New, Like New, or Very Good?
Reputation and Credibility:
Online Appendix

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In Theorem 1 of the main article, we characterise the maximal set of parameters for existence of an honest equilibrium, which is obtained by assuming that $\pi^*_{Bb}(0) = 0$ and thus, $V^*_h(0) = \ell/(1-\delta) = V^*_\ell(0)$. This implies a discontinuity of V^*_h at $\mu = 0$ because i) $V^*_h(\mu) = hp^*_G(\mu) + \delta h V^*_h(\pi^*_{Gg}(\mu)) + \delta(1-h)V^*_h(\pi^*_{Bb}(\mu))$ for $\mu > 0$ and ii) $\lim_{\mu\to 0}\pi^*_{Bb}(\mu) > 0$ as implied by the fact that $p^*_G(\mu) = \delta(V^*_\ell(\pi^*_{Bb}(\mu)) - V^*_\ell(0)) \geq \ell$ for all $\mu \in (0, \bar{\mu})$. However, an honest equilibrium without such a discontinuity also exists if

$$h \ > \ \frac{1+\sqrt{1+4\ell^2+4\ell^3}}{2+2\ell}, \tag{1}$$

which is a sufficient condition[1] that guarantees optimality of truth-telling for an h-type seller when we set $\pi^*_{Bb}(0) = \lim_{\mu\to 0}\pi^*_{Bb}(\mu) > 0$ and thus, an h-type seller would tell the truth even at $\mu = 0$ by continuity. The value functions of such an equilibrium are identical to V^*_ℓ and V^*_h that we obtained for Theorem 1 in the main article, except for the value of $V^*_h(0)$ which is now higher at $\lim_{\mu\to 0} V^*_h(\mu)$. To indicate this difference, we denote h-type's equilibrium value function in the current case as $\tilde{V}^*_h$. We first state and prove this result in Proposition 1 below, which will be used in the main contents of this online Appendix that ensue, namely, the proofs of Theorems 2 and 3 of the main article.

Proposition 1 *If (1) holds and δ is large enough, there exists an honest equilibrium in which an h-type seller announces truthfully even when $\mu = 0$. The value functions V^*_ℓ and $\tilde{V}^*_h$ are continuous on* $[0, 1]$.

Proof. Let y^*, p^* and π^* be the same as those in the proof of Theorem 1 with the following modification: in the off-equilibrium contingency that a bad quality is truthfully announced when $\mu = 0$, the reputation is updated to $\tilde{\pi}^*_{Bb}(0) = \lim_{\mu\to 0}\pi_{Bb}(\mu, y^*(\mu))$. Then, V^*_ℓ is also the same as in Theorem 1 and optimality of an ℓ-type seller's announcement strategy is

[1] An alternative but more tedious proof available upon request shows that a weaker sufficient condition is $h > 1/(2-\ell)$.

verified by the same argument as that in the proof of Theorem 1, except for the case that $q = b$ when $\mu = 0$. In this latter case, lying and telling the truth are equivalent for an ℓ-type seller by continuity because the same is true when $q = b$ for all $\mu \in (0, \bar{\mu})$ as shown in the main article if $\delta > \delta_\ell$, hence $y^*(0) = 1$ is optimal.

The value function of an h-type from always telling the truth is $\tilde{V}_h^*(\mu) = V_h^*(\mu)$ for $\mu > 0$ as obtained in the proof of Theorem 1. So, we use $\tilde{V}_h^*$ and V_h^* interchangeably for $\mu > 0$ for easy reference to earlier results on V_h^*. We set $\tilde{V}_h^*(0) = \lim_{\mu \to 0} V_h^*(\mu)$. We will first prove that

$$V_h^*(\mu) - \tilde{V}_h^*(0) > V_\ell^*(\mu) - V_\ell^*(0) \quad \forall \mu > 0 \tag{2}$$

for sufficiently large $\delta < 1$, then establish optimality of truth-telling for h-type.

Denote the ex-ante payoff from the possibility of always drawing $q = g$ as

$$V_h^o(\mu) := h \sum_{t=0}^{\infty} h^t \delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) \quad \forall \mu > 0,$$

so that

$$V_h^*(\mu) = V_h^o(\mu) + (1-h)\delta \sum_{t=0}^{\infty} h^t \delta^t V_h^*\big(\pi_{Bb}(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big) \quad \forall \mu > 0 \tag{3}$$

and $\frac{dV_h^*(\mu)}{d\mu} > \frac{dV_h^o(\mu)}{d\mu}$ for almost all $\mu > 0$ because V_h^* is strictly increasing[2] and so do $\pi_{Gg}^t(\mu)$ and $\pi_{Bb}^*(\mu)$ (this follows from Lemmas 5 and 6 in the Appendix A.3 of the main article).

In conjunction with (31) in the Appendix A.3, we have

$$V_h^o(\mu) - V_\ell^*(\mu) = \Big[\sum_{t=0}^{\infty} (h^{t+1} - \ell^t)\delta^t p_G\big(\pi_{Gg}^t(\mu), y^*(\pi_{Gg}^t(\mu))\big)\Big] - \delta V_\ell^*(0)\frac{1-\ell}{1-\delta\ell}.$$

For $\mu \geq \bar{\mu}$, since $y^*(\mu) = 1$ we have

$$\frac{dV_h^o(\mu)}{d\mu} - \frac{dV_\ell^*(\mu)}{d\mu} = \sum_{t=0}^{\infty} (h^{t+1} - \ell^t)\delta^t \frac{\partial p_G\big(\pi_{Gg}^t(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^t(\mu)}{d\mu} \tag{4}$$

$$\begin{aligned}
&= \sum_{t=0}^{\infty} \delta^{2t}\Bigg[(h^{2t+1} - \ell^{2t})\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} \\
&\qquad + \delta(h^{2t+2} - \ell^{2t+1})\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t+1}(\mu)}{d\mu}\Bigg] \\
&> \sum_{t=0}^{\infty} \delta^{2t}\ell^{2t}\Bigg[(h-1)\frac{\partial p_G\big(\pi_{Gg}^{2t}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu} \\
&\qquad + \delta(h^2 - \ell)\frac{\partial p_G\big(\pi_{Gg}^{2t+1}(\mu), 1\big)}{\partial \mu}\frac{d\pi_{Gg}^*(\pi_{Gg}^{2t}(\mu))}{d\mu}\frac{d\pi_{Gg}^{2t}(\mu)}{d\mu}\Bigg].
\end{aligned} \tag{5}$$

[2]This is because $p_G^*(\cdot)$ is strictly increasing and $V_h^*(\mu)$ is the same probabilistic sum of $p_G(\cdot)$'s across the posteriors after possible future histories from the current reputation μ, each of which is an increasing function of μ.

As an intermediate step to determining the sign of (5), we calculate

$$(h-1)\frac{\partial p_G\big(\mu,1\big)}{\partial\mu}+(h^2-\ell)\frac{\partial p_G\big(\pi^*_{Gg}(\mu),1\big)}{\partial\mu}\frac{d\pi^*_{Gg}(\mu)}{d\mu}$$
$$=-\frac{h(1-h)(1-\ell)}{(1-(1-h)\mu)^2}+\frac{h^2(h^2-\ell)(1-\ell)\ell}{(\ell(1-\mu)+h^2\mu)^2}, \tag{6}$$

the derivative of which is

$$-2(1-\ell)\frac{h(1-h)^2}{(1-(1-h)\mu)^3}-\frac{\ell(h^3-h\ell)^2}{(\ell(1-\mu)+h^2\mu)^3}<0. \tag{7}$$

It is routinely verified that (6) evaluated at $\mu=1$ is positive if (1) holds and thus, (6) is positive for all μ due to (7). This further implies that (5) is positive for all $\mu\geq\bar\mu$ when $\delta<1$ is sufficiently close to 1 and consequently, because $\frac{dV^*_h(\mu)}{d\mu}>\frac{dV^o_h(\mu)}{d\mu}$ as asserted above,

$$\frac{dV^*_h(\mu)}{d\mu}\geq\frac{dV^*_\ell(\mu)}{d\mu}\quad\forall\mu\geq\bar\mu. \tag{8}$$

Next, let $\mu_m=\min\{\mu|\pi^*_{Gg}(\mu)\geq\bar\mu$ and $\pi_{Bb}(\mu,y^*(\mu))\geq\bar\mu\}$ and consider $\mu\in[\mu_m,\bar\mu]$. Note that $\mu_m<\bar\mu$ due to Lemmas 4 and 5 of the Appendix A.3 of the main article. Since

$$\begin{aligned}V^*_h(\mu)&=hp_G(\mu,y^*(\mu))+\delta\big(hV^*_h(\pi^*_{Gg}(\mu))+(1-h)V^*_h(\pi_{Bb}(\mu,y^*(\mu)))\big)\text{ and}\\ V^*_\ell(\mu)&=p_G(\mu,y^*(\mu))+\delta\big(\ell V^*_\ell(\pi^*_{Gg}(\mu))+(1-\ell)V^*_\ell(0)\big),\end{aligned}$$

we deduce that $\frac{dV^*_h(\mu)}{d\mu}-\frac{dV^*_\ell(\mu)}{d\mu}$, which exists almost everywhere because both $V^*_h(\mu)$ and $V^*_\ell(\mu)$ are continuous and increasing, is equal to the derivative of

$$(1-h)\big(\delta V^*_h(\pi_{Bb}(\mu,y^*(\mu)))-p_G(\mu,y^*(\mu))\big)+\delta\big(hV^*_h(\pi^*_{Gg}(\mu))-\ell V^*_\ell(\pi^*_{Gg}(\mu))\big),$$

which is positive due to (8) because $p_G(\mu,y^*(\mu))=\delta\big(V^*_\ell(\pi_{Bb}(\mu,y^*(\mu)))-V^*_\ell(0)\big)$ for $\mu\leq\bar\mu$. Repeated application of analogous argument establishes that $\frac{dV^*_h(\mu)}{d\mu}>\frac{dV^*_\ell(\mu)}{d\mu}$ for all $\mu>0$ when $\delta<1$ is large enough so long as (1) holds. This proves (2).

Then, as in Theorem 1, an h-type strictly prefers to tell the truth whenever an ℓ-type seller weakly prefers to do so, which is the case when $q=g$ or when $q=b$ and $\mu\in[0,\bar\mu]$. For the cases that $q=b$ and $\mu>\bar\mu$, it suffices to establish that an h-type seller prefers to tell the truth when $\mu=1$, which amounts to showing that $1+\delta\tilde V^*_h(0)\leq\delta h/(1-\delta)$. From

$$\tilde V^*_h(0)=h\ell+\delta h\tilde V^*_h(0)+\delta(1-h)V^*_h(\tilde\pi^*_{Bb}(0))$$

and $V^*_h(\mu)\leq h/(1-\delta)$ for all μ, we obtain

$$1+\delta\tilde V^*_h(0)\leq1+\delta\frac{h\ell+\delta(1-h)h/(1-\delta)}{1-\delta h}=1+\frac{\delta h}{1-\delta}-\frac{\delta h(1-\ell)}{1-\delta h}$$

which is less than $\delta h/(1-\delta)$ if $\delta h(2-\ell)>1$. As (1) implies that $h(2-\ell)>1$, the desired condition is satisfied if $\delta<1$ is sufficiently close to 1. ■

1 Proof of Theorem 2

Recall that we augment our baseline model as follows:

a) In every period, the seller receives a signal $\hat{q}_t \in \{g, b\}$ such that $\Pr(\hat{q}_t = g \mid q_t = g) = 1 > \Pr(\hat{q}_t = b \mid q_t = b) = \alpha$ where α is close to 1.

b) In every period, the seller is replaced by another seller of the other type with a small, exogenous probability $1-\beta$, where $\beta < 1$ is close to 1.

