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à mes parents...

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Part I

Introduction Générale

L'évaluation des actifs financiers est aujourd'hui un domaine de recherche à la fois complexe et dynamique. Il se destine, selon J. Cochrane, à "justifier la valeur actuelle d'une présomption formulée sur des paiements futurs, pour lesquels nous n'avons aucune certitude". En d'autres termes, la détention d'un actif donnant lieu à une rémunération répartie dans le temps, afin d'estimer sa valeur présente, cela suppose de prendre en compte à la fois l'incertitude des flux monétaires mais aussi leur position relative à la date d'évaluation. Ce dernier point découlant du principe établi qu'une unité de consommation aujourd'hui n'équivaut pas à cette même unité consommée demain. Enfin, des deux points énoncés plus haut, l'incertitude liée à la rémunération d'un bien financier semble être le facteur contribuant le plus à la formation du prix.

On distingue deux approches théoriques principales consistant à définir le prix d'un actif, soit de manière absolue en se concentrant sur l'exposition de ses paiements aux risques macro-économiques, soit de manière relative en s'appuyant sur la valeur de marché de ses constituants. Dans le cadre de ce travail de thèse, nous ferons principalement appel à des modèles basés sur l'évolution de la consommation et ainsi notre raisonnement relèvera d'avantage de la première approche.

Tout au long du déploiement de la théorie, plusieurs écueils ont jalonné la démarche des chercheurs, motivant ainsi l'introduction d'hypothèses parfois irréalistes mais nécessaires à la résolution des modèles et à leur conformité aux résultats empiriques. Nous pensons ici aux cortèges de prérequis menant au paradigme d'équilibre général mais aussi aux modèles prônant l'existence d'un *agent représentatif* afin de faciliter la détermination de l'optimum pour l'agent. D'un point de vue plus empirique, de nombreuses études, menées notamment à partir des années 80, ont mis à jour plusieurs incompatibilités entre les prédictions théoriques et les observations réalisées sur les marchés. Nous faisons principalement référence ici aux célèbres travaux de Mehra et Prescott (1985) portant sur l'*equity premium puzzle*. Nous soulignerons également le fait que l'échec des modèles actuels provient en partie de la complexité en termes d'information des marchés financiers, impliquant souvent une forte imprécision dans la mesure de certaines variables telle que la consommation. C'est ainsi qu'il est très précieux de pouvoir parfois se référer à des économies simplifiées et en particulier, à des cas historiques comme celui sous l'Ancien Régime que nous présenterons à l'occasion du premier chapitre.

Afin de remédier aux difficultés techniques ainsi qu'aux défaillances prédictives des modèles actuels, la théorie penche aujourd'hui vers l'intégration de l'hétérogénéité des agents dans le socle de sa logique afin de se défaire définitivement de l'hypothèse d'*agent représentatif*. Cette démarche s'enracine avant tout dans l'idée que des comportements ou interactions locales peuvent potentiellement s'agréger sous forme de mécanismes globaux difficilement prévisibles. C'est ainsi que la compréhension de phénomènes de marché emprunte à la théorie des systèmes complexes et suggère l'utilisation de tout le panel d'outils dédiés à ce domaine. Lorsque l'on permet des différences entre les agents, qu'elles soient informatives, préférentielles, ou même en termes de richesse, il est capital d'être en mesure de contrôler ces hétérogénéités et en particulier de maîtriser le faisceau d'interactions entre les agents. En effet en économie, à moins de considérer des marchés organisés dotés d'un planificateur externe au système qui organiserait les échanges afin d'en garantir l'efficience, il est rare que les agents puissent interagir avec n'importe quel partenaire sans rencontrer la moindre friction. Il est donc impératif de connaître la structure sous-jacente des interactions. Ce point est d'autant plus important dans le cas de comportements stratégiques menant à d'éventuels équilibres de *Nash.* C'est ainsi que certains outils tels que les graphes sont d'une utilité majeure pour visualiser et appréhender le réseau social illustrant les connections entre les agents. Nous verrons alors quelles implications, l'utilisation de ces nouveaux outils peut avoir au niveau de la formation des prix, de la répartition des ressources au sein de l'économie et de la forme globale des échanges sur un marché.

Comme nous l'avons évoqué, la théorie de l'évaluation des actifs se base entièrement sur l'hypothèse que le prix d'un actif est égal à la somme actualisée de ses flux espérés. L'une des premières avancées majeures dans le domaine a été réalisée par R. Lucas (1985) qui propose une structure idéalisée de production purement exogène, c'est-à-dire qui ne dépend pas de la volonté des agents. L'économie se caractérise par un certain nombre d'arbres dont les fruits sont parfaitement identiques et représentent l'unité de consommation. On fait également l'hypothèse que tout stockage étant impossible, la totalité des fruits est consommée. Cette modélisation, appelée *économie de dotations* repose enfin sur le concept *d'agent représentatif* puisque tous ses membres sont supposés semblables. L'auteur déduit l'optimum pour chaque individu et établi la relation qui deviendra le fondement de la théorie de l'évaluation des actifs, basée sur la consommation. Cette équation, parfois présentée comme l'*équation d'Euler* s'écrit :

$$p_t = E_t\left(m_{t,t+1}x_{t+1}\right)$$

avec p_t le prix à la date t de l'actif, E_t l'opérateur espérance conditionnée à l'information détenue à la date t, x_{t+1} le flux généré par l'actif à la date $t+1$ et $m_{t,t+1}$, le facteur stochastique d'actualisation pouvant encore s'écrire sous forme d'utilité marginale $m_{t,t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$. Cette équation peut encore s'exprimer : $p_t u'(c_t) = E_t\left(\beta u'(c_{t+1})x_{t+1}\right)$ et s'interpréter de la manière suivante : *la perte d'utilité générée par l'achat d'une unité additionnelle d'actif doit correspondre au gain espéré et actualisé d'utilité marginale consécutive au flux additionnel* x_{t+1}.

Par la suite, Mehra et Prescott reprendront cette formulation en 1985 dans le cadre d'un modèle avec processus de Markov à deux états, pour montrer dans le cas particulier d'une fonction d'utilité impliquant une élasticité de substitution constante, que les prédictions du modèle de Lucas ne s'accordent pas aux données du marché américain. En effet, celles-ci prises entre 1889 et 1978, montrent qu'à moins de considérer une volatilité de la consommation particulièrement haute ou bien une aversion pour le risque des agents nettement irrationnelle, la théorie ne parvient pas à rendre compte des réalités de marché. En effet, si l'on considère les données actuelles relatives aux dépenses en matière de consommation, nous n'observons pas la volatilité attendue par le modèle. Toutefois, certaines études suggèrent de nouvelles mesures de la consommation comme Savov (2011) qui propose d'utiliser la quantité de déchets des ménages comme alternative et montre que celle-ci s'accorderait d'avantage avec les pré requis théoriques. Par ailleurs, on observe aussi que certaines économies comme celle que nous présentons au cours du premier chapitre, impliquent une consommation bien plus volatile que nos standards actuels.

Lors de sa parution, la découverte de Mehra et Prescott s'est avérée retentissante et motiva au cours des décennies suivantes une multitude de travaux similaires. Nous noterons par ailleurs que R. Shiller fut probablement le premier à s'interroger sur cette inadéquation en 1982. Quelques années plus tard, P. Weil propose en 1989 d'utiliser une nouvelle classe de préférences, dîtes de Kreps-Porteus (1978), dont la spécificité technique consiste à défaire l'interdépendance entre élasticité de substitution et aversion pour le risque, afin de résoudre l'énigme mise à jour par Mehra et Prescott. Il constata que non seulement cette modification ne suffisait pas à rendre les prédictions du modèle conforment aux observations mais qu'elle impliquait aussi un nouveau questionnement : pourquoi le taux sans risque est-il si faible compte tenu de l'aversion des agents à reporter leur consommation dans le temps. En 1991, Hansen et Jagannathan introduisent les premiers éléments de compréhension pour la variabilité du facteur stochastique d'actualisation. Ils établissent que le ratio des deux premiers moments de ce facteur, plus précisément son écart-type sur son espérance est toujours supérieur ou égal au ratio de Sharpe. Ces travaux se prolongent en 1992 par un approfondissement réalisé par Cochrane et Hansen fournissant de nouvelles conditions à imposer aux facteurs d'actualisation afin de rendre les prédictions conformes aux comportements des prix et des flux générés par les actifs sur le marchés.

L'incapacité avérée du modèle de Lucas a rendre compte des données observées a alors motivé une phase de spécialisation des fonctions d'utilité. Plusieurs études se sont efforcées d'inclure de nouveaux paramètres dans les préférences des agents, notamment le temps de loisir qui semblerait participer pour une large part au programme décisionnel des individus. En effet, lorsque l'on accède à certains biens, leur jouissance suppose une certaine disponibilité. Ainsi, pour une personne allouant la majorité de son temps au travail, l'utilité d'un tel bien s'en révélerait presque nulle. Par ailleurs, certains auteurs comme Jagannathan et Wang (1996) ou Reyfman (1997) ont montré que l'intégration ad hoc du revenu du travail comme facteur permettrait d'expliquer les rendements moyens des actions. Ils se basent en particulier sur une version conditionnelle du modèle d'évaluation des actifs financiers (MEDAF). Dans le même esprit Lettau et Ludvingson (2001a) s'intéressent au ratio agrégé consommation-richesse pour expliquer les rendements des actions. Ils montrent plus précisément qu'un tel facteur détient d'avantage de pouvoir prédictif que les variables communément utilisées, à savoir le *rendement de dividendes* et le *taux de distribution des dividendes.*

Dans cet effort de sophistication des préférences des agents, nous retiendrons également les travaux d'Epstein et Zin (1989) ou Hansen, Sargent et Tallarini (1999) qui développent le cas d'utilités non séparables d'un point de vue temporel. En d'autres termes, ils envisagent que l'utilité future espérée puisse affecter le niveau d'utilité actuel. Cette intuition se traduit d'un point de vue technique par une forme récursive qui peut s'écrire de la manière suivante : $U_t = c_t^{1-\gamma} + \beta f\left[E_t\left(f^{-1}U_{t+1}\right)\right]$, le second élément de cette expression, appelé équivalent certain, capture la dimension liée au risque. La propriété majeure de cette fonction réside dans le fait qu'elle permet de défaire le lien existant jusqu'alors dans les formulations standards, entre élasticité intertemporelle de substitution et aversion pour le risque, destinés à être l'inverse l'un de l'autre. La première implémentation de cette innovation technique dans le cadre de l'évaluation des actifs est proposée par Duffie et Epstein (1992). Il est toutefois à noter que les propriétés de ces nouvelles préférences, ne semblent pas révolutionner l'efficience des modèles comme le montre Kocherlakota (1990). Enfin, ces nouvelles spécificités de la fonction d'utilité des agents liée au temps reviennent à considérer des préférences non séparables par rapport à l'état de la nature puisque rien en l'espèce ne distingue le temps des états de la nature. En d'autres termes, on envisage ici que l'utilité marginale associée à un niveau de consommation dans un état donné de l'économie puisse être affectée par ce qu'il se produirait si un autre état s'était réalisé.

Les préférences énoncées plus haut permettent entre autre, de résoudre l'énigme liée au taux sans risque énoncée par Weil mais ne parviennent toujours pas, pour des valeurs modérées d'aversion pour le risque, à expliquer celle mise à jour par Mehra et Prescott. Campbell et Cochrane (1999) proposent une approche légèrement différente où l'utilité dépendrait non seulement du niveau de consommation actuel mais aussi passé. L'intuition principale derrière cette hypothèse est que les individus forment des habitudes de consommation et pourraient ainsi se révéler contre l'idée que leur consommation se réduise d'une période à l'autre. Les préférences des agents peuvent s'exprimer par exemple de la manière suivante : $U_t = E_t\left(\sum_{t=0}^{\infty}\beta^t\frac{(c_t-\lambda c_{t-1})^{1-\sigma}}{1-\sigma}\right)$ avec λ un coefficient qui détermine l'intensité des habitudes. Cette structure permet de capturer un phénomène fondamental en psychologie : la répétition d'un stimulus réduit la perception que l'on en a, ainsi que la réponse qu'il suscite. Plus précisément, elle est à même d'expliquer pourquoi le bien-être ressenti par l'agent semble en général d'avantage lié à un niveau relatif de sa consommation - comparé à son niveau antérieur - plutôt qu'à un niveau absolu. Il est à noter par ailleurs que le concept d'habitudes de consommation s'oppose à celui de durabilité. En effet, si l'on achète un bien durable hier, cela réduit l'utilité marginale d'un achat de ce bien aujourd'hui, alors que pour un bien dont la consommation serait habituelle, nous aurions l'inverse. De plus, si l'on se place à une échelle de temps réduite, on observe que la consommation de la plupart des biens détient une forme de persistance. Il semblerait donc judicieux d'inclure à la fois durabilité et habitudes dans les préférences des agents.

Campbell et Cochrane considèrent toutefois des préférences légèrement différentes de celles évoquées plus haut. En effet, afin de simplifier le problème d'optimisation de l'agent, ils supposent que ses habitudes interviennent de manière purement exogène. En d'autres termes, plutôt que les habitudes d'un individu soient le simple résultat de sa consommation passée, elles dépendraient de l'historique de la consommation agrégée. Cette approche s'inscrit dans l'esprit de travaux antérieurs, notamment d'Abel (1990) ou encore Duesenberry (1949) qui lui, suggère l'importance du revenu relatif. Plus précisément, Duesenberry prétend que chaque agent voit sa richesse en comparaison de celle des autres. Nous reviendrons d'ailleurs sur cette intuition dans le cadre de notre troisième chapitre. Enfin, pour conclure sur la contribution de Campbell et Cochrane, ils supposent par convenance technique que les habitudes répondent avec un retard au niveau de consommation, bien que les études empiriques semblent montrer le contraire. Ils définissent également les habitudes comme non linéairement liées à l'historique de la consommation agrégée afin de garantir une fonction d'utilité toujours finie et positive. Ainsi, leur modèle permet de résoudre l'énigme identifiée par Mehra et Prescott pour des valeurs rationnelles d'aversion pour le risque chez les agents.

Une autre solution aux faiblesses du modèle de Lucas pourrait être apportée par la prise en compte de l'éventualité de désastres. Nous faisons référence ici principalement aux études de Rietz (1988), Barro (2006), Gabaix (2008) et Gourio (2008) pour leur large contribution à cette approche. Rietz fut quelques années après les travaux de Mehra et Prescott, le premier à invoquer le risque de désastre comme solution à l'inadéquation entre prédictions théoriques et observations empiriques. Toutefois, son intuition ne fut alors pas partagée par la communauté des chercheurs et il fallut attendre les travaux de Barro pour remettre à jour ce mouvement de pensée. La contribution majeure de Barro fut de mesurer empiriquement la fréquence et la taille des principaux désastres internationaux, à savoir la *Grande Dépression*, la première et la seconde *Guerre Mondiale*, pour démontrer qu'ils suffisent à étayer les argument de Rietz. Il apporte ainsi la preuve qu'ils peuvent réconcilier les prédictions du modèle de Lucas avec les réalités de marché. En particulier, son approche permet de résoudre le problème soulevé par Weil sur le taux sans risque ainsi que l'excès de volatilité des prix des actions observé par Shiller (1981). Plus précisément sur ce dernier point, Shiller remarqua qu'un modèle supposant un taux d'actualisation constant n'implique pas une variation suffisante des prix. Enfin, Gabaix reprend le raisonnement de Barro-Rietz tout en affinant le modèle. Il autorise les différents désastres impactant l'économie d'être d'ampleur variable afin d'évaluer leur implication sur les prix des actions et des obligations ainsi que la prédictibilité de leur rendement. Ses travaux permettent de résoudre à la fois les difficultés d'ores et déjà surmontées par l'approche de Barro, mais encore une multitude d'autres inconsistances du modèle standard que nous n'énumérerons pas ici. Gourio lui, poursuit la généralisation des approches précédentes en abandonnant l'hypothèse que les désastres sont permanents. Il considère également les phénomènes de reprise économique apparaissant souvent à la suite de ces chocs et les incorpore au modèle de Barro-Rietz en montrant que leur effet repose sur l'élasticité intertemporelle de substitution. Enfin, dans un même esprit, l'histoire des Moulins sur laquelle s'appuie toute la partie empirique de ce travail de thèse est jalonnée de désastres totalement exogènes dont nous pouvons aisément évaluer la fréquence. Cette économie pourrait ainsi parfaitement convenir à l'application d'un des modèles évoqués plus haut d'autant qu'elle ne souffre pas du *peso problem*. Toutefois, comme nous les montrerons au cours du premier chapitre, les propriétés statistiques des données sur la consommation nous permettent justement d'éviter toute sophistication technique, et qu'un modèle standard avec utilité séparable dans le temps suffit à expliquer les observations.

Une importante hypothèse réalisée dans l'approche de Mehra et Prescott est que la consommation d'un agent est égale à la consommation agrégée. Toutefois, il n'est pas vrai que la variabilité de la consommation individuelle corresponde à celle du niveau agrégé puisque la richesse de chaque individu peut être impactée par des chocs qui lui sont propres. Ainsi, ce constat remet fortement en question la véracité de l'hypothèse qu'il existe un *agent représentatif* dans l'économie. Nous savons que dans un marché complet, il est possible de répliquer les flux de n'importe quel actif sur la base d'actifs élémentaires, à savoir ceux payant une unité monétaire dans un état donné de la nature et rien dans les autres états. Ainsi, dans un tel environnement, la consommation de tous les membres de l'économie évolue conjointement. L'intégralité de la population partage les risques, puisque lorsqu'un choc survient, il va impacter chacun de manière équivalente. En revanche, si les marchés sont incomplets, les individus s'exposent à des risques idiosyncratiques purement aléatoires qui vont donc les affecter de manière asymétrique. L'incomplétude du marché implique par ailleurs qu'ils ne pourront pas non plus se couvrir par une détention adaptée d'actifs. C'est en ce sens que les travaux de Constantinides et Duffie (1996) apportent une contribution majeure. Ils construisent un modèle où les risques idiosyncratiques peuvent être ajustés afin de générer n'importe quel profil de consommation agrégée ou de prix. La difficulté principale réside ici dans le fait que lorsque ces risques ne sont pas corrélés avec les rendements, leur implication sur les prix devient nulle ; alors que s'ils le sont, les agents s'organisent pour se couvrir par le marché et le résultat reste le même. Ainsi, une issue demeure d'exploiter la non linéarité de l'utilité marginale. C'est ce que développent Constantinides et Duffie en considérant des chocs non corrélés avec les rendements qui vont, au travers de l'utilité marginale non linéaire, devenir corrélés avec le marché. Par conséquent, les agents ne pourront s'en affranchir par le biais des actifs, et l'impact sur les prix sera préservé. Cette structure permet alors de résoudre à l'instar des modèles précédents, bon nombre d'écueils rencontrés par les formulations standards, d'autant qu'elle ne fait appel à aucune friction particulière et l'aversion pour le risque des agents reste dans des valeurs acceptables. Enfin nous citerons l'étude de Keim et Stambaugh (1986) qui propose une autre approche dans laquelle, une faible quantité de volatilité de la consommation et un haut niveau d'aversion pour le risque permettent de rendre compte de la prédictabilité des rendements.

D'autres études ont tenté d'adapter le modèle de Lucas en invoquant des coûts de transaction ou en introduisant une dimension liée à la production. L'effet de ces deux nouveaux facteurs s'est révélé non convainquant puisque le premier supposerait des frictions nettement irréalistes pour s'accorder avec les données alors que le second dévie de l'objectif de rendre compte de la consommation et non de la façon dont elle est produite.

Afin de compléter au mieux cet bref état de l'art, nous présentons à présent la littérature relative aux risques de long terme. La prise en compte de ces risques dans les modèles permet avant tout de voir certaines implications sur la valeur présente des paiements générés par un actif, puisque celle-ci dépend étroitement de leur exposition aux risques macro-économiques futurs. Plus précisément, il s'agit simplement de la "distance" entre deux points du temps : la date à laquelle on évalue lesdits paiements et l'instant où ils seront véritablement délivrés, cette distance pouvant être considérablement grande. Par ailleurs, rappelons qu'en vertu des standards de la théorie, le prix d'un actif se définit comme la somme pondérée des paiements futurs espérés, ainsi toute information sur ces derniers renseignera également sur l'évolution du prix de l'actif. L'intérêt majeur de ce type d'approche est de s'inscrire dans un raisonnement plus large portant sur la manière dont les prix incorporent l'information et en particulier sur l'impact qu'auront les flux de long terme sur la valeur présente. De plus, cela se prête particulièrement bien au contexte historique utilisé dans ce travail de thèse car dans le cadre des Moulins du Bazacle, les données ont révélé que les dividendes futures contribuaient significativement au prix des parts de la compagnie. D'autant qu'en matière de risque, les Moulins rencontrèrent au cours des siècles de nombreuses crises, d'origine parfois naturelle avec la destruction des infrastructures, politique lors d'événements belliqueux, ou sociale marquées par des grèves, obligeant ainsi ses gestionnaires à étudier les opportunités qui limiteraient les pertes. Nous citerons ainsi dans ce domaine les travaux de Bansal et Yaron (2004) qui exploitent la capacité des risques de long-terme à rendre les prédictions des modèles plus fidèles aux données observées. Ils s'accompagnent des études de Bansal, Dittmar et Lundblad (2005), Hansen, Heaton, Li (2005) et Lettau et Ludvingson (2005) qui montrent que ce type de risques a un fort pouvoir explicatif pour les rendements des actifs.

Comme nous l'avons d'ores et déjà évoqué, l'hypothèse d'existence d'un *agent représentatif* dans l'économie manque fortement de réalisme et se trouve potentiellement à l'origine de nombreux échecs de la théorie. Il suffit pour s'en convaincre, de considérer simplement le fait que les acteurs de marché diffèrent avant tout par leur croyance, leur éducation, leur richesse, leur goût et par d'autres caractéristiques propres qui les incitent à opter pour différents comportements. Ces asymétries génèrent en particulier des positions différentes relatives au risque, qui vont mener en s'agrégeant à des situations économiques difficilement prévisibles par l'étude des actions d'un seul agent. L'une des illustrations les plus frappantes reste ici l'incitation à l'échange puisqu'un des écueils majeurs du modèle de Lucas est son incapacité à justifier les niveaux de volume observés sur les marchés. En effet, si tout le monde à la même évaluation d'un bien, personne ne trouve d'intérêt à l'échange.

Nous reviendrons plus loin sur ce dernier point qui motive le travail théorique présenté lors du second chapitre. Ainsi, permettre aux agents d'être dissemblables en termes de préférences, à défaut d'augmenter la complexité des modèles, détient l'intérêt primordial de les rendre plus à même d'expliquer la réalité. Dumas (1989) proposa une première contribution majeure en décrivant l'interaction entre deux investisseurs ayant des aversions pour le risque différentes et un niveau d'impatience similaire. Il observe que cette asymétrie implique une fluctuation aléatoire de la distribution de richesse parmi les deux agents et constitue une incertitude vis-à-vis de laquelle ces derniers souhaitent se prémunir. Ainsi ce nouveau facteur intervient dans la composition optimale de leur portefeuille. Bien plus tard, Coen-Pirani (2004) s'intéresse aux effets d'une population hétérogène en termes d'aversion pour le risque sur la distribution des richesses parmi les agents sur le long terme, dans le cadre d'une économie de dotation à la Lucas. Plus précisément, le modèle suppose deux individus dont les préférences sont du type Epstein-Zin et délivre le résultat surprenant que pour un certain choix de paramètres, les agents redoutant le moins le risque vont finir par dominer le marché. Plusieurs travaux se sont ainsi intéressés à analyser de manière empirique qu'elle était la répartition réelle de la tolérance pour le risque au sein de différents échantillons d'individus. L'étude de Guvenen (2006) se consacre elle, a identifier comment l'hétérogénéité des préférences couplée à différents types de participation des agents peut améliorer la performance du modèle dans l'évaluation des actifs. Elle suppose en particulier que seuls les agents ayant une haute élasticité intertemporelle de substitution vont détenir des actions alors que les autres n'investiront que dans des obligations. Dans le même esprit mais cette fois sans contrainte sur la participation des agents, nous trouvons les travaux de Bhamraet Uppal (2014), Chabakauri (2013), Gârleanu et Panageas (2008), ainsi que Cozzi (2011) qui eux, se placent d'avantage dans le cadre de marchés complets. En particulier Gârleanu et Panageas proposent un modèle basé sur un continuum de générations qui se chevauchent, composées d'individus détenant différentes dotations et différentes aversions pour le risque. Leurs travaux mènent au résultat intéressant que l'équilibre du prix des actifs correspond à celui qui caractérise une économie peuplée par un seul *agent représentatif*. La majorité de ces études se concentrent toutefois sur le cas de deux agents en présence, et font l'hypothèse que les mécanismes observés sont directement généralisables au cas où il y aurait plus de deux agents. Une telle généralisation n'est pourtant pas si évidente, puisque lorsque l'on augmente la taille d'une population, les interactions deviennent bien plus complexes et génèrent des mécanismes bien plus imprévisibles. Un exemple frappant demeure encore le niveau de volume d'échanges sur les marchés que nous étudions dans le cadre du second chapitre. On observe ainsi que les propriétés obtenues dans le cas simple de deux agents sont très éloignées de ce qui se produirait si le nombre d'agents était plus élevé. Comme nous l'évoquions plus haut, certains modèles basés sur l'hypothèse d'une population hétérogène sont dédiés à la compréhension de la distribution des richesses. Nous relèverons en particulier les travaux de Achdou et. al. (2017) qui reprennent les modèles de Aiyagari (1994), Bewley (1986) et Huggett (1993) en temps continu. Au travers de la résolution d'équations classiques dîtes d'Hamilton-Jacobi-Bellmann et de Kolmogorov, ils apportent ainsi d'importantes contributions théoriques au domaine. Il est enfin important de noter que dans certains cas, l'hétérogénéité injectée dans un modèle à la Lucas ne suffit pas à rendre compte du volume. En effet, comme nous l'avons évoqué, si l'on considère un marché complet ou pouvant se compléter par la détention de portefeuilles adaptés comme dans Kreps (1982), alors les prix évoluent comme s'il existait un agent représentatif dans l'économie et l'équilibre ne nous livre aucune information quant au volume échangé.

Il existe également tout un pan de la littérature basée sur l'hétérogénéité des agents consacrée à l'étude du volume sur les marchés. Varian (1985) établi assez clairement dans son étude les raisons pouvant inciter deux agents à échanger. Il énonce ainsi que les individus interagissent sur les marchés suite à des asymétries de dotations, de préférences ou de croyances. En matière de dotations, nous entendrons également ici les chocs de liquidité pouvant impacter un agent. Il distingue par ailleurs deux grands groupes de croyances : celles générées par des différences d'opinions, reliées aux probabilités-à-priori, et celles issues des déséquilibres au niveau de l'information, capturée par la fonction de vraisemblance. Il montre alors que pour des préférences identiques et une tolérance pour le risque qui ne croît pas trop rapidement, les actifs pour lesquels les opinions sont plus dispersées vont avoir un prix plus bas et faire l'objet d'échanges plus importants. Ses travaux s'inscrivent dans l'esprit d'études précédentes tel que Lintner (1969), Mayshar (1983) ou Rubinstein (1975) qui développent elles aussi des approches basées sur des différences de croyances parmi les individus. En ce qui concerne le cas de dotations asymétriques, nous retiendrons la contribution de Grossman (1976).

Un point important ici est que ces nouveaux modèles intégrant l'hétérogénéité des agents ont désormais la capacité de capturer les caractéristiques relatives au volume. Il convient donc à présent de considérer la littérature qui lui est dédiée. Pour reprendre les arguments de Varian, nous pouvons affirmer que les individus ont l'incitation d'interagir sur un marché dès l'instant qu'ils sont dissemblables. En d'autres termes, leurs différences, peu importe leur nature, impliquent qu'ils formulent chacun une évaluation propre du même bien. Ainsi, puisque les prix respectifs qu'ils assignent ne sont plus égaux, il en résulte immédiatement un gain potentiel à l'échange. Plusieurs auteurs ont en ce sens proposé des structures où les agents n'attribuaient pas la même valeur à un actif. La contribution théorique majeure dans ce domaine, est apportée par Karpoff (1986) qui propose un modèle où les investisseurs repensent à chaque période et de manière idiosyncratique, le prix auquel ils sont prêts à acquérir l'actif. Il montre que deux individus peuvent détenir l'incitation d'acheter ou de vendre même en l'absence d'information. Le seul besoin de liquidités ou la volonté de spéculer suffisent à motiver l'échange. Cette dimension s'associe étroitement aux travaux que l'on présente au cours du second chapitre, puisqu'il s'agit également d'un modèle plongé dans une économie dépourvue de toute information inhérente à l'actif. Comme on le souligne d'ailleurs, l'incorporation de l'information pourrait dans certains cas, impliquer une situation de non-échange en vertu de certains théorèmes issus de la théorie des incitations. Toutefois, bon nombre d'études se concentrant sur des évènements de marché déterminants, postulent l'existence d'un lien entre information et volume. On relève par exemple Kiger (1972) qui analyse les réactions des prix et du volume sur le marché américain à la suite de l'annonce des profits trimestriels mais aussi Morse (1981), Pincus (1983) et Bamber (1985,1986). D'autres cependant ont un avis bien plus réservé sur la nature de ce lien. Par exemple Verrechia (1981), montre que lorsque le niveau de volume ne réagit pas à la divulgation de nouvelles informations, cela implique un parfait consensus entre les investisseurs. Toutefois, il affirme également que la contraposée n'est pas vérifiée. En d'autres termes, il peut y avoir des variations du volume même lorsque les agents interprètent l'information de manière identique, du fait qu'ils n'aient pas nourri les mêmes espérances au préalable par exemple. Tout comme l'étude d'évènements montre qu'il existe souvent une latence dans l'ajustement du prix, Morse (1980) fourni un résultat similaire pour la dynamique du volume dans laquelle on relève une persistance de l'information. Pfleiderer (1984) propose lui, un modèle où l'information agrégée n'est pas entièrement révélée par le prix. Il suppose plus précisément que cette dernière est distribuée à chaque agent sous forme de deux composants, le premier étant commun à tous alors que le second est propre à chacun. Son modèle mène au résultat contre intuitif indiquant que le volume est une fonction décroissante du niveau de désaccord entre les individus. Évidemment, un tel mécanisme n'étant pas observé sur les marchés, il ne mérite pas d'être développé plus avant. Un axe de recherche important est aussi dédié à la relation entre volume et variation du prix. En effet, il semblerait qu'il existe une relation positive entre le nombre d'actifs échangés et l'ampleur de la variation du prix comme le montre une étude de Copeland (1976) proposant un modèle où les investisseurs reçoivent de manière séquentielle des fragments d'information commune. Jenny, Starks et Fellingham reprennent cette approche en autorisant certaines frictions ainsi que les ventes à découvert. Toutefois, ces études se basent systématiquement sur une vision purement dichotomique des investisseurs qui sont alors rangés par catégories comportementales : *optimistes* ou *pessimistes*, *bulls* ou *bears*, *fundamentalist* ou *chartist* ou encore *informés* ou *non informés*. Une telle classification des agents étant pour le moins restrictive, nous avons ainsi choisi dans l'étude théorique présentée au second chapitre, de ne pas la retenir. Il est enfin intéressant de citer la contribution des modèles dits de *distribution mélangées*. Entre autres, Clark (1973), Tauchen et Pitts (1983) et Harris (1983) prédisent que le volume échangé d'un actif est positivement lié à l'ampleur de la variation de son prix au cours d'intervalles de temps fixées ou sur une transaction donnée. Enfin plus récemment, une importante contribution a été apporté par Lo et Wang (2001) qui propose une analyse assez générale du volume. Ils proposent entre autre un nouvel outil de mesure de l'activité d'échange sur les marchés en justifiant sa légitimité et ses avantages vis-à-vis de la mesure classique.

Ils déduisent en outre les nombreuses implications que peut avoir le volume dans le domaine de la théorie actuelle du portefeuille, en particulier sur les *théorèmes de séparation* en deux ou plusieurs fonds mais aussi les implications sur le modèle intertemporel d'évaluation des actifs.

Il est important de noter que l'étude théorique présentée au second chapitre est à la frontière de trois grands axes de recherche. Le premier étant évidemment celui dédié à l'analyse du volume sur les marchés financiers puisque le modèle a vocation de le justifier. Le second étant la grande famille de structures intégrant une population hétérogène d'individus. Enfin, le troisième se définit comme connexe au second puisqu'il s'agit de la littérature émergente consacrée aux échanges et interactions sociales sur des supports de réseaux. C'est de ce dernier que nous allons traiter à présent. Faire l'hypothèse d'une population hétérogène d'agents est une chose, mesurer comment ils interagissent en est une autre. En effet, lorsque l'on introduit des différences, quelque soit leur nature, on motive ainsi l'émergence d'interactions. Comme nous l'avons évoqué plus haut, ces interactions s'illustrent en particulier par la transactions d'actifs. Prenons l'exemple d'une action évaluée à \$10 par un individu A et à \$11 par un individu B. Si l'on considère un marché de gré à gré sans aucune friction et que ces individus sont prêts à vendre ou à acheter l'action au prix qu'ils l'évaluent, en supposant par ailleurs qu'un régulateur fixe le prix de marché à \$10.5, il demeure bel est bien une incitation à échanger dans le seul but spéculatif de réaliser le profit mutuel de \$0.5. Néanmoins, aucune transaction ne pourra être réalisée sans la condition évidente que les agents soient mis en contact ou sans intermédiaire pour réaliser l'opération. En effet, si chacun ignore l'existence de l'autre, bien qu'ils soient potentiellement disposés à échanger, toute interaction demeure impossible. Évidemment, une telle difficulté ne peut survenir sur un marché organisé où, une entité planificatrice - comme une place boursière - se charge d'organiser les échanges. Revenons à présent à notre marché de gré à gré et supposons que les individus A et B soient membres de deux groupes distincts d'investisseurs - c'est-à-dire qu'ils sont en relation avec ces individus - partageant exactement leur évaluation de l'action. Ils n'ont aucune incitation à échanger au sein de leur propre groupe malgré les connections qu'ils y entretiennent. Le nombre d'échanges sera ainsi nul tant que A et B ne seront pas mis en contact. Par conséquent, la structure même du réseau social sous-jacent à ce type de marché est d'une importance capitale pour déterminer le niveau d'interaction. D'autant que lorsqu'on laisse la liberté aux agents de négocier, en indexant par exemple leur pouvoir de négociation sur la place relative qu'ils occupent au coeur du réseau, les mécanismes d'échanges se révèlent bien plus complexes. On peut ainsi étudier la formation des prix et tester certaines conditions largement répandues dans la littérature classique d'évaluation des actifs, comme la loi du prix unique.

Le paradigme de l'équilibre général prévoit dans sa formulation, des économies composées de nombreux agents sans aucun pouvoir de marché. De plus, la résolution des modèles suppose la plupart du temps que chaque individu est en mesure d'échanger avec n'importe quel autre partenaire sans délai ni coût de transaction. Les biens sont enfin considérés comme infiniment divisibles et les agents comme des preneurs de prix. Un tel environnement idéalisé, au delà de sa capacité à générer une allocation efficiente des ressources, demeure parfaitement illusoire. C'est ainsi qu'il est capital d'abandonner progressivement ces hypothèses, bien qu'au risque d'augmenter la complexité des modèles, afin de saisir les mécanismes sous-jacents à la formation des prix et aux autres conséquences de l'interaction entre les agents. En particulier, l'idée que ces derniers soient preneurs de prix est, dans la majorité des cas, non recevable puisqu'elle n'aurait de légitimité que dans le cas extrême d'une large population d'investisseurs sans aucun pouvoir de marché. Ainsi, lorsque l'on considère un nombre plus réduit d'individus, il n'est plus raisonnable de conserver cette hypothèse. Il est à noter également que de nombreux facteurs interviennent dans l'organisation des échanges. En effet, la distance géographique, les liens sociaux ou professionnels, et les régulations locales vont déterminer quel lien portera effectivement une transaction ou non. Un exemple particulièrement édifiant est celui des biens agricoles dans les pays en développement. En général, il existe plusieurs intermédiaires qui achètent auprès des producteurs et revendent aux consommateurs. Toutefois, les limites du système de transport, la périssabilité des produits, la difficulté d'accès au capital, entraînent que producteurs et consommateurs ne peuvent s'adresser qu'à un nombre restreint d'intermédiaires comme le montre Fafchamps et Gabre-Madhin (2006) ou Barrett et Mutambatsere (2008). Ainsi dans un tel environnement, le mécanisme de formation des prix demeurent bien plus complexe et des axiomes comme la loi du prix unique n'ont définitivement plus cours.

Plusieurs études ont ainsi entrepris d'utiliser les outils de la théorie des jeux pour étudier, dans un contexte non coopératif, comment la composition de l'économie et le processus de rencontre entre les agents pouvaient mener à un équilibre général. Nous retiendrons par exemple les travaux de Rubinstein (1982), Rubinstein et Wolinsky (1985, 1990) et Gale (1986, 1987) qui étudient les mécanismes d'échanges bilatéraux sur des marchés dynamiques. Notons également que la structure des marchés réels suppose souvent une profonde hétérogénéité dans l'offre et la demande locales, impliquant des déséquilibres dans les forces de négociation de chaque individu qui vont alors émerger, à la fois de leurs opportunités d'échanges immédiates mais aussi de l'architecture globale du marché. Ces multiples asymétries vont également donner lieu à des allocations sous-efficientes dans le cadre de négociations décentralisées. Afin d'être intégrés dans un modèle, tous ces traits caractéristiques des interactions entre agents nécessitent des outils théoriques capables de les mesurer. Ces outils sont fournis par la théorie des graphes qui permet une représentation très intuitive d'un marché en un faisceau de points matérialisant les agents, et des liens entre ces points indiquant l'existence d'une connection sociale. Deux individus liés par un de ces liens peuvent ainsi potentiellement s'engager dans un échange. Il est toutefois important de remarquer que deux agents connectés ne vont pas obligatoirement réaliser une transaction. En effet, comme nous le développons dans l'étude théorique présentée au second chapitre, d'autres forces sont à l'oeuvre, notamment liées à la topologie locale dans laquelle le lien entre les individus prend place. Par ailleurs, l'apport des graphes dans la représentation en réseaux des marchés est d'autant plus précieux que, pour un modèle supposant l'hétérogénéité des agents, il est souvent déterminant de connaître le schéma des interactions. Prenons à ce titre, l'exemple du prix qui pourrait potentiellement varier selon chaque transaction. On pense également aux fonctions d'utilité basées sur le revenu relatif qui intègrent comme variable, la composition du voisinage de l'agent. Ces fonctions interviennent particulièrement dans l'étude des statuts, qui fera d'ailleurs l'objet des travaux présentés au troisième chapitre de ce travail de thèse. Ainsi, cette nouvelle approche basée sur les réseaux sous-jacents à l'activité de marché, soulève d'importantes questions. Elle interroge par exemple sur la caractéristique topologique responsable de la formation locale des prix, mais aussi sur les conditions permettant la validité de la loi du prix unique. On peut encore se demander comment les pouvoirs de négociation émergent-ils? Pour quel forme de réseau l'organisation des échanges devient-elle efficiente? Quelle serait l'évolution dynamique d'un graphe si ses membres pouvaient influer sur les liens? Et bon nombre d'autres questionnements relatifs à des phénomènes de marché aujourd'hui encore inexpliqués par la théorie classique.

Les contributions majeures dans le domaine de l'utilisation des graphes pour la compréhension des mécanismes de marché débutent avec Corominas-Bosch (2004) et Polanski (2007). Ces auteurs considèrent différentes formes de graphes bipartis, fixés de manière purement exogène, dans lesquels négocie une population d'agents. Évidemment cette population est partitionnée en *vendeurs* et en *acheteurs*, ayant la possibilité d'échanger selon les opportunités d'interaction fournies par le réseau et ce, jusqu'à ce que le marché s'équilibre. Ce modèle s'inscrit donc dans la famille des structures intégrant l'hétérogénéité des agents, entièrement basées sur une vision dichotomique de l'économie. En particulier Corominas-Bosch suppose que chaque joueur fait une série d'offres publiques qui peuvent être acceptées par n'importe quel joueur issue de la classe opposée. Elle relie ainsi la structure du réseau à la distribution des prix à l'équilibre. De plus, et c'est un point qu'elle partage avec Polanski, son modèle établi une relation entre les gains à l'équilibre et la décomposition de Gallai-Edmonds, résultat classique de la théorie des graphes. Abreu et Manea (2012) étudient aussi les mécanismes de marchés décentralisés en s'appuyant sur la richesse de leur dynamique. Ils proposent un modèle où les joueurs sont mis en relation de manière aléatoire mais où chaque association d'individus ne résulte pas toujours par un accord, ils peuvent en effet refuser d'échanger une première fois pour finalement y convenir plus tard, permettant ainsi l'existence d'équilibres multiples. Ils montrent ainsi que la décentralisation entraîne l'inefficience des échanges bien que celle-ci puisse toutefois être évitée dans le cadre d'une politique de sanctions et de récompenses. Nous citerons également les études de Kranton et Minehart (2001), Rahi et Zigrand (2006), Gale et Kariv (2007), Gofman (2011) et encore Nava (2013) comme contribuant de manière substantielle à la littérature. La plupart de ces travaux restreignent le nombre de transactions à une unité d'actif à la fois, ce que nous avons également choisi par commodité dans le modèle présenté au second chapitre de ce mémoire. Babus et Kondor (2018) proposent quant à eux, un cadre où les agents peuvent au contraire fixer des quantités arbitraires. Comme nous l'avons évoqué, les modèles basés sur les structures de réseaux décrivent avant tout le cas de marchés de gré à gré. Ils font ainsi appel également à la littérature consacrée aux mécanismes de recherche de liens, développée entre autres par Duffie et. al. (2005), Vayanos et Weill (2008) ou encore Afonso et Lagos (2012). Au cours de l'évolution de la théorie, les modèles ont donc délaissé progressivement l'hypothèse d'*agent représentatif* pour intégrer les différences entre les individus dans la détermination de l'optimum. Puis, convenant qu'il était nécessaire de comprendre comment ces différences étaient mises en jeu, c'est-à-dire comment allaient-elles interagir, on a introduit la structure décrivant les liens entre les agents comme fondement des mécanismes de marché. Toutefois, une dimension reste encore partialement ignorée, il s'agit de la manière dont ces réseaux se forment. Bien que certains modèles endogénéisent cette formation, la plupart se basent encore sur des structures déterminées de manière exogène. Pourtant, la réalité sociologique nous montre sans cesse le contraire, les connections entre les individus n'apparaissent pas de manière purement aléatoire. Des groupes sociaux émergent dans toute société et se composent de membres, aux caractéristiques communes, d'avantage connectés entre eux qu'avec le reste de la population. C'est ainsi que le modèle présenté au second chapitre de ce mémoire utilise un processus de graphes aléatoires dont la topologie respecte l'intuition que les individus se regroupent sur la base de leur préférences en matière de risque. A notre connaissance, une telle structure n'a encore jamais été traitée.

Il nous reste enfin à évoquer la littérature consacrée aux statuts sociaux. Les économistes se sont longtemps cantonnés à étudier la dimension monétaire qu'ils considéraient être le seul système de récompenses existant alors que les sociologues eux, se sont également intéressés au concept de statut comme nouvel objet du bien-être des agents. Ce concept a été introduit par Max Weber qui le voyait intimement lié à la richesse de la manière suivante : l'argent apporte le statut qui à son tour, apporte la puissance économique. Il est d'avantage compris aujourd'hui comme la position relative occupée par une personne au sein d'un groupe d'individus selon un ordre social communément établi. Par ailleurs, le statut social confère à celui qui le porte le privilège de bénéficier de rapports plus avantageux avec les autres membres de la société. Ainsi un individu possédant une place sociale élevée espère être traité plus favorablement lorsqu'il s'engage dans des interactions sociales ou économiques. Ces avantages peuvent être de différentes natures, soit de purs biens de marché ou services, soit être l'accès à certains biens non échangeables, soit encore correspondre à un transfert d'autorité pour accéder à un rôle prédominant. Ainsi, l'acquisition du statut revêt un véritable intérêt pour les agents qui sont ainsi prêts à réallouer leurs ressources et diminuer la part jusqu'alors consacrée à leur consommation. Il est également à rappeler que lorsqu'un individu accède à un titre particulier, ce dernier n'a de légitimité qu'au sein du groupe le reconnaissant comme tel. De plus, cette reconnaissance sociale n'est en général pas irrévocable puisqu'elle suppose que l'agent observe par la suite un comportement conforme aux honneurs qui lui ont été rendus sous peine d'être destitué. Un exemple particulièrement édifiant est fourni par les hiérarchies simples des corps militaire, professionnel ou encore civil. Le premier attribue des grades sur la base de critères comme l'ancienneté, la bravoure et les risques encourus alors que le second lui, met d'avantage l'accent sur les compétences et l'expérience. Les statuts qu'ils délivrent tous deux impliquent à la fois un gain substantiel d'autorité mais aussi l'accès à certains cercles. Nous reviendrons d'ailleurs sur cette dimension liée aux réseaux par la suite. Enfin, dans le civil, un agencement existe aussi puisqu'un individu peut recevoir une décoration ou un ordre lorsque de par ses actes, il se distingue positivement des membres de la société. Toutefois, chaque membre de ces trois corps accédant à une position particulière, peut faire l'objet d'un limogeage, d'une dégradation ou d'une disgrâce si son comportement s'oppose aux normes associées à l'attribution de leur statut. Plus précisément, il est important de remarquer qu'il existe des pré-requis sociaux sur la base desquels les critères de sélection vont être portés. Prenons pour exemple certaines distinctions exigeant que l'individu pressenti pour les recevoir n'ait jamais fait l'objet de condamnation judiciaire par le passé puisqu'en réalité, la personne honorée par le titre n'est pas le résultat d'un ordre effectué sur l'intégralité de la population mais plutôt sur une sous-population d'ores et déjà constituée par un pré-ordre. Pour s'en convaincre, nous invoquerons le fait que par le passé, la discrimination présente dans de nombreuses sociétés écartait d'emblée les femmes mais aussi les hommes issus de certaines ethnies, de l'accès aux titres honorifiques ou à des positions sociales privilégiées. Il est important de voir également que l'acquisition d'un statut, au même titre que n'importe quel autre critère d'appartenance sociale, garantit l'accès au sous-groupe dont les membres partagent ce même statut. En effet, il est largement accepté en sociologie que les individus ont tendance a se regrouper lorsqu'ils ont des éléments en commun et ce, quelque soit leur nature. Les membres d'un groupe deviennent alors par définition plus liés, plus inter-connectés qu'avec le reste de la population. C'est cet aspect que nous exploitons d'ailleurs dans le second et le troisième chapitre de ce travail de thèse. Nous mettons ainsi l'accent dans notre modélisation sur le fait que deux agents appartenant au même groupe ont potentiellement plus de chance de se connaître que s'ils proviennent de deux groupes différents. Ce mécanisme est également à l'oeuvre pour les statuts puisqu'un individu honoré d'un titre complète son identité d'un élément qu'il partage désormais avec un nouvel ensemble d'individus. Il va ainsi développer les connections sociales associées à ce nouveau groupe. Par ailleurs, il est à noter que tout agent n'a pas la capacité matérielle ou temporelle d'entretenir simultanément un grand nombre de liens au sein d'une population. Certains de leurs liens connaissent alors des dégradations successives jusqu'au point d'être définitivement rompus. C'est pourquoi, il est commun de voir une migration des agents d'un groupe à un autre plutôt qu'une appartenance simultanée à de multiples structures.

Revenons à présent aux principaux travaux composant cette littérature. En économie, Adam Smith (1776) reconnaissait déjà l'importance de l'honneur et de l'estime comme autant de motivations aux actions des agents. Marshall (1890) affirmait lui, que le désire d'obtenir l'approbation et surtout d'éviter le mépris de ses pairs constituait un moteur d'uniformisation des classes sociales. Néanmoins, la théorie a dû, au cours de son développement, justifier avant tout la manière dont les statuts étaient obtenus. Elle a ainsi fait la distinction entre ceux émanant des caractéristiques intrinsèques des individus de ceux générés uniquement par leurs actions. Ensuite, il a fallu définir sous quelle forme un agent bénéficie du statut, à savoir faut-il l'intégrer directement dans sa fonction d'utilité ou bien établir un nouveau mécanisme par lequel le statut améliore l'ensemble de ses opportunités. Veblen (1899) lui, introduit l'idée que le bien-être d'un agent pourrait intimement dépendre de celui des autres. Il affirme en outre que pour susciter l'estime et le respect des hommes, il convient non seulement de détenir richesse et pouvoir mais aussi de les mettre en évidence. Il parle alors de consommation ou de loisir *ostentatoires*. A ce titre, Hirsch (1977) proposera plus tard la distinction entre deux types de biens, les *positionnels* dont la consommation signalent un niveau élevé de statut, et les *non positionnels* qui n'influent en rien sur la position sociale. Le concept de consommation relative développé par Veblen constitue par ailleurs une solution au paradoxe d'Easterlin (1974) que le bonheur apparent des individus est positivement corrélé en tout point du temps au revenu alors que l'augmentation de ce dernier ne semble avoir aucun effet sur le bonheur des agents. Plusieurs travaux ont alors intégré ce concept dans différents modèles. On retient ainsi les études de Hopkins et Kornienko (2004), Layard (2005), Blanchflower et Oswald (2004), Arrow et Dasgupta (2009). Plus proche de l'analyse que nous présentons au troisième chapitre en termes de modélisation, nous trouvons Ghiglino et. al. (2010) ainsi que Immorlica et. al. (2017). En effet, les premiers proposent un modèle où l'utilité des agents est affectée par la consommation de leur voisins. Ils utilisent ainsi une structure de réseau afin de capturer la composition de ce voisinage et démontre qu'à l'équilibre, les prix et les niveaux de consommation sont des fonctions d'un simple paramètre topologique qui est la centralité du graphe sous-jacent. Les derniers se basent quant à eux aussi sur une représentation de l'économie sous forme de réseau où les agents effectuent des actions pour lesquelles est associé un coût. Ces actions engendrent un bénéfice et confère un statut. Ils introduisent ainsi une nouvelle mesure de la connectivité du réseau, appelée *cohésion*, destinée à rendre compte des incitations des individus à rechercher un statut. Le modèle présenté au second chapitre de ce mémoire se distingue toutefois de ces approches sur l'utilité de l'agent dans laquelle nous incluons un élément capturant le gain (ou la perte) psychologique d'échanger avec une personne dont le statut est supérieur (ou inférieur) au nôtre. Ainsi, cette structure permet de considérer l'existence d'une menace tacite d'exclusion lorsqu'un membre dévie du comportement standard de son groupe. A ce titre, nous pouvons citer les travaux de Cole et. al. (1996) qui s'intéressent à des biens non échangeables sur un marché : les mariages. Ils montrent au travers d'un modèle inter-générationnel qu'à l'équilibre, le mariage par classe est maintenu du simple fait de la menace d'être rétrogradé en cas de non respect de la norme sociale établie par le groupe.

Le contexte historique des Moulins de Toulouse est ainsi parfaitement adapté à l'étude de ces phénomènes puisque cette période de l'histoire se caractérise par une délimitation des classes sociales, et en particulier celles liées au statut, bien plus nette qu'aujourd'hui. Nous avions en effet d'un côté, les nobles et les gens d'église dont la position sociale privilégiée demeure indéniable, puis le reste de la population. Il existait également une classe sociale charnière, celle des bourgeois, se trouvant à la frontière entre ceux qui détenaient un statut et ceux qui en étaient dépourvus. Ainsi, il n'est pas rare d'observer une migration de certains membres de la bourgeoisie vers la classe supérieure. Comme nous le décrivons d'ailleurs dans le second chapitre de ce mémoire, il existait une institution à Toulouse servant à anoblir les riches par l'achat d'un certain bien. Ce dernier point convient donc parfaitement aux théories de Veblen et Hirsch évoquées plus haut.

Ce mémoire de thèse, dont le thème global est l'évaluation des actifs et le volume d'échanges sur les marchés financiers, s'organise de la manière suivante. Chaque chapitre contient un article consacré à un objet de recherche différent. Le premier examine le cas de deux des plus anciennes sociétés par actions connues à ce jour en Europe, à savoir l'histoire des Moulins de Toulouse, dont nous avons pu retrouver les données financières dans les archives de la ville. Sur la base de ce support empirique, unique à l'égard de son contexte historique mais aussi de ses propriétés statistiques, nous nous proposons d'utiliser une méthode économétrique basée sur la minimisation de l'entropie pour montrer qu'un modèle à la Lucas avec une fonction d'utilité de type CRRA n'est pas rejeté compte tenu des données. Nous expliquons comment la volatilité de la consommation mesurée pour cette économie nous permet d'arriver à ce résultat pour des niveaux d'aversion pour le risque faibles.

Le second chapitre est une étude théorique visant à justifier le niveau des échanges sur les marchés financiers sur la base de leur composition en termes de groupes sociaux. Plus précisément, on propose un modèle intégrant l'hétérogénéité des agents en regard à leur aversion pour le risque et supposons en outre que ces derniers forment naturellement des groupes sociaux sur la base de cette caractéristique. On utilise ainsi des processus aléatoires intégrant ce mode de rapprochement des individus pour former le réseau sous-jacent à l'économie. Nous introduisons alors le concept de *canal désirable* comme le lien entre deux agents pouvant potentiellement faire l'objet d'un échange et déduisons le nombre espéré de ces canaux ainsi que le niveau espéré d'échanges avérés. Nous proposons ainsi une caractérisation systématique des graphes des connections sociales en termes d'optimalité des échanges qu'ils permettent. Enfin, nous utilisons certains résultats de la théorie des graphes pour traiter la question de taux de participation des agents aux activités de marché dans un tel environnement.

Le dernier chapitre est une étude mixte portant sur l'impact des statuts sociaux sur l'intensité des échanges. Plus précisément, nous proposons un modèle inspiré de la littérature dédiée à la consommation relative ainsi qu'à la volonté des individus de s'élever socialement. On considère ainsi qu'il existe deux biens dans l'économie pouvant être échangés, l'un échappant à tout mécanisme de comparaison sociale et l'autre, permettant de situer chaque individu dans la population. En partitionnant cette dernière selon deux groupes, ceux ayant un statut et ceux qui n'en ont pas, nous incluons en outre les phénomènes de gains et pertes psychologiques associés à chaque interaction et dépendant des statuts relatifs des deux partenaires. Nous montrons dans quel cas les échanges inter groupes prédominent et au contraire sous quelles conditions y a-t-il plus de transactions intra groupes. Puis, nous analysons empiriquement le profil des échanges dans le contexte historique des Moulins de Toulouse selon une stratification sociale bipartite et décrivons leur évolution au cours des siècles. Enfin, nous nous intéressons aux chocs de liquidité éprouvés par chaque membre de l'économie pour montrer comment complétent-ils la justification des échanges observés.

Part II

On Asset Pricing

Asset Pricing in Old Regime France[1]

1 Introduction

In asset pricing theory, the consumption-based models should in principle provide a complete description of the valuation reality and be verified for any form of cash flow (securities, bonds, stocks and derivatives). Of course, this is difficult to test and many of these models work poorly in practice. Over the decades, as the models failed to explain the real data, researchers started to make them more sophisticated by adding new variables of interest and by disentangling others to enhance their capabilities. Extensive literature, from Pratt (1964) and Arrow (1965), Kreps and Porteus (1978), Rietz (1988), Epstein and Zin (1991), Constantinides-Duffie (1996), and Campbell and Cochrane (1999) to Bansal et al. (2004) and Gabaix (2012), has continuously worked to improve the models' validity.

As the theory has developed, many puzzles have arisen; we mostly retain the well-known *equity premium puzzle* (Mehra and Prescott 1985), the *excess volatility puzzle* (Shiller 1981), the *risk-free rate puzzle* (Weil 1989), and the *long-run equity premium puzzle* (Cochrane 2005). As pointed out by Mehra and Prescott (1985), over the period 1889-1978 in the United States, a risk aversion level of 48 is required to reconcile the *equity risk premium* with the observed one, if we use a CRRA model. Thus, a simple consumption-based model with a power utility function is not able to explain the modern U.S. market with rational parameters. Note that the risk-free rate and the mean growth rate of consumption were both approximately equal to one, and the standard deviation of consumption to 0.063. These empirical results generate the high value of the risk aversion coefficient. Even with the refinement of models, we can still make a similar comment: the measured real consumption is too smooth.

However, in our study, we do not face this kind of issue since the economic conditions of the time were much more volatile, making the use of this class of model more realistic, especially since the volatility of consumption is equal to 0.32. Hence, the purpose of this study was to examine the case of a century-old economy, the Toulouse Mill companies, which share many aspects with our modern structures and have the great advantage of covering a very long time series. This empirical case can be viewed as an ideal laboratory for testing certain fundamentals of the asset pricing theory. This paper aims to explain how a very simple consumption-based model, usually rejected with contemporary data, can be verified on a simple real economy. We also assessed whether the general puzzles we mentioned above still exist with our data.

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The mill companies were very long-lived structures that operated on the banks of the Garonne river in Toulouse, a city in southwest France. The Bazacle was mentioned for the first time in 1177 in an official document from a local ecclesiastic institution: the Priory of *Notre-Dame-de-la-Daurade*. A few years later, the Narbonnais Castle was created in 1183 on the land of the Comte of Toulouse Raymond IV. These mills were organized in two corporations at the end of the 14th century. They milled grains for four centuries under this governance structure. The *Honor del Bazacle* was converted into a hydroelectricity generation company in 1888 and nationalized in 1946. The *Castle* was acquired by the municipality around 1900 and definitively destroyed in 1910.

One might wonder whether such an empirical case is relevant for testing asset pricing theory or what these new data could bring to the current analysis.

As we mentioned above, the mill companies exhibit similar characteristics to modern corporations with shareholders, limited liabilities, and transferability of shares and dividends (Sicard 1953). They were administered by a rotating board of directors and the governance remained stable over the centuries. In the same vein, we observe strong stability in the business context with no real change in regulations or legal constraints in the commercial trades.

Le Bris, Goetzmann and Pouget 2017 have already used data from one of these two mills to test the present-value relationship. Here, we complete the data with those from the main competitor of this duopoly to bring new empirical support to the analysis. The primary motivation behind this study involves the following four points.

First, a great advantage of studying the Toulouse mill companies during the Old Regime period is the political and business stability of the time. In spite of significant tensions in Europe with the Thirty Year's war (1618-1648), and the War of the Spanish Succession (1702-1714), there were no major invasions or alterations of the French territory. The political regime was the monarchy, which kept roughly the same business vision. In particular, we observe that during the 14th century, the relationship between the government and private companies was clearly more equitable than what is commonly thought. For instance during this period, the King became a majority shareholder in the Castle's capital and owned 1/7 of the entire company[2]. He was in conflict with the other shareholders' interests because he refused to contribute to the general expenses of the mills. We could easily imagine the King using his supreme authority to reject the shareholders' request. However, a trial was held and a few years later, in 1390, the company finally won against the King. Similar legal solutions are observed in the subsequent centuries (see le Bris, Goetzmann and Pouget *working paper* July 2015 The Development of Corporate Governance in Toulouse: 1372-1946)

Second, over their lifespan, the mills experienced several dramatic events, among which multiple natural disasters such as the major river floods of 1638 for the Castle and of 1709 for the Bazacle.

[2]The royal share of 17 *uchaux*[3] was called the "*septième portion*".

We could also mention other exogenous events such as episodic plagues that ravaged Toulouse, different military campaigns, religious wars and unusual climatic conditions, such as the Little Ice Age, that marked their history. Thus, the mill economy provides an empirical case that is strong suited to testing the Barro (2006) and Reitz (1988) theory, which insists on the impact of rare disasters.

Third, despite all these rare events that strongly impacted the mills, there was no *Peso problem* in this economy. Indeed, the forecasts of the investors do not seem to be significantly biased over the centuries. In particular, during the Old Regime period, we can reasonably state that the political context was rather stable and shareholders could confidently extrapolate information extracted from the past to the future. As regards the other periods, the relative high frequency of wars, outbreaks and natural disasters renders these events likely to occur but such threats do not appear to generate persistent forecasting errors.

Fourth, from the statistical point of view, we have very suitable empirical support in this context, because we detect stationarity in our data. Therefore, we can benefit from real power in statistical tests. More precisely, we find dividends and prices to be stationary for each mill company and for the consumption proxy. Moreover, during the Old Regime period, we have plenty of data and the time series are almost complete.

For each mill, we found records of more than seven hundred transaction prices, sometimes with almost one price per month. We collected data on the grain quantity redistributed to the shareholders and on the annual contributions to the general expenses of the company. Then, by subtracting these time series, we obtained a dividends value. We also benefit from the very meticulous notarial documentation about transaction records to build trading volume time series. Thus, the archives[4] provide us with a large amount of data for share prices and dividends during the Old Regime period, and allow us to rebuild almost complete time series for the period 1590 to 1790. We also take advantage of the fact that we have information about the two main mills in Toulouse that broadly define the market.

By using all recorded data, we can consider a hypothetical portfolio including shares from the two firms. Thus, a hypothetical investor managing such a portfolio was potentially able to diversify the idiosyncratic risk. This assumption is clearly not an artifact since, over the centuries, we found shareholders who indeed held shares in both mill companies (Sicard 1953). We also highlight the fact that during the Old Regime period, the shareholders belong to different social classes. We observe investors from the upper classes, Noble or Bourgeois, but also from the middle classes such as bakers, tailors and other merchants. Of course, we also find some of the richest people living in Toulouse or elsewhere in France, for instance in Paris or Bordeaux. In particular, from the end

[4]Both Haute-Garonne and Toulouse city archives.

of the Middle Ages, a social class composed of rich merchants and "money-makers" is explicitly present in the companies' shareholdings. According to Sicard (1953), this fact reinforces the idea of burgeoning speculative behavior by the investors.

When we use consumption-based models in empirical asset pricing theory, an important aspect is to find a good proxy for consumption. This is more difficult in modern times, where consumption is very diverse and hard to measure. Some progress was made recently by using garbage to assess the consumption level (Savov 2011). Of course in the mill case, such an approach cannot work. However, until the end of the 19th century in France, bread consumption was strongly inelastic and wheat was the basic ingredient of bread production. As a result, wheat was an essential commodity and by using the total milled grain quantity for both mills, we can generate a proxy for local consumption in Toulouse. Importantly, since flour is not a storable good, the milling activity implies local consumption.

In order to avoid the administrative chaos and the consecutive wave of inaccuracies generated by the French Revolution at the end of the 18th century, we chose here to restrict the study to the Old Regime period, more precisely from 1591 to 1788. This choice is also motivated by the fact that during this period, there was no currency changeover or major political upheavals.

In this paper, we use a methodology recently developed in asset pricing by several authors such as Julliard et al. (2015) or Almeida and Garcia (2009). Broadly speaking, it consists in choosing the closest stochastic discount factor (SDF) to a proxy asset pricing model by assessing the distance between them with different classes of measures. More precisely, this approach exploits the fact that we can factorize the SDF into an observable component and an unobservable one. Then, the latter can be estimated by using a relative entropy minimization approach. In particular, the relative entropy measure is chosen because it generates a strictly positive SDF estimate. The primary interest of this methodology is that we assess the SDF non-parametrically directly from prices, dividends and consumption data.

More precisely, if we make covariance appear in the fundamental Euler equation, we split the right hand side into two parts: a first term that evaluates the price in a risk-neutral world, and a second term that captures risk adjustment (Cochrane 2005). According to the theory, it is very challenging to estimate these terms because, on the one hand, the expectations process is quite complex and must include all the future states of nature, and on the other, we need to know a great deal about the investors' preferences to understand the risk compensation. Therefore, if we come back to the reduced form of the Euler equation[5], it would be much more appropriate to assess the SDF as a whole, rather than making a hypothesis about its structure. The entropy approach allows us to do this. It is a highly reliable way of estimating the SDF because in addition to evaluating

[5] $p = E(mx)$.

model-specific unobservable components, we are also able to estimate the entire SDF as an unknown variable.

The remainder of this paper is organized as follows. The first section aims at providing broad insight into the history of the mills. We also discuss the market structure in more detail. We present our data in the second section with a justification of our choice of the consumption proxy, and we explain the dividends calculation. In the third section, we describe the SDF inference methodology, and in the fourth, we analyze the results for a CRRA model. The final section provides a conclusion.

2 Historical Background

Lebris, Goetzmann and Pouget (2014) have already provided a description of the Bazacle mill company and justified with Goetzmann and Pouget (2010) why it can be seen as a corporation. Here, we review only the most important historical facts and shed more light on the *Narbonnais Castel*, its main competitor, and how the two firms interacted on the market. Our study is broadly based on Mot's thesis (1910), which provides some information on the history of the Castle Company and its organization, and Sicard (1953), which constitutes a detailed source of information about both firms during the Middle Ages in Toulouse. In this section, we first introduce the local mill economy. Next, we examine the duopoly structure of the local market. Third, we briefly describe the history of the Castle Company.

2.1 The Mill Economy in Toulouse

At the end of the 12th century, there were three different mill companies in Toulouse: the *Narbonnais Castle*, the *Honor del Bazacle*, and the *Daurade*. The two first were larger and the third was located upstream of the Castle and downstream of the Bazacle. As it was located between the other two mills, its activity suffered from reduced river height of fall. This was probably one of the main causes behind its bankruptcy during the 14th century. Thus, as early as the 15th century, the Castle and the Bazacle shared the market. From the second half of the 14th century, both companies had a very modern structure. They divided their capital and issued shares called "*uchaux*" to shareholders called "*pariers*". The shares were fully transferable and shareholders received dividends. Very early, the companies rigorously kept accounting registers by reporting their main activity indicators. First, we determined raw mill production with the total milled grain quantity, the yield from fishery activities, and the other industries. Then, we found the expenses relative to maintenance repairs and workers' salaries (ordinary expenses), and relative to partial or total destruction of the production tool (extraordinary expenses). As we will see further, these expenses were either directly charged in cash or indirectly charged by absorbing the shareholders returns.

2.2 A True Duopoly Structure

Starting from the 14th century, we observe the gradual establishment of a perfect duopoly structure[6] in the mill market in Toulouse. After the disappearance of the Daurade mill, the Bazacle and the Castle remain the two biggest ones located precisely at each end of the city. According to a 12th century municipal rule, any Toulouse mill could not charge more than 1/16 for milling fees. This fee appears to be among the lowest observed in European history. It was thus highly improbable than someone living in Toulouse would be interested in milling grain outside the city. Thus, there were some outside mills that were far smaller and that tried to penetrate the internal market[7], but during the 17th century, a legal statement denied the right to cross the walls for collecting wheat and definitively sealed the duopoly structure. The market power of each company was based on several factors. First, the number of available millstones. Obviously the greater the production capacity, the larger the demand that can be satisfied. There were also mill workers riding donkeys who were specialized in wheat collection. They had to bring the grain from the client to the mills and take back the flour[8] and they did not have the right to sell the collected grain[9].

Many times, this duopoly structure could have become a monopoly. For example, we found in 1374 (Sicard 1953) an attempt to create a kind of trust between the two companies. The terms of this agreement were to share the gains and the expenses. We do not know whether this trust really worked during the 14th century, but more than four hundred years later, in 1829, a similar attempt is recorded in the shareholders register. This time, the contract states that the firms share the turnover but not the expenses and we know that the agreement was honored for a while at least. Thus, just one year later in 1830, it appears in the accounting registers that each company has dividends of the other. Again in 1843, we find a sheet agreement signed by both firms for a new attempt to establish a trust. These observations show the real desire of both firms to reconcile their economic interests.

2.3 The Narbonnais Castle Company

Although slightly smaller than the Bazacle, the Narbonnais Castle was a very large company located in the south of the city. Like the Bazacle, its main source of revenue was a fee of 1/16 of the grain brought to the mills and its activity was not only dedicated to flour production but also to other industries. The Narbonnais Castle leased unoccupied milling spaces and fishing rights as an

[6] Sicard 1953 (p. 130).

[7] The internal market was located inside the city walls.

[8] The maximum distance reachable for the wheat collection was 15km (Sicard 1953).

[9] Apparently, many workers who collected the grain also played the role of brokers and the companies prohibited the sale of grain outside the mills (Sicard 1953).

additional source of income to pay for part of the operating expenses. It also removed the millstones of some mills to adapt their technology. Thus, we encounter mills working on sheets with two large hammers that hit the textile, mills working on wood with a vertical saw, mills working on iron, and mills producing paper.

