranging from punctual to global influence. Accordingly, this affects the size of the pool of knowledge in which innovations draw themselves on in order to be produced, and therefore the law of motion of knowledge accumulation which depends on the expected scope of diffusion of innovations.

We exploit the formalization of a circular product differentiation model of Salop (1979) in order to generalize the Schumpeterian approach as developed in the seminal models of Grossman & Helpman (1991) and Aghion & Howitt (1992). We extend their analysis to the possibility of stochastic and partial intersectoral knowledge spillovers. As anticipated, the growth rate of the economy depends positively on the expected scope of innovations, that is on the intensity of knowledge spillovers.

Allowing for knowledge to diffuse across the economy's R&D activities with various extents enables us to obtain variegated endogenous growth growth models. In particular, considering a framework in which there are no intersectoral knowledge spillovers at all (*i.e.* considering "inside sector knowledge" only), one gets a model close in spirit to the one developed by Grossman & Helpman (1991). In the polar case in which intersectoral knowledge spillovers are total, one obtains a model *à la* Aghion & Howitt (1992). This framework highlights the complexity of the externality entailed by knowledge spillovers. Indeed, allowing to mitigate the scope of knowledge enables us to apprehend the issue of the funding of research with more parsimony. It has been pointed out, in the traditional literature, that, in endogenous growth models, R&D effort can be suboptimal or over-optimal de-

[1]Mokyr (1990) refers respectively to "macro" and "micro" innovations.

pending on the values of the models parameters. As stated by Benassy (1998), this is in particular true for Schumpeterian models[2].

We show that, depending on the expected scope of diffusion of innovations among R&D activities, the equilibrium growth rate of the economy can either be higher or lower than the Pareto optimal one. In particular, the wider the scope of knowledge, the more likely the R&D effort will be below its optimal level and, therefore, the more likely the economy's growth rate will be suboptimal. This finding is corroborated by the fact that when implementing the first-best, the optimal tool used to correct the externality triggered by knowledge spillovers depends positively on the expected scope of diffusion of knowledge and can either be a subsidy or a tax, depending on the extent to which innovations spread their influence among R&D activities. Moreover, we determine the threshold scope of diffusion above which the growth rate of the economy is suboptimal and below which it is over-optimal.

The paper is organized as follows. In section 2, we present the model; we especially describe the formalization of knowledge accumulation and the way it diffuses among R&D activities. Furthermore, we characterize the optimum. Section 3 focuses on the decentralized economy with creative destruction. We characterize the set of equilibria as a function of the public tools; that is, at each vector of public tools is associated a particular equilibrium. In section 4, we study the distortions that prevent the decentralized economy from being Pareto optimal. This underlines the need for public intervention to sustain appropriate R&D activity as well as the complexity of the task. Finally, we compute the public tools implementing the first-best optimum. In Section 5, we present the two polar cases of endogenous growth models considering respectively no intersectoral knowledge spillovers at all, and total knowledge spillovers. Section 6 deals with the issue of scale-effects. We conclude in Section 7.

2 Model and Welfare

In this section, we present the model. We describe R&D activity, paying particular attention to the process of knowledge creation. We specify the way knowledge spreads among the R&D activities and how the corresponding scope of innovations influences the economy's R&D output. Then, we characterize the first-best optimum.

2.1 R&D Activities, Innovations' Scope of Influence and Knowledge Spillovers

As in the standard Schumpeterian endogenous growth theory[3], there is a continuum Ω of sectors producing intermediate goods. Let N be the measure of this set. Whereas this theory generally considers that these intermediate sectors are lo cated

[2]Benassy (1998) shows that it is also the case for endogenous growth models with expanding product variety such as the one developed by Romer (1990).

[3]See for instance Grossman & Helpman (1991), Aghion & Howitt (1992), Aghion & Howitt (1998), chapters 2 and 3 or Aghion & Howitt (2009), chapter 4.

on a linear space, we differentiate ourselves considering a circular localization. Each intermediate sector ω, $\omega \in \Omega$, is located on a clockwise oriented circle of perimeter N. We assume an uniform distribution; in this case, the intermediate sectors space is completely homogenous.

At each instant t, each intermediate sector ω, $\omega \in \Omega$, is characterized by an intermediate good ω, produced in quantity x_ω, and by a level of knowledge $\chi_{\omega t}$. It has its own R&D activity[4] dedicated to the production of innovations.

As in the seminal endogenous growth literature, every innovation created in a given intermediate sector is embodied in the private intermediate good produced by this sector, upgrading its quality while successively increasing the amount of knowledge inherent in this intermediate sector. This accounts for the high cumulativeness of knowledge, as argued by Green & Scotchmer (1995): "knowledge and technical progress are cumulative in the sense that products are often the result of several steps of invention, modification, and improvement".

Accordingly, we define the whole disposable knowledge in the economy as:

$$\mathcal{K}_t = \int_\Omega \chi_{\omega t}\, d\omega \tag{1}$$

Let us now describe the mechanism at the source of the creation of knowledge. It relies on three core assumptions.

Assumption 1: Poisson arrival rate of innovations.

It is commonly agreed that innovation process is uncertain; R&D activities are subject to stochastic output stream. In this respect, we follow Aghion & Howitt (1992). We assume that, for any intermediate good ω, $\omega \in \Omega$, innovations occurrence follows a Poisson process characterized by an arrival rate $\lambda l_{\omega t}$, where $\lambda > 0$ is a parameter indicating the productivity of the R&D, and $l_{\omega t}$ is the amount of labor devoted to move on to the next generation of intermediate good ω, *i.e.* the overall labor used in research within intermediate sector ω.

Therefore, the overall amount of labor dedicated to research in the economy is:

$$L_t^R = \int_\Omega l_{\omega t}\, d\omega \tag{2}$$

Assumption 2: Miscellaneous scope of innovations.

What follows tackles the issue of knowledge spillovers in research activities, cornerstone of the innovation-based endogenous growth theory developed by Romer (1990) or Aghion & Howitt (1992). This theory relies on the fact that R&D activities influence one another. In this tradition, we take into account the fact that new knowledge spreads across the economy "through a process in which one sector gets ideas from the experience of others"[5]. In the standard theory, whether one considers product-variety models or quality ladder models *à la* Aghion & Howitt, intersectoral knowledge spillovers are basically assumed to be *certain* and *complete*. Indeed, the commonly shared assumption consists in that all innovations

[4]From now on, we will refer to the R&D activity dedicated to improve intermediate good ω as "R&D activity ω".

[5]See Aghion & Howitt (1998), chapter 3, page 85.

diffuse across the whole economy and are used by all R&D activities in order to produce new innovations. Accordingly, R&D activities draw on the same pool of shared technological knowledge; this pool is represented by all the knowledge accumulated so far[6], *i.e.* all the innovations created hitherto.

Our aim, in this model, is to allow for possible *stochastic* and *partial* knowledge spillovers among R&D activities. In this respect, one has to consider two matters. Firstly, one has to deal with the way knowledge diffuses among the economy's R&D activities; that is, with the *scope of influence of innovations.* Secondly, and consequently, one has to give consideration to what is the resulting pool of knowledge which is used by each R&D activity. In the following, the index h, $h \in \Omega$, is used to point out any location from which knowledge χ_h diffuses; the index ω, $\omega \in \Omega$, is used to point out the location of the previously evoked pool of knowledge.

As regards to the scope of influence of the knowledge characteristic to a given intermediate good, in the attempt to consider stochastic and partial knowledge spillovers, we assess that knowledge can spread its influence miscellaneously across the economy's R&D activities. An innovation can either be specific to the intermediate good in which it is embodied, or it can diffuse locally to R&D activities closely located, or propagate more broadly, on a larger set of R&D activities. We will respectively refer to these three types of innovations as to "sector specific innovations" (or "inside sector innovations"), "narrow innovations" and "wide innovations"[7].

We assume that, for any intermediate good h, $h \in \Omega$, an innovation can consist of "inside sector knowledge" with probability p_0, or in knowledge which diffuses on its right and on its left symmetrically over Ω with more or less extent. Formally, denoting respectively by $\underline{\theta}$ and $\overline{\theta}$ the scope of narrow innovations and of wide innovations, knowledge spills on a narrow neighborhood of measure $\underline{\theta}$ with probability p_n (index n for "narrow"), and on a wider neighborhood of measure $\overline{\theta}$ with probability p_W (index W for "wide"), where $p_0 + p_n + p_W = 1$ and $1 < \underline{\theta} \leqslant \overline{\theta} \leqslant N$. The two corresponding neighborhoods of diffusion of knowledge inherent in intermediate sector h, $h \in \Omega$, are $\underline{\Omega^h} \equiv [h - \underline{\theta}/2\,;\, h + \underline{\theta}/2]$ and $\overline{\Omega^h} \equiv \left[h - \overline{\theta}/2\,;\, h + \overline{\theta}/2\right]$, where $\underline{\Omega^h} \subseteq \overline{\Omega^h} \subseteq \Omega$. Basically, in this model, the scope of any innovation is a random variable θ which can take three values: 1, with probability p_0 (in the case of a sector specific innovation), $\underline{\theta}$, with probability p_n (in the case of a narrow innovation), or $\overline{\theta}$, with probability p_W (in the case of a wide innovation). Then, the expected scope of diffusion of any innovation is:

$$\mathbb{E}\left[\theta\right] = p_0 + p_n\underline{\theta} + p_W\overline{\theta} \tag{3}$$

Depending on the nature of knowledge spillovers considered, *i.e.* on the set of parameters $\left(p_0, p_n, p_W, \underline{\theta}, \overline{\theta}\right)$ chosen[8], this formalization enables us to deal with an infinity of cases, stochastic or not, with more or less (or even without) diffusion of innovations among sectors. In particular, in its version with non-stochastic

[6]See for instance Aghion & Howitt (2009), chapters 3 and 4, or Acemoglu (2009) Chapter 13.

[7]In the limit case in which wide innovations spread their influence to the overall economy, one generally talks about general purpose technologies, we will refer to this particular case as to "general knowledge".

[8]Note that this set of parameters comprise in fact only four independent elements; indeed, since $p_0 + p_n + p_W = 1$, once two probabilities are chosen, the third one is given. However, for a purpose of clarity, we decided to include all five elements in the set.

spillovers, one can get a variety of models ranging between two polar cases, each of which echoes back to a seminal endogenous growth model. On one end, for $p_0 = 1$ and $p_n = p_W = 0$, one gets an "inside sector knowledge" endogenous growth model in which there are no intersectoral knowledge spillovers. This model is close, in the spirit, to the quality ladders growth model of Grossman & Helpman (1991). The other polar case, is obtained for $p_0 = p_n = 0$, $p_W = 1$ and $\overline{\theta} = N$. In this model, there are only innovations spreading across the whole economy, being used by all R&D activities. This "general knowledge" endogenous growth model corresponds closely to the seminal model developed by Aghion & Howitt (1992), in the sense that it assumes certain and complete intersectoral knowledge spillovers.