The replacement is not observed by potential buyers. We define δ as the seller's discount factor adjusted for survival, i.e., the discount factor multiplied by the survival rate β.

We fix β and δ close to 1 (this will be made precise below) and show that an honest equilibrium exists for sufficiently large α. Under our assumption that the seller knows for sure that quality is bad when the signal is bad, an ℓ-type seller would never report a good signal as bad because this would reveal his type without short-term gain, but may report a bad signal as good, say with probability y. The equilibrium conditions for ℓ-type are the same as for $\beta = \alpha = 1$ but with different formulae for the equilibrium price and transition functions for the market's update of reputation. In what follows, for conciseness we will use "hat" to denote functions that are evaluated for β and α different from 1. Then, the prices for B and G are $\hat{p}_B^*(\mu) = 0$ and $\hat{p}_G^*(\mu) = \hat{p}_G(\mu, y^*(\mu))$ where:[3]

$$\hat{p}_G(\mu, y) = \frac{\mu h + (1-\mu)\ell}{\mu(h + (1-h)(1-\alpha)) + (1-\mu)(\ell + (1-\ell)(1-\alpha+\alpha y))}.$$

The transition functions are $\hat{\pi}_{Gg}^*(\mu)$, $\hat{\pi}_{Bb}^*(\mu) = \hat{\pi}_{Bb}(\mu, y^*(\mu))$ and $\hat{\pi}_{Gb}^*(\mu) = \hat{\pi}_{Gb}(\mu, y^*(\mu))$ where

$$\begin{aligned}
\hat{\pi}_{Gg}^*(\mu) &= (2\beta - 1)\frac{\mu h}{\mu h + (1-\mu)\ell} + 1 - \beta \\
\hat{\pi}_{Bb}(\mu, y) &= (2\beta - 1)\frac{\mu(1-h)}{\mu(1-h) + (1-\mu)(1-\ell)(1-y)} + 1 - \beta \\
\hat{\pi}_{Gb}(\mu, y) &= (2\beta - 1)\frac{\mu(1-h)(1-\alpha)}{\mu(1-h)(1-\alpha) + (1-\mu)(1-\ell)(1-\alpha+\alpha y)} + 1 - \beta
\end{aligned}$$

These price and transition functions increase in μ. Raising y increases $\hat{p}_G$ and $\hat{\pi}_{Bb}$ but decreases $\hat{\pi}_{Gb}$. The key difference from the baseline model is that when $\alpha < 1$, $\hat{\pi}_{Gb}$ is positive, increases in μ and decreases in y. Note that $\hat{\pi}_{Gb}$ and its derivatives converge to 0 when α tends to 1. More precisely, there exists $\rho > 0$ such that for all α and $\mu \in [1-\beta, \beta]$,

$$\begin{cases} \frac{\partial \hat{\pi}_{Gb}(\mu,y)}{\partial \mu} = \frac{(2\beta-1)(1-\ell)(1-\alpha+\alpha y)(1-h)(1-\alpha)}{[\mu(1-h)(1-\alpha)+(1-\mu)(1-\ell)(1-\alpha+\alpha y)]^2} < \rho(1-\alpha) \\ \left|\frac{\partial \hat{\pi}_{Gb}(\mu,y)}{\partial y}\right| = \frac{(2\beta-1)\mu(1-\mu)(1-\ell)\alpha(1-h)(1-\alpha)}{[\mu(1-h)(1-\alpha)+(1-\mu)(1-\ell)(1-\alpha+\alpha y)]^2} < \rho(1-\alpha). \end{cases} \quad (9)$$

This will imply that a small observation error has a small effect on the strategy of an ℓ-type seller.

[3] In an alternative case that the seller observes the quality perfectly but the buyers mis-report good quality as bad with probability $1-\alpha$, the analysis is the same except that the price is $p_G(\mu, y)$.

To prove Theorem 2, we show first that an ℓ-type seller's strategy and value function exist that satisfy incentive compatibility when α and β are large enough (Section 1.1), then show that these functions converge as α and β tend to 1 (Section 1.2), which is used to verify the incentive compatibility of an h-type seller (Section 1.3).

1.1 Existence of the value function and equilibrium strategy for ℓ-type

We first establish the existence of a value function $V_\ell^{\beta\alpha}$ and a positive policy function $y^{\beta\alpha}$, defined for $\mu \in [1-\beta, \beta]$, that characterize the equilibrium value function and strategy for an ℓ-type seller. We will not claim they are unique but only that they are close to the unique equilibrium of Theorem 1 when β and α are large. In this part, as β and α are fixed (close to 1), we drop the superscript $\beta\alpha$.

For a given $\beta < 1$ close to 1, fix $\varkappa \in \mathbb{R}_+$ large enough so that

$$\varkappa > \frac{1}{1-\delta}\Big(\frac{1-\ell}{h} + \frac{1-\ell}{h}\frac{1}{\beta(1-\beta)}\Big), \tag{10}$$

and define $\mathcal{F}_\beta$ as the set of all non-decreasing $\varkappa$-Lipschitz functions $V : [1-\beta, \beta] \to [\frac{\ell}{1-\delta}, \frac{1}{1-\delta}]$ such that $V(\beta) - V(1-\beta) \leq 1$. In this section we prove that for large enough α, there exists an equilibrium with a value function for an ℓ-type seller in $\mathcal{F}_\beta$.

Define y_V in the same manner as in the proof of Theorem 1:

$$\begin{aligned} y_V(\mu) &= 1 \quad \text{if} \;\; \hat{p}_G(\mu, 1) \geq \delta(V(\beta) - V(\hat{\pi}_{Gb}(\mu, 1))) &(11)\\ \hat{p}_G(\mu, y_V(\mu)) &= \delta(V(\hat{\pi}_{Bb}(\mu, y_V(\mu))) - V(\hat{\pi}_{Gb}(\mu, y_V(\mu)))), \text{ otherwise.} &(12) \end{aligned}$$

Notice that the left-hand side (LHS) of (12) is decreasing in y while the RHS is non-decreasing in y. Let $\underline{y} > 0$ be such that $\hat{p}_G(0, \underline{y})\big|_{\alpha=1} = \frac{\ell}{\ell+(1-\ell)\underline{y}} > \delta \geq \delta(V(\beta) - V(1-\beta))$ and $\check{\mu}$ be close enough to 1 so that $\hat{p}_G(\check{\mu}, 1)\big|_{\alpha=1} > \delta$. Then, if α and β are large enough, $\hat{p}_G(\mu, \underline{y}) > \delta$ for all μ and $\hat{p}_G(\mu, 1) > \delta$ for $\mu > \check{\mu}$ and thus, $y_V(\mu)$ is well-defined and exceeds $\underline{y}$ for all $\mu \in [1-\beta, \beta]$ and is equal to 1 if $\mu \geq \check{\mu}$. Moreover, as $\hat{p}_G(\mu, 1)$ and $\hat{\pi}_{Gb}(\mu, 1)$ increase in μ while $\hat{\pi}_{Bb}(\mu, 1) = \beta$, for any V there exists $\bar{\mu}_V < \check{\mu}$ such that $y_V(\mu) = 1 \Leftrightarrow \mu \geq \bar{\mu}_V$. Therefore, there are $\alpha_{min}, \beta_{min} \in (0,1)$ such that if $\alpha \in (\alpha_{min}, 1)$ and $\beta \in (\beta_{min}, 1)$ then for any $V \in \mathcal{F}_\beta$,

$$y_V(\mu) \geq \underline{y} \text{ for all } \mu \in [1-\beta, \beta] \;\text{ and }\; y_V(\mu) = 1 \Leftrightarrow \mu \in [\bar{\mu}_V, \beta] \;\text{ for some }\; \bar{\mu}_V < \beta. \tag{13}$$

Below we only consider $\alpha \in (\alpha_{min}, 1)$ and $\beta \in (\beta_{min}, 1)$ so that (13) holds. From the same argument as in the perfect signal case, as $\hat{p}_G(\mu, y)$ and $\delta(V(\hat{\pi}_{Gb}(\mu, y)) - V(\hat{\pi}_{Bb}(\mu, y)))$ are continuous, y_V is continuous. In addition, we have

Lemma 1 *For all $V \in \mathcal{F}_\beta$, the function y_V is K-Lipschitz, where K is independent of α.*

Proof. As

$$\left\{ \begin{array}{l} \frac{\partial \hat{p}_G(\mu,y)}{\partial \mu} = \frac{h(1-\ell)(1-\alpha+\alpha y) - \ell(1-h)(1-\alpha)}{(\mu(h+(1-h)(1-\alpha))+(1-\mu)(\ell+(1-\ell)(1-\alpha+\alpha y)))^2} \\ \frac{\partial \hat{p}_G(\mu,y)}{\partial y} = -\frac{(\mu h+(1-\mu)\ell)(1-\mu)(1-\ell)\alpha}{(\mu(h+(1-h)(1-\alpha))+(1-\mu)(\ell+(1-\ell)(1-\alpha+\alpha y)))^2} \end{array} \right. \tag{14}$$

$$\text{and}\quad \begin{cases} \frac{\partial\hat{\pi}_{Bb}(\mu,y)}{\partial\mu} = \frac{(2\beta-1)(1-h)(1-y)(1-\ell)}{[\mu(1-h)+(1-\mu)(1-\ell)(1-y)]^2} \\ \frac{\partial\hat{\pi}_{Bb}(\mu,y)}{\partial y} = \frac{(2\beta-1)\mu(1-h)(1-\mu)(1-\ell)}{[\mu(1-h)+(1-\mu)(1-\ell)(1-y)]^2}, \end{cases} \tag{15}$$

one can find positive numbers k_0 and k_1 such that for $\mu \in [1-\beta, \beta]$ and $y \in [\underline{y}, 1]$:

$$k_0 < \frac{\partial\hat{p}_G(\mu,y)}{\partial\mu}, \frac{\partial\hat{\pi}_{Bb}(\mu,y)}{\partial y}, -\frac{\partial\hat{p}_G(\mu,y)}{\partial y} < k_1; \quad 0 < \frac{\partial\hat{\pi}_{Bb}(\mu,y)}{\partial\mu} < k_1; \quad \rho(1-\alpha_{min}) < k_1. \tag{16}$$