The Castle was organized according to the same legal structure as the Bazacle. It had a rotating board of directors, it issued shares, and ensured limited liability to the shareholders. It also paid dividends each year, first through a redistributed portion of the total grain milled and an individual contribution to the general expenses, then directly with dividends. Like the Bazacle, the Castle faced several exogenous risk sources including flood, fire, and crops yield shortages.

The Castle records provide very useful information about its history, but they are not as complete as those concerning the Bazacle. However, we were able to recover the main events occurring along its lifespan. As we already highlighted, the relationship between the company investors and the highest authorities, i.e. the King, was not a subservient one. Instead, we observe that they operated with an equal legal treatment as the King's 1390 sentence shows, as repeated in the 1417 status.

Even before investing heavily in the firm, the royalty was already interested in the Castle business as land-lord; a property inherited from the Count of Toulouse after the annexation of the county by the King. For instance, in 1346 a huge river flood affected the plant, destroying the dam and all the mills. As France was in the midst of the One Hundred Years War, the King designated new investors to finance plant reconstruction. The aim was to avoid too large a reduction of supply in Toulouse. Indeed, it was too risky to feed the city with only one production establishment. Moreover, the dam at the Castle elevated the river level and thus generated a natural barrier that was difficult to cross for an army.

Finally at the beginning of the 20th century, another major flood destroyed the main dam of the plant and the Narbonnais Castle firm was quickly sold to the municipality. In 1910, a fire ultimately ended the history of the mill.

3 Data

In this section, we describe data recorded in the various registers available for the Castle Company. Le Bris, Goetzmann and Pouget (2014) have already provided a complete description in the Bazacle case. More precisely, we discuss the share prices, the milled grain quantity redistributed to the shareholders (the "*partisons*") and the individual contribution to the general expenses (the "*talhas*"). We also justify the net dividend calculation and the use of the firms' production to build a proxy for consumption.

3.1 Shares

During their lifespan, the mill's activity was reported in official registers. The collection consists of more than 50 registers sorted in two different kinds of documents: the accounting registers and the meeting minutes. We also use notarial records of public sales and a tax register (on notary recorded transactions) called "*centième denier*". We find information about the share transactions, the annual balance sheets, the daily activity of the mill with the quantity of milled grain, the deliberation of the shareholder's council, the reports of maintenance works for machinery, and even legal actions between the mill and its stakeholders. More specifically, we are able to collect the number of shares and the number of shareholders, their names, their profession, and their location, as well as the prices of the shares over the centuries.

As pointed out by Sicard (1953), we observe that the prices of the Bazacle moved roughly together with the prices of the Castle during the Medieval period. Thus, we can reasonably consider these prices as pure market prices. We also found in the contract registers some sales with redemption rights over 2, 4, and 30 years, or even perpetual rights. It was an obscure form of contract that allowed the repurchase of the sold share by its last owner at the same price within a set time period. Perhaps this kind of contract, frequently observed for Toulouse real estate since the Middle Ages, can be viewed as a first step towards what we call "derivatives" today.

For the shares price collection, we essentially used notarial registers of the company, recording transactions available in the municipal archives. They provide us with continuous yearly data from 1590 to 1817. We sometimes have up to 10 recorded transactions for a single year. For each transaction, these registers report the entire agreement sheet that includes all the information about the stakeholders. We also find in this kind of register other notarial acts concerning the mill stocks, such as "inheritance acts", "donation acts", or "wedding acts". Unfortunately, we have a small quantity of data available during the 19th century. At that time, a tax on the value of all notarial transactions was applied. We mostly rely on the registers of this tax which are available in the archives of Haute-Garonne and provide us with prices from 1801 to 1845. These registers indicate the date, contracting parties, nature, object and price. We also tried to find data in local newspapers, as was the case for the Bazacle after 1887, this was not successful. Finally, unlike the Bazacle Company, the Castle was not listed on the Paris Stock Exchange.

The oldest share price of the Narbonnais Castle is given by the archives of Haute-Garonne in 1379 and for the Bazacle by its foundation charter in 1372. During the Old Regime period, the prices were expressed in *livres tournois* (*lt*), which was the official account currency until the creation in 1795 of the *franc germinal*. From 1590 to 1845, we collected a nearly continuous series of approximately 700 prices for the Castle Company. As we mentioned above, it is not unusual for some years to find several transactions irregularly distributed over the year. Thus, we can either take the average price or the last recorded value. We will see further that we need to rely on *partisons* and *talhas* to determine the dividends. The *partisons* were also irregular over the years and in order to obtain a single cash value, we must use the average annual wheat price. Therefore in this study, we only use the average annual price of shares.

Both companies faced the risk of being partially or totally destroyed by a flood or fire. In particular, the dam was a very sensitive part of the plant and was strongly impacted by the vagaries of the river. Usually, this kind of rare event is followed either by a holdback of the usual quantity of grain redistributed to the shareholders and/or by a high *talha* or even by a number of forfeited shares. They can also be followed by strong variations in the trading volume or by the issue of new shares if the activity does not disappear totally. For instance, after the 1643 river flood and the resulting company destruction, many shareholders could not afford to pay for the repairs and lost their shares which were auctioned off to new investors.

Like the Bazacle, the capital structure of the Castle remained relatively stable during the lifespan of the firm and can be entirely reconstructed. During the second half of the 16th century, the company proposed 109.5 *uchaux*, then the number of shares was reduced to 70.3 in 1644, following the 1643 mill destruction. One hundred and fifty years later (1795), the shareholders decided to introduce a new nomenclature with the commonly-known term of "*actions*" rather than *uchaux*. We have to highlight the fact that this change is made according to the following conversion rule : one half of *uchaux* is equivalent to one unit of stock. During the 19th century, we observe that the capital was slightly downsized by 6 shares, and from 1815 to 1900, remained constant and equal to 140 shares. We also notice that both companies allowed fractional ownership of *uchaux* and the most common recorded quantity was 1/2 *uchaux* for the Castle.

As pointed out by Sicard (1953) or Mot (1910), there was a tax for each new shareholder entering the company's shareholding. A new *parier* had to host a dinner for the board of directors and paid the notarial fees. Thus, this tax, which does not exceed 5% of the value of a share, could contribute to the low rate of turnover.

Like the Bazacle, the trading volume of the Castle varies considerably. In particular, the first half of the 17th century and the 18th centuries were marked by high volatility in volumes traded. We sometimes record up to 14 transactions over the year, as in 1638. The average trading volume during the Old Regime period is equal to 2.77.

Finally, we observe during the Middle Ages that there is no real correlation between the wealth of a shareholder and the number of shares he owns. One explanation would be that investing in a mill allows an investor to hedge a fundamental risk which is the risk of famine. Once he owns the required number of shares to ensure his consumption, he does not need to invest further. In this sense, we could say that the investors' behavior deviates from the current concept of capitalist investor. Obviously, this is not always the case. We saw for instance in 1350 that the King owned 1/7 of the whole Castle capital and in 1709 the engineer Abeille brought out half of the Bazacle shares, demonstrating that buying mill shares was clearly not only a hedging transaction.

3.2 *Partisons* and *Talhas*

As already mentioned in the previous section, the quantity of grain redistributed to the shareholders was called the *partisons*. More precisely, the term "*partisons*" refers to the events occurring several times over the year where the mill redistributed the turnover to the *pariers*. There are around 15 registers for the *partisons* for the Castle. These registers were organized as follows: during the period 1583-1598, the registers gather the *partisons* by shareholders and display for each of them the allocated quantity, there is one chapter per year. For the period 1598-1770, the registers gather the distribution to each *parier* by *partison*, each chapter corresponds to one *partison*, and we can have up to 18 *partisons* over a year. From 1687 onward, we no longer use the term *partisons* but rather "sharing". From 1662 to 1770, the registers include the employees' income and the value of the *talhas*. Between 1770 and 1793, the registers quality suddenly deteriorates since we find only separate sheets. As they were not gathered into a book, we are unable to ensure that there is no loss of data. During the 18th century, the Castle Company gradually developed real accounting books by displaying more information, like the annual balance sheets. Until the end of the 18th century, the *partisons* were distributed in wheat quantity, then in 1793 both companies decided to deliver them directly in cash.

The *talha* is the contribution to the general expenses of the mills and it was charged to the shareholders at each *partisons* or once a year. We collected from accounting registers named "*Comptes et recettes du Trésorier*", which display the Castle firm's results. Each year, we find a chapter dedicated to the *talhas*. The accounting officer distinguishes two kinds of values: the "ordinary *talha*" and the "extraordinary" one. The former essentially refers to the cost of the current maintenance works, and the latter to unexpected expenses often due to partial destruction of the production tool. The extraordinary *talha* is usually huge and the mill companies are used to absorbing a part of it into a holdback of *partisons*. These *partisons* were sold for cash and said to be "*brûlées*"[10]. When a shareholder could not afford to pay the *talha* for more than two weeks, the *uchaux* was auctioned off and the balance came back to the *parier*.

Usually, we observe several amounts for *partisons* or *talhas* over the year, so we chose to aggregate them to obtain an annual value. However, from 1830 for the Castle and 1843 for the Bazacle, *partisons* and *talhas* disappeared to be replaced by standard dividends.

[10] This literally means "burned".

3.3 Dividends Calculation

Before 1830, we use the same formula as in the case of the Bazacle: the net dividend is the annual *partison* valued at the prevailing wheat price minus the corresponding annual *talha*. As pointed out by le Bris, Goetzmann and Pouget (2014), the Bazacle did not pay a real dividend from 1816 to 1843, but just a payment made after deduction of operating costs. For the Castle, we are not able to determine this; the term "dividends" is introduced for the first time in the accounting registers in 1837. Then in 1838, we come back to the less clear term "sharing payment" before finally using the concept of "dividends" in the subsequent years.

Our time series of dividends does not suffer from a lack of data except for the end of the 18th where we have two three-year-blanks during the periods 1775-1778 and 1782-1784. We linearly interpolated the missing values to rebuild the time series. Of course, as the first *partisons* register is from 1583 and the last one covers the period 1813-1817, we have no dividend values before 1583 and after 1817. Finally, we are able to build a time series from 1583 to 1817.

As in the Bazacle, dividends are sometimes negative. This usually occurs during periods where the mills are partially destroyed. As we mentioned above, the governance policy starting from the 15th century and before the creation of real dividends was to hold back the portion of revenues normally redistributed (in nature or in cash) to the shareholders to pay a part of the extraordinary expenses. However, this policy was seldom sufficient. During these periods, the *talhas* frequently exceeded the *partisons* and we obtain a negative value for the dividends.

In the time series, we observe far more negative dividends for the Castle than for the Bazacle. During the Old Regime period, we record 10 negative values for the Bazacle and 46 for the Castle. This could be due to the fact that the latter was far more exposed to river floods than the former, but this cannot be established formally. Indeed, as we mentioned above, we do not benefit from the same information quality as for the Bazacle.

We should highlight the very interesting fact that before the introduction of real dividends, the "payments" method of the shareholders provided them with more wealth than a standard dividends payment. As we mentioned above, until 1793 the shareholders received the *partisons* in wheat and paid the *talhas* in cash. Thus, they had the choice to immediately convert the wheat quantity into a currency value by selling it, or to postpone this conversion by keeping the grain for shortage periods. They could also directly consume the grain. Therefore, the shareholders can benefit from the variation of the wheat prices and this type of flexibility grants the *parier* broader power than a modern shareholder.

We also used two additional sources of data : the wheat prices from Wolff (1967) and Frêche and Frêche (1971), and silver prices. The former allows us to convert the wheat quantities into a single cash unit, and the latter allows us to convert data into real terms. In the remainder of this study, share prices, dividends and consumption are expressed in grams of silver in order to adjust for inflation.

3.4 A Proxy for Consumption

Until the end of the 19th century, bread played a leading role in overall consumption in France. In particular, during the Old Regime period, in addition to standard wheat-based bread, people consumed other kinds of bread such as rye-based, oat-based, or corn-based bread. Although used in bakeries, these other cereals were lower quality products and were mostly consumed by the poor[11]. Indeed, we usually refer to rye bread as "*famine bread*" because it was essentially produced during famine periods or food shortages when the wheat price was very high. Therefore, we mostly find wheat, rye, oats and corn in the aggregated consumption of all social classes.

Of course in this study, only the shareholders' consumption is of interest. We have seen that they belong to a social class that mostly consumes wheat and is not particularly interested in other cereals. So, it seems logical to take into account only the aggregated wheat quantity brought to the mills in order to build our proxy. It may also be objected that in this quantity, we also have a share of the non-stockholders' consumption. This is absolutely true, but before we justify this approximation, let us return to asset pricing concepts.

In asset pricing theory, we know that an asset whose payoff covaries positively (negatively) with the consumption will have a lower (higher) price. This is related to the fact that investors fear uncertainty about consumption, so they attempt to smooth it out. They prefer an asset that pays off during a difficult period to an asset that pays off during good times, when their consumption is already high. Therefore, it is the variable component of the consumption that matters the most. This component is captured by the expenditures on nondurable goods. Therefore, during the Old Regime period, we can easily reduce this category to food products.

More precisely, we want to find the component of the shareholders consumption that covaries the most with the assets' payoffs in order to generate the observed prices. Although the shareholders had many other sources of consumption, the one that seems to fulfill this property remains wheat-based bread. Thus, we can reasonably assume that most consumption volatility was driven by this product.

We also assume that this entire quantity was consumed and not exported. Although we observe, as early as the Middle Ages that wheat was exported to Bordeaux, to the region called "*Bas-Languedoc*" (in the South of France) and to Spain (Gandilhon 1941 and Larenaudie 1950), this

[11] As pointed out by Le Bris, Goetzmann and Pouget (2014), the wheat/rye price ratio is a good measure of difficult periods because in bad times, consumption switches more to rye.

concerned unprocessed grain. In fact, milled grain or flour are far more sensitive products that are very difficult to store. Therefore, it seems reasonable to assume that this perishable good (the flour) was consumed exclusively in the local economy.

In empirical asset pricing, we also want to determine whether non-stockholder consumption matters in the determination of the overall level of consumption. In modern markets, for instance, we observed that volatility of non-stockholder consumption differs from the volatility of stockholder consumption (Mankiw and Zeldes 1991). But what about the Old Regime period? This difference is obviously much more marked because the stockholders essentially come from an upper class (Sicard 1953); they were elected representatives of Toulouse[12], rich merchants, bourgeois, judges and lawyers, physicians and nobles. Moreover, the shareholders also had other kinds of consumption goods such as meat, fish, wine, and vegetables, as well as all the non-perishable and real estate goods.

As we mentioned above, we have registers that provide us with the exact quantity of milled grain redistributed to the shareholders by year, the *partisons*. We also know the ratio between this share and the total milled grain quantity because it was an official fee stated by the local authorities. Moreover, we also take into account the fact that only 9/10 is really redistributed to the shareholders because 1/10 of the fee was usually paid to the employees of the mills. Thus, we are able to reconstruct the time series of the quantity of grain brought to each mill. Thanks to the wheat prices collected by Wolff (1967) and Frêche and Frêche (1971) we can give a cash value to our quantitative data. Therefore, we estimated the aggregated consumption in our economy on the basis of the *partisons* data aggregated from the two mills.

To use dividends rather than *partisons* as a consumption proxy would be inconsistent because the shareholders definitely had other sources of income. It is also likely that a share of the received dividends was saved or reinvested into another industry, so it does not seem reasonable to assume that the dividends were entirely consumed as a single income. Another issue concerns the fact that during the lifespan of the firms, we encounter shareholders who own large proportions of the capital. The most significant cases are for the Bazacle, the engineer M. Abeille, who bought half of the *uchaux*, and for the Castle, the merchant Pierre Romestas who held 13 and a half *uchaux* and the famous royal share of 1/7 of the capital. This is why we chose to work directly on consumed quantity rather than on shareholders' wealth.

4 Descriptive Statistics

In this section, we compare all available data about the two companies. We compare the time series of the prices and the dividends. We also test the structural features of the time series and the

[12] *Capitouls.*

potential links between them. As mentioned above, all the values are inflation-adjusted by silver prices.

[Insert Figure 1 here]

The **Figure 2** and **Figure 1** shows the dividends and the prices, respectively for both mill companies. Over the Old Regime period, the average share price for the Castle was equal to 19,968 *lt*. This is consistent with the value found for the Bazacle (16,397 *lt*). Starting from the second half of the 17th century, the share value was higher for the Castle than for the Bazacle. The price volatilities are fairly close for both firms during the Old Regime Period. The average paid dividend was 309 *lt* for the Castle and 774 *lt* for the Bazacle in real terms. This is obviously due to the extreme values recorded in 1637 and 1638, and from 1642 to 1647 for the Castle. In particular, we find the extreme value of -9330 in 1637. This period is somewhat obscure for us because we do not know what exactly happened at the Castle. We know simply that a major event also impacted the Bazacle since both firms were inoperable in 1637. We are fairly sure that the plant was significantly damaged and unlike the Bazacle, the Castle did not recover rapidly. We also observe during this period that the volume of transactions increases critically and the share prices drop at the same time. It is no surprise that dividend volatility is higher for the Castle (1314.1) than for the Bazacle (896.1) during the Old Regime period.

The prices are strongly autocorrelated until a lag of 5 years for the two companies. The dividends are much less autocorrelated and display only significant persistence until a lag of 1 year.

[Insert Figure 2 here]

The number of outstanding shares was roughly constant between the 16th and the 18th centuries for both mill companies. As we mentioned above, a major reduction of shares is recorded for the Narbonnais Castle in 1644 (from 109.5 to 70.3). For the Bazacle, we find a similar gap with its dam destruction in 1709 and the issue of 28 new shares 12 years later (from 100 to 128 shares). Given the average shares price for both mills and the average share numbers, we compute the market value of the two firms and obtain 1 795 201 *lt* for the Castle and 1 869 259 *lt* for the Bazacle, in real terms. Therefore, it seems that the two structures were closely valuated.

[Insert Table 12 here]

For the capital gains, we compute the first difference between the prices at date t and $t+1$, i.e. $(P_{t+1} - P_t)/P_t$. **Table 12** shows for the Castle an average real capital gains equal to 12.2% during the period 1590-1845 and 4.97% during the Old Regime period (1591-1788). Although these results are lower than those for the Bazacle (16% and 11.5% respectively), they remain consistent with them. During the Old Regime period, we also observe that the capital gains are much more volatile for the Bazacle (93.4%) than for the Castle (33.4%). We explain this by the dramatic drop in price recorded for the Bazacle between 1709 and 1710 followed by a huge rebound in 1711. Over the year 1711, the prices jumped from 1482.3 to 18,299 *lt*, i.e. a growth of 1134.5% in real terms and this is not due to the variation of silver values, which remained constant during this period. Historically, the price drop in 1709-1710 is due to the dam destruction that occurred in 1709.

Le Bris, Goetzmann and Pouget (2014) report an average dividend yield for the Bazacle that is close to 5% over the period 1372-1946. For the Castle, we are not able to study the same time horizon since we only have values between 1590 and 1845. During the Old Regime period, we find a dividend yield very close to zero or even negative, while for the Bazacle, the value is still above 4%. This result is consistent with the fact that we observe many more negative dividends during this period for the Castle than for the Bazacle. The negative values strongly push the average down towards zero. The **Table 12** also shows that the standard deviation of the dividend yields is higher for the Castle (22.2%) than for the Bazacle (10.7%). Like for the Bazacle, the prices of the Castle shares seem to adjust to different levels of expected dividends. We observe for example in 1622 a real dividend of 1397.7 and one hundred years later in 1723 a dividend of 601.61, while the dividend yield remains the same (5.6%).

All these results are consistent with the dividend policy of both firms to pay out all earnings. We chose to perform a Phillips-Perron test in order to check time series stationarity. We used this test rather than the standard augmented Dickey-Fuller test because it is non-parametric and works well in the case of large samples. Like for the Bazacle, we found that dividends, prices and consumption are stationary[13].

We find a very low correlation between the dividends in level of the two companies (4.6%), but the prices in level recorded for the two mills are highly correlated (30%). Indeed we can see in **Figure 1** that the prices move roughly together. Despite a low correlation between the dividends, **Figure 2** shows that values from both time series are consistent. We also find a positive correlation (by considering an immediate conversion of the *partisons* into cash) between the returns and consumption (10.5% for the Bazacle and 4.5% for the Castle). To hold a share in one mill thus increases the consumption volatility and according to the theory, the mill would have to offer higher expected returns to motivate investors to hold them.

[13]Note that when we perform the augmented Dickey-Fuller test, we find that the prices are slightly (with a lag of 5 years) non-stationary for both mills; this issue can easily be solved by choosing to work on the period 1591-1798.

5 Methodology

In order to use aggregate consumption data in the SDF specifications we assume in this study the existence of a representative agent.

We also assume that the model-implied SDF is generated by a CRRA model. Thus the representative consumer maximizes the following utility program

$$E_t \sum \beta^j \left(\frac{C_{t+j}}{C_t}\right)^{\gamma} \tag{1}$$

where β is called the subjective discount factor and captures the *impatience* of investors (e.g. Cochrane 2005), C_t is the consumption level at time t and E_t is the conditional expectation operator.

Obviously in our empirical case, the market is not complete, so the SDF is not unique.

In this section we describe the general methodology we use to extract parametrically and non-parametrically the SDF from our time series. We first present a general view of the theory before to explain how we apply it in the case of our data. Then, we show how this approach can be also used to build reliable bounds for the SDF estimators.

5.1 The relative entropy minimization approach

In asset pricing, the theory states that if the *law of one price* holds and if there is no arbitrage opportunities in the economy, there exists a strictly positive SDF $M_{t,t+1}$ such that the following Euler equation is verified for each asset $i = 0, ..N$

$$P_t^i = E_t^{\mathbb{P}}(M_{t,t+1} X_{t+1}^i) \tag{2}$$

where $\mathbb{P}$ is the empirical probability of our sample, E_t the expectation operator conditional on the information available at time t, P_t^i is the price of the asset i at time t, $M_{t,t+1}$ is the pricing kernel for the period $[t, t+1]$ and X_{t+1}^i the payoff of the asset i at time $t+1$.

By taking the unconditional expectation, the previous equation becomes

$$E^{\mathbb{P}}(P_t^i) = E^{\mathbb{P}}(M_{t,t+1} X_{t+1}^i)$$

We can also express (2) with respect to the returns

$$1 = E^{\mathbb{P}}(M_{t,t+1} R_{t,t+1}^i) \tag{3}$$

or with respect to the excess returns by assuming that a risk-free asset does exist

$$0 = E^{\mathbb{P}}(M_{t,t+1}R^{ei}_{t,t+1}) \tag{4}$$

where $R^{ei}_{t,t+1} = R^{i}_{t,t+1} - R^{f}_{t,t+1} = \frac{X^i_{t+1}}{P^i_t} - R^{f}_{t,t+1}$ is the excess return and $R^{i}_{t,t+1}$ the gross return of the asset i between the dates t and $t+1$. Here we removed the time subscripts in (3) and (4) because the unconditional expectation is equal to the conditional one. That is definitely not the case when we consider the Euler equation written directly on the payoffs because the left hand side P^i_t is time-varying. We discuss further how this characteristic could explain a part of our empirical results.

We consider a decomposition (Julliard et al. 2015) of the SDF into an observable time-varying component $m(\theta, t)$, strictly positive and depending on the parameter vector $\theta \in \Theta \subseteq \mathbb{R}^k$ and an unobservable one π_t also time-varying. Thus, the SDF can be rewritten as

$$M_{t,t+1} = m(\theta, t) \times \pi_t \tag{5}$$

In this study, the considered model (CRRA) restricts the vector θ to be a one-dimensional parameter and $m(\theta, t)$ is just a function of the consumption growth $\frac{C_{t+1}}{C_t}$. At the equilibrium the expected price is

$$E(P^i_t) = \int m(\theta, t)\pi_t X^i_{t+1} d\mathbb{P}$$

so we can obtain the following pricing restriction

$$E(P^i_t)\overline{\pi}^{-1} = \int m(\theta, t)\frac{\pi_t}{\overline{\pi}} X^i_{t+1} d\mathbb{P} = \int m(\theta, t) X^i_{t+1} d\Pi = E^{\Pi}(m(\theta, t)X^i_{t+1})$$

where $\frac{\pi_t}{\overline{\pi}} = \frac{d\Pi}{d\mathbb{P}}$ is the Radon-Nikodym derivative of Π with respect to $\mathbb{P}$. Because $d\Pi$ is a probability and $d\mathbb{P} = 1/T$ is the observation frequencies in the sample, we need to normalize by $\overline{\pi}$ in order to obtain $\int d\Pi = 1$.

Here, we are tackling the heart of the method. We make a change of measure in our Euler equation in order to get rid off the unobservable component π_t. Then, we measure the distance between the empirical probability $\mathbb{P}$ and every probability Π for which the pricing restriction holds and we choose the closest one. That is called a *minimum discrepancy problem.* Thus, for determining this distance, we use the relative entropy also called the Kullback-Leibler Information Criterion (KLIC) Divergence

$$D(A||B) = \int \ln \frac{dA}{dB} dA \quad \text{for two measures A and B} \tag{6}$$

and $D(A||A) = 0$. The existence of the logarithm function in the relative entropy ensures a strictly positive SDF. In the case of our probabilities, we have

$$\Pi^*(\theta) = arg\min_{\Pi} D(\mathbb{P}||\Pi) = arg\min_{\Pi} \int \ln \frac{d\mathbb{P}}{d\Pi} d\mathbb{P} \quad \text{s.t.} \quad E^{\Pi}\left[m(\theta,t)X^i_{t+1}\right] = E(P^i_t)\overline{\pi}^{-1} \tag{7}$$

Thus as pointed out by Julliard et al. (2015), Π^* is just the probability that adds the minimum of new information relative to $\mathbb{P}$ such that the pricing restriction holds. However, the KLIC divergence is just a special case of a more general class of functions called the Cressie Read functions (Almeida Garcia 2011) which provides with a large range of distance measures. The probability Π^* captures what is missed for the asset pricing conditions being satisfied. From the probability point of view, we could say that the relative entropy measures how much we need to distort the empirical probability (Π is the distorted probability) in order to verify the minimization constraint $E^{\Pi}\left[m(\theta,t)X^i_{t+1}\right] = E(P^i_t)\overline{\pi}^{-1}$. Note that in (7) we identify Π^* up to a positive scale constant and we will see further how we can recover it thanks to the risk-free asset.

As the estimator Π^* depends on the parameter θ, we can proceed to a new minimization program over the set of parameters Θ in order to find the θ value for which we obtain the closest probability to the empirical one in the set $\{\Pi^*(\theta),\ \theta \in \Theta\}$

$$\theta^* = arg\min_{\theta} D(\mathbb{P}||\Pi^*) \tag{8}$$

this estimator belongs to the broader class of the Generalized Minimum Contrast (GMC) estimators.

The KLIC divergence in (6) is asymmetric, so usually we have $D(A||B) \neq D(B||A)$. We exploit this mathematical property to benefit from another distance measure for estimating Π^*. Thus, we can invert the roles of Π and $\mathbb{P}$ in (7) and estimate the probabilities Π^* through another minimization procedure

$$\Pi^*(\theta) = arg\min_{\Pi} D(\Pi||\mathbb{P}) = arg\min_{\Pi} \int \ln \frac{d\Pi}{d\mathbb{P}} d\Pi \quad \text{s.t.} \quad E^{\Pi}\left[m(\theta,t)X^i_{t+1}\right] = E(P^i_t)\overline{\pi}^{-1} \tag{9}$$

We could also chose not to factorize the SDF by assuming that we don't observe any component. We consider the whole SDF as an unobservable variable $M_{t,t+1}$. Thus, the Euler equation (2) becomes

$$E(P^i_t)\overline{M}^{-1} = E^Q(X^i_{t+1})$$

with the Radon-Nikodym derivative $\frac{M_t}{\overline{M}} = \frac{dQ}{d\mathbb{P}}$ and the minimum discrepancy problem

$$Q^* = arg\min_Q D(\mathbb{P}||Q) = arg\min_Q \int \ln \frac{d\mathbb{P}}{dQ} d\mathbb{P} \quad \text{s.t.} \quad E^Q\left[X^i_{t+1}\right] = E(P^i_t)\overline{M}^{-1} \tag{10}$$

the new probability is named Q in reference to the risk-neutral probability.

Indeed, we know that the price of an asset in a risk-neutral world can be written

$$p = \frac{E^*(x)}{R_f} \tag{11}$$

with E^* the unconditional expectation operator under the risk neutral probability $*$. Thus, when $T \to \infty$ we have $\overline{M} = E(M_{t,t+1}) = E\left(\frac{1}{R_f}\right)$ and by taking the unconditional expectation under the probability $\mathbb{P}$ of the equation (11) we find $E(p) = E^*(x)E\left(\frac{1}{R_f}\right)$ which is equivalent to $E(p)\overline{M}^{-1} = E^*(x)$, so Q is the risk neutral probability $*$.

Thanks to equation (10), we are able to extract non-parametrically the SDF of the economy from the prices and the payoffs. This is a very valuable way to estimate it because we don't need to make any hypothesis about the structure of the SDF and in particular about the preferences of the representative agent, it's a model-free estimation.

Solving the minimum discrepancy problem (7) leads to the empirical likelihood (EL) estimator of the unknown part of the SDF

$$\hat{\pi}_t = \frac{1}{T\left(1 + \lambda^*(\theta)'\left(m(\theta,t)X^i_{t,t+1} - \overline{\pi}^{-1}E(P^i_t)\right)\right)} \tag{12}$$

where $\lambda^*(\theta) \in \mathbb{R}^N$ is the solution of

$$\lambda^*(\theta) = arg\min_\lambda - \sum_{t=1}^T \log\left(1 + \lambda^*(\theta)'\left(m(\theta,t)X^i_{t,t+1} - \overline{\pi}^{-1}E(P^i_t)\right)\right) \tag{13}$$

And the solution of the exponential titling (ET) discrepancy problem (9) is

$$\hat{\pi}_t = \frac{e^{\lambda^*(\theta)'m(\theta,t)X^i_{t,t+1}}}{\sum_{t=1}^T e^{\lambda^*(\theta)'m(\theta,t)X^i_{t,t+1}}} \tag{14}$$

where $\lambda^*(\theta) \in \mathbb{R}^N$ is the solution of

$$\lambda^*(\theta) = arg\min_\lambda \frac{1}{T}\sum_{t=1}^T e^{\lambda^*(\theta)'\left[m(\theta,t)X^i_{t,t+1} - \overline{\pi}^{-1}E(P^i_t\right]} \tag{15}$$

In particular when we consider the risk neutral probability, the solution of the problem (10) is the following EL estimator

$$\hat{Q}_t = \frac{1}{T\left(1 + \lambda^*(\theta)'\left(X^i_{t,t+1} - \overline{M}^{-1} E(P^i_t)\right)\right)} \tag{16}$$

where $\lambda^*(\theta) \in \mathbb{R}^N$ is the solution of

$$\lambda^*(\theta) = arg \min_{\lambda} - \sum_{t=1}^{T} \log\left(1 + \lambda^*(\theta)'\left(X^i_{t,t+1} - \overline{M}^{-1} E(P^i_t)\right)\right) \tag{17}$$

Of course we can also estimate Q_t through the exponential titling form of the minimization problem and obtain another estimator. To make this section easier to read, we don't report this result here.

The formulae show that the variables $\overline{\pi}$ and $\overline{M}$ must now be estimated. To assess $\overline{M}$ we just use the fact that $\overline{M} \simeq E(M_{t,t+1}) = E\left(\frac{1}{R_f}\right)$ and in Appendix we present an iterative procedure developed by Julliard et al. (2015) to recover $\overline{\pi}$.

5.2 Entropy bounds

The minimization of the relative entropy also provide bounds beyond which a potential SDF cannot be eligible. Thus we use the theoretic approach presented above to built constraints for the estimators.

In asset pricing a large literature is dedicated to developing tools to assess the plausibility of any potential SDF. For instance Hansen and Jagannathan (HJ) develop in 1997 a canonical bound for measuring the degree of misspecification of asset pricing models. It's a benchmark which states a lower bound for the variance of every admissible SDF. In the case where the minimization constraint is written on the payoffs, the HJ bound is the following

$$M^*_{t,t+1} = \min_{M_{t,t+1}} Var(M_{t,t+1}) \quad \text{s.t.} \quad E(P^i_t) = E\left[M_{t,t+1} X^i_{t+1}\right] \tag{18}$$

for any SDF $M_{t,t+1}$. Therefore the variance of any SDF must be higher or equal to the variance of $M^*_{t,t+1}$.

As pointed out by Almeida and Garcia (2009) the HJ-bound is just a special case of estimators obtained from a minimization procedure based on a quadratic norm. However this distance measure does not ensure that the estimator will be strictly positive. As we assume a no-arbitrage setting we need this property, so we rely more on the entropy-bounds described in this section for which the logarithmic form imposes the non-negativity of the SDF. It should also be noted that the HJ-bound mirrors the quality of the bound obtained from a risk-neutral constraint (Q-bound) to provide a model-free estimator.

Thus, we use the relative entropy measure to build bounds using the same principle that for the HJ bound and we obtain benchmarks from the solution of the problem (10) with risk-neutral constraints

1. Q-bounds (EL) :

$$D\left(\mathbb{P}||\frac{M_t}{\overline{M}}\right) \geq D(\mathbb{P}||Q^*)$$

   Q-bounds (ET) :

$$D\left(\frac{M_t}{\overline{M}}||\mathbb{P}\right) \geq D(Q^*||\mathbb{P})$$

Remember that Q^* is the closest probability (by definition) to the empirical one from which we non-parametrically extract the whole SDF. Consequently any probability built from a SDF must have a relative entropy higher than $D(\mathbb{P}||Q^*)$. Roughly speaking Q^* is the "*best SDF*" in the probability distortion sense and any eligible SDF cannot do better.

From the solution of the problem (7) with asset pricing constraints based on the payoffs we form the following bounds

2. M-bounds (EL) :

$$D\left(\mathbb{P}||\frac{m(\gamma,t)\pi_t}{\overline{m(\gamma,t)\pi_t}}\right) \geq D\left(\mathbb{P}||\frac{m(\gamma,t)\pi_t^*}{\overline{m(\gamma,t)\pi_t^*}}\right)$$

   M-bounds (ET) :

$$D\left(\frac{m(\gamma,t)\pi_t}{\overline{m(\gamma,t)\pi_t}}||\mathbb{P}\right) \geq D\left(\frac{m(\gamma,t)\pi_t^*}{\overline{m(\gamma,t)\pi_t^*}}||\mathbb{P}\right)$$

Here given the SDF decomposition mentioned above any SDF cannot do better than $m(\gamma,t)\pi_t^*$ since by definition π_t^* is the unobservable component estimate that brings the minimum of additional information needed to price assets. The next bound just follows from the definition of the minimization procedure.

3. Π-bounds (EL) :

$$D\left(\mathbb{P}||\frac{\pi_t}{\overline{\pi}_t}\right) \geq D\left(\mathbb{P}||\frac{\pi_t^*}{\overline{\pi}_t^*}\right)$$

   Π-bounds (ET) :

$$D\left(\frac{\pi_t}{\overline{\pi}_t}||\mathbb{P}\right) \geq D\left(\frac{\pi_t^*}{\overline{\pi}_t^*}||\mathbb{P}\right)$$

Here we develop successively a model-free bound (Q-bounds) for the SDF as a whole and parametric bounds (M-bounds and Π-bounds) for the decomposed SDF and its unobservable component.

As showed in Julliard et al (2005) to a second order approximation of the relative entropy measure, the HJ-bound and the Q-bounds structures are equivalent. These bounds just need information about the payoffs and the prices, we don't make any assumption about the structure of the SDF and we don't use consumption data. Instead, the M-bounds and the Π-bounds depend on the underlying model that determines the form of the SDF. Although in this study we only consider a CRRA model with a power utility function, this approach allows to test a more sophisticated class of asset pricing models which includes habit formation models (Campbell and Cochrane 1999), long-run risk models (Bansal et al. 2004), etc..

It's also important to note that by definition, the M-bounds are tighter than the Q-bounds since $D(\mathbb{P}||Q^*)$ determines the minimum distance for the SDF as a whole without any condition on its structure. Thus, these bounds verify the following condition

$$D(\mathbb{P}||Q^*) \leq D\left(\mathbb{P}||\frac{m(\gamma,t)\pi_t^*}{m(\gamma,t)\pi_t^*}\right) \quad \text{and} \quad D(Q^*||\mathbb{P}) \leq D\left(\frac{m(\gamma,t)\pi_t^*}{m(\gamma,t)\pi_t^*}||\mathbb{P}\right)$$

the M-bound is also more informative since it depends on consumption data and on the structure of the SDF.

As pointed out by Julliard et al. (2005) the main interest of the Π-bounds is to provide a way to test the *entropy contribution* of each component of the SDF. Indeed we can compare the distances $D\left(\mathbb{P}||\frac{\pi_t}{\pi_t}\right)$ and $D\left(\mathbb{P}||\frac{m(\gamma,t)\pi_t^*}{m(\gamma,t)\pi_t^*}\right)$ in order to check the informative power of the unobservable component in the whole SDF. We also could compare the individual contribution of each component with the distance $D(\mathbb{P}||Q^*)$.

By using the relative entropy approach when the SDF is taken as a single unobservable variable (no decomposition), we are able to extract the SDF of the economy from the prices and the payoffs non-parametrically. This is a highly valuable way to estimate it because there is no need to make a hypothesis about the structure of the SDF and, in particular, about the preferences of the representative agent; it is a model-free estimation. Finally, the theory also provides bounds (see Julliard et al. 2015) beyond which a potential SDF cannot be eligible, we use them in our analysis.

6 Empirical Results

In this section, we describe the results of the entropy analysis based on the pricing restriction (2). We focus here on the Old Regime period, more precisely on the period 1591-1788. First, we discuss the results obtained about γ^* for both companies considered separately and jointly in the same economy. Second, we study the different bounds for each mill. Third, we compare the time series of the filtered SDF with those of the model-free SDF and we check the links between consumption data and the non-parametrically extracted SDFs.

We use here the nomenclature developed in Julliard et al. (2016) by naming *filtered SDFs* the SDF $m(\gamma, t)\pi_t^*$, where $m(\gamma, t)$ is the observable component and π_t^* is the solution of the problem (7). The choice of the consumption-based asset pricing model decides the form of $m(\gamma, t)$ and uses the local curvature of the utility function (γ in the CRRA case) as a parameter. So, at the equilibrium, the unobservable component π_t^* is strongly related to the structure of $m(\gamma, t)$ and of course to γ since both of them intervene in the constraint of the minimization problem. In the following entropy analysis, we assume both the subjective discount factor δ and the risk-free rate[14] are constant and equal to one.

6.1 The Absolute Risk Aversion Estimate

We performed the relative entropy methodology for different values of the risk aversion coefficient, usually within a range from 0 to 20 and for each value of γ, we estimated π_t^*. Then we selected the γ that solves the problem (8). **Table 6** show the optimum value γ^* obtained for the Bazacle, the Castle, and the overall economy for the pricing restriction (6).

When each company is considered separately or when both of them interact on a same market, we find low values of γ^*, namely lower than 10. In fact, we have in **Table 6** for the Bazacle $\gamma^* = 6$, for the Castle $\gamma^* = 8.8$ and for the overall economy $\gamma^* = 6.2$. Thus, the risk aversion coefficient is lower for the Bazacle than for the Castle, and in the case of a portfolio including both firms, the overall risk aversion level is closer to the risk aversion displayed for the Bazacle. So, it seems that the Castle's shareholders were more risk averse than those of the Bazacle. This could be explained by the fact that during the Old Regime period, the former was far more affected by rare events than the latter. This is also demonstrated by the number of strictly negative dividends recorded for the Castle. If we consider the overall economy, the trust of investors in the Bazacle firm appears to offset the risk related to the Castle's activity.

We also compute the standard deviation of the γ^* estimate and we find $\sigma_{\gamma^*} = 0.41$ for the Bazacle, $\sigma_{\gamma^*} = 0.51$ for the Castle, and $\sigma_{\gamma^*} = 0.4$ for the overall economy. We also computed confidence intervals and we observed that the risk aversion coefficient for the Bazacle lies in a range between 5 and 7, for the Castle the range is between 7 and 10, and for the overall economy between 5 and 7. Consequently, we can state that the companies do not generate the same perceived risk even though the coefficients are close. Once again, we observe that the overall economy inherits the risk aversion coefficient more from the Bazacle than from the Castle.

[14] This assumption is supported by the empirical results obtained in Le Bris, Goetzmann and Pouget (2014) but will be released in further research.

6.2 The Entropy Bounds

As we explained above, we perform three different kinds of bounds: the Q-bound, the M-bound, and the Π-bound. The Q-bound is obtained by performing the difference between the relative entropy of the SDF candidate (in our case the CRRA-implied SDF) and the relative entropy of Q^*. Then, we pick up the lowest value of γ for which this difference is positive and we obtain the minimum risk aversion coefficient for which the SDF is eligible. We find a bound $\gamma^Q = 1$ for the Bazacle, $\gamma^Q = 0.4$ for the Castle and $\gamma^Q = 1$ for the overall economy. As the bound is lower for the Castle, its set of candidate SDFs is broader. This can be explained by the number of strictly negative returns recorded for the Castle compared to the Bazacle, which reduce the required distortion of the empirical probability to price assets.

The M-bound is obtained by computing the difference between the relative entropy of the observable part of our SDF candidate and the relative entropy of $m(\gamma, t)\pi_t^*$. The value of γ for which this difference is positive provided us with a new condition for the model-implied SDF to be eligible. When we use the pricing restriction (**??**), we find that this difference is strictly positive on the following intervals $[1.1, 6]$ for the Bazacle, $[0.6, 8.8]$ for the Castle, and $[1, 6.2]$ for the overall economy. We thus deduce the M-bounds in each case $\gamma^M = 1.1$, $\gamma^M = 0.6$ and $\gamma^M = 1$ and we find that the result is fairly close to the Q-bounds. We notice that the upper bounds of these intervals are exactly equal to γ^*. This is easy to understand if we consider the values of $\pi^*(\gamma^*)$. We observe that $\pi^*(\gamma^*)$ is very close to the unit vector so the relative entropy of the product $m(\gamma, t) \times \pi^*(\gamma)$ is almost equal to the relative entropy of $m(\gamma, t)$ at the point γ^*. Therefore, it is clear that the difference between these two quantities is close to zero at these points and γ^* coincides with the upper bounds of the M-bound intervals. In accordance with the theory presented in section **IV**, we also observed that the M-bound is tighter than the Q-bound.

For the Π-bound, we noted that in the case of a CRRA model with constant subjective discount factor ($\delta = \pi_t$ in the SDF decomposition), the relative entropy $D\left(\mathbb{P}||\frac{\pi_t}{\overline{\pi}_t}\right)$ is equal to zero. This is due to the ratio $\frac{d\Pi}{d\mathbb{P}}$ in the entropy formula. Consequently, to compute the Π-bound, we only have to examine whether the value of $D\left(\mathbb{P}||\frac{\pi_t^*}{\overline{\pi}_t^*}\right)$ is significantly different from zero. This is never the case, so the bound is always fixed to zero.

6.3 The SDF Time Series

As we mentioned above, we are able to generate two kind of SDF time series. We alternatively extracted the model-free SDF (Q^*) by solving the problem based on the pricing restriction (10) (if we consider the SDF as a single unobservable variable) and the *filtered* SDF by solving (7). The former uses no information about consumption or inner structure; it only needs price and dividend time series. This approach is particularly powerful since it generates a very general pricing kernel.

Table 3 shows the correlation between the model-free SDF and different consumption growth levels. We observed that the free-model SDF Q^* is always negatively correlated with consumption or consumption growth. We found for example a correlation of -20% when we consider the firms jointly. This makes perfect sense if we rely on a CRRA structure for the SDF. It also makes sense in a more general context. If we consider the fundamental pricing equation $pu'(c_t) = E(\delta u'(c_{t+1})x_{t+1})$ from which the consumption-based asset pricing theory derives, we observe that the SDF $m = \delta \frac{u'(c_{t+1})}{u'(c_t)}$ is negatively correlated with consumption because of the concavity of the utility function $u(.)$. This result is perhaps the largest contribution of this article since it validates a fundamental intuition of asset pricing theory.

Table 4 shows a very low correlation between the model-free SDFs of the two companies for each kind of constraint. This correlation is also negative, so we have some difficult periods for the Bazacle investors that could be interpreted as good ones for the Castle investors. This is logical if we refer to the dividends dynamic displayed in **Figure 2**, where the curves vary exactly in opposite directions for some years. In fact, over the centuries we observe a principle of communicating vessels between the firms since when an event shuts down the activity of one company, the other suddenly benefits from a monopoly situation and all the demand spills over to one production facility. We also find that the Bazacle covaries more with the overall economy than the Castle does, i.e. $\rho(Q_b^*, Q_{bc}^*) > \rho(Q_c^*, Q_{bc}^*)$. This is consistent with the observations made previously about the risk aversion coefficient. For the filtered SDFs, we can see that the results are extremely sensitive to the risk aversion level. Unlike the case of a non-parametric SDF extraction, the Castle covaries more with the overall economy than the Bazacle does. We also find a negative correlation between the filtered SDFs of the two companies, equal to -0.69. Finally, we find that the model-free SDFs covary with the model-implied ones. This is due to the fact that Q^* already covaries with consumption and by definition the model-implied SDF depends on the consumption growth. This can also be explained by the fact that there is a correlation between Q^* and the unobservable component π_t.

7 Conclusion

We tested the reliability of a CRRA model on this economy. Our main contributions are summarized as follows. We built the longest time series of both accounting and market data for a modern corporation structure in Europe. We used this unique empirical field as a simplified version of modern markets. The proxy we used for recovering local consumption is much more volatile than any modern consumption level. By performing a relative entropy minimization, we extracted the SDF of the economy non-parametrically and parametrically. We observed that a standard consumption-based asset pricing model with a power utility function is not rejected for a very low risk aversion coefficient.

8 Appendix

8.1 Descriptive Statistics

	Capital Gain		**Dividend Yield**		**Price Change**	
1591-1788	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
Castle	4.9%	33.4%	-0.5%	22.2%	86.5	4412.4
Bazacle	11.5%	93.4%	4.45%	10.7%	-37.2	4795.7
1590-1845						
Castle	12.2%	140.2%	-	-	-40.8	4542.8
Bazacle	16%	129.2%	4.6%	9.6%	-61.3	5610.8
1591-1798						
Castle	14.3%	155.1%	1.6%	21.6%	-39.8	4618.2
Bazacle	18.6%	142.3%	4.5%	10.5%	-68.5	5548

Table 1: : Descriptive Statistics (all data are inflation adjusted by silver prices)

8.2 Entropy analysis

	EL		
	Bazacle	Castle	Baz.-Cas.
γ^*	6	8.8	6.2
Q-bound	1	0.4	1
M-bound	6	8.8	6.2
Π-bound ($< 0.1\%$)	0	0	0
$\sigma(\gamma^*)$	0.41	0.51	0.4
C.I.	[5.20;6.80]	[7.78;9.81]	[5.40;6.99]

Table 2: : The table reports the different values of γ which ensure the model reliability in the case we consider the Euler equation $\overline{\pi}^{-1}E(P_t^i) = E^{\Pi}[m(\gamma,t)X_{t+1}^i)]$. These values include γ^*, i.e. the minimization (8), the **Q**-bound, the **M**-bound and the Π-bound. We also indicate the standard error of γ^* and a confidence interval for this optimum value.

	ET		
Correlation	Bazacle	Castle	Bazacle-Castle
$\rho\,(Q^*, C_{t+1})$	-0.06	-0.08	-0.10
$\rho\,(Q^*, C_t)$	-0.10	-0.007	-0.10
$\rho\left(Q^*, \frac{C_{t+1}}{C_t}\right)$	-0.21	-0.03	-0.20
$\rho\left(Q^*, (\frac{C_{t+1}}{C_t})^{-\gamma^*}\right)$	0.12	0.02	0.12
$\rho\left(Q^*, (\frac{C_{t+1}}{C_t})^{-\gamma_{QB}}\right)$	0.16	0.04	0.17
$\rho\left(Q^*, (\frac{C_{t+1}}{C_t})^{-\gamma_{MB}}\right)$	0.16	0.04	0.17

Table 3: : The table reports the correlation values between the non-parametrically extracted SDF and the CRRA-implied SDF for different values of γ, i.e. consumption growth taken at different exponent. These results include the cases where the exponent is 1, γ^*, the **Q**-bound and the **M**-bound.

Correlation	**EL**
$\rho(Q_c^*, Q_b^*)$	-0.020
$\rho(Q_c^*, Q_{bc}^*)$	0.30
$\rho(Q_b^*, Q_{bc}^*)$	0.93
$\rho(\Pi_c^*(\gamma_c^*), \Pi_b^*(\gamma_b^*))$	-0.69
$\rho(\Pi_c^*(\gamma_c^*), \Pi_{bc}^*(\gamma_{bc}^*))$	-0.68
$\rho(\Pi_b^*(\gamma_b^*), \Pi_{bc}^*(\gamma_{bc}^*))$	0.01
$\rho(Q_c^*, \Pi_c^*(\gamma_c^*))$	-0.05
$\rho(Q_b^*, \Pi_b^*(\gamma_b^*))$	0.06
$\rho(Q_{bc}^*, \Pi_{bc}^*(\gamma_{bc}^*))$	-0.08

Table 4: : The table reports the correlation between the different extracted SDFs. These values include the correlation between non-parametrically extracted SDF and filtered SDFs.

Figure 1. Moving Average (5 years) of Mills Prices in Silver During The Old Regime Period

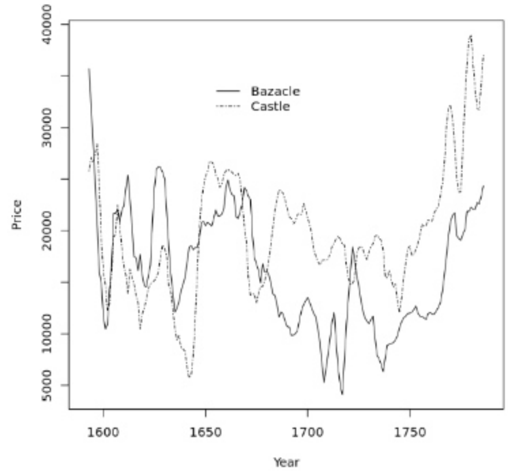


Figure 2. Mills Dividends in Silver During The Old Regime Period

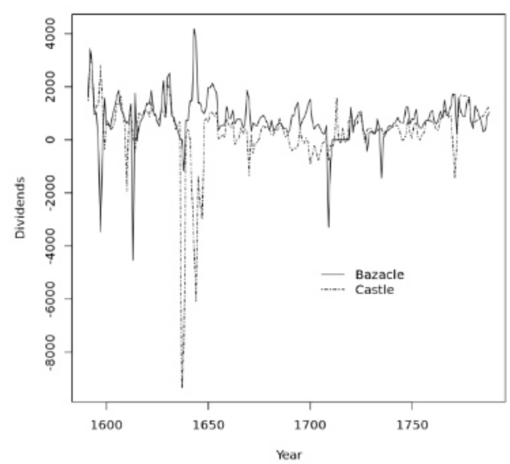
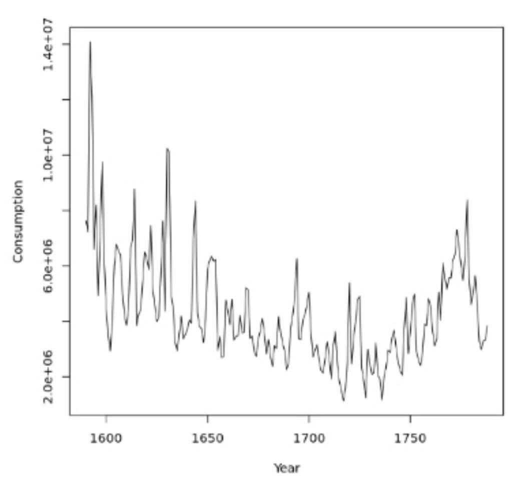


Figure 3. Consumption in Silver During The Old Regime Period



8.3 One-period Model

We assume the Gaussian vector $\begin{pmatrix} X_B \\ X_C \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_B \\ \mu_C \end{pmatrix}, \begin{pmatrix} \sigma_B^2 & \rho \\ \rho & \sigma_C^2 \end{pmatrix}\right)$ and each agent maximizes :

$$\underset{q_B,q_C}{max} E\left[U(\tilde{W})\right] = \underset{q_B,q_C}{max} E\left[-e^{-A\tilde{W}}\right]$$

where $\tilde{W}$ is the wealth of the agent and q_B, q_C the number of share from he firm B and C respectively with the convention $\sum_{i\in\mathbb{I}} q_k^i = 1,\ k = A, B$.
We assume there exists a Pareto equilibrium $\Rightarrow$ representative agent. We have :

$$\tilde{W} = q_B X_B + q_C X_C - [q_B P_B + q_C P_C](1 + R^f)$$

$\tilde{W} \sim \mathcal{N}\left(E(\tilde{W}), Var(\tilde{W})\right)$, so

$$\begin{aligned} E\left(-e^{-A\tilde{W}}\right) &= -e^{-AE(\tilde{W})+\frac{A^2}{2}Var(\tilde{W})} \\ &= -e^{-A\left(E(\tilde{W})-\frac{A}{2}Var(\tilde{W})\right)} \end{aligned}$$

As

$$\begin{aligned} &\underset{q_B,q_C}{max} && -e^{-A\left(E(\tilde{W})-\frac{A}{2}Var(\tilde{W})\right)} \\ &\underset{q_B,q_C}{min} && e^{-A\left(E(\tilde{W})-\frac{A}{2}Var(\tilde{W})\right)} \\ \mathcal{M} = &\underset{q_B,q_C}{max} && E(\tilde{W}) - \frac{A}{2}Var(\tilde{W}) \end{aligned}$$

And

$$\begin{cases} E(\tilde{W}) &= q_B\mu_B + q_C\mu_C - [q_B P_B + q_C P_C](1 + R^f) \\ Var(\tilde{W}) &= q_B^2\sigma_B^2 + q_C^2\sigma_C^2 + 2q_C q_B\rho \end{cases}$$

So

$$\begin{cases} \frac{\partial E(\tilde{W})}{\partial q_B} &= \mu_B - P_B(1 + R^f) \\ \frac{\partial Var(\tilde{W})}{\partial q_B} &= 2q_B\sigma_B^2 + 2q_C\rho \end{cases}$$

And symetrically

$$\begin{cases} \frac{\partial E(\tilde{W})}{\partial q_C} &= \mu_C - P_C(1 + R^f) \\ \frac{\partial Var(\tilde{W})}{\partial q_C} &= 2q_C\sigma_C^2 + 2q_B\rho \end{cases}$$

The FOC is

$$\begin{cases} 0 &= \mu_B - P_B(1+R^f) - \frac{A}{2}\left(2q_B\sigma_B^2 + 2q_C\rho\right) \\ 0 &= \mu_C - P_C(1+R^f) - \frac{A}{2}\left(2q_C\sigma_C^2 + 2q_B\rho\right) \end{cases}$$

$$\Leftrightarrow \begin{cases} q_B &= \frac{\mu_B - P_B(1+R^f)}{A\sigma_B^2} - \frac{q_C\rho}{\sigma_B^2} \\ q_C &= \frac{\mu_C - P_C(1+R^f)}{A\sigma_C^2} - \frac{q_B\rho}{\sigma_C^2} \end{cases}$$

$$\Leftrightarrow \begin{cases} P_B &= \frac{\mu_B - A(q_B\sigma_B^2 + q_C\rho)}{1+R^f} \\ P_C &= \frac{\mu_C - A(q_C\sigma_C^2 + q_B\rho)}{1+R^f} \end{cases}$$

The quantity of interest are

$$\begin{aligned} E(R^i) &= E\left(\frac{X_i - P_i}{P_i}\right) = \frac{\mu_i - P_i}{P_i} \\ Var(R^i) &= Var\left(\frac{X_i - P_i}{P_i}\right) = \frac{\sigma_i^2}{P_i^2} \end{aligned}$$

and the clearing condition $q_k^m = \sum_{i\in\mathbb{I}} q_k^i = 1$, $k = A, B$ leads to

$$\begin{aligned} E(R^m) &= E\left(\frac{X_B + X_C - P_B - P_C}{P_B + P_C}\right) = \frac{\mu_B + \mu_C - P_B - P_C}{P_B + P_C} \\ Var(R^m) &= Var\left(\frac{X_B + X_C - P_B - P_C}{P_B + P_C}\right) = \frac{\sigma_B^2 + \sigma_C^2 + 2\rho}{(P_B + P_C)^2} \\ Cov(R^i, R^m) &= \frac{Cov(X_B + X_C, X_i)}{P_i(P_B + P_C)} = \frac{\rho + \sigma_i^2}{P_i(P_B + P_C)} \end{aligned}$$

Thus we have

$$\begin{aligned} \beta_{i,m} &= \frac{Cov(R^i, R^m)}{Var(R^m)} = \frac{\rho + \sigma_i^2}{P_i(P_B + P_C)} \times \frac{(P_B + P_C)^2}{\sigma_B^2 + \sigma_C^2 + 2\rho} \\ &= \frac{Cov(R^i, R^m)}{Var(R^m)} = \frac{(\rho + \sigma_i^2)(P_B + P_C)}{P_i(\sigma_B^2 + \sigma_C^2 + 2\rho)} \end{aligned}$$

and

$$\begin{aligned} P_m = P_B + P_C &= \frac{\mu_B + \mu_C - A(\sigma_B^2 + \sigma_C^2 + 2\rho)}{1 + R^f} \\ \Leftrightarrow R^f &= \frac{\mu_B + \mu_C - A(\sigma_B^2 + \sigma_C^2 + 2\rho)}{P_B + P_C} - 1 \end{aligned}$$

or equivalently in the case $q_B = q_C = 1$, $R^f = \frac{\mu_i - A(\sigma_i^2 + \rho)}{P_i} - 1$.

Now, we have $P_i = \frac{\mu_i}{1 + \underbrace{\frac{\mu_i - P_i}{P_i}}_{E(R^i)}}$ so according to the CAPM we could write

$$P_i = \frac{\mu_i}{1 + R^f + \beta_{i,m}(E(R_m) - R^f)}$$

We show that $E(R^i) - R^f = \beta_{i,m}(E(R_m) - R^f)$:

$$\begin{aligned}
E(R^i) - R^f &= \frac{\mu_i}{P_i} - 1 - \left(\frac{\mu_i - A(\sigma_i^2 + \rho)}{P_i} - 1\right) \\
&= \frac{\mu_i}{P_i} - 1 - \frac{\mu_i}{P_i} + \frac{A(\sigma_B^2 + \rho)}{P_i} + 1 \\
&= \frac{A(\sigma_B^2 + \rho)}{P_i}
\end{aligned}$$

and

$$\begin{aligned}
E(R^m) - R^f &= \frac{\mu_B + \mu_C}{P_B + P_C} - 1 - \left(\frac{\mu_B + \mu_C}{P_B + P_C} - \frac{A(\sigma_B^2 + \sigma_C^2 + 2\rho)}{P_B + P_C} - 1\right) \\
&= \frac{A(\sigma_B^2 + \sigma_C^2 + 2\rho)}{P_B + P_C} \\
\Rightarrow \beta_{i,m}(E(R^m) - R^f) &= \frac{(\rho + \sigma_i^2)(P_B + P_C)}{P_i(\sigma_B^2 + \sigma_C^2 + 2\rho)} \times \frac{A(\sigma_B^2 + \sigma_C^2 + 2\rho)}{P_B + P_C} \\
&= \frac{A(\sigma_B^2 + \rho)}{P_i}
\end{aligned}$$

Finally we just have to replicate the average payoff μ_C, we form a portfolio $\mathcal{P}$ such that $X_{\mathcal{P}} = X_B + \delta$ with $\delta = \mu_C - \mu_B$ and δ bonds. We have $\mu_{\mathcal{P}} = E(X_B + \delta) = \mu_C$ (Do we have to add this bond to the market portfolio?). The law of one price gives us the price of $\mathcal{P}$: $P_{\mathcal{P}} = P(X_{\mathcal{P}} + \delta) = P_B + \frac{\delta}{1+R^f}$. So according to the equation (19) $P_{\mathcal{P}} < P_C$ should implies that $\beta_{\mathcal{P},m} > \beta_{C,m}$.

8.4 Multi-period Model

The framework

We consider here an overlapping generation model. Each generation lives only one period. At $t = 0$, N individuals (Young) are born, there is no consumption but they borrow at the interest rate R_f to invest into two different trees at the prices P_t^B and P_t^C. At $t = 1$, these N individuals (Old) use the payoff of the trees to reimburse their loan and to consume before to died. At the same time, a new generation of N individuals born and we repeat the sequence over and over again. As only Old individuals consume, there is no tradeoff between consuming today or tomorrow and the risk-free rate is assumed to be constant.

0	1	2	3	T
N young born	N old died N young born	N old died N young born	N old died N young born	N old died N young born

Each period, a new generation has the opportunity to invest (q_t^B, q_t^C) into the two different trees.

	t	t+1
Young	borrow $q_t^B P_t^B + q_t^C P_t^C$ buy (q_t^B, q_t^C) tree	borrow $q_t^B P_t^B + q_t^C P_t^C$ buy (q_t^B, q_t^C) tree
Old	$\emptyset$	earn $(D_{t+1}^B + P_{t+1}^B, D_{t+1}^C + P_{t+1}^C)$ per tree and reimburse loan

We also assume the following dynamic for the dividends

$$\begin{cases} D_{t+1}^B &= \alpha^B + \gamma^B D_t^B + \epsilon_{t+1}^B \\ D_{t+1}^C &= \alpha^C + \gamma^C D_t^C + \epsilon_{t+1}^C \end{cases}$$

where $\epsilon_{t+1}^C|_t \sim \mathcal{N}(0, \sigma_{\epsilon^C}^2)$ and $\epsilon_{t+1}^B|_t \sim \mathcal{N}(0, \sigma_{\epsilon^B}^2)$ and $cov_t(\epsilon_{t+1}^B, \epsilon_{t+1}^C) = \rho$. We make the following conjecture for the price :

$$\begin{cases} P_t^B &= a^B + b^B D_t^B \\ P_t^C &= a^C + b^C D_t^C \end{cases}$$

The payoff each period is equal to $X_{t+1}^i = P_{t+1}^i + D_{t+1}^i$ and the Young solve the following program[15] :

$$\max_{q_t^B, q_t^C} E_t \left[U(\tilde{W}_{t+1}) \right] = \max_{q_B, q_C} E_t \left[-e^{-A\tilde{W}_{t+1}} \right]$$

[15]Because for each new generation at time t, $W_t = 0$

The equilibrium prices

We have :

$$\begin{aligned}
\tilde{W}_{t+1} &= q_t^B X_{t+1}^B + q_t^C X_{t+1}^C - \left[q_t^B P_t^B + q_t^C P_t^C\right](1+R^f) \\
&= q_t^B (P_{t+1}^B + D_{t+1}^B) + q_t^C (P_{t+1}^C + D_{t+1}^C) - \left[q_t^B P_t^B + q_t^C P_t^C\right](1+R^f) \\
&= q_t^B \left[a^B + (b^B+1) D_{t+1}^B\right] + q_t^C \left[a^C + (b^C+1) D_{t+1}^C\right] - \left[q_t^B P_t^B + q_t^C P_t^C)\right](1+R^f) \\
&= q_t^B \left[a^B + (b^B+1)(\alpha^B + \gamma^B D_t^B + \epsilon_{t+1}^B)\right] + q_t^C \left[a^C + (b^C+1)(\alpha^C + \gamma^C D_t^C + \epsilon_{t+1}^C)\right] \\
&\qquad - \left[q_t^B P_t^B + q_t^C P_t^C\right](1+R^f)
\end{aligned}$$

And at the end of each period $\tilde{W}_{t+1} = C_{t+1}$. So the two first conditional moments are

$$\left\{\begin{aligned}
E_t(\tilde{W}_{t+1}) &= q_t^B \left[a^B + (b^B+1)(\alpha^B + \gamma^B D_t^B)\right] + q_t^C \left[a^C + (b^C+1)(\alpha^C + \gamma^C D_t^C)\right] \\
&\qquad - \left[q_t^B P_t^B + q_t^C P_t^C)\right](1+R^f) \\
Var_t(\tilde{W}_{t+1}) &= q_t^{B^2} (b^B+1)^2 \sigma_{\epsilon^B}^2 + q_t^{C^2} (b^C+1)^2 \sigma_{\epsilon^C}^2 + 2 q_t^B q_t^C (b^B+1)(b^C+1)\rho
\end{aligned}\right.$$

So

$$\left\{\begin{aligned}
\frac{\partial E_t(\tilde{W}_{t+1})}{\partial q_t^B} &= a^B + (b^B+1)(\alpha^B + \gamma^B D_t^B) - P_t^B(1+R^f) \\
\frac{\partial Var_t(\tilde{W}_{t+1})}{\partial q_t^B} &= 2 q_t^B (b^B+1)^2 \sigma_{\epsilon^B}^2 + 2 q_t^C (b^B+1)(b^C+1)\rho
\end{aligned}\right.$$

and

$$\left\{\begin{aligned}
\frac{\partial E_t(\tilde{W}_{t+1})}{\partial q_t^C} &= a^C + (b^C+1)(\alpha^C + \gamma^C D_t^C) - P_t^C(1+R^f) \\
\frac{\partial Var_t(\tilde{W}_{t+1})}{\partial q_t^C} &= 2 q_t^C (b^C+1)^2 \sigma_{\epsilon^C}^2 + 2 q_t^B (b^B+1)(b^C+1)\rho
\end{aligned}\right.$$

The FOC is

$$\left\{\begin{aligned}
0 &= a^B + (b^B+1)(\alpha^B + \gamma^B D_t^B) - P_t^B(1+R^f) - \frac{A}{2}\left(2 q_t^B (b^B+1)^2 \sigma_{\epsilon^B}^2 + 2 q_t^C (b^B+1)(b^C+1)\rho\right) \\
0 &= a^C + (b^C+1)(\alpha^C + \gamma^C D_t^C) - P_t^C(1+R^f) - \frac{A}{2}\left(2 q_t^C (b^C+1)^2 \sigma_{\epsilon^C}^2 + 2 q_t^B (b^B+1)(b^C+1)\rho\right)
\end{aligned}\right.$$

Since each period, the last generation must resell all the trees to the new one, the market clearing condition is $q_t^B = q_t^C = 1$, so

$$\Leftrightarrow \left\{\begin{aligned}
P_t^B &= \frac{a^B - (b^B+1)\left[A\left((b^B+1)\sigma_{\epsilon^B}^2 + (b^C+1)\rho\right) - \alpha^B\right]}{1+R^f} + \frac{(b^B+1)\gamma^B D_t^B}{1+R^f} \\
P_t^C &= \frac{a^C - (b^C+1)\left[A\left((b^C+1)\sigma_{\epsilon^C}^2 + (b^B+1)\rho\right) - \alpha^C\right]}{1+R^f} + \frac{(b^C+1)\gamma^C D_t^C}{1+R^f}
\end{aligned}\right.$$

By identification we obtain

$$\Leftrightarrow \begin{cases} a^B &= -\frac{(b^B+1)}{R^f}\left[A\left((b^B+1)\sigma^2_{\epsilon^B}+(b^C+1)\rho\right)-\alpha^B\right] \\ b^B &= \frac{\gamma^B}{1+R^f-\gamma^B} \\ a^C &= -\frac{(b^C+1)}{R^f}\left[A\left((b^C+1)\sigma^2_{\epsilon^C}+(b^B+1)\rho\right)-\alpha^C\right] \\ b^C &= \frac{\gamma^C}{1+R^f-\gamma^C} \end{cases}$$

As $b^i+1=\frac{\gamma^i}{1+R^f-\gamma^i}+1=\frac{1+R^f}{1+R^f-\gamma^i}$ we can rewrite the prices as

$$\begin{cases} P^B_t &= -\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{(1+R^f-\gamma^B)^2}\sigma^2_{\epsilon^B}+\frac{1+R^f}{(1+R^f-\gamma^B)(1+R^f-\gamma^C)}\rho\right)-\frac{\alpha^B}{1+R^f-\gamma^B}\right]+\frac{\gamma^B D^B_t}{1+R^f-\gamma^B} \\ \\ P^C_t &= -\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{(1+R^f-\gamma^C)^2}\sigma^2_{\epsilon^C}+\frac{1+R^f}{(1+R^f-\gamma^C)(1+R^f-\gamma^B)}\rho\right)-\frac{\alpha^C}{1+R^f-\gamma^C}\right]+\frac{\gamma^C D^C_t}{1+R^f-\gamma^C} \end{cases}$$

$$\begin{cases} P^B_t &= \frac{\alpha^B(1+R^f)}{T_B R^f}+\frac{\gamma^B D^B_t}{T_B}-\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_B^2}\sigma^2_{\epsilon^B}+\frac{1+R^f}{T_B T_C}\rho\right)\right] \\ \\ P^C_t &= \frac{\alpha^C(1+R^f)}{T_C R^f}+\frac{\gamma^C D^C_t}{T_C}-\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_C^2}\sigma^2_{\epsilon^C}+\frac{1+R^f}{T_B T_C}\rho\right)\right] \end{cases}$$

where $T_i=1+R^f-\gamma^i$. Notice that in the case $\gamma^i=1$ we have $T_i=R^f$. Here we don't have the $p=\frac{E(x)}{1+R^f}+cov(m,x)$ formula yet.