Now that we have introduced some uncertainty and incompleteness in the way knowledge diffuses among the different R&D activities, let us move on to the characterization of the resulting pool of knowledge in which innovations are drawn. In this respect, we consider that each R&D activity ω, $\omega \in \Omega$, in order to produce knowledge $\chi_{\omega t}$, draws from a pool of knowledge, $\mathcal{P}_t^\omega$, which is composed of all the knowledge reaching the location of intermediate good ω. In other words, any given R&D activity ω, $\omega \in \Omega$, can only make use of innovations created by R&D activities h, $h \in \Omega$, whose scopes of influence include its location. Accordingly, and under assumption 2, one gets the expression of $\mathcal{P}_t^\omega$, given in Lemma 1 below.

Lemma 1: *At each instant t, in any intermediate sector ω, $\omega \in \Omega$, the expected pool of knowledge in which the corresponding R&D activities draw from in order to produce innovations is:*

$$\mathcal{P}_t^\omega = (1 - p_n - p_W)\chi_{\omega t} + p_n \int_{\underline{\Omega^\omega}} \chi_{ht}\, dh + p_W \int_{\overline{\Omega^\omega}} \chi_{ht}\, dh \, , \, \forall \omega \in \Omega \tag{4}$$

The proof is the following. As stated above, any given innovation h, $h \in \Omega$, consists either in a sector specific innovation with probability $p_0 = 1 - p_n - p_W$, in a narrow innovation with probability p_n, or in a wide innovation with probability p_W.

In the first case, the only sector specific innovation h, $\forall h \in \Omega$, reaching the location of R&D activity ω is innovation ω itself. The corresponding amount of knowledge captured by R&D activity ω is then $\chi_{\omega t}$.

In the second case, solely narrow innovations h which are located in the nearby neighborhood $\underline{\Omega^\omega}$ can get to R&D activity ω. The consequent amount of knowledge related to narrow innovations which can be used by R&D activity ω is then $\int_{\underline{\Omega^\omega}} \chi_{ht}\, dh$.

Finally, all wide innovations h which are located in the neighborhood $\overline{\Omega^\omega}$ attain R&D activity ω. Thus, the amount of knowledge emitted by wide innovations and received by R&D activity ω is $\int_{\overline{\Omega^\omega}} \chi_{ht}\, dh$.

Consequently, the expected total amount of knowledge used by any R&D activity $\omega \in \Omega$ is given by the expression (4) above.□

Allowing for uncertainty and partialness in the diffusion of knowledge, we depart from what is generally done in the standard literature on several respects. As exhibited in Lemma 1, introducing miscellaneous scopes of influence of innovations has a critical aftermath on the composition of the pool of knowledge in which a given R&D activity draws new innovations from. Moreover, given that the knowledge inherent in each intermediate sector varies from one sector to the other, these pools are possibly heterogenous. Finally, this formalization constitutes a general framework of endogenous growth, enabling us to consider a large collection of models, according to the set of parameters $\left(p_0, p_n, p_W, \underline{\theta}, \overline{\theta}\right)$ chosen.

Assumption 3: The third assumption concerns the magnitude of the knowledge increase when a new quality of good is achieved. For any intermediate good ω, $\omega \in \Omega$, if an innovation occurs at instant t, the induced increase in knowledge, $\Delta\chi_{\omega t}$, is a linear function of $\mathcal{P}_t^{\omega}$:

$$\Delta\chi_{\omega t} = \sigma \mathcal{P}_t^{\omega} \text{ , } \forall \omega \in \Omega, \ \sigma > 0$$

This specification implies that there is a quality ladder for each intermediate good; each innovation takes the intermediate good quality up by one rung on this ladder. At instant t, the quality improvement of a given intermediate good is proportional to the current size of the expected pool of knowledge in which this sector's R&D activities draw from in order to produce innovations. The parameter σ is a measure of the productivity of R&D activities in the economy in the sense that, the larger its value, the higher the increase in quality for a given size of the pool.

From the previous assumptions, one derives the law of motion of knowledge inherent in each intermediate good as expressed in Proposition 1 below.

Proposition 1: *The law of motion of the average knowledge characterizing any intermediate good ω is:*

$$\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \mathcal{P}_t^{\omega} \text{ , } \forall \omega \in \Omega \tag{5}$$

Proof, see Appendix 8.1.[9]

In order to illustrate expressions (4) and (5), let us get back to the two polar cases mentioned above[10]:

- In the "inside sector knowledge" endogenous growth model ($p_0 = 1$ and $p_n = p_W = 0$), the pool of knowledge used by R&D activity in any sector ω, $\omega \in \Omega$, is $\mathcal{P}_t^{\omega} = \chi_{\omega t}$, that is, only the knowledge produced within the sector. The corresponding law of motion of the average knowledge characterizing any intermediate good ω, $\omega \in \Omega$, is:

$$\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \chi_{\omega t}$$

- In the "general knowledge" endogenous growth model ($p_0 = p_n = 0$, $p_W = 1$ and $\overline{\theta} = N$), the pool of knowledge used by R&D activity in any sector ω,

[9]This methodology is similar to the one used in Grimaud & Rouge (2004).

[10]We will go back to those particular cases in section 5.

$\omega \in \Omega$, is $\mathcal{P}_t^\omega = \int_{\omega-N/2}^{\omega+N/2} \chi_{ht}\, dh = \int_\Omega \chi_{ht}\, dh = \mathcal{K}_t$, that is, the whole disposable knowledge in the economy. Replacing in (5), one gets:

$$\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \mathcal{K}_t \; , \; \forall \omega \in \Omega \tag{6}$$

It is noteworthy that this law of motion, which is endogenously derived from assumptions made in a stochastic quality ladders model, leads to a law of motion of the whole disposable knowledge which is formally identical to the one initially assumed by Romer (1990). Indeed, differentiating (1) with respect to time and using (6) yields:

$$\dot{\mathcal{K}}_t = \int_\Omega \dot{\chi}_{\omega t}\, d\omega = \lambda\sigma \left(\int_\Omega l_{\omega t} d\omega \right) \mathcal{K}_t \Leftrightarrow \dot{\mathcal{K}}_t = \lambda \sigma L_t^R \mathcal{K}_t \; ,$$

where L_t^R is defined in (2) above.

2.2 Labor, Final Good, Intermediate Goods, and Preferences

Each household is endowed with one unit of labor that is supplied inelastically. Total labor, L, is assumed to be constant and has two competing uses at each instant t: it can be used to produce the final good, Y_t, and in R&D activities. This yields the following labor constraint:

$$L = L_t^Y + L_t^R \; , \tag{7}$$

where, L_t^Y is the amount of labor used in the final good sector, and L_t^R, the overall amount of labor dedicated to research in the economy, as defined in (2).

The production of the final good makes use of the particular knowledge of every intermediate goods, according to:

$$Y_t = (L_t^Y)^{1-\alpha} \int_\Omega \chi_{\omega t} (x_{\omega t})^\alpha d\omega \; , \; 0 < \alpha < 1, \tag{8}$$

where, as mentioned above, $x_{\omega t}$ is the quantity of intermediate good ω used at instant t and $\chi_{\omega t}$ the corresponding quality.

The final good is used for the households consumption and for the production of intermediate goods. Denoting by $y_{\omega t}$ the quantity of final good used to produce $x_{\omega t}$ units of intermediate good ω, and by c_t the consumption of the representative household, one gets the following output resource constraint:

$$Y_t = Lc_t + \int_\Omega y_{\omega t} d\omega \tag{9}$$

Each intermediate good is produced using final good as an input along with:

$$x_{\omega t} = \frac{y_{\omega t}}{\chi_{\omega t}} \; , \; \omega \in \Omega \tag{10}$$

This formulation illustrates the increasing complexity in the production of intermediate goods. That is, it takes into account the fact that, as the quality of a given intermediate good increases, more resources need to be devoted to its production.

Finally, intertemporal preferences of the representative household are given by:

$$\mathcal{U} = \int_0^{\infty} \ln(c_t) e^{-\rho t} dt \quad , \tag{11}$$

where $\rho > 0$ denotes the rate of time preferences[11].

2.3 Optimum

Let us now compute the optimum of the model, that is, the solution of the social planner's program. He maximizes (11) subject to (1), (4), (5), (7), (8), (9) and (10). As in the standard literature, we consider the symmetric case in which intermediate sectors firms are identical. We denote by g_{z_t} the rate of growth, $\dot{z}_t / z_t$, of any variable z_t. The steady-state optimum is characterized in Proposition 2 below.

Proposition 2: *At the steady-state optimum, the repartition of labor is*:

$$L^{Ropt} = L - \frac{\rho N}{\lambda \sigma \mathbb{E}[\theta]} \text{ and } L^{Y opt} = L - L^{Ropt} = \frac{\rho N}{\lambda \sigma \mathbb{E}[\theta]} \ .$$

The quantity of each intermediate good $\omega \in \Omega$ is:

$$x_{\omega}^{opt} = x^{opt} = \alpha^{\frac{1}{1-\alpha}} L^{Y opt} = \frac{\alpha^{\frac{1}{1-\alpha}} \rho N}{\lambda \sigma \mathbb{E}[\theta]} \ , \ \forall \omega \in \Omega \ ,$$

and the growth rates are:

$$g_c^{opt} = g_Y^{opt} = g_{\mathcal{X}}^{opt} = g_{\mathcal{K}}^{opt} = g^{opt} = \frac{\lambda \sigma \mathbb{E}[\theta] L}{N} - \rho \ ,$$

where, $\mathbb{E}[\theta]$ is given by (3), and g^{opt} denotes the optimal steady-state rate of growth of the economy.

Proof, see Appendix 8.2.

Let us give some comments on the results obtained in Proposition 2:

- For consistency, one has to assume that the parameters of the model are such that $\mathbb{E}[\theta] > \frac{\rho N}{\lambda \sigma L}$. This is to ensure that there is positive growth at the first-best optimum (*i.e.* that $g^{opt} > 0$).
- As anticipated, the optimal growth rate of the economy depends positively on the expected scope of innovations $\mathbb{E}[\theta]$, that is on the intensity of knowledge spillovers which plays the same role as the parameters λ and σ. All three of them account for the level of productivity of R&D activities in the economy, each of which referring to a particular dimension. Indeed, λ gives the efficiency of the labor devoted to R&D activity in developing new innovations,

[11] The results are robust if one considers a C.E.S. instantaneous utility function of parameter $\varepsilon \in]0;1[$, $u(c_t) = \frac{c_t^{1-\varepsilon}}{1-\varepsilon}$.