Hence, $H(\mu,y) = \hat{p}_G(\mu,y) + \delta(V(\hat{\pi}_{Gb}(\mu,y)) - V(\hat{\pi}_{Bb}(\mu,y)))$ is k-Lipschitz in both components, where $k = k_1 + 2\varkappa k_1$. Consider $\mu, \mu' \in (1-\beta, \bar{\mu}_V)$ so that $y_V(\mu) < 1$ and $y_V(\mu') < 1$. Then, $H(\mu, y_V(\mu)) = H(\mu', y_V(\mu'))$ by (12), which implies

$$H(\mu, y_V(\mu)) - H(\mu', y_V(\mu)) = H(\mu', y_V(\mu')) - H(\mu', y_V(\mu))$$

and consequently,

$$k\,|\mu - \mu'| \geq |H(\mu', y_V(\mu')) - H(\mu', y_V(\mu))|\,.$$

As H is absolutely continuous and $\frac{\partial}{\partial y}H(\mu,y) \leq \frac{\partial\hat{p}_G(\mu,y)}{\partial y} < -k_0 < 0$ where the first inequality follows from $\frac{\partial\hat{\pi}_{Gb}(\mu,y)}{\partial y} < 0$ and $\frac{\partial\hat{\pi}_{Bb}(\mu,y)}{\partial y} > 0$,

$$k\,|\mu - \mu'| \geq |H(\mu', y_V(\mu')) - H(\mu', y_V(\mu))| > k_0\,|y_V(\mu') - y_V(\mu)|\,.$$

Hence, y_V is K-Lipschitz on $(1-\beta, \bar{\mu}_V)$ where $K = k/k_0$. As $y_V(\mu) = 1$ for all $\mu \in [\bar{\mu}_V, \beta]$, it is K-Lipschitz on the entire domain. ■

We now define an operator T on the set $\mathcal{F}_\beta$ as

$$TV(\mu) = \hat{p}_G(\mu, y_V(\mu)) + \delta(\ell V(\hat{\pi}_{Gg}(\mu)) + (1-\ell)V(\hat{\pi}_{Gb}(\mu, y_V(\mu)))). \tag{17}$$

Lemma 2 *There are $\bar{\alpha} < 1$ and $\kappa > 0$ such that if $\alpha > \bar{\alpha}$ then for all $V \in \mathcal{F}_\beta$ and $\mu \in [1-\beta, \beta]$, (i) $\ell \leq (1-\delta)TV(\mu) \leq 1$, (ii) $TV(\beta) - TV(1-\beta) < 1$, (iii) TV is $\varkappa$-Lipschitz, and (iv) $TV'(\mu) > -\kappa(1-\alpha)$ for a.e. μ. In addition, (v) T is continuous on $\mathcal{F}_\beta$.*

Proof. (i) That $\ell/(1-\delta) \leq TV \leq 1/(1-\delta)$ obtains because $\ell/(1-\delta) \leq V(\mu) \leq 1/(1-\delta)$ and $\ell \leq \hat{p}_G(\mu,y) \leq 1$.

(ii) Using $y_V(\beta) = 1$, we have

$$\begin{aligned} TV(\beta) &= 1 + \delta(\ell V(\beta) + (1-\ell)V(\hat{\pi}_{Gb}(\beta,1))) \ \text{ and} \\ TV(1-\beta) &\geq \ell + \delta V(1-\beta). \end{aligned}$$

For α large, we have (using the fact that V is $\varkappa$-Lipschitz):

$$\begin{aligned} TV(\beta) - TV(1-\beta) &\leq 1 - \ell + \delta\ell(V(\beta) - V(1-\beta)) + \delta(1-\ell)(V(\hat{\pi}_{Gb}(\beta,1)) - V(1-\beta)) \\ &\leq 1 - \ell + \delta\ell + \delta(1-\ell)\varkappa(\hat{\pi}_{Gb}(\beta,1) - (1-\beta)) \\ &= 1 - \ell + \delta\ell + \delta(1-\ell)\varkappa(2\beta-1)\frac{\beta(1-h)(1-\alpha)}{\beta(1-h)(1-\alpha) + (1-\beta)(1-\ell)} \end{aligned}$$

where the equality follows from writing out $\hat{\pi}_{Gb}(\beta, 1)$. Hence, $TV(\beta) - TV(1-\beta) < 1$ for large enough α as $-\ell + \delta\ell < 0$.

(iii) We first show that $\hat{\pi}_{Gb}(\mu, y_V(\mu))$ is increasing in μ. This is straightforward from $\frac{\partial \hat{\pi}_{Gb}(\mu,y)}{\partial \mu}$ when $y_V(\mu) = 1$. When $y_V(\mu) < 1$, we have $\hat{p}_G(\mu, y_V(\mu)) + \delta V(\hat{\pi}_{Gb}(\mu, y_V(\mu))) = \delta V(\pi_{Bb}(\mu, y_V(\mu)))$ from (12). The value of this equality is non-decreasing in μ, because if it were decreasing then $y_V(\mu)$ would have to increase for the LHS to decrease as μ increases while $y_V(\mu)$ would have to decrease for the RHS to decrease. Note that

$$\frac{\frac{\partial \hat{\pi}_{Gb}(\mu,y)}{\partial \mu}}{-\frac{\partial \hat{\pi}_{Gb}(\mu,y)}{\partial y}} = \frac{1-\alpha+\alpha y}{\mu(1-\mu)\alpha} > \frac{\frac{\partial \hat{p}_G(\mu,y)}{\partial \mu}}{-\frac{\partial \hat{p}_G(\mu,y)}{\partial y}} = \frac{h(1-\ell)(1-\alpha+\alpha y) - \ell(1-h)(1-\alpha)}{(\mu h + (1-\mu)\ell)(1-\mu)(1-\ell)\alpha} > 0.$$

If $\pi_{Gb}(\mu, y_V(\mu))$ were non-increasing, then y_V would be increasing and the above inequality implies that $\hat{p}_G(\mu, y_V(\mu))$ would be decreasing. But this would imply that $\hat{p}_G(\mu, y_V(\mu)) + \delta V(\hat{\pi}_{Gb}(\mu, y_V(\mu)))$ is decreasing, a contradiction to the assertion above. Hence, we conclude that $\pi_{Gb}(\mu, y_V(\mu))$ is increasing in μ.

We now show that TV is $\varkappa$-Lipschitz. As the function y_V, $\hat{p}_G$, $\hat{\pi}_{Bb}$, $\hat{\pi}_{Gb}$, TV are Lipschitz, they are absolutely continuous and we can apply the rule of composition for derivatives. From (17) we get

$$\begin{aligned} TV'(\mu) \;\; = \;\; & \frac{\partial \hat{p}_G(\mu, y_V(\mu))}{\partial \mu} + \frac{\partial \hat{p}_G(\mu, y_V(\mu))}{\partial y}\frac{dy_V(\mu)}{d\mu} + \delta\Bigg[\ell V'(\hat{\pi}_{Gg}(\mu))\frac{d\hat{\pi}_{Gg}(\mu)}{d\mu} \\ & + (1-\ell)V'(\hat{\pi}_{Gb}(\mu, y_V(\mu)))\Big(\frac{\partial \hat{\pi}_{Gb}(\mu, y_V(\mu))}{\partial \mu} + \frac{\partial \hat{\pi}_{Gb}(\mu, y_V(\mu))}{\partial y}\frac{dy_V(\mu)}{d\mu}\Big)\Bigg]. \end{aligned} \tag{18}$$

Notice that

$$\frac{(2\beta-1)h\ell}{(\beta h + (1-\beta)\ell)^2} < \frac{d\hat{\pi}_{Gg}(\mu)}{d\mu} = \frac{(2\beta-1)h\ell}{(\mu h + (1-\mu)\ell)^2} < \frac{\ell}{h} < 1. \tag{19}$$

Using, in addition, the fact that $\hat{\pi}_{Gg}(\mu)$ and $\hat{\pi}_{Gb}(\mu, y_V(\mu))$ increase in μ, we obtain (from now on we omit the arguments for partial derivatives):

$$TV'(\mu) \leq \frac{\partial \hat{p}_G}{\partial \mu} + \frac{\partial \hat{p}_G}{\partial y}\frac{dy_V(\mu)}{d\mu} + \delta\varkappa + \delta(1-\ell)\varkappa\Big(\frac{\partial \hat{\pi}_{Gb}}{\partial \mu} + \frac{\partial \hat{\pi}_{Gb}}{\partial y}\frac{dy_V(\mu)}{d\mu}\Big). \tag{20}$$

Recall that $\frac{\partial \hat{p}_G}{\partial y}$ and $\frac{\partial \hat{\pi}_{Gb}}{\partial y}$ are negative. Moreover,

$$\frac{\partial \hat{p}_G(\mu, y)}{\partial \mu} < \frac{h(1-\ell)}{(\mu h + (1-\mu)\ell)^2} < \frac{1-\ell}{h} \quad \text{and} \quad \left|\frac{\partial \hat{p}_G}{\partial y}\right| < \frac{1-\ell}{h}. \tag{21}$$

We consider two cases as below.

a) Suppose that $\frac{dy_V(\mu)}{d\mu} < 0$. Then, the LHS of (12) increases in μ, which implies that $\hat{\pi}_{Gb}(\mu, y_V(\mu))$ increases and thus,

$$\frac{dy_V(\mu)}{d\mu} > -\frac{\partial \pi_{Bb}}{\partial \mu}\Big/\frac{\partial \pi_{Bb}}{\partial y} = -\frac{1-y_V(\mu)}{\mu(1-\mu)} > -\frac{1}{\beta(1-\beta)}.$$

Thus, for α large enough, using (9) and (21) we obtain from (20) that

$$TV'(\mu) < \frac{1-\ell}{h} + \frac{1-\ell}{h}\frac{1}{\beta(1-\beta)} + \delta\varkappa + \delta(1-\ell)\varkappa\Big(1 + \frac{1}{\beta(1-\beta)}\Big)\rho(1-\alpha) < \varkappa$$

and that

$$TV'(\mu) > \frac{d\hat{p}_G(\mu, y_V(\mu))}{d\mu} > 0.$$

b) Suppose that $\frac{dy_V(\mu)}{d\mu} \geq 0$. Then, from (20) we have

$$TV'(\mu) < \frac{1-\ell}{h} + \delta\varkappa + \delta(1-\ell)\varkappa\rho(1-\alpha) < \varkappa$$

for large enough α. Moreover, as $\hat{p}_G(\mu, y_V(\mu))$ and $V(\hat{\pi}_{Gg}(\mu))$ increase in μ, for large enough α we obtain from (18) that

$$TV'(\mu) \geq \delta(1-\ell)\varkappa\frac{\partial\hat{\pi}_{Gb}}{\partial y}\frac{dy_V}{d\mu} > -\delta(1-\ell)\varkappa\rho(1-\alpha)K > -\varkappa. \tag{22}$$

(iv) As $TV'(\mu) > 0$ has been shown when $\frac{dy_V(\mu)}{d\mu} < 0$ above, $TV'(\mu) > -\kappa(1-\alpha)$ by (22) whenever $TV'(\mu)$ is defined if we set $\kappa = \delta(1-\ell)\varkappa\rho K$.

(v) The proof is omitted. It is the same as in the case $\alpha = \beta = 1$, except that V is continuous which simplifies the proof. Notice that the image of $\mathcal{F}_\beta$ is also a set of $\varkappa$-Lipschitz function. Thus, the topology of pointwise convergence is the same as the topology of uniform convergence. ■

Lemma 2 shows that T preserves all properties of V except for monotonicity. Hence, we define $\widetilde{T} : \mathcal{F}_\beta \to \mathcal{F}_\beta$ as

$$\widetilde{T}V(\mu) \;=\; \max_{m\in[1-\beta,\mu]} TV(m)$$

and show in the next two lemmas that $\widetilde{T}$ has a fixed point and that it is a fixed point of T as well.