The returns

The quantity of interest are

$$\begin{aligned} R^i &= \frac{P^i_{t+1}+D^i_{t+1}-P^i_t}{P^i_t} \\ &= \frac{1}{P^i_t}\left(-\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_i^2}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_B T_C}\rho\right)-\frac{\alpha^i}{T_i}\right]+\frac{1+R^f}{T_i}D^i_{t+1}\right) \\ &\qquad\qquad +\frac{1}{P^i_t}\left(\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_i^2}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_B T_C}\rho\right)-\frac{\alpha^i}{T_i}\right]-\frac{\gamma^i}{T_i}D^i_t\right) \\ &= \frac{\left(1+R^f\right)D^i_{t+1}-\gamma^i D^i_t}{P^i_t T_i} \\ &= \frac{\left(1+R^f\right)(\alpha^i+\gamma^i D^i_t+\epsilon^i_{t+1})-\gamma^i D^i_t}{P^i_t T_i} \\ &= \frac{\left(1+R^f\right)\alpha^i+\gamma^i R^f D^i_t+\left(1+R^f\right)\epsilon^i_{t+1}}{P^i_t T_i} \end{aligned}$$

And in particular by using the fact that $q_m^k = \sum_{i\in\mathbb{I}} q_i^k = 1$, $k = A, B$

$$\begin{aligned}
R^m &= \frac{P_{t+1}^B + P_{t+1}^C + D_{t+1}^B + D_{t+1}^C - P_t^B - P_t^C}{P_t^B + P_t^C} \\
&= \frac{P_{t+1}^B + D_{t+1}^B - P_t^B}{P_t^B + P_t^C} + \frac{P_{t+1}^C + D_{t+1}^C - P_t^C}{P_t^B + P_t^C} \\
&= \frac{\left(1+R^f\right) D_{t+1}^B - \gamma^B D_t^B}{(P_t^B + P_t^C)T_B} + \frac{\left(1+R^f\right) D_{t+1}^C - \gamma^C D_t^C}{(P_t^B + P_t^C)T_C}
\end{aligned}$$

The betas

As $E_t(D_{t+1}^i) = \alpha^i + \gamma^i D_t^i$, we have

$$\begin{aligned}
E_t(R^i) &= \tfrac{(1+R^f)\alpha^i + R^f\gamma^i D_t^i}{P_t^i T_i} \\
Var_t(R^i) &= \tfrac{\left(1+R^f\right)^2 \sigma_{\epsilon^i}^2}{P_t^{i2} T_i^2} \\
E_t(R^m) &= \tfrac{R^f}{P_t^B + P_t^C}\left[\tfrac{\gamma^B D_t^B}{T_B} + \tfrac{\gamma^C D_t^C}{T_C}\right] + \tfrac{1+R^f}{P_t^B + P_t^C}\left[\tfrac{\alpha^B}{T_B} + \tfrac{\alpha^C}{T_C}\right] \\
Var_t(R^m) &= \left(\tfrac{1+R^f}{P_t^B + P_t^C}\right)^2 \left[\tfrac{\sigma_{\epsilon^B}^2}{T_B^2} + \tfrac{\sigma_{\epsilon^C}^2}{T_C^2} + \tfrac{2\rho}{T_B T_C}\right] = \left(\tfrac{1+R^f}{P_t^B + P_t^C}\right)^2 \left[\tfrac{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}{T_B^2 T_C^2}\right]
\end{aligned}$$

And for the covariance

$$\begin{aligned}
Cov_t(R^i, R^m) &= \frac{Cov_t\left(X_{t+1}^B + X_{t+1}^C, X_{t+1}^i\right)}{P_t^i(P_t^B + P_t^C)} \\
&= \frac{Cov_t\left(P_{t+1}^B + D_{t+1}^B + P_{t+1}^C + D_{t+1}^C, P_{t+1}^i + D_{t+1}^i\right)}{P_t^i(P_t^B + P_t^C)} \\
&= \frac{Cov_t\left((b^B+1)\epsilon_{t+1}^B + (b^C+1)\epsilon_{t+1}^C, (b^i+1)\epsilon_{t+1}^i\right)}{P_t^i(P_t^B + P_t^C)} \\
&= \frac{(b^i+1)^2\sigma_{\epsilon^i}^2 + (b^C+1)(b^B+1)\rho}{P_t^i(P_t^B + P_t^C)} \\
&= \frac{(\frac{1+R^f}{T_i})^2\sigma_{\epsilon^i}^2 + (\frac{1+R^f}{T_C})(\frac{1+R^f}{T_B})\rho}{P_t^i(P_t^B + P_t^C)} \\
&= \frac{(1+R^f)^2}{P_t^i(P_t^B + P_t^C)}\left[\frac{\sigma_{\epsilon^i}^2}{T_i^2} + \frac{\rho}{T_B T_C}\right] \\
&= \frac{(1+R^f)^2}{P_t^i(P_t^B + P_t^C)}\left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2 + \rho T_C T_B}{T_B^2 T_C^2}\right]
\end{aligned}$$

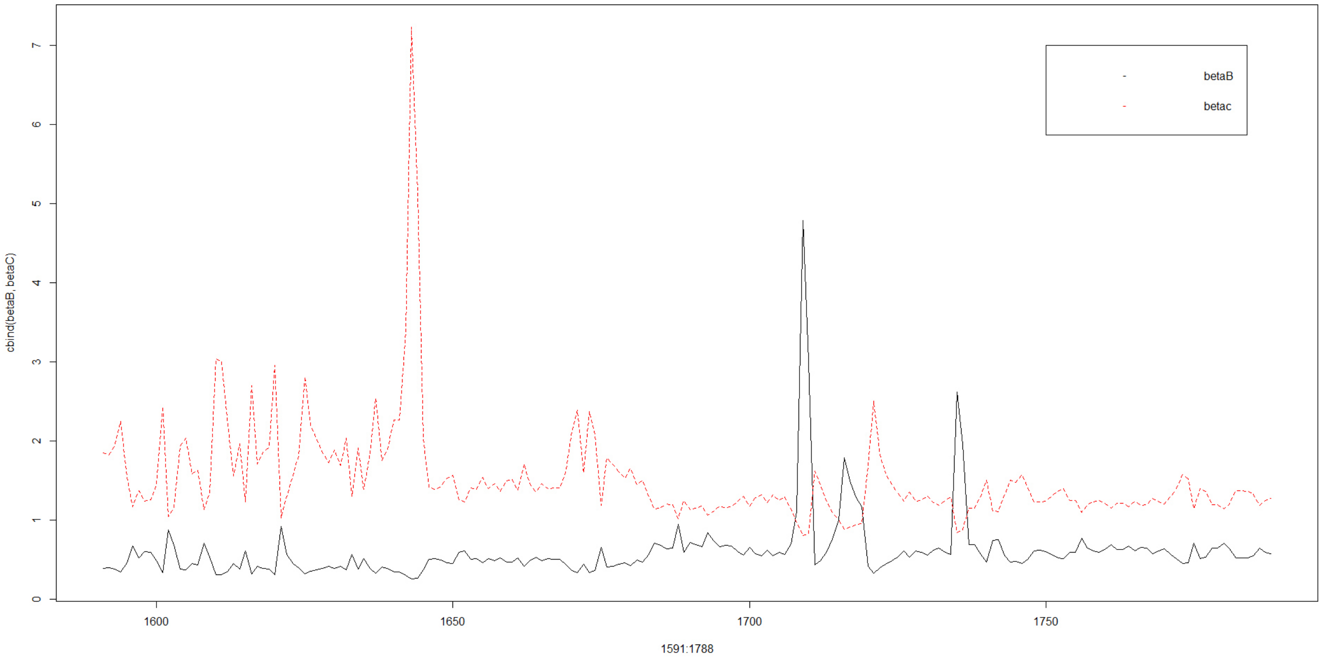
Thus

$$\begin{aligned}
\beta_t^{i,m} &= \frac{Cov_t(R^i, R^m)}{Var_t(R^m)} \\
&= \frac{(1+R^f)^2}{P_t^i(P_t^B + P_t^C)} \left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2 + \rho T_C T_B}{T_B^2 T_C^2}\right] \left(\frac{P_t^B + P_t^C}{1+R^f}\right)^2 \left[\frac{T_B^2 T_C^2}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}\right] \\
&= \frac{P_t^B + P_t^C}{P_t^i} \left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2 + \rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}\right] \\
&= \left(\frac{P_t^{-i}}{P_t^i} + 1\right) \left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2 + \rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}\right]
\end{aligned}$$

Finally we can compare the betas

$$\begin{aligned}
\beta_t^{B,m} - \beta_t^{C,m} &= \left(\frac{P_t^C}{P_t^B} + 1\right) \left[\frac{\sigma_{\epsilon^B}^2 T_C^2 + \rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}\right] - \left(\frac{P_t^B}{P_t^C} + 1\right) \left[\frac{\sigma_{\epsilon^C}^2 T_B^2 + \rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C}\right] \\
&= \frac{1}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C} \left[\left(\frac{P_t^C}{P_t^B} + 1\right)(\sigma_{\epsilon^B}^2 T_C^2 + \rho T_C T_B) - \left(\frac{P_t^B}{P_t^C} + 1\right)(\sigma_{\epsilon^C}^2 T_B^2 + \rho T_C T_B)\right] \\
&= \frac{1}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C} \left[\left(\frac{P_t^C}{P_t^B} + 1\right)\sigma_{\epsilon^B}^2 T_C^2 - \left(\frac{P_t^B}{P_t^C} + 1\right)\sigma_{\epsilon^C}^2 T_B^2 + \left(\frac{P_t^C}{P_t^B} - \frac{P_t^B}{P_t^C}\right)\rho T_C T_B\right] \\
&= \frac{1}{\sigma_{\epsilon^B}^2 T_C^2 + \sigma_{\epsilon^C}^2 T_B^2 + 2\rho T_B T_C} \left[(r+1)\,\sigma_{\epsilon^B}^2 T_C^2 - \left(\frac{1}{r} + 1\right)\sigma_{\epsilon^C}^2 T_B^2 + \left(r - \frac{1}{r}\right)\rho T_C T_B\right]
\end{aligned}$$

with r the price ratio $\frac{P_t^C}{P_t^B}$.



The CAPM

We have

$$\begin{aligned}
\beta_t^{i,m}(E_t(R^m)-R^f) &= \frac{P_t^B+P_t^C}{P_t^i}\left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2+\rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2+\sigma_{\epsilon^C}^2 T_B^2+2\rho T_B T_C}\right] \\
&\times\frac{1}{P_t^B+P_t^C}\left(\frac{\gamma^B R^f D_t^B}{T_B}+\frac{\gamma^C R^f D_t^C}{T_C}-R^f(P_t^B+P_t^C)+\frac{1+R^f}{P_t^B+P_t^C}\left[\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right]\right) \\
&= \frac{R^f}{P_t^i}\left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2+\rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2+\sigma_{\epsilon^C}^2 T_B^2+2\rho T_B T_C}\right]\left[\frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}-(P_t^B+P_t^C)\right] \\
&\quad +\frac{R^f}{P_t^i}\left[\frac{\sigma_{\epsilon^i}^2 T_{-i}^2+\rho T_C T_B}{\sigma_{\epsilon^B}^2 T_C^2+\sigma_{\epsilon^C}^2 T_B^2+2\rho T_B T_C}\right]\frac{1+R^f}{R^f}\left[\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right]
\end{aligned}$$

From

$$\begin{cases}
P_t^B = -\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_B^2}\sigma_{\epsilon^B}^2+\frac{1+R^f}{T_B T_C}\rho\right)-\frac{\alpha^B}{T_B}\right]+\frac{\gamma^B D_t^B}{T_B} \\
P_t^C = -\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_C^2}\sigma_{\epsilon^C}^2+\frac{1+R^f}{T_B T_C}\rho\right)-\frac{\alpha^C}{T_C}\right]+\frac{\gamma^C D_t^C}{T_C}
\end{cases}$$

We have

$$\begin{aligned}
\frac{A(1+R^f)^2}{R^f}\left(\frac{\sigma^2_{\epsilon^B}}{T_B^2}+\frac{\sigma^2_{\epsilon^C}}{T_C^2}+\frac{2\rho}{T_BT_C}\right) &= \frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}+\frac{1+R^f}{R^f}\left(\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right)-(P_t^B+P_t^C) \\
\Leftrightarrow \sigma^2_{\epsilon^B}T_C^2+\sigma^2_{\epsilon^C}T_B^2+2\rho T_BT_C &= \frac{R^f(T_BT_C)^2}{A(1+R^f)^2}\left[\frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}+\frac{1+R^f}{R^f}\left(\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right)\right] \\
&\quad -\frac{R^f(T_BT_C)^2}{A(1+R^f)^2}\left(P_t^B+P_t^C\right)
\end{aligned}$$

So

$$\begin{aligned}
\beta_t^{i,m}(E_t(R^m)-R^f) &= \frac{R^f}{P_t^i}\left[\frac{\sigma^2_{\epsilon^i}T^2_{-i}+\rho T_CT_B}{\sigma^2_{\epsilon^B}T_C^2+\sigma^2_{\epsilon^C}T_B^2+2\rho T_BT_C}\right]\left[\frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}-(P_t^B+P_t^C)\right] \\
&\quad +\frac{R^f}{P_t^i}\left[\frac{\sigma^2_{\epsilon^i}T^2_{-i}+\rho T_CT_B}{\sigma^2_{\epsilon^B}T_C^2+\sigma^2_{\epsilon^C}T_B^2+2\rho T_BT_C}\right]\frac{1+R^f}{R^f}\left[\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right] \\
&= \frac{R^f}{P_t^i}\left[\frac{\sigma^2_{\epsilon^i}T^2_{-i}+\rho T_CT_B}{\frac{R^f(T_BT_C)^2}{A(1+R^f)^2}\left[\frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}+\frac{1+R^f}{R^f}\left(\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right)-P_t^B-P_t^C\right]}\right] \\
&\quad \times\left[\frac{\gamma^B D_t^B}{T_B}+\frac{\gamma^C D_t^C}{T_C}-(P_t^B+P_t^C)+\frac{1+R^f}{R^f}\left(\frac{\alpha^B}{T_B}+\frac{\alpha^C}{T_C}\right)\right] \\
&= \frac{R^f}{P_t^i}\left(\sigma^2_{\epsilon^i}T^2_{-i}+\rho T_CT_B\right)\frac{A(1+R^f)^2}{R^f(T_BT_C)^2} \\
&= \left(\frac{\sigma^2_{\epsilon^i}}{T_i^2}+\frac{\rho}{T_CT_B}\right)\frac{A(1+R^f)^2}{P_t^i}
\end{aligned}$$

Given that

$$\begin{aligned}
E_t(R^i)-R^f &= \frac{(1+R^f)\alpha^i+R^f\gamma^i D_t^i}{P_t^iT_i}-R^f \\
&= \frac{R^f}{P_t^i}\left(\frac{1+R^f}{R^f}\frac{\alpha^i}{T^i}+\frac{\gamma^i D_t^i}{T^i}-P^i\right) \\
&= \frac{R^f}{P_t^i}\left(\frac{1+R^f}{R^f}\right)A\left(\frac{1+R^f}{T_B^2}\sigma^2_{\epsilon^B}+\frac{1+R^f}{T_BT_C}\rho\right) \\
&= \frac{A(1+R^f)^2}{P_t^i}\left(\frac{\sigma^2_{\epsilon^B}}{T_B^2}+\frac{\rho}{T_BT_C}\right)
\end{aligned}$$

we have shown that $\beta_t^{i,m}(E_t(R^m) - R^f) = E_t(R^i) - R^f$

Finally, we have $P_t^i = \frac{\frac{1}{T_i}[(1+R^f)\alpha^i + R^f\gamma^i D_t^i]}{\frac{(1+R^f)\alpha^i + R^f\gamma^i D_t^i}{P_i T_i}} = \frac{(1+R^f)\alpha^i + R^f\gamma^i D_t^i}{T_i E(R^i)}$. So, according to the CAPM we can write :

$$\begin{aligned}
P_t^i &= \frac{(1+R^f)\alpha^i + R^f\gamma^i D_t^i}{T_i E_t(R^i)} \\
&= \frac{(1+R^f)\alpha^i + R^f\gamma^i D_t^i}{T_i(R^f + \beta_t^{i,m}(E_t(R^m) - R^f))}
\end{aligned}$$

Fundamental Euler Equation

Now, we want to rewrite the price as $p = \frac{E(x)}{R^f} + cov(m, x)$. We have

$$\begin{aligned}
E_t(D_{t+1}^i + P_{t+1}^i) &= E_t\left[a^i + (b^i+1)(\alpha^i + \gamma^i D_t^i + \epsilon_{t+1}^i)\right] \\
&= E_t\left[a^i + b^i D_t^i + (b^i+1)\alpha^i + (b^i+1)\gamma^i D_t^i - b^i D_t^i + (b^i+1)\epsilon_{t+1}^i)\right] \\
&= P_t^i + (b^i+1)\alpha^i + (b^i+1)\gamma^i D_t^i - b^i D_t^i \\
&= P_t^i + (b^i+1)\alpha^i + \left[(b^i+1)\gamma^i - b^i\right] D_t^i \\
&= P_t^i + (b^i+1)\alpha^i + \left[(\gamma^i - 1)b^i + \gamma^i\right] D_t^i
\end{aligned}$$

As $(\gamma^i - 1)b^i + \gamma^i = (\gamma^i - 1)\frac{\gamma^i}{T_i} + \gamma^i = \frac{\gamma^i}{T_i}\left(\gamma^i - 1 + T_i\right)$, we have

$$\begin{aligned}
P_t^i &= -\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_i^2}\sigma_{\epsilon^i}^2 + \frac{1+R^f}{T_i T_{-i}}\rho\right) - \frac{\alpha^i}{T_i}\right] + \frac{\gamma^i D_t^i}{T_i} \\
\Leftrightarrow \left[\gamma^i - 1 + T_i\right] P_t^i &= -\left[\gamma^i - 1 + T_i\right]\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_i^2}\sigma_{\epsilon^i}^2 + \frac{1+R^f}{T_i T_{-i}}\rho\right) - \frac{\alpha^i}{T_i}\right] \\
&\quad + \underbrace{\left[\gamma^i - 1 + T_i\right]\frac{\gamma^i D_t^i}{T_i}}_{[(\gamma^i-1)b^i + \gamma^i]D_t^i}
\end{aligned}$$

Therefore

$$
\begin{aligned}
E_t(D^i_{t+1}+P^i_{t+1}) &= P^i_t+(b^i+1)\alpha^i+\left[\gamma^i-1+T_i\right]\frac{1+R^f}{R^f}\left[A\left(\frac{1+R^f}{T_i^2}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_iT_{-i}}\rho\right)-\frac{\alpha^i}{T_i}\right] \\
&\qquad +\left[\gamma^i-1+T_i\right]P^i_t \\
\Leftrightarrow (\gamma^i+T_i)P^i_t &= E_t(D^i_{t+1}+P^i_{t+1})-\underbrace{(b^i+1)\alpha^i}_{\frac{1+R^f}{T_i}\alpha^i}-(1+R^f)\left[A\left(\frac{1+R^f}{T_i^2}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_iT_{-i}}\rho\right)-\frac{\alpha^i}{T_i}\right] \\
\Leftrightarrow (\gamma^i+T_i)P^i_t &= E_t(D^i_{t+1}+P^i_{t+1})-\frac{1+R^f}{T_i}\left[A\left(\frac{1+R^f}{T_i}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_{-i}}\rho\right)\right] \\
\Leftrightarrow P^i_t &= \frac{E_t(D^i_{t+1}+P^i_{t+1})}{1+R^f}-\frac{A}{T_i}\left(\frac{1+R^f}{T_i}\sigma^2_{\epsilon^i}+\frac{1+R^f}{T_{-i}}\rho\right)
\end{aligned}
$$

Asset's payoff replication

We observe that the first term on the right hand side of the equation (14.1) determine the price of the tree in a risk-neutral world (or standard discounted PV formula) and the last term is a risk adjustment which depends on the risk aversion coefficient A, the dividend risk $\sigma^2_{\epsilon^i}$ and the covariance between the dividend shocks ρ.

As we have $E(X^B_{t+1})=a^B+(b^B+1)\alpha^B+(b^B+1)\gamma^B E(D^B_t)$, we form a portfolio

$$
\mathcal{P}=\left\{\underbrace{\frac{\gamma^C(b^C+1)E(D^C_t)}{\gamma^B(b^B+1)E(D^B_t)}}_{W_1}\;;\;\underbrace{-\frac{\gamma^C(b^C+1)E(D^C_t)}{\gamma^B(b^B+1)E(D^B_t)}(a^B+(b^B+1)\alpha^B)+a^C+(b^C+1)\alpha^C}_{W_2}\right\}
$$

on the Bazacle's asset and the risk-free asset such that

$$
\begin{aligned}
E(X^{\mathcal{P}}_{t+1}) &= E\left(W_1X^B_{t+1}+W_2\right) \\
&= W_1E(X^B_{t+1})+W_2 \\
&= \frac{\gamma^C(b^C+1)E(D^C_t)}{\gamma^B(b^B+1)E(D^B_t)}\left[a^B+(b^B+1)\alpha^B+(b^B+1)\gamma^B E(D^B_t)\right] \\
&\qquad -\frac{\gamma^C(b^C+1)E(D^C_t)}{\gamma^B(b^B+1)E(D^B_t)}(a^B+(b^B+1)\alpha^B)+a^C+(b^C+1)\alpha^C \\
&= a^C+(b^C+1)\alpha^C+\gamma^C(b^C+1)E(D^C_t) \\
&= E(X^C_{t+1})
\end{aligned}
$$

So we just replicated the payoff of the Castle. Because of the Law of One Price we have

$$
\begin{aligned}
E(P_t^{\mathcal{P}}) &= E(W_1 P_t^B + W_2 B_t(1,t+1)) \\
&= W_1 E\left[\frac{E_t(X_{t+1}^B)}{1+R^f} - \frac{A}{T_B}\left(\frac{1+R^f}{T_B}\sigma_{\epsilon^B}^2 + \frac{1+R^f}{T_{-B}}\rho\right)\right] + \frac{W_2}{1+R^f} \\
&= \frac{E\left(W_1 X_{t+1}^B + W_2\right)}{1+R^f} - \frac{W_1 A}{T_B}\left(\frac{1+R^f}{T_B}\sigma_{\epsilon^B}^2 + \frac{1+R^f}{T_{-B}}\rho\right) \\
&= \frac{E(X_{t+1}^C)}{1+R^f} - \frac{W_1 A}{T_B}\left(\frac{1+R^f}{T_B}\sigma_{\epsilon^B}^2 + \frac{1+R^f}{T_{-B}}\rho\right)
\end{aligned}
$$

with $B_t(1,t+1))$ the price of the risk-free asset that pays 1 unit of consumption next period. Finally, given that the equation (14.1) also holds for Castle's price, we have

$$
\begin{aligned}
E(P_t^{\mathcal{P}}) - E(P_t^C) &= \frac{A}{T_C}\left(\frac{1+R^f}{T_C}\sigma_{\epsilon^C}^2 + \frac{1+R^f}{T_{-C}}\rho\right) - \frac{W_1 A}{T_B}\left(\frac{1+R^f}{T_B}\sigma_{\epsilon^B}^2 + \frac{1+R^f}{T_{-B}}\rho\right) \\
&= A(1+R^f)\left(\frac{\sigma_{\epsilon^C}^2}{T_C^2} + \frac{\rho}{T_C T_{-C}} - W_1\left[\frac{\sigma_{\epsilon^B}^2}{T_B^2} + \frac{\rho}{T_B T_{-B}}\right]\right) \\
&= A(1+R^f)\left(\frac{\sigma_{\epsilon^C}^2}{T_C^2} + \frac{\rho}{T_C T_B} - \frac{\gamma^C(b^C+1)E(D_t^C)}{\gamma^B(b^B+1)E(D_t^B)}\left[\frac{\sigma_{\epsilon^B}^2}{T_B^2} + \frac{\rho}{T_B T_C}\right]\right)
\end{aligned}
$$

The spread between these two prices should be only due to differences in the risk adjustments.

The issue of R^f

We could also consider a time-varying risk-free rate R_t^f and at the equilibrium we would obtain the following parameters :

$$
\Leftrightarrow \left\{
\begin{array}{lcl}
a_{t+1}^B &=& -\frac{(b_{t+1}^B+1)}{R_t^f}\left[A\left((b_{t+1}^B+1)\sigma_{\epsilon^B}^2 + (b_{t+1}^C+1)\rho\right) - \alpha^B\right] \\
b_{t+1}^B &=& \frac{\gamma^B}{1+R_t^f-\gamma^B} \\
a_{t+1}^C &=& -\frac{(b_{t+1}^C+1)}{R_t^f}\left[A\left((b_{t+1}^C+1)\sigma_{\epsilon^C}^2 + (b_{t+1}^B+1)\rho\right) - \alpha^C\right] \\
b_{t+1}^C &=& \frac{\gamma^C}{1+R_t^f-\gamma^C}
\end{array}
\right.
$$

with the following conjecture on prices :

$$
\left\{
\begin{array}{lcl}
P_t^B &=& a_t^B + b_t^B D_t^B \\
P_t^C &=& a_t^C + b_t^C D_t^C
\end{array}
\right.
$$

Now, we need an estimate for R_t^f. We know that $R_t^f = \frac{1}{E_t(m_{t,t+1})}$ with $m_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)}$ the pricing kernel of the economy.

Here, we have $u(c_t) = -e^{-Ac_t}$, so

$$m_{t,t+1} = \beta e^{-A(c_{t+1}-c_t)}$$

and usually for each generation as $c_t = 0$ and $c_{t+1} = \tilde{W}_{t+1}$, we obtain $m_{t,t+1} = \beta e^{-A\tilde{W}_{t+1}}$. As $\{c_t\}$ is normally distributed we can write : $E_t(m_{t,t+1}) = \beta e^{-AE_t(c_{t+1}-c_t)+\frac{A^2}{2}Var_t(c_{t+1}-c_t)}$ or for each generation $E_t(m_{t,t+1}) = \beta e^{-AE_t(\tilde{W}_{t+1})+\frac{A^2}{2}Var_t(\tilde{W}_{t+1})}$. So, we need to deal with the conditional moments E_t and Var_t.

Mathematically, the conditional expectation of a variable x_{t+1} given an information set I_t, $E(x_{t+1}|I_t)$ is equal to a regression forecast of x_{t+1} using every variable $z_t \in I_t$ and their nonlinear transformations. We can choose to restrict the set of forecasting variables $z_t \in I_t$ to their linear transformations by only considering $proj(x_{t+1}|z_t)$ and we could also assume in our case that this set is only composed by all the observed state variables at time t, i.e. the talhas T_t, the partisons A_t, the prices P_t, the volume V_t and consumption C_t. And finally, on the basis of all the information available at time t, we use the projection theorem to determine the conditional moments in the multidimensional case.

Consider a n-dimensional random variable $(X, Z) \sim \mathcal{N}(\mu, \sigma)$, with $\mathbf{x}$ a vector of n_X random variables and $\mathbf{z}$ is a vector of $n_Z = n - n_X$ random variables. The projection theorem gives us :

$$\begin{cases} E_t(x_{t+1}|I_t) \approx E_t(x_{t+1}|\mathbf{z}_t) = \mu_X + \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}(\mathbf{z}_t - \mu_Z) \\ Var_t(x_{t+1}|I_t) \approx Var_t(x_{t+1}|\mathbf{z}_t) = \Sigma_{X,X} - \Sigma_{X,Z}\Sigma_{Z,Z}^{-1}\Sigma_{Z,X} \end{cases}$$

And in the particular case of a single random variable X, i.e. $X = C$, given 5 signals $\{Z^i\}_{i=1..5} = \{P_t, A_t, T_t, V_t, C_t\}$.

$$\begin{cases} E_t(x_{t+1}|I_t) \approx E_t(x_{t+1}|z_t^1, z_t^2, ..., z_t^N) = \mu_X + \frac{1}{\tau_X + \sum_{i=1}^N \tau_{z^i}} \sum_{i=1}^N \tau_{z^i}(z_t^i - \mu_Z) \\ Var_t(x_{t+1}|I_t) \approx Var_t(x_{t+1}|\mathbf{z}_t) = \frac{1}{\tau_X + \sum_{i=1}^N \tau_{z^i}} \end{cases}$$

with $\tau_Y = \frac{1}{Var(Y)}$ is the precision of the variable Y.

References

[1] Almeida, Garcia 2012, ”Assessing misspecified asset pricing models with empirical likelihood estimators”, *Journal of Econometrics* 170(2):519–537

[2] E. Asparouhova, P. Bossaerts, N. Roy 2016, ”“Lucas” in the Laboratory”, *Journal of Finance*

[3] Barro R. J 2009, ”Rare Disasters, Asset Prices, and Welfare Costs”, *American Economic review*

[4] Campbell, John Y. et John H. Cochrane, 1999, ”By force of habit : A Consumption-Based Explanation of Aggregate Stock Market Behavior”, *Journal of Political Economy*

[5] Campbell, John Y. et Robert J. Shiller, 1988, ”The Dividend-Price Ratio and Expectations of Future Dividends and Discount Factors”, *Review of Financial Studies*

[6] Cochrane, J.H 2005, ”Asset Pricing”, *Princeton University Press*, Book

[7] Epstein, Lawrence et Stanley Zin, 1989, ”Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns : A theoetical Framework”, *Econometrica*

[8] Epstein, Lawrence et Stanley Zin, 1991, ”Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns : An Empirical Investigation”, *Journal of Political Economy*

[9] Ghosh, Julliard et Alex P. Taylor, 2017, ”What is the Consumption-CAPM missing?
An Information-Theoretic Framework for the Analysis of Asset Pricing Models” *The Review of Financial Studies*

[10] Hansen, Lars Peter, John C. Heaton et Nan Li, 2008, ”Consumption Strikes Back? Measuring Long-Run Risk”, *Journal of Political Economy*

[11] Hansen, Lars Peter et Kenneth J. Singleton, 1983, ”Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns”, *Journal of Political Economy*

[12] Hansen, Lars Peter et José A. Scheinkman, 2009, ”Long-Term Risk : An Operator Approach”, *Econometrica*

[13] Kim Oosterlinck, Loredana Ureche-Rangau, 2014, ”Baring, Wellington and the Resurrection of French Public Finances Following Waterloo”, *The Journal of Economic History*

[14] Mehra, Rajnish et Edward C. Prescott, 1985, ”The equity Premium : A Puzzle”, *Journal of Monetary Economics*

[15] Mot, G. (1910), Le Moulin du Château Narbonnais. Toulouse

[16] Pouget, Goetzmann et David Le Bris, 2017, "The present-value relation over six centuries : The case of the Bazacle Company", *Journal of Financial Economics*.

[17] Pouget, Goetzmann et David Le Bris, 2017, "The Development of Corporate Governance in Toulouse: 1372-1946", *Working Paper*.

[18] Savov 2011, "Asset Pricing with Garbage", *The Journal of Finance* 66 (1), 177-201

[19] Sicard, G. Aux origines des sociétés anonymes. Les moulins de Toulouse au Moyen Age 1953, *Armand Colin*, Paris. (French)

Part III
On Trading Volume

Trading Volume and Networks

9 Introduction

"there is nothing that living things do that cannot be understood from the point of view that they are made of atoms acting according to the laws of physics"

— R. P. Feynman, *Lectures on Physics*

Volume information is of critical importance on markets. This why it is regularly associated to price data in the financial media. So far, the literature cannot fully explain the mechanism behind the observed trading activity. Several empirical works shed some light on the relationships between volume and other financial variable but few theoretical studies are dedicated to this topic. The main reason for this, is the difficulty to account for heterogeneity among market players. In this paper, we use graph theory to make the differences between agents more tractable. Our approach also allows us to rely volume to exogenous shock and to capture local dependencies of individual preferences. Hence, we take a cross-disciplinary approach mixing networks and incentives theory.

It is common knowledge that interactions are at the roots of any system evolution and governs the physical world. Once we bring at least two elements together, a two-way causality can potentially occur between them, thereby modifying their current steady state. For instance, in the case of stable elementary particles, they are not altered if they remain isolated but could interact when they are embedded within the same field. By making them more dissimilar, we can even magnify the interaction that will become more complex. Such stylized fact can be also observed in other disciplines as Finance.

Thus, we could analogize with what happens in social science by depicting market interactions as a corpuscular system of point-like objects connected by causality channels. We could for instance, extend the physical metaphor to the Newton's law of gravitation by interpreting the differences of taste among the agents as a product of two masses and use the geographical distance between them for assessing the probability they know each other. Thus, the more dissimilar they are, the stronger the force - the further apart they are, the less likely they are socially connected. The mutual incentives to exchange would be described by an *attractive force* proportional to asymmetry of preferences and inversely proportionate to the distance. Thus, if we consider people acting on a Market, we can see their trades as the output of pure social interactions, which are ruled by their relative individual preferences. Of course, this description is somewhat caricatural and would require more financial insight, this is precisely what our study is about.

In our model, we randomly generate the social connections between the agents, which amount to distributing them in space. That leads us to address a first question : how trading take place on a Market? In asset pricing, a long-established tradition of canonical models is entirely based on the assumption that there exists a representative agent in the economy (see for instance Lucas (1978)). Obviously, that leads to the following conclusion : as everyone is the same in terms of preferences, endowment and information, every agent will have the same valuation and consequently, there will not be any incentives to trade on the Market. Despite many model refinements as Epstein and Zin (1989) or Campbell and Cochrane (1999) for explaining a cortege of puzzles generated by this classical approach, the representative agent assumption still seems unrealistic. Moreover, it does not allow to properly describe market trading since it predicts volume is always equal to zero.

It is almost trivial to say that an exchange occurs only if agents have an incentive to do it. More precisely, people are willing to buy or sell a specific asset if they have different valuations of it. By definition, a representative agent assumption does not allow any difference among the market players since they are supposed to behave in the same way. Thus, if we want to understand trading volume on Markets, we have to relax this strong assumption. One way to address this challenge is to consider instead some recent theory developments which embed heterogeneity accross economy players. We are thinking here in terms of a new class of models that is of growing interest in the current macro-economy and finance literature, the heterogeneous agents models (HAM).

There are several kinds of heterogeneity sources that can be potentially accounted for by this new class of models. First, we can suppose differences in endowment by setting asymmetric wealth or income. In our paper for instance, we assume that every social group experiences a liquidity shock with a given probability that pushes its members to become *pure sellers* to the non-impacted part of the Market. As we shall develop further, even if this shock cannot be tracked in the static case resolution, we can fairly assume that it does happen and has consequences for the trading pattern. We can also assume heterogeneity in beliefs due either to different opinions or different information. In order to describe these beliefs, we can use either prior probabilities or likelihood. Finally, we can also assume differences in terms of preferences by considering for instance that the agents have different risk aversions. In the literature, some progress has already been made on beliefs by Lintner (1969), Mayshar (1983), Rubinstein (1975) and Milgrom-Stokey (1982). Endowments and preferences has been addressed by Grossman's work (1976), (1978), extended by Varian (1990) but without focusing on volume formation.

Another strand of the literature using HAM also sorts agents with respect to their behavior. For instance, some frameworks state that agents are of two types : *fundamentalists* and *chartists* as in Zeeman (1974), Frankel and Froot (1986), or *noise traders* and *informed traders* as in Kyle (1985), Black (1986) or DeLong et al. (1990). We shall see that a substantial advantage of our approach is to not impose a exogenous partition of the population.

Some HAM has been also dedicated to stochastic interactions accross agents. This literature, which refers to the related field of complex system theory, highlights the fact that even weak local interactions can generate strong dependencies and large deviations at the macroeconomic level. The main references are Kirman (1991), (1993) for his work on local interactions and Brock and Durlauf (2001) for their work on social interactions. Also under this heading, an interesting model that shares a similar vision with our paper, has been introduced by Föllmer (1974). In his model, the author assumes random preferences whose randomness is governed by a probability law that depends upon the agents' environment. This last intuition is very much in line with our theoretical framework. Indeed, as we shall highlight later, we are concerned by the impact of the Market composition on the global trades pattern by allowing people to band together within social groups. Thus, we must account for theses different subsets by distinguishing the social group of each agent. One way to do this would consist of allowing the individual preferences to depend on the local topology to which the agents belong. More precisely, we assume that people from the same social group share on average the same preferences and the heterogeneity within a group is proportional to its size. Föllmer's paper essentially aims to describe imitation and herding behavior while here, our ultimate purpose is to address the trading volume formation.

Although we can easily deal with an "homogeneous" heterogeneity without any dedicated support, it is more difficult to handle the case where we allow local variations in randomness to happen. This difficulty has prompted us to seek tools for capturing the underlying framework of our economy.

A growing number of studies dedicated to trades in networks could potentially remedy the following pitfalls. First of all, it is common knowledge that the old paradigm of general equilibrium is not suitable to describe real Market mechanisms. The asymmetries among traders position within a Market clearly lead to differences in bargaining powers and thereby to strong deviations from the law of one price. Moreover, in the case of small Market size, we cannot reasonably consider that investors have a price-taking behavior. Finally, the assumption that the Market is perfectly liquid, that is agents can freely trade with every partners without any friction is not reliable either. Thus, a appropriate way to describe all these asymmetries in the context of social interactions is to use network structures and more precisely, the underlying graph theory. Let us first distinguish two kind of models. Those that *ex ante* assign a type to every trader, for instance *pure buyers* and *pure sellers*, and those that consider a stochastic assignment. For the former, we think about the Kranton and Minehart (2001) or Corominas-Bosch (2004) studies that consider bipartite networks with only buyers and sellers to derive the equilibrium of the bargaining game. Kranton and Minehart propose a model where buyers and sellers can act strategically to form a network that maximizes the total welfare. In turn, the main result of Corominas-Bosch is that given the Gallai-Edmonds decomposition, there exists for every discount factor a subgame perfect equilibrium which is efficient.

The author's predictions are tested through an experimental setting in Charness, Corominas-Bosch and Frechette (2007).

Let us now take a look on what has been done so far in the literature on trading volume. Several studies support the idea that a price-volume relation exists. For instance, Osborne proposes a model where the price follows a diffusion process with variance depending on the trading volume. Clark, Tauchen and Pitts, and Harris show that the volume is positively correlated to the absolute price changes in models where transaction time intervals are variable. Empirical findings also show that volume is negatively related to the bid-ask spread. On the theoretical side, Epps (1976) proposes a model where the agents are sorted into two groups, "potential buyers" and "potential sellers". He found that the expected volume from an exogenous shock is a decreasing function of transaction costs, and in particular of the bid-ask spread. Another important strand of the literature is also dedicated to the "mixture of distributions" models which address the leptokurtosis in the empirical distribution of speculative prices. The volume can also be studied through its relation to information as in Copeland where a common part of information arrives sequentially to agents. In Pfleiderer, the author considers that every agent receives information about the true value of an asset which has a common and an individual component. The major drawback of this model is that it predicts a negative relationship between the volume and the investors disagreement. Another approach proposed by Varian consists to say that trading volume is only due to differences of opinions which is also inconsistent with most empirical evidence. Karpoff (1986) derives a model where people frequently revise their valuations of an asset and are randomly paired with potential partners. Finally Wang and Lo (2006) introduce *turnover* as a new measure of trading volume, derive volume implications of basic portfolio theory and propose a model where volume is endogenously generated by liquidity needs and risk-sharing motives. All these papers do not take into account for explaining volume, the local frictions due to social connections in the Market and the resulting asymmetries in investor positions.

As we mentioned above, we consider in this paper that the agents are heterogeneous in terms of preferences. In order to understand how exchanges can emerge on the Market, let us proceed step by step by focusing on the different conditions that led to it. First of all, the agents must know each other, that is a social link must exist between them to allow a trade. Then, they must have an incentive to buy or to sell, that is their preferences must be such that they have a mutual advantage to exchange. These two conditions only ensure that the interaction is desirable but do not warrant that trade will do occur. To let it really happen, an additional condition is required. Indeed, given their feasible action sets, two agents will do trade if they are the best choice of each other. In other terms, they look at their neighborhood and select among all the available partners, the one that maximizes their gain. We think here in terms of best price comparing to their own valuation of the asset. Thus, a player will always choose the neighbor with the furthest preferences. Finally, we claim that the volume is the combination of these three conditions : a social link must exists, agents must have an incentive to trade in terms of their mutual preferences and they have to be the optimal choice of each other regarding their feasible action set. Notice that in our model, agents that reach an arrangement exit the market and the game is over when no more trade is feasible on the Market.

Here, a question arises : why does graph theory has a critical contribution in this study ? As one preliminary condition to let an exchange happen, a social link must exist between the potential partners - the most natural way to model social links in a population remains vertices and edges. We have made the choice here to use graphs to describe the fundamental structure on which the interactions take place. Moreover, our model also accounts for liquidity shocks experienced by the different social groups. These shocks fundamentally alter the social connections within the Market and we need an underlying framework for tracking it. As we shall highlight in the model resolution, it is not an easy task to derive closed formulas for the volume in general networks. Thus, we have to control for randomness on which the graph is based to obtain a tractable topology.

The main advantage of our setting is to do not specify *ex ante* any bipartite structure. Once the preferences are randomly determined for the entire economy, only two agents will be of a single type, the ones with the highest and the lowest risk aversion. The others are both buyers and sellers, their type being defined relatively to their neighbors preferences. The key is that the agents' type is endogenously determined. Moreover, we assume pure random pairing as we generate random graph rather than given topology on which the trades would take place. Our setting includes the idea that the agent preferences depend on his environment in order to capture the impact of groups heterogeneity on volume and price formation. Finally, the model also captures pure exogenous motivations to trade by specifying a shock probability for each group of investors.

Our main findings are that a link exists between the Market composition and the number of desirability channels, which are the combination of the conditions based on social connection and incentive to trade. More precisely, when the number of groups increases - their size being equal - the incentives to trade decreases and the same applies for the volume. However, when the relative differences between the groups sizes increases - the number of them remained equal - new desirability channels appear and the volume potentially increases. We also establish a nonlinear relationship between the number of incentives and the shock probability. We derive closed formula for the expected volume in the case where the Market is composed by a single group that cannot experience a liquidity shock. Finally, from a social planner perspective, we propose to characterize any networked Market regarding to his efficiency. In other terms, given the individual preferences, we measure how the graph of social connections maximizes the number of trades.

The remainder of this paper is organized as follows. The first section is dedicated to a general presentation of the model and its main assumptions. We establish a necessary condition for trading to occur based on the risk aversions ratio of the agents enrolled in the exchange. Then, we extend this condition by adding the social connections. We also describe how the graph and the individual preferences are randomly generated. In the second section, we introduce the concept of desirability channels before looking at how it is related to the graph topology. We present in the third section some important results from Graph theory that are a very useful material for the discussion of the volume in section four. In the light of what we found concerning desirability channels, we are able to determine the expected number of trades in the Market. Then, we assess how this quantity evolves when the main parameters of the model that drive the graph architecture are altered. In section five, we address the pricing dimension of our economy by emphasizing how our results support the standards of the literature dedicated to the price-volume relation. We also examine the contribution of individual preferences to the average market price through the agents participation to trading. Finally, we provide some results about the price formation when a shock occurs. The section six concludes.

10 The Model

10.1 Assumptions

(H1) We consider a static exchange Market composed by n rational agents where social groups or communities are *ex ante* determined. Each agent is allowed to trade a unique asset but can only buy or sell a fixed quantity of share per transaction.

(H2) Agents are heterogeneous in terms of preferences, more precisely they don't share the same risk aversion which is randomly determined. Moreover, this randomness depends on the agents environment, that is the social group to which they belong.

Thus, they don't have the same valuation of the asset and we can state a condition based on their respective preferences that allows a trade to occur.

(H3) Each agent belongs to exactly one social group and is more likely connected to compatriots than to members of other groups. A connection represents a social link in the Market.

There is no explicit transaction cost in this Economy although assumption (1) could suggest a huge constraint that limits the number of tradable shares per transaction. Moreover, the combination of assumptions (2) and (3) can also be viewed as a real friction that does not permit to freely trade with every partner.

(H4) Each community randomly and independently experiences a liquidity shock. In this case, all the members suddenly have the same preferences and get pure sellers for the non impacted part of the Market.

10.2 Trade condition

Let us consider a non-informational static economy with n agents heterogeneous in terms of risk aversion who can exchange a single asset. We don't pay attention here to differences in endowments by considering that all the agents begin with one unit of share. They have CARA preferences and maximize their wealth according to the alternative to buy, to sale or to don't trade a fixed quantity $\epsilon \in [0,1]$ of share at once. We argue that the constraint on the number of tradable shares per transaction could come from the fact that takes time to make a trade or equivalently that there are huge transaction costs in the Market.

Notice that in the case of a dynamic setting with information, we would face a *no trade* situation here. Indeed, as pointed out by Tirole (1982), if we consider rational players with the common knowledge that to have an incentive to trade each of them should expect a positive gain given his information, no one should be willing to trade. Thus, in our case, after a round based on hedging and speculative desires there would not be any reason that new information motivates further exchanges. That why we don't take into account information here, we only consider a snapshot of the Market after each investor revised his preferences to look at the volume that could be generated by the potential interactions.

We denote p the price of the asset and we assume its real value $\tilde{v}$, whose the distribution is common knowledge, is normally distributed with mean μ and variance σ^2. Thus, the wealth[16] of an agent can be written as

$$W = \begin{cases} W_B = (\tilde{v} - p)\epsilon + \tilde{v} = (1+\epsilon)\tilde{v} - p\epsilon & \text{if he buys } \epsilon \text{ share} \\ W_S = (p - \tilde{v})\epsilon + \tilde{v} = (1-\epsilon)\tilde{v} + p\epsilon & \text{if he sells } \epsilon \text{ share} \\ W_{NT} = \tilde{v} & \text{if he does not trade} \end{cases}$$

Therefore, the maximization program for every agent i is here

$$\max_{\iota \in \{-1,0,1\}} -\mathbb{E}(e^{-A_i(\iota(\tilde{v}-p)\epsilon + \tilde{v})})$$

where A_i is the risk aversion coefficient. Thus, an agent will be willing to buy ϵ share when $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_S))$. Similarly, he will be willing to sell the same quantity when $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_B))$. By using the moment

[16] We propose in Appendix a dynamic version of this model where we endogenize the probability that a shock occurs.

properties of a log-normally distributed random variable, we can state the following proposition.

Proposition : *There is an incentive between agent i and agent j to trade ϵ share if and only if*

$$\frac{A_i}{A_j} \notin \left[\frac{1-\frac{\epsilon}{2}}{1+\frac{\epsilon}{2}}, \frac{1+\frac{\epsilon}{2}}{1-\frac{\epsilon}{2}}\right] \underset{\epsilon\to 0}{\to} \{1\}$$

with A_i and A_j the risk aversion coefficients of agents i and j.

Notice that this proposition describes a necessary condition for a trade to occur but it is not a sufficient one. We will detail this point in section (13). In the case[17] the ratio $\frac{A_i}{A_j}$ would be drawn from a continuous distribution of probabilities, we would have whatever the law that governs the randomness, $\mathbb{P}\left(\frac{A_i}{A_j} \neq 1\right) = 1$ when ϵ tends towards zero by definition. In the next of this paper, we will consider that the agents can only trade one share at once, that is $\epsilon = 1$ and we have the following trade rule.

Corollary : *There is an incentive between agent i and agent j to trade one unit of share if and only if $\frac{A_i}{A_j} \notin \left[\frac{1}{3}, 3\right]$. More precisely, i will sale to j if and only if $\frac{A_i}{A_j} > 3$ and i will buy from j if and only if $\frac{A_i}{A_j} < \frac{1}{3}$.*

One drawback of our model is that it does not allow heterogeneity in terms of endowment since this quantity disappears within the resolution. Our purpose is now to determine the probability that two agents are willing to trade or not[18] but first, let us introduce the underlying framework that describes the social connections in the Market.

10.3 An underlying network

In this economy, we consider that agents are gathered into K social groups which are *ex ante* chosen[19]. Thus, we consider a K-partition of the players set as $\{1, ..., n\} = \{i_1, ..., i_{n_1}\} \cup \{i_{n_1+1}, ..., i_{n_2}\} \cup ... \cup \{i_{n_{(K-1)}}, ..., i_n\}$. Then, to describe how these players interact, we base our framework on the sociological definition of a group by stating that *two members of a social group are more likely to know each other than if they come from different areas.* This hypothesis, which is strongly supported by most of studies on social connections, leads to the following interpretation : the group to which an agent belongs, can be viewed here as his primary social circle composed by those he frequently meet and those he can easily reach.

[17]See section (10.4).
[18]We will see how we can decompose this probability in section (11)
[19]We will see further how we use group belonging to draw the risk aversion parameter of each agent.

We also assume that people tend to come together in affinity groups, that is they share on average the same preferences in accordance with the old adage that "Birds of a feather flock together". However, there exists another important feature of real networks we want to capture here, it is the *small-world phenomenon*[20]. This phenomenon states that the World seems generally "small" when you count the number of intermediate acquaintances it takes from any individual to reach anyone else. In other terms, individuals can have long-range connection with people who have very little in common with them in terms of culture, status, geographical environment, consumption habits, investment behaviour etc... Notice here that we are not claiming they mostly differ in their preferences since two people from different groups could have close preferences but these preferences could evolve in a very different way. Thus, in the dynamic version of our model we propose in Appendix, we consider a law of motion for the risk aversion coefficient of every agent which is contingent to his neighborhood. Indeed, when an individual has been involved in a trade, we just replace his risk aversion by the average one resulting from the preferences of his partner. Thus, it is possible to assess the speed at which the market reach a unique price.

In order to represent the social connections which are not known with certainty but should be based on (H3), we need to set up an underlying random framework. As we argued above, we consider that two agents are randomly paired but the group belonging must play an important role in the randomness. Thus, we propose to use a random graph representation with vertices and edges to describe the pairing process in the Market. Every vertex represents an agent and every edge a social connection. As some individuals know each other but others don't, our framework intuitively describes the shape of an *OTC Market* and we will see how we can make progressively tend this setting towards an *Organized Market*. In this version of the model, we don't specify the weights of the links but we could reasonably think about this additional parameter as a mean to introduce the concept of trust or other bargaining features in the economy. Let us now introduce a first definition from Graph theory to formalize in mathematical terms our Market structure.

Definition : *In Graph theory, the two main models for describing random graphs are :* $\mathcal{G}(n, M)$ *which consists of all graphs with vertex set* $V = \{1, 2, .., n\}$ *having* M *edges, and* $\mathcal{G}(n, (p_{ij}))$ *for all graphs with the same vertex set where the edges* $x_i x_j$ *are chosen independently with probability* p_{ij}*. In* $\mathcal{G}(n, M)$*, all the graphs have the same probability* $\binom{M}{N}^{-1}$*.*

In this paper, we only consider the model $\mathcal{G}(n, (p_{ij}))$ where $p_{ii} = 0$, that is loops are not allowed since an agent cannot trade with himself. The simplest random graph formulation is the case where

[20] A very famous experiment of this phenomenon has been carried out by Milgram in the 1960s. He picked up randomly 296 individuals to try forwarding a letter to a target people who was leaving very far from the other ones. Thus, each people who received the letter had to forward it to someone they knew on a first-name basis, then the recipient had to proceed according to the same instructions until the letter finally reaches the target. Milgram observed that the median length of the required paths to reach the final destination was six and deduced that there exists very short paths in the social networks.

all the vertices pairs have the same probability to be linked by an edge, that is when $\forall i, j$, $p_{ij} = p$. This special case is known as the Erdös-Rényi standard model $\mathcal{G}(n, p)$ and will be regularly invoked in the next sections. Instead, when we allow different pairing probabilities, we say that the resulting structure belongs to the class of *inhomogeneous random graphs*. Moreover, when we consider a K-partition of the vertex set $V(G)$ as a collection of disjoint subsets $\{H_1, H_2, ..., H_K\}$ with the property that p_{ij} depends on the subsets the vertices i and j belong, we obtain the subclass of the stochastic block models. Thus, given that every vertex belongs to exactly one group and that the probability of a link between two vertices depends on which groups the vertices lie in, these models are perfectly fitted to describe our market interactions. The subsets here can be viewed as communities or social groups and in regards to the assumptions we made on the connections between agents, it is coherent to set that the probability is higher within a group than across them. Therefore, we consider that $\forall i \neq j$

$$p_{ij} = \begin{cases} p \text{ if } x_i, x_j \text{ belong to the same group} \\ 1 \text{ if only } x_i \text{ or } x_j \text{ belongs to a group impacted by a shock} \\ \frac{p}{c} \text{ otherwise} \end{cases}$$

with c a constant. In the next of the paper and unless otherwise specified, we will take $c = 2$ to get the computations more tractable. Nevertheless, we are aware that from the sociological point of view, c should be much more higher than 2. Notice that if $c = 1$, the model would describe the case of an organized Market where a regulatory body would match players regardless of their group belonging. As we mentioned above, in the case where $\forall i, j$, $p_{ij} = p$ or equivalently when $K = 1$, if the shock probability is zero, we would obtain the Erdös-Rényi standard model $\mathcal{G}(n, p)$.

We propose here a slight refinement of the stochastic block model by introducing an additional dimension to capture the shocks that can be independently experienced by each group. Let S^k be the event "the group composed by the vertices in $V(H_k)$ is impacted by a shock". We assume $\forall k \in [\![1, K]\!]$, $\mathbb{P}(\{S^k\}) = q$ and $\forall k \neq l$ $\mathbb{P}(\{S^k\}|S^l) = \mathbb{P}(\{S^k\})$, that is each shock occurs independently but with the same probability for every group. Thus, regarding the probabilities p_{ij} we introduced above, when a such event does happen, all the members of the impacted group become fully connected with the rest of the graph.

Intuitively, we could interpret this phenomenon as any kind of shock that potentially pushes an impacted segment of the market to suddenly spend a lot of money for searching the best partner to trade. That could be for instance a local liquidity shock which makes the agents very impatient as they need cash. In line with this idea, a growing literature is dedicated to contagion in financial networks as in Acemoglu et al. (2015) or Gai and Kapadia (2010). We could also think about market regulations which would suddenly impose a sharp reduction of the risk exposure for a type of investors, let say the banks. Thus, all of them must quickly re-balanced their portfolios by selling their most risky asset. Finally, that could also simply suggest herding behaviours, that is when the members of a group mimic the moves of some leader or when the number of sellers reaches a threshold which triggers a local panic.

We are now able to combine these results to what we stated in section (10.2) to introduce the following definitions :

Definition : *Let $\delta(x)$ be the degree of the vertex x, that is the number of its adjacent vertices, and $N(x)$ his neighborhood*[21]*. Any agent in our economy belongs exactly to one of the following categories :*

- *An agent who has no social connection in the graph is said isolated. Formally x is isolated if and only if $\delta(x) = 0$.*
- *An agent who has only buyers in its neighborhood is called pure buyer. Formally, x is a pure a buyer if and only if $\nexists y \in N(x) : \frac{A_x}{A_y} > 3$.*
- *An agent who has only sellers in its neighborhood is called pure seller. Formally, x is a pure a seller if and only if $\nexists y \in N(x) : \frac{A_x}{A_y} < \frac{1}{3}$.*
- *An agent who has at least one buyer and one seller in its neighborhood is called a trader. Formally, x is a trader if and only if $\exists y, z \in N(x) : \frac{A_x}{A_y} > 3$ and $\frac{A_x}{A_z} < \frac{1}{3}$.*

[INSERT FIGURE HERE OF A GRAPH WITH COMMUNITIES]

Notice that the connectivity of a graph $G \in \mathcal{G}(n, p)$, and so the neighborhood of its vertices, will strongly depend on the value of p. We will further develop this point in section (14.4). Now, let us address how we are dealing with the randomly determined preferences of every agent.

10.4 The preferences of the agents

In section (10.2), we have seen that one condition to observe a trade between two agents is that the ratio of their risk aversion does not belong to the interval $[\frac{1}{3}, 3]$. However, we did not mention how these coefficients are chosen.

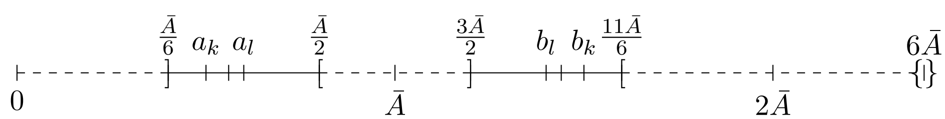
Let us assume now that they are drawn from uniform distributions whose the support depends on the size of the group to which each agent belongs. As we mentioned above, this choice is in line with some previous work as in Föllmer (1973) where the state of each agent, that is his preferences, is affected by his environment. For instance, if an agent (or a vertex x) belongs to a group $V(H_k)$ of size n_k, his risk aversion A_x would be a random variable with density of probability $U_{[a,b]}$ and the size of the interval $[a, b]$ would be proportional to n_k. Formally, we write $\lambda([a, b]) = f(n_k)$ with

[21]These notions are detailed in section (12).

f a strictly increasing function of n_k and λ the classical Lebesgue measure. Notice that generates one drawback of our model, if we consider two equally sized groups, they are necessarily similar in terms of risk aversion distribution[22].

This setting has been motivated by several reasons. First, we want to emphasize the fact that in large groups, we expect to find more heterogeneity or a higher dispersion than in smaller groups in terms of preferences. Thus, the most natural way to formally describe this heterogeneity, is to allow the support of the uniform distribution to depend on the size of each group since $\forall x_i \in V(H_k)$, we would have $Var(A_i) = \frac{(b_k - a_k)^2}{12}$. Moreover, we want to avoid the case where a group is composed by two agents with risk aversions located at the two bounds of the preferences set. It is also rational to guess that two people from a small community are more likely to have closer preferences than if they come from a big one.

Second, we want to make a distinction among the different groups and rely the randomness of the agent risk aversion to his environment. More precisely, we claim that social pressure is stronger in small structures than in big ones, so the members of a small group would be more influenced by their entourage, implying that their preferences would be more closely located around a central value. Instead in large group, people are less constrained and could behave more freely. This idea is supported by several sociological works, in particular on electoral participation as in Funk (forthcoming). In this paper, the author claims that social pressure is the main ingredient in charge of the positive relationship between community size and civic involvement since in smaller community the ability of each member to monitor his neighbors behavior is higher than in larger structures, leading to higher social pressure. One good example can be found in the cities analogy[23] as it is more likely to find lifestyles released from the social norm in big agglomerations than in small town. We also assume that all the agents of this economy have on average the same risk aversion[24], denoted $\bar{A}$. Thus we construct the bounds of our intervals centered on $\bar{A}$ as follows.



Thus when there is no shock, the lower bounds can only be located in the set $]\frac{\bar{A}}{6}, \frac{\bar{A}}{2}[$ and the upper bounds must belong to the interval $]\frac{3\bar{A}}{2}, \frac{11\bar{A}}{6}[$. Let N_k denote the size of the interval from which are drawn the risk aversion parameter of every agent who lies in the group $V(H_k)$. We impose the following form

[22] Regarding to this issue, we paid attention in the simulations to only generate random heterogeneous sizes for the groups.

[23] See also Oliver (2000).

[24] Instead, if we consider intervals centered on growing risk aversion levels, we would have some groups that always are seller or buyer with respect to others. As we do not observe this pattern on real Markets, we avoid to set up the model on this way.

$$\forall k \in [\![1, K]\!], \quad N_k = \bar{A}\left(1 + \frac{2}{3n}\left(n_k - \frac{1}{n}\right)\right)\left(1 - \mathbb{1}_{\{S^k\}}\right)$$

with $\mathbb{1}_{\{S^k\}}(.)$ an indicator function. Clearly here, N_k is a random variable which entirely depends on the occurrence of the event S^k. We observe that the length of any interval is an increasing function of its associated group size n_k and is bounded by $N_k \in \{0\} \cup \left]\bar{A}, \frac{5}{3}\bar{A}\right[$. The choice of the form of N_k has been motivated by technical concerns for solving the model, as detailed in next section and Appendix, but does not diminish the scope of the results. The bounds of any interval can be expressed as

$$a_k = \bar{A}\left(1 + 5\mathbb{1}_{\{S^k\}}\right) - \frac{N_k}{2} \quad \text{and} \quad b_k = \bar{A}\left(1 + 5\mathbb{1}_{\{S^k\}}\right) + \frac{N_k}{2}$$

Thus, when a group H_k is impacted by a shock, we observe that the interval $[a_k, b_k]$ is both contracted and shifted to the singleton $\{6\bar{A}\}$. Therefore, all the risk aversion parameters of the impacted members will be equal to $6\bar{A}$. Consequently, no agents from this group will be willing to trade anymore since in regards to section (10.2) results, as $\forall x_i, x_j \in V(H_k)$, $\frac{A_i}{A_j} = 1 \in \left[\frac{1}{3}, 3\right]$, there is no incentive to exchange locally in this group. Moreover, they become *pure sellers* for the rest of the graph since $\forall y \in V(G \backslash H_k)$ and $x \in V(H_k)$, we have $\frac{A_x}{A_y} > 3$. These properties of our setting are perfectly coherent with the economic and financial insights we presented in previous section.

By construction of our intervals, we can easily derive a first result

Proposition : *A necessary and sufficient condition for an agent x to be a trader is given by the following intersection :*

$$\left\{A_x \in \left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]\right\} \bigcap \left\{\exists y, z \in N(x) : \frac{A_x}{A_y} > 3, \frac{A_x}{A_z} < \frac{1}{3}\right\}$$

where A_x is the risk aversion parameter of the agent and $\mathbb{P}\left(A_x \in \left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]\right) = \frac{\bar{A}}{9N_x}$.

Let A_y, A_x and A_z be the risk aversions of agents x, y and z with $A_y < A_x < A_z$. The condition $\frac{A_z}{A_x} > 3$ implies that there exists a seller z from who x would be willing to buy if and only if A_x is strictly lower than $\frac{11\bar{A}}{18}$. Similarly, $\frac{A_x}{A_y} > 3$ implies that A_x must be higher than $\frac{\bar{A}}{2}$ to provide an incentive to sell. Notice here that the probability is a decreasing function of the size of the group the agent lie in. However, people must be connected to partners that belong to a big group to have incentives to trade with them. Thus, we expect to find traders only in well-connected small groups or in big ones. We also notice that our setting implies that two traders cannot exchange in the economy, they don't have "*space* enough in terms of preferences. We will see in section (13) a direct implication of this proposition.

Now that we have set up the support on which the preferences will be randomly chosen, let us go further in the computation of the no trade probability.

10.5 The probability of no trade

Regarding to the results of section (10.2), the probability of no trade is given by the quantity $\mathbb{P}\left(\frac{A_i}{A_j} \in \left[\frac{1}{3}, 3\right]\right)$. Here, both A_i and A_j are random variables uniformly distributed, so we must first define the cumulative density function of the ratio of two uniformly distributed variables. Let x_i and x_j be two vertices from the group $V(H_k)$, we denote $Z_{kk} = \frac{A_i}{A_j}$ the ratio of their risk aversion parameters. Notice that Z_{kk} is only indexed with respect to the group to which these vertices belong since its distribution will be the same whatever the pair of vertices considered in $V(H_k)$. For instance, if x_i and x_j come from the sugraphs $V(H_k)$ and $V(H_l)$ respectively, we would write the ratio Z_{kl}. Hence, the probability that x_i and x_j do not trade can be written $\mathbb{P}\left(Z_k \in \left[\frac{1}{3}, 3\right]\right)$. Clearly, the ratio distribution will depend both on the size of the group the vertices lie in and if they belong two different groups or not. As shown in Appendix, Z_k is a random variable with the following density functions.

$\forall (x_i, x_j) \in V(H_k)^2$, we have

$$f_Z(z)_{kk} = \frac{b_k^2 - \left(\frac{a_k}{z}\right)^2}{2(b_k - a_k)^2} \mathbb{1}_{z \in \left[\frac{a_k}{b_k}, 1\right]} + \frac{\left(\frac{b_k}{z}\right)^2 - a_k^2}{2(b_k - a_k)^2} \mathbb{1}_{z \in \left[1, \frac{b_k}{a_k}\right]}$$

$\forall (x_i, x_j) \in V(H_k) \times V(H_l)$ with $|V(H_l)| \neq |V(H_k)|$, we have

- For $N_k < N_l$,

$$f_Z(z)_{k<l} = \frac{b_l^2 - \left(\frac{a_k}{z}\right)^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{a_k}{b_l}, \frac{b_k}{b_l}\right]\right\}} + \frac{\left(\frac{b_k}{z}\right)^2 - \left(\frac{a_k}{z}\right)^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{b_k}{b_l}, \frac{a_k}{a_l}\right]\right\}} + \frac{\left(\frac{b_k}{z}\right)^2 - a_l^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{a_k}{a_l}, \frac{b_k}{a_l}\right]\right\}}$$

- For $N_k > N_l$,

$$f_Z(z)_{k>l} = \frac{b_l^2 - \left(\frac{a_k}{z}\right)^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{a_k}{b_l}, \frac{a_k}{a_l}\right]\right\}} + \frac{b_l^2 - a_l^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{a_k}{a_l}, \frac{b_k}{b_l}\right]\right\}} + \frac{\left(\frac{b_k}{z}\right)^2 - a_l^2}{2N_k N_l} \mathbb{1}_{\left\{z \in \left[\frac{b_k}{b_l}, \frac{b_k}{a_l}\right]\right\}}$$

As we need the cumulative distribution function of Z_k to find the probability of no trade, we must take the integral of these functions over the set $[\frac{1}{3}, 3]$. Obviously, we don't know *a priori* the intersection of this set with the support of the density function, that is with the support of the different indicators functions.

Thus, we require that the condition $\frac{a_k}{b_k} < \frac{1}{3}$ holds since if it doesn't, we can easily show that no one has an incentive to trade and the probability is equal to one. We can show[25] that this inequality implies for all k, l, $\frac{a_k}{b_l} < \frac{1}{3}$. Moreover, it is also straightforward[26] to prove that regardless of this condition, we always have for all k, l, $\frac{b_k}{b_l} > \frac{1}{3}$.

That leads to different constraints on the intervals set that will shape the structure of N_k we introduced in previous section. Indeed, we can intuitively set an upper bound for N_k since the risk aversion parameter cannot be negative, that is $N_k < 2\bar{A}$. However, this condition is not sufficient as the ratio $\frac{a_k}{a_l}$ becomes arbitrarily large when a_l tends to zero. Therefore, to refine this upper bound and to keep the model in a tractable way, we further assume for all that k, l, $\frac{a_k}{a_l} < 3$. Finally, that implies the minimum and maximum size[27] of N_k we mentioned above.

In light of these results, we are now able to define the probability of no trade in the case both when vertices belong to the same group and when they belong to different groups.

10.5.1 The Intragroup and Intergroup Probabilities

Let us first introduce the *intragroup* probability, that is the case where we consider vertices from the same subgraph.

Proposition : *Let x_i and x_j be two agents from the group $V(H_k)$, their probability of no trade is given by*

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[25]Indeed, if $N_k > N_l$ we have $\frac{a_k}{b_l} = \underbrace{\frac{a_l}{b_l}}_{<1/3}\underbrace{\frac{a_k}{a_l}}_{<1} < \frac{1}{3}$ and if $N_k < N_l$ we have $\frac{a_k}{b_l} = \underbrace{\frac{a_k}{b_k}}_{<1/3}\underbrace{\frac{b_k}{b_l}}_{<1} < \frac{1}{3}$.