σ indicates to which extent the pool of knowledge is used in the quality improvements, whereas $\mathbb{E}[\theta]$ stands for the average influence of innovations on the various R&D activities.

Moreover, L^{Ropt} is increasing in the overall level of productivity of research activities, while and $L^{Y\,opt}$ and x^{opt} are decreasing in it: the more efficient the R&D activities are, the more resources are allocated to research.

It appears clearly here that the engine of economic growth is technological progress in which the diffusion of knowledge within the R&D activities plays a key part.

- An important feature of seminal endogenous growth models is the presence of the property of scale-effects[12]: the larger the size of the population, the greater the growth rate. It has been argued by Jones (1995) and others, that this property is undesirable since this prediction is strongly at odds with twentieth century observed stylized facts[13]. The model we develop here also exhibits this non-desirable property. However, we will see in section 6 that it is possible to have a simple variant of this model in which scale-effects do not appear.

3 Decentralized Economy and Characterization of the Set of Equilibria

In this section, we study a decentralized economy which is in direct line with the analysis conducted by Aghion & Howitt (1992). We consider a Schumpeterian equilibrium with incomplete markets: there is no market for knowledge. Instead, R&D activities are privately and indirectly funded by monopoly profits on the sale of intermediate goods embodying the knowledge. We normalize the price of final good to one, and denote respectively by w_t, r_t and $q_{\omega t}$ ($\omega \in \Omega$) the wage, the interest rate and the price of intermediate good ω at instant t.

The final good market, the labor market and the financial market are perfectly competitive. Once invented, an intermediate good can be modified, improved as the result of several steps of innovations. Regarding their markets, we consider Schumpeter's well known "creative destruction" mechanism. It involves that, in a given intermediate sector, the firm that succeeds in innovating is granted a patent and can monopolize the intermediate good production and sale until replaced by the next innovator[14].

Because of the considered decentralized economy, there is potentially a divergence between the equilibrium allocation and the first-best optimal one. Indeed, in the presented framework, there are two sources of inefficiency. The first one results from the presence of monopolies on the production and sale of intermediate goods.

[12] In Gray & Grimaud (2010), we propose a double differentiation growth model which allows to shed a new light on the issue of scale-effects in endogenous growth theory.

[13] For an excellent overview and very accurate exposition of the growth theory related body of literature, see Jones (1999) or Dinopoulos & Sener (2007).

[14] This is the standard framework considered in the seminal endogenous growth developed by Aghion & Howitt (See for instance Aghion & Howitt (1992) or (1998, chapter 2)).

The second is related to the incompleteness of markets. There is thus a possibility of Pareto improving public policy interventions:

- The distortion entailed by monopolist behaviors can be mitigated by an add valorem subsidy ψ, $\psi \in [0;1]$, on each intermediate good demand. For convenience and to simplify expressions, let $s = 1 - \psi$. By abuse of notation, we will refer to the subsidy as to s. Accordingly, note that, $s = 1$ corresponds to no subsidy (*i.e.* to a *"laisser faire"* policy regarding the monopoly distortion, as seen in expression of the final sector profit (12)) and that the subsidy increases as s decreases.

- The externality triggered by the fact that there is no market for knowledge can be corrected by a public tool φ devoted to R&D activity, which can be positive or negative, depending on whether the R&D effort is suboptimal or over-optimal. For convenience, let $\mathfrak{T} = 1 + \varphi$. $\mathfrak{T}$ can thus consist in a subsidy on the monopoly profit (if $\mathfrak{T} > 1$), as well as in a tax imposed on it (if $\mathfrak{T} < 1$).

This section is organized as follows. Firstly, we describe the behavior of the different agents of the economy. Then, we characterize the set of steady-state symmetric equilibria as a function of the public tools vector $(s, \mathfrak{T})$.

The purpose of this section is that it will enable us, in the next section, to study the distortions and public policies mentioned above as well as to compute the vector of the tools which implement the first-best optimum within this decentralized economy.

3.1 Agents Behavior

We are now to analyse the individual behavior in order to characterize the set of equilibria in the decentralized economy. At each vector of public policies $(s, \mathfrak{T})$ is associated a particular equilibrium.

Formally, given that the price of final good is normalized to one, an *equilibrium* is represented as time paths of set of prices $\left\{\left(\{q_{\omega t}\}_{\omega\in\Omega}, w_t, r_t\right)\right\}_{t=0}^{\infty}$ and of quantities $\left\{\left(c_t, Y_t, \{x_{\omega t}\}_{\omega\in\Omega}, L_t^Y, \{l_{\omega t}\}_{\omega\in\Omega}, \{\chi_{\omega t}\}_{\omega\in\Omega}, \mathcal{K}_t\right)\right\}_{t=0}^{\infty}$, such that: the final good market, the labor market and the financial market are perfectly competitive and clear; on each intermediate good market, the incumbent monopolizes the production and sale until replaced by the next innovator; there is free entry on each R&D activity (*i.e.* the zero profit condition holds for each R&D activity); firms maximize their profits and the representative household maximizes her utility.

In the final sector, the competitive firm maximizes its profit given by:

$$\pi_t^Y = (L_t^Y)^{1-\alpha} \int_\Omega \chi_{\omega t}(x_{\omega t})^\alpha d\omega - w_t L_t^Y - \int_\Omega s q_{\omega t} x_{\omega t} d\omega \qquad (12)$$

The first-order conditions yield:

$$w_t = (1-\alpha)\frac{Y_t}{L_t^Y} \quad \text{and} \quad q_{\omega t} = \frac{\alpha(L_t^Y)^{(1-\alpha)}\chi_{\omega t}(x_{\omega t})^{\alpha-1}}{s}\ ,\ \forall\omega\in\Omega \qquad (13)$$

In each intermediate good sector ω, $\omega \in \Omega$, the incumbent monopoly maximizes its profit $\pi_t^{x_\omega} = q_{\omega t}x_{\omega t} - y_{\omega t} = (q_{\omega t} - \chi_{\omega t})x_{\omega t}$, where the demand for intermediate good ω, is given by (13). After maximization, one obtains the standard symmetric use of intermediate goods in the final good production and the mark-up on the price of intermediate goods, as seen below in (14):

$$x_{\omega t} = x_t = \left(\frac{\alpha^2}{s}\right)^{\frac{1}{1-\alpha}} L_t^Y \quad \text{and} \quad q_{\omega t} = \frac{\chi_{\omega t}}{\alpha} \quad , \forall \omega \in \Omega \tag{14}$$

All intermediate good producers thus produce the same amount and charge the same price.

Together with the definition of the whole disposable knowledge in the economy (1), (14) allows us to rewrite the final good production function (8), the wage expression given in (13), and the instantaneous monopoly profit on the sale of each intermediate good ω, respectively as:

$$Y_t = \left(\frac{\alpha^2}{s}\right)^{\frac{\alpha}{1-\alpha}} L_t^Y \mathcal{K}_t \ , \ w_t = (1-\alpha)\left(\frac{\alpha^2}{s}\right)^{\frac{\alpha}{1-\alpha}} \mathcal{K}_t \ \text{ and}$$
$$\pi_t^{x_\omega} = \frac{1-\alpha}{\alpha}\left(\frac{\alpha^2}{s}\right)^{\frac{1}{1-\alpha}} \chi_{\omega t} L_t^Y \ , \ \forall \omega \in \Omega \tag{15}$$

The final good resource constraint (9) becomes $Y_t = Lc_t + (\alpha^2/s)^{\frac{1}{1-\alpha}} L_t^Y \mathcal{K}_t$. Dividing both sides by Y_t and using the previous expressions of x_t and Y_t gives: $Lc_t/Y_t = 1 - \alpha^2/s$. Therefore, the growth rates of per capita consumption and of the final good equalize:

$$g_{Y_t} = g_{c_t} \tag{16}$$

Moreover, log-differentiating with respect to time the expression of the final good production function given in (15), one obtains:

$$g_{Y_t} = g_{L_t^Y} + g_{\mathcal{K}_t} \tag{17}$$

Let us now consider any R&D activity ω, $\omega \in \Omega$, and derive the innovators' arbitrage condition. Given the governmental intervention on behalf of research activities, the incumbent innovator, having successfully innovated at instant t, receives, at any instant $\tau > t$, the net profit[15] $\tilde{\pi}_\tau^{x_\omega} = \mathfrak{T}\pi_\tau^{x_\omega}$ with probability $e^{-\int_t^\tau \lambda l_{\omega u} du}$ (*i.e.* provided that there is no innovation upgrading intermediate good ω between t and τ, since $l_{\omega u}$ is the amount of labor devoted to research in sector ω at instant u). The sum of the present values of the incumbent's expected net profits on the sale of intermediate good ω, at instant t, is therefore:

$$\tilde{\Pi}_t^\omega = \int_t^\infty \tilde{\pi}_\tau^{x_\omega} e^{-\int_t^\tau (r_u + \lambda l_{\omega u})du} d\tau, \tag{18}$$

Differentiating (18) with respect to time gives the standard arbitrage condition in each R&D activity ω:

$$r_t + \lambda l_{\omega t} = \frac{\dot{\tilde{\Pi}}_t^\omega}{\tilde{\Pi}_t^\omega} + \frac{\tilde{\pi}_t^{x_\omega}}{\tilde{\Pi}_t^\omega} \ , \ \forall \omega \in \Omega \tag{19}$$

[15] As mentioned above, if $\mathfrak{T} > 1$ (resp. $\mathfrak{T} < 1$) then the monopoly's net profit is larger (resp. lower) than the *"laisser faire"* profit; this corresponds to subsidizing (resp. taxing) the monopoly to foster appropriate research effort.

The latter arbitrage conditions state that, in equilibrium, the rate of return is the same on the financial market as well as on all R&D activities.