Lemma 3 *There exists a fixed point $V_\ell^{\beta\alpha}$ of $\widetilde{T}$.*

Proof. As $\mathcal{F}_\beta$ is a compact set by Ascoli-Arzelà Theorem,[4] it suffices to show that $\widetilde{T}$ is continuous. Suppose that V_n converges to V, where convergence is uniform. Suppose that $\widetilde{T}V_n(\mu)$ converges to $z > \widetilde{T}V(\mu)$. Then there exists a sequence $\mu_n < \mu$ such that $TV_n(\mu_n)$ converges to z. Assume without loss of generality that μ_n converges to $\hat{\mu} \leq \mu$. Since TV_n is $\varkappa$-Lipschitz and TV_n converges to TV, it must be the case that $\lim_{n\to+\infty}\widetilde{T}V_n(\mu) = TV(\hat{\mu})$. Hence $TV(\hat{\mu}) = z > \widetilde{T}V(\mu)$, a contradiction with $\hat{\mu} \leq \mu$. Suppose that $\widetilde{T}V_n(\mu)$ converges to $z < \widetilde{T}V(\mu)$, then there is some $\hat{\mu} \leq \mu$ such that $TV(\hat{\mu}) > z$. But then $TV_n(\hat{\mu})$ converges to $TV(\hat{\mu})$ so that $\lim_{n\to+\infty}\widetilde{T}V_n(\mu) \geq TV(\hat{\mu}) > z$, a contradiction. Hence, it must be the case that $\widetilde{T}V_n(\mu)$ converges to $\widetilde{T}V(\mu)$ for all μ. ■

[4]See Royden H.L.(1988), *Real Analysis,* 3rd ed., Macmillan Publishing Company, New York.

Lemma 4 *For α sufficiently close to 1, if V is a fixed point of $\widetilde{T}$, then V is a fixed point of T, i.e., V is increasing.*

Proof. Recall from (13) that if α and β are sufficiently close to 1, then $y_V(\mu) \geq \underline{y}$ and $y_V(\mu) = 1$ if $\mu \geq \bar{\mu}_V$. Moreover, the reasoning for (13) has shown that $\bar{\mu}_V$ is uniformly bounded above by $\check{\mu}$. For the proof we will need to bound some slopes. For any given $c > 0$, one can set $\eta > 0$ small enough so that

$$\check{\mu}\eta < (\beta - \check{\mu})c/2. \tag{23}$$

Then Lemma 2, properties (9) and (16) and the definition (18) imply that there exist $\xi > 0$ and $c > 0$ such that if we choose $\alpha_0 \geq \underline{\alpha}$ so that $\kappa(1 - \alpha_0) < \eta$, then the following holds:

$$\text{For } \alpha > \alpha_0 \text{ and } \mu \in [1-\beta, \beta]\,,\ TV'(\mu) > -\eta \ \text{ and } TV'(\mu) > c \text{ if } \frac{dy_V(\mu)}{d\mu} < \xi. \tag{24}$$

To prove the lemma, suppose V is a fixed point of $\widetilde{T}$ for $\alpha > \alpha_0$.

First, note from (24) that $TV'(\mu) > c$ for all $\mu \geq \check{\mu}$ so that $TV(\beta) > TV(\check{\mu}) + (\beta - \check{\mu})c$. Together with (23), this further implies that

$$\max_{m \in [1-\beta, \check{\mu}]} TV(m) < TV(\check{\mu}) + \check{\mu}\eta < TV(\check{\mu}) + (\beta - \check{\mu})c/2 \ < \ TV(\beta) - (\beta - \check{\mu})c/2.$$

Then, by choosing $\mu^\dagger \in (\check{\mu}, \beta)$ such that $\varkappa(\beta - \mu^\dagger) < (\beta - \check{\mu})c/2$, we have $TV(\mu) > \max_{m \in [1-\beta, \check{\mu}]} TV(m)$ for $\mu \geq \mu^\dagger$ and thus $TV(\mu) = \max_{m \in [\check{\mu}, \mu]} TV(m)$ by (24). Hence,

$$\widetilde{T}V(\mu) = TV(\mu) = V(\mu) \textit{ and } \widetilde{T}V'(\mu) = TV'(\mu) = V'(\mu) > c \textit{ for all } \mu \geq \mu^\dagger. \tag{25}$$

Notice that $\mu^\dagger$ is independent of $\alpha > \alpha_0$. For each $\alpha > \alpha_0$, define

$$\mu^\circ(\alpha) = \sup\{\mu | \widetilde{T}V(\mu) \neq TV(\mu)\} \leq \mu^\dagger$$

if $\widetilde{T}V(\mu) \neq TV(\mu)$ for some $\mu \in [1 - \beta, \beta]$, and let $\mu^\circ(\alpha) = 1 - \beta$ otherwise. Observe from (19) that, independently of α, there exist $\epsilon > 0$ and $\nu > 0$ such that

$$\textit{For } \mu \in [1-\beta, \mu^\dagger],\ \hat{\pi}_{Gg}(\mu) > \mu + \epsilon \textit{ and } \frac{d\hat{\pi}_{Gg}(\mu)}{d\mu} > \nu. \tag{26}$$

We assume $\nu < 1$ without loss of generality. From this we establish the following:

Claim There exist finite sequences $c_n > 0$ and $\alpha_n < 1$ for $n = 1, \cdots, N$, with $\mu^\dagger - N\epsilon < 1 - \beta$, such that if $\alpha > \alpha_n$ and $\mu^\dagger - (n-1)\epsilon > \mu^\circ(\alpha)$, then $\widetilde{T}V'(\mu) = TV'(\mu) = V'(\mu) \geq c_n$ for all $\mu \in [\max\{\mu^\dagger - n\epsilon, \mu^\circ(\alpha)\}, \beta]$.

Proof of Claim: We proceed by induction. The property holds for $n = 1$ with $c_1 = c$. Suppose that it holds for $n - 1$ and c_k is non-increasing in $k \leq n - 1$.

Consider $\alpha \in (\alpha_{n-1}, 1)$ such that $\mu^\dagger - (n-1)\epsilon > \mu^\circ(\alpha)$. If such α does not exist, then the property of the Claim holds trivially for n by setting $c_n = c_{n-1}$ and $\alpha_n = \alpha_{n-1}$. Hence, suppose such α exists. Then $\widetilde{T}V'(\mu) = TV'(\mu) = V'(\mu) \geq c_{n-1}$ if $\mu \geq \mu^\dagger - (n-1)\epsilon$ by induction hypothesis.

Define $I^n = [\max\{\mu^\dagger - n\epsilon, \mu^\circ(\alpha)\}, \mu^\dagger - (n-1)\epsilon]$. For $\mu \in I^n$, we have $TV'(\mu) > c$ if $\frac{dy_V(\mu)}{d\mu} < \xi$ from (24). Suppose now that $\frac{dy_V(\mu)}{d\mu} \geq \xi$, then condition (12) implies that

$$\begin{aligned}\frac{d\hat{p}_G(\mu, y_V(\mu))}{d\mu} &+ \delta V'(\hat{\pi}_{Gb}(\mu, y_V(\mu)))\Big(\frac{\partial \hat{\pi}_{Gb}}{\partial \mu} + \frac{\partial \hat{\pi}_{Gb}}{\partial y}\frac{dy_V(\mu)}{d\mu}\Big) \\ &= \delta V'(\hat{\pi}_{Bb}(\mu, y_V(\mu)))\Big(\frac{\partial \hat{\pi}_{Bb}}{\partial \mu} + \frac{\partial \hat{\pi}_{Bb}}{\partial y}\frac{dy_V(\mu)}{d\mu}\Big) \geq 0. \qquad (27)\end{aligned}$$

Moreover, for $\mu \in I^n$, $\hat{\pi}_{Gg}(\mu) \geq \mu^\dagger - (n-1)\epsilon$ implies that $V'(\hat{\pi}_{Gg}(\mu)) \geq c_{n-1}$. These properties imply from (18) that

$$TV'(\mu) \geq \ell \frac{d\hat{p}_G(\mu, y_V(\mu))}{d\mu} + \delta \ell V'(\hat{\pi}_{Gg}(\mu))\frac{d\hat{\pi}_{Gg}(\mu)}{d\mu} \geq \ell \frac{d\hat{p}_G(\mu, y_V(\mu))}{d\mu} + \delta \ell c_{n-1}\nu. \qquad (28)$$

If the second term of the LHS of (27) is negative then $\frac{d\hat{p}_G(\mu, y_V(\mu))}{d\mu}$ is positive and $TV'(\mu) \geq \delta \ell c_{n-1}\nu$ by (28).

Consider the alternative case that the second term of the LHS of (27) is positive. Then, from Lemma 1 and Lemma 2, we can choose $\alpha_n \geq \alpha_{n-1}$, so as to ensure that for $\alpha \geq \alpha_n$ the second term of the LHS of (27) is less than $\delta c_{n-1}\nu/2$. Thereby, $\frac{d\hat{p}_G}{d\mu} \geq -\delta c_{n-1}\nu/2$ so that $TV'(\mu) \geq -\delta \ell c_{n-1}\nu/2 + \delta \ell c_{n-1}\nu = \delta \ell c_{n-1}\nu/2$ by (28).

Thus, we have shown that the property of the Claim holds for α_n and $c_n = \delta \ell c_{n-1}\nu/2$, completing the induction argument. $\square$

The Claim above establishes that there is $\alpha^* < 1$ and $c^* > 0$ such that:

$$\widetilde{T}V'(\mu) = TV'(\mu) = V'(\mu) \geq c^* \text{ for all } \mu \in [\mu^\circ(\alpha), \beta] \text{ if } \alpha > \alpha^*. \qquad (29)$$

To prove the lemma, it remains to show that $\mu^\circ(\alpha) = 1 - \beta$ for sufficiently large α. To do this, for $\alpha > \alpha^*$ such that $\mu^\circ(\alpha) > 1 - \beta$, by applying the induction step in the proof of the Claim one more time on $J = [\mu^\circ(\alpha) - \epsilon, \mu^\circ(\alpha)] \cap [1-\beta, \beta]$, we deduce that there exists $\alpha^{**} < 1$ such that (notice that we cannot claim $\widetilde{T}V'(\mu) = TV'(\mu)$ on J):

$$TV'(\mu) \geq \delta \ell c^* \nu/2 \text{ for all } \mu \in J = [\mu^\circ(\alpha) - \epsilon, \mu^\circ(\alpha)] \cap [1-\beta, \beta] \text{ if } \alpha > \alpha^{**}. \qquad (30)$$

The last step is to show that $\mu^\circ(\alpha) - \epsilon \leq 1 - \beta$ for α large. Suppose that $\mu^\circ(\alpha) - \epsilon > 1 - \beta$ for some $\alpha > \alpha^{**}$. One can choose $\alpha^{***} \geq \alpha^{**}$ such that $\kappa(1 - \alpha^{***}) < \epsilon \delta \ell c^* \nu/4$. For $\alpha > \alpha^{***}$, as $TV'(\mu) > -\epsilon \delta \ell c^* \nu/4$ from Lemma 2, we have for any $\mu < \mu^\circ(\alpha) - \epsilon$,

$$TV(\mu) < TV(\mu^\circ(\alpha) - \epsilon) + \epsilon \delta \ell c^* \nu/4(\mu^\circ(\alpha) - \epsilon - \mu) < TV(\mu^\circ(\alpha)) - \epsilon \delta \ell c^* \nu/4. \qquad (31)$$

In conjunction with (30), this means that there is $\mu' < \mu^\circ(\alpha)$ such that $TV'(\mu) > 0$ if $\mu \geq \mu'$ and $TV(\mu) < TV(\mu')$ if $\mu < \mu'$, which would contradict the definition of $\mu^\circ(\alpha)$. Thus, we conclude that $\mu^\circ(\alpha) - \epsilon \leq 1 - \beta$ for all $\alpha > \alpha^{***}$. Then for $\alpha > \alpha^{***}$, properties (29) and (30) imply that TV is increasing on $[1-\beta, \beta]$ and thus that $\mu^\circ(\alpha) = 1 - \beta$.