[26]Let us assume that $\frac{b_k}{b_l} < \frac{1}{3}$. If $\frac{a_k}{a_l} > 3$ that implies

$$\begin{aligned}
b_k &< \frac{1}{3}b_l \\
2\bar{A} - a_k &< \frac{1}{3}(2\bar{A} - a_l) \\
6\bar{A} - 3a_k &< 2\bar{A} - a_l \\
4\bar{A} - 4a_k &< -4a_l \\
\bar{A} &< a_k - a_l
\end{aligned}$$

which is not possible by construction. And similarly if $\frac{a_k}{a_l} < 3$ we would have

$$\begin{aligned}
4\bar{A} &< 3a_k - a_l \\
4\bar{A} + a_k &< 3a_k + 2a_l \\
4\bar{A} &< 2a_k + 2a_l \\
2\bar{A} &< a_k + a_l
\end{aligned}$$

which is not possible by construction.

[27]Indeed, we have $N_k = b_k - a_k > b_k - \frac{1}{3}b_k \Leftrightarrow N_k > \frac{2}{3}(\bar{A} + \frac{N_k}{2}) \Leftrightarrow N_k > \bar{A}$. But also a maximum size since we have $N_k > \bar{A} \Rightarrow a_k < \frac{\bar{A}}{2}$ and as $\frac{a_k}{a_l} < 3$, that implies $a_k > \frac{\bar{A}}{6}$. By symmetry, we obtain $b_k < \frac{11\bar{A}}{6}$ and finally $\bar{A} < N_k < \frac{5}{3}\bar{A}$.

$$\mathbb{P}_{kk} = \begin{cases} 1 - \frac{4}{3}\left(\frac{\bar{A}}{N_k} - 1\right)^2 & \text{no shock} \\ 1 & \text{otherwise} \end{cases}$$

where N_k is the size of the interval associated to the group $V(H_k)$.

Here in the probability, we distinguish the case where a shock occurs from the case nothing happens. We have already seen in sections (10.2) and (10.4) that by construction, when a segment of the Market is impacted, every preference is shifted in the singleton $6\bar{A}$ such that no one is willing to trade anymore in the group. Hence, the resulting probability of no trade is equal to one. Notice we can easily check that the quantity presented in this proposition does fulfill the properties of a probability measure[28].

As we have[29] $\forall k \in [\![1, K]\!]$, $\frac{\bar{A}}{N_k} \in [\frac{1}{2}, 1]$ and N_k is a increasing function of n_k, smaller the size of the group, greater the probability $\mathbb{P}_{kk}$. This result is coherent with the intuition with mentioned in section (10.4). Indeed, we claimed there is potentially less heterogeneity in small groups than in large ones and consequently, it is more likely to find two agents with close preferences, that is with roughly the same asset valuation, who are not willing to trade in a small community. Notice here that this probability only depends on the parameters n and n_k.

Let us now introduce the *intergroup* probability, that is the case where we consider vertices from different subgraphs.

Proposition : *Let x_i and x_j be two agents from the groups $V(H_k)$ and $V(H_l)$ respectively with $n_k \neq n_l$, their probability of no trade is given by*

$$\mathbb{P}_{kl} = \begin{cases} \frac{4}{3} - \frac{1}{6N_kN_l}\left(\frac{8}{5}\bar{A}^2 + \frac{5}{2}\left(N_k + N_l - \frac{8}{5}\bar{A}\right)^2\right) & \text{no shock} \\ 0 & \text{one shock} \\ 1 & \text{two shocks} \end{cases}$$

where N_k and N_l the sizes of the intervals associated to the groups $V(H_k)$ and $V(H_l)$ respectively.

Again, it is straightforward[30] to show that this quantity does belong to the interval $[0, 1]$. Notice that[31] $\mathbb{P}\left(\frac{1}{3} < Z_{kl} < 3 \mid S^k, S^l\right) = \mathbb{P}\left(\frac{1}{3} < Z_{kk} < 3 \mid S^k\right) = 1$. Here, we consider separately the case where no shock happens, only one group is impacted, and both groups are impacted. As we

[28]Indeed, $\mathbb{P}(\frac{1}{3} < Z_{kk} < 3) \in [0, 1]$ since $\frac{2}{3}b_k^2 - 2a_k^2 - (b_k - a_k)^2 = -\frac{1}{3}b_k^2 - 3a_k^2 + 2a_kb_k = -\frac{1}{3}(b_k - 3a_k)^2 < 0$ and $\frac{a_k}{b_k} < \frac{1}{3} \Rightarrow \frac{2}{3}b_k^2 - 2a_k^2 > 0$.

[29]See Appendix.

[30]We have $\mathbb{P}(\frac{1}{3} < Z_{kl} < 3) = \frac{2(b_kb_l + a_ka_l) - 3(a_k^2 + a_l^2) - \frac{1}{3}(b_k^2 + b_l^2)}{2N_kN_l}$ and $2(b_kb_l + a_ka_l) - 3(a_k^2 + a_l^2) - \frac{1}{3}(b_k^2 + b_l^2) - 2(b_kb_l - b_ka_l - a_kb_l + a_ka_l) = -\frac{1}{3}\left[(3a_k - b_l)^2 + (3a_l - b_k)^2\right] < 0$ which implies $\mathbb{P}(\frac{1}{3} < Z_{kl} < 3) < 1$ and by definition of the integral, we also have $\mathbb{P}(\frac{1}{3} < Z_{kl} < 3) > 0$.

[31]If H^k and H^l are impacted, $N_k = N_l = 0$ and if H^k is impacted but H^l doesn't, we have $3 < \frac{b_k}{b_l} < \frac{b_k}{a_l}$ and the integral is equal to zero.

mentioned above, the singleton $6\bar{A}$ has been chosen such that when one segment is in distress, all of his member become *pure sellers* for the rest of the Market. Consequently, the probability of no trade between two agents when only one of them belongs to an impacted group is equal to zero.

In this proposition, we observe that the probability is clearly a decreasing function of the aggregated size $N_k + N_l$ and only depends on the parameters n and n_k. We show in appendix that $N_k < N_l$ implies $\mathbb{P}_{kl} < \mathbb{P}_{kk}$, that is each member of a group is more likely to be willing to trade with someone who belongs to a larger group rather than with a member of his own community. But notice that the reverse is not always true, $N_k > N_l \Rightarrow \mathbb{P}_{kl} > \mathbb{P}_{kk}$ if and only if $N_k - N_l < \frac{16\bar{A}}{5}\left(1 - \frac{\bar{A}}{N_k}\right)$. Finally, we can also state the following results[32] : $N_k < N_l < N_m$ implies $\mathbb{P}_{km} < \mathbb{P}_{kl}$ and for any triple (n_k, n_l, n_m) such that $n_k < n_l < n_m$ and $\frac{1}{3n}(n_m - n_k) > \frac{8}{5}\left(1 - \frac{1}{1+\frac{2}{3n}\left(n_m - \frac{1}{n}\right)}\right)$, we have $\mathbb{P}_{mk} < \mathbb{P}_{mm}$.

Thus, we are now able to state with which probability two agents are willing to trade or not, according to their group belonging. In next section, we highlight how that can help to define the expected volume in the Market.

11 Desirability Channel

Let us sum up what we did so far for developing the model. First, we have endogenously generated the social links between the agents and then, we have deduced their preferences by drawing the risk aversion parameters from uniform distributions whose the support depend on each group size. Finally, we have determined the probability that two agents are willing to trade with respect to their relative position in the graph, that is if they lie into the same community or not.

We now introduce the concept of desirability channel between two agents as the case where they know each other and their preferences are such that a trade is *desirable* (even if it finally does not occur). Here, we are just reasoning in terms of social connections and risk aversions to state that two individuals must be linked to interact on the Market and their respective preferences must be sufficiently distinct to generate an incentive to exchange. However, we didn't address so far the optimization program of each agent according to his budget constraint, we will deepen this point in section (13).

Let consider DC_{ij} the binary variable that takes one if a desirability channels exists between x_i and x_j and zero otherwise. We denote $\{x_i \sim x_j\}$ the event "there exists an edge between x_i and x_j" or equivalently, "agents i and j know each other". Thus, $\forall x_i, x_j \in V(G)$, the probability that a desirability channel exists is given by the following expression

[32]See Appendix for the proof.

$$\mathbb{P}(DC_{ij}=1)=\mathbb{E}(DC_{ij})=\mathbb{P}\left(\left\{\frac{A_i}{A_j}\notin\left[\frac{1}{3},3\right]\right\}\bigcap\{x_i\sim x_j\}\right)$$

Notice here that the events $\left\{\frac{A_i}{A_j}\notin\left[\frac{1}{3},3\right]\right\}$ and $\{x_i\sim x_j\}$ are independent. If we assume $q=0$, this probability becomes

$$\mathbb{P}(DC_{ij}=1)=\begin{cases}(1-\mathbb{P}_{kk})p & \forall(x_i,x_j)\in V(H_k)^2\\ (1-\mathbb{P}_{kl})\frac{p}{2} & \forall(x_i,x_j)\in V(H_k)\times V(H_l)\}\end{cases}$$

Thus, we are able to compute the expected number of desirability channels for the whole Market by considering separately the incentives to trade intragroup and intergroup as $\mathbb{E}(DC)\sum_{k=1}^{K}\mathbb{E}(DC_k)+\frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\mathbb{E}(DC_{kl})$.

Proposition : *An exchange Market with n agents gathered into K unequally sized groups exhibits the following expected number of desirability channels*

$$\mathbb{E}(DC)|_K=\frac{p(1-q)}{2}\left(\frac{1}{2}\left[\left(1-q+\frac{4q}{p}\right)n^2-2n+\left(1+q-\frac{4q}{p}\right)\sum_{k=1}^{K}n_k^2\right]-Q_1-\frac{(1-q)}{2}Q_2\right)$$

where Q_1 and Q_2 are functions of K and $(n_i)_{i\in[\![1,K]\!]}$.

As detailed in Appendix, this quantity depends on the parameters p, q, K and $(n_i)_{i\in[\![1,K]\!]}$. Obviously, $\mathbb{E}(DC)|_K$ is an increasing function of p and n. Indeed, if the probability p increases, the size of every neighborhood gets larger and each agent is more likely to find someone to trade with. Similarly, if the size of the population increases, the expected number of desirability channels increases too. As the probabilities $\mathbb{P}_{kl}$ and $\mathbb{P}_{kk}$ do not depend on $\bar{A}$, it is the same for $\mathbb{E}(DC)$. Notice that this proposition is only verified for unequally sized groups. Moreover, there is of course a maximal number of unequally sized groups for a given population. Let $K^{\max}$ be this optimum, by considering n agents in the Market, to ensure there is no two groups with the same size the condition[33] $K^{\max}=\lfloor\frac{\sqrt{1+8n}-1}{2}\rfloor$ has to hold. Regarding to the interpretation of hypothesis (H3), p could also contribute to transaction costs. More precisely, if we consider search costs, when it is more costly to create social connections - due to geographical or cultural concerns for instance - the probability p diminishes and the same applies for the expected number of desirability channels.

Let us now examine how the shock probability alters the expected number of desirability channels.

[33] Indeed, notice that $K^{\max}$ verifies $n=1+2+...+K^{\max}$, that is we cannot increase the size of any group without changing the number of groups of without obtaining two groups with the same size. We can rewrite this equality as $n=\frac{k(k+1)}{2}\Leftrightarrow k^2+k-2n=0$ and by solving this polynomial function, we obtain $K^{\max}=\lfloor\frac{\sqrt{1+8n}-1}{2}\rfloor$.

11.1 Desirability channels and shock probability

First, we consider the simple case of K unequally sized groups where only one of them, let say with size n_k, is impacted by a shock. The global size of the graph being n, we denote $(n_i)_{\substack{i \in \\ i \neq k}[\![1,K-1]\!]}$ the size of the other groups and we have

$$\mathbb{E}(DC/\{\bar{S}^k\}) - \mathbb{E}(DC/\{S^k\}) = \binom{n_k}{2}p(1-\mathbb{P}_{kk}) + n_k \sum_{\substack{j=1 \\ j \neq k}}^{K} n_j \left(\tfrac{p}{2}(1-\mathbb{P}_{kj}) - 1\right)$$

Thus, depending on how large the quantity $n_k \sum_{j=1}^{K} n_j \left(\frac{p}{2}(1-\mathbb{P}_{kj}) - 1\right)$ is, the shock increases the number of desirability channels or not. It is clear that if $n_k = 1$, the expected number of desirability channels increases since $\binom{n_k}{2}p(1-\mathbb{P}_{kk})$ becomes zero and the initial number of intergroups channels was necessarily lower than $n-1$. As we raise the size of the impacted group, we lose a growing number of intragroup channels to the point where $\mathbb{E}(DC/\{S^k\}) < \mathbb{E}(DC/\{\bar{S}^k\})$. For instance, if we consider the case where a giant component of size $n-1$ is impacted, we swap $\binom{n-1}{2}p(1-\mathbb{P}_{11})+(n-1)\frac{p}{2}(1-\mathbb{P}_{12})$ initial channels against $n-1$ new ones and the resulting value of $\mathbb{E}(DC/\{S^k\})$ could be lower. Therefore, smaller the size of the impacted group, more likely the shock will increase the number of desirability channels.

As the relationship between $\mathbb{E}(DC)$ and the probability q is highly non linear, we have simulated the number of expected desirability channels for different values of $q \in [0,1]$ with $K = 6$. We obtain the graph displayed on figure (5) where the variations of $E(DC)$ are described by an inverted parabola. Thus, it appears there exists an optimal value $E(DC)^*$ for $q \approx 0.5$. This result makes sense if we think about the "gain" in terms of incentives, generated by a shock. Let n_1 be the aggregated size of the impacted groups, we know there are $n_1(n-n_1)$ desirability channels newly created from these shocks[34]. It is straightforward to show that this quantity is maximal for $n_1 = \frac{n}{2}$, that is when the half of the Market is impacted. As $q = 0.5$ implies, in expectation, that the half of the groups is impacted by a shock, it is not surprising that the corresponding number of expected desirability channels is close to the optimum[35].

[INSERT FIGURE (5)]

Of course, many groups can be jointly impacted and if all of them are experiencing a shock, there is no trade anymore and every vertex becomes isolated. We will readdress the impact on the Market of this kind of event in section (14.3) through the concept of absorption ability.

In next section, we explain how other parameters, as the number of groups K, can also alter the expected number of desirability channels.

[34] As two impacted groups have no incentives to trade anymore.

[35] Of course this result also depends on the groups sizes distribution.

11.2 Desirability channels and groups number

Partition with equally sized components : Let us consider the very simple case where there would be only one group $K = 1$ and $q = 0$. The resulting expected number of edges is $\mathbb{E}(E(G))|_{K=1} = \binom{n}{2}p$ and so the expected number of desirability channels can be written $\binom{n}{2}p(1-\mathbb{P})$. Now, if we halve this group, that is $K = 2$ such that the graph would be composed by two equally sized components with $n_1 = n_2 = \frac{n}{2}$, it is straightforward to show that $\mathbb{E}(DC)$ becomes
$\mathbb{E}(E(G))|_{K=2}(1 - \mathbb{P}') = \left[2\binom{n/2}{2}p + \left(\frac{n}{2}\right)^2 \frac{p}{2}\right](1 - \mathbb{P}')$ with $\mathbb{P}_{11} = \mathbb{P}_{22} = \mathbb{P}_{12} = \mathbb{P}'$ the new probability of no trade. Thus, we observe that the expected number of desirability channels has decreased since $\mathbb{P}' > \mathbb{P}$ and $\mathbb{E}(E(G))|_{K=2} < \mathbb{E}(E(G))|_{K=1}$.

More generally, if we divide over and over again each group into two equally sized groups, we show in Appendix that for any pair of divisors (d, d') of n such that $d < d'$, we have $\mathbb{E}(E(G))|_{K=d} > \mathbb{E}(E(G))|_{K=d'}$. As $\mathbb{P}_{dd}$ is an increasing function of d (when d increases, the size of each group decreases), we conclude[36] that $\mathbb{E}(DC)$ decreases towards the limit value, $\mathbb{E}(DC)|_{K=n} = \binom{n}{2}\frac{p}{2}(1 - \bar{\mathbb{P}})$ with $\bar{\mathbb{P}}$ the probability of no trade in an atomistic Market, that is when each agent is a group by himself. We report on the following figure $\mathbb{E}(DC)$ for all the divisors of $N = 60$.

[INSERT FIGURE EQUALLY SIZED GROUPS E(DC)=f(K)]

General case for a random partition : From the results obtained in next section, we are able to state that the expected number of desirability channels decreases when we pass from $K = 1$ to $K = 2$ whatever the respective sizes of the two newly formed groups. For higher groups numbers, we observe on the simulations that the quantity $\mathbb{E}(DC)$ still decreases. More specifically, we observe a sharp decrease for the first values of K that mimics an hyperbola, as we can see on the following figure

[INSERT FIGURE E(DC) W.R.T. K]

Thus, it appears that on average when we increase the number of groups, we decrease the expected number of desirability channels. That makes sense when you think in terms of differences between the groups. Indeed, if we keep constant the global population, when K increases, the groups size will progressively tend to the limit $\frac{1}{n}$. Thus, as the sizes differences are removed, every individual start to share the same environment and the preferences get closer. Finally, the probability to have an incentive to trade decreases in the same way that $\mathbb{E}(DC)$. Now, let us examine how the differences between the group sizes affect the expected number of desirability channels.

[36] Notice that in the case of $K = l$ equally sized groups with $1 < l < n$ a divisor of n and $q = 0$, we would have $\mathbb{E}(DC)|_{K=l} = l\binom{n_l}{2}(1 - \mathbb{P})p + \binom{l}{2}n_l^2(1 - \mathbb{P})\frac{p}{2} = \left(l\binom{n_l}{2} + \frac{1}{2}\binom{l}{2}n_l^2\right)(1 - \mathbb{P})p$ with $\mathbb{P}$ the probability of no trade based on the unique group size $\frac{n}{l}$.

11.3 Desirability channels and groups sizes distribution

We want to highlight here the relationship between the variance in the group sizes distribution and $\mathbb{E}(DC)$. We start again with a very simple case of two equally sized groups ($K = 2$) of size $\frac{n}{2}$ with $q = 0$ and we progressively move agents from one group to another to obtain new sizes setting $(\frac{n}{2} - i, \frac{n}{2} + i)$ with $i = 1, ..., \frac{n}{2}$. In order to understand the evolution of $\mathbb{E}(DC)$, we write down analytically the equations where i agents migrate from one group to another

$$\left[2\binom{n/2}{2}p + \left(\frac{n}{2}\right)^2 \frac{p}{2}\right](1-\mathbb{P}) \quad \to \quad \binom{n/2-i}{2}(1-\mathbb{P}_{11})p + \binom{n/2+i}{2}(1-\mathbb{P}_{22})p + \left(\frac{n}{2}-i\right)\left(\frac{n}{2}+i\right)(1-\mathbb{P}_{12})\frac{p}{2}$$

Of course, we have $\mathbb{P}_{11}\left(\frac{n}{2} - i\right) > \mathbb{P}\left(\frac{n}{2}\right) > \mathbb{P}_{22}\left(\frac{n}{2} + i\right)$ as we know it is a decreasing function of the group size. However, we show in Appendix that under the condition $\frac{2i}{3n} > \frac{8}{5}\left(1 - \frac{1}{\frac{4}{3}+\frac{2}{3n}\left(i-\frac{1}{n}\right)}\right)$, we have $\mathbb{P}_{12}\left(\frac{n}{2} - i, \frac{n}{2} + i\right) < \mathbb{P}\left(\frac{n}{2}\right)$. Hence, it is no easy process to conclude on the evolution of $\mathbb{E}(DC)$ in this case as some quantities potentially offset others. Therefore, we don't prove it analytically but we run simulations to conclude on this relationship.

Thus, it seems that on average, the expected number of desirability channels increases when the group size volatility increases. We report the results of the following figure.

[INSERT FIGURE $\neq$ SIZED GROUPS E(DC)=f(volat)]

This point is in accordance with the widely accepted positive relationship between volume and Market heterogeneity. Indeed, as we claimed in section (10.4), the size of each group is a state variable that describes the environment of each agent. We already argued how the size of a group affects the preferences of its members, so it is clear that if we reinforce the differences between the Market environments, we potentially generate more heterogeneity among the agents.

In light of these results, we are now able to rely our analysis to the volume but, before we get into this point, let us introduce the required material to describe random graphs.

12 Some graph theory results

In this section we present some basics of graph theory to address the volume in the case $q = 0$. The most of the following definitions and theorems come from Bollobas (1985), Diestel (2005) and Frieze and Karonski(2015).

Definition : *For any vertex $x \in V(G)$, the set of all the vertices incident to x is called its neighborhood and we denote it by $N(x) = \{y \in G : xy \in E(G)\}$. Notice here that x is not included in $N(x)$ since the loops are not allowed in the graph.*

Definition : *Let $G \in \mathcal{G}(n,(p_{ij}))$ be a graph with (V,E) its vertex and edge sets respectively. A graph $G' \in \mathcal{G}(n,(p_{ij}))$ is a subgraph of G if and only if its vertex and edge sets are subset of V and E respectively, that is if $V' \subset V$ and $E' \subset E$.*

Definition : *Let $G' = (V', E')$ such that $G' \subseteq G$ and G' contains all the edges $e = xy \in E$ with $x, y \in V'$, then G' is called an induced subgraph of G. Thus, for any vertex x, the set of all the vertices adjacent to x, that is linked to x by an edge, is a subgraph of G which is called the neighborhood of x.*

Definition : *A path is a non-empty graph (V,E) of the form :*

$$V = \{x_0, ..., x_k\} \qquad\qquad E = \{x_0x_1, x_1x_2, ..., x_{k-1}x_k\}$$

where $\forall i,j \in [\![0,k]\!]$, $x_i \neq x_j$. The length of the path is the number of its edges.

Definition : *A property $\mathcal{Q}$, subset of $\mathcal{G}(n,(p_{ij}))$, is said monotone increasing if whenever $G_1 \in \mathcal{Q}$ (the graph G_1 has the property) and $G_1 \subset G_2$ then also $G_2 \in \mathcal{Q}$.*

Definition : *Given a monotone increasing property $\mathcal{Q}$, a function $t(n)$ is said to be a threshold function for $\mathcal{Q}$ if*

$$\frac{p(n)}{t(n)} \to 0 \text{ implies that almost no graph has } \mathcal{Q}$$
$$\frac{p(n)}{t(n)} \to \infty \text{ implies that almost every graph has } \mathcal{Q}$$

Definition : Let $m(G) = \max\{d(F) \text{ such that } F \subset G\}$ be the maximum average degree of a graph G with $d(F) = \frac{2|E(F)|}{|V(F)|}$ the average degree of the graph. The graph G is said balanced if $m(G) = d(G)$ and strictly balanced if $F \subset G$ and $m(G) = d(G) \Rightarrow F = G$.

12.1 Special case K=1

Now, we introduce some results that only hold for the standard Erdös-Renyi model, that is in the special case of our model where $K = 1$.

Theorem : (Erdös and Rényi 1960) : *If H is a balanced graph with k vertices and $l \geq 1$ edges, then $t(n) = n^{-k/l}$ is a threshold function for $\mathcal{Q}_H$.*

Corollary : *Considering the standard Erdös-Renyi model $\mathcal{G}(n,p)$, we can derive the threshold function for several properties*

- *For the emergence of the first link, $t(n) = \frac{1}{n^2}$*
- *To observe an isolated component of three vertices and only two edges, $t(n) = \frac{1}{n^{3/2}}$*
- *To observe a k-tree with $k \geq 2$, $t(n) = \frac{1}{n^{k/(k-1)}}$*

- *To observe a cycle,* $t(n) = \frac{1}{n}$
- *To obtain a k-complete graph* K^k *with* $k \geq 2$, $t(n) = n^{-2/(k-1)}$

In regards to this corollary, we can say that as long as the probability p belongs to $\left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$, whp (with high probability) a graph should be only composed by pairs and isolated vertices. Similarly for values in a range from $\frac{1}{n^{3/2}}$ to $\frac{1}{n^{4/3}}$, whp a graph should be only composed by isolated components of three vertices and only two edges, by pairs and isolated vertices. Thus, as we detail in Appendix, that leads to the following corollary.

Corollary : *In the case of our setting with* $K = 1$ *and* $q = 0$*, the expected number of desirability channels can be asymptotically written, that is when* $n \to \infty$*, as*

$$\mathbb{E}(DC) = \begin{cases} \mathbb{E}(\#(\text{pairs}))(1 - \mathbb{P}_{NN}) & \text{for } p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right] \\ \left[\mathbb{E}(\#(\text{pairs})) + 2\mathbb{E}(\#(\text{triples}))\right](1 - \mathbb{P}_{NN}) & \text{for } p \in \left[\frac{1}{n^{3/2}}, \frac{1}{n^{4/3}}\right] \end{cases}$$

Notice here that for $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$, the expected number of desirability channels is exactly the expected volume in the graph[37]. Thus, we first determine $\mathbb{E}(DC)$ for a pair and a triple and then, given the expected number of these components, we can deduce $\mathbb{E}(DC)$ for the whole graph.

We can easily state that

$$\begin{aligned} \mathbb{E}(\{\#\text{ of isolated pairs}\}) &= \binom{n}{2} p(1-p)^{2n-4} \\ \mathbb{E}(\{\#\text{ of isolated triples}\}) &= \binom{n}{3} p^2(1-p)^{3n-8} \\ \mathbb{E}(\{\#\text{ of isolated triangles}\}) &= \binom{n}{3} p^3(1-p)^{3n-9} \\ \mathbb{E}(\{\#\text{ of isolated } K^k\}) &= \binom{n}{k} p^{\binom{k}{2}}(1-p)^{k(n-k)} \end{aligned}$$

We now introduce a fundamental theorem that provides the threshold function for a graph to exhibit the connectivity property.

Theorem (Erdös and Rényi 1961) : *The function* $t(n) = \frac{\log n}{n}$ *is a threshold function for the connectivity of the standard Erdös-Renyi model* $\mathcal{G}(n, p)$*. More formally, let consider* $p(n) = \nu \frac{log(n)}{n}$*, we have*

$$\text{if } \nu < 1,\ \mathbb{P}(\mathcal{G} \text{ is connected}) \to 0$$
$$\text{if } \nu > 1,\ \mathbb{P}(\mathcal{G} \text{ is connected}) \to 1$$

[37]We will go into further detail on this point in next section.

Notice about graph connectivity that, in our model for $q \neq 0$, whatever the size of the impacted component when a shock occurs, the whole graph of the social connections gets fully connected.

Definition : *A random graph process on V with n nodes is a Markov chain $\tilde{G} = (G_t)_0^\infty$, whose states are graphs on V. We have $G_0 = (V, \emptyset)$ the empty graph and $\forall k \geq 1$, the graph G_{k+1} is obtained from G_k by adding a random edge which is chosen uniformly among the missing edges, we have the inclusion sequence $\forall k$, $G_k \subset G_{k+1}$.*

This last definition will be fully meaningful in the dynamic version of our model but it is interesting to observe here that the model $\mathcal{G}(n, M)$ we introduced in section (10.3) can be view as a state of a random graph process.

12.2 General case $K \geq 1$

Now, we present some useful results in the more general case where $K \geq 1$ and $q = 0$. Let consider the model $\mathcal{G}(n, (p_{ij}))$, we denote $\forall i \neq j$, $\rho_{ij} = 1 - p_{ij}$ and

$$Q_i = \prod_{j=1}^{n} \rho_{ij},\ \lambda_n = \sum_{i=1}^{n} Q_i \text{ and } R_{ik} = \min_{1 \leq j_1 \leq ... \leq j_k \leq n} \rho_{ij_1} ... \rho_{ij_k}$$

Let us suppose that the probabilities p_{ij} are chosen such that the three following conditions hold simultaneously as $n \to \infty$:

$$\max_{1 \leq i \leq n} Q_i \to 0 \tag{19}$$

$$\lim_{n \to \infty} \lambda_n = \lambda = \text{constant} \tag{20}$$

$$\lim_{n \to \infty} \sum_{l=1}^{n/2} \frac{1}{l!} \left(\sum_{i=1}^{n} \frac{Q_i}{R_{il}} \right)^l = e^\lambda - 1 \tag{21}$$

The following theorem allows us to rely the Market participation to the price formation mechanism.

Theorem (Kovalenko 1971) : *Let X_0 denote the number of isolated vertices in the random graph $G \in \mathcal{G}(n, (p_{ij}))$. If conditions (19), (20) and (21) hold, then the number of isolated vertices is asymptotically distributed, that is $\forall k = 0, 1, ...$* we have

$$\mathop{lim}_{n \to \infty} \mathbb{P}(X_0 = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

Theorem (Kovalenko 1971) : *If the conditions (19), (20) and (21) hold, then*

$$\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{G}(n,(p_{ij})) \text{ is connected}) = e^{-\lambda}$$

Now that we introduced the concept of desirability channels and the different tools to control for the graph topology, let us address the volume determination.

13 The Volume

As we mentioned above, a desirability channel only describes a *pre-trade* situation but does not ensure that an exchange really occurs. It is a necessary but not sufficient condition, except for the very special case of a pair component. In order to determine the volume, we have to consider both the desirability and the optimality of a trade in terms of choice among potential partners. Indeed, we assume here that given his budget constraints, an agent can interact with one buyer and one seller at most, so he has to make a choice[38]. Therefore, if a desirability channel does exists between two agents, a trade will really happen if and only if they are the best choice of each other[39] given their feasible actions set. We emphasize here the fact that the choice entirely depends on the available partners set since every agent will not always be able to trade at his first best alternative. For instance, if the best partner of an individual X is already committed in an exchange with his own best partner who is not actually X, X will try with his second best partner, then with his third best, etc... He will span his whole neighborhood until he finds someone to trade with. Let V_{ij} be the binary variable that takes one if an exchange does exist between x_i and x_j and zero otherwise, we can now introduce the following proposition.

Proposition : *Let $N(x_i)$ and $N(x_j)$ be the neighborhood of agents x_i and x_j respectively, a necessary and sufficient condition for a trade to occur between them is the following intersection*

$$\{V_{ij} = 1\} \quad = \quad \{DC_{ij} = 1\} \cap \{x_i = x_j^*\} \cap \{x_j = x_i^*\}$$

where for a first-best-matching, $x_i^* = \underset{x_j \in N(x_i)}{\arg\max} |A_i - A_j|$ and $x_j^* = \underset{x_i \in N(x_j)}{\arg\max} |A_i - A_j|$.

Thus, the information every vertex has in the graph, is the preferences of his neighbors. As we said, a trade is not always generated by the matching of two first best choices, it could also be due to a first and a second best choice, to a third and a fifth best choice, etc... Everything depends on the neighborhood of each agent, on the neighborhood of his neighbors, etc... Thus, the expected volume can be decomposed according to the optimal order at which each agent commits in the

[38]This hypothesis will be fully meaningful when we will present a dynamic version of our model in section (16.1.2).

[39]More formally we could consider an automorphism ϕ on the vertex set $V(G)$ which returns the best partner of any agent x_i, that is $\phi(x_i) = \underset{x_j \in N(x_i)}{\arg\max} |A_i - A_j| = x_i^*$. Thus, provided that a desirability channel does exist, we are looking for the second order fixed points x such that $\phi(\phi(x)) = x$ as the equivalent of $\{x_i = x_j^*\} \bigcap \{x_j = x_i^*\}$.

trade. Thus, we can write $\mathbb{E}(V_{ij}) = \sum_{m,n}^{|N(x_i)|,|N(x_j)|} \mathbb{E}(V_{ij} = 1|(m,n)^{\text{th}})\mathbb{P}((m,n)^{\text{th}})$ with $\mathbb{P}((m,n)^{\text{th}})$ the probability that the exchange takes place as the m^{th} best choice of agent x_i and the n^{th} best choice of agent x_j. Therefore, $\mathbb{E}(V_{ij})$ has a recursive form since $\mathbb{P}((m,n)^{\text{th}})$ can be expressed as a product of expected volumes and depends on the neighbors of some neighbors of x_i and x_j and so forth.

Since all these conditional expectations have the same form and for a sake of simplicity, let us focus here on the simple case where only the first best partners are matching. We provide in Appendix the expression of the expected volume in the more general case where an agent would play his k^{th} best choice and his partner, his l^{th} best choice.

As the event $\{x_i = x_j^*\}$ and $\{x_j = x_i^*\}$ depend on the neighborhoods of x_j and x_i respectively, they also depend on p that decides the connectivity of the graph. Thus with high probability, we could be able to control the architecture of the graph by constraining the probability p. This result is essential as it will allow us for some range of p, to reduce the expression of the expected volume to a first-best-matching problem. More precisely, we can state the following proposition.

Proposition : *Let G be the underlying graph of a trades network, the expected volume determination is equivalent to a first-best-matching problem if G is only composed by stars and/or 3-cycles*[40].

Indeed, a *star* is a complete bipartite graph with a singleton partition which is called the star's center, surrounded by its *leaves*. The degree of each leaf is equal to one, so they can only trade with their best partner and the central vertex will be sure to trade at his optimal choice too. Notice here that isolated pairs and isolated triples are stars with one and two leaves respectively. Finally, in the case of a 3-cycle, whatever if people are allowed to trade once or twice at most, the trades can only take place at a first-best order.

Finally, in order to determine the expected volume for the whole graph, we just aggregate every pair of vertices by separating the *intergroup* and the *intragroup* pairings. As the probability of a desirability channel is not the same according to whether the agents belong to the same group or not, the same applies for the probability that a trade occurs. So, in the case of K groups we have

$$\mathbb{E}(V)|_K = \sum_{k=1}^{K} \mathbb{E}(V_k/\bar{S}^k)(1-q) + \frac{1}{2}\sum_{\substack{k,l \\ k\neq l}} \left\{2\mathbb{E}(V_{kl}/S^k,\bar{S}^l)q(1-q) + \mathbb{E}(V_{kl}/\bar{S}^k,\bar{S}^l)(1-q)^2\right\}$$

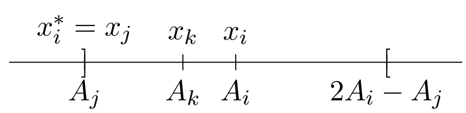
[40]Notice that a *cycle* is just a path whose the first node and the last one are the same. Moreover, in the special case of a 3-cycle, we observe that is also a *triangle* or a *complete* graph of order 3, denoted K^3. In the case of random graphs, there is a very small probability to obtain a topology displaying only stars of order higher than 3 or 3-cycles with only isolated vertices, pairs or triples. This is due to the fact that the threshold function required to observe a cycle is too high to not generate before whp more complex structures than triples or pairs.

Notice that we can either assume that an agent only trade with his best partner (among all his neighbors) or that he can both trade with his best buyer and his best seller to realize an arbitrage opportunity. With this second assumption, we would obtain the following Market rules : the *pure buyers* and the *pure sellers* can have one partner at most and only the *traders* can have two partners at most. As a budget constraint argument would not be relevant here to limit the number of arbitrage opportunities a trader can use, we rather claimed in section (10.2) that takes time to make a trade.

13.1 One trade at most

First of all, let us consider the case where only one trade is permitted. An agent x_i looks at his best choice x_i^* among his entire neighborhood such that $x_i^* = \underset{x_j \in N^*(x_i)}{\arg\max} |A_i - A_j|$, where $N^*(x_i) = \left\{ y \in N(x_i) : \frac{A_{x_i}}{A_y} \notin \left[\frac{1}{3}, 3\right] \right\}$ denotes the set of his potential partners. Notice that the dimension of x_i^* is usually lower than 2 but could be much more higher if a shock occurs. Indeed, let $V(H_k)$ be an impacted group of size n_k to which x_i does not belong, that implies $|x_i^*| \geq n_k$. Thus, in the case where two or more of his neighbors would be identically optimal, we assume that x_i is randomly paired with one of them by a regulatory body with equal probability $\frac{1}{|x_i^*|}$. Obviously, once all the possible trades has been made in the Market, if we remove every edge that does not carry one of them, the resulting graph should be only composed by isolated pairs and isolated vertices.

Formally, if x_i is buyer and x_j is seller, $x_j = x_i^*$ implies that the risk aversion parameter of any other neighbors $x_k \in N^*(x_i)$ must lie into the interval $[A_j, 2A_i - A_j]$ as shown on the following figure.



As we claimed in section (10.4), agents have incentives to act as traders if and only if their risk aversions belong to the set $\left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]$. Considering that when an individual has the choice among different partners he will always prefer the one with the furthest preferences, we can state the following corollary.

Corollary : *In the case where only one trade is permitted, a trader will always prefer to buy the share.*

Indeed, let us consider again the case with three agents x, y and z such that $A_y < A_x < A_z$ and $A_x \in \left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]$. The condition $\frac{A_z}{A_x} > 3$ implies $A_z - A_x > 2A_x$, so for $A_x = \frac{\bar{A}}{2}$, we have $A_z - A_x > \bar{A} > \frac{\bar{A}}{3} > A_x - A_y$ by construction. While A_x is walking up the interval to $\frac{11\bar{A}}{18}$, the quantity $A_z - A_x$ grows linearly with respect to A_x and the quantity $A_x - A_y$ grows linearly with respect to A_y. As A_y is always lower than A_x, the preferences of z will be always farther from x than the preferences of y.

As we mentioned above, the best partner of any agent is a random variable that entirely depends on the state of the graph process. In order to obtain a closed-form formula of the expected volume, we need to restrict the number of groups to $K = 1$ and to consider the optimized graph $G^* = (V^*, E^*)$ where $V^* = V(G)$ and $E^* = E(G) \backslash \left\{ xy \in E(G) : \frac{A_x}{A_y} \in [\frac{1}{3}, 3] \right\}$. The probability that two vertices are linked by an edge in this graph is exactly $\mathbb{P}(\{DC_{ij} = 1\})$. Given that the events $\{x_i = x_j^*\}$, $\{x_j = x_i^*\}$ and $\{x_i \sim x_j\}$ are totally independent, we can derive the following result.

Proposition : *The expected volume for a first-best-matching between any pair of agents* (x_i, x_j) *from* G^* *when they can only trade once with* $q = 0$ *and* $K = 1$*, can be written as*

$$\mathbb{E}(V_{ij}|(1,1)^{st}) = \left(\left(\frac{1}{2}\right)^{|N^*(x_i)|-2} + \left(\frac{1}{2}\right)^{|N^*(x_j)|-2} - \left(\frac{1}{2}\right)^{N^*_{ij}-3} \right) \left(\frac{2}{3} \left(1 - \frac{\bar{A}}{N_I}\right)^2 \right)^{N^*_{ij}-1} p$$

with $N^*_{ij} = |N^*(x_i)| + |N^*(x_j)|$.

Notice here that $N^*_{ij} \geq 2$ since $\mathbb{E}(V_{ij}) = \mathbb{E}(V_{ij}|\{DC_{ij} = 1\})\mathbb{P}(\{DC_{ij} = 1\})$, that is a conditional probability on the event "x_i and x_j are linked in the optimized graph". If we want to state the equivalent of this proposition in the original graph G, we have to consider every possible setting of the x_i and x_j neighborhoods[41]. For the whole graph, we deduce that $\mathbb{E}(V|(1,1)^{st}) = \binom{n}{2}\mathbb{E}(V_{ij}|(1,1)^{st})$. Clearly, the conditional expected volume is maximal when every vertex has exactly one partner to trade with, that is $\forall i \in [\![1, n]\!]$, $|N^*(x(i))| = 1$. In the particular case where $\epsilon \to 0$, we have $N^*(x_i) = N(x_i)$ and the kind of graph that maximizes this quantity is a 1-regular graph. For this topology, the expected volume is equal to $\mathbb{E}(DC_{ij})$, which is in accordance with the fact that only desirability channels matter in the volume computation for isolated pairs. Notice also that a 1-regular graph can only be obtained in the case there is an even number of agents in the Market. Under the assumption that players can only trade once, this topology provides us with an upper bound for the volume since the number of exchanges cannot exceed $\frac{n}{2}$.

In this proposition, the expected volume is a random quantity which depends on the neighborhood size of each agent enrolled in the exchange. As we argued earlier, we know that every neighborhood is shaped by the value of p. More precisely, when the probability p increases, the graph becomes more connected and the neighborhood size of each vertex increases. Since $N(x_i)$ is

[41] Indeed, we would have

$$\mathbb{E}(V_{ij})|_G = \sum_{x,y}^{|N(x_i)|,|N(x_j)|} \binom{|N(x_i)|}{x}\binom{|N(x_j)|}{y}\mathbb{E}\left[V_{ij}|(|N^*(x_i)|, |N^*(x_j)|) = (x,y)\right]\mathbb{P}\left[(|N^*(x_i)|, |N^*(x_j)|) = (x,y)\right]$$

related to $N^*(x_i)$, the probability to observe a first-best-matching pairs must also decreases. Of course, that does not mean that the expected volume is a decreasing function of p since when people have more neighbors, they have more potential partners and consequently more chance to find someone to trade with.

So far, we only discussed the volume in the case where the Market is composed by one group of n agents. However, as we will see, a big part of the volume can be attributed to the presence of groups with different sizes. When we allow K to be higher than one, it is not an easy task to keep the model in a tractable way, especially because we don't know *a priori* the composition of every neighborhood in the graph. Thus, by running simulations, we try to overcome these issues and we observe that the volume in a more general setting, is a non-monotone function of p as we can see on the following figure.

[INSERT GRAPH E(V)=f(p)]

In regards to this global pattern, it seems that the volume is positively related to p, that is when the agents have more connections in the Market, we expect to observe a higher number of trades. In order to better understand the non linear mechanism behind this result, especially when $K \neq 1$, let us now consider the very simple case where we start with three players whose one is isolated, then we connect him to the two other ones. Thus, we will be able to analytically track the volume evolution when we add a new social link for a very simple topology and we will see how the output can be related to the group belonging of the different agents. Here, we consider a two state deterministic graph process on three vertices which can be written as follow : $(G_t)_1^2$ with G_1 a graph with edge $E(G_1) = \{xy\}$ and G_2 a graph with edges $E(G_2) = \{xy, xz\}$.

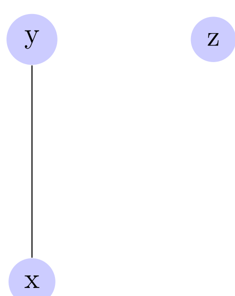


Figure 1: G_1

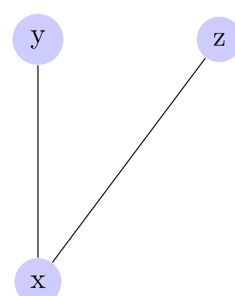


Figure 2: G_2

We show in Appendix that this graph process leads to the following proposition.

Proposition : *Consider any isolated graph component with three different vertices* $\{x, y, z\}$

where x and y are already connected by an edge. We connect the third vertex z to x and we assume that the agents can trade once at most.

- *If x and y belong to the same group N_k, the expected volume increases (respectively decreases) if and only if z belongs to a larger group N_l (respectively a smaller group such that $N_l < N_k < N_l + \frac{16\bar{A}}{5}\left(1 - \frac{\bar{A}}{N_k}\right)$) than y.*
- *If x and y belong to different groups such that $N_x < N_y$, the expected volume increases (respectively decreases) if and only if z belongs to a larger (respectively smaller) group than y.*
- *If x, y and z belong to the same group, the expected volume doesn't change.*

Thus, depending on the groups to which the different agents belong, adding a new edge will not systematically lead to increase the expected volume.

More importantly is the relationship between volume and desirability channels. We already highlighted the fact that when the number of desirability channels increases, people are more likely to find someone to trade with and the expected volume (not conditioned on a first-best matching) will increase too. Thus, we can rely to the results of section (11) to describe the evolution of the unconditional expected volume. We found that $\mathbb{E}(DC_{ij})$ is negatively correlated to the number of groups K and positively correlated to the groups size distribution. By running simulations, we found similar results for the volume.

[INSERT FIGURE V=f(K) and V=f(var)]

Notice that we can also compute the maximum volume allowed by a specific topology of G^* with the following theorem.

Theorem (Tutte-Berge formula) : *For any graph $G = (V, E)$, the maximum matching size is given by*

$$\nu(G) = \frac{1}{2} \min_{U \subseteq V} \left(|U| - o(G - U) + |V| \right)$$

where $o(H)$ is the number of connected components with odd number of vertices.

This theorem is a generalization of the classic König's theorem for bipartite graphs. This provides us with an upper bound on the volume for any graph whose all the edges that do not carry a trade have been removed. Thus, for any optimized graph[42] G^*, the maximum number of trades that can occur on G^* is given by $\nu(G^*)$ if the agents are allowed to trade once at most.

[42] Notice that any optimized graph cannot includes cycles since inequalities are not symmetric. Indeed, a cycle is a path such that the first vertex and the last one are the same, so by definition we would have for three vertices x, y, z, $A_x < A_y < A_z < A_x$ which is not possible.

Let us now take another tack. If we consider a social planner who observes the agents preferences and aims to only maximize the volume in the Market with respect to a constraint on the number of social connections, what should be the networks that allow to reach this optimum? Let $\mathcal{G}^*(m,n) = \{G^* : G \in \mathcal{G}(m,n)\}$ be the set of the optimized graphs with m edges and n vertices, we introduce the following definition.

Definition : *Given the preferences of all the agents, a m-suboptimal (respectively m-optimal) setting is a graph H of order*[43] *m on which the volume is strictly lower than (respectively equal to) the maximum volume allowed on $\mathcal{G}^*(m,n)$.*

If we denote V_H the volume associated to the underlying graph H, H is m-suboptimal means that $V_H < \max_{U \in \mathcal{G}^*(m,n)} \nu(U)$ since the maximum cardinality of a matching in a graph U is exactly the maximum volume allowed on this topology. For instance, if we consider four agents with preferences $(A_i)_{i \in [\![1,4]\!]}$ such that $A_1 < \frac{A_2}{3} < \frac{A_3}{9} < \frac{A_4}{27}$ enrolled within the following network.

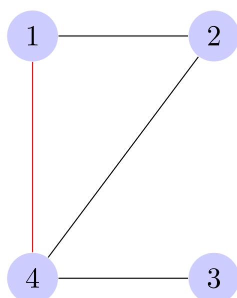


Figure 3: H with $V_H = 1$

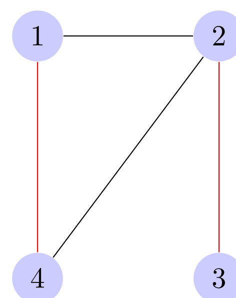


Figure 4: H' with $V_{H'} = 2$

We observe on H that x_2 is strictly dominated by x_4 to trade with x_1 or by x_1 to trade with x_4. Similarly x_3 is strictly dominated by x_1 to trade with x_4. As x_2 and x_3 are not connected by an edge, the only possible trade is here (x_1, x_4) and H is 4-suboptimal. Instead, in H' the topology allows them to trade and the volume is maximal, so H' is 4-optimal. Notice that an optimal graph does not necessarily maximizes the total surplus of the economy.

Finally, from the results of section (10.4) we can state the following proposition.

Proposition : *In our model, any optimized graph is either bipartite or tripartite.*

Indeed, if there is no *trader* in the economy, that is if we assume $\nexists x \in G$ such that the both conditions $A_x \in \left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]$ and $\exists y, z \in N(x) : \frac{A_x}{A_y} > 3, \frac{A_x}{A_z} < \frac{1}{3}$ hold, then we immediately conclude that G has a bipartite structure since its vertices can be divided into two disjoint independent sets : the *pure buyers* and the *pure sellers*. Now, if *traders* do exist in the Market, we know that

[43] This means with m edges.

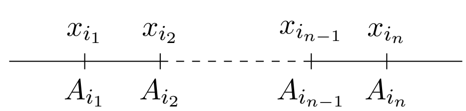
their risk aversions belong to the interval $\left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]$. However, a desirability channel cannot exist between them since the length of the interval is not sufficient to ensure that the trade condition holds. Indeed, we have $\frac{11\bar{A}}{18} \times \frac{2}{\bar{A}} \in \left[\frac{1}{3}, 3\right]$, so no trade can take place between two *traders*. Thus, no edge has its two ends in the same set of vertices and the graph is tripartite.

As we mentioned above, in most cases the expected volume for a first-best-matching is not the expected volume in the Market since we don't take into account the other best-matching orders at which people could also trade. However, in some special cases, we are able to fix this issue.

13.1.1 Special cases

In this section, we show that for some range of p, we can analytically compute the expected volume.

Let us consider for instance the special case of a frictionless Market. More precisely, we assume $K = 1$, $q = 0$, $p = 1$ and $\epsilon \to 0$, that is an organized Market without shock where everyone knows each other and where we can exchange a fraction of share as small as we want. As we have seen in section (10.2), we would have $\mathbb{P} = 0$ and a complete graph for the social connections, that is all the trades are desirable and $\mathbb{E}(DC) = \mathbb{E}(E(G)) = \binom{n}{2}$. In this case, we cannot directly determine the volume since several agents only exchange a fraction ϵ of share but we can definitely determine the number of transactions in the economy. Thus, we just use the fact that after all the preferences have been drawn, there is an order on the set of the risk aversion parameters as shown on the following figure.



Here everyone knows each other, so each agent wants to trade with x_{i_1} or x_{i_n}. As only one trade is allowed, at the first best choice, the only matching pair is (x_{i_1}, x_{i_n}). Then, at the second best, the only matching pair is $(x_{i_2}, x_{i_{n-1}})$, etc... Every trade consists here in matching the highest outstanding offer to buy (current bid) with the lowest outstanding offer to sell (current ask). Finally, this suggests that the number of transactions is maximal and is equal[44] to $NT = \lfloor \frac{n}{2} \rfloor$, given that the size of the population is even or not.

Let us now highlight the case $p \neq 1$. As we mentioned above, for some graph topology, volume is only generated by first-best pairings.

Thus, it is interesting to see how these components emerge in random graphs. When we control for the value of p, we are able to generate almost surely a specific architecture of G. For instance,

[44] Notice in the case $\epsilon \not\to 0$, $\frac{A_{\min}}{A_{\max}}$ could belong to $\left[\frac{1}{3}, 3\right]$, that would imply $V = 0$.

as we mentioned in section (12) when $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$ and $n \to \infty$, whp the resulting graph is only composed by pairs and isolated vertices. Similarly, when $p \in \left[\frac{1}{n^{3/2}}, \frac{1}{n^{4/3}}\right]$, whp the resulting graph is only composed by pairs, triples and isolated vertices. As we already know the expected number of pairs and triples in the graph, by computing the expected volume for these components individually, we can deduce the expected volume for the whole graph. That leads to the following proposition.

Proposition : *In a Market with n agents who can only trade once, we assume $K = 1$ and $q = 0$. The expected volume can be expressed in the asymptotic case ($n \to \infty$) when $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$ as*

$$\mathbb{E}(V) = \frac{4}{3}\binom{n}{2}p(1-p)^{2n-4}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2$$

And when $p \in \left[\frac{1}{n^{3/2}}, \frac{1}{n^{4/3}}\right]$ as

$$\mathbb{E}(V) = \frac{4}{3}\binom{n}{2}p(1-p)^{2n-4}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2 + \binom{n}{3}p^2(1-p)^{3n-8}\frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\left(2-\frac{16}{9}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^4\right)$$

Of course, we recover here $\mathbb{E}(V) \underset{n\to\infty}{\to} \infty$ in both cases. In light of these results, we now relax the assumption that agents can only trade once by letting them realize their best arbitrage opportunity.

13.2 Two trades at most

In this section, we still assume $K = 1$ for technical reasons but every agent is now allowed to trade with both his best seller and his best buyer. More precisely, a *pure buyer* or a *pure seller* can only trade once, while a *trader* can trade twice. Our main motivation behind this new hypothesis is to allow people to transmit shares through social connections by introducing the concept of graph connectivity. Thus, everyone can potentially act on the Market during the same session and each individual preferences will affect the average market price. We will further develop the pricing implications of our model in section (14). Thus, we have

Proposition : *The expected volume for a pair of agents from G^* when the traders are allowed to realize their best arbitrage opportunity can be expressed in the case $q = 0$ and $K = 1$ as*

$$\mathbb{E}(V_{ij}|(1,1)^{st}) = 8\left(1-\left(\frac{1}{2}\right)^{|N^*(x_i)|}\right)\left(1-\left(\frac{1}{2}\right)^{|N^*(x_j)|}\right)\left(\frac{2}{3}\left(1-\frac{\bar{A}}{N_I}\right)^2\right)^{N^*_{ij}-1}$$

with $N^*_{ij} = |N^*(x_i)| + |N^*(x_j)|$.

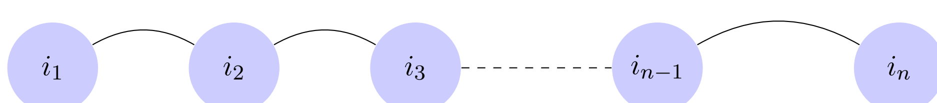
Again, $N^*_{ij} \geq 2$ since the result is based on the condition "x_i and x_j are linked in the optimized graph". In order to find the expected volume for the whole graph, we just aggregate over all the individual pairs as $\mathbb{E}(V|(1,1)^{st}) = \binom{n}{2}\mathbb{E}(V_{ij}|(1,1)^{st})$. Of course, regarding to the case where people can only trade once, the conditional expected volume is higher for any pairs of agents when they can realize their best arbitrage opportunity. We observe now that when each agent has exactly one partner to trade with, that is $\forall i \in [\![1,n]\!]$, $|N^*(x(i))| = 1$, this quantity is not always maximal.

Let us now examine some special cases. First of all, we want to determine the longest path in terms of share transmission observable in the graph after all the edges that do not carry a trade has been removed. As we claimed in section (10.4), a necessary condition for an agent to be a *trader* is given by $A_x \in \left[\frac{\bar{A}}{2}, \frac{11\bar{A}}{18}\right]$. By construction, it is clear that a share can be transmitted twice at most. Indeed, we cannot find more than three people whose the risk aversions are ordered in a way there is an incentive to trade from one to another, that is we cannot have $A_x < A_y < A_z < A_t$ such that $DC_{xy} = DC_{yz} = DC_{zt} = 1$. So the same share cannot be bought more than twice and the longest path is 2. As we will see in next section, there exist an upper bound for the volume that is equal to $V = n - 1$.

13.2.1 Special cases

Let us consider again the case where $q = 0$, $p = 1$ and $\epsilon \to 0$, that is an organized Market without shock where everyone knows each other and where we can exchange a fraction of share as small as we want. As in section (13.1.1), after the preferences have been drawn, the order on the set of the risk aversion parameters $A_{i_1} < ... < A_{i_n}$ leads to the following reasoning. Any agent x_{i_k} with $k \neq \{1, n\}$ in the Market want to trade with both x_{i_1} and x_{i_n}. These two people are *pure buyer* and *pure seller* respectively, so they can trade only once. The resulting first-best-matching pair is (x_{i_1}, x_{i_n}). Then, every agent will consider his second best choice, that is $\forall k \neq \{2, (n-1)\}$, x_{i_k} is willing to trade with x_{i_2} and $x_{i_{n-1}}$. However, x_{i_2} and $x_{i_{n-1}}$ cannot trade with x_{i_1} and x_{i_n}, so they become *pure buyer* and *pure seller* respectively and the only second-best-matching pair is $(x_{i_2}, x_{i_{n-1}})$ and so forth. Finally, we obtain the same result here than in the case people can only trade once and the number of transactions is equal to $NT = \lfloor \frac{n}{2} \rfloor$.

Now, we consider another special case where we keep $q = 0$ and $\epsilon \to 0$ but we relax the assumption $p = 1$ and $K = 1$. Given the order $A_{i_1} < ... < A_{i_n}$, we only assume the following underlying structure for the social connections between the agents.



Thus, as there exists a desirability channels between every pair of agents, everyone is a *trader* in this Market except for the vertices x_{i_1} and x_{i_n}. Given the degree of each node, that is $\forall k \neq \{1, n\}$,

$\delta(x_{i_k}) = 2$, all the *traders* can realize their best arbitrage opportunity and the number of transactions will be equal to $NT = n - 1$.

Let us consider again the two states deterministic graph process on three vertices we have introduced in previous section : $(G_t)_1^2$ with G_1 a graph with edge $E(G_1) = \{xy\}$ and G_2 a graph with edges $E(G_2) = \{xy, xz\}$. When a *trader* is allowed to realize his best arbitrage opportunity, we can state the proposition

Proposition : *Consider any isolated graph component with three different vertices $\{x, y, z\}$ where x and y are already connected by an edge. We connect the third vertex z to x and we assume that the agents can realize their best arbitrage opportunity.*

- *If y and z are of the same type (e.g. they are two buyers), the expected volume behaves exactly in the same way that we described in the case where people are allowed to trade only once and the proposition of previous section applies.*

- *If y and z are not of the same type, the expected volume always increases since x is potentially able to trade with both of them.*

Let us now address the asymptotic case where we control for the topology of the graph by constraining the probability p. As previously mentioned, we can state the following proposition.

Proposition : *In a Market with n agents who can trade at most twice, we assume $K = 1$ and $q = 0$. For $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$, the asymptotic expected volume ($n \to \infty$) is the same that in the case where an agent can only trade once. For $p \in \left[\frac{1}{n^{3/2}}, \frac{1}{n^{4/3}}\right]$, it can be expressed as*

$$\mathbb{E}(V) = \frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2 p(1-p)^{2n-4}\left[\binom{n}{2} + \binom{n}{3} p(1-p)^{n-4}\left(\frac{8}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^4 + 2 - \frac{8}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\right)\right]$$

It is straightforward to show here that the expected volume has increased compared to the case where the agents are allowed to only trade once.

Let us now address the pricing implications of our model. In next section, we define the price and its two first moments for each trade.

14 Asset pricing and graph connectivity

14.1 The price

As we mentioned above, an agent will always prefer to trade with the neighbor who has the furthest preferences compared to their own ones. Once he found his best partner, if a share can be traded, we assume that the transaction price $P(ij)$ is the output of a bargaining game that takes place between the two agents. Our setting suggests multiple equilibrium prices and regarding to the results of section (10.2), we propose the following price definition.

Definition : *Let x_i and x_j be a seller and a buyer respectively. If $V_{ij} = 1$, the price at which the trade occurs is*

$$P(ij) = f(A_i, A_j) = w_i \left(\mu - \frac{A_i}{2}\sigma^2\right) + w_j \left(\mu - \frac{3}{2}A_j\sigma^2\right)$$

where w_i and w_j are the weights associated to the individuals preferences and verify $w_i + w_j = 1$.

The function f could be chosen such that the relative neighborhood composition of each player has an effect on the price determination. For instance, in an alternating offers bargaining game such as in Corominas-Bosch, we would have $w_i = \mathbb{1}_{|N^s(x_j)|>|N^b(x_i)|}$ and $w_j = 1 - w_i$ where N^s and N^b describe the number of sellers and buyers respectively an agent has in his neighborhood. In order to capture the bargaining power of an agent, we could also define

$$f(A_i, A_j) = \frac{N^s(x_j)}{N^b(x_i)+N^s(x_j)} \left(\mu - \frac{A_i}{2}\sigma^2\right) + \frac{N^b(x_i)}{N^b(x_i)+N^s(x_j)} \left(\mu - \frac{3}{2}A_j\sigma^2\right)$$

Thus, higher the number of buyers in the neighborhood of x_i compared to the number of sellers in the neighborhood of x_j, stronger the bargaining power of x_i with respect to x_j. Therefore, the weight associated to the x_i minimum ask $\mu - \frac{A_i}{2}\sigma^2$ will be lower and the resulting price of the trade will be closer to the x_j maximum bid $\mu - \frac{3}{2}A_j\sigma^2$. Notice that in the case where x_i belongs to a group impacted by a shock, if the size of this group is higher than $\frac{n}{2}$, x_i has less bargaining power than x_j. The reverse is not always true.

However, in the rest of this paper, we will simply assume that $\forall(i,j)$, $w_i = w_j = \frac{1}{2}$. In other terms, when two agents are willing to trade, the price is entirely built on their preferences regardless of their respective positions in the graph. This could be due to a regulatory body or social planner that sets the price only with respect to the individual risk aversions to ensure a perfectly balanced bargaining power. This is the classical perfect equilibrium payoff of the two players game developed by Rubinstein (1982) in the case of an isolated pair of very impatient agents ($\delta = 1$). Therefore $\forall x_i, x_j \in V(G)$ the price can be expressed as follow

$$P(ij) = \begin{cases} \mu - \frac{1}{4}\left(3A_i + A_j\right)\sigma^2 & \text{for } \frac{A_i}{A_j} > 3 \\ \mu - \frac{1}{4}\left(A_i + 3A_j\right)\sigma^2 & \text{for } \frac{A_i}{A_j} < \frac{1}{3} \end{cases}$$

Notice that when x_i is impacted by a shock, the price becomes $P(ij) = \mu - \frac{1}{4}\left(18\bar{A} + A_j\right)\sigma^2$ which is lower than the price in the case where no shock occurs. As we mentioned in section (13), when some agent x_i has many first best partners to trade with, he will simply randomly pick up one of them with probability $\frac{1}{|x_i^*|}$. Thus, when a group of size $n_k \leq \frac{n}{2}$ is impacted by a shock, all of his members will be able to trade with the n_k best buyers of the non-impacted part of the graph but will be randomly paired with them. However, if $n_k > \frac{n}{2}$, only $n - n_k$ impacted agents, randomly chosen, will be allowed to exchange a share in the Market. We propose in Appendix an alternative mechanism by considering an auction that would take place when a group is impacted by a shock.

Before we move into the properties of the price, let us introduce a first proposition.

Proposition : *Let assume that n agents can trade a fraction of share as small as they want, that is $\epsilon \to 0$. Moreover, we consider $K = 1$ and $p = 1$ to generate a complete graph. Then for $n \to \infty$, the law of one price holds in the Market.*

Indeed, when n tends to infinity, the individual risk aversion coefficients would span the entire support of the single uniform distribution from which each agent preferences are drawn in this economy. Regarding to the results of section (13.1.1), we know that whatever the rules about the number of trades permitted for each player, the matching process will always associate agents with the furthest preferences. The almost continuum of individuals in the Market implies that every price is generated from risk aversion parameters symmetrically located around $\bar{A}$. Finally, by using the results obtained in section (10.2), we know that the price can be expressed as $P(ij) = \mu - \frac{1}{2}(A_i + A_j)$ and $\frac{1}{2}(A_i + A_j)$ should be always equal to $\bar{A}$. Therefore, each transaction price should be the same and the *law of one price* should be verified.

14.2 The price distribution

As we introduced the price $P(ij)$ in previous section (13), we are now able to express its two first moments. First, consider the general case where $q \geq 0$, $\forall (x_i, x_j) \in V(H_k) \times V(H_l)$ we have

$$\mathbb{E}(P(ij)|\{V_{ij} = 1\}) = \mathbb{E}(P(ij)|\bar{S}^k, S^l)q(1-q) + \mathbb{E}(P(ij)|S^k, \bar{S}^l)q(1-q) + \mathbb{E}(P(ij)|\bar{S}^k, \bar{S}^l)(1-q)^2$$

Notice that a trade cannot occur between two impacted groups. Thus, we have $\forall (x_i, x_j) \in V(H_k) \times V(H_l)$

$$\mathbb{E}(P(ij)|\{V_{ij} = 1\}) = (1 - q^2)\mu - (1 - q)\left(\frac{17}{2}q + 1\right)\bar{A}\sigma^2$$

Here, the expected price is a non-monotone function of the shock probability q and for $q = 0$

$$\mathbb{E}\left(P(ij)\right)) = \mu - \bar{A}\sigma^2$$

We observe that the expected price is always the same whatever the group to which the agents belong and whatever the direction of the trade. As we mentioned above, if we assume individual bargaining power based on the agents relative positions in the graph, clearly the price will strongly depend on the value of p which determines the neighborhood of each vertex.

For the variance, we obtain $\forall (x_i, x_j) \in V(H_k) \times V(H_l)$ with x_j is a buyer and x_i is a seller

$$\mathrm{Var}(P(ij)|\{V_{ij}=1\})) = (1-q)^2 \left(\frac{5-4q}{96} N_k^2 \sigma^4 + \frac{N_l^2 - N_k^2}{16} \frac{\sigma^4}{12} \right)$$

Clearly here, this quantity is a decreasing function of the shock probability q and for $q = 0$

$$\mathrm{Var}(P(ij)|\{V_{ij}=1\})) = \frac{5}{96} N_k^2 \sigma^4 + \frac{N_l^2 - N_k^2}{16} \frac{\sigma^4}{12}$$

Notice that in the case $|V(H_k)| = |V(H_l)|$, the x_i and x_j preferences are drawn from the same distribution and the formula is reduced to $\frac{5}{96} N_k^2 \sigma^4$. Here, the variance of the price does depend on the group belonging of each agent enrolled in the trade. Symmetrically when x_j sells and x_i buys, we have $\mathrm{Var}(P(ij)|\{V_{ij}=1\})) = \frac{5}{96} N_l^2 \sigma^4 + \frac{N_k^2 - N_l^2}{16} \frac{\sigma^4}{12}$. Hence, depending on the sign of $N_l^2 - N_k^2$ or $N_k^2 - N_l^2$, the variance for a trade intra-group will be lower or higher than the variance for a trade inter-group. For instance, when x_j buys and x_i sells, if x_j belongs to a larger group than x_i, the variance increases.

Let us now consider the mean price of the graph $\bar{P} = \frac{1}{\mathrm{card}(\mathcal{V})} \sum\limits_{x_i x_j \in \mathcal{V}} P(ij)$ with $\mathcal{V} = \{x_i x_j \in E(G) : V_{ij} = 1\}$ the set of the edges that actually carry a trade. As $P(ij)$ and $\mathcal{V}$ are independent, we can easily derive the two first moments of the mean price.

$$\mathbb{E}(\bar{P}) = \mathbb{E}(\mathbb{E}(\bar{P}|\mathcal{V})) = \mathbb{E}\left(\frac{1}{\mathrm{card}(\mathcal{V})} \sum_{x_i x_j \in \mathcal{V}} \mathbb{E}(P(ij)) \right) = \mathbb{E}\left(\frac{1}{\mathrm{card}(\mathcal{V})} \mathrm{card}(\mathcal{V}) \mathbb{E}(P(ij)) \right) = \mathbb{E}(P(ij))$$

For its variance, let $\mathcal{B}_{kl} = \{x_i x_j \in E(G),\, x_i \in V(H_k),\, x_j \in V(H_l) : V_{i,j} = 1\}$ be the set of the trades inter-group and when $l = k$, $\mathcal{B}_{kk}$ the set of the trades inter-group. Hence we have $\bigcup\limits_{\substack{k,l \\ k \leq l}} \mathcal{B}_{kl} = \mathcal{V}$ and for $q = 0$

$$\mathrm{Var}(\bar{P}|\mathcal{V}) = \frac{1}{V} \left[\frac{5}{96} \sigma^4 \left(\sum_{k=1}^{K} V_k + \sum_{k<l} V_{kl} \right) N_k^2 + \sum_{k<l} \left(\frac{N_l^2 - N_k^2}{16} V_{kl}^{\mathrm{out}} + \frac{N_k^2 - N_l^2}{16} V_{kl}^{\mathrm{in}} \right) \right]$$

Where V_{kl}^{out} (respectively V_{kl}^{in}) is the number of sales from the group H_k to the group H_l (respectively purchases by the group H_k from the group H_l). Hence, the variance of the mean price clearly depends on the volume and on the graph topology.

[INSERT FIGURE $\mathrm{Var}(\bar{P}) = f(p)$]

Let us now address an alternative mechanism that could take place when a shock occurs in the Market.

14.3 The auction's mechanism

We consider here the case $c = 1$, that is when the social connections are drawn regardless of the group belonging of each agent and we generate a pure homogeneous random graph.

We assume that a group H_k of size n_k is impacted by a shock. Every member of this group is instantaneously connected with the non impacted part of the graph and has a risk aversion equal to $6\bar{A}$. Thus, any agent from a non impacted group has among his neighbors n_k partners with the same preferences. In the case where the matching process would not be ensured by a regulatory body as we assumed in section (13) and (14.1), we need new decision rules to elect the best choice of such agent. Hence, we consider a first-price simultaneous auction mechanism with $n - n_k$ bidders and n_k identical objects since all the agents from the impacted group want to sell their shares. Of course, we require $n_k < n/2$ to ensure a non-zero price. Because of the trade rules mentioned above, every potential buyer demands only one object. Let $\mathbb{B} = \{b_1, ..., b_{n-n_k}\}$ and $\mathbb{S} = \{s_1, ..., s_{n-n_k}\}$ be the sets of the bids and the individual signals respectively. $\forall i \in [\![1, n - n_k]\!]$, $b_i = b(s_i)$ and $s_i = \min\limits_{y \in N^s(x_i)\backslash V(H_k)} \left(\mu - \frac{A_y}{2}\sigma^2\right)$, that is the signal of each vertex is the lowest price at which an agent is willing to sell among his neighbors. The profit Π_i is expressed as a function of both b_i and s_i. For every vertex x_i such that $\delta^{\text{in}}(x_i) \geq 1$, with $\delta^{\text{in}}(x) = |N^s(x)\backslash V(H_k)|$ the number of sellers an agent has in his non impacted neighborhood, we fairly expect that $b(s_i) < s_i$. Let $s^{n-n_k}(1), ..., s^{n-n_k}(n - n_k)$ be the order statistics[45] of the $n - n_k$ draws.