Since we have assumed that innovations of each intermediate good ω occur along with a Poisson arrival rate of $\lambda l_{\omega t}$, if one unit of labor is invested in R&D activity ω, the probability to obtain one innovation is λ. Once created, its value, taking into account the R&D public policy, is $\tilde{\Pi}_t^\omega$. Hence, $\lambda\tilde{\Pi}_t^\omega$ is the expected revenue when investing one unit of labor in R&D. Therefore, the free-entry condition in each R&D activity ω is[16] $w_t = \lambda\tilde{\Pi}_t^\omega$, where w_t, the cost of one unit of labor, is given in (15). Consequently,

$$\tilde{\Pi}_t^\omega = \tilde{\Pi}_t = \frac{1-\alpha}{\lambda}\left(\frac{\alpha^2}{s}\right)^{\frac{\alpha}{1-\alpha}}\mathcal{K}_t\ ,\ \forall\omega\in\Omega$$

which implies that $\dot{\tilde{\Pi}}_t^\omega/\tilde{\Pi}_t^\omega = g_{\mathcal{K}_t}$ and $\tilde{\pi}_t^{x_\omega}/\tilde{\Pi}_t^\omega = \frac{\lambda\alpha\mathfrak{T}\chi_{\omega t}L_t^Y}{s\mathcal{K}_t}$, $\forall\omega\in\Omega$. The arbitrage condition (19) can thus be rewritten as:

$$r_t + \lambda l_{\omega t} = g_{\mathcal{K}_t} + \frac{\lambda\alpha\mathfrak{T}\chi_{\omega t}L_t^Y}{s\mathcal{K}_t}\ ,\ \forall\omega\in\Omega \tag{20}$$

As in the standard literature, we focus on a symmetric equilibrium[17], in which $\chi_{\omega t} = \chi_t$ and $l_{\omega t} = l_t$, $\forall\omega\in\Omega$. Hence $L_t^R = Nl_t$ and $\mathcal{K}_t = N\chi_t$. Accordingly, one obtains that the growth rate of the knowledge inherent in any intermediate sector and the growth rate of the whole disposable knowledge in the economy are the same:

$$g_{\chi_{\omega t}} = g_{\mathcal{K}_t}\ ,\ \forall\omega\in\Omega \tag{21}$$

Moreover, the pool of knowledge used by each R&D activity ω, $\omega\in\Omega$, at each instant t is:

$$\mathcal{P}_t^\omega = \mathcal{P}_t = \left(p_0 + p_n\underline{\theta} + p_W\overline{\theta}\right)\chi_t = \frac{\mathbb{E}\left[\theta\right]\mathcal{K}_t}{N}, \forall\omega\in\Omega \tag{22}$$

Plugging this expression in (5) and using the symmetric labor assumption, one gets, for any intermediate good sector ω, the following law of motion of knowledge:

$$\dot{\chi}_{\omega t} = \dot{\chi}_t = \frac{\lambda\sigma\mathbb{E}\left[\theta\right]L_t^R}{N}\chi_t\ ,\ \forall\omega\in\Omega \Leftrightarrow g_{\chi_{\omega t}} = \frac{\lambda\sigma\mathbb{E}\left[\theta\right]L_t^R}{N}\ ,\ \forall\omega\in\Omega$$

[16] Note that, an alternative methodology could be used here. It is used, for instance, in Barro & Sala-i-Martin (1995) and consists in subsidizing (or possibly taxing) the R&D labor demand. Denoting by γ the public tool targeted to the R&D labor, the net cost of one unit of labor is $(1-\gamma)w_t$. Subsidizing (resp. taxing) labor demand of R&D activity corresponds to $\gamma > 0$ (resp. to $\gamma < 0$). The free-entry condition, according to that method, is $(1-\gamma)w_t = \lambda\Pi_t^\omega$, where $\Pi_t^\omega = \int_t^\infty \pi_\tau^{x_\omega} e^{-\int_t^\tau (r_u + \lambda l_{\omega u})du} d\tau$.

The two methods are equivalent here since the public tool does not depend on time. Indeed, according to our method, the free-entry condition in each R&D activity ω is $w_t = \lambda\tilde{\Pi}_t^\omega$ and $w_t = \lambda\tilde{\Pi}_t^\omega \Leftrightarrow w_t = \lambda\mathfrak{T}\Pi_t^\omega \Leftrightarrow \frac{1}{\mathfrak{T}}w_t = \lambda\Pi_t^\omega \Leftrightarrow (1-\gamma)w_t = \lambda\Pi_t^\omega$, where $1-\gamma = 1/\mathfrak{T}$ (note that $\gamma > 0 \Leftrightarrow \mathfrak{T} > 1$ and $\gamma < 0 \Leftrightarrow \mathfrak{T} < 1$). It is hence equivalent here to target the public policy to the monopoly profit or to the R&D labor demand.

[17] see Aghion & Howitt (1992) for instance. This point is discussed in details in Cozzi, Giordani & Zamparelli (2007).

Consequently, from (21), the growth rate of the whole disposable knowledge in the economy is:

$$g_{\mathcal{K}_t} = \frac{\lambda\sigma\mathbb{E}\left[\theta\right] L_t^R}{N} \tag{23}$$

Finally, we can rewrite (20), the arbitrage condition in any R&D activity ω, $\omega \in \Omega$, as:

$$r_t + \lambda\frac{L_t^R}{N} = \frac{\lambda\sigma\mathbb{E}\left[\theta\right] L_t^R}{N} + \frac{\lambda\alpha\mathfrak{T}L_t^Y}{sN} \tag{24}$$

The representative household maximizes her intertemporal utility given by (11) subject to her budget constraint: $\dot{b}_t = w_t + r_t b_t - c_t - \frac{T_t}{L}$, where b_t is the stock of bonds and T_t is a lump-sum tax charged by the government in order to finance public policies. This yields the usual Keynes-Ramsey condition:

$$r_t = g_{c_t} + \rho \tag{25}$$

3.2 Steady-State Symmetric Equilibrium

The equilibrium quantities, growth rates and prices are characterized by equations (7), (14), (15), (16), (17), (21), (23), (24) and (25).

At steady-state, all variables grow at constant rate. In particular $g_{\mathcal{K}_t}$ must be constant. Therefore, at steady-state, L_t^R is constant (*cf.* (23)) and so is L_t^Y (*cf.* (7)). Let us denote by Z^e the steady-state equilibrium of any variable Z_t. Given that $g_{L^Y}^e = 0$, using (16), (17), (21) and (23), one gets:

$$g_c^e = g_Y^e = g_{\mathcal{K}}^e = g_{\chi_\omega}^e = \frac{\lambda\sigma\mathbb{E}\left[\theta\right] L^{Re}}{N}, \ \forall\omega \in \Omega$$

The steady-state symmetric equilibrium is thus characterized by the following system of equations:

$$\begin{cases} g_c^e = g_Y^e = g_{\mathcal{K}}^e = g_{\chi_\omega}^e = \frac{\lambda\sigma\mathbb{E}[\theta]L^{Re}}{N}, \ \forall\omega \in \Omega & (l1) \\ L = L^{Ye} + L^{Re} & (l2) \\ r^e = g_c^e + \rho & (l3) \\ r^e + \lambda\frac{L^{Re}}{N} = \frac{\lambda\sigma\mathbb{E}[\theta]L^{Re}}{N} + \frac{\lambda\alpha\mathfrak{T}L^{Ye}}{sN} & (l4) \\ w_t^e = (1-\alpha)\left(\frac{\alpha^2}{s}\right)^{\frac{\alpha}{1-\alpha}} \mathcal{K}_t^e & (l5) \\ q_{\omega t}^e = q_t^e = \frac{\chi_t^e}{\alpha} = \frac{\mathcal{K}_t^e}{\alpha N} & (l6) \\ x_\omega^e = x^e = \left(\frac{\alpha^2}{s}\right)^{\frac{1}{1-\alpha}} L^{Ye} & (l7) \end{cases}$$

Solving this system[18], one obtains Proposition 3, below.

[18]From ($l1$), ($l3$) and ($l4$), one gets $\rho + \lambda L^{Re}/N = \lambda\alpha\mathfrak{T}L^{Ye}/sN$.

Using ($l2$) yields $L^{Re} = \frac{\lambda\alpha L\mathfrak{T} - \rho N s}{\lambda(\alpha\mathfrak{T}+s)}$ and $L^{Ye} = \frac{(\lambda L + \rho N)s}{\lambda(\alpha\mathfrak{T}+s)}$. Replacing in ($l1$) gives the steady-state growth rates. □

Proposition 3: *Given a vector of public policies $(s, \mathfrak{T})$, the steady-state symmetric equilibrium in the decentralized economy is characterized by the following repartition of labor, quantity of intermediate goods, growth rates and prices:*

$$L^{Re}(s,\mathfrak{T}) = \frac{\lambda\alpha L\mathfrak{T} - \rho N s}{\lambda(\alpha\mathfrak{T}+s)} \quad , \quad L^{Ye}(s,\mathfrak{T}) = \frac{(\lambda L + \rho N)\, s}{\lambda(\alpha\mathfrak{T}+s)}$$

$$x^e_\omega(s,\mathfrak{T}) = x^e(s,\mathfrak{T}) = \left(\frac{\alpha^2}{s}\right)^{\frac{1}{1-\alpha}} L^{Ye}(s,\mathfrak{T}) = \left(\frac{\alpha^2}{s}\right)^{\frac{1}{1-\alpha}} \frac{(\lambda L + \rho N)\, s}{\lambda(\alpha\mathfrak{T}+s)} \,,\ \forall \omega \in \Omega \,,$$

$$g^e_c = g^e_Y = g^e_{\mathcal{K}} = g^e_{\chi_\omega} = g^e(s,\mathfrak{T}) = \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda\alpha L\mathfrak{T} - \rho N s)}{N(\alpha\mathfrak{T}+s)} \,,$$

$$r^e = g^e(s,\mathfrak{T}) + \rho \,,$$

$$w^e_t = (1-\alpha)\left(\frac{\alpha^2}{s}\right)^{\frac{\alpha}{1-\alpha}} \mathcal{K}^e_t \ ,\ q^e_{\omega t} = q^e_t = \frac{\mathcal{K}^e_t}{\alpha N} \,,\ \forall \omega \in \Omega \,,$$

where $g^e(s,\mathfrak{T})$ denotes the rate of growth of the economy at the steady-state symmetric equilibrium as a function of the vector of public policies.

4 Pareto Optimality and Public Policies

As mentioned above, there are two distortions in this decentralized economy. One is induced by the presence of a monopoly on the production of intermediate goods and can be removed by a subsidy on each intermediate good demand. The other is related to the incompleteness of markets which implies knowledge spillovers effects. In this section, we study those distortions. We shed a new light on Pareto suboptimality. We underline the role of public policies to foster appropriate research activity. We characterize the public tools implementing the first-best optimum within the steady-state symmetric equilibrium.