In conjunction with (29), this proves that $V'(\mu) > 0$ for all μ if V is a fixed point of $\widetilde{T}$ when $\alpha > \alpha^{***}$. Hence, $V = TV$. $\blacksquare$

1.2 Continuity of $V_\theta^{\beta\alpha}$ in β and α

Recall that given β, a fixed point of T and the corresponding policy function, $V_\ell^{\beta\alpha}$ and $y^{\beta\alpha}$, are Lipschitz with uniformly bounded Lipschitz constant for α close to 1. For a fixed value of β, consider a sequence α_k that converges to 1. As the set $\mathcal{F}_\beta$ is compact, we can assume without loss of generality that the value function $V_\ell^{\beta\alpha_k}$ converges uniformly as $k \to \infty$. Let V_ℓ^β denote this limit. Define y^β as the solution of equations (11) and (12) for V_ℓ^β, which is unique. It must be the case that $y^{\beta\alpha_k}$ converges to y^β. We thus conclude that the limits V_ℓ^β and y^β correspond to a fixed point and policy function pair $(V_\ell^{\beta 1}, y^{\beta 1})$ for β and $\alpha = 1$.

Now consider a sequence β_n that converges to 1. Again as the set of increasing Lipschitz functions is compact we can assume without loss of generality that $V_\ell^{\beta_n}(\mu)$ converges to a limit denoted by $V_\ell^\infty(\mu)$ for all $\mu \in (0,1)$. Define $V_\ell^\infty(0) = \lim_{n\to\infty} V_\ell^{\beta_n}(1-\beta_n)$ and $V_\ell^\infty(1) = \lim_{n\to\infty} V_\ell^{\beta_n}(\beta_n)$. By continuity we have (here superscript n of $\hat{p}_G^n$ and $\hat{\pi}_{Gg}^n$ means evaluated at β_n and $\alpha = 1$)

$$\begin{aligned} y^{\beta_n}(\mu) &= 1 \text{ for } \mu > \bar{\mu}_n \\ \hat{p}_G^n(\mu, y^{\beta_n}(\mu)) &= \delta(V_\ell^{\beta_n}(\hat{\pi}_{Bb}^n(\mu, y^{\beta_n}(\mu))) - V_\ell^{\beta_n}(1-\beta_n)) \text{ for } \mu \le \bar{\mu}_n, \quad \text{and} \\ V_\ell^{\beta_n}(\mu) &= \hat{p}_G^n(\mu, y^{\beta_n}(\mu)) + \delta(\ell V_\ell^{\beta_n}(\hat{\pi}_{Gg}^n(\mu)) + (1-\ell)V_\ell^{\beta_n}(1-\beta_n)) \end{aligned}$$

where $\bar{\mu}_n$ is $\bar{\mu}_V$ of (13) obtained for the given β_n and $\alpha = 1$ relative to $V = V_\ell^{\beta_n}$.

We now show that the limit is an equilibrium.

i) For any $\mu \in (0,1)$, let y^∞ be the unique solution to equations (11) and (12) evaluated at $\beta = \alpha = 1$. Recall that $\check{\mu}$ is independent of β and $\check{\mu} > \bar{\mu}_n$. For $\mu > \check{\mu}$, we have $V_\ell^{\beta_n}(\mu) = \sum_{t=0}^\infty (\delta\ell)^t \hat{p}_G^n(\hat{\pi}_{Gg}^n(\mu)^t, 1) + \phi_n$ where $\hat{\pi}_{Gg}^n(\mu)^t$ is the t-iteration of $\hat{\pi}_{Gg}^n(\mu)$, and ϕ_n is a constant. Given that $\hat{\pi}_{Gg}^n(\cdot)$ and $\hat{p}_G^n$ have uniformly bounded slopes on $[\check{\mu}, 1]$, it follows that V^{β_n} has a uniformly bounded slope on $\mu > \check{\mu}$. The same argument as in Lemma 2 (of this Appendix) shows that for any $\mu_0 > 1-\beta_n$ arbitrarily close to $1-\beta_n$, the slope of $V_\ell^{\beta_n}$ is uniformly bounded on $[\mu_0, \beta_n]$ (by choosing $\varkappa > \frac{1}{1-\delta}(\frac{1-\ell}{h} + \frac{1-\ell}{h}\frac{1}{\mu_0(1-\mu_0)})$ one can ensure that the bound of the slope on $(\mu_0, \check{\mu}]$ is independent of β_n). This implies that for any $\mu \in (0,1)$, $y^{\beta_n}(\mu)$ converges to $y^\infty(\mu)$ and

$$V_\ell^\infty(\mu) = p_G(\mu, y^\infty(\mu)) + \delta(\ell V_\ell^\infty(\pi_{Gg}(\mu)) + (1-\ell)V_\ell^\infty(0)).$$

ii) Moreover

$$\begin{aligned} V_\ell^\infty(1) &= \lim_{n\to\infty} \left\{\hat{p}_G^n(\beta_n, 1) + \delta(\ell V_\ell^{\beta_n}(\hat{\pi}_{Gg}^n(\mu)) + (1-\ell)V_\ell^{\beta_n}(1-\beta_n))\right\} \\ &= p_G(1,1) + \delta(\ell V_\ell^\infty(1) + (1-\ell)V_\ell^\infty(0)) \end{aligned}$$

iii) To conclude we show that $V_\ell^\infty(0) = \ell/1-\delta$. By continuity of V_ℓ^∞, we have

$$V_\ell^\infty(0) = \lim_{\mu\to 0} \left\{p_G(\mu, y^\infty(\mu)) + \delta(\ell V_\ell^\infty(\pi_{Gg}(\mu)) + (1-\ell)V_\ell^\infty(0))\right\}.$$

Also, by continuity of V_ℓ^∞, $\lim_{\mu\to 0} \delta(V_\ell^\infty(\pi_{Bb}(\mu, y)) - V_\ell^\infty(0)) = 0$ for $y < 1$. Hence, $y^\infty(\mu)$ converges to 1 when μ goes to zero, which implies that the $V_\ell^\infty(0) = \ell + \delta V_\ell^\infty(0) = \ell/1-\delta$.

We thus conclude from i), ii) and iii) that the limit (V^{∞}, y^{∞}) is the unique equilibrium (V_ℓ^*, y^*) for $\beta = \alpha = 1$ characterized in Proposition 1 of this Appendix.

To show uniform convergence, we extend $V_\ell^{\beta_n}$ on $[0,1]$ by postulating $V_\ell^{\beta_n}(\mu) = V_\ell^{\beta_n}(1-\beta_n)$ for $\mu < 1-\beta_n$ and $V_\ell^{\beta_n}(\mu) = V_\ell^{\beta_n}(\beta_n)$ for $\mu > \beta_n$. Then $V_\ell^{\beta_n}(\mu)$ converges to $V_\ell^*(\mu)$ for all $\mu \in [0,1]$. Notice that $V_\ell^{\beta_n}$ is increasing while the limit V_ℓ^* is continuous on $[0,1]$. According to Polya's Theorem[5] this implies that the extension of $V_\ell^{\beta_n}$ to $[0,1]$ converges uniformly to V_ℓ^*. Excluding 0 and 1 as they are not in the support $[1-\beta_n, \beta_n]$, we obtain uniform convergence of $V_\ell^{\beta_n}$ to V_ℓ^* on $(0,1)$.

In the proof of Theorem 3 (Claim 2 of Lemma 8 of this Appendix), we show V_ℓ^* to be Lipschitz-continuous on any interval $[\mu_0, 1]$ with $\mu_0 > 0$. From the argument of Lemma 1 of this Appendix, y^* is also Lipschitz-continuous on $[\mu_0, 1]$.

By the same argument, we can show that the value function $V_h^{\beta_n}$ converges to V_h^*, uniformly on $(0,1)$ (again convergence follows because all the functions involved are Lipschitz and converge on any interval $[\mu_0, 1]$, with uniformly bounded Lipschitz constant, and uniform convergence follows from Polya's Theorem).

1.3 Incentive compatibility for h-type

Consider first $\beta < 1$ and $\alpha = 1$ and let y^β be the equilibrium policy. Let V_h^β denote the value function of an h-type seller who truthfully announces the signal received.

Lemma 5 *Suppose that (1) holds and δ is large. For β sufficiently close to 1, $\hat{p}_G(\mu, y^\beta(\mu)) < \delta(V_h^\beta(\hat{\pi}_{Bb}(\mu, y^\beta(\mu))) - V_h^\beta(1-\beta))$ for all $\mu \in [1-\beta, \beta]$*

Proof. Suppose this is not the case. Then we can find sequences $\beta_n < 1$ and $\mu_n \in [1-\beta_n, \beta_n]$ such that (where the superscript n means evaluated at β_n)

$$\hat{p}_G^n(\mu_n, y^{\beta_n}(\mu_n)) \geq \delta(V_h^{\beta_n}(\hat{\pi}_{Bb}^n(\mu_n, y^{\beta_n}(\mu_n))) - V_h^{\beta_n}(1-\beta_n)) \tag{32}$$

where $\beta_n \to 1$. Assume without loss of generality that μ_n and $y^{\beta_n}(\mu_n)$ converge and let μ_∞ and $y^*(\mu_\infty)$ denote their respective limits.

Recalling that the value function $V_h^{\beta_n}$ converges to V_h^* uniformly on $(0,1)$, we focus on the equilibrium of the game when $\beta = \alpha = 1$ where the value function $\tilde{V}_h^*$ of the h-type is continuous at $\mu = 0$, that we derived in Proposition 1 of this Appendix when (1) holds and δ is large. Recall that $\tilde{V}_h^*(0) = \inf_{\mu > 0} V_h^*(\mu)$ and $\tilde{\pi}_{Bb}^*(0) = \lim_{\mu \to 0} \pi_{Bb}^*(\mu)$.