14.3.1 Case $n_k = 1$

First, we consider the case where only one object is auctioned, that is when a group of size 1 is impacted. We have $\forall x_i \in V(G)\backslash V(H_k)$

$$\begin{aligned}
\mathbb{E}(\Pi_i(b_i, s_i)) &= (s_i - b_i)\mathbb{P}(\forall x_j \neq x_i \in V(G)\backslash V(H_k),\, b(s_j) < b_i) \\
&= (s_i - b_i)\mathbb{P}(\forall x_j \neq x_i \in V(G)\backslash V(H_k),\, s_j < b^{-1}(b_i)) \\
&= (s_i - b_i) \prod_{\substack{x_j \in V(G)\backslash V(H_k) \\ x_j \neq x_i}} F_{s_j}(b^{-1}(b_i)) \times \frac{1}{F_{s_i}(b^{-1}(b_i))}
\end{aligned}$$

[45] For instance, there are $j - 1$ realizations in the sample strictly lower than $s^{n-n_k}(j)$.

Now, we need to determine the distribution of the variable $s_i = \min_{y \in N^s(x_i)} \left(\mu - \frac{A_y}{2}\sigma^2\right)$. We have[46] $F_{s_i}(z) = \prod_{j \neq k}^{K} \left(\frac{\bar{A} + N_i/2 - \frac{2}{\sigma^2}(\mu - z)}{N_i}\right)^{|N^s(x_i) \cap V(H_j)|}$ with $N^s(x_i) = \{y \in N(x_i) : \frac{A_y}{A_i} > 3\}$.

Thus, the optimal bid of every agent can be expressed as

$$b_i^* = \arg\max_{b_i} \left(\ln \mathbb{E}(\Pi_i(b_i, s_i))\right)$$

with $\ln \mathbb{E}(\Pi_i(b_i, s_i)) = \ln(s_i - b_i) - \ln F_{s_i}(b^{-1}(b_i)) + \sum_{\substack{x_j \in V(G)\setminus V(H_k) \\ x_j \neq x_i}} \ln F_{s_j}(b^{-1}(b_i))$.

14.3.2 Case $n_k > 1$

In order to solve for the optimal bid, we have to control for the topology of the graph and more precisely for the neighborhood of each agent. Thus, we consider the case where $n \to \infty$ and $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$, that is whp the non impacted part of the graph is only composed by pairs and isolated vertices. Thus, for any non impacted vertex, we have whp

$$s_i = \begin{cases} \mu - \frac{A_i}{2}\sigma^2 \text{ if } |N^s(x_i)\setminus V(H_k)| = 0 \\ \mu - \frac{A_{-i}}{2}\sigma^2 \text{ if } |N^s(x_i)\setminus V(H_k)| = 1 \end{cases}$$

where A_{-i} is the risk aversion parameter of the single non impacted neighbor of x_i when $|N^s(x_i)\setminus V(H_k)| = 1$. We denote I the event "*the non impacted part of the graph is only composed by pairs and isolated vertices*", so

$$\begin{aligned} \mathbb{E}(s_i|I) &= \mu - \frac{\mathbb{E}(A)\sigma^2}{2} \\ &= \mu - \frac{\bar{A}\sigma^2}{2} \end{aligned}$$

Notice here that $A \sim U_{[a,b]} \Rightarrow s_i \sim U_{[\mu - \frac{\sigma^2}{2}b, \mu - \frac{\sigma^2}{2}a]}$. Therefore, as we have n_k objects to sell in this auction, we are interested by the k-th order statistics of $n - n_k$ draws from a uniform distribution on $[\mu - \frac{\sigma^2}{2}b, \mu - \frac{\sigma^2}{2}a]$. To make the resolution more readable, we choose $\mu = \frac{\sigma^2}{2}\left(\bar{A} + \frac{1}{2}N_n\right)$ and we have $s_i \sim U_{[0, \frac{\sigma^2}{2}N_n]}$. As we show in Appendix, the expected value of $s^{n-n_k}(k)$ is

$$\mathbb{E}(s^{n-n_k}(k)|I) = \frac{k}{2(n - n_k + 1)}\sigma^2 N_n$$

[46] See the Appendix for more details.

Therefore the individual profit can be expressed as

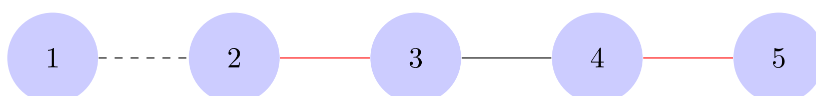
$$\mathbb{E}(\Pi_i(b_i, s_i)) = (s_i - b_i)\mathbb{P}\left(\frac{k}{2(n - n_k + 1)}\sigma^2 N_n < b_i\right)$$

14.4 Graph connectivity and agent participation

Depending on the probability p, every vertex or agent in the Market could be potentially involved in a trade. Indeed, we have seen that the implication $\{x_i \sim x_j\} \Rightarrow \{V_{ij} = 1\}$ does not systematically hold. However, its negation $\{x_i \not\sim x_j\} \Rightarrow \{V_{ij} = 0\}$ is always verified. In other terms, if two agents do not know each other, even if their preferences are such that they have an incentive to trade, no trade will take place.

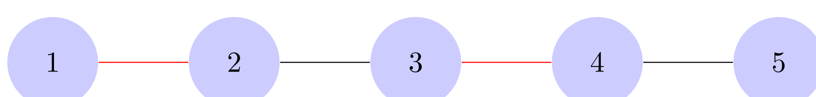
In section (11) and (13), we already discussed the relationship between the connectivity of the graph and the number of desirability channels and the volume respectively. Here, our purpose is more to understand how the impact of individual preferences potentially propagates through the graph. We will focus on connected component, whatever the connectivity level, to examine how the preferences of one agent can potentially affect the trade of another.

Thus, the participation of an agent could potentially modify the setting of the trades and so the expected mean price of the Market, especially when this agent has one of the lowest or the highest preferences. For instance, let us consider the simple case of a Market composed by the individuals $\{x_1, x_2, ..., x_5\}$ where we have the following trades pattern.



We assume here that a desirability channel exists between any connected pair of agents. The red edges carry a trade while the black ones don't. The dashed line between the vertices x_1 and x_2 just describes the fact that we want to connect them. Let us assume that people can exchange once at most and we have the optimal choices : $x_2^* = x_1$, $x_3^* = x_2$, $x_4^* = x_3$ and $x_5^* = x_4$. When $x_2x_1 \notin \mathcal{V}$, the trades take place as $(1,2)^{th}$ order pairings.

Now, if we connect x_1 and x_2, we get the following new trades pattern



where (x_1, x_2) is a $(1,1)^{st}$ order pairing and (x_3, x_4) is a $(1,2)^{th}$ order pairing. Therefore, the participation of x_1 in the Market, modifies both the trades pattern and the price at which x_2 can exchange. That leads to the next proposition.

Proposition : *Considering any pair of agents embedded in a Graph, there exists at least one topology for which they are the best choice of each other.*

This will be especially the case for the dynamic version of our model presented in Appendix as everyone learns from his neighbors. We would go even further by saying that the price at which an agent is able to trade, indirectly depends on the preferences of every member of the connected component to which he belongs.

Proposition : *Let $\mathcal{G}(n,(p_{ij}))$ be a random graph model such that $(p_{ij})_{ij} = \left\{\frac{p}{c}, p\right\}$ and X_0 denote the number of isolated vertices in the random graph $G \in \mathcal{G}(n,(p_{ij}))$. For any probability $p(n)$ higher than $\frac{c\log(n)}{n}$, we have*

$$\mathop{lim}_{n\to\infty} \mathbb{P}(X_0 = k) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \mathbb{P}(G \in \mathcal{G}(n,(p_{ij})) \text{ is connected}) = 1$$

So the graph is almost surely connected.

This proposition obviously describes our model for $c = 2$ and $q = 0$. Thus, we can state the following corollary.

Corollary : *In our model, there is no isolated group in the Market almost surely when $p \geq \frac{2\log(n)}{n}$, so there exists a path of acquaintances which connects any pair of agents.*

This corollary provides us with the necessary condition for obtaining a maximal volume in a Market with n agents who can trade at most twice. Indeed, as we illustrated in section (13), the volume is maximal only for some classes of topology, that is the underlying graph must be connected to allow each agent to be enrolled in a trade.

15 Conclusion and Extensions

In this paper, we rely the social architecture to the agents preferences and we found a relationship between Market composition and trading activity. Every social group in this economy can be impacted by a shock with equal probability. This shock can be interpreted as any exogenous motives to become a *pure seller* for the rest of the Market. We mostly think here in terms of liquidity shock for instance. We derive closed formula for the expected number of desirability channels and for the expected volume in different special cases of our model. We found that in the case of equal-sized groups, an increase of their number generates a decrease of the incentives to trade and thereby the volume potentially diminishes. Symmetrically, for a given number of groups, if we make them more dissimilar, new desirability channels appear and the volume potentially increases. We also found a nonlinear relationship between the expected number of desirability channels and the shock probability. This can be easily explained by arguing that given our setting, the incentives generated by one or many shocks non linearly depend on the size of the impacted part of the Market. As this size is entirely determined by the shock probability value, the result follows. Finally, we used some graph theory results to characterize the optimality of any network. Thus, from a social planner perspective, we are able to state which class of topology maximizes the number of trades on the Market.

As a natural extension of our model, we could allow shocks to be of different kinds. More specifically, we could allow the interval from which the preferences are drawn to be shifted into the both directions such that all the members of an impacted group can become pure sellers or pure buyers with respect to the rest of the graph. The dynamic version of our model presented in Appendix would also be improved and developed to understand the evolution of a such Market over time.

16 Appendix

16.1 Trade condition

16.1.1 Static case

With CARA preferences, we have for any agent who would decide to buy, $U(W) = -e^{-A((1+\epsilon)\tilde{v}-p\epsilon)}$ and $\mathbb{E}(U(W)) = -e^{-A((1+\epsilon)\mu-p\epsilon)+\frac{A^2}{2}(1+\epsilon)^2\sigma^2}$. So the corresponding expected utilities are

$$\begin{cases} \mathbb{E}(U(W_B)) & = & -e^{-A((1+\epsilon)\mu-p\epsilon)+\frac{A^2}{2}(1+\epsilon)^2\sigma^2} \\ \mathbb{E}(U(W_S)) & = & -e^{-A((1-\epsilon)\mu+p\epsilon)+\frac{A^2}{2}(1-\epsilon)^2\sigma^2} \\ \mathbb{E}(U(W_{NT})) & = & -e^{-A\mu+\frac{A^2}{2}\sigma^2} \end{cases}$$

Therefore, an agent is willing to buy ϵ share when $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_S))$. Symmetrically he sells this quantity when $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_B))$. To sum up

$$\text{he buys iff} \begin{cases} p < \mu - A\left(1+\frac{\epsilon}{2}\right)\sigma^2 \\ p < \mu - A\sigma^2 \end{cases} \qquad \text{he sells iff} \begin{cases} p > \mu - A\left(1-\frac{\epsilon}{2}\right)\sigma^2 \\ p > \mu - A\sigma^2 \end{cases}$$

Finally, there is an incentive to trade between agent i and agent j if and only if

$$
\begin{aligned}
\mu - A_i\left(1+\frac{\epsilon}{2}\right)\sigma^2 &\gtrless \mu - A_j\left(1-\frac{\epsilon}{2}\right)\sigma^2 \\
A_i\left(1+\frac{\epsilon}{2}\right)\sigma^2 &\lessgtr A_j\left(1-\frac{\epsilon}{2}\right)\sigma^2 \\
\frac{A_i}{A_j} &\notin \left[\frac{1-\frac{\epsilon}{2}}{1+\frac{\epsilon}{2}}, \frac{1+\frac{\epsilon}{2}}{1-\frac{\epsilon}{2}}\right] \underset{\epsilon\to 0}{\to} \{1\}
\end{aligned}
$$

with $\mathbb{P}\left(\frac{A_i}{A_j} \neq 1\right) = 1$ when ϵ tends towards zero by definition.

16.1.2 Dynamic case

Without additional costs : We keep here our original framework by adding a law of motion for the risk aversion parameter of each agent. Thus, according to his decision to buy or sale the asset, or to do not trade, an agent will have the following wealth

$$
W_{t+1} = \begin{cases} W_{t+1}^B = (\tilde{v}-p)\epsilon_{t+1} + \tilde{v} + W_t = (\epsilon_{t+1}-1)\tilde{v} + p\epsilon_{t+1} + W_t & \text{if he buys } \epsilon \text{ share} \\ W_{t+1}^S = (p-\tilde{v})\epsilon_{t+1} + \tilde{v} + W_t = (1-\epsilon_{t+1})\tilde{v} + p\epsilon_{t+1} + W_t & \text{if he sells } \epsilon \text{ share} \\ W_{t+1}^{NT} = \tilde{v} + W_t & \text{if he does not trade} \end{cases}
$$

Therefore, the maximization program for every agent i becomes

$$
\max_{\iota_t \in \{-1,0,1\}} -\mathbb{E}_t\left(e^{-A_{i,t+1}(\iota_{t+1}(\tilde{v}-p)\epsilon_{t+1}+\tilde{v}+W_t)}\right)
$$

where $\mathbb{E}_t()$ is the conditional operator based on the filtration $\mathcal{F}_t = \{W_\tau : \tau \leq t\}$. Of course, we obtain the same trade condition that in the static case $\frac{A_{i,t+1}}{A_{j,t+1}} \notin \left[\frac{1-\frac{\epsilon_{t+1}}{2}}{1+\frac{\epsilon_{t+1}}{2}}, \frac{1+\frac{\epsilon_{t+1}}{2}}{1-\frac{\epsilon_{t+1}}{2}}\right] \underset{\epsilon_{t+1}\to 0}{\to} \{1\}$. Notice here that the risk aversion is time-varying and has the following dynamic

$$
A_{i,t+1} = \begin{cases} \frac{A_{i,t}+A_{y,t}}{2} & \text{if } \exists y \in V(G) : V_{iy}^t = 1 \\ A_{i,t} & \text{otherwise} \end{cases}
$$

Thus, at the end of each period when we matched all the compatible pairs, people who has been enrolled in a trade revise their preferences in the light of the risk aversion of his partner. In other terms, there is a *learning effect*. Then, a new period starts, we replicate the game and so forth. Everything stops when there is no incentive to exchange in the Market anymore.

It is very interesting to see here at which speed the preferences converge. We claim that this speed will strongly depend on the graph topology.

With additional costs : Let us further assume now that is costly to stay on the Market. At the beginning of each period, the agents must pay a fixed cost $C > \mu$ to continue to trade. Thus, according to his decision to buy or sale the asset, or to do not trade, an agent will have the following wealth

$$W_{t+1} = \begin{cases} W_{t+1}^B = (\tilde{v} - p)\epsilon_{t+1} + \tilde{v} - C + W_t = (\epsilon_{t+1} - 1)\tilde{v} + p\epsilon_{t+1} + W_t & \text{if he buys } \epsilon \text{ share} \\ W_{t+1}^S = (p - \tilde{v})\epsilon_{t+1} + \tilde{v} - C + W_t = (1 - \epsilon_{t+1})\tilde{v} + p\epsilon_{t+1} + W_t & \text{if he sells } \epsilon \text{ share} \\ W_{t+1}^{NT} = \tilde{v} - C + W_t & \text{if he does not trade} \end{cases}$$

Therefore, it is costly now for an agent to do not trade during one period. The maximization program for every agent i becomes

$$\max_{\iota_t \in \{-1,0,1\}} - \mathbb{E}_t\big(e^{-A_{i,t+1}(\iota_{t+1}(\tilde{v}-p)\epsilon_{t+1} - C + \tilde{v} + W_t)}\big)$$

By using the law of motion mentioned above for the risk aversion, we obtain the same trade condition than before since both W_t and C disappear in the resolution.

However, we are able to determine the probability of an idiosyncratic liquidity shock, that is for every agent $q_{i,t+1} = \mathbb{P}(W_t^i < 0 | \mathcal{F}_{t-1}) = \Phi_{\nu_t, \chi_t^2}(0)$ with

$$(\nu_t, \chi_t^2) = \begin{cases} ((\mu - p_t)\epsilon_t - C + \mu + W_{t-1}, (1 + \epsilon_t)^2\sigma^2) & \text{if he buys} \\ ((p_t - \mu)\epsilon_t - C + \mu + W_{t-1}, (1 + \epsilon_t)^2\sigma^2) & \text{if he sells} \\ (-C + \mu + W_{t-1}, \sigma^2) & \text{if he does not trade} \end{cases}$$

Where $p_t = P(ij)$ is the price described in section (14). As this price depends on the group belonging of the agents enrolled in the trade x_i and x_j, the same applies for the probability $q_{i,t+1}$. We can deduce the probability of a collective shock on a specific group $V(H_k)$, that is when every member of $V(H_k)$ simultaneously experience a liquidity shock, we have $q_{t+1}(n_k) = \prod_{x_i \in V(H_k)} q_{i,t+1} = \big(\Phi_{\nu,\chi^2}(0)\big)^{n_k}$. Notice here that every group will not have the same probability to be impacted by a shock anymore.

In this version of the model, we assume that both the social connections and the preferences of each agent are time-varying. Thus, the game will evolve as follows.

1. We generate the random graph of the social connections and the individual preferences in the same way than described in section (10.4) and (10.3).

2. The game starts and we first match all the possible pairs.

3. Given the value of C and the different realized trades, we compute the wealth of each agent and detect those who are insolvent.

4. According to the individual wealth signs, we fully connect the agents who has experienced a liquidity shock to the non impacted agents.

5. A new period starts, we match all the possible pairs and we remove all the previously impacted vertices from the graph. Then we replicate exactly the same procedure than before and so forth.

6. The game stops when there is no feasible trade during one period.

16.2 Expected number of desirability channels w.r.t. p and q

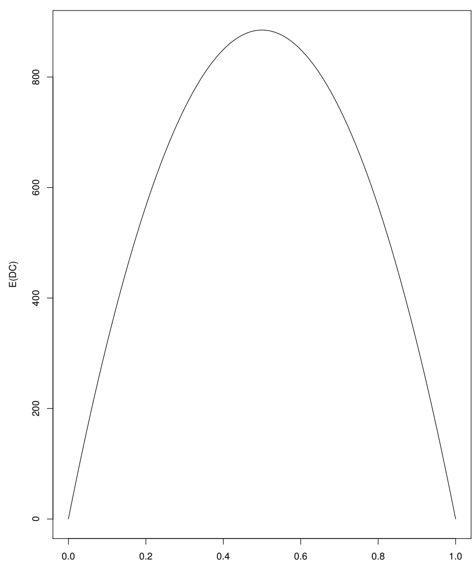


Figure 5: The expected number of desirability channels as a function of q

16.3 Random graphs and corresponding prices

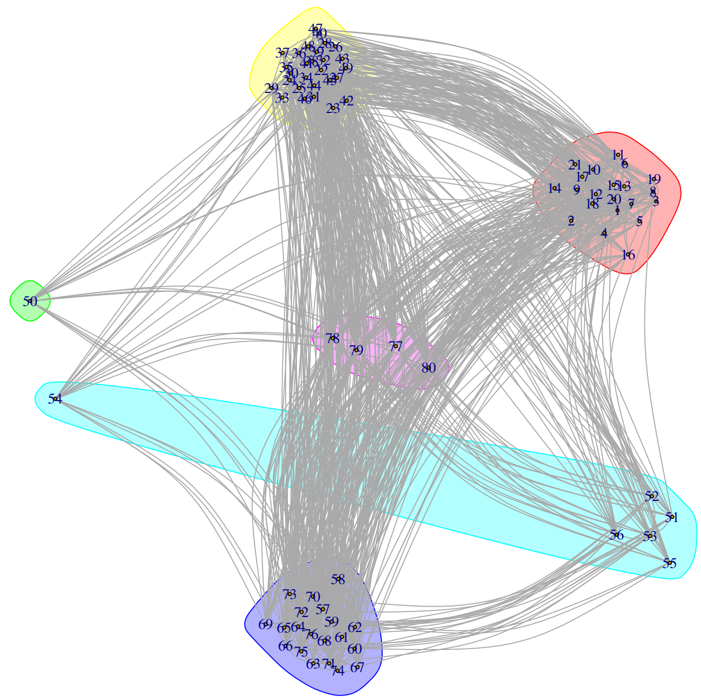


Figure 6: The social connections between the agents for $N = 80$, $K = 6$, $p = 0.4$, $q = 0.1$. Here the group $\{77, 78, 79, 80\}$ is impacted and so fully connected with the rest of the graph.

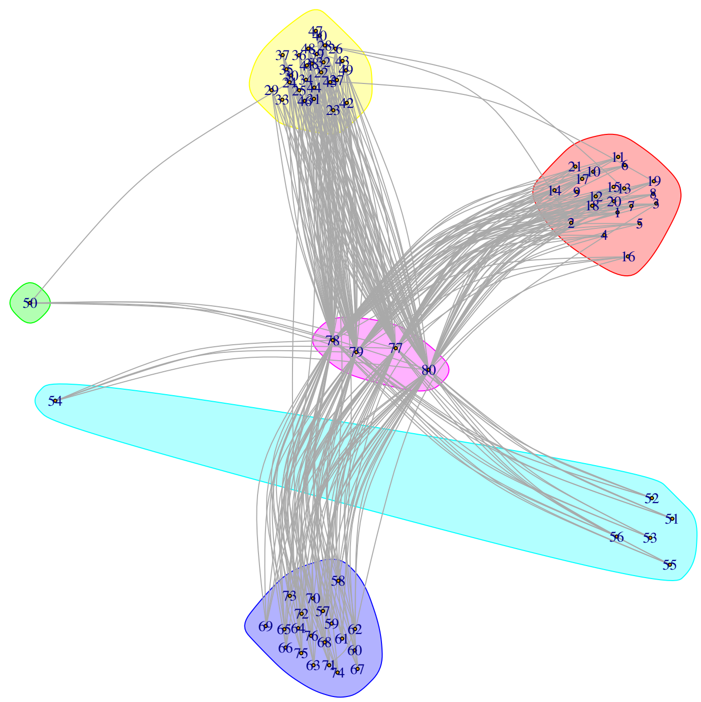


Figure 7: The resulting desirability channels for $N = 80$, $K = 6$, $p = 0.4$ and $q = 0.1$. Here the group $\{77, 78, 79, 80\}$ is impacted and so fully connected with the rest of the graph.

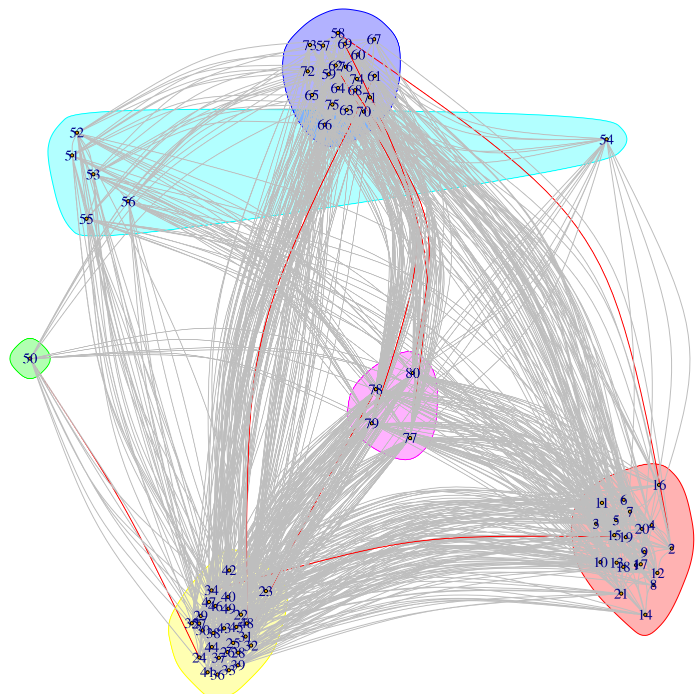


Figure 8: In red the realized trades for $N = 80$, $K = 6$, $p = 0.4$ and $q = 0.1$. Here the group $\{77, 78, 79, 80\}$ is impacted and so fully connected with the rest of the graph.

The auction prices have a *.

16.4 Volume Code

In order to simulate the volume in the general case of K groups and $q \neq 0$, we have implemented in R language an algorithm that proceeds as follows :

1. We generate the random sizes of the K different groups by adding the opportunity to only draw unequally sized groups.

Table 5: : Trades with corresponding prices for $\bar{A} = 10$, $\mu = 90$ and $\sigma = 1$

Buyers	**Sellers**	**Price**
"2"	"58"	83.212
"19"	"34"	83.296
"24"	"50"	82.873
"48"	"61"	83.459
"28"	"78"	34.232*
"62"	"77"	34.232*
"68"	"80"	34.232*
"69"	"79"	34.232*

2. We take a partition of the global population given the sizes of the groups.

3. According to the size of the group an agent belongs to, we draw his risk aversion coefficient from the appropriate uniform distribution.

4. We detect which group(s) will be impacted by a shock by generating a random binary vector.

5. We generate the random graph that describes the social connections between agents. For any pair of vertices, we set the probability that there exists an edge between them according to their group belonging and to the probability that one of these groups are impacted by a shock.

6. According to the rules we presented in the previous sections, we define the different functions that return the neighborhood of each vertex, his preference and his best partner .

7. In the particular case where at least one group is impacted by a shock, we first consider the output of the auction, that is the number of share sold and the corresponding prices. If the number of auctioned "objects" is lower than the number of potential buyers, we keep the highest losing valuation among the non impacted agents. Otherwise, we set the price to zero.

8. In order to compute the volume, we first count the first-best-matching pairs, then we remove all the nodes that have been involved in a trade and we compute again the first-best-matching pairs on the remaining graph, over and over until no trade is feasible. Finally, we just have to add the number of shares sold during a potential previous auction.

16.5 Distribution of $Z_{ij} = \frac{A_i}{A_j}$

Let X and Y be two random variables, we denote f_X and f_Y their probability density functions. The distribution of the ratio $Z = X/Y$ is

$$
\begin{aligned}
F_Z(z) = \mathbb{P}(Z \le z) &= \mathbb{P}(X \le Yz \mid Y > 0) + \mathbb{P}(X \ge Yz \mid Y < 0) \\
&= \int_0^{\infty} \left(\int_{-\infty}^{yz} f_X(x)dx \right) f_Y(y)dy + \int_{-\infty}^{0} \left(\int_{yz}^{\infty} f_X(x)dx \right) f_Y(y)dy
\end{aligned}
$$

By differentiating, we obtain the probability density function of Z

$$
\begin{aligned}
f_Z(z) = \frac{d}{dz} F_Z(z) &= \int_{-\infty}^{0} \left(-y f_X(yz)\right) f_Y(y)dy + \int_0^{\infty} y f_X(yz) f_Y(y)dy \\
&= \int_{-\infty}^{+\infty} |y| f_X(yz) f_Y(y)dy
\end{aligned}
$$

In the case where X and Y are uniformly distributed on the intervals $[a_I, b_I]$ and $[a_J, b_J]$ respectively, we have $Z \sim \frac{U_{[a_I,b_I]}}{U_{[a_J,b_J]}}$. We are still looking for the distribution of Z.

First of all, we assume $I = J$.

$$
\begin{aligned}
f_X(yz) f_Y(y) &= U_{[a_I,b_I]}(yz) U_{[a_I,b_I]}(y) \\
&= \left(\frac{1}{b_I - a_I} \right)^2 \mathbb{1}_{\left\{ [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_I, b_I] \right\}}(y)
\end{aligned}
$$

Since $U_{[a_I,b_I]}(yz) = 1$ if $yz \in [a_I, b_I] \Leftrightarrow y \in [\frac{a_I}{z}, \frac{b_I}{z}]$. Thus, we distinguish two cases :

- $z \in \left[\frac{a_I}{b_I}, 1\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_I, b_I] = \left[\frac{a_I}{z}, b_I\right]$
- $z \in \left[1, \frac{b_I}{a_I}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_I, b_I] = \left[a_I, \frac{b_I}{z}\right]$

Therefore

$$
f_Z(z) = \begin{cases} \int_0^{\infty} \frac{|y|}{(b_I - a_I)^2} \mathbb{1}_{\left\{ [\frac{a_I}{z}, b_I] \right\}}(y)dy = \frac{b_I^2 - \left(\frac{a_I}{z}\right)^2}{2(b_I - a_I)^2} \text{ if } z \in \left[\frac{a_I}{b_I}, 1\right] \\ \int_0^{\infty} \frac{|y|}{(b_I - a_I)^2} \mathbb{1}_{\left\{ \left[a_I, \frac{b_I}{z}\right] \right\}}(y)dy = \frac{\left(\frac{b_I}{z}\right)^2 - a_I^2}{2(b_I - a_I)^2} \text{ if } z \in \left[1, \frac{b_I}{a_I}\right] \end{cases}
$$

Now we consider the case where $I \neq J$

$$
f_X(yz) f_Y(y) = \frac{1}{(b_I - a_I)(b_J - a_J)} \mathbb{1}_{\left\{ [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] \right\}}(y)
$$

Thus when $N_I < N_J$

- $z \in \left[\frac{a_I}{b_J}, \frac{b_I}{b_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = \left[\frac{a_I}{z}, b_J\right] \Rightarrow f_Z(z) = \frac{b_J^2 - \left(\frac{a_I}{z}\right)^2}{2N_I N_J}$

- $z \in \left[\frac{b_I}{b_J}, \frac{a_I}{a_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = \left[\frac{a_I}{z}, \frac{b_I}{z}\right] \Rightarrow f_Z(z) = \frac{\left(\frac{b_I}{z}\right)^2 - \left(\frac{a_I}{z}\right)^2}{2N_I N_J}$

- $z \in \left[\frac{a_I}{a_J}, \frac{b_I}{a_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = \left[a_J, \frac{b_I}{z}\right] \Rightarrow f_Z(z) = \frac{\left(\frac{b_I}{z}\right)^2 - a_J^2}{2N_I N_J}$

and when $N_I > N_J$

- $z \in \left[\frac{a_I}{b_J}, \frac{a_I}{a_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = \left[\frac{a_I}{z}, b_J\right] \Rightarrow f_Z(z) = \frac{b_J^2 - \left(\frac{a_I}{z}\right)^2}{2N_I N_J}$
- $z \in \left[\frac{a_I}{a_J}, \frac{b_I}{b_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = [a_J, b_J] \Rightarrow f_Z(z) = \frac{b_J^2 - a_J^2}{2N_I N_J}$
- $z \in \left[\frac{b_I}{b_J}, \frac{b_I}{a_J}\right] \Rightarrow [\frac{a_I}{z}, \frac{b_I}{z}] \cap [a_J, b_J] = \left[a_J, \frac{b_I}{z}\right] \Rightarrow f_Z(z) = \frac{\left(\frac{b_I}{z}\right)^2 - a_J^2}{2N_I N_J}$

16.6 Evolution of $\mathbb{E}(E(G))$ with respect to q

In the case of two groups, if the group of size n_1 is impacted, we have

$$\begin{aligned}
\mathbb{E}(DC \ / \ \{\bar{S}^1\}) - \mathbb{E}(DC \ / \ \{S^1\}) &= \frac{n_1(n_1-1)p}{2}(1-\mathbb{P}_{11}) + \frac{n_2(n_2-1)p}{2}(1-\mathbb{P}_{22}) + n_1 n_2 \frac{p}{2}(1-\mathbb{P}_{12}) \\
&\qquad - \frac{n_2(n_2-1)p}{2}(1-\mathbb{P}_{22}) - n_1 n_2 \\
&= \frac{n_1(n_1-1)p}{2}(1-\mathbb{P}_{11}) + n_1 n_2 \left(\frac{p}{2}(1-\mathbb{P}_{12}) - 1\right) \\
&= n_1 \left((n_1-1)\frac{p}{2}(1-\mathbb{P}_{11}) + (n-n_1)\left(\frac{p}{2}(1-\mathbb{P}_{11}-\mathbb{P}_{12}+\mathbb{P}_{11}) - 1\right)\right) \\
&= n_1 \left((n-1)\frac{p}{2}(1-\mathbb{P}_{11}) + n_2 \left(\underbrace{\frac{p}{2}(\mathbb{P}_{11}-\mathbb{P}_{12}) - 1}_{<0}\right)\right)
\end{aligned}$$

Similarly, in the case of three groups, If two of them (n_1 and n_2) are impacted, we have

$$
\begin{aligned}
\mathbb{E}(DC/\{\bar{S}^1,\bar{S}^2\}) - \mathbb{E}(DC/\{S^1,S^2\}) &= \sum_{k=1}^{3} \frac{n_k(n_k-1)p}{2}(1-\mathbb{P}_{kk}) + \frac{1}{2}\sum_{l,k=1}^{3} n_l n_k \frac{p}{2}(1-\mathbb{P}_{kl}) \\
&\qquad - \frac{n_3(n_3-1)p}{2}(1-\mathbb{P}_{33}) - n_1 n_3 - n_2 n_3 \\
&= \sum_{k=1}^{2} \frac{n_k(n_k-1)p}{2}(1-\mathbb{P}_{kk}) + n_1 n_2 \frac{p}{2}(1-\mathbb{P}_{12}) \\
&\qquad + \left(\frac{p}{2}-1\right)\left(n_1 n_3(1-\mathbb{P}_{13}) + n_2 n_3(1-\mathbb{P}_{23})\right) \\
&= \frac{n_1}{2}\left[(n_1-1)p(1-\mathbb{P}_{11}) + n_2\frac{p}{2}(1-\mathbb{P}_{12}) + \underbrace{(p-2)}_{<0} n_3(1-\mathbb{P}_{13})\right] \\
&\quad + \frac{n_2}{2}\left[(n_2-1)p(1-\mathbb{P}_{22}) + n_1\frac{p}{2}(1-\mathbb{P}_{12}) + \underbrace{(p-2)}_{<0} n_3(1-\mathbb{P}_{23})\right]
\end{aligned}
$$

Again, as $n_3 = n - (n_1 + n_2)$, the sign of this difference entirely depends on the size of the impacted part of the graph. Lower the size of $n_1 + n_2$, more likely the shock will increase the number of desirability channels.

16.7 Evolution of $\mathbb{E}(E(G))$ in the case of equally sized groups

We want to show that $\forall k < k'$, $\mathbb{E}(E(G))|_k > \mathbb{E}(E(G))|_{k'}$:

$$
\begin{aligned}
\mathbb{E}(E(G))|_k - \mathbb{E}(E(G))|_{k'} &= \binom{\frac{n}{k}}{2}p + \binom{k}{2}\left(\frac{n}{k}\right)^2\frac{p}{2} - \binom{\frac{n}{k'}}{2}p - \binom{k'}{2}\left(\frac{n}{k'}\right)^2\frac{p}{2} \\
&= n\left(\frac{n}{k}-1\right)\frac{p}{2} + (k-1)\frac{n^2}{k}\frac{p}{4} - n\left(\frac{n}{k} + \left(\frac{n}{k'}-\frac{n}{k}\right) - 1\right)\frac{p}{2} - (k'-1)\frac{n^2}{k'}\frac{p}{4} \\
&= -n\left(\frac{n}{k'}-\frac{n}{k}\right)\frac{p}{2} + n^2\frac{p}{4}\left(\frac{1}{k'}-\frac{1}{k}\right) \\
&= n^2\frac{p}{4}\left(\frac{1}{k}-\frac{1}{k'}\right) > 0
\end{aligned}
$$

16.8 The probabilities

The different probabilities are determined here for the case no group is impacted by a shock. When a group k is impacted by a shock, we know that $\mathbb{P}_{kk} = 1$ and $\forall l \neq k$, $\mathbb{P}_{kl} = 0$.

The intragroup probability is

$$
\begin{aligned}
\mathbb{P}(\frac{1}{3} < Z_{kk} < 3) &= \frac{\frac{2}{3}b_k^2 - 2a_k^2}{N_k^2} \\
&= \frac{1}{N_k^2}\left(\frac{2}{3}\left(\bar{A}^2 + \left(\frac{N_k}{2}\right)^2 + N_k\bar{A}\right) - 2\left(\bar{A}^2 + \left(\frac{N_k}{2}\right)^2 - N_k\bar{A}\right)\right) \\
&= \frac{1}{N_k^2}\left(-\frac{4}{3}\bar{A}^2 + \frac{8}{3}\bar{A}N_k - \frac{1}{3}N_k^2\right) \\
&= -\frac{4}{3}\frac{\bar{A}^2}{N_k^2} + \frac{8}{3}\frac{\bar{A}}{N_k} - \frac{1}{3} \\
&= -\frac{4}{3}\left(\frac{\bar{A}}{N_k} - 1\right)^2 + 1
\end{aligned}
$$

The intergroup probability is

$$
\mathbb{P}(\frac{1}{3} < Z_{kl} < 3) = \frac{1}{2N_kN_l}\left(2\underbrace{(b_kb_l + a_ka_l)}_{B} - \left[\underbrace{3(a_k^2 + a_l^2) + \frac{1}{3}(b_k^2 + b_l^2)}_{C}\right]\right)
$$

$$
\begin{aligned}
N_lN_k &= (b_k - a_k)(b_l - a_l) \\
&= b_kb_l - b_ka_l - a_kb_l + a_ka_l \\
\Leftrightarrow B &= N_lN_k + b_ka_l + a_kb_l \\
&= N_lN_k + \left(\bar{A} + \frac{N_k}{2}\right)\left(\bar{A} - \frac{N_l}{2}\right) + \left(\bar{A} - \frac{N_k}{2}\right)\left(\bar{A} + \frac{N_l}{2}\right) \\
&= N_lN_k + 2\bar{A}^2 - \frac{N_kN_l}{2} \\
&= \frac{N_kN_l}{2} + 2\bar{A}^2
\end{aligned}
$$

And,

$$
\begin{aligned}
C &= 3\left(2\bar{A}^2 + \frac{N_k^2 + N_l^2}{4} - \bar{A}(N_k + N_l)\right) + \frac{1}{3}\left(2\bar{A}^2 + \frac{N_k^2 + N_l^2}{4} + \bar{A}(N_k + N_l)\right) \\
&= \frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_k^2 + N_l^2) - \frac{8}{3}\bar{A}(N_k + N_l)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathbb{P}_{kl} = \mathbb{P}(\frac{1}{3} < Z_{kl} < 3 — \bar{S}^k, \bar{S}^l) &= \frac{1}{2N_kN_l}\left(2(b_kb_l + a_ka_l) - \left[3(a_k^2 + a_l^2) + \frac{1}{3}(b_k^2 + b_l^2)\right]\right) \\
&= \frac{1}{2N_kN_l}\left(N_kN_l + 4\bar{A}^2 - \left[\frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_k^2 + N_l^2) - \frac{8}{3}\bar{A}(N_k + N_l)\right]\right) \\
&= \frac{4}{3} - \frac{1}{6N_kN_l}\left(\frac{8}{5}\bar{A}^2 + \frac{5}{2}\left(N_k + N_l - \frac{8}{5}\bar{A}\right)^2\right) \\
&= \frac{1}{2} - \frac{1}{N_kN_l}\frac{4}{3}\bar{A}^2 - \frac{5}{12}\left(\frac{N_k}{N_l} + \frac{N_l}{N_k}\right) + \frac{4}{3}\left(\frac{\bar{A}}{N_l} + \frac{\bar{A}}{N_k}\right)
\end{aligned}
$$

16.9 Probabilities properties

We have the following properties

1. $\forall k, l \in [\![1, K]\!]$

$$
\begin{aligned}
\mathbb{P}_{kl} &= \frac{1}{2} - \frac{1}{N_kN_l}\frac{4}{3}\bar{A}^2 - \frac{5}{12}\left(\frac{N_k}{N_l} + \frac{N_l}{N_k}\right) + \frac{4}{3}\left(\frac{\bar{A}}{N_l} + \frac{\bar{A}}{N_k}\right) \\
&= -\frac{4}{3}\frac{\bar{A}^2}{N_k^2} + \frac{8}{3}\frac{\bar{A}}{N_k} - \frac{1}{3} + \frac{1}{2} - \frac{4\bar{A}^2}{3}\left(\frac{1}{N_kN_l} - \frac{1}{N_k^2}\right) - \frac{5}{12}\left(\frac{N_k}{N_l} + \frac{N_l}{N_k}\right) + \frac{4}{3}\left(\frac{\bar{A}}{N_l} - \frac{\bar{A}}{N_k}\right) + \frac{1}{3} \\
&= \mathbb{P}_{kk} + \frac{5}{6} - \frac{4\bar{A}^2}{3}\left(\frac{1}{N_kN_l} - \frac{1}{N_k^2}\right) - \frac{5}{12}\left(\frac{N_k}{N_l} + \frac{N_l}{N_k}\right) + \frac{4}{3}\left(\frac{\bar{A}}{N_l} - \frac{\bar{A}}{N_k}\right) \\
&= \mathbb{P}_{kk} - \frac{4\bar{A}^2}{3}\left(\frac{1}{N_kN_l} - \frac{1}{N_k^2}\right) + \frac{5}{12}\left(\frac{2N_kN_l - N_k^2 - N_l^2}{N_kN_l}\right) + \frac{4\bar{A}}{3}\left(\frac{N_k - N_l}{N_kN_l}\right) \\
&= \mathbb{P}_{kk} - \frac{4\bar{A}^2}{3}\left(\frac{1}{N_kN_l} - \frac{1}{N_k^2}\right) + \frac{(N_k - N_l)}{N_kN_l}\left(\frac{4\bar{A}}{3} - \frac{5}{12}(N_k - N_l)\right) \\
\Leftrightarrow \mathbb{P}_{kl} - \mathbb{P}_{kk} &= \frac{(N_k - N_l)}{N_kN_l}\left(\frac{4\bar{A}}{3}\underbrace{\left(1 - \frac{\bar{A}}{N_k}\right)}_{>0} - \frac{5}{12}\underbrace{(N_k - N_l)}_{<0}\right)
\end{aligned}
$$

Therefore in the case $N_k < N_l$, we have $\mathbb{P}_{kk} > \mathbb{P}_{kl}$. Notice that in the case $N_k > N_l$, we have $\mathbb{P}_{kl} > \mathbb{P}_{kk}$ if and only if we have $N_k - N_l < \frac{16\bar{A}}{5}\left(1 - \frac{\bar{A}}{N_k}\right) \Leftrightarrow \frac{1}{3n}(n_k - n_l) < \frac{8}{5}\left(1 - \frac{1}{1+\frac{2n_k}{3n}}\right)$.

2. Moreover, we have

$$\begin{aligned}
\mathbb{P}_{km} - \mathbb{P}_{kl} &= \frac{4}{3}\bar{A}^2\left(\frac{1}{N_kN_l} - \frac{1}{N_kN_m}\right) - \frac{5}{12}\left(\frac{N_k}{N_m} + \frac{N_m}{N_k} - \frac{N_k}{N_l} - \frac{N_l}{N_k}\right) + \frac{4}{3}\left(\frac{\bar{A}}{N_m} - \frac{\bar{A}}{N_l}\right) \\
&= \frac{4\bar{A}^2}{3N_k}\left(\frac{N_m - N_l}{N_lN_m}\right) - \frac{5}{12N_lN_m}\left(\frac{N_k^2N_l + N_m^2N_l - N_k^2N_m - N_l^2N_m}{N_k}\right) + \frac{4\bar{A}}{3}\left(\frac{N_l - N_m}{N_mN_l}\right) \\
&= \frac{4\bar{A}}{3}\left(\frac{N_m - N_l}{N_lN_m}\right)\left(\frac{\bar{A}}{N_k} - 1\right) + \frac{5}{12}\frac{(N_m - N_l)}{N_lN_m}\left(N_k - \frac{N_lN_m}{N_k}\right) \\
&= \left(\frac{N_m - N_l}{N_lN_m}\right)\left[\frac{4\bar{A}}{3}\underbrace{\left(\frac{\bar{A}}{N_k} - 1\right)}_{<0} + \frac{5}{12}\left(N_k - \frac{N_lN_m}{N_k}\right)\right] \qquad (22)
\end{aligned}$$

Therefore, $N_k < N_l < N_m \Rightarrow \mathbb{P}_{km} < \mathbb{P}_{kl}$.

3. Finally, we want to show that under specific conditions $n_k < n_l < n_m$ implies $\mathbb{P}_{mk} < \mathbb{P}_{ll}$.

   We have for any triple (n_k, n_l, n_m) such that $n_k < n_l < n_m$ and $\frac{1}{3n}(n_m - n_k) > \frac{8}{5}\left(1 - \frac{1}{1+\frac{2}{3n}\left(n_m - \frac{1}{n}\right)}\right)$, we have $\mathbb{P}_{mk} < \mathbb{P}_{mm}$ and as the probability $\mathbb{P}_{xx}$ is a decreasing function of n_x, we have $\forall x$ such that $n_x < n_m$, $\mathbb{P}_{xx} > \mathbb{P}_{mm}$. So in particular, we have $\mathbb{P}_{ll} > \mathbb{P}_{mm}$ and that implies $\mathbb{P}_{mk} < \mathbb{P}_{ll}$.

16.10 Expected number of desirability channels

16.10.1 Determination of $\sum_{k=1}^{K} n_k(n_k-1)\mathbb{P}_{kk}$

$$
\begin{aligned}
\sum_{k=1}^{K} n_k(n_k-1)\mathbb{P}_{kk} &= \sum_{k=1}^{K} n_k(n_k-1)\left(-\frac{4}{3}\frac{\bar{A}^2}{N_k^2}+\frac{8}{3}\frac{\bar{A}}{N_k}-\frac{1}{3}\right)\\
&= n^2\sum_{k=1}^{K}\frac{3}{2}\left(\frac{N_k}{\bar{A}}-1\right)\frac{3}{2}\left(\frac{N_k}{\bar{A}}-1-\frac{2}{3n}\right)\left(-\frac{4}{3}\frac{\bar{A}^2}{N_k^2}+\frac{8}{3}\frac{\bar{A}}{N_k}-\frac{1}{3}\right)\\
&= n^2\frac{9}{4}\sum_{k=1}^{K}\left(\left(\frac{N_k}{\bar{A}}\right)^2-\frac{N_k}{\bar{A}}\left(2+\frac{2}{3n}\right)+1+\frac{2}{3n}\right)\left(-\frac{4}{3}\frac{\bar{A}^2}{N_k^2}+\frac{8}{3}\frac{\bar{A}}{N_k}-\frac{1}{3}\right)\\
&= n^2\frac{9}{4}\sum_{k=1}^{K}\left\{-\frac{4}{3}+\frac{8N_k}{3\bar{A}}-\frac{1}{3}\left(\frac{N_k}{\bar{A}}\right)^2+\frac{\bar{A}}{N_k}\frac{4}{3}\left(2+\frac{2}{3n}\right)-\frac{8}{3}\left(2+\frac{2}{3n}\right)+\frac{1}{3}\left(2+\frac{2}{3n}\right)\frac{N_k}{\bar{A}}\right\}\\
&\qquad +n^2\frac{9}{4}\sum_{k=1}^{K}\left\{-\frac{4}{3}\left(1+\frac{2}{3n}\right)\left(\frac{\bar{A}}{N_k}\right)^2+\frac{8}{3}\left(1+\frac{2}{3n}\right)\frac{\bar{A}}{N_k}-\frac{1}{3}\left(1+\frac{2}{3n}\right)\right\}\\
&= n^2\frac{9}{4}\sum_{k=1}^{K}\left\{-7-\frac{2}{n}+\frac{1}{3}\left(10+\frac{2}{3n}\right)\frac{N_k}{\bar{A}}-\frac{1}{3}\left(\frac{N_k}{\bar{A}}\right)^2+\frac{4}{3}\left(4+\frac{2}{n}\right)\frac{\bar{A}}{N_k}\right\}\\
&\qquad +n^2\frac{9}{4}\sum_{k=1}^{K}\left\{-\frac{4}{3}\left(1+\frac{2}{3n}\right)\left(\frac{\bar{A}}{N_k}\right)^2\right\}\\
&= n^2\frac{9}{4}\left\{-\left(7+\frac{2}{n}\right)K+\frac{1}{3}\left(10+\frac{2}{3n}\right)(K+\frac{2}{3})-\frac{1}{3}\left(K+\frac{4}{3}+\frac{4}{9}\sum_{k=1}^{K}\frac{n_k^2}{n^2}\right)\right\}\\
&\qquad +n^2\frac{9}{4}\left\{\frac{4}{3}\left(4+\frac{2}{n}\right)\sum_{k=1}^{K}\frac{\bar{A}}{N_k}-\sum_{k=1}^{K}\frac{4}{3}\left(1+\frac{2}{3n}\right)\sum_{k=1}^{K}\left(\frac{\bar{A}}{N_k}\right)^2\right\}\\
&= n^2\frac{9}{4}\left\{-\left(4+\frac{16}{9n}\right)K+\left(\frac{16}{9}+\frac{4}{27n}\right)-\frac{4}{27}\sum_{k=1}^{K}\frac{n_k^2}{n^2}+\frac{4}{3}\left(4+\frac{2}{n}\right)\sum_{k=1}^{K}\frac{\bar{A}}{N_k}\right\}\\
&\qquad +n^2\frac{9}{4}\sum_{k=1}^{K}\left\{-\frac{4}{3}\left(1+\frac{2}{3n}\right)\sum_{k=1}^{K}\left(\frac{\bar{A}}{N_k}\right)^2\right\}\\
&= Q_1(K,(n_i)_{i\in[\![1,K]\!]})
\end{aligned}
$$

with $\mathbb{P}(\frac{1}{3}<Z_{kk}<3)=\mathbb{P}_{kk}$. Note that we have $\forall k\in[\![1,K]\!], \frac{1}{n}<\frac{n_k}{n}<1 \Rightarrow \frac{K}{n^2}<\frac{1}{n^2}\sum_{k=1}^{K} n_k^2<1$.

For the determination of the quantity $\sum_{\substack{k,l\\k\neq l}} n_k n_l \mathbb{P}_{kl}$, we assume $\forall (k,l)\in[\![1,n]\!]^2,\ N_k\neq N_l$ since

$N_k = N_l \Rightarrow \mathbb{P}_{kl} = \mathbb{P}_{kk} = \mathbb{P}_{ll}$. We have

$$\begin{aligned}
\sum_{\substack{k,l \\ k\neq l}} n_k n_l \mathbb{P}_{kl} &= n^2 \frac{9}{4} \sum_{\substack{k,l \\ k\neq l}} \left(\frac{N_k}{\bar{A}} - 1\right)\left(\frac{N_l}{\bar{A}} - 1\right)\mathbb{P}_{kl} \\
&= n^2 \frac{9}{4} \sum_{\substack{k,l \\ k\neq l}} \left(\frac{N_k N_l}{\bar{A}^2} - \frac{N_k + N_l}{\bar{A}} + 1\right)\mathbb{P}_{kl} \\
&= n^2 \frac{9}{4} \left\{ \underbrace{\sum_{\substack{k,l \\ k\neq l}} \frac{N_k N_l}{\bar{A}^2}\mathbb{P}_{kl}}_{E} - \underbrace{\sum_{\substack{k,l \\ k\neq l}} \frac{N_k + N_l}{\bar{A}}\mathbb{P}_{kl}}_{F} + \sum_{\substack{k,l \\ k\neq l}} \mathbb{P}_{kl} \right\}
\end{aligned}$$

With $\mathbb{P}(\frac{1}{3} < Z_{kl} < 3) = \mathbb{P}_{kl}$.

16.10.2 Determination of E

We have

$$\begin{aligned}
E &= \sum_{\substack{k,l \\ k\neq l}} \frac{N_k N_l}{\bar{A}^2}\mathbb{P}_{kl} \\
&= \underbrace{\sum_{\substack{k,l \\ k\neq l}} \left(\frac{N_k N_l}{2\bar{A}^2} + 2\right)}_{L} - \underbrace{\frac{1}{2}\sum_{\substack{k,l \\ k\neq l}} \left[\frac{20}{3} + \frac{5}{6}\left(\frac{N_k^2}{\bar{A}^2} + \frac{N_l^2}{\bar{A}^2}\right) - \frac{8}{3}\left(\frac{N_k}{\bar{A}} + \frac{N_l}{\bar{A}}\right)\right]}_{M}
\end{aligned}$$

Where[47]

[47] Notice that $\sum\limits_{\substack{k,l \\ k\neq l}} = \sum\limits_{l}\sum\limits_{\substack{k \\ k\neq l}} = \sum\limits_{l}(K-1) = K(K-1)$ and $\sum\limits_{\substack{k,l \\ k\neq l}} n_k = \sum\limits_{l}\sum\limits_{\substack{k \\ k\neq l}} n_k = \sum\limits_{l} n - n_l = (K-1)n$ and $\sum\limits_{\substack{k,l \\ k\neq l}} n_k^2 = \sum\limits_{l}\sum\limits_{\substack{k \\ k\neq l}} n_k^2 = \sum\limits_{l}\sum\limits_{k} n_k^2 - n_l^2 = (K-1)\sum\limits_{k} n_k^2$ and $\sum\limits_{\substack{k,l \\ k\neq l}} n_k n_l = \sum\limits_{l}\sum\limits_{\substack{k \\ k\neq l}} n_k n_l = \sum\limits_{l}(n - n_l)n_l = n^2 - \sum\limits_{k} n_k^2$ and $\sum\limits_{\substack{k,l \\ k\neq l}} n_k n_l (-1)^{\sigma} = 0$ (remind we have $\mathbb{P}_{kl}$ associated to this quantity and we alternatively consider $\mathbb{P}_{kl}$, then $\mathbb{P}_{lk}$, you can take the simple case with n_1, n_2, n_3 and $\sum\limits_{\substack{k\neq l \\ k,l}}^{3} n_k n_l$.

$$
\begin{aligned}
L &= \sum_{\substack{k,l \\ k \neq l}} \frac{N_k N_l}{2\bar{A}^2} + 2 \sum_{\substack{k,l \\ k \neq l}} \\
&= \frac{1}{2} \left(\sum_{\substack{k,l \\ k \neq l}} + \frac{2}{3n} \sum_{\substack{k,l \\ k \neq l}} (n_k + n_l) + \frac{4}{9n^2} \sum_{\substack{k,l \\ k \neq l}} n_k n_l \right) + 2K(K-1) \\
&= \frac{1}{2} \left(K(K-1) + \frac{4}{3}(K-1) + \frac{4}{9} - \frac{4}{9} \sum_k \frac{n_k^2}{n^2} + 4K(K-1) \right) \\
&= \frac{1}{2} \left(\left(5K + \frac{4}{3} \right) (K-1) + \frac{4}{9} - \frac{4}{9} \sum_k \frac{n_k^2}{n^2} \right)
\end{aligned}
$$

Since $N_k N_l = \bar{A}^2 \left(1 + \frac{2n_k}{3n}\right)\left(1 + \frac{2n_l}{3n}\right) = \bar{A}^2 \left(1 + \frac{2}{3} \frac{n_k + n_l}{n} + \frac{4 n_k n_l}{9n^2}\right)$. And

$$
\begin{aligned}
M &= \frac{10}{3} \sum_{\substack{k,l \\ k \neq l}} + \frac{5}{12} \left(\sum_{\substack{k,l \\ k \neq l}} \frac{N_k^2}{\bar{A}^2} + \sum_{\substack{k,l \\ k \neq l}} \frac{N_l^2}{\bar{A}^2} \right) - \frac{8}{6} \left(\sum_{\substack{k,l \\ k \neq l}} \frac{N_k}{\bar{A}} + \sum_{\substack{k,l \\ k \neq l}} \frac{N_l}{\bar{A}} \right) \\
&= \frac{10}{3} K(K-1) + \frac{5}{6} \left(\sum_{\substack{k,l \\ k \neq l}} + \frac{4}{3} \sum_{\substack{k,l \\ k \neq l}} \frac{n_k}{n} + \frac{4}{9} \sum_{\substack{k,l \\ k \neq l}} \left(\frac{n_k}{n} \right)^2 \right) - \frac{8}{3} \left(\sum_{\substack{k,l \\ k \neq l}} + \frac{2}{3} \sum_{\substack{k,l \\ k \neq l}} \frac{n_k}{n} \right) \\
&= \frac{10}{3} K(K-1) + \frac{5}{6} \left(K + \frac{4}{3} \right) (K-1) - \frac{8}{3} \left(K + \frac{2}{3} \right) (K-1) + \frac{10}{27} \sum_{\substack{k,l \\ k \neq l}} \left(\frac{n_k}{n} \right)^2 \\
&= \frac{K-1}{3} \left(10K - \frac{11}{2} K - 2 \right) + \frac{10}{27} (K-1) \sum_k \left(\frac{n_k}{n} \right)^2 \\
&= \frac{K-1}{3} \left(\frac{9}{2} K - 2 + \frac{10}{9} \sum_k \left(\frac{n_k}{n} \right)^2 \right)
\end{aligned}
$$

Finally

$$
\begin{aligned}
E &= \frac{1}{2}\left((5K+\frac{4}{3})(K-1)+\frac{4}{9}-\frac{4}{9}\sum_k \frac{n_k^2}{n^2}\right)-\frac{K-1}{3}\left(\frac{9}{2}K-2+\frac{10}{9}\sum_k\left(\frac{n_k}{n}\right)^2\right)\\
&= (K-1)\left[\frac{1}{2}(5K+\frac{4}{3})-\frac{1}{3}\left(\frac{9}{2}K-2\right)\right]-\left(\frac{10}{27}(K-1)+\frac{2}{9}\right)\sum_k \frac{n_k^2}{n^2}+\frac{2}{9}\\
&= (K-1)\left(K+\frac{4}{3}\right)-\frac{1}{27}\left(10K-4\right)\sum_k \frac{n_k^2}{n^2}+\frac{2}{9}
\end{aligned}
$$

16.10.3 Determination of F

Here, we have[48]

---

[48]Since $\sum_{\substack{k,l\\k\neq l}} \frac{N_l}{N_k} = \frac{\sum_k N_k - N_1}{N_1} + \frac{\sum_k N_k - N_2}{N_2} + \ldots = \sum_j \frac{\sum_k N_k - N_j}{N_j} = (K+\frac{2}{3})\sum_k \frac{\bar{A}}{N_k} - K$. Similarly $\sum_{\substack{k,l\\k\neq l}} \frac{N_l^2}{N_k} = \sum_j \frac{\sum_k N_k^2 - N_j^2}{N_j} = \sum_j \frac{\sum_k N_k^2}{N_j} - \sum_j N_j = \bar{A}(K+\frac{4}{3}+\frac{4}{9}\sum_k\left(\frac{n_k}{n}\right)^2)\sum_k \frac{\bar{A}}{N_k} - \bar{A}(K+\frac{2}{3})$.

$$
\begin{aligned}
F &= \sum_{\substack{k,l\\k\neq l}} \frac{1}{2N_kN_l}\left(\frac{N_k+N_l}{\bar{A}}\right)\left(2\left(\frac{N_kN_l}{2}+2\bar{A}^2\right)-\left[\frac{20}{3}\bar{A}^2+\frac{5}{6}(N_k^2+N_l^2)-\frac{8}{3}\bar{A}(N_k+N_l)\right]\right)\\
&= \frac{1}{2\bar{A}}\sum_{\substack{k,l\\k\neq l}}\left(\frac{1}{N_k}+\frac{1}{N_l}\right)\left(N_kN_l+4\bar{A}^2-\frac{20}{3}\bar{A}^2-\frac{5}{6}(N_k^2+N_l^2)+\frac{8}{3}\bar{A}(N_k+N_l)\right)\\
&= \frac{1}{2\bar{A}}\left[\sum_{\substack{k,l\\k\neq l}}\left(1-\frac{5}{6}\right)(N_k+N_l)-\frac{8}{3}\bar{A}^2\sum_{\substack{k,l\\k\neq l}}\left(\frac{1}{N_k}+\frac{1}{N_l}\right)-\frac{5}{6}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_l^2}{N_k}+\frac{N_k^2}{N_l}\right)+\frac{8}{3}\bar{A}\left(\frac{N_l}{N_k}+\frac{N_k}{N_l}\right)\right]\\
&\qquad +\frac{8}{3}\sum_{\substack{k,l\\k\neq l}}\\
&= \frac{1}{6}\left(\sum_{\substack{k,l\\k\neq l}}+\frac{2}{3}\sum_{\substack{k,l\\k\neq l}}\frac{n_k}{n}\right)-\frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\left(\frac{\bar{A}}{N_k}+\frac{\bar{A}}{N_l}\right)-\frac{5}{12\bar{A}}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_l^2}{N_k}+\frac{N_k^2}{N_l}\right)+\frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_l}{N_k}+\frac{N_k}{N_l}\right)\\
&\qquad +\frac{8}{3}K(K-1)\\
&= \left(\left(\frac{8}{3}+\frac{1}{6}\right)K+\frac{1}{9}\right)(K-1)-\frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\left(\frac{\bar{A}}{N_k}+\frac{\bar{A}}{N_l}\right)+\frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_l}{N_k}+\frac{N_k}{N_l}\right)-\frac{5}{12\bar{A}}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_l^2}{N_k}+\frac{N_k^2}{N_l}\right)\\
&= \left(\frac{17}{6}K+\frac{1}{9}\right)(K-1)-\frac{8}{3}(K-1)\sum_k\frac{\bar{A}}{N_k}+\frac{8}{3}\sum_{\substack{k,l\\k\neq l}}\frac{N_l}{N_k}-\frac{5}{6}\sum_{\substack{k,l\\k\neq l}}\frac{N_l}{N_k}\frac{N_l}{\bar{A}}\\
&= \left(\frac{17}{6}K+\frac{1}{9}\right)(K-1)-\frac{8}{3}(K-1)\sum_k\frac{\bar{A}}{N_k}+\frac{8}{3}\left(\left(K+\frac{2}{3}\right)\sum_k\frac{\bar{A}}{N_k}-K\right)\\
&\qquad -\frac{5}{6\bar{A}}\left(\bar{A}\left(K+\frac{4}{3}+\frac{4}{9}\sum_k\left(\frac{n_k}{n}\right)^2\right)\sum_k\frac{\bar{A}}{N_k}-\bar{A}(K+\frac{2}{3})\right)\\
&= \left(\frac{17}{6}K+\frac{1}{9}\right)(K-1)-\frac{8}{3}K+\left[\frac{8}{3}\left(K+\frac{2}{3}-K+1\right)-\frac{5}{6}\left(K+\frac{4}{3}+\frac{4}{9}\sum_k\left(\frac{n_k}{n}\right)^2\right)\right]\sum_k\frac{\bar{A}}{N_k}\\
&\qquad -\frac{5}{6}\left(K+\frac{2}{3}\right)\\
&= \frac{17}{6}K^2-\frac{41}{9}K+\frac{4}{9}+\left(-\frac{5}{6}K+\frac{10}{3}-\frac{10}{27}\sum_k\left(\frac{n_k}{n}\right)^2\right)\sum_k\frac{\bar{A}}{N_k}
\end{aligned}
$$

16.10.4 Determination of the probabilities sum

$$
\begin{aligned}
\sum_{\substack{k,l\\k\neq l}} \mathbb{P}_{kl} &= \sum_{\substack{k,l\\k\neq l}} \frac{1}{2N_lN_k}\left(N_lN_k + 4\bar{A}^2 - \left[\frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_l^2+N_k^2) - \frac{8}{3}\bar{A}(N_l+N_k)\right]\right)\\
&= \frac{1}{2}\sum_{\substack{k,l\\k\neq l}} - \frac{4}{3}\sum_{\substack{k,l\\k\neq l}} \frac{\bar{A}}{N_k}\times\frac{\bar{A}}{N_l} - \frac{5}{12}\sum_{\substack{k,l\\k\neq l}}\left(\frac{N_k}{N_l}+\frac{N_l}{N_k}\right) + \frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\left(\frac{\bar{A}}{N_l}+\frac{\bar{A}}{N_k}\right)\\
&= \frac{1}{2}K(K-1) - \frac{4}{3}\sum_{\substack{k,l\\k\neq l}} \frac{\bar{A}}{N_k}\times\frac{\bar{A}}{N_l} - \frac{5}{6}\sum_{\substack{k,l\\k\neq l}}\frac{N_k}{N_l} + \frac{8}{3}(K-1)\sum_k \frac{\bar{A}}{N_k}\\
&= \frac{1}{2}K(K-1) - \frac{4}{3}\sum_{\substack{k,l\\k\neq l}} \frac{\bar{A}}{N_k}\times\frac{\bar{A}}{N_l} - \frac{5}{6}\left((K+\frac{2}{3})\sum_k\frac{\bar{A}}{N_k} - K\right) + \frac{8}{3}(K-1)\sum_k \frac{\bar{A}}{N_k}\\
&= \frac{1}{2}K^2 + \frac{1}{3}K + \left(\frac{11}{6}K - \frac{29}{9}\right)\sum_k\frac{\bar{A}}{N_k} - \frac{4}{3}\sum_{\substack{k,l\\k\neq l}} \frac{\bar{A}}{N_k}\times\frac{\bar{A}}{N_l}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{\substack{k,l\\k\neq l}} n_kn_l\mathbb{P}_{kl} &= n^2\frac{9}{4}\left(E - F + \sum_{\substack{k,l\\k\neq l}}\mathbb{P}_{kl}\right)\\
&= n^2\frac{9}{4}\left(-\frac{4}{3}K^2 + \frac{47}{9}K - \frac{14}{9} + \left[\frac{8}{3}K - \frac{59}{9} + \frac{10}{27}\sum_k\left(\frac{n_k}{n}\right)^2\right]\sum_k\frac{\bar{A}}{N_k} - \frac{1}{27}(10K-4)\sum_k\frac{n_k^2}{n^2}\right)\\
&\qquad - n^2\frac{9}{4}\left(\frac{4}{3}\sum_{\substack{k,l\\k\neq l}}\frac{\bar{A}}{N_k}\times\frac{\bar{A}}{N_l}\right)\\
&= Q_2(K,(n_i)_{i\in[\![1,K]\!]})
\end{aligned}
$$

16.10.5 Determination of $\mathbb{E}(DC)|_K$

$$
\begin{aligned}
\mathbb{E}(DC)|_K &= \sum_{k=1}^{K}\mathbb{E}(DC_k) + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\mathbb{E}(DC_{kl}) \\
&= \sum_{k=1}^{K}\left\{\underbrace{\mathbb{E}(DC_k/S^k)}_{=0}q + \mathbb{E}(DC_k/\bar{S}^k)(1-q)\right\} \\
&\quad +\frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\left\{\underbrace{\mathbb{E}(DC_{kl}/S^k,S^l)}_{=0}q^2 + 2\mathbb{E}(DC_{kl}/S^k,\bar{S}^l)q(1-q) + \mathbb{E}(DC_{kl}/\bar{S}^k,\bar{S}^l)(1-q)^2\right\} \\
&= \sum_{k=1}^{K}\binom{n_k}{2}(1-\mathbb{P}_{kk})\,p(1-q) + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}2n_kn_l\underbrace{\frac{p}{2}}_{\to 1}q(1-q) + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}n_kn_l\frac{p}{2}(1-\mathbb{P}_{kl})(1-q)^2 \\
&= \sum_{k=1}^{K}\frac{n_k(n_k-1)p}{2}(1-\mathbb{P}_{kk})(1-q) + q(1-q)\sum_{\substack{k,l\\k\neq l}}n_kn_l + \frac{1}{2}(1-q)^2\sum_{\substack{k,l\\k\neq l}}n_kn_l\frac{p}{2}(1-\mathbb{P}_{kl}) \\
&= (1-q)\left[\underbrace{\sum_{k=1}^{K}\frac{n_k(n_k-1)p}{2} + \sum_{\substack{k,l\\k\neq l}}\frac{n_kn_l}{2}\frac{p}{2}(1-q) + q\sum_{\substack{k,l\\k\neq l}}n_kn_l}_{G}\right] - \frac{p(1-q)}{2}\sum_{k=1}^{K}n_k(n_k-1)\mathbb{P}_{kk} \\
&\qquad - \frac{p(1-q)^2}{4}\sum_{\substack{k,l\\k\neq l}}n_kn_l\mathbb{P}_{kl}
\end{aligned}
$$