4.1 *"Laisser Faire"* Equilibrium

Let us start by comparing the growth rate of the *"laisser faire"* economy with the optimal one. From Proposition 3, one can easily characterize the equilibrium in which there is no public intervention *i.e.* when the vector of public policies is $(s,\mathfrak{T}) = (1,1)$.[19]

[19] The *"laisser faire"* steady-state equilibrium is characterized by the following repartition of labor, quantity of intermediate goods, growth rates and prices:

$$L^{R^{lf}} = \frac{\lambda\alpha L - \rho N}{\lambda(\alpha+1)} \ ,\ L^{Y^{lf}} = \frac{\lambda L + \rho N}{\lambda(\alpha+1)} \ ,\ x^{lf}_\omega = x^{lf} = \alpha^{\frac{2}{1-\alpha}} \frac{\lambda L + \rho N}{\lambda(\alpha+1)} \ ,\ \forall \omega \in \Omega$$

$$g^{lf}_c = g^{lf}_Y = g^{lf}_{\mathcal{K}} = g^{lf}_{\chi_\omega} = g^{lf} = \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda\alpha L - \rho N)}{N(\alpha+1)}$$

$$r^{lf} = g^{lf} + \rho \ ,\ w^{lf}_t = (1-\alpha)\alpha^{\frac{2\alpha}{1-\alpha}} \mathcal{K}^{lf}_t \ ,\ q^{lf}_{\omega t} = q^{lf}_t = \frac{\mathcal{K}^{lf}_t}{\alpha N} \ ,\ \forall \omega \in \Omega$$

Corollary 1: *The steady-state growth rate of the* "laisser faire" *economy is:*

$$g^{lf} = g^e(1,1) = \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda\alpha L - \rho N)}{N\,(\alpha+1)} \tag{26}$$

This result revives the one obtained in Aghion & Howitt (1992), stating that the provision of R&D effort can either be suboptimal or over-optimal[20]. Indeed, it appears clearly here that the *"laisser faire"* growth rate can be either above or below the optimal growth rate, depending on the value of the parameters of the model. Accordingly, there exists particular cases in which, for certain values of the parameters of the model, the *"laisser faire"* growth rate is optimal, suboptimal or over-optimal:

$$\begin{aligned} g^{lf} \lesseqqgtr g^{opt} \Leftrightarrow \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda\alpha L - \rho N)}{N\,(\alpha+1)} &\lesseqqgtr \frac{\lambda\sigma\mathbb{E}[\theta]\,L}{N} - \rho \\ \Leftrightarrow \mathbb{E}[\theta]\,\sigma\,(\lambda\alpha L - \rho N) &\lesseqqgtr (\alpha+1)\,(\lambda\sigma\mathbb{E}[\theta]\,L - \rho N) \\ \Leftrightarrow (\alpha+1)\rho N \lesseqqgtr \mathbb{E}[\theta]\,\sigma\,(\lambda L + \rho N) \Leftrightarrow \frac{(\alpha+1)\rho N}{\sigma\,(\lambda L + \rho N)} &\lesseqqgtr \mathbb{E}[\theta] \end{aligned}$$

The following proposition summarizes these results.

Proposition 4:

- $g^{lf} = g^{opt}$ *if and only if the set of parameters* $(L, N, \alpha, \lambda, \rho, \sigma, \mathbb{E}[\theta])$ *verifies* $\mathbb{E}[\theta] = \frac{(\alpha+1)\rho N}{\sigma(\lambda L+\rho N)} \equiv \widetilde{\mathbb{E}[\theta]}^{lf}$.
- *For expected scopes of diffusion of knowledge above (resp. below) the threshold* $\widetilde{\mathbb{E}[\theta]}^{lf}$*, the* "laisser faire" *growth rate is suboptimal (resp. over-optimal).*

The more innovations diffuse within the R&D activities of the economy, the more likely the *"laisser faire"* policy will entail sub-optimality; that is the more likely the R&D effort will be below its optimal level. This underlines the need for public intervention to sustain appropriate R&D activity.

4.2 Distortions and Implementation of the First-Best

Let us now implement the optimum within the steady-state symmetric equilibrium. We compute the public tools which correct the two distortions inherent to the considered decentralized economy: the optimal subsidy on each intermediate good demand (s^*) and the optimal tool dedicated to R&D activities ($\mathfrak{T}^*$). Meanwhile, we investigate the issue of appropriate R&D effort. This will allow us to shed a new light on the fact that, in Schumpeterian growth theory, the equilibrium growth rate of the economy is Pareto non optimal: it can be suboptimal as well as over-optimal. The explanation we propose here deals with the scope of diffusion of knowledge.

[20] This is a standard result in endogenous growth theory see for instance, Aghion & Howitt (1998), Benassy (1998), Acemoglu (2009) or Aghion & Howitt (2009).

4.2.1 Monopoly Distortion

Proposition 5: *The instantaneous optimal subsidy on each intermediate good demand removing the distortion entailed by presence of a monopoly on the production and sale on each of them is* $s^* = \alpha$.

This result is standard and analog to the one found in Aghion & Howitt (1992). The proof is straightforward. Identification of the equilibrium quantity of each intermediate good ω (given in Proposition 3) with the optimal one (given in Proposition 2) implies that the instantaneous optimal subsidy on each intermediate good demand enabling to remove the distortion induced by the presence of a monopoly, s^*, must satisfy the following equation: $\left(\frac{\alpha^2}{s^*}\right)^{\frac{1}{1-\alpha}} L^{Y\,opt} = \alpha^{\frac{1}{1-\alpha}} L^{Y\,opt} (= x_\omega^{opt})$, $\forall \omega \in \Omega$. Therefore, one has $s^* = \alpha$. □

4.2.2 Distortion Entailed by Knowledge Spillovers

Before computing the optimal tool dedicated to research, let us first investigate with more parsimony the distortion entailed by knowledge spillovers. In this respect, let us get rid of the monopoly distortion, assuming that the level of the subsidy on each intermediate good demand is $s^* = \alpha$.

From Proposition 3, one can easily characterize the equilibrium in which the monopoly distortion has been removed. As expressed in Corollary 2 below, the corresponding growth rates are function of $\mathfrak{T}$, the public tool dedicated to foster appropriate R&D effort.

Corollary 2: *Once the monopoly distortion has been corrected, the steady-state growth rate of the economy is:*

$$g^e(\alpha, \mathfrak{T}) = g^m(\mathfrak{T}) = \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda L \mathfrak{T} - \rho N)}{N\,(\mathfrak{T} + 1)} \tag{27}$$

Let us first consider that there is no public policy dedicated to research activity *i.e.* that the vector of public policies is $(s, \mathfrak{T}) = (\alpha, 1)$. The corresponding growth rate of the economy is:

$$g^e(\alpha, 1) = g^m(1) = g^m = \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda L - \rho N)}{2N} \tag{28}$$

As in the case of the *"laisser faire"* economy studied above in subsection 4.1, the growth rate of the economy when the public intervention aims solely at dealing with the monopoly distortion can still be either equal, above or below the optimal growth rate, depending on the value of the parameters of the model:

$$\begin{aligned} g^m \lesseqqgtr g^{opt} &\Leftrightarrow \frac{\mathbb{E}[\theta]\,\sigma\,(\lambda L - \rho N)}{2N} \lesseqqgtr \frac{\lambda\sigma\mathbb{E}[\theta]\,L}{N} - \rho \\ &\Leftrightarrow \mathbb{E}[\theta]\,\sigma\,(\lambda L - \rho N) \lesseqqgtr 2\lambda\sigma\mathbb{E}[\theta]\,L - 2\rho N \\ \Leftrightarrow 2\rho N \lesseqqgtr \mathbb{E}[\theta]\,\sigma\,(\lambda L + \rho N) &\Leftrightarrow \frac{2\rho N}{\sigma\,(\lambda L + \rho N)} \lesseqqgtr \mathbb{E}[\theta] \end{aligned}$$

The following proposition summarizes these results.

Proposition 6:

- *$g^m = g^{opt}$ if and only if the set of parameters $(L, N, \alpha, \lambda, \rho, \sigma, \mathbb{E}[\theta])$ verifies $\mathbb{E}[\theta] = \frac{2\rho N}{\sigma(\lambda L+\rho N)} \equiv \widetilde{\mathbb{E}[\theta]}^m$.*

- *For expected scopes of diffusion of knowledge above (resp. below) the threshold $\widetilde{\mathbb{E}[\theta]}^m$, the growth rate of the economy when the public policy aims solely at correcting the monopoly distortion is suboptimal (resp. over-optimal).*

Note that, since $\widetilde{\mathbb{E}[\theta]}^m > \widetilde{\mathbb{E}[\theta]}^{lf}$ (because $0 < \alpha < 1$), removing the monopoly distortion rises the threshold of expected scope of diffusion of knowledge above which the growth rate of the economy is suboptimal. Accordingly, once the monopoly distortion has been removed, it is more likely that R&D effort is over optimal for large expected scope of influence of innovations.

This result is supported by the comparison of g^m and g^{lf}; indeed, one finds that the growth rate of the economy is higher once the monopoly distortion is removed[21].

Those findings show that the possibility for the growth rate of the economy to be over-optimal is resulting from the externality triggered by knowledge spillovers.

4.3 Optimal R&D Policy

The results obtained above underline the need for public intervention on behalf of research to sustain appropriate R&D activity. The optimal tool dealing with the distortion resulting from knowledge spillovers is given in Proposition 7 below.

Proposition 7: *The optimal tool used to deal with the externality inherent in R&D activity is constant over time and can consist either in a subsidy or in a tax. Furthermore, it depends positively on the expected scope of influence of innovations:*

$$\mathfrak{T}^* = \frac{\mathbb{E}[\theta]\,\sigma(\lambda L + \rho N)}{\rho N} - 1 \tag{29}$$

The proof is as follows. Given that $s = s^* = \alpha$, identifying the equilibrium growth rate of the economy (given in Proposition 3) with the optimal one (given in Proposition 2), one gets that $\mathfrak{T}^*$ has to verify:

$$g^e(\alpha, \mathfrak{T}^*) = g^{opt} \Leftrightarrow \mathbb{E}[\theta]\,\frac{\sigma(\lambda\alpha L\mathfrak{T}^* - \rho N\alpha)}{N(\alpha\mathfrak{T}^* + \alpha)} = \mathbb{E}[\theta]\,\frac{\lambda\sigma L}{N} - \rho$$

$$\Leftrightarrow \mathbb{E}[\theta]\,\sigma(\lambda L\mathfrak{T}^* - \rho N) = (\mathbb{E}[\theta]\,\lambda\sigma L - \rho N)(\mathfrak{T}^* + 1)$$

$$\Leftrightarrow \rho N\mathfrak{T}^* + \rho N = \mathbb{E}[\theta]\,\sigma(\lambda L + \rho N) \quad \square$$

In this section, we see that, as traditionally in Schumpeterian growth theory, the decentralized equilibrium is not Pareto optimal: the Schumpeterian equilibrium

[21]Indeed, $g^m - g^{lf} = \frac{\mathbb{E}[\theta]\sigma(\lambda L-\rho N)}{2N} - \frac{\mathbb{E}[\theta]\sigma(\lambda\alpha L-\rho N)}{N(\alpha+1)} = \frac{\mathbb{E}[\theta]\sigma}{N}\left[\frac{(\alpha+1)(\lambda L-\rho N)-2(\lambda\alpha L-\rho N)}{2(\alpha+1)}\right]$
$= \frac{\mathbb{E}[\theta]\sigma}{N}\left[\frac{(1-\alpha)\lambda L+(1-\alpha)\rho N}{2(\alpha+1)}\right] = (1-\alpha)\frac{\mathbb{E}[\theta]\sigma(\lambda L+\rho N)}{2(\alpha+1)N} > 0$ since $\alpha < 1$.

growth rate can be too high or too low relative to the optimal one. We provide a new explanation for this caveat: we argue that, depending on the level of the expected scope of influence of innovations, the equilibrium may have a higher or a lower rate of growth than the Pareto optimal allocation.