Notice that $\lim_{n\to\infty} V_h^{\beta_n}(1-\beta_n) \leq V_h^*(\mu)$ for any $\mu > 0$, implying that $\lim_{n\to\infty} V_h^{\beta_n}(1-\beta_n) \leq \tilde{V}_h^*(0)$. Thus, taking the limit of condition (32) we obtain (notice that this hold even if $\mu_\infty = 0$ because $\hat{\pi}_{Bb}^n(\mu_n, y^{\beta_n}(\mu_n))$ converges to $\tilde{\pi}_{Bb}^*(0) > 0$)

$$p_G(\mu_\infty, y^*(\mu_\infty)) \geq \delta(V_h^*(\pi_{Bb}(\mu_\infty, y^*(\mu_\infty))) - \tilde{V}_h^*(0)),$$

which contradicts the fact the h-type strictly prefers to announce truthfully for all $\mu \in [0,1]$ as shown in the proof of Proposition 1 of this Appendix. This proves the lemma. ■

Finally, we have

[5]See R. Rao (1962), "Relations between Weak and Uniform Convergence of Measures with Applications," *The Annals of Mathematical Statistics,* Vol. 33, No. 2, pp. 659-680.

Lemma 6 *Suppose that (1) holds and δ is large. For β close to 1, the incentive constraint of an h-type seller is satisfied for all $\mu \in [1-\beta, \beta]$ when α is close to 1.*

Proof. Fix β sufficiently large so that Lemma 5 holds. To reach a contradiction, suppose that the conclusion of this lemma fails. Then, we can find a sequence α_n converging to 1 such that for each fixed point $V_\ell^{\beta\alpha_n}$ with policy function $y^{\beta\alpha_n}$, there exists μ_n where the incentive constraint of the h-type is violated. One can ensure δ is large enough so that for any $\mu \in [0, \beta]$ if $y^{\beta\alpha_n}(\mu) = 1$ then an h-type seller strictly prefers to tell the truth when the signal is bad because the long-term loss would exceed the maximum possible short-term gain, 1, if n is large enough. Then, it must be the case that $y^{\beta\alpha_n}(\mu_n) < 1$ for large enough n. By an argument analogous to before, the value function $V_h^{\beta\alpha_n}$ converges uniformly to V_h^β on $[1-\beta, \beta]$ (again this follows because all the functions involved are Lipschitz with uniformly bounded Lipschitz constant). Thus, at the limit the incentive constraint of the h-type must hold with equality at any limit value of μ_n, which would contradict Lemma 5. ■

Lemmas 3, 4, 5 and 6 (in this Appendix) imply Theorem 2.

2 Negative social value

We establish here that an honest equilibrium exist when sellers incur a small distribution cost $c < \ell$. The equilibrium has the following features:

i) An h-type seller announces the quality truthfully and doesn't sell when quality is bad.

ii) An ℓ-type seller announces truthfully when $q = g$ but lies when $q = b$ with probability y and sells.

The characterization of the ℓ-type seller strategy and value function is the same as in the baseline model except that $p_G(\mu, y)$ has to be replaced by $p_G(\mu, y) - c$. At $\mu = 0$, ℓ-type seller "babbles" and sell at a price equal to $\ell - c$.

The proof that an h-type seller always announces bad quality truthfully is the same as before if

$$\delta > \delta_h^c = \frac{1-c}{1 + h(1-c) - \ell}.$$

However, we need to ensure that an h-type seller does not lie when $q = g$ to boost reputation. As the reputation monotonically increases from an initial level μ_1 for h-type seller, it suffices to show that

$$p_G^*(\mu) - c + \delta V_h^*(\pi_{Gg}^*(\mu)) \geq \delta V_h^*(\pi_{Bb}^*(\mu)) \text{ for all } \mu \geq \mu_1.$$

As $V_h^*(\pi_{Gg}^*(\mu))$ converges to $V_h^*(1)$ when μ tends to 1 by continuity, this inequality holds for large enough μ because as $\mu \to 1$,

$$p_G^*(\mu) - c \;\to\; h - c > \delta\big(V_h^*(1) - V_h^*(\pi_{Gg}^*(\mu))\big) \geq \delta\big(V_h^*(\pi_{Bb}^*(\mu)) - V_h^*(\pi_{Gg}^*(\mu))\big).$$

Therefore, an honest equilibrium exists if μ_1 is large enough.

3 Proof of Theorem 3

To prove the existence of an honest equilibrium in the model where sellers can start afresh with a new identity, we proceed in two steps. First, we characterise the equilibrium of an instrumental "auxiliary model" in which a newborn seller with initial reputation $\mu \in [0,1]$ must trade in the platform in the first period but may exit in any future period for an exogenous outside option value $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$. Then, by incorporating equilibrium restart decisions into this analysis, we establish the existence of a stationary equilibrium of the model with an option to restart, where v_1 is endogenously determined as the value of an ℓ-type seller deciding to restart.

First, we show that the auxiliary model described above has an equilibrium. It is worth pointing out that this result is of independent interest to the extent that in the proof of Theorem 3 we only exploit the existence of an incentive compatible value function of the auxiliary model and the optimality of the h-type seller strategy is obtained differently.

Lemma 7 *In the auxiliary model with an exit value $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, an honest equilibrium exists if $\delta(\frac{h}{1-\delta} - v_1) > 1$. All honest equilibria coincide on the equilibrium path.*

Proof. It is straightforwardly verified that an ℓ-type seller truthfully announces when $q = g$ by the same argument as before. Thus, as in the baseline model, an equilibrium strategy is described by the probability of lying by an ℓ-type seller upon drawing $q = b$, denoted by $y^\dagger(\mu)$ to distinguish it from that in Section 4 of the main article. Let $V^\dagger_{v_1}$ denote the value function of an ℓ-type seller in an honest equilibrium (presuming one exists). Adopting the convention, for the same reason as before, that an ℓ-seller always announce G when $\mu = 0$, we have $V^\dagger_{v_1}(0) = p^*_G(0) + \delta \max\{v_1, V^\dagger_{v_1}(0)\} = \ell + \delta v_1 > \frac{\ell}{1-\delta}$; and, analogously to (25) of the main article, $V^\dagger_{v_1}(\mu) > V^\dagger_{v_1}(0)$ is immediate for all $\mu > 0$. In addition, an argument analogous to that leading to (13) in the main article, establishes that without loss of generality we may set $\pi^\dagger_{Gb}(1) = 0$. Then, the boundary values are routinely computed to be

$$
\begin{aligned}
V^\dagger_{v_1}(0) \;&=\; \ell + \delta v_1 \in (\frac{\ell}{1-\delta}, v_1) \quad \text{and} \quad V^\dagger_{v_1}(1) = v_1 + \Delta_{v_1} \;>\; V^*_\ell(1) \qquad (33)\\
\text{where} \quad \Delta_{v_1} \;&:=\; \frac{1-(1-\delta)v_1}{(1-\delta\ell)} \;<\; \Delta < 1\,.
\end{aligned}
$$

Moreover, the relevant steps in the proof for Lemma 2 of the main article, extend straightforwardly to verify that $V^\dagger_{v_1}$ is continuous and strictly increasing in μ.

Define $\bar{\mathcal{F}}_{v_1}$ to be the set of all non-decreasing and right-continuous functions V on $[0,1]$ such that $V(0) = V^\dagger_{v_1}(0)$ and $V(1) = V^\dagger_{v_1}(1)$. Define $y^\dagger_V(\mu)$ in the same manner as in (41) and (42) of the main article, with $V(0)$ replaced by v_1 and $\bar{\mu}$ replaced by $\bar{\mu}^\dagger := \min\{\mu \mid p_G(\mu,1) \geq \delta\Delta_{v_1}\} < \bar{\mu}$ where the last inequality follows from $\Delta_{v_1} < \Delta$. As long as $\delta\Delta_{v_1} > \ell$ so that $y^\dagger_V(\mu) < 1$ for some μ, which we assume below (else, $y^\dagger_V(\mu) \equiv 1$ and the proof is simpler), we have $y^\dagger_V(\mu) \in (0,1)$ for $\mu \in (0, \bar{\mu}^\dagger)$ with $\lim_{\mu\to 0} y^\dagger_V(\mu) = 1$ because $\delta\big(\max\{V(\pi_{Bb}(\mu,y)), v_1\} - v_1\big)$ approaches $\delta\Delta_{v_1} > \ell$ as $y \to 1$ while it approaches

0 as $\mu \to 0$ for all $y < 1$.[6] Furthermore, $y^\dagger_V(\mu)$ is clearly continuous and equal to 1 for $\mu \geq \bar{\mu}^\dagger$. Define $T_{v_1} : \bar{\mathcal{F}}_{v_1} \to \bar{\mathcal{F}}_{v_1}$ as

$$T_{v_1}(V)(\mu) := p_G(\mu, y^\dagger_V(\mu)) + \delta(\ell \max\{v_1, V(\pi^*_{Gg}(\mu))\} + (1-\ell)v_1). \tag{34}$$

It is straightforward to verify that $T_{v_1}(V) \in \bar{\mathcal{F}}_{v_1}$.

Then, the proof for Proposition 1 of Section 4.2 (of the main article) extend to T_{v_1}, establishing that, for any $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, there is a unique fixed point of T_{v_1} and it is continuous and strictly increasing. We omit the proofs because they are analogous with straightforward changes due to the seller opting to exit whenever his reputation level is so low that the continuation value falls short of v_1.[7]

Since the outside option value is v_1 for an h-type seller as well, optimality of truth-telling for h-type seller can be verified by an argument analogous to that leading to Theorem 1, with δ_h replaced by the threshold δ_{v_1} that solves

$$\delta_{v_1}\Big(\frac{h}{1-\delta_{v_1}} - v_1\Big) = 1.$$

Thus, we have shown that an honest equilibrium exists if $\delta(\frac{h}{1-\delta} - v_1) > 1$, when sellers can exit for an outside option value v_1. ■

3.1 Existence of an honest equilibrium with restart

Recall that μ_1 and χ_1 denote the default reputation level and stationary mass of new sellers, respectively; and v_1 denotes ℓ-seller's default value. In equilibrium, an ℓ-seller's strategy $y^\dagger_{v_1}$ and value function and $V^\dagger_{v_1}$ are given by Lemma 7 above.

Let $\rho_\theta(q)$ denote the probability that a seller of type θ draws $q \in \{g, b\}$, i.e., $\rho_\theta(g) = \theta = 1 - \rho_\theta(b)$. For any k-period quality history $\mathbf{h}^k = (q_1, \cdots, q_k) \in H^k := \{g, b\}^k$, let $\rho_\theta(\mathbf{h}^k)$ be the ex-ante probability that $\mathbf{h}^k$ realises for a seller of type θ. We use $\mathbf{h}^k_j = (q_1, \cdots, q_j)$ to denote the first j-entry truncation of $\mathbf{h}^k$.