Since $\mathbb{P}\left(\left\{\frac{A_i}{A_j}\in[\frac{1}{3},3] \text{ with } A_i\sim U_{[a_k,b_k]},\ A_i\sim U_{[a_l,b_l]}\right\}/S^k\right)=0$. We have[49]

[49] In our R code $Afun=\sum_{k=1}^{K}n_k(n_k-1)\mathbb{P}_{kk}$, $Bfun=\sum_{\substack{k,l\\k\neq l}}n_kn_l\mathbb{P}_{kl}$.

$$
\begin{aligned}
G &= \frac{p}{2}\sum_{k=1}^{K} n_k^2 - \frac{p}{2}\sum_{k=1}^{K} n_k + \frac{p(1-q)}{4}\sum_{\substack{k,l\\k\neq l}} n_k n_l + q\sum_{\substack{k,l\\k\neq l}} n_k n_l \\
&= \frac{p}{2}\sum_{k=1}^{K} n_k^2 - \frac{np}{2} + \left(\frac{p(1-q)}{4} + q\right)\sum_{\substack{k,l\\k\neq l}} n_k n_l \\
&= \frac{p}{2}\sum_{k=1}^{K} n_k^2 - \frac{np}{2} + \left(\frac{p(1-q)}{4} + q\right)\left(n^2 - \sum_{k=1}^{K} n_k^2\right) \\
&= \left(\frac{p(1-q)}{4} + q\right) n^2 - \frac{np}{2} + \left(\frac{p(1+q)}{4} - q\right)\sum_{k=1}^{K} n_k^2 \\
&= \frac{p}{4}\left[\left(1 - q + \frac{4q}{p}\right) n^2 - 2n + \left(1 + q - \frac{4q}{p}\right)\sum_{k=1}^{K} n_k^2\right]
\end{aligned}
$$

To conclude

$$
\begin{aligned}
\mathbb{E}(DC)|_K &= \frac{p(1-q)}{4}\left[\left(1 - q + \frac{4q}{p}\right) n^2 - 2n + n^2\left(1 + q - \frac{4q}{p}\right)\sum_{k=1}^{K} \frac{n_k^2}{n^2}\right] \\
&\qquad - \frac{p(1-q)}{2}\left(\sum_{k=1}^{K} n_k(n_k - 1)\mathbb{P}_{kk} + \frac{(1-q)}{2}\sum_{\substack{k,l\\k\neq l}} n_k n_l \mathbb{P}_{kl}\right)
\end{aligned}
$$

16.11 $\mathbb{E}(DC)$ in the asymptotic case

We know that in the special case of our model where $K = 1$ and $q = 0$, the expected number of desirability channels is equal to $E(DC) = \binom{n}{2}p(1 - \mathbb{P}_{NN})$. Notice from the definition of a threshold function that

$$
\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{Q}) = \begin{cases} 0 \text{ if } p/t \underset{n\to\infty}{\rightarrow} 0 \\ 1 \text{ if } p/t \underset{n\to\infty}{\rightarrow} \infty \end{cases}
$$

Thus, for specific ranges of p, when $n \to \infty$, the probability that the graph has a given topology tends to one and let $\mathcal{Q}_1 = \{\max \delta(G) \leq 1\}$ and $\mathcal{Q}_2 = \{\max \delta(G) \leq 2\} \cap \{\text{longest path} \leq 2\}$ be the two properties that consist in having only isolated vertices and pairs, and isolated vertices, pairs and triples respectively. For instance, we have $\mathbb{E}(DC) = \mathbb{E}(DC|\mathcal{Q}_1)\mathbb{P}(G \in \mathcal{Q}_1) + \mathbb{E}(DC|\bar{\mathcal{Q}}_1)(1 - \mathbb{P}(G \in$

$\mathcal{Q}_1$)). As $\lim_{n\to\infty} \mathbb{P}(G \in \mathcal{Q}_1) = 1$ for $p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right]$, we have $\lim_{n\to\infty} \mathbb{E}(DC) = \lim_{n\to\infty} \mathbb{E}(DC|\mathcal{Q}_1)$. We make the same reasoning for $\mathcal{Q}_2$ and we can write

$$\lim_{n\to\infty} \mathbb{E}(DC) = \begin{cases} \lim_{n\to\infty} \mathbb{E}(\#(\text{pairs}))(1-\mathbb{P}_{NN}) \text{ for } p \in \left[\frac{1}{n^2}, \frac{1}{n^{3/2}}\right] \\ \lim_{n\to\infty} \left[\mathbb{E}(\#(\text{pairs})) + 2\mathbb{E}(\#(\text{triples}))\right](1-\mathbb{P}_{NN}) \text{ for } p \in \left[\frac{1}{n^{3/2}}, \frac{1}{n^{4/3}}\right] \end{cases}$$

16.12 Determination of the probability when $I = J$

$$\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j} < 1\right)_{II} &= \int_{\frac{a_I}{b_I}}^{1} \frac{b_I^2 - \left(\frac{a_I}{z}\right)^2}{2(b_I - a_I)^2} dz \\
&= \frac{1}{2N_I^2}\left[b_I^2\left(1 - \frac{a_I}{b_I}\right) + \left[\frac{a_I^2}{z}\right]_{\frac{a_I}{b_I}}^{1}\right] \\
&= \frac{1}{2N_I^2}\left[b_I^2 - a_I b_I + a_I^2 - a_I b_I\right] \\
&= \frac{1}{2N_I^2}(b_I - a_I)^2 \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right)_{II} &= \int_{\frac{a_I}{b_I}}^{\frac{1}{3}} \frac{b_I^2 - \left(\frac{a_I}{z}\right)^2}{2(b_I - a_I)^2} dz \\
&= \frac{1}{2N_I^2}\left[b_I^2\left(\frac{1}{3} - \frac{a_I}{b_I}\right) + \left[\frac{a_I^2}{z}\right]_{\frac{a_I}{b_I}}^{\frac{1}{3}}\right] \\
&= \frac{1}{2N_I^2}\left[\frac{b_I^2}{3} - a_I b_I + 3a_I^2 - a_I b_I\right] \\
&= \frac{1}{6N_I^2}(b_I - 3a_I)^2 \\
&= \frac{1}{6N_I^2}(2\bar{A} - 4a_I)^2 \\
&= \frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2
\end{aligned}$$

$$
\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j} > 3\right)_{II} &= 1 - \mathbb{P}\left(\frac{1}{3} < \frac{A_i}{A_j} < 3\right)_{II} - \mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right)_{II} \\
&= 1 - \left(-\frac{4}{3}\left(\frac{\bar{A}}{N_I} - 1\right)^2 + 1\right) - \frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2 \\
&= \frac{4}{3}\left(\frac{\bar{A}}{N_I} - 1\right)^2 - \frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2 \\
&= \frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2 \\
&= \mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right)_{II}
\end{aligned}
$$

16.13 Determination of the probability when $I \neq J$

When $N_I < N_J$, we have

$$
\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j} < 1\right)_{IJ} &= \int_{\frac{a_I}{b_J}}^{\frac{b_I}{b_J}} \frac{b_J^2 - \left(\frac{a_I}{z}\right)^2}{2N_I N_J} dz + \int_{\frac{b_I}{b_J}}^{1} \frac{\left(\frac{b_I}{z}\right)^2 - \left(\frac{a_I}{z}\right)^2}{2N_I N_J} dz \\
&= \frac{1}{2N_I N_J}\left(b_J^2\left(\frac{b_I}{b_J} - \frac{a_I}{b_J}\right) + \left[\frac{a_I^2}{z}\right]_{\frac{a_I}{b_J}}^{\frac{b_I}{b_J}} + \left[\frac{a_I^2}{z}\right]_{\frac{b_I}{b_J}}^{1} - \left[\frac{b_I^2}{z}\right]_{\frac{b_I}{b_J}}^{1}\right) \\
&= \frac{1}{2N_I N_J}\left(b_I b_J - a_I b_J + a_I^2 \frac{b_J}{b_I} - a_I b_J + a_I^2 - a_I^2 \frac{b_J}{b_I} - b_I^2 + b_I b_J\right) \\
&= \frac{1}{2N_I N_J}\left(2b_I b_J - 2b_J a_I + a_I^2 - b_I^2\right) \\
&= \frac{1}{2N_I N_J}(a_I - b_I)(a_I + b_I - 2b_J) \\
&= -\frac{N_I(2\bar{A} - 2b_J)}{2N_I NJ} \\
&= \frac{b_J - \bar{A}}{NJ} \\
&= \frac{1}{2}
\end{aligned}
$$

and similarly when $N_J < N_I$

$$
\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j}<1\right)_{IJ} &= \int_{\frac{a_I}{b_J}}^{\frac{a_I}{a_J}} \frac{b_J^2-\left(\frac{a_I}{z}\right)^2}{2N_IN_J}dz + \int_{\frac{a_I}{a_J}}^{1} \frac{b_J^2-a_J^2}{2N_IN_J}dz \\
&= \frac{1}{2N_IN_J}\left(b_J^2\left(\frac{a_I}{a_J}-\frac{a_I}{b_J}\right)+\left[\frac{a_I^2}{z}\right]_{\frac{a_I}{b_J}}^{\frac{a_I}{a_J}}+\left(b_J^2-a_J^2\right)\left(1-\frac{a_I}{a_J}\right)\right) \\
&= \frac{1}{2N_IN_J}\left(b_J^2\frac{a_I}{a_J}-a_Ib_J+a_Ia_J-a_Ib_J+b_J^2-a_J^2-b_J^2\frac{a_I}{a_J}+a_Ia_J\right) \\
&= \frac{1}{2N_IN_J}\left(2a_Ia_J-2b_Ja_I+b_J^2-a_J^2\right) \\
&= \frac{1}{2N_IN_J}(b_J-a_J)(a_J+b_J-2a_I) \\
&= \frac{N_J(2\bar{A}-2a_I)}{2N_IN_J} \\
&= \frac{\bar{A}-a_I}{N_I} \\
&= \frac{1}{2}
\end{aligned}
$$

Whatever the sign of N_I-N_J we have

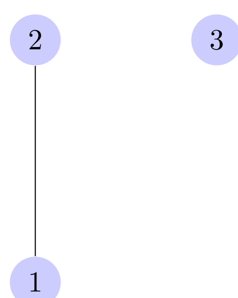
$$
\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j}<\frac{1}{3}\right)_{IJ} &= \int_{\frac{a_I}{b_J}}^{\frac{1}{3}} \frac{b_J^2-\left(\frac{a_I}{z}\right)^2}{2N_IN_J}dz \\
&= \frac{1}{2N_IN_J}\left(b_J^2\left(\frac{1}{3}-\frac{a_I}{b_J}\right)+\left[\frac{a_I^2}{z}\right]_{\frac{a_I}{b_J}}^{\frac{1}{3}}\right) \\
&= \frac{1}{2N_IN_J}\left(\frac{1}{3}b_J^2-a_Ib_J+3a_I^2-a_Ib_J\right) \\
&= \frac{1}{2N_IN_J}\left(\frac{1}{3}\left(b_J-3a_I\right)^2\right) \\
&= \frac{1}{6N_IN_J}\left(\frac{N_J+3N_I}{2}-2\bar{A}\right)^2
\end{aligned}
$$

Be careful, here $\mathbb{P}\left(\frac{A_i}{A_j}<\frac{1}{3}\right)_{IJ} \neq \mathbb{P}\left(\frac{A_i}{A_j}<\frac{1}{3}\right)_{JI}$. And whatever the sign of N_I-N_J we have

$$
\begin{aligned}
\mathbb{P}\left(\frac{A_i}{A_j} > 3\right)_{IJ} &= 1 - \mathbb{P}\left(\frac{1}{3} < \frac{A_i}{A_j} < 3\right)_{IJ} - \mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right)_{IJ} \\
&= 1 - \frac{1}{2N_I N_J}\left(N_I N_J + 4\bar{A}^2 - \left[\frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_I^2 + N_J^2) - \frac{8}{3}\bar{A}(N_I + N_J)\right]\right) \\
&\qquad - \frac{1}{6N_I N_J}\left(\frac{N_J + 3N_I}{2} - 2\bar{A}\right)^2 \\
&= \frac{1}{2N_I N_J}\left(2N_I N_J - N_I N_J - 4\bar{A}^2 + \left[\frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_I^2 + N_J^2) - \frac{8}{3}\bar{A}(N_I + N_J)\right]\right) \\
&\qquad - \frac{1}{2N_I N_J}\frac{1}{3}\left(\frac{N_J + 3N_I}{2} - 2\bar{A}\right)^2 \\
&= \frac{1}{2N_I N_J}\left(N_I N_J - 4\bar{A}^2 + \frac{20}{3}\bar{A}^2 + \frac{5}{6}(N_I^2 + N_J^2) - \frac{8}{3}\bar{A}(N_I + N_J)\right) \\
&\qquad - \frac{1}{2N_I N_J}\left(\frac{1}{12}N_J^2 + \frac{3}{4}N_I^2 + \frac{N_I N_J}{2} + \frac{4}{3}\bar{A}^2 - \frac{2}{3}\bar{A}(N_J + 3N_I)\right) \\
&= \frac{1}{2N_I N_J}\left(\frac{N_I N_J}{2} + \frac{4}{3}\bar{A}^2 + \frac{1}{12}N_I^2 + \frac{3}{4}N_J^2 - \frac{2}{3}\bar{A}(N_I + 3N_J)\right) \\
&= \frac{1}{6N_I N_J}\left(\frac{N_I + 3N_J}{2} - 2\bar{A}\right)^2 \\
&= \mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right)_{JI}
\end{aligned}
$$

16.14 Volume and graph connectivity

In this section we still assume that every agent can only trade once and we try to understand how the volume evolves with respect to the graph connectivity. Let us first consider the following case

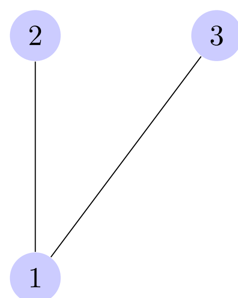


Here, there are three agents but only one link between agent 1 and agent 2. We can compute the expected value of the volume for this graph as

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\}) &= \mathbb{P}(DC_{12} = 1|\{x_1 \sim x_2\}) \\
&= \mathbb{P}\left(\frac{A_1}{A_2} \notin \left[\frac{1}{3}, 3\right]\right) \\
&= 1 - \mathbb{P}_{kk} \text{ or } 1 - \mathbb{P}_{kl}
\end{aligned}$$

So the expected volume is equal to the probability that agent 1 and agent 2 are willing to trade. Depending on if they belong to the same group, this probability will be $1 - \mathbb{P}_{kk}$ or $1 - \mathbb{P}_{kl}$.

Let us add a new link between agent 2 and agent 3 as follows



The expected volume becomes

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) &= \mathbb{E}(V_{12}|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) + \mathbb{E}(V_{13}|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) \\
&= \mathbb{P}(DC_{12} = 1)\mathbb{P}(x_2 = x_1^*) + \mathbb{P}(DC_{13} = 1)\mathbb{P}(x_3 = x_1^*) \\
&= [\mathbb{P}(DC_{12} = 1) - \mathbb{P}(DC_{13} = 1)]\,\mathbb{P}(x_2 = x_1^*) + \mathbb{P}(DC_{13} = 1)
\end{aligned}$$

Notice here that $\mathbb{P}(x_3 = x_1^*) = \mathbb{P}(|A_3 - A_1| > |A_2 - A_1|) = 1 - \mathbb{P}(x_2 = x_1^*)$

If $x_1, x_2 \in V(H_k)$ and $x_3 \in V(H_l)$

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) &= (1 - \mathbb{P}_{kk})\mathbb{P}(x_2 = x_1^*) + (1 - \mathbb{P}_{kl})\mathbb{P}(x_3 = x_1^*) \\
&= (1 - \mathbb{P}_{kk})(1 - \mathbb{P}(x_3 = x_1^*)) + (1 - \mathbb{P}_{kl})\mathbb{P}(x_3 = x_1^*) \\
&= (1 - \mathbb{P}_{kk}) + \mathbb{P}(x_3 = x_1^*)(\mathbb{P}_{kk} - \mathbb{P}_{kl})
\end{aligned}$$

In the case where $N_k < N_l$, we have $\mathbb{P}_{kk} > \mathbb{P}_{kl}$ and $\mathbb{E}(V|\{x_1 \sim x_2\} \cup \{x_1 \sim x_3\}) > 1 - \mathbb{P}_{kk}$, so the expected volume has increased compared to the setup where the agents x_1 and x_3 didn't know each other. If[50] $N_l < N_k < N_l + \frac{16\bar{A}}{5}\left(1 - \frac{\bar{A}}{N_k}\right)$, the expected volume is lower than $1 - \mathbb{P}_{kk}$,

[50]See Appendix for the proof.

so it is lowered compared to the former setup. Symmetrically, If $(x_1, x_2) \in V(H_k) \times V(H_l)$ and $x_3 \in V(H_k)$, we would have $\mathbb{E}(V|\{x_1 \sim x_2\} \cup \{x_1 \sim x_3\}) = (1 - \mathbb{P}_{kl}) + \mathbb{P}(x_3 = x_1^*)(\mathbb{P}_{kl} - \mathbb{P}_{kk})$.

If $x_1, x_2, x_3 \in V(H_k)$

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) &= (1 - \mathbb{P}_{kk})\mathbb{P}(x_2 = x_1^*) + (1 - \mathbb{P}_{kk})\mathbb{P}(x_3 = x_1^*) \\
&= (1 - \mathbb{P}_{kk})(1 - \mathbb{P}(x_3 = x_1^*)) + (1 - \mathbb{P}_{kk})\mathbb{P}(x_3 = x_1^*) \\
&= (1 - \mathbb{P}_{kk})
\end{aligned}$$

Obviously, we can extend[51] this result by saying that $\forall x$ such that $\delta(x) \neq 0$, the expected volume of $N(x)$ is the same if and only if the whole set $N(x)\backslash\{x\}$ is included into the same group.

Of course here, we recover the same expected volume than in the first case. Symmetrically, If $(x_1, x_2) \in V(H_I) \times V(H_l)$ and $x_3 \in V(H_l)$, we have $\mathbb{E}(V|\{x_1 \sim x_2\} \cup \{x_1 \sim x_3\}) = (1 - \mathbb{P}_{kl})$.

If $x_1, x_2, x_3 \in V(H_k), V(H_l), V(H_m)$

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\}, \{x_1 \sim x_3\}) &= (1 - \mathbb{P}_{kl})\mathbb{P}(x_2 = x_1^*) + (1 - \mathbb{P}_{km})\mathbb{P}(x_3 = x_1^*) \\
&= (1 - \mathbb{P}_{kl})(1 - \mathbb{P}(x_3 = x_1^*)) + (1 - \mathbb{P}_{km})\mathbb{P}(x_3 = x_1^*) \\
&= 1 - \mathbb{P}_{kl} + \mathbb{P}(x_3 = x_1^*)(\mathbb{P}_{kl} - \mathbb{P}_{km})
\end{aligned}$$

So in the case where $N_k < N_l < N_m$, we have $\mathbb{P}_{kl} > \mathbb{P}_{km}$ and $\mathbb{E}(V|\{x_1 \sim x_2\} \cup \{x_1 \sim x_3\}) > 1 - \mathbb{P}_{kl}$, so the expected volume has increased compared to the setup where the agents x_1 and x_3 didn't know each other.

16.15 Expected Volume when only one trade is permitted

Here, the neighborhood composition of each agent is totally random and we don't know *a priori* the desirability channels in the graph. Therefore, in order to get the computations more tractable, we must assume that every non desirable channels has already been removed and we have $K = 1$. Thus, we consider the optimized graph $G^* = (V^*, E^*)$ where $V^* = V(G)$ and $E^* = E(G)\backslash\left\{xy \in E(G) : \frac{A_x}{A_y} \in [\frac{1}{3}, 3]\right\}$. The probability that two vertices are linked by an edge in this graph is exactly $\mathbb{P}(\{DC_{ij} = 1\})$. If we consider a partition $\{]-\infty, \frac{1}{3}[,]3, +\infty[\}$, the total probability law leads to

[51] See Appendix for having more details.

$$
\begin{aligned}
\mathbb{E}(V_{ij}) &= \mathbb{E}(V_{ij}|\{DC_{ij}=1\})\mathbb{P}(\{DC_{ij}=1\}) + \mathbb{E}(V_{ij}|\{DC_{ij}=0\})\mathbb{P}(\{DC_{ij}=0\}) \\
&= \mathbb{P}(\{V_{ij}=1\}|\{DC_{ij}=1\})\mathbb{P}(\{DC_{ij}=1\}) \\
&= \mathbb{P}\left(\{V_{ij}=1\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) \mathbb{P}\left(\frac{A_i}{A_j} < \frac{1}{3}\right) p \\
&\qquad\qquad + \mathbb{P}\left(\{V_{ij}=1\} — \left\{\frac{A_i}{A_j} > 3\right\}, \{x_i \sim x_j\}\right) \mathbb{P}\left(\frac{A_i}{A_j} > 3\right) p
\end{aligned}
$$

Now, as the events $\{x_i = x_j^*\}$ and $\{x_i^* = x_j\}$ are conditionally independent on the event $\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}$, we have

$$
\begin{aligned}
\mathbb{P}\left(\{V_{ij}=1\} — \{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) &= \mathbb{P}\left(\{x_i = x_j^*\} \cap \{x_i^* = x_j\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) \\
&= \mathbb{P}\left(\{x_i = x_j^*\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) \\
&\qquad \times \mathbb{P}\left(\{x_i^* = x_j\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right)
\end{aligned}
$$

As we mentioned in section (13), when only one trade is permitted, traders always prefer to exchange with sellers rather than with buyers. Thus, the probability that a buyer is elected as a first best choice is zero when at least one seller exists in the neighborhood. More precisely, the only way a buyer can be chosen by an agent is when the agent is a pure seller. Similarly, the probability that a seller is elected only depends on the number of other sellers there are in the neighborhood.

$$
\begin{aligned}
\mathbb{P}\left(\{x_i = x_j^*\}|\{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) &= \mathbb{P}\left(\forall x_k \in N^*(x_j)\backslash\{x_i\},\, x_k \prec x_i \text{ for } x_j \;/\; \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) \\
&= \mathbb{P}\left(\nexists y \in N^*(x_j)\backslash\{x_i\} : \frac{A_y}{A_j} > 3\right) \mathbb{P}(A_k > A_i)^{|N^*(x_j)|-1} \\
&= \prod_{\substack{y \in N^*(x_j) \\ y \neq x_i}} \mathbb{P}\left(\frac{A_y}{A_j} < \frac{1}{3}\right) \left(\frac{1}{2}\right)^{|N^*(x_j)|-1} \\
&= \left(\frac{1}{2}\right)^{|N^*(x_j)|-1} \left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_j)|-1}
\end{aligned}
$$

And

$$
\begin{aligned}
\mathbb{P}\left(\{x_j = x_i^*\}|\{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) &= \mathbb{P}\left(\forall x_k \in N^*(x_i) \backslash \{x_j\},\, x_k \prec x_j \text{ for } x_i \;/\; \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) \\
&= \sum_{m=1}^{|N^*(x_i)|} \mathbb{P}\left(\{x_j = x_i^*\}|\{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{|N^s(x_i)| = m\}\right) \\
&\qquad \times \mathbb{P}(\{|N^s(x_i) \backslash \{x_j\}| = m-1\}) \\
&= \sum_{m=1}^{|N^*(x_i)|} \mathbb{P}(A_k < A_j)^{m-1} \mathbb{P}\left(\frac{A_i}{A_k} < \frac{1}{3}\right)^{m-1} \mathbb{P}\left(\frac{A_i}{A_l} > 3\right)^{|N^*(x_i)|-m} \\
&= 2\left(1 - \left(\frac{1}{2}\right)^{|N^*(x_i)|}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_i)|-1}
\end{aligned}
$$

with $N^s(x_i) = \left\{y \in N(x_i) : \frac{A_y}{A_i} > 3\right\}$. Therefore, we have

$$
\mathbb{P}\left(\{V_{ij} = 1\}\text{—}\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) = \left(\left(\frac{1}{2}\right)^{|N^*(x_j)|-2} - \left(\frac{1}{2}\right)^{N_{ij}^*-2}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{N_{ij}^*-2}
$$

where $N_{ij}^* = |N^*(x_j)| + |N^*(x_i)|$. Symetrically

$$
\mathbb{P}\left(\{V_{ij} = 1\}\text{—}\left\{\frac{A_i}{A_j} > 3\right\}, \{x_i \sim x_j\}\right) = \left(\left(\frac{1}{2}\right)^{|N^*(x_i)|-2} - \left(\frac{1}{2}\right)^{N_{ij}^*-2}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{N_{ij}^*-2}
$$

And

$$
\mathbb{E}(V_{ij}) = \left(\left(\frac{1}{2}\right)^{|N^*(x_i)|-2} + \left(\frac{1}{2}\right)^{|N^*(x_j)|-2} - \left(\frac{1}{2}\right)^{N_{ij}^*-3}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{N_{ij}^*-1} p
$$

16.15.1 Expected volume in the asymptotic case

As $\mathbb{P}_{II} = \mathbb{P}_{IJ} = \mathbb{P}_{NN}$, we have $\mathbb{P}(V_{ij} = 1) = \mathbb{P}(V_{ij} = 1)_{NN}$ and the volume for an isolated pair is[52] the same whatever if the agents are allowed to trade once or twice at most.

[52]Notice, we have $\frac{\bar{A}}{N_N} = \frac{3}{5-2/n}$.

$$\begin{aligned}
\mathbb{E}(V_{ij}=1|\{x_i\sim x_j\})|_{\text{pair}} &= \mathbb{E}(DC_{ij}|\{x_i\sim x_j\}) \\
&= \left(\mathbb{P}\left(\frac{A_i}{A_j}<\frac{1}{3}\right)+\mathbb{P}\left(\frac{A_i}{A_j}>3\right)\right) \\
&= \frac{4}{3}\left(1-\frac{\bar{A}}{N_I}\right)^2 \\
&= \frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2 \underset{n\to\infty}{\to} \frac{16}{75}
\end{aligned} \tag{23}$$

As we mentioned in section (12) we have for $p\in\left[\frac{1}{n^2},\frac{1}{n^{3/2}}\right]$, $\lim\limits_{n\to\infty}\mathbb{E}(V)=\lim\limits_{n\to\infty}\mathbb{E}(DC|\mathcal{Q}_1)$, so in the asymptotic case, the expected volume is

$$\begin{aligned}
\mathbb{E}(V) &= \mathbb{E}(\#(\text{pairs}))\mathbb{E}(V_{ij}=1|\{x_i\sim x_j\})|_{\text{pair}} \\
&= \frac{4}{3}\binom{n}{2}p(1-p)^{2n-4}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2
\end{aligned} \tag{24}$$

Since $p\in\left[\frac{1}{n^2},\frac{1}{n^{3/2}}\right]$, we have $\mathbb{E}(\#(\text{pairs}))\underset{n\to\infty}{\to}+\infty$, so as $\mathbb{P}(V_{ij}=1)\to\frac{16}{75}$, we have $\mathbb{E}(V)\underset{n\to\infty}{\to}+\infty$. Indeed, we have $\lim\limits_{n\to\infty}\binom{n}{2}p(1-p)^{2n-4}=\frac{p}{2}\lim\limits_{n\to\infty}n^2(1-p)^{2n}$ where $\lim\limits_{n\to\infty}(1-p)^{2n}=\lim\limits_{n\to\infty}e^{-2np(n)}$, so for $p(n)=\lambda\ln n/2n$, we have $\lim\limits_{n\to\infty}(1-p)^{2n}=n^{-\lambda}$ and we can easily show that $\lim\limits_{n\to\infty}n^2(1-p)^{2n}=n^{2-\lambda}$. Thus, as $p\in\left[\frac{1}{n^2},\frac{1}{n^{3/2}}\right]\Rightarrow\lambda<2$, we have $\lim\limits_{n\to\infty}\binom{n}{2}p^2(1-p)^{2n-4}=+\infty$.

Let us now consider the case where we also have isolated triples in the graph, that is when $p\in\left[\frac{1}{n^{3/2}},\frac{1}{n^{4/3}}\right]$. When the agents are allowed to trade once, the expected volume for a pair embedded within a triple is

$$\begin{aligned}
\mathbb{E}(V_{ij}|\{x_i\sim x_j\},\{x_i\sim x_k\})|_{\text{triples}} &= \left[2\left(\frac{1-\mathbb{P}}{2}\right)^2(1-\mathbb{P})+(1-\mathbb{P})\mathbb{P}\right] \\
&= (1-\mathbb{P})\left[2\left(\frac{1-\mathbb{P}}{2}\right)^2+\mathbb{P}\right]
\end{aligned}$$

where $1-\mathbb{P}=\mathbb{P}\left(\frac{A_i}{A_j}\notin\left[\frac{1}{3},3\right]\right)$. That implies

$$
\begin{aligned}
\mathbb{E}(V|\{x_i \sim x_j\},\{x_i \sim x_k\})|_{\text{triples}} &= 2(1-\mathbb{P})\left[2\left(\frac{1-\mathbb{P}}{2}\right)^2+\mathbb{P}\right] \\
&= (1-\mathbb{P})(1+\mathbb{P}^2) \\
&= \frac{4}{3}\left(1-\frac{\bar{A}}{N_I}\right)^2\left(2-\frac{16}{9}\left(1-\frac{\bar{A}}{N_I}\right)^4\right) \\
&= \frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\left(2-\frac{16}{9}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^4\right) \\
&\underset{n\to\infty}{\to} \frac{4}{3}\left(\frac{2}{5}\right)^2\left(2-\frac{16}{9}\left(\frac{2}{5}\right)^4\right) \qquad (25)
\end{aligned}
$$

If the agents are allowed to trade twice at most, this quantity becomes $\mathbb{E}(V|\{x_i \sim x_j\},\{x_i \sim x_k\})|_{\text{triples}} = (1-\mathbb{P})\left[\frac{3}{2}(1-\mathbb{P})^2+2\mathbb{P}\right]$. Thus, the expected volume for the whole graph in the asymptotic case, can be expressed as

$$
\begin{aligned}
\mathbb{E}(V) &= \mathbb{E}(\#(\text{pairs}))\mathbb{E}(V_{ij}|\{x_i \sim x_j\})|_{\text{pair}}+\mathbb{E}(\#(\text{triples}))\mathbb{E}(V|\{x_i \sim x_j\},\{x_i \sim x_k\})|_{\text{triples}} \\
&= \frac{4}{3}\binom{n}{2}p(1-p)^{2n-4}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2+\binom{n}{3}p^2(1-p)^{3n-8}\frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\left(2-\frac{16}{9}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^4\right)
\end{aligned}
$$

Or if the agents are allowed to trade twice at most,

$$
\mathbb{E}(V)=\frac{4}{3}\binom{n}{2}p(1-p)^{2n-4}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2+\binom{n}{3}p^2(1-p)^{3n-8}\frac{4}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\left(\frac{8}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^4+2-\frac{8}{3}\left(\frac{2-\frac{2}{n}}{5-\frac{2}{n}}\right)^2\right)
$$

16.16 Expected volume when agents can trade at most twice

Here, we assume that $K=1$ and traders are allowed to benefit from their best arbitrage opportunity.

$$
\begin{aligned}
\mathbb{E}(V_{ij}) &= \mathbb{P}\left(\{x_i = x_j^*\}\bigcap\{x_j = x_i^*\}\} \text{ — } \left\{\frac{A_i}{A_j}<\frac{1}{3}\right\},\{x_i \sim x_j\}\right)\mathbb{P}\left(\left\{\frac{A_i}{A_j}<\frac{1}{3}\right\}\right)\mathbb{P}\left(\{x_i \sim x_j\}\right) \\
&\quad +\mathbb{P}\left(\{x_i = x_j^*\}\bigcap\{x_j = x_i^*\} \text{ — } \left\{\frac{A_i}{A_j}>3\right\},\{x_i \sim x_j\}\right)\mathbb{P}\left(\left\{\frac{A_i}{A_j}>3\right\}\right)\mathbb{P}\left(\{x_i \sim x_j\}\right)
\end{aligned}
$$

As the events $\{x_i = x_j^*\}$ and $\{x_i^* = x_j\}$ are conditionally independent on the events $\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}$ and $\left\{\frac{A_i}{A_j} > 3\right\}$, we can examine separately

$$\begin{aligned}
\mathbb{P}\left(\{x_i = x_j^*\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) &= \mathbb{P}\left(\forall x_k \in N^*(x_j)\setminus\{x_i\}, x_k \prec x_i \text{ for } x_j —\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) \\
&= \sum_{m=1}^{N^*(x_j)} \mathbb{P}\left(\{x_i = x_j^*\}|\{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{|N^b(x_j)| = m\}\right) \\
&\qquad \times \mathbb{P}(\{|N^b(x_j)\setminus\{x_j\}| = m-1\}) \\
&= \sum_{m=1}^{N^*(x_j)} \mathbb{P}(A_k > A_i)^{m-1} \mathbb{P}\left(\frac{A_j}{A_k} > 3\right)^{m-1} \mathbb{P}\left(\frac{A_j}{A_l} < \frac{1}{3}\right)^{|N^*(x_j)|-m} \\
&= 2\left(1 - \left(\frac{1}{2}\right)^{|N^*(x_j)|}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_j)|-1}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{P}\left(\{x_j = x_i^*\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) &= \mathbb{P}\left(\forall x_k \in N^*(x_i)\setminus\{x_j\}, x_k \prec x_j \text{ for } x_i —\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) \\
&= \sum_{m=1}^{N^*(x_i)} \mathbb{P}\left(\{x_j = x_i^*\}|\{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{|N^s(x_i)| = m\}\right) \\
&\qquad \times \mathbb{P}(\{|N^s(x_i)\setminus\{x_j\}| = m-1\}) \\
&= \sum_{m=1}^{N^*(x_i)} \mathbb{P}(A_k < A_j)^{m-1} \mathbb{P}\left(\frac{A_i}{A_k} < \frac{1}{3}\right)^{m-1} \mathbb{P}\left(\frac{A_i}{A_l} > 3\right)^{|N^*(x_i)|-m} \\
&= 2\left(1 - \left(\frac{1}{2}\right)^{|N^*(x_i)|}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_i)|-1}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{P}\left(\{V_{ij} = 1\}—\left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{x_i \sim x_j\}\right) &= 4\left(1 - \left(\frac{1}{2}\right)^{|N^*(x_i)|}\right)\left(1 - \left(\frac{1}{2}\right)^{|N^*(x_j)|}\right)\left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{N_{ij}^*-2} \\
&= \mathbb{P}\left(\{V_{ij} = 1\}—\left\{\frac{A_i}{A_j} > 3\right\}, \{x_i \sim x_j\}\right)
\end{aligned}$$

Notice here that the conditional expected volume is the same whatever they are *sellers* or *buyers*.

We observe that these quantities are maximal[53] for $N^*_{ij} = N^*_{ij} = 2$, that is when (x_i, x_j) have only one potential partner in their neighborhood.

Finally, the expected volume is

$$\mathbb{E}(V_{ij}) = 8\left(1-\left(\frac{1}{2}\right)^{|N^*(x_i)|}\right)\left(1-\left(\frac{1}{2}\right)^{|N^*(x_j)|}\right)\left(\frac{2}{3}\left(1-\frac{\bar{A}}{N_I}\right)^2\right)^{N^*_{ij}-1}$$

However, in the case where $K > 1$ is

$$\begin{aligned}
\mathbb{E}(V)|_K &= \sum_{k=1}^{K}\mathbb{E}(V_k) + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\mathbb{E}(V_{kl}) \\
&= \sum_{k=1}^{K}\left\{\underbrace{\mathbb{E}(V_k/S^k)}_{=0}q + \mathbb{E}(V_k/\bar{S}^k)(1-q)\right\} \\
&\quad + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\left\{\underbrace{\mathbb{E}(V_{kl}/S^k, S^l)}_{=0}q^2 + 2\mathbb{E}(V_{kl}/S^k,\bar{S}^l)q(1-q) + \mathbb{E}(V_{kl}/\bar{S}^k,\bar{S}^l)(1-q)^2\right\} \\
&= \sum_{k=1}^{K}\mathbb{E}(V_k/\bar{S}^k)(1-q) + \frac{1}{2}\sum_{\substack{k,l\\k\neq l}}\left\{2\mathbb{E}(V_{kl}/S^k,\bar{S}^l)q(1-q) + \mathbb{E}(V_{kl}/\bar{S}^k,\bar{S}^l)(1-q)^2\right\}
\end{aligned}$$

16.17 Determination of the expected volume in the general case

When a lth and kth best choice are matching, we assume that the agents can only trade once and we have

$$\begin{aligned}
\mathbb{E}(V_{ij})|_{l,k \text{ best}} &= \mathbb{P}(V_{ij} = 1)|_{l,k \text{ best}} \\
&= \mathbb{P}\left(\{x_i = x_j^{l*}\}\bigcap\{x_j = x_i^{k*}\}\right)\mathbb{P}(DC_{ij} = 1)
\end{aligned}$$

Let $\Omega_l(x) = \{x^{(l-1)*}, x^{(l-2)*}, ..., x^*\} \subseteq N(x_i)$ be the set of the $(l-1)$ first best partners of x and $N^*_{-l}(x_j) = N^*(x_j)\backslash(\{x_i\}\cup\Omega_l(x_j))$ the set of the $N^*(x_i) - l$ remaining best partners.

[53] In this case we would have $\mathbb{P}(V_{ij} = 1)_{II} = \frac{10}{3}\left(1-\frac{\bar{A}}{N_I}\right)^2 p$.

$$
\begin{aligned}
\mathbb{P}\left(\{x_i = x_j^{l*}\} \mid \{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) &= \mathbb{P}\left[\forall x_k \in N^*_{-l}(x_j),\, x_k \prec x_i \prec x^{(l-1)*} \text{ for } x_j \;/\; \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right] \\
&= \mathbb{P}\left(\nexists y \in N^*_{-l}(x_j) : \frac{A_y}{A_j} > 3\right) \mathbb{P}(A_k > A_i)^{|N^*_{-l}(x_j)|} \\
&\qquad \times \mathbb{P}(\forall x_l \in \Omega_l(x_j) : x_i \prec x_l) \\
&= \left(\frac{1}{2}\right)^{|N^*_{-l}(x_j)|} \left(\frac{A_y}{A_j} < \frac{1}{3}\right)^{|N^*_{-l}(x_j)|} \mathbb{P}(\{|\Omega^s_l(x_j)| = m\}) \\
&\qquad \times \sum_{m=0}^{l-1} \mathbb{P}\left(\forall x_l \in \Omega_l(x_j) : x_i \prec x_l | \{|\Omega^s_l(x_j)| = m\}\right) \\
&= \left(\frac{1}{2}\right)^{|N^*(x_j)|-l} \left(\frac{A_y}{A_j} < \frac{1}{3}\right)^{|N^*(x_j)|-l} \\
&\qquad \times \sum_{m=0}^{l-1} \mathbb{P}\left(A_l < A_i\right)^{l-1-m} \mathbb{P}\left(\frac{A_l}{A_j} > 3\right)^m \mathbb{P}\left(\frac{A_l}{A_j} < \frac{1}{3}\right)^{|\Omega_l(x_j)|-m} \\
&= \left(\frac{1}{2}\right)^{|N^*(x_j)|-l} \left(\frac{A_y}{A_j} < \frac{1}{3}\right)^{|N^*(x_j)|-l} \\
&\qquad \times \left(\frac{1}{2}\right)^{l-1} \mathbb{P}\left(\frac{A_l}{A_j} < \frac{1}{3}\right)^{|\Omega_l(x_j)|} \sum_{m=0}^{l-1} 2^m \\
&= \left(\left(\frac{1}{2}\right)^{|N^*(x_j)|-l-1} - \left(\frac{1}{2}\right)^{|N^*(x_j)|-1}\right) \left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_j)|-1}
\end{aligned}
$$

with $|N^*_{-l}(x_j)| = |N^*(x_j)| - |\Omega^b_l(x_j)| - 1 = |N^*(x_j)| - l$ and $\Omega^s_k(x_j) = \left\{y \in \Omega_k(x_j) : \frac{A_y}{A_j} > 3\right\}$.

Similarly with $N^*_{-k}(x_i) = N^*(x_i) \backslash \left(\{x_j\} \cup \Omega_k(x_i)\right)$ the set of the $N^*(x_i) - k$ remaining best partners, we have

$$
\begin{aligned}
\mathbb{P}\left(\{x_j = x_i^{k*}\} \mid \{x_i \sim x_j\}, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) &= \mathbb{P}\left[\forall x_l \in N^*_{-k}(x_i),\, x_l \prec x_j \prec x^{(k-1)*} \text{ for } x_i \;/\; \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right] \\
&= \sum_{m=0}^{|N^*(x_i)|-k} \mathbb{P}\left(\{x_j = x_i^{k*}\}|, \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}, \{|N^s_{-k}(x_i)| = m\}\right) \\
&\qquad \times \mathbb{P}(\{|N^s_{-k}(x_i)| = m\})\mathbb{P}(\forall x_t \in \Omega_k(x_i) : x_j \prec x_t) \\
&= \sum_{m=0}^{|N^*(x_i)|-k} \mathbb{P}(A_y < A_j)^m \left(\frac{A_y}{A_i} > 3\right)^m \mathbb{P}\left(\frac{A_y}{A_i} < \frac{1}{3}\right)^{|N^s_{-k}(x_i)|-m} \\
&\qquad \times \mathbb{P}\left(\nexists t \in \Omega_k(x_i) : \frac{A_t}{A_i} < \frac{1}{3}\right) \mathbb{P}(A_t > A_j)^{|\Omega_k(x_i)|} \\
&= \left(\frac{A_y}{A_i} < \frac{1}{3}\right)^{|N^*(x_i)|-1} \left(\frac{1}{2}\right)^{k-1} \sum_{m=0}^{|N^*(x_i)|-k} \left(\frac{1}{2}\right)^m \\
&= \left(\left(\frac{1}{2}\right)^{k-2} - \left(\frac{1}{2}\right)^{|N^*(x_i)|-1}\right) \left(\frac{2}{3}\left(1 - \frac{\bar{A}}{N_I}\right)^2\right)^{|N^*(x_i)|-1}
\end{aligned}
$$

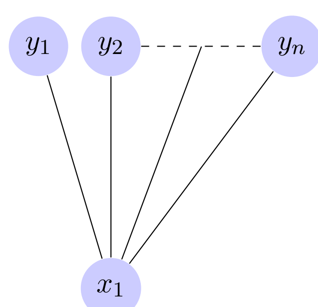
so

$$
\mathbb{P}\left(\{V_{ij} = 1\} — \left\{\frac{A_i}{A_j} < \frac{1}{3}\right\}\right) = \left(\left(\frac{1}{2}\right)^{|N^*(x_j)|+k-2} - \left(\frac{1}{2}\right)^{N^*_{ij}-1}\right) \left(2^{1+l} - 2\right) \left(\frac{1-\mathbb{P}}{2}\right)^{N^*_{ij}-2}
$$

Of course, we will find a symmetrical result for $\mathbb{P}\left(\{V_{ij} = 1\} — \left\{\frac{A_i}{A_j} > 3\right\}\right)$.

16.18 Additional simple cases

Let us consider a vertex x_1 located at the center of a star in topological terms. His neighborhood would be composed by n neighbors $\{y_1, ..., y_n\}$ such that $\forall i \in [\![1, n]\!]$, $\delta(y_i) = 1$. We assume here that x_1 can be a *trader*, that is he can realize his best arbitrage opportunity. The social network is as follows



If $x_1 \in V(H_k)$ and $N(x_1) \subseteq V(H_k)$, we have

$$\begin{aligned}
\mathbb{E}(V|\{x_1 \sim x_2\} \cap ... \cap \{x_1 \sim x_n\}) &= \sum_{i=2}^{n} \mathbb{E}(V_{1i}|\{x_1 \sim x_2\} \cap ... \cap \{x_1 \sim x_n\}) \\
&= n\mathbb{E}(V_{1i}|\{x_1 \sim x_2\} \cap ... \cap \{x_1 \sim x_n\})
\end{aligned}$$

As we are considering a star, we know that the expected volume determination is equivalent to a firs-best-matching problem, so we can use directly the proposition of section (13).

$$\mathbb{E}(V_{1i})|_G = \sum_{x=0}^{n-1} \binom{n-1}{x} \mathbb{E}\left(V_{1i} — \{DC_{1i} = 1\}, |N^*(x_1)| = x+1\right) \mathbb{P}\left(|N^*(x_1) \backslash \{DC_{1i} = 1\}| = x\right)$$

with

$$\mathbb{E}\left(V_{1i} — \{DC_{1i} = 1\}, |N^*(x_1)| = x+1\right) = 8\left(1 - \left(\frac{1}{2}\right)^{x+1}\right)\left(1 - \left(\frac{1}{2}\right)\right)\left(\frac{1 - \mathbb{P}_{kk}}{2}\right)^{x+1}$$

We remind here that these quantities are based on the condition $\{DC_{1i} = 1\}$. Therefore,

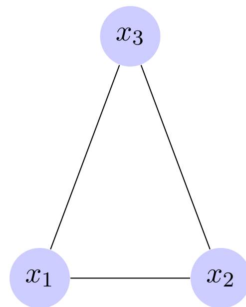
$$\mathbb{E}(V_{1i})|_G = \sum_{x=0}^{n-1} \binom{n-1}{x} \left(1 - \left(\frac{1}{2}\right)^{x+1}\right) (1 - \mathbb{P}_{kk})^{2x+1} \, \mathbb{P}_{kk}^{n-x-1} 2^{1-x}$$

And,

$$\mathbb{E}(V| \bigcap_{i \in [\![2,n]\!]} \{x_1 \sim x_i\}) = n \sum_{x=0}^{n-1} \binom{n-1}{x} \left(\left(\frac{1}{2}\right)^{x-1} - \left(\frac{1}{2}\right)^{2x}\right) (1 - \mathbb{P}_{kk})^{2x+1} \, \mathbb{P}_{kk}^{n-x-1}$$

So when the neighborhood of x belongs to the same group, the expected volume for the component $N(x) \cup \{x\}$ only depends on the size of $N(x)$. From this result, it is straightforward to recover the expression of the expected volume obtained for $n = 1, 2$.

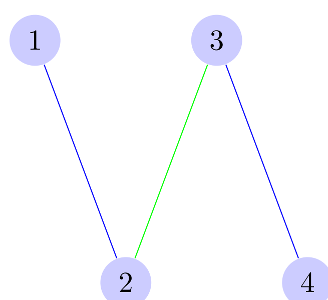
Now, let consider the following pattern



This is the topology of a complete graph K^3. As each agent can at most realize his best arbitrage opportunity but cannot trade with two neighbors of the same type (for instance two buyers), the volume will be equal to 0 or 1. Indeed, if we assume $DC_{12} = DC_{23} = DC_{13} = 1$, one agent exactly can be a *trader*, let say x_2, while x_1 and x_3 are the best choice of each other. Therefore, a *trader* will never trade in this topology as the others will always prefer to realize their single trade between them. The expected volume can be expressed as[54]

$$\begin{aligned}
\mathbb{E}(V_{ij} — \{(x_1, x_2, x_3) \sim \Delta\}) &= \frac{9}{2}\left(\frac{1-\mathbb{P}_{kk}}{2}\right)^3 (1-\mathbb{P}_{kk})^2 + 2 \times 3\left(\frac{1-\mathbb{P}_{kk}}{2}\right)^2 (1-\mathbb{P}_{kk})\mathbb{P}_{kk} \\
&= \frac{(1-\mathbb{P}_{kk})^3}{2}\left[\frac{9}{8}(1-\mathbb{P}_{kk})^2 + 3\mathbb{P}_{kk}\right]
\end{aligned}$$

Therefore, the expected volume for a K^3 topology is $3\frac{(1-\mathbb{P}_{kk})^3}{2}\left[\frac{9}{8}(1-\mathbb{P}_{kk})^2 + 3\mathbb{P}_{kk}\right]$. And



We have

[54]We denote here $\{(x_1, x_2, x_3) \sim \Delta\}$ the event the vertices belong to the complete graph K^3.

$$
\begin{aligned}
\mathbb{E}(V — (1,1)^{st}) &= \frac{9}{2}\left(\frac{1-\mathbb{P}_{kk}}{2}\right)^3(1-\mathbb{P}_{kk})^2 + 2\times 3\left(\frac{1-\mathbb{P}_{kk}}{2}\right)^2(1-\mathbb{P}_{kk})\mathbb{P}_{kk} \\
&\qquad + 2\times\left(3\left(\frac{1-\mathbb{P}_{kk}}{2}\right)^2(1-\mathbb{P}_{kk}) + 2\left(\frac{1-\mathbb{P}_{kk}}{2}\right)\mathbb{P}_{kk}\right) \\
&= \frac{(1-\mathbb{P}_{kk})}{2}\left[\frac{9}{8}(1-\mathbb{P}_{kk})^3 + 3(1-\mathbb{P}_{kk})^2\mathbb{P}_{kk} + \frac{3}{2}(1-\mathbb{P}_{kk})^2 + 2\mathbb{P}_{kk}\right] \\
&= \frac{(1-\mathbb{P}_{kk})}{2}\left[\frac{3}{2}(1-\mathbb{P}_{kk})^2\left(\frac{3}{4}(1-\mathbb{P}_{kk}) + 2\mathbb{P}_{kk} + 1\right) + 2\mathbb{P}_{kk}\right]
\end{aligned}
$$

Of course here, we would have to also consider second-order-matching to obtain the general expected volume.

16.19 Variance of the price

For all $x_i, x_j \in V(H_k)$ we have for $q = 0$

$$
\begin{aligned}
\mathrm{Var}(P(ij)|\{V_{ij}=1\}) &= \mathrm{Var}\left(\mu - \frac{1}{4}(3A_i + A_j)\sigma^2\right) \text{ or } \left(\mathrm{Var}\left(\mu - \frac{1}{4}(A_i + 3A_j)\sigma^2\right)\right) \\
&= \frac{5}{96}N_k^2\sigma^4
\end{aligned}
$$

Notice that we obtain exactly the same expression whatever who buys and sells since we consider here that all the agents belong to the same group, so their preferences are drawn from the same distribution.

Now, let consider the case where $x_i, x_j \in V(H_k) \times V(H_l)$, we have for $q = 0$

$$
\begin{aligned}
\mathrm{Var}(P(ij)|x_j \text{ buys and } x_i \text{ sells}) &= \mathrm{Var}\left(\mu - \frac{1}{4}(3A_i + A_j)\sigma^2\right) \\
&= \frac{1}{16}(9\mathrm{Var}(A_i) + \mathrm{Var}(A_j))\sigma^4 \\
&= \frac{1}{16}(9N_k^2 + N_l^2)\frac{\sigma^4}{12} \\
&= \frac{5}{96}N_k^2\sigma^4 + \frac{N_l^2 - N_k^2}{16}\frac{\sigma^4}{12}
\end{aligned}
$$

16.20 Distribution of s_i

We have $s_i = \min_{y \in N^s(x_i)\setminus V(H_k)}(Z_i)$ with $Z_i = \mu - \frac{A_y}{2}\sigma^2$

$$
\begin{aligned}
\mathbb{P}(Z_i \le z_i) &= \mathbb{P}\left(\mu - \frac{\sigma^2}{2} A_i \le z_i\right) \\
&= \mathbb{P}\left(\mu - z \le \frac{\sigma^2}{2} A_i\right) \\
&= 1 - \mathbb{P}\left(A_i \le \frac{2}{\sigma^2}(\mu - z_i)\right) \\
&= 1 - \frac{\frac{2}{\sigma^2}(\mu - z_i) - (\bar{A} - N_i/2)}{N_i} \\
&= \frac{\bar{A} + N_i/2 - \frac{2}{\sigma^2}(\mu - z_i)}{N_i}
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\mathbb{P}(s_i \le u) &= \mathbb{P}\left(\min_{y \in N^s(x_i) \setminus V(H_k)} \left(\mu - \frac{A_y}{2}\sigma^2\right) \le u\right) \\
&= \mathbb{P}(\forall y \in N^s(x_i) \setminus V(H_k), Z_i \le u) \\
&= \prod_{j \ne k}^{K} \left(\frac{\bar{A} + N_i/2 - \frac{2}{\sigma^2}(\mu - u)}{N_i}\right)^{|N^s(x_i) \cap V(H_j)|} \\
&= F_{s_i}(u)
\end{aligned}
$$

16.21 Expected value of $s^{n-n_k}(m)$

Let $\{X_1, X_2, ..., X_n\}$ be n independent and identically distributed random variables with cumulative distribution function F. We know that the density of the m-order statistic from a sample of size n is $f_{(m,n)}(x) = n\binom{n-1}{m-1}(1 - F(x))^{(n-1)-(m-1)} F(x)^{m-1} f(x)$. Hence, in the case of a uniform distribution on $[0, u]$, we obtain

$$
f_{(m,n)}(x) = n\binom{n-1}{m-1}\left(1 - \frac{x}{u}\right)^{n-m} \left(\frac{x}{u}\right)^{m-1} \frac{1}{u}
$$

And we can write

$$
\begin{aligned}
1 &= \int_0^u f_{(m,n)}(x)dx \\
\Leftrightarrow 1 &= n\binom{n-1}{m-1}\int_0^u \left(1-\frac{x}{u}\right)^{n-m}\left(\frac{x}{u}\right)^{m-1}\frac{1}{u}dx \\
\Leftrightarrow \frac{(n-m)!(m-1)!}{n!} &= \int_0^1 (1-t)^{n-m}\, t^{m-1}dt
\end{aligned}
$$

with $t = \frac{x}{u}$. Now, let us denote $r = m$ and $v = n - r + 1$, we have $\int_0^1 (1-t)^{n-m}\, t^{m-1}dt = \int_0^1 (1-t)^{v-1}\, t^{r-1}dt = \frac{(v-1)!(r-1)!}{(v+r-1)!} = \frac{\Gamma(v)\Gamma(r)}{\Gamma(r+v)}$, by the properties of the Gamma function. Finally, as the beta(r,v) distribution has the following density $\forall x \in [0,1]$, $f_{\text{beta}(r,v)}(x) = \frac{\Gamma(r+v)}{\Gamma(v)\Gamma(r)}(1-x)^{v-1}x^{r-1}$, we can write

$$
\begin{aligned}
E(s^{n-n_k}(m)) &= n\binom{n-1}{m-1}\int_0^1 x\left(1-\frac{x}{u}\right)^{n-m}\left(\frac{x}{u}\right)^{m-1}\frac{1}{u}dx \\
&= \frac{\Gamma(r+v)}{\Gamma(v)\Gamma(r)}u\int_0^1 t\,(1-t)^{v-1}\, t^{r-1}dt \\
&= u\int_0^1 t f_{\text{beta}(r,v)}(t)dt \\
&= \frac{ur}{r+v} = \frac{uk}{n+1}
\end{aligned}
$$

Since the mean of a beta(r,v) distribution is $\frac{r}{r+v}$.

16.22 Connectivity of the graph in the case $k \neq 1$

We want to show that for $p(n) > \frac{c\log(n)}{n}$, the conditions (19),(20) and (21) holds and by the theorem of Kovalenko (1971), the graph is almost surely connected when n tends to infinity.

First, we compute the main quantities involved in the three conditions. For the probability that a vertex $x_i \in V(H_k)$ is an isolated vertex, we have

$$
Q_i = \prod_{j=1}^{n}\rho_{ij} = (1-p)^{n_k-1}\left(1-\frac{p}{c}\right)^{n-n_k}
$$

For the expected number of isolated vertices, we have

$$\lambda_n = \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} (1-p)^{n_k-1}\left(1-\frac{p}{c}\right)^{n-n_k} = \sum_{k=1}^{K} n_k (1-p)^{n_k-1}\left(1-\frac{p}{c}\right)^{n-n_k} \tag{26}$$

For R_{il}, we have $\forall l \leq n$

$$R_{il} = \min_{1\leq j_1 \leq \ldots \leq j_k \leq n} \rho_{ij_1}\ldots\rho_{ij_k} = \begin{cases} (1-p)^l \text{ if } l \leq n_k \\ (1-p)^{n_k}\left(1-\frac{p}{c}\right)^{l-n_k} \; otherwise \end{cases} = (1-p)^{\min(l,n_k)}\left(1-\frac{p}{c}\right)^{l-\min(l,n_k)}$$

Now, we are able to compute the limits.

$$\begin{aligned} \max_{1\leq i\leq n} Q_i &= \left(1-\frac{p}{c}\right)^{n-\min\limits_{(n_i)_{i\leq K}}(n_i)} (1-p)^{\min\limits_{(n_i)_{i\leq K}}(n_i)-1} \\ &= \left(1-\frac{p}{c}\right)^{n} \underbrace{\left(\frac{1-p}{1-\frac{p}{c}}\right)^{\min\limits_{(n_i)_{i\leq K}}(n_i)}}_{<1} \times \frac{1}{1-p} \end{aligned}$$

We have $\left(1-\frac{p}{c}\right)^n = e^{n\log(1-\frac{p}{c})} \approx e^{-n\frac{p}{c}}$ so for $p(n) = \frac{\nu c \log(n)}{n}$, we have $\left(1-\frac{p}{c}\right)^n \approx e^{-\nu\log(n)} = n^{-\nu}$ and we have $\forall \nu > 0$, $n^{-\nu} \underset{n\to\infty}{\to} 0$ so $\max\limits_{1\leq i\leq n} Q_i \underset{n\to\infty}{\to} 0$.

Similarly, we have

$$\begin{aligned} \lim_{n\to\infty} \lambda_n &= \lim_{n\to\infty} \sum_{k=1}^{K} n_k (1-p)^{n_k-1}\left(1-\frac{p}{c}\right)^{n-n_k} \\ &= \lim_{n\to\infty} \sum_{k=1}^{K} n_k \left(\frac{1-p}{1-\frac{p}{c}}\right)^{n_k}\left(1-\frac{p}{c}\right)^{n} \times \frac{1}{1-p} \\ &= (1-p)^{-1} \lim_{n\to\infty} \sum_{k=1}^{K} (n - \sum_{j\neq k}^{K} n_j) \left(\frac{1-p}{1-\frac{p}{c}}\right)^{n_k}\left(1-\frac{p}{c}\right)^{n} \\ &= (1-p)^{-1} \lim_{n\to\infty} \left(\sum_{k=1}^{K} n \left(\frac{1-p}{1-\frac{p}{c}}\right)^{n_k}\left(1-\frac{p}{c}\right)^{n} - \sum_{k=1}^{K}\sum_{j\neq k}^{K} n_j \left(\frac{1-p}{1-\frac{p}{c}}\right)^{n_k}\left(1-\frac{p}{c}\right)^{n}\right) \end{aligned}$$

As we mentioned above for $p(n) = \frac{\nu c\log(n)}{n}$, $\left(1-\frac{p}{c}\right)^n \approx n^{-\nu}$, so $n\left(1-\frac{p}{c}\right)^n \approx n^{1-\nu} \underset{n\to\infty}{\to} 0$ if and only if $\nu > 1$. Therefore, $\lim\limits_{n\to\infty} \lambda_n = 0$ with $\nu > 1$.

Finally, we have $\forall x_i \in V(H_k)$

$$\frac{Q_i}{R_{il}} = \frac{(1-p)^{n_k-1}\left(1-\frac{p}{c}\right)^{n-n_k}}{(1-p)^{\min(l,n_k)}\left(1-\frac{p}{c}\right)^{l-\min(l,n_k)}} = \begin{cases} (1-p)^{n_k-l-1}\left(1-\frac{p}{c}\right)^{n-n_k} & \text{if } l \leq n_k \\ (1-p)^{-1}\left(1-\frac{p}{c}\right)^{n-l} & otherwise \end{cases}$$

So,

$$\sum_{i=1}^{n} \frac{Q_i}{R_{il}} = \sum_{k=1}^{K} n_k \left[(1-p)^{n_k-l-1}\left(1-\frac{p}{c}\right)^{n-n_k} \mathbb{1}_{\{l\leq n_k\}} + (1-p)^{-1}\left(1-\frac{p}{c}\right)^{n-l} \mathbb{1}_{\{l>n_k\}}\right]$$

On the basis of the results we obtained for the two first conditions, it is straightforward to show that with $p(n) > \frac{c\log(n)}{n}$, $\forall l \leq n/2$, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{Q_i}{R_{il}} = 0$ so $\lim_{n\to\infty} \sum_{l=1}^{n/2} \frac{1}{l!} \left(\sum_{i=1}^{n} \frac{Q_i}{R_{il}}\right)^l = 0$.

Thus, we have shown that for any probability p higher than $\frac{c\log(n)}{n}$, the three conditions (19),(20) and (21) holds and we can apply the Kovalenko theorem. Of course, this result is verified for any value of c, in particular for $c = 2$.

References

[1] D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi. Systemic risk and stability in financial networks. *American Economic Review*, 105(2):564–608, 2015-02.

[2] L. Adamic, C. Brunetti, J. H. Harris, and A. Kirilenko. Trading networks: Trading networks. *The Econometrics Journal*, 20(3):S126–S149, 2017-10.

[3] L. S. Bamber, O. E. Barron, and T. L. Stober. Differential interpretations and trading volume. *The Journal of Financial and Quantitative Analysis*, 34(3):369, 1999-09.

[4] D. Barczyk and M. Kredler. Inequality and asset prices. *Working Paper*, 2012.

[5] J. G. Brida, D. Matesanz, and M. N. Seijas. Network analysis of returns and volume trading in stock markets: The euro stoxx case. *Physica A: Statistical Mechanics and its Applications*, 444:751–764, 2016-02.

[6] J. Chae. Trading volume, information asymmetry, and timing information. *The Journal of Finance*, 60(1):413–442, 2005-02.

[7] T. Chordia, R. Roll, and A. Subrahmanyam. Market liquidity and trading activity. *The Journal of Finance*, 56(2):501–530, 2001-04.

[8] E. Cohen-Cole, A. Kirilenko, and E. Patacchini. Trading networks and liquidity provision. *Journal of Financial Economics*, 113(2):235–251, 2014-08.

[9] T. E. Copeland. A model of asset trading under the assumption of sequential information arrival. *The Journal of Finance*, 31(4):1149, 1976-09.

[10] M. Corominas-Bosch. Bargaining in a network of buyers and sellers. *Journal of Economic Theory*, 115(1):35–77, 2004-03.

[11] T. W. Epps. Security price changes and transaction volumes: Theory and evidence. *The American Economic Review*, 65(4):586–597, 1975.

[12] H. Föllmer. Random economies with many interacting agents. *Journal of Mathematical Economics*, 1(1):51–62, 1974-03.

[13] P. Gai and S. Kapadia. Contagion in financial networks. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 466(2120):2401–2423, 2010-08-08.

[14] C. Gollier. Wealth inequality and asset pricing. *The Review of Economic Studies*, 68(1):181–203, 2001.

[15] T. C. Green. Economic news and the impact of trading on bond prices. *The Journal of Finance*, 59(3):1201–1233, 2004-06.

[16] H. Hong, J. D. Kubik, and J. C. Stein. Thy neighbor's portfolio: Word-of-mouth effects in the holdings and trades of money managers. *The Journal of Finance*, 60(6):2801–2824, 2005-12.

[17] K. L. Judd, F. Kubler, and K. Schmedders. Asset trading volume with dynamically complete markets and heterogeneous agents. *The Journal of Finance*, 58(5):2203–2217, 2003-10.

[18] J. M. Karpoff. A theory of trading volume. *The Journal of Finance*, 41(5):1069, 1986-12.

[19] J. M. Karpoff. The relation between price changes and trading volume: A survey. *The Journal of Financial and Quantitative Analysis*, 22(1):109, 1987-03.

[20] B. Karrer and M. E. J. Newman. Random graph models for directed acyclic networks. *Physical Review E*, 80(4), 2009-10-15.

[21] A. A. Kirilenko. A network model of market prices and trading volume. *Working Paper*, page 24, 2008.

[22] C. M. Lee and B. Swaminathan. Price momentum and trading volume. *The Journal of Finance*, 55(5):2017–2069, 2000-10.

[23] A. Lo and J. Wang. Trading volume: Implications of an intertemporal capital asset pricing model. *The Journal of Finance*, page 36, 2006.

[24] A. W. Lo, H. Mamaysky, and J. Wang. Asset prices and trading volume under fixed transactions costs. *journal of political economy*, page 37, 2004.

[25] A. W. Lo and J. Wang. Stock market trading volume. *Working paper*, page 102, 2001.

[26] M. Manea. Models of bilateral trade in networks. *The Oxford Handbook of the Economics of Networks*, page 35, 2016.

[27] S. Nagel. Trading rules and trading volume. *Working Paper*, page 53, 2003.

[28] M. E. J. Newman. Mixing patterns in networks. *Physical Review E*, 67(2), 2003-02-27.

[29] T. Odean. Volume, volatility, price, and profit when all traders are above average. *Journal of Finance*, page 48, 1998.

[30] M. Pagano. Trading volume and asset liquidity. *The Quarterly Journal of Economics*, 104(2):255, 1989-05.

[31] C. Panagopoulos. Social pressure, surveillance and community size: Evidence from field experiments on voter turnout. *Electoral Studies*, 30(2):353–357, 2011-06.

[32] D. J. Pearce and P. H. J. Kelly. A dynamic topological sort algorithm for directed acyclic graphs. *Journal of Experimental Algorithmics*, 11:1.7, 2007-02-09.

[33] P. Pfleiderer and A. R. Admati. A theory of intraday patterns volume and price variability. *The Review of Financial Studies*, 1988.

[34] B. Pittel and R. Tungol. A phase transition phenomenon in a random directed acyclic graph. *Random Structures and Algorithms*, 18(2):164–184, 2001-03.

[35] G. D. Sarolli. Pump up the volume: Income risk and counter-cyclical asset trading. *Eastern Economic Journal*, 42(4):594–610, 2016-09.

[36] J. A. Scheinkman and L. Weiss. Borrowing constraints and aggregate economic activity. *Econometrica*, 54(1):23, 1986-01.

[37] H. R. Varian. Divergence of opinion in complete markets. *Journal of Finance*, 1985.

[38] J. Wang. A model of competitive stock trading volume. *Journal of Political Economy*, 1994.

Part IV

On Social Status

Social status, liquidity shocks and trading Volume

17 Introduction

"In the land of the blind, the one-eyed man is king"

Most of people has been concerned at least once in their life by a phenomenon of social comparison. They look at their entourage and measure how far they seem to be in terms of happiness compared to their own condition. When they realize that their position is strongly unbalanced compared to the rest of the population, they feel disadvantaged and suffer from this situation. Thus, in order to be more at ease among their relatives, they try to turn any comparison to their advantage by reallocating their resources to reduce the gap between them and their environment. Thus, a such phenomenon is of crucial interest for understanding social interactions and is definitely at play in economics.

In the literature, happiness is usually interpreted in terms of utility. For instance, the neoclassical view assumes that the utility of an agent only depends on the absolute level of his consumption. Of course, this assumption contradicts the idea that social comparisons play a role in the optimal choice of the consumer. Thus, many efforts have been made over the decades, to develop new preferences forms. For instance, Veblen (1899) was the first to propose that the happiness of an individual could depend on the apparent happiness of his acquaintances. More precisely, Veblen introduced the concept of conspicuous consumption by stating that people pay more attention to goods which allow to signal their wealth to the others and thereby obtain a higher social status. Thus, the main concern of an agent is how he can get the esteem of his relatives and one way to do it, is to regularly display is social rank to his fellows. An extreme case, which called the Veblen's effect, is when an individual is willing to pay higher price for functionally equivalent goods only for distinguishing itself (see Frank 1985a and Heffetz 2004). Finally, Veblen makes the distinction between two types of behaviors motivated by status needs. First, an agent with a high social rank will consume conspicuously to show that he does not belong to a lower class. Second, an agent with a low social rank will consume conspicuously to imitate the behavior of the upper class. The concept of conspicuous consumption is particularly relevant in our historical context since during the Old Regime period, a specific good was consumed to obtain greater status. Let us now shed some light on the concept of social status.

Social status is a widely recognized ranking of individuals within a social organization. People have intrinsic characteristics that bestow them with a specific position relative to the others. For instance, some of them may have higher status due to their wealth while others, could owe their status to intelligence or leadership. The key point here is that the status of an individual does exist only for those who share the same criteria of social positioning than him. Indeed, if a wealthy agent is living in a society that does not recognize wealth as impacting hierarchy, he will not benefit from any preferred position. Of course many other skills or accomplishments can generate social status and every sub-population can have different ones.