This result is obviously corroborated when looking at the expression of the optimal tool dealing with the externality inherent in R&D activity. Indeed, as seen in (29), $\mathfrak{T}^*$ is increasing in the expected scope of diffusion of knowledge and can consist either in a subsidy or in a tax: as stated in Proposition 8 below, the wider the expected scope of influence of innovations, the more likely it will be necessary to subsidize R&D.

Corollary 3: *$\mathfrak{T}^*$ is a subsidy (resp. a tax) if and only if $\mathfrak{T}^*$ is larger (resp. lower) than one,* i.e. *if and only if* [22] *the expected scope of diffusion of knowledge, $E[\theta]$, is above (resp. below) the threshold $\widetilde{\mathbb{E}[\theta]}^m \equiv \frac{2\rho N}{\sigma(\lambda L+\rho N)}$, characterized above*[23].

5 Polar Cases

As mentioned in subsection 2.1, the model is a general framework permitting to consider a variety of Schumpeterian endogenous growth models. We have introduced some uncertainty and partiality in the diffusion of knowledge across the economy's R&D activities in the sense that the scope of innovations can be randomly more or less extensive.

In this section, we get back over two particular cases. Removing the uncertainty, one obtains a general endogenous growth model in which innovations' scope of influence is certain but possibly partial. This framework allows to consider various extents to which knowledge diffuses across the economy's R&D activities[24], ranging from no intersectoral knowledge spillovers at all (*i.e.* considering "inside sector knowledge" only) to total intersectoral knowledge spillovers (*i.e.* considering "general knowledge" only). In the former case, one obtains an endogenous growth model close in spirit to the one developed by Grossman & Helpman (1991), and in the latter, an endogenous growth model *à la* Aghion & Howitt (1992).

5.1 "Inside Sector Knowledge" Schumpeterian Growth Model

The first polar case is a model in which there are no intersectoral knowledge spillovers. Formally, setting $p_0 = 1$ (and thus, $p_n = p_W = 0$) allows to consider only innovations that do not diffuse across the economy R&D activities. In this case, knowledge produced in a given sector is specific to this one.

Accordingly, at each instant t, the pool of knowledge used by each R&D activities in sector ω, $\omega \in \Omega$, is just composed of the amount of knowledge inherent in

[22] $\mathfrak{T}^* \lesseqqgtr 1 \Leftrightarrow \frac{\mathbb{E}[\theta]\sigma(\lambda L+\rho N)}{\rho N} - 1 \lesseqqgtr 1 \Leftrightarrow \mathbb{E}[\theta] \lesseqqgtr \frac{2\rho N}{\sigma(\lambda L+\rho N)} \equiv \widetilde{\mathbb{E}[\theta]}^m$

[23] Note that, in the particular case in which the values of the parameter are such that $\mathbb{E}[\theta] = \widetilde{\mathbb{E}[\theta]}^m$, then $\mathfrak{T}^* = 1$. Accordingly, the growth rate of the decentralized economy is then optimal even though there is no public policy dedicated to R&D activity.

[24] See Appendix 8.3 for the presentation of the non-stochastic general case model. It corresponds to a version of the model in which θ is a constant random variable (*i.e.* in which $\mathbb{E}[\theta] = \theta$).

the corresponding intermediate good, *i.e.* $\mathcal{P}_t^\omega = \chi_{\omega t}$, $\forall \omega \in \Omega$. Hence, for a given vector of public policies $(s, \mathfrak{T})$, the steady-state symmetric equilibrium growth rate of the economy comprising only "inside sector knowledge" is:

$$g^e(s, \mathfrak{T}) = \frac{\sigma\left(\lambda\alpha L\mathfrak{T} - \rho N s\right)}{N\left(\alpha\mathfrak{T} + s\right)} \; .$$

The optimal public policies vector is:

$$(s^*, \mathfrak{T}^*) = (\alpha, \frac{\lambda\sigma L}{\rho N} + \sigma - 1) \; ,$$

and the steady-state optimal growth rate of the economy is:

$$g^{opt} = \frac{\lambda\sigma L}{N} - \rho \; .$$

5.2 "General Knowledge" Schumpeterian Growth Model

Consider now the other polar case in which any innovation spreads its influence across all R&D activities within the economy (*i.e.* in which there are only general purpose innovations). Setting $\theta = N$, one obtains a model with complete intersectoral knowledge spillovers. The corresponding expression of the neighborhood of diffusion of an innovation related to any intermediate good h is $\underline{\Omega^h} = \overline{\Omega^h} \equiv [h - N/2 \, ; \, h + N/2] = \Omega, \forall h \in \Omega$. Replacing in (4), one gets that, for any R&D activity ω, $\omega \in \Omega$, the resulting pool of knowledge is the whole disposable knowledge in the economy:

$$\mathcal{P}_t^\omega = \mathcal{P}_t = \int_\Omega \chi_{ht} \, dh = \mathcal{K}_t \; , \; \forall \omega \in \Omega$$

This endogenous growth model with certain and complete diffusion of knowledge is very close to the one proposed by Aghion & Howitt (1992).

For a given vector of public policies $(s, \mathfrak{T})$, the steady-state symmetric equilibrium growth rate of the economy is:

$$g^e(s, \mathfrak{T}) = \frac{\sigma\left(\lambda\alpha L\mathfrak{T} - \rho N s\right)}{\alpha\mathfrak{T} + s} \; ,$$

the optimal public policies vector is:

$$(s^*, \mathfrak{T}^*) = (\alpha, \frac{\lambda\sigma L}{\rho} + \sigma N - 1) \; ,$$

and, the steady-state first-best growth rate of the economy is:

$$g^{opt} = \lambda\sigma L - \rho \; .$$

6 Intersectoral Knowledge Spillovers and Scale-Effects

A non desirable feature of the first generation of models of endogenous growth is the presence of scale-effects[25]. This, whether one considers the Romer (1990) model or Schumpeterian growth models considering only vertical knowledge accumulation. This property predicts that the larger the population level L, the greater the growth rate of the economy. As in this seminal innovation-based theory, our model displays this non-desirable property. As observed in Propositions 2 and 3, it exhibits scale-effects at any steady-state equilibrium. Moreover, if one allows for constant population growth, steady-states do not exist. "This prediction is implied because increased population raises the size of the market that can be captured by a successful entrepreneur and also because it raises the supply of potential researchers"[26]. The specialists all agree on the fact that this property is undesirable since this prediction is strongly at odds with twentieth century observed stylized facts[27]. It has been shown, in the literature entitled "Endogenous Growth without Scale-Effects theory", that this counterfactual property can be eliminated from the theory by allowing for both horizontal and vertical knowledge accumulation[28].

In this section, inspired by the formalization used by Jones (1999), Dinopoulos & Sener (2007) or Aghion & Howitt (2009, chapter 4, section 4.4.), we show how scale-effects can be removed from our model by incorporating Young's (1998) insight that, as population grows, the diversification of the intermediate sectors reduces the efficiency of R&D activities in improving the quality of the intermediate goods. As stated by Aghion & Howitt, R&D efforts are "spread more thinly over a larger number of different sectors, thus dissipating the effect on the overall rate of productivity growth".

Two modifications of the standard framework are necessary. First, one needs to modify the final good production function, assuming that it does not depend on the absolute labor input L_t^Y but on the labor per sector L_t^Y/N. Secondly, one needs to introduce in the model a process by which the number of intermediate goods sectors increases.

We depart from this methodology, in that we do not have to modify the final good production function. We only have to consider intermediate goods variety increases; in this respect, we adapt the methodology proposed by Aghion & Howitt (2009). We introduce a very simple scheme in which it is assumed that the probability of inventing a new intermediate good is a linear function of the population, κL In particular, no R&D expenditure is introduced. Moreover, we also suppose that an exogenous fraction ξ of intermediate goods becomes obsolete and vanishes at each instant t. As in the standard methodology, if population is constant, then the variation of the number of sectors is given by:

$$\dot{N}_t = \kappa L - \xi N_t \ , \ \forall t$$

[25]For an excellent overview and very accurate exposition of the growth theory related body of literature, see Jones (1999) and Dinopoulos & Sener (2007).

[26]Aghion & Howitt (2009), chapter 4, page 96.

[27]Jones (1995) has been the first to point out this matter.

[28]Gray & Grimaud (2010) deals extensively with this matter.

The solution of this autonomous non-homogenous first-order linear differential equation is:

$$N_t = \frac{\kappa L}{\xi} + \left(N_0 - \frac{\kappa L}{\xi} \right) e^{-\xi t} \ , \ \forall t$$

At steady-state, the number of intermediate goods will stabilize at:

$$N = \frac{\kappa L}{\xi} \tag{30}$$

Plugging (30) in the expression of the steady-state equilibrium growth rate of the economy (given in Proposition 3), one gets:

$$g_c^e = g_Y^e = g_{\mathcal{K}}^e = g_{\chi_\omega}^e = g^e(s, \mathfrak{T}) = \frac{\mathbb{E}\left[\theta\right] \sigma \left(\frac{\xi}{\kappa} \lambda \alpha \mathfrak{T} - \rho s \right)}{\alpha \mathfrak{T} + s}$$

Under this specification, the economy's rate of growth at any steady-state equilibrium is independent of the scale of the economy, as measured by the level of the population L. In particular, this result is true for the first-best of the model (indeed, under this specification, $g^{opt} = \frac{\mathbb{E}[\theta]\lambda\sigma\xi}{\kappa} - \rho$ is independent of L). One should however remark that since $\mathbb{E}\left[\theta\right] \in [1; N]$, $\mathbb{E}\left[\theta\right]$ is defined on a set whose size depends on N. Since N and L are proportional, as seen in (30), the growth rate of the economy implicitly depends on the level of the population via the definition set of the expected scope of knowledge.

7 Conclusion

Exploiting the formalization of a circular product differentiation model of Salop (1979) in an attempt to generalize the standard Schumpeterian approach, we proposed an endogenous growth quality ladder model in which the knowledge inherent in any intermediate sector spills variously across the R&D activities of the economy. The assumption that innovations' influence ranges from punctual to global shapes the pool of knowledge in which innovations draw themselves on in order to be produced. Consequently, the law of motion of knowledge accumulation depends on the expected scope of diffusion of innovations and so does the growth rate of the economy.