Given a default reputation $\mu_1 > 0$, let $\pi(\mathbf{h}^k_j)$ denote the posterior reputation for a seller who has survived the history $\mathbf{h}^k_j$ without cheating, updated according to $y^\dagger_{v_1}$. Setting $\pi(\mathbf{h}^k_0) = \mu_1$, we can define $\pi(\mathbf{h}^k_j)$ recursively by:

$$\pi(\mathbf{h}^k_j) = \frac{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j)}{\pi(\mathbf{h}^k_{j-1})\rho_h(q_j) + (1-\pi(\mathbf{h}^k_{j-1}))\rho_\ell(q_j)(1-y^\dagger_{v_1}(\pi(\mathbf{h}^k_{j-1}), q_j))}, \tag{35}$$

where $y^\dagger_{v_1}(\mu, g) = 0$ and $y^\dagger_{v_1}(\mu, b) = y^\dagger_{v_1}(\mu)$ for all μ. Then, the ex-ante probability that an ℓ-seller remains in the market without having cheated after k-period history $\mathbf{h}^k$ is

$$\Pr(\mathbf{h}^k) = \prod_{j=1}^{k}[\rho_\ell(q_j)(1-y^\dagger_{v_1}(\pi(\mathbf{h}^k_{j-1}), q_j))(1-\chi)]. \tag{36}$$

[6] Note that this implies $V(\pi_{Bb}(\mu, y^\dagger_V(\mu))) - v_1 > p_G(\mu, y^\dagger_V(\mu))$ for all $\mu \in (0, \bar{\mu}^\dagger)$. Thus, an h-seller with any reputation $\mu > 0$ does not restart after trading a bad quality item because the value of updated reputation exceeds v_1 as this inequality shows. However, both types of seller may exit after trading a good quality item in the initial period if the value of the updated reputation, $V^\dagger_\theta(\pi^*_{Gg}(\gamma))$, falls short of v_1.

[7] In Appendix A.3 of the main article, (31) becomes $V^\dagger_{v_1}(\mu) = \sum_{t=0}^\infty \delta^t \ell^t (p_G(\pi^t_{Gg}(\mu), y_V(\pi^t_{Gg}(\mu))) - \ell) + \frac{\delta v_1(1-\ell)+\ell}{1-\delta\ell}$ and thus, (37) becomes $V^\dagger_\ell(\tilde{\mu}) - v_1 < \sum_{t=0}^\infty (p_G(\pi^t_{Gg}(\tilde{\mu}), \hat{y}) - \ell)\delta^t\ell^t$ because $\frac{\delta v_1(1-\ell)+\ell}{1-\delta\ell} < v_1$.

Consequently, in a stationary state, the measure of nominally k-period old ℓ-sellers who restart in period $k+1$ for $k \geq 1$, is

$$\chi_1(1-\mu_1)\Big(\sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k)(1-\ell)y^{\dagger}_{v_1}(\pi(\mathbf{h}^k))(1-\chi)\Big).$$

This implies that the total measure of old ℓ-sellers who restart in an arbitrary period is $\chi_1(1-\mu_1)\Lambda(v_1,\mu_1)$ where

$$\Lambda(v_1,\mu_1) := (1-\ell)(1-\chi)\sum_{k=1}^{\infty}\sum_{\mathbf{h}^k \in H^k} \Pr(\mathbf{h}^k)y^{\dagger}_{v_1}(\pi(\mathbf{h}^k)). \tag{37}$$

Now, as verified in the discussion preceding Theorem 3 in the main article, the values of v_1 and μ_1 in a stationary equilibrium constitute a fixed point that solves

$$v_1 = V^{\dagger}_{v_1}(\mu_1) \text{ and } \mu_1 = \frac{\gamma - \gamma\Lambda(v_1,\mu_1)}{1-\gamma\Lambda(v_1,\mu_1)}. \tag{38}$$

To show that such a fixed point exists, we need the next result.

Lemma 8 *Let $\psi : (\frac{\ell}{1-\delta}, \frac{1}{1-\delta}) \to \mathcal{C}_{[0,1]}$ be a mapping such that $\psi(v_1) = V^{\dagger}_{v_1}$ where $\mathcal{C}_{[0,1]}$ is the set of all continuous functions on $[0,1]$. Then, ψ is continuous in v_1 under the sup norm at any $v_1 > \frac{\ell}{1-\delta}$.*

Proof. Since continuity under the sup norm requires uniform convergence, the possibility of a fixed point having unbounded derivative poses a potential problem. The bulk of the proof evolves around how to circumvent this problem. We start with two preliminary claims asserting that $p^*_G(\mu)$ is of bounded variation on $[\varepsilon,1]$ for any $\varepsilon > 0$ (Claim 1) and consequently, so is the fixed point $V^{\dagger}_{v_1}$ (Claim 2).

Claim 1 *For any $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that $\forall v_1 \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, $\forall V \in \bar{\mathcal{F}}_{v_1} \cap C_{[0,1]}$, $\forall \mu$ and $\mu' \in (\varepsilon, \bar{\mu}^{\dagger}_{v_1})$,*

$$\frac{p_G(\mu', y^{\dagger}_{v_1}(\mu')) - p_G(\mu, y^{\dagger}_{v_1}(\mu))}{\mu' - \mu} \leq M_\varepsilon. \tag{39}$$

Proof of Claim 1: Note that we can find $k > 0$ such that $\frac{\partial p_G}{\partial \mu} > 0$ is bounded above uniformly by k, and $\frac{\partial p_G}{\partial y} < 0$ is bounded below uniformly by $-k$. Suppose $\mu < \mu' < \bar{\mu}^{\dagger}_{v_1}$ without loss of generality. If $y^{\dagger}_{v_1}(\mu') \geq y^{\dagger}_{v_1}(\mu)$, then $\frac{p_G(\mu', y^{\dagger}_{v_1}(\mu')) - p_G(\mu, y^{\dagger}_{v_1}(\mu))}{\mu'-\mu} < k$ because p_G decreases in y, proving (39).

Now suppose that $y^{\dagger}_{v_1}(\mu') < y^{\dagger}_{v_1}(\mu)$. Note that one can find k_ε, $\tilde{k}_\varepsilon > 0$ such that

$$\begin{aligned}
\frac{\partial \pi_{Bb}(\mu,y)}{\partial \mu} &= \frac{(1-h)(1-\ell)(1-y)}{[\mu(1-h)+(1-\mu)(1-\ell)(1-y)]^2} < k_\varepsilon \\
\frac{\partial \pi_{Bb}(\mu,y)}{\partial y} &= \frac{(1-h)(1-\ell)(1-\mu)\mu}{[\mu(1-h)+(1-\mu)(1-\ell)(1-y)]^2} > \tilde{k}_\varepsilon
\end{aligned}$$

for all $\mu > \varepsilon$ and $y \in [0,1]$. Thus, recalling that $\pi_{Bb}(\mu, y^{\dagger}_{v_1}(\mu))$ is nondecreasing, we deduce that (using the facts that $y^{\dagger}_{v_1}(\mu') < y^{\dagger}_{v_1}(\mu)$ and $\mu < \mu'$)

$$0 \le \pi_{Bb}(\mu', y^{\dagger}_{v_1}(\mu')) - \pi_{Bb}(\mu, y^{\dagger}_{v_1}(\mu)) < k_\varepsilon(\mu' - \mu) + \tilde{k}_\varepsilon(y^{\dagger}_{v_1}(\mu') - y^{\dagger}_{v_1}(\mu)),$$

and consequently,

$$y^{\dagger}_{v_1}(\mu') - y^{\dagger}_{v_1}(\mu) > -\frac{k_\varepsilon}{\tilde{k}_\varepsilon}(\mu' - \mu).$$

Therefore, we have

$$\begin{aligned} p_G(\mu', y^{\dagger}_{v_1}(\mu')) - p_G(\mu, y^{\dagger}_{v_1}(\mu)) &< k(\mu' - \mu) - k\big(y^{\dagger}_{v_1}(\mu') - y^{\dagger}_{v_1}(\mu)\big) \\ &< k\Big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\Big)(\mu' - \mu). \end{aligned}$$

We complete the proof by setting $M_\varepsilon = k\big(1 + \frac{k_\varepsilon}{\tilde{k}_\varepsilon}\big)$. $\square$

Claim 2 *For any $\varepsilon > 0$ and $v_1 \in [\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$,*

$$D^+ V^{\dagger}_{v_1}(\mu) := \limsup_{\mu' \downarrow \mu} \frac{V^{\dagger}_{v_1}(\mu') - V^{\dagger}_{v_1}(\mu)}{\mu' - \mu} \le \frac{M_\varepsilon}{1 - \delta h} \quad \forall \mu > \varepsilon.$$

Proof of Claim 2: For given v_1 there exists $\underline{\mu} > 0$ defined by $V^{\dagger}_{v_1}(\pi^*_{Gg}(\underline{\mu})) = v_1$, so that

$$V^{\dagger}_{v_1}(\mu) = \begin{cases} p_G(\mu, y^{\dagger}_{v_1}(\mu)) + \delta v_1 \quad \text{if} \quad \mu \le \underline{\mu} \\ p_G(\mu, y^{\dagger}_{v_1}(\mu)) + \delta(\ell V^{\dagger}_{v_1}(\pi^*_{Gg}(\mu)) + (1-\ell)v_1) \quad \text{if} \quad \mu \ge \underline{\mu}. \end{cases} \tag{40}$$

To prove the claim we first show that $D^+ V^{\dagger}_{v_1}(\mu)$ is uniformly bounded. To reach a contradiction, suppose that for any $K > 0$ one can find $\mu_K > \varepsilon$ such that $D^+ V^{\dagger}_{v_1}(\mu_K) > K$. Then, since $\pi^*_{Gg}(\mu)$ is differentiable and $\frac{\ell}{h} \le \frac{\partial \pi^*_{Gg}(\mu)}{\partial \mu} \le \frac{h}{\ell}$, (39) and (40) would imply that $\mu_K > \underline{\mu}$ when K is sufficiently large and that one can construct a sequence $\mu_n \to 1$ where $\mu_n = \pi^*_{Gg}(\mu_{n-1})$ and $\mu_2 = \pi^*_{Gg}(\mu_K)$. Consider $\tau < \infty$ such that $\pi^{\tau}_{Gg}(\varepsilon) > \bar{\mu}$. By choosing K arbitrarily large, one can ensure that $D^+ V^{\dagger}_{v_1}(\pi^{\tau}_{Gg}(\mu_K))$ is arbitrarily large while $\pi^*_{Gg}(\mu_{n-1}) > \bar{\mu}^{\dagger}$. But, this is impossible because $D^+ V^{\dagger}_{v_1}(\mu)$ is bounded for $\mu > \bar{\mu}^{\dagger}$ as can be verified from

$$V^{\dagger}_{v_1}(\mu) = \Big[\sum_{t=0}^{\infty} \ell^t \delta^t p_G(\pi^t_{Gg}(\mu), 1)\Big] + \delta v_1 (1-\ell) \sum_{t=0}^{\infty} \ell^t \delta^t, \tag{41}$$

a formula derived from $y^{\dagger}_{v_1}(\mu) = 1$ for $\mu > \bar{\mu}^{\dagger}$. Hence, we conclude that $D^+ V^{\dagger}_{v_1}(\mu)$ is uniformly bounded for $\mu > \varepsilon$ and thus, (39) and (40) imply

$$\begin{aligned} D^+ V^{\dagger}_{v_1}(\mu) &\le M_\varepsilon + \ell\delta\Big(\sup_{\mu > \varepsilon} D^+ V^{\dagger}_{v_1}(\mu)\Big)\Big(\max_{\mu} \frac{\partial \pi^*_{Gg}(\mu)}{\partial \mu}\Big) \\ &\le M_\varepsilon + h\delta\Big(\sup_{\mu > \varepsilon} D^+ V^{\dagger}_{v_1}(\mu)\Big) \end{aligned}$$

for $\mu > \varepsilon$ where the second inequality follows from $\frac{\partial \pi^*_{Gg}(\mu)}{\partial \mu} \leq h/\ell$. Thus, $D^+ V^\dagger_{v_1}(\mu) \leq \frac{M_\varepsilon}{1-\delta h}$ if $\mu > \varepsilon$. $\square$