Another interesting aspect related to ranking is when we publicly sort people according to some established criteria, we make them more dissimilar and thereby magnify attraction or repulsion between them. Indeed, it is common knowledge that status entitles their owner to certain privileges and modifies the way the others behave. More specifically, we observe two effects, the first is that low status individuals are more willing to share with higher status people but the reverse is not true. Usually, people with low social position are more respectful with upper class members while people with status are more dismissive with lower class members. For instance, during the Old Regime period in France, the *Nobles* frown upon to get married with commoners. The second is an indirect effect based on a comparison with the neighborhood since an agent will suffer from a disutility when his position is dominated by his neighbors ones. Therefore, he will have a strong incentive to deviate in order to align with his entourage.

When we consider groups of people sharing the same status, we notice that the cohesion between them is of crucial interest in terms of individual behavior. Indeed, the action of one agent can potentially affects the status of his relatives. Thus, each of them monitor his neighbors and thereby a social pressure is exerted on the group members. In particular, some sociological studies support the idea that this pressure is enhanced by the size of the structure since the smaller the group, the higher the social pressure. This point is particularly relevant in the context of our empirical support as we can fairly assume that when high ranking people as *Nobles* start to deviate from established standards, they affects the whole group. Hence, their fellows have a strong incentive to convince them to comply with the social rules, otherwise they would incur the risk of being banished.

In economical terms, we can say that if status is desirable, any agent would be willing to reallocate his current consumption to have it. Therefore, the relative social positions also affect the allocation of resources in the economy and thereby the trades pattern. Indeed, if the quest for status push people to get closer to some partners rather than to others or to modify their preferences, this will potentially shut down channels commonly used to exchange, to open new ones which would be less efficient. More precisely, an agent could deviate from his best choice in terms of trade partners for a specific good to prefer someone else who is entitled by status. This point is of particular interest for explaining trading volume on markets and it is partially what our study is about.

Another interesting point is how the quest for status can alter the agents preferences in terms of risk. Indeed, as it was suggested in Friedman and Savage (1948), the marginal utility of wealth rises, and thereby the risk aversion decreases, when we reach a higher social position. In their paper, the authors investigate how broad observations about the behavior of agents who are facing alternatives involving risk, are consistent with the hypothesis of von Neumann and Morgenstern utility theory. They found that a special shape of the utility function is required to allow a such matching. A direct implication of this result is the interpretation we made above about the relationship between socioeconomic classes and risk aversion. More precisely, the author claim that people are willing to take risks to distinguish themselves. A caricatural example would be here, the case of any extreme sports where participants put their life on the line to become famous. Thus, the higher the status of an agent, the lower his risk aversion. In our historical context, this would suggest that the nobility members are less risk averse than people from the lower class. We will see this point can be tested empirically as we are able to detect the bad state of this old economy and thereby to establish the periods during which investing was more risky due to high uncertainty.

Finally, another strand of the literature initiated by Postlewaite (1998), considers that status would not be intrinsic preferences but rather an instrumental concern. In other terms, people do not value status itself but seek it because this allows to have an access to exclusive goods or to better consumption opportunities. To illustrate this last point with our empirical support, we would say that when a *bourgeois* for instance, is willing to get married with a *Noble*, he must rise his status to make this marriage possible. Thus, the status is only an instrument that provide an access to some privileges.

In this paper, we propose a model with heterogeneous agents that describes the main features of the Toulouse mills economy. More precisely, our model is in the spirit of several previous works on social status, starting from Hirsch (1977) which introduced a distinction between *positional* and *non positional* goods. More precisely, Hirsch claimed that *positional* goods derive their value from their social scarcity. Indeed, if everyone receive a medal, a such distinction cannot be used to distinguish people anymore as it becomes a common good. This concept of *positional* good and relative consumption has been extensively used in many studies as in Frank (1985b), Blanchflower and Oswald (2004), Duesenberry (1949), Frey and Stutzer (2002) or in Hopkins and Kornienko (2004). In this last paper, the authors investigate the strategical choice of an agent in terms of conspicuous consumption as his utility depends on the consumption of the others. They found at the equilibrium, that agents spend an inefficiency high amount on the *positional* good.

Another important feature of our model is that it relies on network theory to build the underlying structure that describes the social interactions. Indeed, as we mentioned above, each agent is concerned about the consumption of the others. More precisely, as he does not have information about all the population members, we assume that he only cares about the conspicuous consumption of his acquaintances. Thus, the most natural way to display these connections is to use nodes and edges as material of Graph theory. A very interesting paper and closely related to our present work is Ghiglino and Goyal (2010). They consider a networked market where individuals have private endowments and feel a disutility when their relative consumption compared to their neighbors is low for the conspicuous good. They found that equilibrium prices and consumption depend on a centrality measure of the underlying network. In our model, we consider a partition of the population composed by agents with status and agents without status. This setting fairly reproduces the two main social groups existing during the Old Regime period in Toulouse. An important difference between Ghiglino's paper and this one is that he considers arbitrary given networks while we use heterogeneous random graphs for highlighting the fact that people are more likely to know each others if they share the same status than if they belong to different social groups. This implies that the neighborhood of each agent will be more homogeneous in our model. For instance, a *Noble* will be more likely connected to other *Nobles* and a *merchant* has more low class members in his neighborhood than *Nobles*.

We also randomly draw the preferences of the agents by considering that non status people have on average a higher risk aversion than those who have a status. This is of course in accordance with the Friedmann and Salvage conclusion. Another interesting paper which also investigates how risk-taking behavior can emerge from the assumption that people care about status is Ray and Robson (2012). They show how risk-taking behavior might coexists with risk-averse behavior and derive the same findings than Friedmann and Savage (1948) without making ad hoc specifications about the utility function. In this study, we first consider a static version of our model where the agents can only exchange a *non positional* good. We assume they can only choose to buy or to sell one unit of good or to don't trade but cannot act on the *positional* good. In other terms, the status are exogenously determined and we only focus on their impact on the trades pattern. We further assume that the agent's utility is impacted by the relative status of his trade partner, positively for a higher status and negatively for a lower status. We found that although individuals would be more willing to trade with individuals who have different status levels, since such partners maximize on average their profit, the utility's component capturing their status concern lowers this incentive. Hence, we show that in some cases, the intra groups trades predominate. Second, we propose a dynamic version of our model where the agents are allowed to choose their optimum consumption in both the *non positional* and the *positional* one. Here, we slightly modify the utility function by adding a component dedicated to the relative consumption concerns. Thus, a low status agent who has a higher ranking neighbor, will have an incentive to increase his conspicuous consumption. Obviously, in order to be coherent with our historical context, the price of the *positional* good is assumed higher than the price of the *non positional* good and every agent faces a budget constraint that does not allow him to freely migrate to the upper class. Moreover, we include in the individual wealth dynamic a binary random variable that captures the disasters occurrence and can be interpreted as a liquidity shock. Thus, our model is able to explain trading volume on both agents heterogeneity in terms of status and liquidity shock.

By using the Toulouse mills data and in particular specific sales contracts to test our theoretical predictions, we found very interesting stylized facts. First, it seems that a part of the trading volume could be explained by the occurrence of the multiple exogenous shocks that have impacted the companies over the centuries. More precisely, we show that the huge *talhas* required for repairing the mills, which indicate the periods where some shareholders could experience a liquidity shock, significantly explain the trading volume. We further investigate this relationship by focusing on the different social groups, that is the *Religious*, the *Nobles*, the *Merchants* and the *Bourgeois*. We found that the religious institutions was more likely to default during such periods and thereby to sell their shares. Then, we performed a network analysis in order to highlight the interactions between the different social groups and in particular, between people who has a status and those from lower social position. We observe that majority of trades take place between the *Nobles* and the *Merchants*. We finally test the results of Ball (2001) obtained in an experimental setting. They found that average prices are higher when higher-status sellers face lower-status buyers, and lower when buyers have higher status than sellers.

This paper is organized as follows. In a first section, we present the historical context in which the Toulouse mill companies take place. More precisely, we outline the events which have played a key role in the Castle mills story and the main social groups to which every agent belong. Then, we describe the data we use in this study to test the volume dynamics. In particular, we highlight our dividends computation from the companies output and the shareholders contribution to general expenses. We also depict the different types of sales contracts available at that time and we explain how they can be used to detect liquidity shocks. In the third section, we propose a model with heterogeneous agents that capture the main features of this old economy. We justify how different individual social status lead to preferences skewness and impact the global trading pattern. Then, we introduce some descriptive statistics and stylized facts about volume and social position for this old economy before testing our predictions in the next section. Section 6 concludes.

18 Historical Background and data

A complete description of the *Bazacle* company is already provided in Le Bris et al. (2014). Le Bris, Goetzmann, Pouget and Wavasseur (2017) gives some insights into its main competitor, the *Narbonnais Castle* mills. Sicard (1953) and Goetzmann and Pouget (2010) describe why these firms can be seen as corporations. Thus, we just bring here additional historical facts about the *Narbonnais Castle* company and we depict how the shareholders behave on the market according to their social category. Our study is broadly based on the plenty of data provided by the Municipal archives and all the historical information about Mill companies from Sicard's monograph (1953). In this section, we present the data we found about the trading volume, the shareholders identity and the social class to which they belong, before to introduce the times series we use as a proxy for assessing the liquidity shocks experienced by the investors.

18.1 The economic and social environment

Let us now describe the main features of the Toulouse society during the 17th and the 18th century. The content of this section is largely inspired by the works of Godechot (1966), Thoumas-Schapira (1955) and Wolff (1958).

It seems that the population in Toulouse has decreased between the 15th and the 16th century, then remained roughly constant until 1750 before to increase by 30% over the next half-century. Here, the real question is what were the living standards of the citizens at that time and how many social groups existed. There were four main categories of people represented in Toulouse, the *nobles*, the *bourgeois*, the *religious* and the *lower classes*. The examination of the marriage contracts allows to both assess the size of the different social groups and the wealth of each of them. It appears that the *nobles* make up less than 2% of the population in 1749 and in 1785 and receive the third of the total dowries of the year. About 15% of the population is *bourgeois* and perceives 45% of the dowries. In this category, we find the dealers, the merchants, the liberal professions and the annuitants. The wealth of some of them is similar to a *noble*'s one. The majority (85%) of marriages is celebrated by people from *lower classes*, including masters manufacturers, journeymen, farmers, commoners and servants. They obtain 20% of the total dowries. Thus, during the 18th century, the inequalities are great since a small portion of the population (17%) owns the three quarters of the capital.

Following the golden age of trade in Pastel, that is starting from the second-half of the 16th century, the economy is declining while a new social class is emerging, the magistrates. At the end of the 17th century, people from lower classes are willing to belong to the *bourgeoisie* while its own members want to become *nobles*. As we mentioned above, some *bourgeois* had a life very similar to the *nobles* one. Moreover, we observe that the most of *religious* people actually belong to these upper classes since they are often *capitouls*, magistrates or *nobles*. There was also a predominance of the religious institutions among the property owners and the capital. This why we found in 1637 for the Narbonnais Castle that almost half of the shareholding is composed by ecclesiastics.

At that time, the main concern of any wealthy people is to look like a *noble*. The typical way to mimic the Lord's life was to proceed in the same manner as during the Medieval period, that is by buying a smallholding although this activity was not very profitable. There were also other alternatives that bestowed a bourgeois with titles. The most common one was to buy a position at the Toulouse Parliament. Indeed, this institution had a crucial role in the procedure to acquire nobility. During the 17th century, we notice in the *bourgeoisie*, a decrease of the number of merchants to make room for people who owe their rank to judicial or administrative posts. The context was conductive for the emergence of this new social class since a large part of the "*sword nobility*" has been decimated by the *Albigensian crusade* some centuries earlier. Moreover, many *nobles* also chose to enter the Parliament ranks and thereby increased its size. Thus, at the end of the 17th century, this community was the most influential one in Toulouse society. Here, a natural question to ask would be how people acceded to this judicial institution? The Parliament of Toulouse, which was created in 1442, was composed by chairmen, counsellors and King's representatives. To become member, we had to fulfill some conditions. First of all, this requires to apply for an official position in it. This position can be inherited from a family member or directly bought at an expensive price that varies between 15,000 and 150,000 *livres*. Moreover, each applicant must be over 25, has worked as a lawyer for at least three years and does not have any relative in the Parliament. Over decades, we observe that the composition of this social group remained quite stable since most positions were inherited through generations. This why that takes time and money to access to a such closed community. It appears that only people from the *high bourgeoisie* had a chance to become a Parliament member. Therefore at that time, there were few ways to rise in society and the social ladder was very demanding. We further conclude that a part of the investment decisions of some people was only motivated by the willingness to rise in society. Such historical fact is of particular interest to support our model.

18.2 Trading Volume

As we mentioned above, the contract registers available at the municipal archives, are very well documented and provides the reader with a plenty of valuable data. Thus, most of the transactions have been recorded and for each of them, we are able to determine the profession or the social category to which the seller and the buyer belong, if the seller have lost her shares because of liquidity constraints and the contract's form used (irrevocable or with redemption rights). We also find the expiry date of the option in the case of an agreement with redemption rights. In several cases, the deeds even indicate the path followed by the share until the current transaction. Therefore, we are able sometimes to rebuild the trade network of buyers-sellers over many years. Thus, we built a time series for the Bazacle that covers the period 1511-1874 and for the Castle, the period 1590-1845. We believe that the registers are fairly complete and that only few transactions are missing, especially during the Old Regime period. The main difficulty to fully follow a share's path over the multiple transactions is due to the fact that the shares are often transmitted through inheritance or a marriage and it is difficult to perfectly document these kind of notarial acts without the corresponding contracts.

18.3 The "*Recrobit*'s" form

In the contract registers, we distinguish two main forms of agreements : the binding and irrevocable sales[55] and the sales with redemption rights[56] which was called "*recrobit*". The former is a standard deed of sale while the latter allows the seller to repurchase her sold shares within a set time period, usually between 2 and 30 years, or even with perpetual rights.

For the shareholders, the "*recrobit*" form provides the opportunity to avoid to be expropriated and to quickly cash out the value of their shares in order to honour potential debts or to satisfy liquidity needs. A very stylized fact is that during the periods where the mills have experienced a disaster, more contracts with redemption rights are documented[57]. The most likely explanation is the following, when the plant was partially or totally destroyed, huge fees were required to repair it and to restore the mills activity. Some shareholders faced liquidity constraints and cannot afford these extraordinary expenses, so they lost their shares, constrained by the company to auction them off to new investors. Besides these rare events of disasters, any shareholder could experience shocks on his private income and use the "*recrobit*"'s right to keep a control on his holding the time to regain solvency.

Thus this contract's form, frequently observed for Toulouse real estate since the Middle Ages, is a very valuable tool for detecting periods where a set of shareholders were facing borrowing constraints, either due to extraordinary huge *talhas* or to idiosyncratic shocks on their private income. Moreover, we can fairly associate these liquidity shocks to the social category the shareholders belong. As we mentioned above, we observe that investors usually come from the upper classes such as Noble or Bourgeois, but also from lower classes such as bakers, tailors and other merchants who don't have the same individual wealth and can potentially lose their shares during bad periods. Thus, even though social inequalities are not really correlated with the portfolio's composition of individual investors, we find that the state of nature at which they trade can potentially determine the social category they belong. Indeed, the poorest shareholders or the most exposed ones should sell while only the wealthiest or the less exposed ones should be able to buy during a bad state.

Finally and more specifically, the "recrobit"'s form provides the seller with the right but not the obligation to buy back its shares during the set time period at the price they were sold (at the contract date) plus the aggregate *talhas* the buyer had to spend during the holding period. Therefore, this can be viewed as a kind of *forward start option* for which the strike would not be known at the beginning. Moreover, since the owner can exercise the option at any time prior to and including its maturity date, it is more fair to compare it to an American option.

[55]"vente pure et à jamais irrévocable"

[56]"vente sous faculté de rachat"

[57]It would be also very interesting to check in the modern markets during the recession periods what kind of contracts is the more traded (stocks, stock option, derivatives, forward, etc...) and where the Volume comes from

18.4 Expropriation

As we mentioned above, when a huge *talha* was required to repair the mills, some shareholders cannot afford the cost involved. In such a case, the company board gave to the investors a payment term of 15 days at the risk of being expropriated. More precisely, the official rule included into the mills charter of 1417 (Mot 1910) states that the shares of a *parier* can be auctioned when his debt exceeds 12 *livres tournois*. We observe these expropriations many times over the mills lifespan, in particular during periods where the dam was partially destroyed by a river flood. It is important to emphasize that the liquidity shock also depends on the size of the ownership. Indeed, bigger shareholders were more exposed to the default risk as they had to spend an amount proportional to the number of share they owned.

18.5 Rare events and disasters

Every event associated to the exogenous risks incurred by the mills companies are gathered into the registers of shareholders' deliberations. For example, when the dam of the plant was damaged or destroyed, the shareholders hold a general meeting to take stock of the situation and carry out a review of the available alternatives by computing their associated costs. The governance board informed all the investors and the decisions were voted. We find in these registers all the issues the companies encountered about plant's damages, unfair competition, or disagreement with local policies. They also document the new investment opportunities, the changes in the governance structure, the *partisons*, and the *talhas*.

19 The model

19.1 Exogenous status

In this simple version of our model, we assume that people cannot act on their own status which are exogenously determined. We will propose further a more general setting where each agent will be able to both choose his optimal action to buy or sell an asset and the status that maximizes his utility.

We consider a non-informational economy with n agents heterogeneous in terms of social position. They are gathered into two groups depending on whether or not they have a status. As the status is positively related to marginal utility, the higher the rank of an agent, the lower his risk aversion. This last point is in accordance with the idea that preferences are partially determined by the relative position of an individual with respect to his environment. Every player is assumed to maximize the following pairing-dependent utility function :

$$u_{ij}(W_i, \eta_j) = -e^{-A_i\left(W_i + (\alpha[\eta_j - \eta_i]^+ - \beta[\eta_i - \eta_j]^+)|\iota_i|\right)}$$

where W_i is the wealth of the agent, A_i the risk aversion, $\alpha, \beta > 0$ and $\eta_i, \eta_j = \{0, 1\}$ are binary variables that describe status. More precisely, $\eta_i = 1$ describes the case where the agent i has a status, and $\eta_i = 0$ otherwise. Notice here that the utility function can only be optimized on the basis of individual wealth and trade partner status η_j since people cannot decide their social position but can elect their preferred neighbor. Clearly here, u is increasing in η_j, that is for a given value of W_i, $u_{ij}(W_i, \eta_j^{'}) > u_{ij}(W_i, \eta_j)$ with $\eta_j^{'} > \eta_j$. In other words, the agent's utility raises when he exchanges with someone with higher status, only depends on wealth if the trade takes place within the same social class and is lowered when his partner comes from a lower class. Here, the coefficients α (respectively β) can be viewed as psychological gain (respectively cost) generated by the trade partner status. We chose to split the impact of relative social positions into two components to avoid a symmetric effect for the different partners. More precisely, we assume $\alpha > \beta$, that is the disutility experienced by some status players exchanging with lower class partners is lower than the rise in utility experienced in the opposite situation. This specification will be of crucial interest in our model to elect the best choice of a player among his neighborhood.

The agents can only trade a single asset x in the market whose the real value $\tilde{v}$ is normally distributed with mean μ and variance σ^2. They start with the same endowment of one unit of share and maximize their wealth W_i according to the alternative to buy, to sale or to don't trade a fixed quantity $\epsilon \in [0, 1]$ of share at once. We can write $W_i = \iota_i(\tilde{v} - p)\epsilon + \tilde{v}$ with $\iota_i = \{-1, 0, 1\}$ with p the price of at which the x is traded. We can easily derive the following proposition.

Proposition : *Let x_i and x_j be two agents with risk aversions A_i and A_j respectively and let Ξ_{xy} be a random variable such that $\forall x \neq y$, $\Xi_{xy} = A_x\left(\frac{\epsilon}{2} - 1\right) + A_y\left(1 + \frac{\epsilon}{2}\right)$. By assuming a tradable asset x exists on the market and can be divided into ϵ shares, there is an incentive to trade between the agents i and j if and only if*

$$\Xi_{ij} < \frac{\alpha - \beta}{\epsilon\sigma^2} \text{ or } \Xi_{ji} < \frac{\alpha - \beta}{\epsilon\sigma^2}$$

As we mentioned above, we sort people into status groups by considering a 2-partition of the population set as $\{1, ..., n\} = \{i_1, ..., i_{n_1}\} \cup \{i_{n_1+1}, ..., i_n\} = I_1 \cup I_2$. For instance, I_1 would be composed by members with status and I_2 by those who don't have one. In order to draw the social connections between the agents, we rely on the sociological definition of a group to state that there is a stronger intra-group cohesion than an inter-group one. Indeed, most studies on social connections agree on the fact that people are more likely to know each other within a group than if they belong to two different groups.

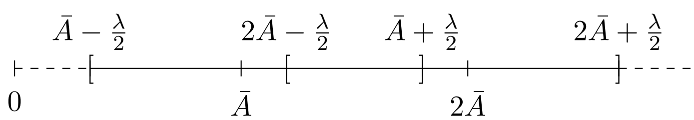
In order to capture these features, we set up an underlying random network. More precisely, we generate a random graph with vertices and edges whose the randomness entirely depends on the group to which each node belongs. Thus, we consider the following model.

$$p_{ij} = \begin{cases} p \text{ if } x_i, x_j \text{ belong to the same group} \\ \\ \frac{p}{c} \text{ otherwise} \end{cases}$$

with c a constant. Notice here for $c = 1$, we would obtain the Erdös-Rényi standard model, that is the case of an organized Market where a regulatory body would match players regardless of their status.

This setting allows to capture another important feature of real networks, the *small-world* phenomenon. This phenomenon states that, in addition to their primary social circle, individuals also have some long-range connections with people who have very little in common with them in terms of preferences. Thus, an agent from lower classes, that is without status, can know someone else bestowed with status.

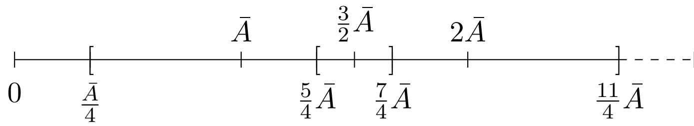
Let us now describe how the preferences are generated in this economy. We assume here that the individual risk aversions are drawn from uniform distributions whose the mean depends on the status of the agents. Indeed, as we mentioned above, people with higher status are more likely to be risk seekers than those with a low status. Thus we construct the bounds of our intervals as follows.



Here, the support of the uniform distribution is entirely based on two parameters $\bar{A}$ and λ. We assume that people from lower classes are twice as much risk averse than agents with status and the size of each interval is the same. In order to avoid negative values and thereby to keep the model in a tractable way, we further assume $\lambda < 2\bar{A}$. Moreover, by making vary the value of λ, we increase or decrease the length of the intersection of the two intervals. In other terms, λ is controlling the social diversity between groups.

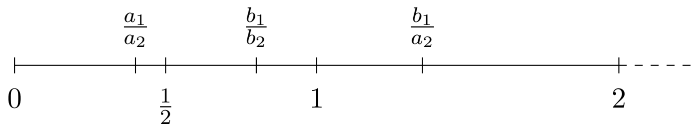
As we mentioned above, the probability of no trade is given by the quantity $\mathbb{P}\left(\left\{\Xi_{ij} \geq \frac{\alpha+\beta}{\sigma^2}\right\} \cap \left\{\Xi_{ji} \geq \frac{\alpha+\beta}{\sigma^2}\right\}\right)$. We also made the assumption that the individual risk aversions are uniformly distributed on intervals which depend on the group to which each agent belongs. Thus, we must determine the distribution of Ξ_{ij} to derive the probability that two players are willing to exchange. Let $a_1 = \bar{A} - \frac{\lambda}{2} = b_1 - \lambda$ and $a_2 = 2\bar{A} - \frac{\lambda}{2} = b_2 - \lambda$, as shown in Appendix, Ξ_{ij} is a continuous random variable with the density function $f_\Xi(\xi)$.

We propose here to set up $\lambda = \frac{3}{2}\bar{A}$ such that we obtain the following intervals.



Where $a_1 = \frac{\bar{A}}{4}, b_1 = \frac{7}{4}\bar{A}, a_2 = \frac{5}{4}\bar{A}$ and $b_2 = \frac{11}{4}\bar{A}$. This setting implies that $\frac{a_1}{b_1} < \frac{1}{3}$ but $\frac{3}{5} > \frac{a_2}{b_2} > \frac{1}{3}$. Moreover by construction, we have[58] $\frac{a_1}{a_2} < \frac{1}{2}$ and symmetrically $\frac{b_1}{b_2} > \frac{1}{2}$. More precisely, we have $\frac{a_1}{a_2} < \frac{1}{3}$ and $\frac{b_1}{b_2} < \frac{2}{3}$. Finally, as we imposed the condition $\frac{\lambda}{2} < \bar{A}$, we have $\frac{b_1}{a_2} < 2$.

We deduce from these results that



Thus, we can determine the probability of no trade for every fraction of share ϵ.

19.2 Dynamic case

So far, we have presented a static game where players can only act on their consumption of the *non positional* good to reach an optimum. This version was very convenient to highlight that the existence of status directly affects the trading volume. However, as we *ex ante* set up the partition of the population, that is we decide who has a status and who does not have one, this setting does not allow to track the evolution of each social group. Indeed, that would be interesting to look at closely when higher class members are constrained to leave their social position or when non status people have an incentive to rise in society. Obviously, in a static context a such analysis cannot be conducted.

Thus, we propose now a dynamic version of our original model that should be able to capture all these features. We still consider a non-informational economy with n agents who are assumed to be heterogeneous in terms of endowment. Agents can now trade both a positional good x and a non-positional one y. In order to raise in society, they buy the former while the latter is dedicated to speculative desires. We still start again on an *ex ante* partitioned population with n_1 players who only have one unit of the *positional* good and n_2 players who have one unit of both the *positional* and *non-positional* good. This time, they can select their optimal consumption for the both goods. More precisely, by keeping the same notation than before, we assume there are two tradable assets on the market, a *non-positional* good x which is still divisible and a *positional* one y which can only be exchanged per single unit. The real value of x is still denoted by $\tilde{v}$ and is normally distributed such that $\tilde{v} \sim \mathcal{N}(\mu_v, \sigma_v^2)$. Similarly, the real value of y is denoted $\tilde{\eta}$ which is normally distributed

[58] We have $\frac{a_1}{a_2} = \frac{\bar{A}-\frac{\lambda}{2}}{2\bar{A}-\frac{\lambda}{2}} < \frac{\bar{A}-\frac{\lambda}{2}}{2\bar{A}-\lambda} = \frac{1}{2}$ and $\frac{b_1}{b_2} = \frac{\bar{A}+\frac{\lambda}{2}}{2\bar{A}+\frac{\lambda}{2}} > \frac{\bar{A}+\frac{\lambda}{2}}{2\bar{A}+\lambda} = \frac{1}{2}$

with $\tilde{\eta} \sim \mathcal{N}(\mu_\eta, \sigma_\eta^2)$. Let q be the price of one unit of good y and we assume $q >> p$. Of course, we make the same connection between the risk aversion of an agent and his social status than before, and every player is assumed to maximize the following utility function :

$$u_{ij}(W_t^i, \eta_t^i, \{\eta_t^k\}_{x_k \in N(x_i)}) = -e^{-A_t^i \left(W_t^i + (\alpha[\eta_t^j - \eta_t^i]^+ - \beta[\eta_t^i - \eta_t^j]^+)|\iota_t^i| + \gamma \sum\limits_{x_k \in N(x_i)} (\eta_{t-1}^i - \eta_{t-1}^k) \right)}$$

Here, we add a new ingredient to the utility function we developed in the static version of our model. Indeed, as people are now able to directly act on their status, we let them be concerned about the social position of each of their neighbors. More precisely, when the consumption of the *positional* good of an individual is lower than the average consumption of the same good by its neighborhood, he is negatively impacted by this difference through the coefficient γ. This specification is in line with the previous work of Immorlica (2017) and Ghiglino (2010). Thus, when someone have mostly relatives with a higher status, he has an incentive to increase his consumption of the *positional* good.

As we mentioned above, n_1 players start with one unit of x while n_2 players start with both one unit of x and one unit of y. In fact, the stock of shares an agent has at time t of the *positional* good entire determines the group to which he belongs. Notice that in our model and unlike previous work on status, the social position of an individual will alter his social connections as we generate random graphs based on group belonging.

The wealth of an agent can be written as follow

$$W_t^i = W_{t-1}^i + (\tilde{v} - p)\iota_t + \left(2\sum_{k=0}^{t-1} \nu_k - 1 \right) \nu_t q + \tilde{\eta} \left(\sum_{k=0}^{t-1} \nu_k + \left(1 - 2\sum_{k=0}^{t-1} \nu_k \right) \nu_t \right) - \tilde{b}\xi \sum_{k=0}^{t-1} \iota_k$$

where $\tilde{b}$ is a binary variable that takes 1 with probability $\mathbb{P}$ and 0 otherwise. When $\tilde{b} = 1$, that describes the case where the agents are experiencing a collective liquidity shock. Such component is important in our model as it captures the players incentive to sell the *positional* good. Moreover, notice that the shock magnitude is proportional to the stock of good x they have, $\xi \sum\limits_{k=0}^{t-1} \iota_k$. From a historical perspective, this specification can be interpreted as the negative dividends experienced by the Toulouse mill shareholders when a major disaster impacted the economy. Indeed, as we mentioned in section (**??**), when the plant or the production tool was damaged, each investor had to contribute to the repairs by paying an amount proportional to his ownership. The quantity $\sum\limits_{k=0}^{t-1} \nu_k$ indicates the group to which an agent belongs and his consumption of good y, denoted ν_t, can only take the values 0 or 1. Notice that if an agent has a status at time $t-1$, that is $\sum\limits_{k=0}^{t-1} \nu_k = 1$, his wealth can be expressed as $W_t^i = W_{t-1}^i + (\tilde{v} - p)\iota_t + \nu_t q + \tilde{\eta}\,(1 - \nu_t) - \tilde{b}\xi \sum_{k=0}^{t-1} \iota_k$. Thus, as $\nu_t = \{0, 1\}$, the feasible actions of a such agent are only to sell y or to keep it. Symmetrically, when $\sum_{k=0}^{t-1} \nu_k = 1$, that is for a lower class member, his wealth becomes $W_t^i = W_{t-1}^i + (\tilde{v} - p)\iota_t - \nu_t q + \tilde{\eta}\nu_t - \tilde{b}\xi \sum_{k=0}^{t-1} \iota_k$ and he can only buy the *positional* good. Finally, when $W_t^i < 0$, the player i must leave the game.

For a given price, a lower class member always prefers to exchange with someone from the upper class rather than with one of his fellows. Conversely, for a given price, an agent with a status always prefers to exchange with someone from the same rank. Notice that when two persons from the same social group make a trade, the status component vanishes and the profit takes a standard form.

20 Descriptive statistics

As we mentioned above, we have collected data about volume that covers the Old Regime period for the both mill companies.

Let us first examine the two time series displayed on the figure (14). We observe that the two curves roughly evolve within the same range of values although their correlation of 0.03 suggests a weak relationship between them. We also notice that the volume is more volatile[59] for the Bazacle than for the Castle and the Bazacle shares have been more traded at some dates during the 18th century. In particular, there is a sharp peak in 1714 of 64 shares. Indeed, the *Honor del Bazacle* has experienced the worse period of its lifespan between 1709 and 1720 where the dam was entirely destroyed by the ice flows on the Garonne river. During this period, the mills activity was shut down because of the very high cost of repair which cannot be afforded by the shareholders. Finally, the engineer Abeille purchased the half of the company shares and started the plant reconstruction in 1714 with the support of some Genevans bankers. As the very high *talha* recorded in 1709 has been really paid later when Abeille became the majority shareholder, we chose to postpone this expense to its effective date in 1714. This explains why we observe a such peak in the graph. Other abnormal values for the volume have been recorded in 1639, 1731, 1732, 1735, 1753 and 1756 for the Bazacle. Unfortunately, the registers do not provide enough information to justify all of these events. We only notice that in 1638 and 1735, the Bazacle mills were damaged. Thus, this could explain an increase of the trading volume. In 1752, we also document a famine in Toulouse which has potentially altered the company attractiveness during the following year and could justify the corresponding peak we find on the graph. For the Castle, we observe two main dates at which we have a high volume, 1593 and 1638. Although we don't have specific information about what

[59] The standard deviation is equal to 5.9 for the Bazacle and 1.5 for the Catsle.

happens around 1593, we know that in 1637, a part of the plant has been destroyed. Thus, this would explain the peak of 1638 with some lag.

The tables (15) and (18) display both the dividends and the trading volume recorded during the Old Regime period. For the Bazacle, almost every time a huge negative dividend is associated to a large trading volume, during the same year or with one year lag. In particular, we observe this phenomenon in 1597-1598, 1638-1639, 1709-1714 and 1735. This result is in line with what we mentioned above about the disasters events. When the mills expenses are higher than the production, that is when the *partisons* do not compensate the *talhas*, the shareholders must re-inject money in the firm if they want to keep their shares. Moreover, these individual contributions are proportional to the ownership. In other terms, if we own 10 shares, we must pay 10 times the *talha*. Therefore, either an investor cannot afford this amount and his liquidity shock forces him to sell or, the subsequent alteration of the firm value will motivate speculative desires for other agents. The only result we cannot explain here is the huge negative dividend experienced in 1613 which is not associated with any movement in the trading volume. For the Castle, we make similar comments about the relationship between negative dividends and volume increase for the periods 1637-1638 and 1643-1645. Moreover, we also document the same implication in the case of a positive dividend, which is less obvious for the Bazacle. Indeed, during the period 1592-1593, we observe that an important increase of the mills production is followed one year later, by a rise of the number of traded shares in 1593.

Now let us turn to the tables (17) and (20) that break down the dividends into the *talhas* and the *partisons*. These graphs confirm for the Bazacle and the Castle what we discussed above. Indeed, we observe here that large trading volumes usually follow great *talhas* and in particular for the Castle, it seems that the high number of traded shares of 1593 is associated to the high *partisons* of 1592.

Finally, the tables (16) and (19) gather volume and price data. Here, the relationship between price and volume is not really obvious. Indeed, even if the firm value and its shares price are related to the disaster occurrence, they are not clearly connected to the number of shares exchanged in the market.

21 Stylized Facts About Volume and Social Status

21.1 Volume and liquidity shock

As we mentioned above, we expect to find a relationship between volume and liquidity shock. More precisely in the case of the Toulouse mill companies, when the dam or the plant was destroyed, a huge *talha* was required and potentially several shareholders cannot afford such expenses. In particular, the religious institutions were the most vulnerable to this kind of shock as they usually owned large numbers of shares. For instance, if we look at the pie charts displayed on figures (13), we observe the evolution of the Castle shareholding between 1637 and 1648. As we mentioned above, during this period, the mills have experienced major destruction in 1637 and in 1643, but the region has also been impacted by a drought in 1642. Thus, in 10 years, there have been substantial changes in the social group distribution of the shareholders. More specifically, we notice the religious institutions that owned almost half of the outstanding shares in 1637, drastically reduced their ownership mostly in favour to other investors as *merchants*, *bourgeois* and *religious members*. Indeed, we will see further that the *merchants* formed the most active group in the trading of mills shares over the centuries, so it is not surprising to find them here. The *bourgeois* were wealthy investors able to provide liquidity to the economy when a bad state occurs. Finally, many religious members was also nobles and had funds to buy back the shares that their institutions cannot afford anymore. Thus, we expect to find a positive correlation between the *talhas* and the number of transactions involving a religious institution.

First, we propose to run the following regression :

$$V_t = a + bPV_t + cTalhas_t + dTalhas_{t-1} + e(Partisons_t - \overline{Partisons}) + \epsilon_t$$

where PV_t describes the present value of all the cash flows that take place starting from date t. More precisely, this variable can be expressed as $PV_t = \sum_{i=t}^{T} \frac{D_i}{(1+R^f)^{i-t}}$ with D_i the dividends and R^f the risk-free rate. Here, we test if the value of the company, computed on the basis of the standard present value formula, affects the trading volume. We also split the dividends into their two sub-components : the *talhas* and the *partisons*. More precisely, we include into the regression the *talhas* from both date t and date $t+1$ in order to check if a latency effect exists and thereby to explain why a current shock does not impact the current number of shares traded. We normalized the *partisons* time series by removing its mean value to focus the analysis on the abnormal values. Moreover, notice that volume data collected in the archives suffers from missing values and incomplete information about the exact date at which it takes place. Thus, the recorded transactions could occur sometimes before the natural disaster but the registers inaccuracy does not allow us to rebuild the real chronological order of the different events. The regression estimations are gathered into the tables (8) and (9) where we only focus on the transactions involving ecclesiastics and merchants respectively.

[INSERT FIGURE HERE]

We observe in the case of religious institutions that only the normalized *partisons* and the present value significantly explain the trading volume. Moreover, the estimations of b and e are positive and negative respectively. Therefore, when the sum of future expected payoffs increases, the volume increases as well. Obviously, if the company value rises, that increases the shares price and thereby the incentive to trade them. In the case of *partisons*, we observe that they are negatively correlated to the volume. This effect is a bit more ambiguous since it suggests that the volume decreases when the quantity of grain redistributed to the shareholders increases. Unfortunately, we will not be able to provide more insight on this point.

Now, let us refine our regression to capture the effect of dividends on trading volume. For the Castle mill company, we run the following regression :

$$V_t^i = a + bND_t^+ + cND_t^- + dV_t^B + ePV_t + \epsilon_t$$

where $i = \{\text{rel, brg, lowercl, nble}\}$, $D_t = ND_t^+ \mathbb{1}_{D_t > 0} \; - \; |ND_t^-| \mathbb{1}_{D_t < 0}$, that is ND_t^+ and ND_t^- are the positive and negative dividends respectively. We also include into this regression the volume recorded for the main competitor of the Castle company : the Bazacle, denoted V^B. Again, we use the present value of the future expected discounted cash flow to control for the firm value.

The estimation results are gathered into the tables (10) and (11). We observe for the *religious* that only PV_t seems to have a significant impact on the volume over the centuries and is negatively related to it. Thus, when the value of the firm increases, the volume decreases. We could interpret this phenomenon by saying that at this time, a large part of the shareholders and in particular the religious institutions, were very long-term investors as they used the product of their investment for their own consumption of wheat.

On the *merchants* side, we observe exactly the opposite. Indeed, it is very interesting to see that the volume is positively related to PV_t for this social class. Thus, when the value of the firm increases, the *merchants* buy or sell more. This fact is in accordance with the network analysis we perform in next section which shows that they were the most active investors and had a central position in the trades over the companies lifespan. Moreover, some of them displayed several times a speculative behavior.

[INSERT FIGURE HERE]

21.2 Volume and social group

In figure (21), we display from 1591 to 1788 the proportion of each social group in the total number of trades for the Castle each year. The shareholders are sorted among four different categories : the *Bourgeois*, the *Nobles*, the *Religious* and the *Lower Classes*. The content of each of these sub-populations is detailed in Appendix and follows the classification introduced in Godechot (1966) and Thoumas-Schapira (1955).

Thus, we observe that the market seems to be broadly balanced around two social groups : the *Nobles* and the *Lower Classes*, although the former slightly dominates the trades. More precisely during the 18th century, the *Nobles* are more frequently involved in transactions that the other groups. This is coherent with the fact that this category of shareholders was almost always present at the board of directors. We also notice that these two social groups were several times the only ones to trade during the Old Regime period. The *Lower Classes* members are the second most active shareholders in the market, they frequently appear over the centuries and in particular around 1700. Then, we find the *Bourgeois* and the *Religious* which are in a minority in this diagram. This is due first to the fact that these two groups have historically less members than the *Lower Classes* and the *Nobles*. Moreover, the religious institutions were long-term investors that bought mills shares for feeding their members and did not show any speculative desires.

22 Descriptive results from a network analysis

Let us now introduce a first definition from the standards of the network analysis.

Definition : *Let G be a simple, (strongly) connected graph. Let $S(x,y)$ be the set of the shortest paths between two vertices $x, y \in V(G)$ and $S(x,u,y) \subseteq S(x,y)$ the ones that pass through vertex $u \in V(G)$. The betweenness centrality $c_B(u)$ of vertex u is defined as*

$$c_B(u) = \sum_{x,y} \frac{|S(x,u,y)|}{|S(x,y)|(N-1)(N-2)}$$

Here we normalize by $(N-1)(N-2)$, the possible pairs of vertices that u can connect.

This quantity is particularly useful to understand here which social category played a role of marketmaker in this old economy. We observe in the table (6) that the *merchants* and the *Nobles of the Robe* had a crucial position at this time. In particular, the *merchants* were on more than 50% of all shortest paths in the network. In other terms, for each pair of node in this graph, when we look at the shortest sequence of intermediaries between them, we have fifty-fifty chance to find a *merchant* on the way.

Let us now sort the shareholders into two categories, those who have a status and those who don't have one. By considering subperiods of 10 years between 1590 and 1788, we compute for each of them the ratio of the number of intra group trades over the total number of trades recorded for this period and we find the results summarized in table (12). Here we observe that there were mostly more intra group trades than inter group ones during the Old regime period. However, for the periods 1641-1648, 1661-1668, 1682-1688, 1691-1698 and 1780-1787 this ratio is lower than 0.5 and the inter group trades dominate. Thus, over the centuries, this suggests that different mechanisms are at play in the social interactions. In our model, we distinguish two main forces which have two opposite effects. The first is based on a pure comparison of the individual preferences to elect the best partner of each agent and generate on average more incentives to trade with someone with a different status. The second mostly reflects the psychological gain or lost generated by the relative status of the paired players. As we highlighted, sometimes the disutility related to some trade can offset the preferences-based incentives and lead to a trade pattern where the exchanges mostly take place intragroup.

We also sort all the transactions into three categories, those for which the buyer has a status and the seller does not have one, those for which we observe the opposite and those for which buyer and seller have the same social position. Then, we compute the mean of the deflated prices for each of these categories and we found the following results. It seems that on average the price is higher for the first category than for the second. The second and the third ones does not display large differences. This result is in accordance with the experimental work of Ball et. al. (2001) that observe the same phenomenon in a laboratory market.

23 Conclusion

We examine here the case of an old economy where the social distinctions based on status were very clear. More precisely, we are able to sort the whole population into four main classes, the *religious*, the *nobles*, the *bourgeois* and the *lower class*. The two first ones definitely had a social privileged position and can be viewed as the status group while the other classes are the non status group. We are mostly interested to explain the evolution of the trades pattern over the centuries. Thus, we propose a model which captures two forces with potentially antagonist effects, the preferences-based incentives and the status-based ones. The model predicts that under specific conditions, the former force can be offset by the latter one and partially explain the empirical network analysis we performed for the mills companies in Toulouse. We also introduce a dynamic version of our model where the agents can experience liquidity shocks. This new feature describes the impact of the rare disasters which have occurred over the firms lifespan. We also show empiricaly why we can fairly relate the abnormal trade observed during some periods to these liquidity shocks.

A Trade condition

A.1 Static case

If the trade partner has a strictly higher status than the agent, we have the following expected utility: $\mathbb{E}(u_{ij}(W_i, \eta_j)|\eta_j - \eta_i = 1) = -e^{-A_i(\iota_i(\mu-p)\epsilon+\mu+\alpha|\iota_i|)+\frac{A_i^2}{2}(1+\iota_i\epsilon)^2\sigma^2)}$. For the opposite case, we obtain : $\mathbb{E}(u_{ij}(W_i, \eta_j)|\eta_j - \eta_i = -1) = -e^{-A_i(\iota_i(\mu-p)\epsilon+\mu-\beta|\iota_i|)+\frac{A_i^2}{2}(1+\iota_i\epsilon)^2\sigma^2)}$. Thus, we have

$$
\begin{cases}
\mathbb{E}(U(W_B)) & = & -e^{-A\left((1+\epsilon)\mu-p\epsilon+\alpha\mathbb{1}_{\eta_j>\eta_i}-\beta\mathbb{1}_{\eta_j<\eta_i}\right)+\frac{A^2}{2}(1+\epsilon)^2\sigma^2} \\
\mathbb{E}(U(W_S)) & = & -e^{-A\left((1-\epsilon)\mu+p\epsilon+\alpha\mathbb{1}_{\eta_j>\eta_i}-\beta\mathbb{1}_{\eta_j<\eta_i}\right)+\frac{A^2}{2}(1-\epsilon)^2\sigma^2} \\
\mathbb{E}(U(W_{NT})) & = & -e^{-A\mu+\frac{A^2}{2}\sigma^2}
\end{cases}
$$

Therefore, an agent is willing to buy ϵ share when $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_B)) > \mathbb{E}(U(W_S))$. Symmetrically he sells this quantity when $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_{NT}))$ and $\mathbb{E}(U(W_S)) > \mathbb{E}(U(W_B))$. To sum up

$$
\text{he buys iff } p < \mu - A\left(1+\tfrac{\epsilon}{2}\right)\sigma^2 + \tfrac{\alpha\mathbb{1}_{\eta_j>\eta_i}-\beta\mathbb{1}_{\eta_j<\eta_i}}{\epsilon}
$$

$$
\text{he sells iff } p > \mu + A\left(\tfrac{\epsilon}{2}-1\right)\sigma^2 - \tfrac{\alpha\mathbb{1}_{\eta_j>\eta_i}-\beta\mathbb{1}_{\eta_j<\eta_i}}{\epsilon}
$$

Finally, let $\Xi_{xy} = A_x(\frac{\epsilon}{2}-1) + A_y(1+\frac{\epsilon}{2})$, we have

$$
i \text{ sells to } j \text{ iff } \Xi_{ij} < \tfrac{\alpha-\beta}{\epsilon\sigma^2}
$$

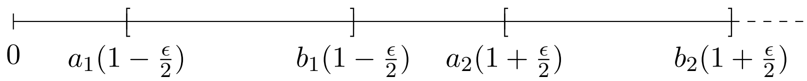
$$
i \text{ buys from } j \text{ iff } \Xi_{ji} < \tfrac{\alpha-\beta}{\epsilon\sigma^2}
$$

A.2 Distribution of Ξ

Let A_j and A_i be two independent uniform variables drawn from the intervals $[a, b]$ and $[a', b']$ respectively. We know that the density of Ξ is equal to the convolution product of the $Y_i^{(1)}$ and $Y_j^{(2)}$ densities where $Y_j^{(1)} = A_j(\frac{\epsilon}{2}-1)$ and $Y_i^{(2)} = A_i(1+\frac{\epsilon}{2})$.

$$
\begin{aligned}
f_\Xi(\xi) &= (f_{Y_j^{(1)}} * f_{Y_i^{(2)}})(\xi) \\
&= \int_{-\infty}^{\infty} f_{Y_j^{(1)}}(\xi-u) f_{Y_i^{(2)}}(u)du \\
&= \int_{-\infty}^{\infty} U_{\left[b(\frac{\epsilon}{2}-1),a(\frac{\epsilon}{2}-1)\right]}(\xi-u)U_{\left[a'(1+\frac{\epsilon}{2}),b'(1+\frac{\epsilon}{2})\right]}(u)du \\
&= \int_{-\infty}^{\infty} U_{\left[a(1-\frac{\epsilon}{2})+\xi,b(1-\frac{\epsilon}{2})+\xi\right]}(u)U_{\left[a'(1+\frac{\epsilon}{2}),b'(1+\frac{\epsilon}{2})\right]}(u)du
\end{aligned} \tag{27}
$$

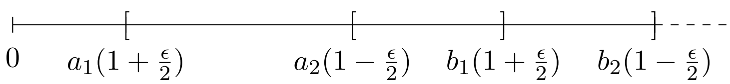
In the case $[a, b] = [a_1, b_1] = \left[\frac{\bar{A}}{4}, \frac{7}{4}\bar{A}\right]$ and $[a', b'] = [a_2, b_2] = \left[\frac{5}{4}\bar{A}, \frac{11}{4}\bar{A}\right]$, we have



Thus by denoting $\bigcap$ the intersection $\left[a_1(1-\frac{\epsilon}{2})+\xi, b_1(1-\frac{\epsilon}{2})+\xi\right] \cap \left[a_2(1+\frac{\epsilon}{2}), b_2(1+\frac{\epsilon}{2})\right]$, we have

- $a_2(1+\frac{\epsilon}{2}) - b_1(1-\frac{\epsilon}{2}) < \xi < a_2(1+\frac{\epsilon}{2}) - a_1(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_2(1+\frac{\epsilon}{2}), b_1(1-\frac{\epsilon}{2})+\xi\right]$
- $a_2(1+\frac{\epsilon}{2}) - a_1(1-\frac{\epsilon}{2}) < \xi < b_2(1+\frac{\epsilon}{2}) - b_1(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_1(1-\frac{\epsilon}{2})+\xi, b_1(1-\frac{\epsilon}{2})+\xi\right]$
- $b_2(1+\frac{\epsilon}{2}) - b_1(1-\frac{\epsilon}{2}) < \xi < b_2(1+\frac{\epsilon}{2}) - a_1(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_1(1-\frac{\epsilon}{2})+\xi, b_2(1+\frac{\epsilon}{2})\right]$

In the case $[a', b'] = [a_1, b_1] = \left[\frac{\bar{A}}{4}, \frac{7}{4}\bar{A}\right]$ and $[a, b] = [a_2, b_2] = \left[\frac{5}{4}\bar{A}, \frac{11}{4}\bar{A}\right]$, we have



Thus by denoting $\bigcap$ the intersection $\left[a_2(1-\frac{\epsilon}{2})+\xi, b_2(1-\frac{\epsilon}{2})+\xi\right] \cap \left[a_1(1+\frac{\epsilon}{2}), b_1(1+\frac{\epsilon}{2})\right]$, we have

- $a_1(1+\frac{\epsilon}{2}) - b_2(1-\frac{\epsilon}{2}) < \xi < a_1(1+\frac{\epsilon}{2}) - a_2(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_1(1+\frac{\epsilon}{2}), b_2(1-\frac{\epsilon}{2})+\xi\right]$
- $a_1(1+\frac{\epsilon}{2}) - a_2(1-\frac{\epsilon}{2}) < \xi < b_1(1+\frac{\epsilon}{2}) - b_2(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_2(1-\frac{\epsilon}{2})+\xi, b_2(1-\frac{\epsilon}{2})+\xi\right]$
- $b_1(1+\frac{\epsilon}{2}) - b_2(1-\frac{\epsilon}{2}) < \xi < b_1(1+\frac{\epsilon}{2}) - a_2(1-\frac{\epsilon}{2}) \Rightarrow \bigcap = \left[a_2(1-\frac{\epsilon}{2})+\xi, b_1(1+\frac{\epsilon}{2})\right]$

Therefore, we have the following density function :

$$
\begin{aligned}
f_{\Xi}(\xi) \quad = \quad & \frac{1}{\lambda^2(1-\frac{\epsilon}{2})(1+\frac{\epsilon}{2})} \int_{a_2(1+\frac{\epsilon}{2})}^{b_1(1-\frac{\epsilon}{2})+\xi} \mathbb{1}_{\left[a_2(1+\frac{\epsilon}{2})-b_1(1-\frac{\epsilon}{2}), a_2(1+\frac{\epsilon}{2})-a_1(1-\frac{\epsilon}{2})\right]}(\xi) du \\
& + \frac{1}{\lambda^2(1-\frac{\epsilon}{2})(1+\frac{\epsilon}{2})} \int_{a_1(1-\frac{\epsilon}{2})+\xi}^{b_1(1-\frac{\epsilon}{2})+\xi} \mathbb{1}_{\left[a_2(1+\frac{\epsilon}{2})-a_1(1-\frac{\epsilon}{2}), b_2(1+\frac{\epsilon}{2})-b_1(1-\frac{\epsilon}{2})\right]}(\xi) du \\
& + \frac{1}{\lambda^2(1-\frac{\epsilon}{2})(1+\frac{\epsilon}{2})} \int_{a_1(1-\frac{\epsilon}{2})+\xi}^{b_2(1+\frac{\epsilon}{2})} \mathbb{1}_{b_2(1+\frac{\epsilon}{2})-b_1(1-\frac{\epsilon}{2}), b_2(1+\frac{\epsilon}{2})-a_1(1-\frac{\epsilon}{2})}(\xi) du
\end{aligned}
$$

B Graph analysis

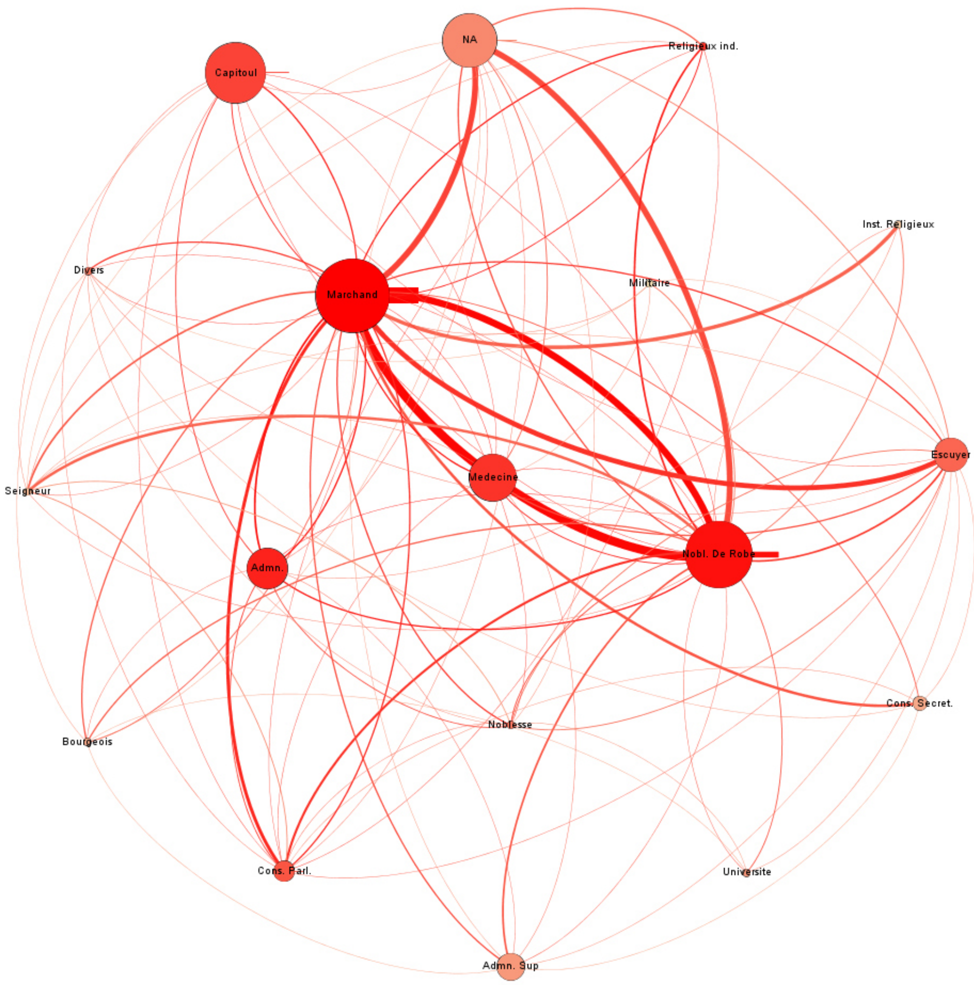


Figure 9: The trades network for the *Castle* company during the period 1590-1788. The size of each node depends on its degree

Table 6: Betweeness measure

Inst. Religieux	Marchand	NA	Nobl. De Robe	Bourgeois	Divers
0.0	0.5144	0.0122	0.3215	0.0	0.0
Cons. Parl.	Cons. Secret.	Noblesse	Seigneur	Admn. Sup	Religieux ind.
0.0019	0.0004	0.0	0.0	0.0042	0.0
Escuyer	Admn.	Medecine	Capitoul	Militaire	Université
0.0052	0.0072	0.0086	0.02941	0	0.0

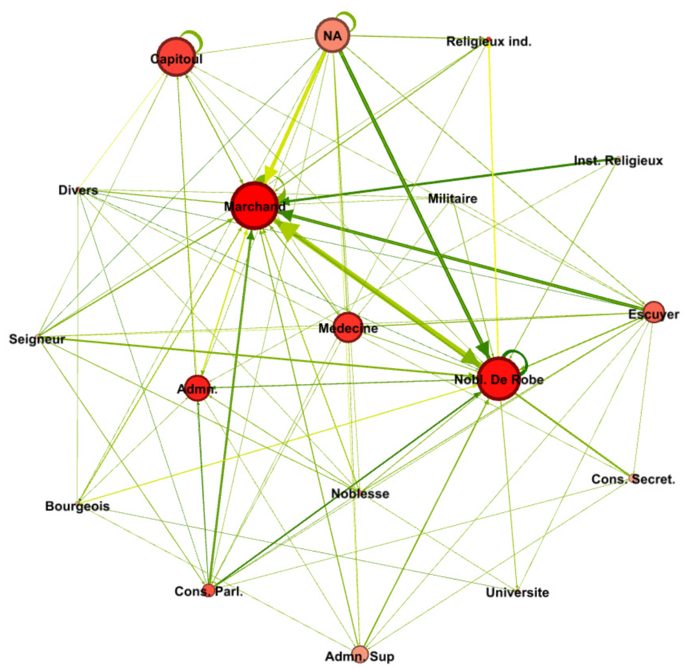


Figure 10: A directed network of the trades for the *Castle* company during the period 1590-1788.

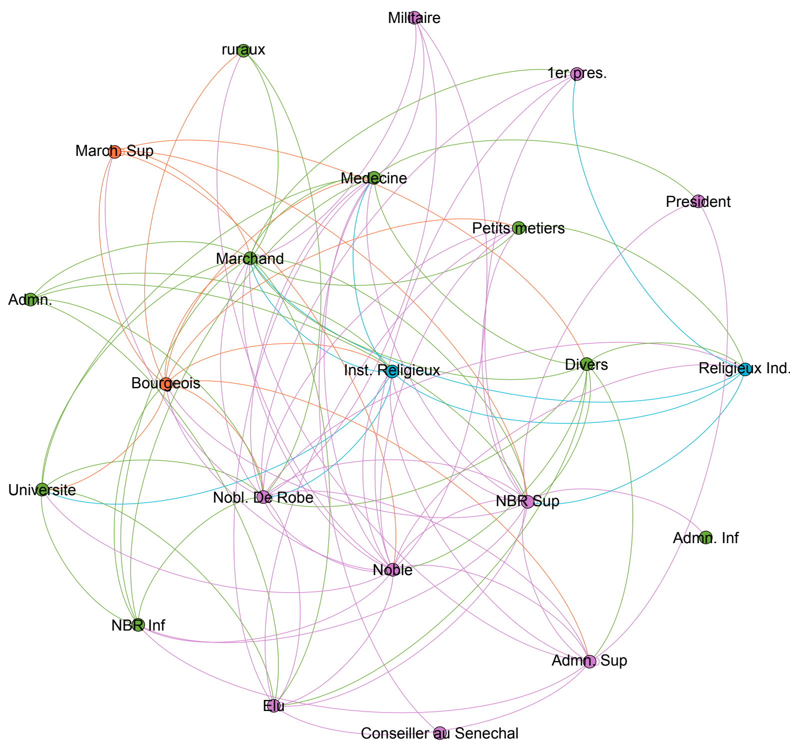


Figure 11: A visualization of the trades sorted by group for the *Castle* company. The groups are the *religious*, the *nobles*, the *merchants* and the *lower class*

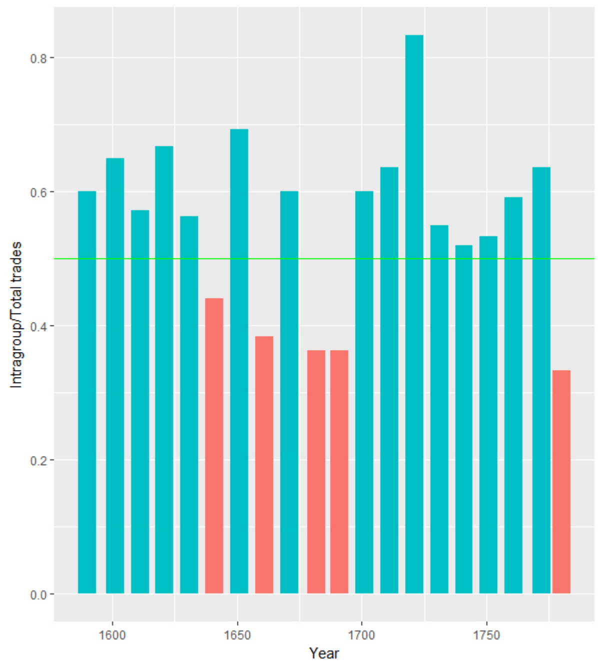


Figure 12: The ratio between the number of intra group trades over the total number of them during subperiod of 10 years for the *Castle* mills company

Table 7: Composition of each social class

Nobles	Bourgeois	Lower class	Religious
Nobl. of the Robe Sup	Bourgeois	Marchand	Religieux Ind.
Nobl. of the Robe	March. Sup	Medecine	Inst. Religieux
Elu		Universite	
President		Admn. Inf	
Noble		NBR Inf	
1er pres.		Admn.	
Admn. Sup		Petits metiers	
Militaire		ruraux	
Conseiller au Senechal			

Table 8: Regression Religious I

	Dependent variable :
	Vol^{rel}
$Talhas_t$	1.830e-07
	(0.85102)
$Talhas_{t-1}$	5.759e-06
	(0.549792)
$Partisons_t - \overline{Partisons}$	4.064e-05**
	(0.022670)
PV_t	$-$4.103e-06***
	(0.003289)
Constant	0.060***
	(0.017)
Observations	198
R^2	0.066
Adjusted R^2	0.047
Residual Std. Error	0.136 (df = 193)
F Statistic	3.435*** (df = 4; 193)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 9: Regression Merchant I

	Dependent variable:
	Vol^{mar}
$Talhas_t$	$-$2.892e-06
	(0.91549)
$Talhas_{t-1}$	1.939e-05
	(0.46894)
$Partisons_t - \overline{Partisons}$	$-$1.562e-04***
	(0.00173)
PV_t	1.071e-05***
	(0.00573)
Constant	0.269***
	(0.048)
Observations	198
R^2	0.079
Adjusted R^2	0.060
Residual Std. Error	0.378 (df = 193)
F Statistic	4.161*** (df = 4; 193)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 10: Regression Religious II

	Dependent variable:
	Vol^{rel}
D_t^+	2.829e-05
	(0.1183)
D_t^-	5.589e-06
	(0.5782)
V_t^B	−3.535e-04
	(0.8316)
PV_t	−3.388e-06**
	(0.0187)
Constant	0.044**
	(0.020)
Observations	198
R^2	0.051
Adjusted R^2	0.031
Residual Std. Error	0.137 (df = 193)
F Statistic	2.580** (df = 4; 193)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 11: Regression Merchant II

	Dependent variable:
	Vol^{mar}
D_t^+	−1.020e-04**
	(0.0445)
D_t^-	5.646e-06
	(0.8407)
V_t^B	−3.132e-03
	(0.5005)
PV_t	9.242e-06**
	(0.0217)
Constant	0.360***
	(0.056)
Observations	198
R^2	0.053
Adjusted R^2	0.033
Residual Std. Error	0.384 (df = 193)
F Statistic	2.693** (df = 4; 193)
Note:	*p<0.1; **p<0.05; ***p<0.01

C Comments about the Mills collection

- We observe into the register of partisons 1636-1659, ph.1756 that when the wheat was redistributed to the shareholders, if some of them were missing, their quantity was stocked into a specific storage place under the authority of the treasurer to give them time to retrieve it later. Moreover, they had a delay of 3 days to pick up the wheat (cf. ph. 4234 partisons 1761)
- We observe that an official member of the *Bourse de Toulouse* Daupiac became a shareholder in 1763 ph.654.

D The estimations

We also perform the following regressions :

$$V_t^i = \beta_0 + \beta_1 ND^+ + \beta_2 ND^- + \beta_3 V_t^{-i} + PV_t + \epsilon_t$$

$$V_t^i = \beta_0 + \beta_1 d_t + \beta_2 D_t^- + PV_t + \epsilon_t$$

where V^{-i} is the volume recorded for the main competitor, PV_t the present value of the future dividends, d_t the dividends for which the partisons were low and D_t^- the negative dividends corresponding to the very huge talhas episodes. The estimation results are gathered in the following tables.