Extending Aghion & Howitt (1992) analysis to the possibility of stochastic and partial intersectoral knowledge spillovers, allowed us to apprehend the issue of the funding of research with more parsimony. We emphasized that the fact that the Schumpeterian equilibrium may have a higher or a lower rate of growth than the Pareto optimal allocation depends on the level of the expected scope of influence of innovations. Accordingly, we underlined that the public tool dealing with the externality inherent in R&D activity could consist in a subsidy as well as in a tax. In particular, the wider the expected scope of knowledge, the more likely it will be necessary to subsidize R&D activity.

This framework constitutes an attempt in the design of public policies aiming at mitigating the externality triggered by R&D activity. A possible step forward would be to introduce a public tool which distinguish between innovations with narrow influence and those with a wide impact on the economy. This tool should ideally depend on the actual scope of influence of innovations.

8 Appendix

8.1 Law of Motion of Intermediate Good ω Inherent Knowledge

When an innovation of intermediate good ω occurs, the induced increase in knowledge is a linear function of $\mathcal{P}_t^\omega$: $\Delta\chi_{\omega t} = \sigma\mathcal{P}_t^\omega$. Furthermore we assume that the Poisson arrival rate of innovations for every intermediate good ω is $\lambda l_{\omega t}$, $\lambda > 0$. Consider a time interval $(t, t+\Delta t)$ and that, at instant t, the knowledge associated to intermediate good ω is $\chi_{\omega t}$. Thus $\chi_{\omega\, t+\Delta t}$, the knowledge at instant $t+\Delta t$, is a random variable which can take two values: $\chi_{\omega\, t+\Delta t} = \chi_{\omega t} + \sigma\mathcal{P}_t^\omega$, with probability $\lambda l_{\omega t}\Delta t$ (in this case, one innovation occurs during the time interval) or $\chi_{\omega\, t+\Delta t} = \chi_{\omega t}$, with probability $1 - \lambda l_{\omega t}\Delta t$ (in this case, no innovation occurs during the time interval). Hence we have that $\mathbb{E}\left[\chi_{\omega\, t+\Delta t}\right] = \lambda l_{\omega t}\Delta t\left(\chi_{\omega t} + \sigma\mathcal{P}_t^\omega\right) + \left(1 - \lambda l_{\omega t}\Delta t\right)\chi_{\omega t} = \lambda\sigma l_{\omega t}\mathcal{P}_t^\omega\Delta t + \chi_{\omega t}$. We can rewrite this equality exhibiting the Newton quotient of $\mathbb{E}\left[\chi_{\omega t}\right]$:

$$\frac{\mathbb{E}\left[\chi_{\omega\, t+\Delta t}\right] - \chi_{\omega t}}{\Delta t} = \lambda\sigma l_{\omega t}\mathcal{P}_t^\omega$$

Letting Δt tend to zero we have $\mathbb{E}\left[\dot{\chi}_{\omega t}\right] = \lambda\sigma l_{\omega t}\mathcal{P}_t^\omega$, which gives the law of motion of the average knowledge for any intermediate good $\omega \in \Omega$, as given by (5).$\square$

8.2 Optimum

The social planner maximizes (11) subject to (1), (4), (5), (7), (8), (9) and (10). The maximisation program can be written as follows:

$$\begin{array}{l}\text{Max} \quad \text{U}=\int_0^\infty \ln(c_t)e^{-\rho t}dt \text{ subject to} \\ \{\text{c}_t\}_{t\in[0,\infty[} \\ \{\text{L}_t^Y\}_{t\in[0,\infty[} \\ \{\text{l}_{\omega t}\}_{t\in[0,\infty[,\, \omega\in\Omega} \\ \{x_{\omega t}\}_{t\in[0,\infty[,\omega\in\Omega}\end{array} \quad \begin{cases} Y_t = (L_t^Y)^{1-\alpha}\int_\Omega \chi_{\omega t}(x_{\omega t})^\alpha d\omega \\ x_{\omega t} = \frac{y_{\omega t}}{\chi_{\omega t}},\ \omega\in\Omega \\ Y_t = Lc_t + \int_\Omega y_{\omega t}d\omega \\ L = L_t^Y + \int_\Omega l_{\omega t}\, d\omega \\ \dot{\chi}_{\omega t} = \lambda\sigma l_{\omega t}P_t^\omega\ ,\ \omega\in\Omega \\ \mathcal{P}_t^\omega = p_0\chi_{\omega t} + p_n\int_{\underline{\Omega^\omega}}\chi_{ht}dh \\ \qquad\qquad +p_W\int_{\overline{\Omega^\omega}}\chi_{ht}dh, \forall\omega\in\Omega \end{cases}$$

where c_t, L_t^Y, $l_{\omega t}$ and $x_{\omega t}$, $\omega \in \Omega$, are the control variables, and $\chi_{\omega t}$, $\omega \in \Omega$, the continuum of state variables of the dynamic optimization problem[29].

Before writing the corresponding hamiltonian, let us first combine the constraints. We reduce the problem to maximizing (11) subject to the three following constraints:

1. Final output resource constraint (plugging (8) and (10) in (9)):

$$(L_t^Y)^{1-\alpha}\int_\Omega \chi_{\omega t}(x_{\omega t})^\alpha d\omega = Lc_t + \int_\Omega \chi_{\omega t}x_{\omega t}d\omega \tag{31}$$

[29]Accordingly, note that the constraint relative to the law of motion of knowledge is in fact a continuum of constraints.

2. Labor constraint:

$$L = L_t^Y + \int_\Omega l_{\omega t}\, d\omega \tag{32}$$

3. A continuum of dynamic constraints relative to state variable:

$$\dot{\chi}_{\omega t} = \lambda \sigma l_{\omega t} \mathcal{P}_t^\omega \ , \ \omega \in \Omega \tag{33}$$

where $\mathcal{P}_t^\omega = p_0 \chi_{\omega t} + p_n \int_{\underline{\Omega^\omega}} \chi_{ht} dh + p_W \int_{\overline{\Omega^\omega}} \chi_{ht} dh, \forall \omega \in \Omega$
and $p_0 = 1 - p_n - p_W$.

Denoting respectively by μ_t, ν_t and $\zeta_{\omega t}$ ($\omega \in \Omega$), the co-state variables associated to constraints (31), (32) and (33), the Hamiltonian can be written as:

$$\begin{aligned} \mathcal{H} = \ln(c_t) e^{-\rho t} + \mu_t \left[(L_t^Y)^{1-\alpha} \int_\Omega \chi_{\omega t} (x_{\omega t})^\alpha d\omega - L c_t - \int_\Omega \chi_{\omega t} x_{\omega t} d\omega \right] \\ + \nu_t \left[L - L_t^Y - \int_\Omega l_{\omega t} d\omega \right] \\ + \int_\Omega \zeta_{\omega t} \left[\lambda \sigma l_{\omega t} \left(p_0 \chi_{\omega t} + p_n \int_{\underline{\Omega^\omega}} \chi_{ht} dh + p_W \int_{\overline{\Omega^\omega}} \chi_{ht} dh \right) \right] d\omega \end{aligned}$$

The first-order conditions $\frac{\partial \mathcal{H}}{\partial c_t} = 0$, $\frac{\partial \mathcal{H}}{\partial L_t^Y} = 0$, $\frac{\partial \mathcal{H}}{\partial l_{it}} = 0$ ($i \in \Omega$), $\frac{\partial \mathcal{H}}{\partial x_{it}} = 0$ ($i \in \Omega$) and $\frac{\partial \mathcal{H}}{\partial \chi_{it}} = -\dot{\zeta}_{it}$ ($i \in \Omega$) respectively yield[30]:

$$c_t^{-1} e^{-\rho t} = \mu_t L \tag{34}$$

$$\mu_t (1-\alpha) \frac{Y_t}{L_t^Y} = \nu_t \tag{35}$$

$$\zeta_{it} \lambda \sigma \mathcal{P}_t^i = \nu_t, \forall i \in \Omega \tag{36}$$

$$\mu_t \left[\alpha (L_t^Y)^{1-\alpha} \chi_{it} (x_{it})^{\alpha - 1} - \chi_{it} \right] = 0 \ , \forall i \in \Omega \tag{37}$$

$$\begin{aligned} \mu_t \left[(L_t^Y)^{1-\alpha} (x_{it})^\alpha - x_{it} \right] \\ + \lambda \sigma \left(p_0 \zeta_{it} l_{it} + p_n \int_{\underline{\Omega^i}} \zeta_{ht} l_{ht} dh + p_W \int_{\overline{\Omega^i}} \zeta_{ht} l_{ht} dh \right) = -\dot{\zeta}_{it}, \forall i \in \Omega \end{aligned} \tag{38}$$

From (37), one gets:

$$x_{it} = x_t = \alpha^{\frac{1}{1-\alpha}} L_t^Y \ , \ \forall i \in \Omega \tag{39}$$

Plugging (39) in (8), and using the definition of the whole disposable knowledge in the economy (given by (1)), the final good production function can be rewritten as:

$$Y_t = \alpha^{\frac{\alpha}{1-\alpha}} L_t^Y \mathcal{K}_t, \tag{40}$$

[30] Plus the usual transversality conditions.

which gives:

$$g_{Y_t} = g_{L_t^Y} + g_{\mathcal{K}_t} \tag{41}$$

Moreover, plugging (39) in the final good resource constraint, (9) becomes $Y_t = Lc_t + \alpha^{\frac{1}{1-\alpha}} L_t^Y \mathcal{K}_t$. Dividing both sides by Y_t and using the previous expressions of x_t and Y_t (respectively given by (39) and (40)), one obtains $Lc_t/Y_t = 1-\alpha$, yielding:

$$g_{Y_t} = g_{c_t} \tag{42}$$

Finally, the first-order conditions (35) and (38) become respectively:

$$\mu_t(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\mathcal{K}_t = \nu_t \tag{43}$$

$$\text{and} \quad \frac{\mu_t}{\zeta_{it}}(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} L_t^Y + \lambda\sigma\left(p_0 l_{it} + p_n \int_{\underline{\Omega^i}} \frac{\zeta_{ht}}{\zeta_{it}} l_{ht} dh + p_W \int_{\overline{\Omega^i}} \frac{\zeta_{ht}}{\zeta_{it}} l_{ht} dh\right) = -g_{\zeta_{it}}, \forall i \in \Omega \tag{44}$$

As in the standard literature, we consider the symmetric case in which the quality of the intermediate goods and the quantities of labor used in each R&D activity are the same, *i.e.* $\chi_{\omega t} = \chi_t$ and $l_{\omega t} = l_t$, $\forall \omega \in \Omega$. Accordingly, one has $\mathcal{K}_t = N\chi_t$ and $L_t^R = Nl_t$. Hence, under the symmetry assumption, $g_{\chi_{\omega t}} = g_{\mathcal{K}_t}$, $\forall \omega \in \Omega$ and the pool of knowledge used by each R&D activity $w \in \Omega$ at each instant t is:

$$\mathcal{P}_t^\omega = \mathcal{P}_t = \left(p_0 + p_n\underline{\theta} + p_W\overline{\theta}\right)\chi_t = \frac{\mathbb{E}\left[\theta\right]\mathcal{K}_t}{N}, \forall \omega \in \Omega \tag{45}$$

Replacing in (5), one derives the following law of motion of particular knowledge inherent in any intermediate good ω:

$$\dot{\chi}_{\omega t} = \dot{\chi}_t = \lambda\sigma \frac{L_t^R}{N}\frac{\mathbb{E}\left[\theta\right]\mathcal{K}_t}{N}\ , \ \forall \omega \in \Omega$$

Consequently, one has $g_{\chi_{\omega t}} = g_{\chi_t} = \frac{\lambda\sigma\mathbb{E}[\theta]L_t^R}{N}$, $\forall \omega \in \Omega$. Moreover, since $g_{\chi_{\omega t}} = g_{\mathcal{K}_t}, \forall \omega \in \Omega$, the growth rate of the whole disposable knowledge in the economy is:

$$g_{\mathcal{K}_t} = \frac{\lambda\sigma\mathbb{E}\left[\theta\right]L_t^R}{N} \tag{46}$$

Furthermore, (36) can be rewritten as:

$$\zeta_{it}\frac{\lambda\sigma\mathbb{E}\left[\theta\right]\mathcal{K}_t}{N} = \nu_t, \forall i \in \Omega \tag{47}$$

Therefore, $\zeta_{it} = \zeta_t$, $\forall i \in \Omega$. From (43) and (47) one gets $\frac{\mu_t}{\zeta_t} = \frac{\lambda\sigma\mathbb{E}[\theta]}{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}N}$. Log-differentiating this expression yield:

$$g_{\zeta_t} = g_{\mu_t} \tag{48}$$

Finally, using the previous results and the labor constraint (7), one can rewrite (44) as:

$$-g_{\zeta_t} = \lambda\sigma\mathbb{E}\left[\theta\right]\frac{L_t^Y}{N} + \lambda\sigma\mathbb{E}\left[\theta\right]\frac{L_t^R}{N} \Leftrightarrow -g_{\zeta_t} = \lambda\sigma\mathbb{E}\left[\theta\right]\frac{L}{N} \tag{49}$$

Log-differentiating (34) yield $g_{c_t} + \rho = -g_{\mu_t}$; combining with (48) and (49), we derive the optimal growth rate of per-capita consumption:

$$g_c^{opt} = \frac{\lambda\sigma\mathbb{E}[\theta] L}{N} - \rho \tag{50}$$

The optimum is given by equations (7), (39), (41), (42), (46) and (50). At steady-state, all variables grow at constant rate.

In particular $g_{\mathcal{K}_t}$ is constant at the steady-state, therefore L_t^R is constant (*cf.* (46)) and, so is L_t^Y (since $L_t^Y + L_t^R = L$). The steady-state optimum is thus characterized by the following system of equations:

$$\begin{cases} g_c^{opt} = \frac{\lambda\sigma\mathbb{E}[\theta]L}{N} - \rho \\ g_c^{opt} = g_Y^{opt} = g_\chi^{opt} = g_{\mathcal{K}}^{opt} \\ g_{\mathcal{K}}^{opt} = \lambda\sigma\mathbb{E}[\theta] \frac{L^{Ropt}}{N} \\ L^{Y\,opt} + L^{Ropt} = L \\ x^{opt} = \alpha^{\frac{1}{1-\alpha}} L^{Y\,opt} \end{cases}$$

After some trivial computations, one gets:

$$\begin{cases} g_c^{opt} = g_Y^{opt} = g_\chi^{opt} = g_{\mathcal{K}}^{opt} = \frac{\lambda\sigma\mathbb{E}[\theta]L}{N} - \rho \\ L^{Ropt} = L - \frac{\rho N}{\lambda\sigma\mathbb{E}[\theta]} \\ L^{Y\,opt} = \frac{\rho N}{\lambda\sigma\mathbb{E}[\theta]} \\ x^{opt} = \frac{\alpha^{\frac{1}{1-\alpha}}\rho N}{\lambda\sigma\mathbb{E}[\theta]} \end{cases}$$

□

8.3 A General Framework with Non-Stochastic and (Possible) Partial Intersectoral Knowledge Spillovers

Removing the uncertainty in the scope of diffusion of innovations and allowing for (possibly partial) intersectoral knowledge spillovers one gets a general framework of Schumpeterian growth in which it is possible to consider various extent to which knowledge diffuses across the economy R&D activities. Formally, it corresponds to a version of the model in which θ is a constant random variable (thus, in which $\mathbb{E}[\theta] = \theta$). Accordingly, if one set $p_0 = 0$ and $\underline{\theta} = \overline{\theta} = \theta \in \,]1; N]$[31], there is only one type of neighborhood of diffusion of innovations: $\underline{\Omega^h} = \overline{\Omega^h} = \Omega^h \equiv [h - \theta/2\,;\, h + \theta/2]\,, \forall h \in \Omega$. Consequently, rewriting (4), one obtains that, for any R&D activity $\omega \in \Omega$, the pool of knowledge in which innovations draw themselves on at each instant t reduces to:

$$\mathcal{P}_t^\omega = \int_{\omega-\theta/2}^{\omega+\theta/2} \chi_{ht}\, dh\;,\; \forall\omega \in \Omega\;,\; \theta \in \,]1; N] \tag{51}$$

[31]Instead of $\left(p_0 = 0, \underline{\theta} = \overline{\theta} = \theta\right)$ one can equivalently choose the sets of parameters $\left(p_n = 0, p_W = 1, \overline{\theta} = \theta\right)$ or $(p_n = 1, p_W = 0, \underline{\theta} = \theta)$.

In the symmetric case, one has:

$$\mathcal{P}_t^{\omega} = P_t = \int_{\omega-\theta/2}^{\omega+\theta/2} \frac{\mathcal{K}_t}{N}\, d\omega = \frac{\theta \mathcal{K}_t}{N}\ ,\ \forall \omega \in \Omega\ ,\ \theta \in]1; N]$$

In the model with (possible) partial intersectoral spillovers, the steady-state symmetric equilibrium growth rate of the economy is increasing in the innovations' scope of diffusion, that is in the intensity of knowledge spillovers:

$$g_c^e = g_Y^e = g_{\mathcal{K}}^e = g_{\chi}^e = g^e(s, \mathfrak{T}) = \frac{\sigma\theta\left(\lambda\alpha L\mathfrak{T} - \rho N s\right)}{N\left(\alpha\mathfrak{T} + s\right)}\ ,\ \theta \in]1; N]\,.$$

If $s = \alpha$ and $\mathfrak{T} = \mathfrak{T}^* = \theta\left(\frac{\lambda\sigma L}{\rho N} + \sigma\right) - 1$, then the steady-state symmetric equilibrium is optimal. The repartition of labor is:

$$L^{Ropt} = L - \frac{\rho N}{\lambda\sigma\theta} \text{ and } L^{Y opt} = L - L^{Ropt} = \frac{\rho N}{\lambda\sigma\theta}\ ,$$

the quantity of each intermediate good ω used is:

$$x_{\omega}^{opt} = x^{opt} = \alpha^{\frac{1}{1-\alpha}} L^{Y opt} = \frac{\alpha^{\frac{1}{1-\alpha}}\rho N}{\lambda\sigma\theta}\ ,\ \forall\omega \in \Omega\ ,$$

and the growth rates are:

$$g_c^{opt} = g_Y^{opt} = g_{\mathcal{K}}^{opt} = g_{\chi}^{opt} = g^{opt} = \frac{\lambda\sigma\theta L}{N} - \rho\ .$$

References

[1] Acemoglu, D. (2009), *Modern Economic Growth*. Princeton. NJ: Princeton University Press.

[2] Aghion, P. and Howitt, P. (1992), "A Model of Growth through Creative Destruction". Econometrica 60: 323-351.

[3] Aghion, P. and Howitt, P. (1998), *Endogenous Growth Theory*. Cambridge. MA: MIT Press.

[4] Aghion, P. and Howitt, P. (2009), *The Economics of Growth*. Cambridge. MA: MIT Press.

[5] Benassy, J.-P. (1998), "Is there always too little research in endogenous growth with expanding product variety?". European Economic Review 42: 61-69.

[6] Barro, R. and Sala-i-Martin, X. (1995), *Economic Growth*. New York: McGraw-Hill.

[7] Cozzi, G., Giordani, P.E. and Zamparelli, L. (2007), "The Refoundation of the Symmetric Equilibrium in Schumpeterian Growth Models". Journal of Economic Theory, vol. 136(1), (September): 788-797.

[8] Dinopoulos, E. and Sener, F. (2007), "New Directions in Schumpeterian Growth Theory". In Hanusch H. and A. Pyka (eds), Edgar Companion to Neo-Schumpeterian Economics, Edward Elgar.

[9] Gray, E. and Grimaud, A. (2010), "Double-Differentiation and Scale-Effects in Endogenous Growth Theory". Working paper.

[10] Green, J. and Scotchmer, S. (1995), "On the Division of Profit in Sequential Innovation". RAND Journal of Economics, The RAND Corporation, vol. 26(1): 20-33.

[11] Grimaud, A. and Rouge, L. (2004), "Polluting Non Renewable Resources, Tradeable Permits and Endogenous Growth". International Journal of Global Environmental Issues, vol. 4: 38-57.

[12] Grossman, G. and Helpman, E. (1991), "Quality Ladders in the Theory of Growth". Review of Economic Studies 58: 43-61.

[13] Helpman, E. (1998). General Purpose Technologies and Economic Growth: Introduction. In Helpman, E. (ed.) General Purpose Technologies and Economic Growth , Cambridge: MIT Press, 1998, 1-13.

[14] Helpman, E. and Trajtenberg, M. (1998). Diffusion of General Purpose Technologies. In Helpman, E. (ed.) General Purpose Technologies and Economic Growth , Cambridge: MIT Press, 1998, 85-119.

[15] Jones, C. (1995), "Time Series Tests of Endogenous Growth Models". Quarterly Journal of Economics 110: 495-525.

[16] Jones, C. (1999), "Growth: With or Without Scale Effects?". American Economic Review Papers and Proceedings 89: 139-144.

[17] Mokyr, J. (1990), *The Lever of Riches: Technological Creativity and Economic Progress*. New York: Oxford University Press.

[18] Romer, P. (1990), "Endogenous Technological Change". Journal of Political Economy 98, 5: S71-S102.

[19] Salop, S. (1979), "Monopolistic Competition with Outside Goods". Bell Journal of Economics 10, 141-156.

[20] Young, A. (1998), "Growth Without Scale Effects. Journal of Political Economy 106: 41-63.