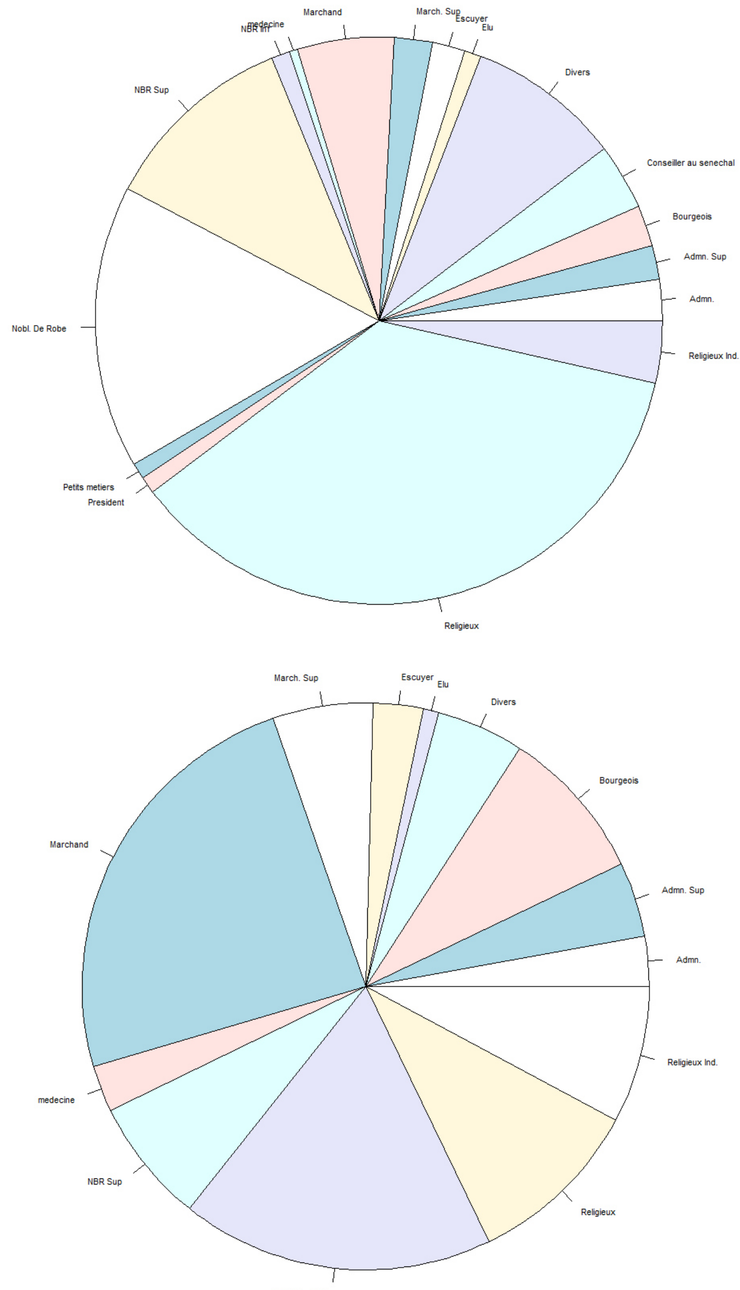


Figure 13: Shareholding evolution between 1637 and 1648 for the Castle company

Bazacle		Castle		General	
Year	Description	Year	Description	Year	Description
1597	Dam damaged	1610	Dam damaged	1528	Languedoc famine
1613	Dam damaged	1611	Dam damaged	1621	Montauban assault
1637	Mills destroyed	1613	Dam damaged	1636	Both mills inoperable
1638	Dam damaged	1637	Dam destroyed	1641	Dryness
1709-1720	Mills destroyed	1642	Dam damaged	1690	Artisans Riot
1717	Lackeys Riot	1670	Dam damaged	1691	Tax Revolt
1724	Lackeys Riot	1673	Dam damaged	1692	Famine
1728	Mills destroyed	1743	Dam Damaged	1693	Famine
1736	Dam destroyed	1745	Dam destroyed	1694	Grain Riots
1802	Mills destroyed	1746	Dam destroyed	1709	Languedoc Famine
1814	Fire	1795	Dam damaged	1710	Languedoc Famine
		1884	Fire	1713	Grain Riot
		1910	Mills destruction	1721	Students Riot
				1737	Students Riot
				1739	Several Riots
				1740	Students Riot
				1742	Grain Riot
				1747	Riots and Seizure
				1750	Students Riot
				1751	Rise of prices
				1758	Prisoners Riot
				1766	Trade Constraints
				1771	Prisoners Riot
				1773	Grain Riot
				1776	Grain Riot
				1778	Grain Riot
				1782	Grain Riot
				1787	Famine
				1788	Famine
				1789	Grain Riot
				1799	Toulouse battle
				1814	Toulouse battle
				1816	Bad Weather
				1817	Famine

Table 12: : The table reports the different rare events recorded for each Mills and the disasters impacting the whole economy (Schneider 1989, Sicard 1953, Mot 1910)

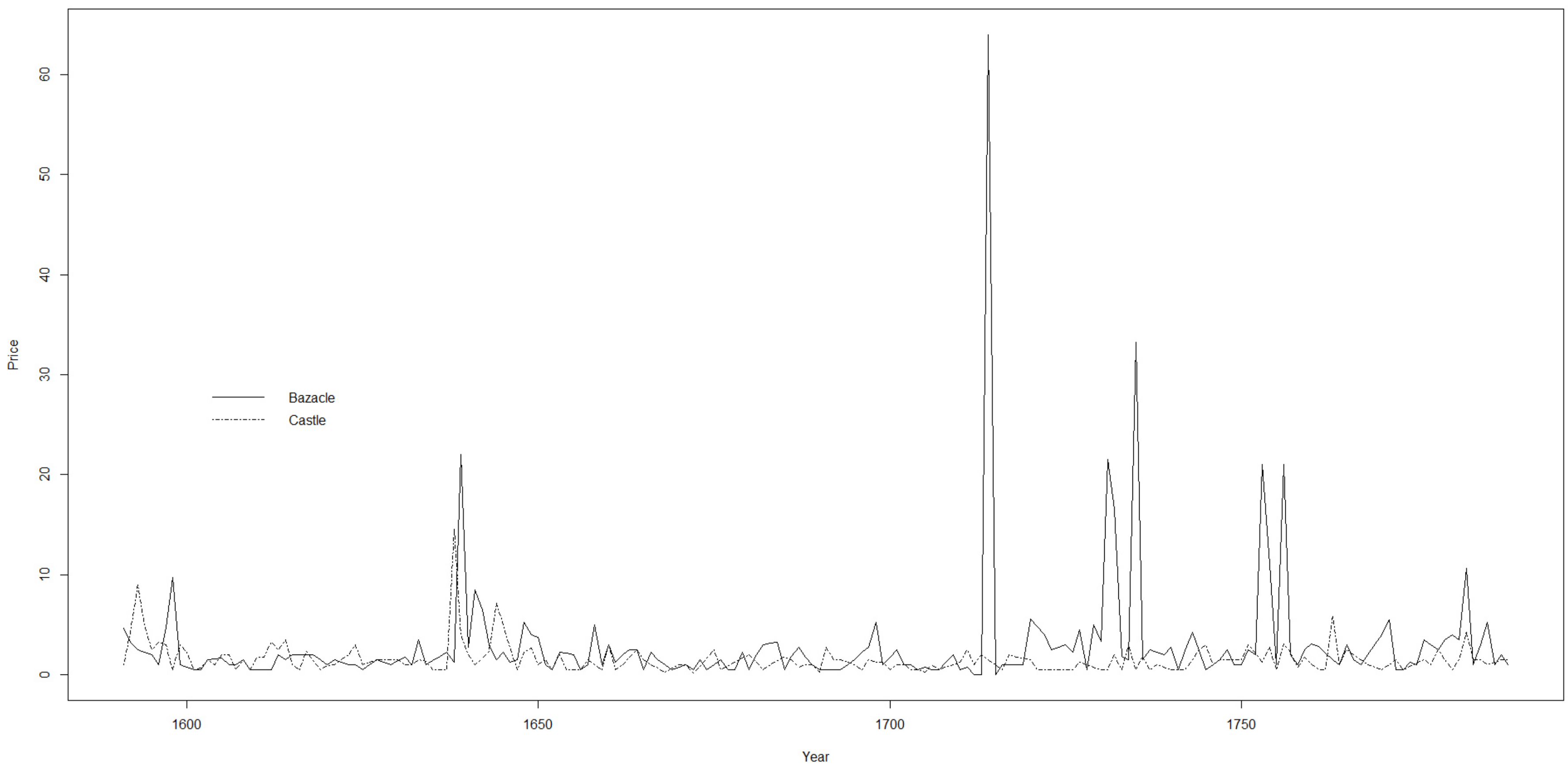


Figure 14: Volume

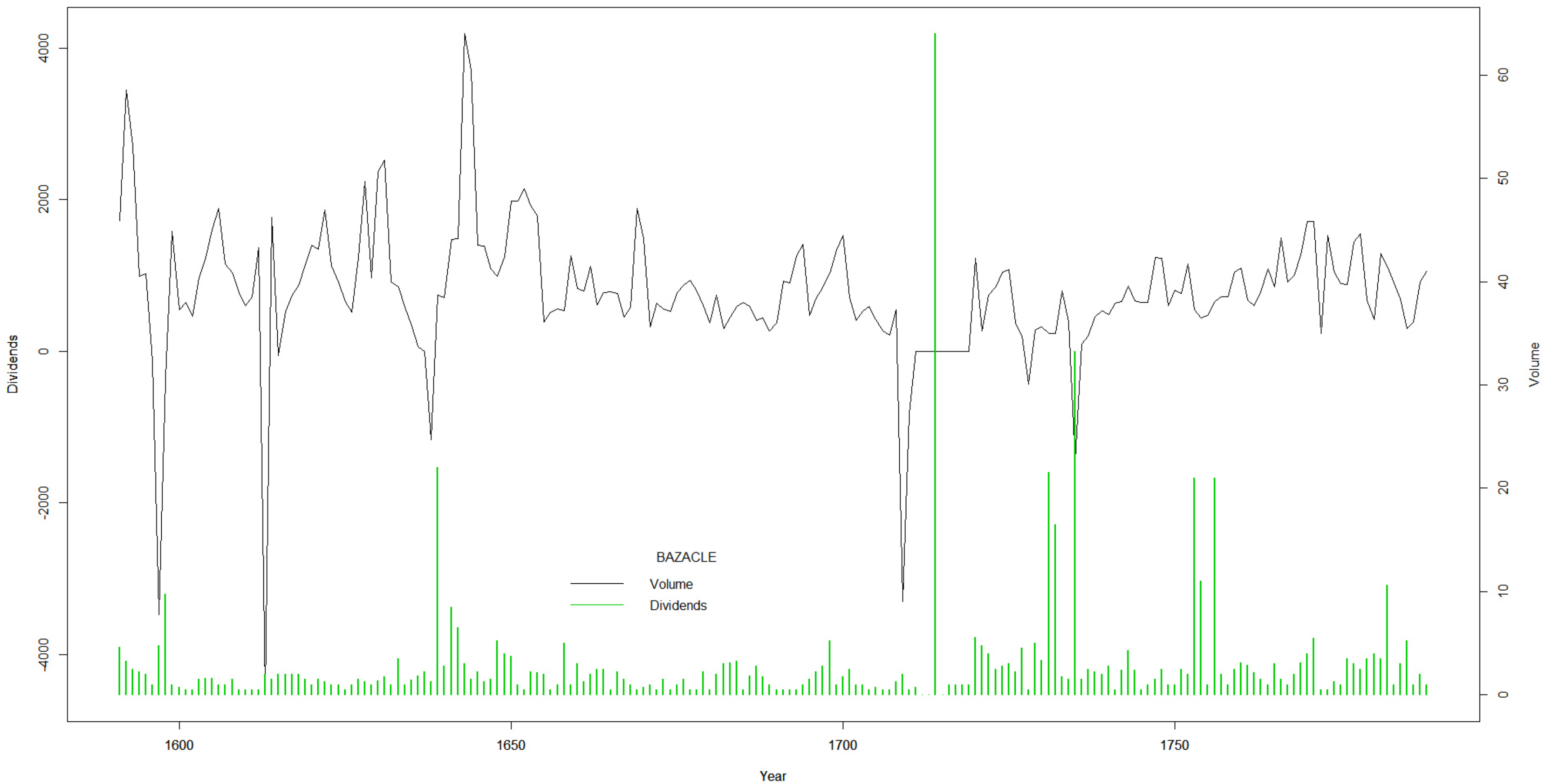


Figure 15: BAZACLE : Volume and Dividends

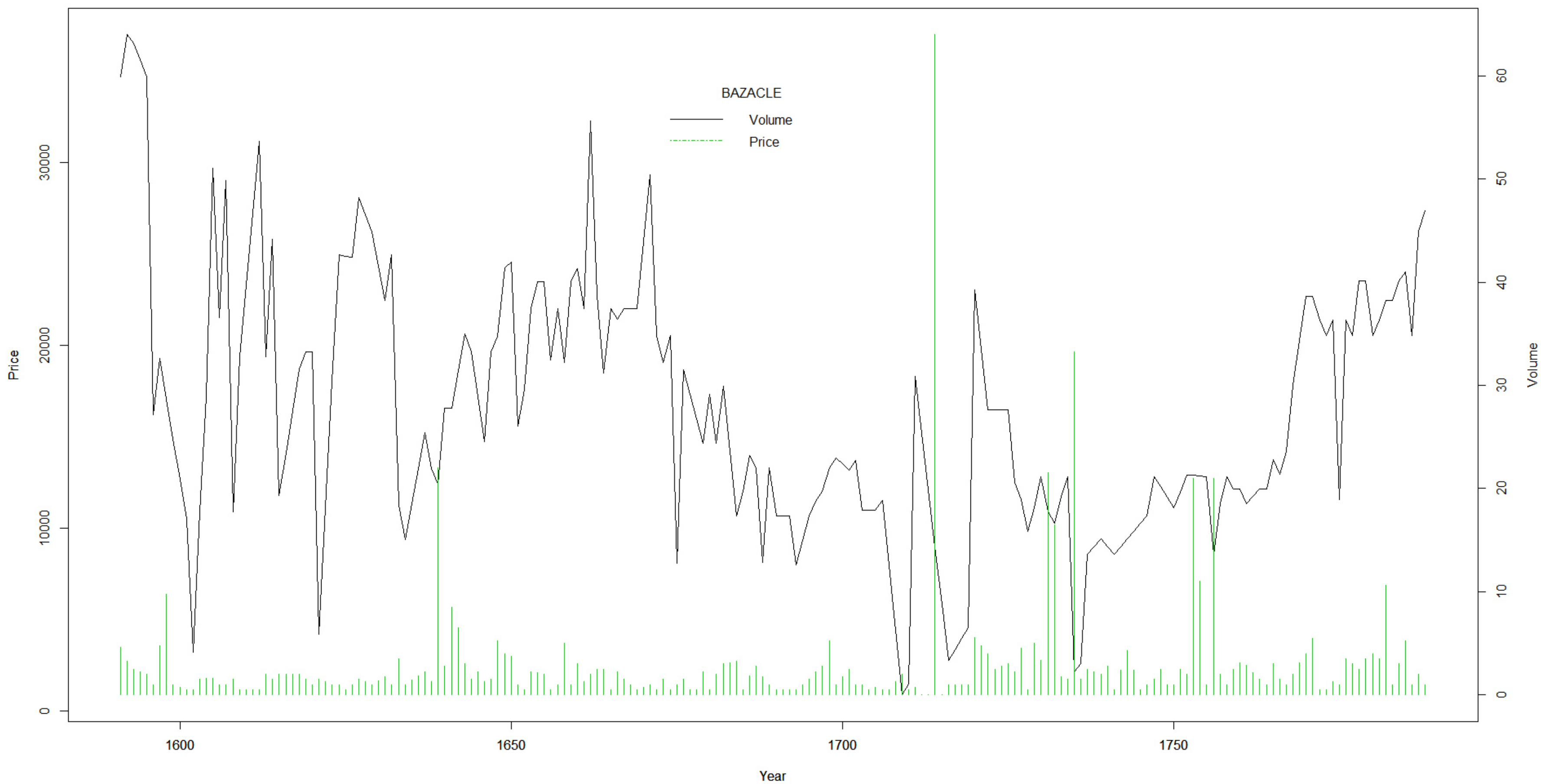


Figure 16: BAZACLE : Volume and Price

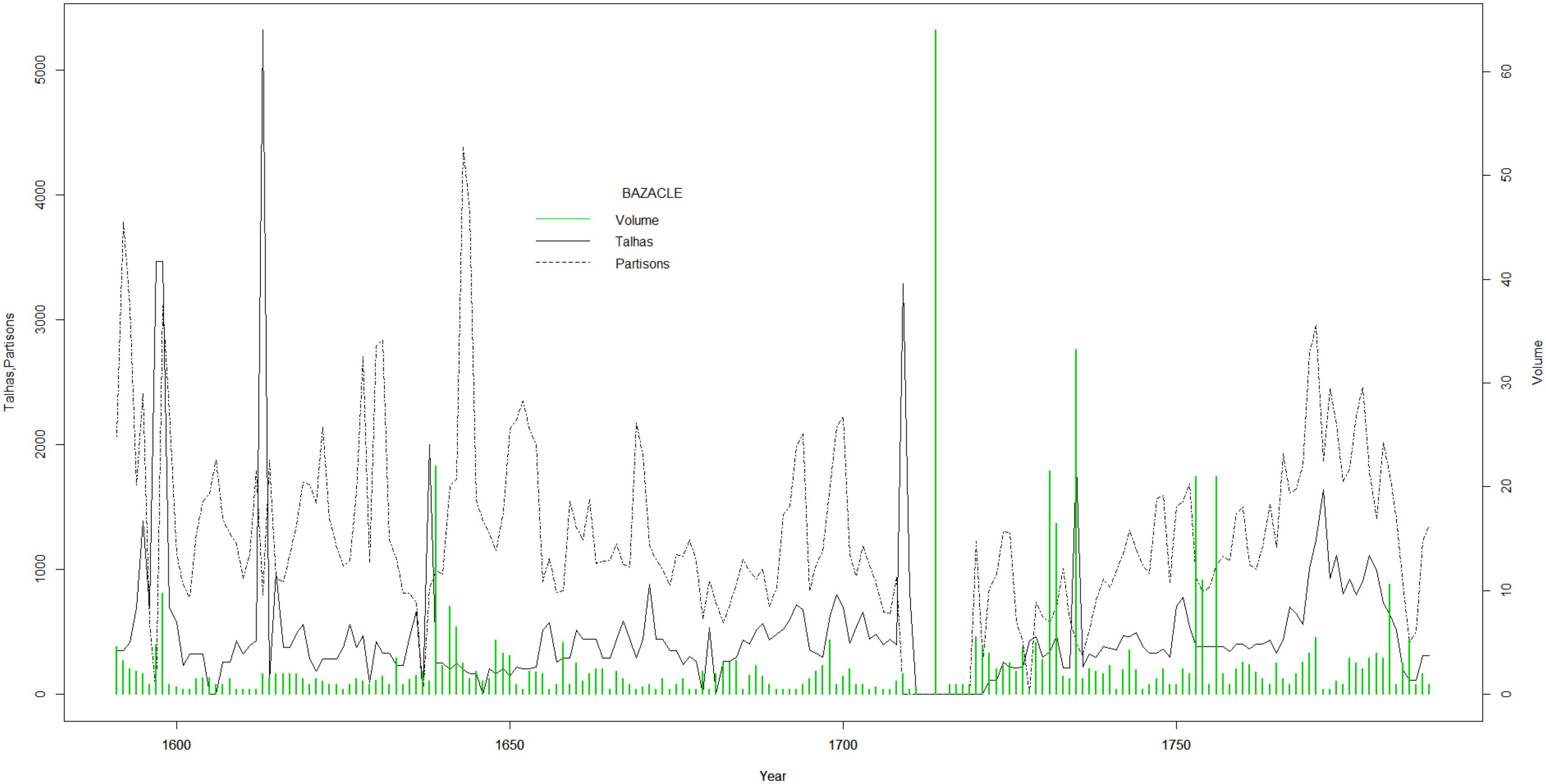


Figure 17: BAZACLE : Volume, Talhas and Partisons

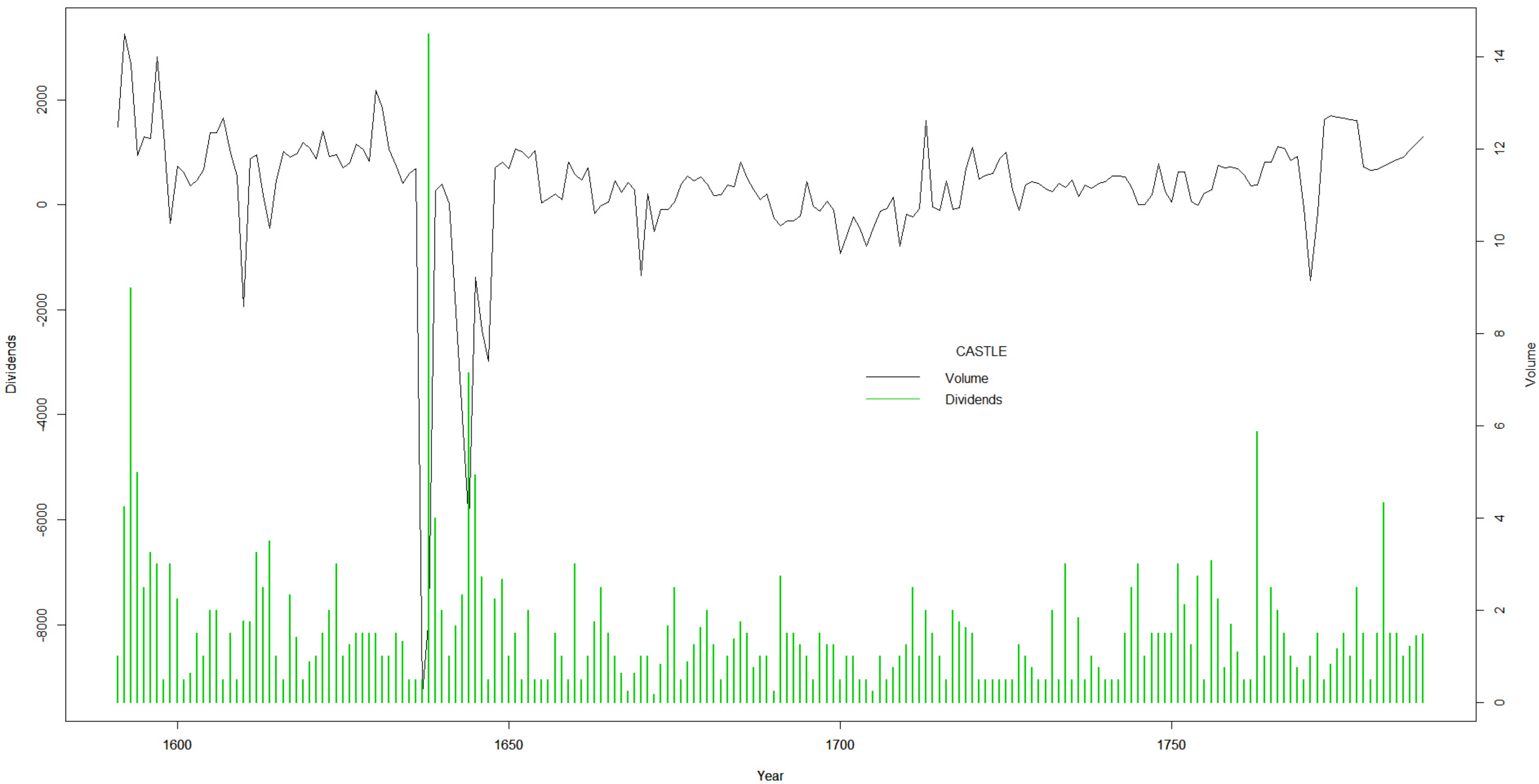


Figure 18: CASTLE : Volume and Dividends

Figure 19: CASTLE : Volume and Price

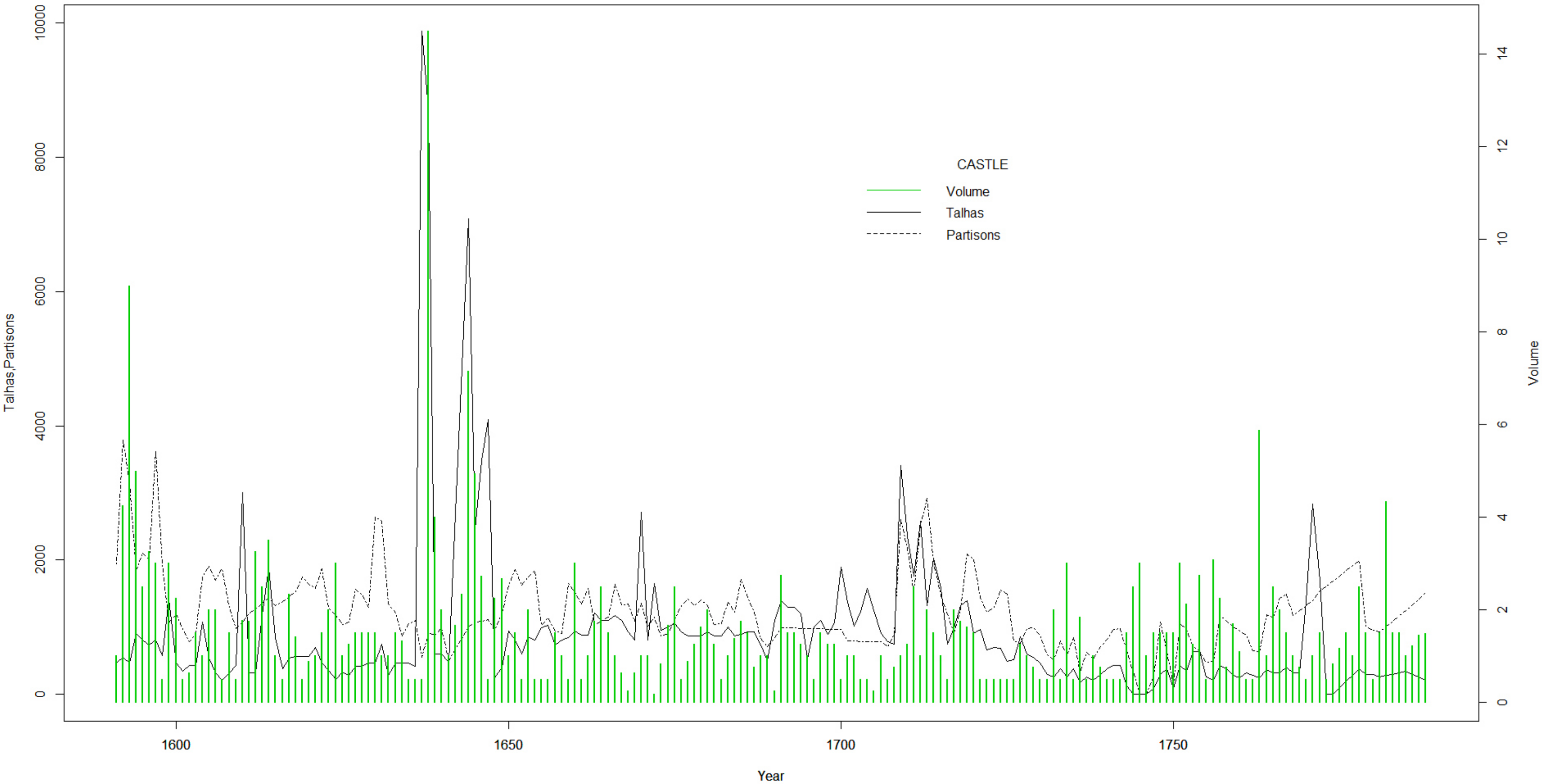


Figure 20: CASTLE : Volume, Talhas and Partisons

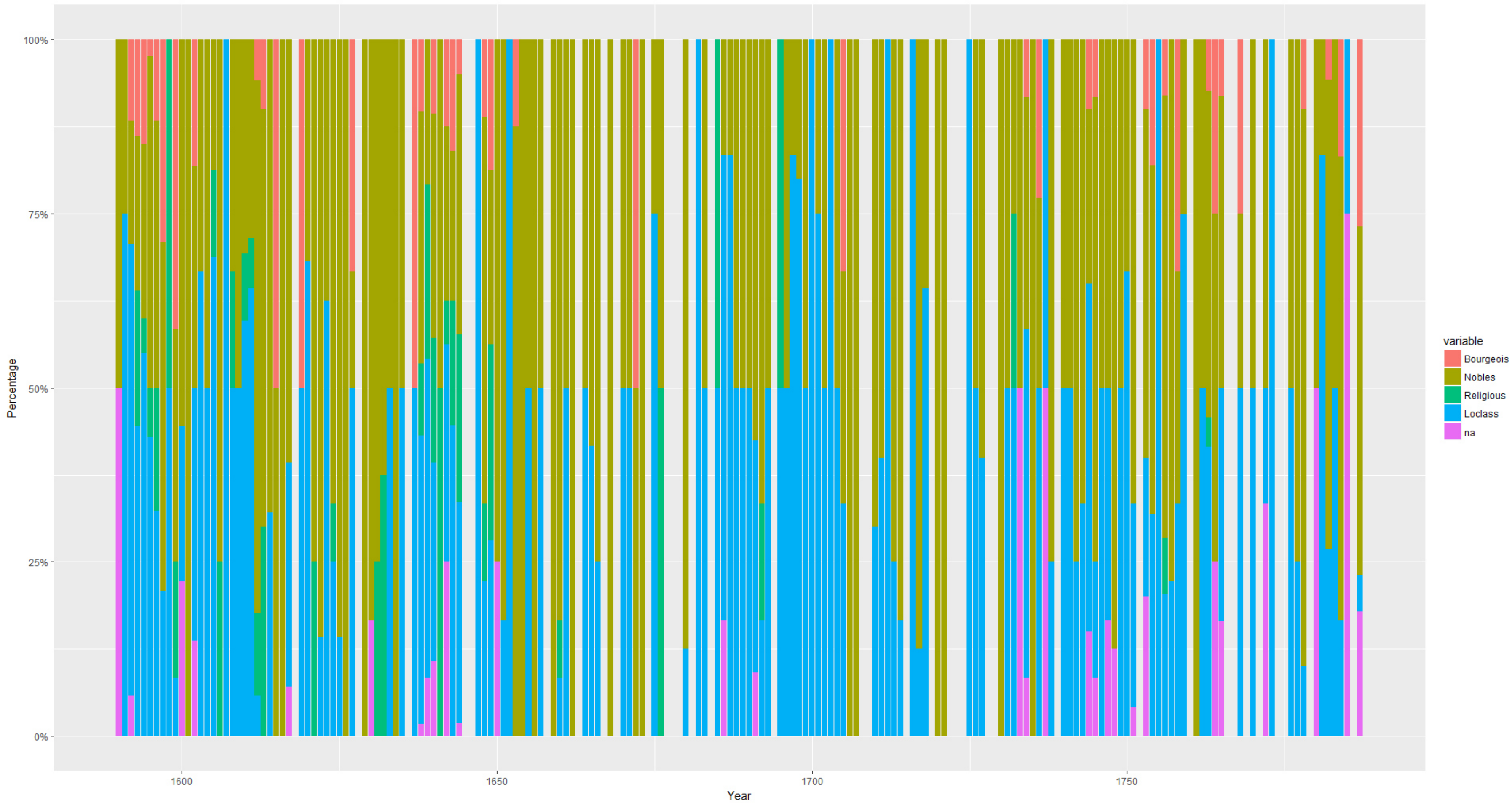


Figure 21: CASTLE : Relative volume per social groups

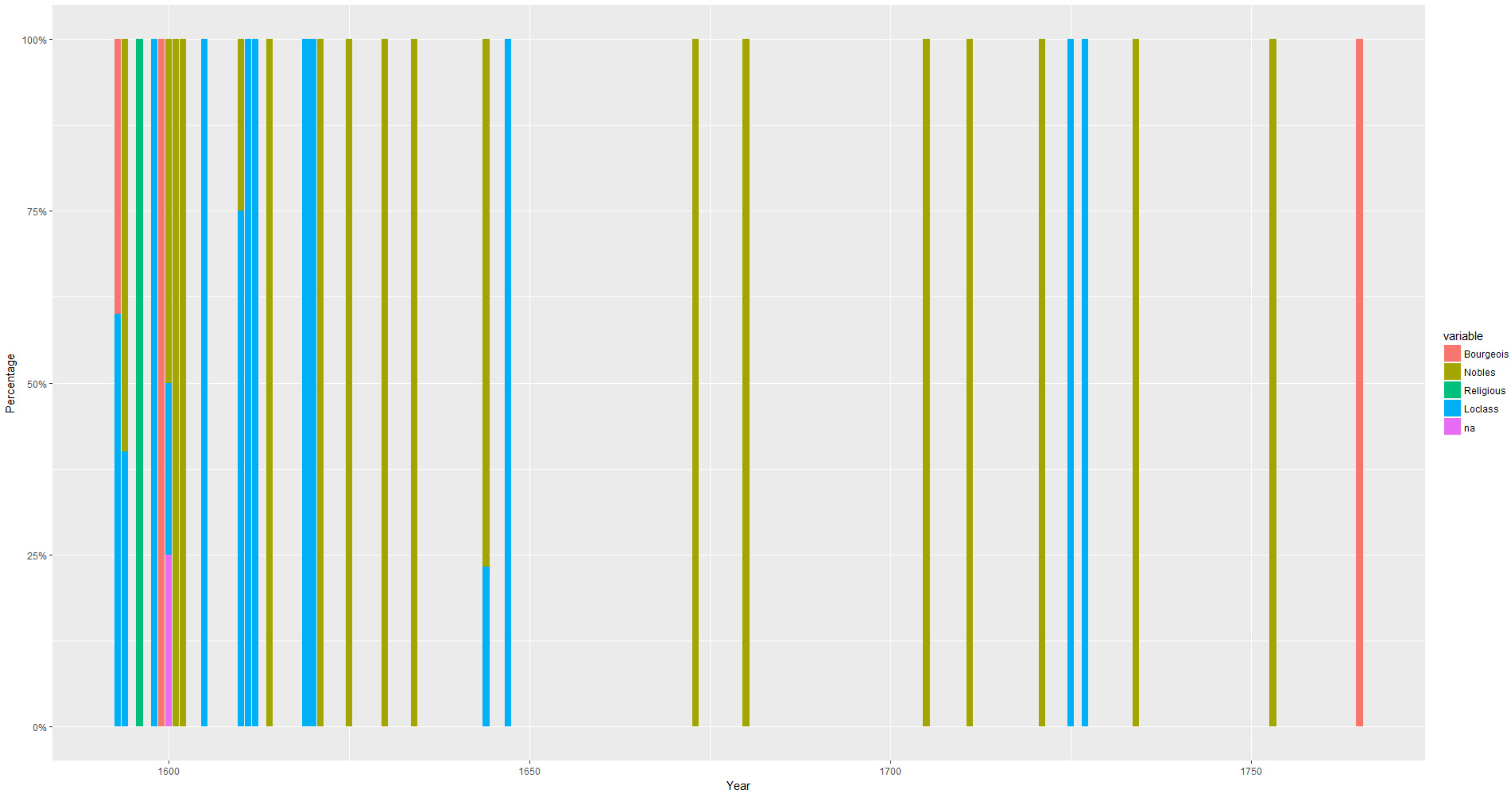


Figure 22: CASTLE : Relative volume per social groups of buyers in the case of *recrobit* contracts

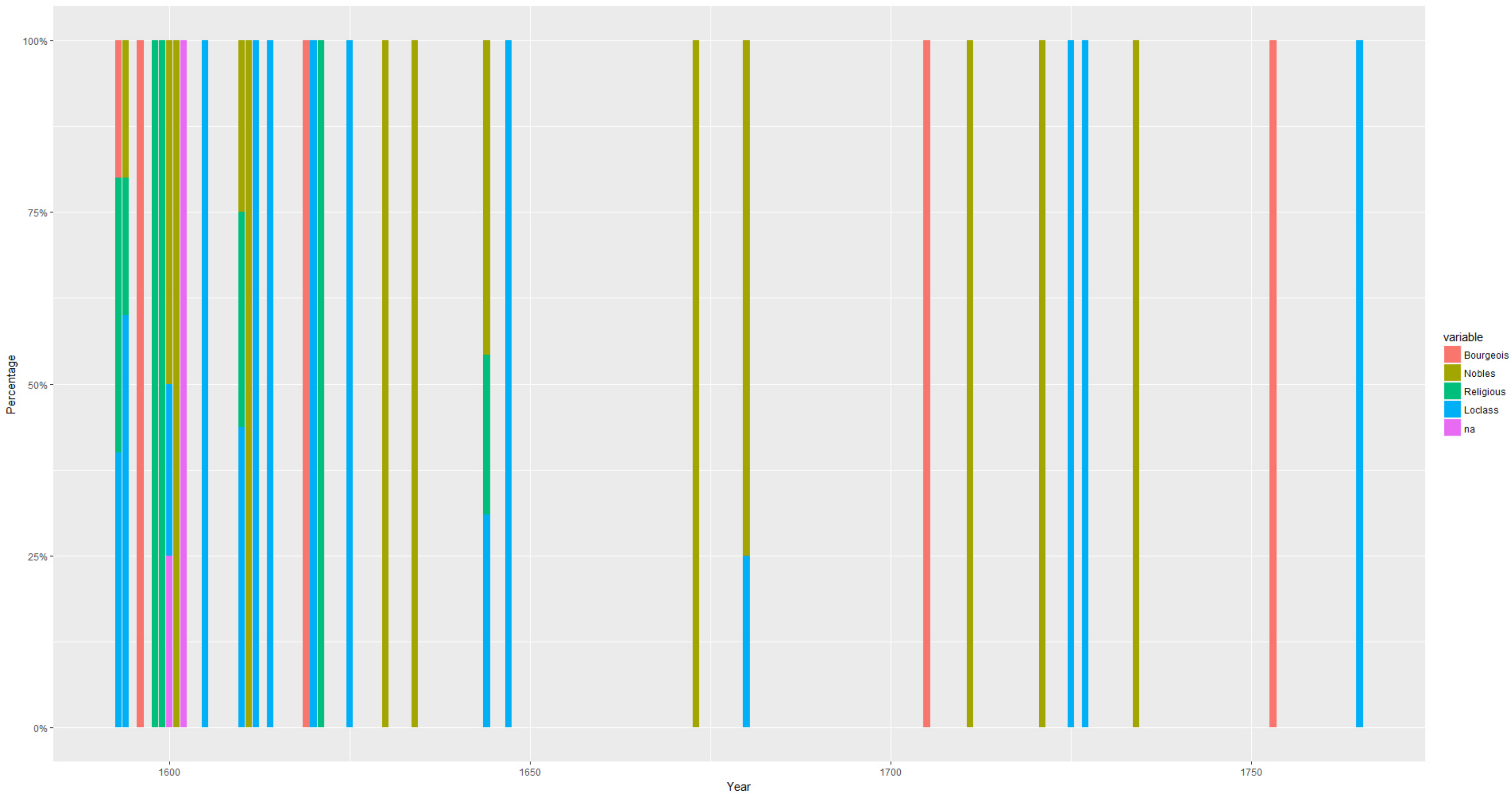


Figure 23: CASTLE : Relative volume per social groups of sellers in the case of *recrobit* contracts

Figure 24: CASTLE : Relative volume per social groups of buyers in the case of *expropriation* contracts

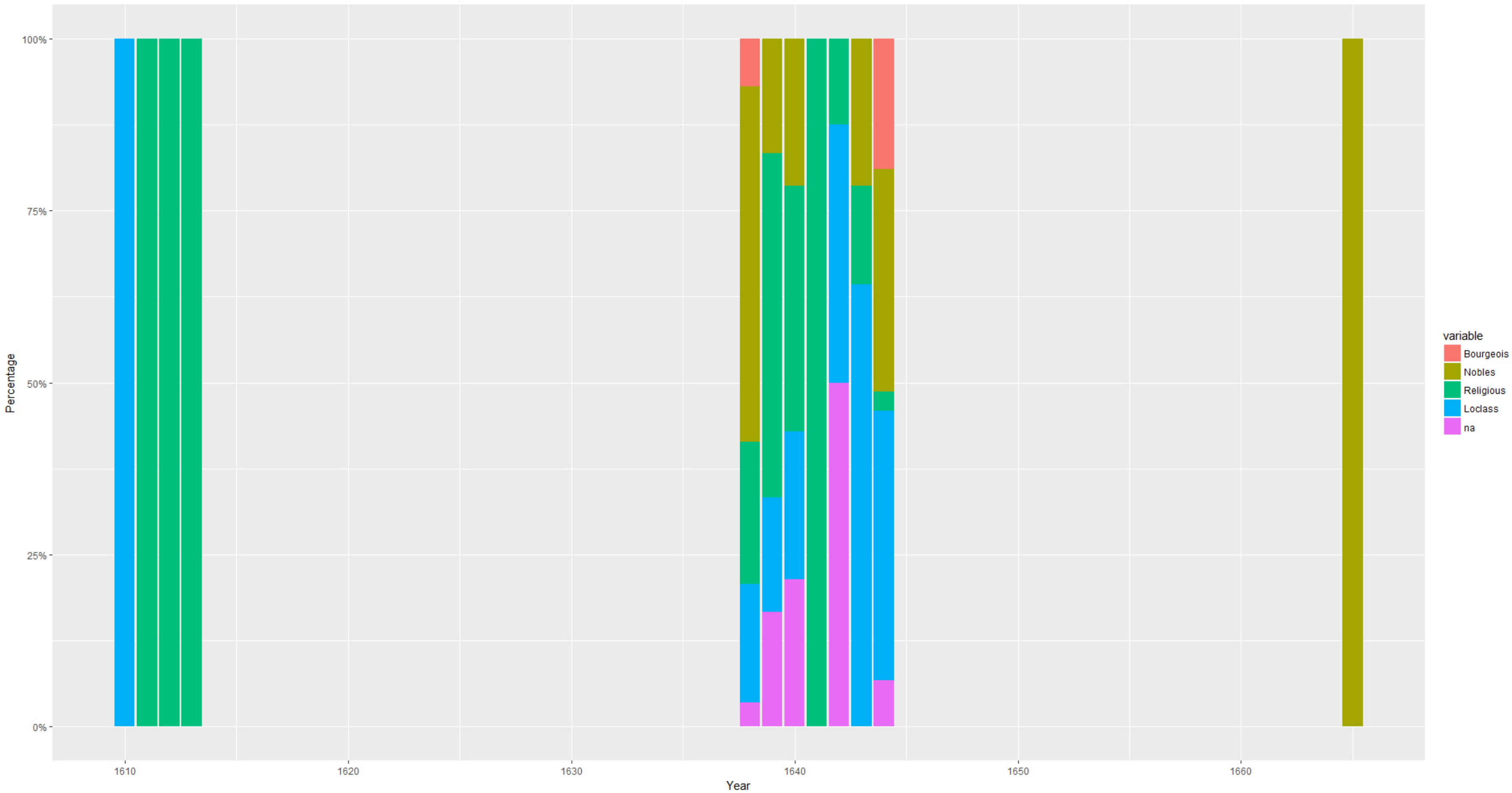


Figure 25: CASTLE : Relative volume per social groups of sellers in the case of *expropriation* contracts

Table 13: . δ_t^B (or δ_t) is a dummy variable measuring if a rare event has occured for the Castle (or in general) at time t and $\{\theta_t^i\}_{i=1,..,4}$ are different temperature measures.

	Dependent variable:	
	$BazT_t$	
	(1)	(2)
δ_{t-1}^B	84.921	
	(107.872)	
δ_t	421.312***	
	(107.872)	
θ_t^1		11.369
		(39.170)
θ_t^4		−41.711
		(49.789)
Constant	399.298***	487.640***
	(46.944)	(62.613)
Observations	198	198
R^2	0.081	0.004
Adjusted R^2	0.071	−0.006
Residual Std. Error (df = 195)	566.166	589.309
F Statistic (df = 2; 195)	8.579***	0.410
Note:	*p<0.1; **p<0.05; ***p<0.01	

Table 14:

	Dependent variable:
	$BazP_t$
θ_t^1	30.466
	(80.594)
θ_t^2	209.745
	(154.596)
θ_t^3	−216.367
	(266.608)
θ_t^4	102.166
	(64.572)
δ_{t-1}^B	−33.578
	(138.967)
Constant	−603.247
	(2,459.190)
Observations	198
R^2	0.023
Adjusted R^2	−0.002
Residual Std. Error	730.211 (df = 192)
F Statistic	0.910 (df = 5; 192)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 15:

	Dependent variable:	
	V_t^B $*1000$	V_t^B
	(1)	(2)
$BazT_t$	1.158	
	(0.801)	
$BazP_t$	0.019	
	(1.001)	
$BazT_{t-1}$	-0.277	
	(0.776)	
log(CoSil)	$-1{,}993.261$	
	(1,979.875)	
P_b	-0.045	
	(0.079)	
δ_t	$-2{,}114.028^{*}$	
	(1,193.273)	
δ_{t-1}^B	$2{,}071.922^{*}$	
	(1,185.079)	
$\frac{P_{t+1}^B - P_t^B}{P_t^B}$		-0.436
		(0.454)
Constant	33,524.120	3.446^{***}
	(28,401.300)	(0.658)
Observations	198	198
R^2	0.057	0.005
Adjusted R^2	0.022	-0.0004
Residual Std. Error	5,860.624 (df = 190)	5.928 (df = 196)
F Statistic	1.638 (df = 7; 190)	0.922 (df = 1; 196)
Note:		*p<0.1; **p<0.05; ***p<0.01

Table 16:

	Dependent variable:	
	$Shturnover^B_t$ $*10\hat{}6$	$Shturnover^B_t$
	(1)	(2)
$BazT_t$	9.901	
	(6.403)	
$BazP_t$	0.124	
	(8.001)	
$BazT_{t-1}$	−0.852	
	(6.204)	
log(CoSil)	−14,753.360	
	(15,831.400)	
P^B_t	−0.323	
	(0.632)	
δ_t	−18,496.690*	
	(9,541.603)	
δ^B_{t-1}	16,066.940*	
	(9,476.077)	
$\frac{P^B_{t+1}-P^B_t}{P^B_t}$		−0.004
		(0.004)
Constant	250,911.200	0.030***
	(227,101.300)	(0.005)
Observations	198	198
R^2	0.055	0.005
Adjusted R^2	0.020	−0.0001
Residual Std. Error	46,862.470 (df = 190)	0.047 (df = 196)
F Statistic	1.576 (df = 7; 190)	0.978 (df = 1; 196)
Note:		*p<0.1; **p<0.05; ***p<0.01

Table 17:

	Dependent variable:	
	CasT	
	(1)	(2)
δ^C_{t-1}	250.998	
	(233.921)	
δ^C_t	42.022	
	(231.046)	
θ^1_t		−94.797
		(79.692)
θ^4_t		−67.939
		(101.299)
Constant	875.048***	798.504***
	(98.986)	(127.388)
Observations	198	198
R^2	0.006	0.009
Adjusted R^2	−0.004	−0.001
Residual Std. Error (df = 195)	1,200.527	1,198.973
F Statistic (df = 2; 195)	0.630	0.885
Note:	*p<0.1; **p<0.05; ***p<0.01	

Table 18:

	Dependent variable:
	$CasP_t$
θ_t^1	14.655
	(59.065)
θ_t^2	-57.135
	(113.592)
θ_t^3	-135.049
	(195.493)
θ_t^4	-171.153^{***}
	(47.438)
δ_{t-1}^C	6.423
	(104.934)
Constant	$3{,}310.738^{*}$
	(1,807.935)
Observations	198
R^2	0.087
Adjusted R^2	0.063
Residual Std. Error	535.466 (df = 192)
F Statistic	3.644^{***} (df = 5; 192)
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 19:

	Dependent variable:	
	V_t^C $*1000$	V_t^C
$CasT_t$	0.153*	
	(0.088)	
$CasP_t$	−0.321	
	(0.211)	
$CasT_{t-1}$	0.568***	
	(0.087)	
CoSil	0.000***	
	(0.000)	
P_t^C	0.000	
	(0.013)	
δ_t^C	−19.260	
	(236.057)	
δ_{t-1}^C	−229.948	
	(238.833)	
$\frac{P_{t+1}^C - P_t^C}{P_t^C}$		0.689**
		(0.315)
Constant	125.166	0.823**
	(329.651)	(0.347)
Observations	198	198
R^2	0.354	0.024
Adjusted R^2	0.330	0.019
Residual Std. Error	1,219.334 (df = 190)	1.476 (df = 196)
F Statistic	14.882*** (df = 7; 190)	4.781** (df = 1; 196)
Notes:	***Significant at the 1 percent level.	
	**Significant at the 5 percent level.	
	*Significant at the 10 percent level.	

Table 20:

	Dependent variable:	
	$Shturnover_t^C$ $*10\hat{}6$	$Shturnover_t^C$
	(1)	(2)
$CasT_t$	1.629	
	(1.016)	
$CasP_t$	-4.980^{**}	
	(2.439)	
$CasT_{t-1}$	5.994^{***}	
	(1.011)	
CoSil	0.002^{***}	
	(0.001)	
P_t^C	0.167	
	(0.146)	
δ_t^C	1,273.015	
	(2,729.196)	
δ_{t-1}^C	-723.597	
	(2,761.292)	
$\frac{P_{t+1}^C - P_t^C}{P_t^C}$		0.008^{**}
		(0.003)
Constant	4,862.613	0.011^{***}
	(3,811.290)	(0.004)
Observations	198	198
R^2	0.281	0.027
Adjusted R^2	0.254	0.022
Residual Std. Error	14,097.430 (df = 190)	0.016 (df = 196)
F Statistic	10.587^{***} (df = 7; 190)	5.480^{**} (df = 1; 196)
Note:		*p<0.1; **p<0.05; ***p<0.01

Table 21: . Poisson regression model BAZACLE

	Dependent variable:
	V_t^B
$Cons_t$	-0.014^{***}
	(0.0003)
Constant	6.258^{***}
	(0.011)
Observations	198
Log Likelihood	−42,923.550
Akaike Inf. Crit.	85,851.090
Note:	*p<0.1; **p<0.05; ***p<0.01

Table 22: . Poisson regression model CASTLE

	Dependent variable:
	V_t^C
$Cons_t$	0.011^{***}
	(0.0002)
Constant	4.550^{***}
	(0.013)
Observations	198
Log Likelihood	−8,792.050
Akaike Inf. Crit.	17,588.100
Note:	*p<0.1; **p<0.05; ***p<0.01

1610	**1614**	**1637**	**1638**	**1642**	**1643**	**1644**	**1647**	**1670**
Medecine	Marchand	Bourgeois	Inst. Religieux	Marchand	1er pres.	Petits metier	Marchand	Nobl. De Robe
President	Marchand		NBR Sup	Divers	Marchand	Nobl. De Robe		
Nobl. De Robe	Marchand		Nobl. De Robe	Divers	Marchand	Marchand		
Marchand	Noble		Nobl. De Robe	Divers	Inst. Religieux	Nobl. De Robe		
Religieux Ind.	Noble		Bourgeois	Divers	Marchand	Divers		
Universite	Marchand		NBR Sup	Inst. Religieux	Marchand	Nobl. De Robe		
			Medecine	Marchand		Nobl. De Robe		
			Elu			Inst. Religieux		
			Divers			Marchand		
			Inst. Religieux			Marchand		
			Nobl. De Robe			Bourgeois		
			NBR Sup			Nobl. De Robe		
			Medecine			Inst. Religieux		
			Nobl. De Robe			ruraux		
			Bourgeois			Nobl. De Robe		
			Medecine					
			Nobl. De Robe					
			Marchand					
			Marchand					
			Inst. Religieux					
			Nobl. De Robe					
			Elu					

1672	**1700**	**1709**	**1710**	**1711**	**1712**	**1714**	**1771**	**1772**
March. Sup	Medecine		Nobl. De Robe	Marchand	Marchand	Noble		Noble
			NBR Sup	NBR Sup	Marchand	Marchand		Noble
				Medecine		NBR Sup		
				Noble				
				Medecine				

Table 23: : The table reports the professions of the various sellers during the most important sales recorded for the Castle Mills (Archives départementales)

1593		1638		1644	
Name	Profession	Name	Profession	Name	Profession
Vedelle	Docteur Régent	Salles		de Latanerie	Chanoine
Dutel	Bourgeois	Pelissier	Marchand	de Latanerie	Chanoine
Nolet		Pelissier	Marchand	Bajordan	Tailleur
St étienne		Figueres	Maître batteur d'or	de Lafour	Chanoine
Barthélémy	Religieux	Figueres	Maître batteur d'or	de Tousin	Chanoine
Fargues		Miquel	Marchand	De Lausluisier	Docteur
Dalbanir	Docteur régent	Miquel	Marchand	Fanier	Religieux
Canalié		Miquel	Marchand	Grangeron	Apothicaire
Precheur		Miquel	Marchand	de Bunard	Docteur et advocat
Galien	Docteur	Olivier	Trésorier général	de Bunard	Docteur et advocat
		Delpech	Bourgeois		
		de Druille	Conseiller du roi		
		Aliguier	Marchand du port Garaud		
		Aliguier	Marchand du port Garaud		
		Barthes	Marchand du port Garaud		
		Barthes	Marchand du port Garaud		
		Doudal	Marchand du port Garaud		
		Doudal	Marchand du port Garaud		
		Doudal	Marchand du port Garaud		
		Declergeault	Bourgeois		
		Denemy	Marchand du port Garaud		
		Durell et tanlanra	Marchand		

Table 24: : The table reports the names and professions of the various sellers during the three most important sales recorded for the Castle Mills (Archives départementales)

The other list

[New1] L. S. Bagwell and B. D. Bernheim. veblen effects in a theory of conspicuous consumption, 1996.

[New2] G. S. Becker, K. M. Murphy, and I. Werning. Status, lotteries and inequality. *Working Paper*, page 48, 2000.

[New3] F. Bloch and N. Quérou. Pricing in networks. *Working Paper*, page 33, 2008.

[New4] B. Cooper, C. Garcia-Penalosa, and P. Funk. Status effects and negative utility growth. *The Economic Journal*, 111(473):642–665, 2001-07.

[New5] R. Debraj and R. Arthur. Status, intertemporal choice, and risk-taking. *Econometrica*, 80(4):1505–1531, 2012.

[New6] C. Fershtman and Y. Weiss. Social rewards, externalities and stable preferences. *Journal of Public Economics*, 70(1):53–73, 1998-10.

[New7] R. H. Frank. The demand for unobservable and other nonpositional goods. *The American Economic Review*, 75(1):101–116, 1985.

[New8] M. Friedman and L. J. Savage. The utility analysis of choices involving risk. *Journal of Political Economy*, 56(4):279–304, 1948.

[New9] H. Fu, X. Xie, Y. Rui, D. Lian, G. Sun, and E. Chen. Predicting social status via social networks: A case study on university, occupation, and region. *arXiv:1604.02694 [physics]*, 2016-04-10.

[New10] J. Gali. Keeping up with the joneses: Consumption externalities, portfolio choice, and asset prices. *Journal of Money, Credit and Banking*, 26(1):1, 1994-02.

[New11] C. Ghiglino and S. Goyal. Keeping up with the neighbors: Social interaction in a market economy. *Journal of the European Economic Association*, 8(1):90–119, 2010-01.

[New12] I. Hasani-Mavriqi, F. Geigl, S. C. Pujari, E. Lex, and D. Helic. The influence of social status and network structure on consensus building in collaboration networks. *Social Network Analysis and Mining*, 6(1), 2016-12.

[New13] O. Heffetz. Conspicuous consumption and expenditure visibility : Measurement and applications, 2007.

[New14] E. Hopkins and T. Kornienko. Running to keep in the same place: Consumer choice as a game of status. *The American Economic Review*, 94(4):1085–1107, 2004.

[New15] C. S. Kumru and L. Vesterlund. The effect of status on charitable giving. *Journal of Public Economic Theory*, 12(4):709–735, 2010-07-19.

[New16] E. F. Luttmer. Neighbors as negatives: relative earnings and well-being, 2004.

[New17] N. Roussanov. Diversification and its discontents: Idiosyncratic and entrepreneurial risk in the quest for social status. *The Journal of Finance*, 65(5):1755–1788, 2010-10.

[New18] B. Sheryl. Status in markets. *The Quarterly Journal of Economics*, 116(1):161–188, 2001-02-01.

[New19] H. van de Stadt, A. Kapteyn, and S. van de Geer. The relativity of utility: Evidence from panel data. *The Review of Economics and Statistics*, 67(2):179, 1985-05.

[New20] Y. Weiss and C. Fershtman. Social status and economic performance: A survey. *European Economic Review*, page 20, 1998.

Part V

Conclusion Générale

Au cours de ce mémoire de thèse, deux thèmes majeurs ont été abordés : l'évaluation des actifs financiers et le volume des échanges sur les marchés.

Le premier à fait l'objet d'une étude empirique visant à démontrer que des modèles simples basés sur la consommation des agents peuvent expliquer les niveaux de prix observés. Pour ce faire, nous nous sommes rapportés au cas historique de deux compagnies multiséculaires : les moulins de Toulouse, dont les documents officiels ont été conservés aux archives municipales et départementales de la ville. Ainsi, nous avons pu constituer une base de données unique sur ces structures et le marché dans lequel elles évoluent. Unique avant tout par sa dimension, ce support empirique couvre près de cinq siècles d'activité et se révèle ainsi être l'un des plus long actuellement disponible en Europe. Mais aussi unique par sa richesse comme il regroupe des informations précises et individuelles sur les actionnaires.

Dans la tradition des modèles canoniques d'évaluation des actifs, nous nous sommes intéressés à l'équation fondamentale sur laquelle repose la théorie, encore appelée *équation d'Euler*, présentée en introduction de ce mémoire. Au sein de cette équation, intervient un outil macroéconomique de première importance, le facteur stochastique d'actualisation, dont le rôle est de détecter les mauvais états de la nature pour les intégrer dans le processus de détermination du prix. Ce facteur fait depuis longtemps couler beaucoup d'encre puisque sa structure est souvent désignée comme étant à l'origine de la mauvaise performance prédictive des modèles actuels. Afin de surmonter les différents échecs théoriques avérés, toute une génération de chercheur n'a cessé de sophistiquer son expression et relativement peu d'études ont envisagé que c'est l'estimation empirique même qui pouvait être en cause. Nous montrons ici que dans le cas d'une économie simplifiée où les variables d'intérêt tel que la consommation sont facilement mesurables, de simples structures avec fonction d'utilité séparable dans le temps, permettent de justifier les niveaux de prix observés. Plus précisément, nous trouvons que la volatilité de la consommation est suffisamment élevée pour qu'un modèle simple ne soit pas rejeté avec des niveaux d'aversion pour le risque sensiblement bas. De plus, nous identifions une corrélation significative entre le facteur stochastique d'actualisation, dont la structure n'a pas été spécifiée au préalable, et le taux de croissance de la consommation. Un tel résultat soutient ainsi l'intuition fondamentale de la théorie.

L'une des conséquences directes de cette étude porte sur le coût du capital calculé pour les entreprises. En effet, le taux de rentabilité espéré par les actionnaires étant calculé par le modèle d'évaluation des actifs, la célèbre énigme de l'*equity premium* rendait jusqu'alors son estimation peu fiable. Notre étude réhabilite ici ce modèle en montrant qu'il fonctionne sur une économie simplifiée dont les caractéristiques principales restent en accord avec les fondements de nos marchés modernes.

Le second thème abordé dans ce mémoire de thèse fait l'objet d'un traitement théorique d'une part et d'un traitement empirique d'autre part. La première approche vise a proposer une structure adaptée à la description des interactions sur les marchés financiers. Plus précisément, nous avons essayé de justifier le niveau de volume observé en considérant une décomposition de la population des agents selon différents groupes sociaux. Il est entre autre supposé que ces groupes se constituent sur la simple base des accointances de leurs membres. Deux personnes auront ainsi plus de chance de se connaître s'ils sont issus du même sous-ensemble que s'ils étaient issus de deux agrégats distincts. Nous avons également introduit le risque de choc de liquidité endossé par chaque groupement d'individus présent dans l'économie comme facteur des échanges.

L'étude des interactions sur les marchés de gré à gré suscite un intérêt croissant d'autant que l'on sait aujourd'hui que ces interactions forment souvent des systèmes complexes pour lesquels des comportements locaux, même minimes, peuvent engendrer de larges anomalies au niveau macroéconomique. Il convient donc de s'intéresser à la dimension incitative qui détermine l'organisation des échanges entre les agents. Il est par ailleurs important d'indiquer que la compréhension des interactions entre les acteurs d'un marché, lorsque l'on maîtrise leur caractéristiques propres, permet d'apporter des éléments de réponse quant à la formation des prix.

En utilisant la théorie des graphes comme support sous-jacent aux connexions sociales entre les individus, nous avons montré qu'il existe une relation non linéaire entre le risque d'un choc de liquidité et les incitations à l'échange. Nous observons également que dans le cas de groupes de même taille, lorsque leur nombre augmente, les incitations a échanger décroissent et ainsi le volume potentiellement décroît lui aussi. De manière similaire, pour un nombre donné de groupes, lorsqu'on accentue leur différences en matière de taille, on augmente sensiblement les incitations à échanger et par conséquent le volume espéré des échanges. Enfin, notre modèle permet aussi de caractériser toute configuration de réseau associé à un marché en fonction de sa capacité à maximiser les interactions entre les agents.

Ces résultats sont donc d'un intérêt tout particulier dans l'étude des risques de liquidité et des phénomènes de contagion qu'ils supposent au sein d'un marché globalisé. Par ailleurs, ils possèdent également de fortes implications dans le domaine de la composition de l'actionnariat des entreprises, puisqu'ils définissent selon les caractéristiques de chaque actionnaire et le nombre de groupes sociaux qu'ils constituent, qu'elle sera le niveau espéré des échanges. Ainsi, ils éclairent la relation ambivalente entre connexion sociale et préférences puisque des agents issus d'un même groupe ont plus de chance de se connaître, donc d'avoir l'opportunité d'échanger, alors que leurs préférences sont elles, bien plus proches et donc minimisent les incitations à échanger.

Le volume des échanges a également été traité dans ce mémoire au travers d'une étude empirique basée sur les données du moulin du Château Narbonnais. Cette étude est avant tout motivée par le contexte historique dans lequel la compagnie prend place puisque celui-ci a le précieux avantage d'être caractérisé par une division sociale nette de la population en termes de statut. Il y avait en effet d'un côté les nobles et les religieux qui bénéficiaient d'une place de choix dans la hiérarchie sociale puis de l'autre, les bourgeois et les classes populaires dont la position était inférieure. L'histoire des moulins a par ailleurs été marquée par des épisodes de désastres purement exogènes qui ont impacté ces derniers constituant souvent autant de chocs de liquidité pour les actionnaires. Une telle particularité nous permet ainsi d'évaluer empiriquement le rôle de ces chocs dans l'organisation des échanges au cours de ces périodes. Enfin nous mettons aussi en place un modèle où les agents peuvent investir dans un bien positionnel leur conférant un statut, et un bien non positionnel pour une consommation standard. Une telle structure nous permet de montrer comment le désir de s'élever dans la hiérarchie sociale peut affecter le schéma des échanges basé à l'origine sur les simples connexions sociales et les préférences individuelles des individus.

Nous montrons ainsi que la survenue de chocs de liquidité permet d'expliquer une part substantielle du volume présent dans l'économie des moulins. Une étude minutieuse et systématique des échanges lors des périodes de désastre révèle que la majorité des transactions prend place entre les groupes sociaux plutôt qu'au sein de ces derniers.

E Asset Pricing in Old Regime France

In this study, we examine the Toulouse mill companies, the oldest shareholding corporations known in Europe, to test asset pricing models. We collected data on dividends and share prices for the *Honor del Bazacle* and the *Narbonnais Castle* from 1591 to 1788. The total milled grain quantity in Toulouse was also used to build a proxy for local consumption. In accordance with the consumption-based asset pricing theory, we describe the prices by using a stochastic discount factor (SDF) in the Euler equation. More specifically, we decompose the SDF into an observable component and an unobservable one (as in Julliard et al. 2016), and we perform a relative entropy analysis to estimate the pricing kernel of the economy parametrically and non-parametrically. We observe that the model-free SDF correlates with the model-implied one and with consumption. A CRRA-based model with power utility is not rejected by the data for very low risk aversion levels, and some classic issues in finance, such as the *equity premium puzzle*, do not arise thanks to high consumption growth volatility.

Dans cette étude, nous proposons d'utiliser le cas historique des Moulins de Toulouse - une des plus vieilles sociétés par actions connue à ce jour en Europe - afin de tester les modèles fondamentaux de l'évaluation des actifs. Ainsi, nous avons collecté des données sur les dividendes et sur le prix des parts détenues à la fois à l'*Honor del Bazacle* et aux *Moulins du Château Narbonnais* entre 1591 et 1788. La quantité totale de grains moulus à Toulouse nous permet de construire un *proxy* pour la consommation locale. En accord avec la théorie de l'évaluation des actifs basée sur la consommation, nous décrivons les prix en utilisant un facteur stochastique d'évaluation (FSE) qui intervient dans l'équation d'Euler. Plus spécifiquement, nous décomposons le FSE en deux éléments : l'un observable et l'autre non (selon la méthodologie proposée par Julliard et. al. 2016), et nous réalisons une analyse basée sur la minimisation de l'entropie relative pour estimer le *pricing kernel* de l'économie de manière paramétrique et non paramétrique. Nous constatons que le modèle pour lequel aucune structure n'est spécifiée à priori sur le FSE, est liée à la consommation. Par ailleurs, un modèle simple basé sur une fonction d'utilité puissance n'est pas rejeté par les données et ce, même pour des niveaux d'aversion pour le risque bas. Enfin, certaines incohérences bien connues de la théorie financière telles que l'*equity premium puzzle*, n'apparaissent pas ici du fait de la forte volatilité de la consommation.

F Trading Volume And Networks

In this paper, we study the relationship between Market composition and trading volume. More precisely, we rely the existence of social groups to trading activity. In our setting, the preferences of an agent depend on his environment and an exogenous shock can be collectively experienced by the group members. We also introduce the concept of desirability channels as a preliminary outcome that leads to the volume determination. We are able to generate closed formula for both of these expected quantities and we measure how the social architecture affects them. We also rely the topology of the underlying network to the occurrence of exogenous shocks. Thus, we find that for equally sized groups, if their number increases, the incentives to trade and thereby the volume decrease. However, for a given number of groups, when they become more dissimilar in terms of size, new desirability channels are created and the volume potentially increases. We also show that a nonlinear relationship exists between the incentives to trade and the shock probability. Finally, given the agents preferences, we propose a characterization of any network regarding to its capacity to maximize the trading activity.

Dans cette étude, nous nous intéressons au lien entre la structure de marché en termes de groupes sociaux et le volume des échanges. Plus précisément, nous tentons de relier l'existence de sous-agrégats dans la population à l'activité de marché. Dans notre modèle, les préférences d'un agent dépendent de son environnement et un choc exogène peut impacter de manière collective les membres de chaque groupe. Nous introduisons également le concept de *canal de désirabilité* comme condition nécessaire menant à la détermination du volume. Nous sommes ainsi capables d'exprimer à la fois le volume espéré mais aussi le nombre espéré de canaux de désirabilité sous forme analytique pour plusieurs cas particuliers. Ces premiers résultats nous permettent alors de mesurer comment la structure du réseau de marché va affecter ces quantités. En termes plus mathématiques, on parlera de topologie, laquelle pourra être modifiée par la survenue d'un choc sur un ou plusieurs groupes. Ainsi, nous trouvons que, pour des groupes de tailles identiques, si leur nombre augmente, les incitations à échanger et par conséquent le volume, décroissent. De manière symétrique, pour un nombre de groupes donné, lorsque ces derniers se différencient d'avantage par leur taille, de nouveaux canaux de désirabilité apparaissent et le volume peut augmenter. Par ailleurs, nous montrons également qu'il existe une relation non linéaire entre les incitations à échanger et la probabilité d'observer un choc. Enfin, le cadre théorique que nous proposons nous permet également de caractériser n'importe quel réseau selon sa capacité à maximiser l'activité de marché.

G Social Status, Liquidity Shocks And Trading Volume

This paper investigates the relationship between trading volume and social status concern. Following the Hirsch (1977) setting, we propose a model where heterogeneous agents can trade two types of goods, a *positional* and a *non positional* one. The former can be owned in binary quantities only, that is an agent holds one unit of good or nothing and the latter can be freely accumulated over time. We use a pairing-dependent utility function that both captures the absolute utility of the agent and determines his best partner through a status comparison. The economy is splitted into two social groups, those who have a status and those who don't have one. We randomly generate the social connections between the agents according to their group belonging and we assume that every individual can be impacted by a liquidity shock. We depict how the trades take place over time regarding to the groups and justify in which case the inter group trades dominate. Finally, we test our predictions on a very suitable historical support, the Toulouse mills companies.

Nous étudions ici le lien entre niveau d'échange sur un marché et quête de statuts des investisseurs. Suivant les travaux de Hirsch (1977), nous proposons ici un modèle avec agents hétérogènes pouvant échanger deux types de biens, l'un dit *positionnel* et l'autre *non positionnel*. Le premier peut être détenu uniquement par lot unitaire, c'est-à-dire que pour chaque transaction, les investisseurs ont le choix d'acheter ou vendre exactement une unité du bien ou ne rien faire. Le second peut quant à lui être librement échangé sans restriction. Nous utilisons ici une fonction d'utilité dépendant de l'identité de la contrepartie dans l'échange, elle capture à la fois l'utilité absolue des agents et permet de définir son meilleur partenaire en termes de statut. La population est partitionnée selon différent groupes sociaux, ceux qui ont un statut et ceux qui n'en ont pas. Nous générons alors aléatoirement les connections sociales entre les agents selon leur appartenance sociale et nous supposons que chaque individu peut être impacté par un choc de liquidité. Nous décrivons ainsi comment les échanges prennent place au cours du temps selon la structure des groupes et justifions dans quel cas les échanges inter-groupes dominent. Enfin, nous testons nos prédictions sur des données historiques dont le cadre est particulièrement bien adapté à cette étude : l'économie des moulins de Toulouse.

Résumé

Ce mémoire de thèse est organisé en trois articles. Le premier est dédié au cas des moulins de Toulouse dont les données nous permettent de tester certains points de la théorie de l'évaluation des actifs. Plus précisément, nous proposons une mesure de la consommation locale et réalisons une analyse basée sur l'entropie relative pour extraire le facteur stochastique d'actualisation de cette économie. Nous observons que ce dernier est lié à la consommation et qu'un modèle simple à la Lucas n'est pas rejeté pour des niveaux d'aversion pour le risque bas. Dans le second article, nous décrivons de manière purement théorique la relation entre le volume d'échange et la composition du marché par le biais d'un modèle où les préférences d'un agent dépendent de son environnement et où un choc de liquidité peut survenir de manière collective pour tous les membres d'un même groupe. Nous introduisons alors le concept de *canal désirable* comme condition nécessaire à la réalisation d'un échange et lions la topologie du réseau au volume espéré des échanges. Le troisième article porte sur le rôle des statuts sociaux dans la dynamique de marché. Nous proposons un modèle où deux types de biens sont disponibles, un bien *positionnel* et un bien *non positionnel*. En distinguant dans l'économie ceux possédant un statut et ceux qui n'en possèdent pas nous justifions comment les échanges prennent place au cours du temps par rapport à cette distinction sociale. Les prédictions du modèle sont alors testées sur les données historiques des moulins de Toulouse.

Abstract

This doctoral thesis is organized in three articles. In the first one, we use the Toulouse mills companies data as a suitable testbed for asset pricing theory. More precisely, we provide a proxy for local consumption and perform a relative entropy analysis to extract the stochastic discount factor of this old economy. We found that the model-free pricing kernel correlates with consumption and a standard CRRA-model is not rejected by the data, even for very low risk aversion levels. In the second article, we describe the relationship between trading volume and market composition through a pure theoretical approach. We build a model where the agent preferences depend on his environment and a liquidity shock is collectively experienced by the members of each social group in the economy. We introduce the concept of *desirability channel* as a necessary condition for a trade to occur and we rely the topology of the network to the expected volume. The third article focus on the role of social status concern in the exchanges dynamic. We propose a setting where two types of goods are available, a *positional* and a *non positional* one. By splitting the economy into two social groups, we depict how trades take place over time regarding to these social groups. The model predictions are finally tested on the historical support of the Toulouse mills companies.