Next, choose $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$. Notice that for a sufficiently small $\eta > 0$, in particular smaller than $v_1 - \frac{\ell}{1-\delta}$, the operator T_{v_1} can be extended to $\bar{\mathcal{F}}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ where $\bar{\mathcal{F}}^\eta_{v_1} := \cup_{v_1-\eta \leq v \leq v_1+\eta} \bar{\mathcal{F}}_v$. As an intermediate step, we need:

Claim 3 *For $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$, the operator*

$$T_{v_1} : \bar{\mathcal{F}}^\eta_{v_1} \cap \mathcal{C}_{[0,1]} \to \mathcal{C}_{[0,1]} \text{ is continuous in sup norm.} \tag{42}$$

Proof of Claim 3: Consider $V, V' \in \bar{\mathcal{F}}^\eta_{v_1} \cap \mathcal{C}_{[0,1]}$ such that $\max_{\mu \in [0,1]} |V'(\mu) - V(\mu)| < \epsilon$. Since $y^\dagger_V(\mu)$ and $y^\dagger_{V'}(\mu)$ are, by construction, the solutions to

$$\min_{0 \leq y \leq 1} \left| p_G(\mu, y) - \delta\big(\max\{v_1, V(\pi_{Bb}(\mu, y))\} - v_1 \big) \right| \tag{43}$$

and the same equation with V' instead of V, respectively, it follows that $|p_G(\mu, y^\dagger_{V'}(\mu)) - p_G(\mu, y^\dagger_V(\mu))| < \epsilon$. From (34), therefore, we deduce that

$$\max_{\mu \in [0,1]} |T_{v_1}(V')(\mu) - T_{v_1}(V)(\mu)| < \epsilon + \delta\epsilon,$$

which establishes (42). $\square$

Given $v_1 \in (\frac{\ell}{1-\delta}, \frac{1}{1-\delta})$ and η small as specified above, consider small $|\kappa| < \eta/2$ and any $V \in \bar{\mathcal{F}}^\eta_{v_1} \cap \bar{\mathcal{F}}^\eta_{v_1+\kappa} \cap \mathcal{C}_{[0,1]}$. By (43), the value of $p_G(\mu, y^\dagger_V(\mu))$ differs when calculated for T_{v_1} and when calculated for $T_{v_1+\kappa}$, and the difference is at most $\delta\kappa$. Thus, from (34),

$$T_{v_1}(V)(\mu) - 2|\delta\kappa| \leq T_{v_1+\kappa}(V)(\mu) \leq T_{v_1}(V)(\mu) + 2|\delta\kappa| \quad \forall \mu \in [0,1]. \tag{44}$$

In particular, observe that

$$T_{v_1}(V^\dagger_{v_1+\kappa})(\mu) - 2|\delta\kappa| \leq T_{v_1+\kappa}(V^\dagger_{v_1+\kappa})(\mu) = V^\dagger_{v_1+\kappa}(\mu) \leq T_{v_1}(V^\dagger_{v_1+\kappa})(\mu) + 2|\delta\kappa|.$$

Finally, to prove continuity of ψ at v_1, we decompose the argument into two parts: First, we prove uniform convergence of functions $\psi(v_1 + \kappa) = V^\dagger_{v_1+\kappa}$ to $\psi(v_1) = V^\dagger_{v_1}$ as $\kappa \to 0$ on intervals $[\varepsilon, 1]$, then do the same separately on $[0, 2\varepsilon]$. The continuity will be established by combining the two parts.

a) We know from Claim 2 that on the interval $[\varepsilon, 1]$, the function $V^\dagger_{v_1+\kappa}$ is K_ε-Lipschitz where $K_\varepsilon = \frac{M_\varepsilon}{1-\delta h}$. Then from Ascoli-Arzelà Theorem (see Royden (1988)), the subset consisting of all K_ε-Lipschitz function of $\bar{\mathcal{F}}^\eta_{v_1}$ is compact under the sup norm. Hence, there exists a sequence of fixed points $V^\dagger_{v_1+\kappa}$ such that, when restricted to the domain $[\varepsilon, 1]$, it converges as $\kappa \to 0$ to a limit, denoted by $W^{[\varepsilon,1]}_{v_1}$, continuous on $[\varepsilon, 1]$ and

$$V^\dagger_{v_1+\kappa} \xrightarrow{unif} W^{[\varepsilon,1]}_{v_1} \text{ under the sup norm on } [\varepsilon, 1] \text{ for any } \varepsilon > 0. \tag{45}$$

Let $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote $V^{\dagger}_{v_1+\kappa}$ restricted on $[\varepsilon, 1]$ and let $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ denote the continuous linear extension of $V^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ on $[0, \varepsilon]$. Then, by (42) and (44),

$$T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq \lim_{\kappa\to 0} T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu) \leq T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu). \tag{46}$$

Note that $T_{v_1}(\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})(\mu)$ for each μ is fully determined by $\lim_{\kappa\to 0} \widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa}$ restricted on $[\mu, 1]$ according to (34), and the same is true for $T_{v_1+\kappa}(\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa})$. Since $\widetilde{V}^{\dagger[\varepsilon,1]}_{v_1+\kappa} = V^{\dagger}_{v_1+\kappa}$ on $[\varepsilon, 1]$ by definition, therefore, (45) and (46) imply that

$$T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \leq \widetilde{W}^{[\varepsilon,1]}_{v_1}(\mu) \leq T_{v_1}(\widetilde{W}^{[\varepsilon,1]}_{v_1})(\mu) \quad \text{for all} \quad \mu \in [\varepsilon, 1],$$

where $\widetilde{W}^{[\varepsilon,1]}_{v_1}$ is the continuous linear extension of $W^{[\varepsilon,1]}_{v_1}$ on $[0, \varepsilon]$. Since $\varepsilon > 0$ is arbitrary and $V^{\dagger}_{v_1}$ is the only function V that satisfies $T_{v_1}(V)(\mu) = V(\mu)$ on $[\varepsilon, 1]$ for all $\varepsilon \in (0, 1)$ by uniqueness of the fixed point of T_{v_1}, it further follows that $\widetilde{W}^{[\varepsilon,1]}_{v_1} = V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$, i.e., $W^{[\varepsilon,1]}_{v_1}$ coincides with $V^{\dagger}_{v_1}$ on $[\varepsilon, 1]$. From (45), therefore,

$$V^{\dagger}_{v_1+\kappa} \xrightarrow{unif} V^{\dagger}_{v_1} \text{ under the sup norm on } [\varepsilon, 1] \text{ for any } \varepsilon > 0. \tag{47}$$

b) Note, however, that this is not sufficient for uniform convergence on $[0, 1]$. Hence, choose $\varepsilon > 0$ small enough such that $V^{\dagger}_{v_1}(\pi^*_{Gg}(2\varepsilon)) < v_1$. Then, because $V^{\dagger}_{v_1+\kappa}$ converges to $V^{\dagger}_{v_1}$ under the sup norm on $[\varepsilon, 1]$ by (47), we have $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(2\varepsilon)) < v_1 + \kappa$ for sufficiently small κ. But this implies that $V^{\dagger}_{v_1+\kappa}(\pi^*_{Gg}(\mu)) < v_1 + \kappa$ for all $\mu \leq 2\varepsilon$ for sufficiently small κ, and consequently, $V^{\dagger}_{v_1+\kappa}(\mu) = p_G(\mu, 1) + \delta(v_1 + \kappa)$ on $[0, 2\varepsilon]$, which converges uniformly to $V^{\dagger}_{v_1}(\mu) = p_G(\mu, 1) + \delta v_1$. Thus,

$$V^{\dagger}_{v_1+\kappa} \xrightarrow{unif} V^{\dagger}_{v_1} \text{under the sup norm on } [0, 2\varepsilon]. \tag{48}$$

Combining (47) and (48), we obtain uniform convergence under the sup norm on the entire domain $[0, 1]$, which proves continuity of ψ at v_1 and thus, Lemma 8. ■

Recall that $y^{\dagger}_{v_1}(\mu)$ uniquely solves

$$\begin{aligned} p_G(\mu, y^{\dagger}_{v_1}(\mu)) &= \delta\big(V^{\dagger}_{v_1}(\pi_{Bb}(\mu, y^{\dagger}_{v_1}(\mu))) - v_1\big) \text{ if } p_G(\mu, 1) \leq \delta\Delta_{v_1}, \text{ and} \\ y^{\dagger}_{v_1}(\mu) &= 1 \text{ otherwise.} \end{aligned}$$

By an analogous argument as in the proof of Theorem 1 (of the main article), $y^{\dagger}_{v_1}(\mu)$ is continuous in v_1 and in μ.

To complete the proof of Theorem 3, consider a mapping from $[\frac{\ell+\varepsilon}{1-\delta}, \frac{1-\varepsilon}{1-\delta}] \times [0, \gamma]$ into itself as below for sufficiently small $\varepsilon > 0$:

$$\begin{pmatrix} v_1 \\ \mu_1 \end{pmatrix} \longmapsto \begin{pmatrix} \max\left\{V^{\dagger}_{v_1}(\mu_1), \frac{\ell+\varepsilon}{1-\delta}\right\} \\ \frac{\gamma-\gamma\Lambda(v_1,\mu_1)}{1-\gamma\Lambda(v_1,\mu_1)} \end{pmatrix}.$$

Since this mapping is continuous as explained above, there is a fixed point by Brouwer's Fixed Point Theorem.

Note that, for given v_1 close to $\frac{\ell}{1-\delta}$, the solution value to $\mu_1 = \frac{\gamma-\gamma\Lambda(v_1,\mu_1)}{1-\gamma\Lambda(v_1,\mu_1)}$ remains bounded away from zero (because $\Lambda(v_1,\mu_1)$ is bounded away from 1 by (37)) and satisfies $V^{\dagger}_{v_1}(\mu_1) > \frac{\ell+\varepsilon}{1-\delta}$. Moreover, $V^{\dagger}_{v_1}(\mu_1) < V^{\dagger}_{v_1}(\gamma) < p_G(\gamma, \frac{h-\ell}{1-\ell}) + \delta(v_1 + \Delta_{v_1}) < \frac{1-\varepsilon}{1-\delta}$ by (33). Hence the fixed point value of v_1 satisfies $v_1 \in (\frac{\ell+\varepsilon}{1-\delta}, \frac{1-\varepsilon}{1-\delta})$.

Let μ_1 and v_1 denote a fixed point of the mapping. To establish the existence of an equilibrium, we still need to show that it is optimal for h-type sellers to always report truthfully as long as $\mu \geq \mu_1$. Since the continuation value of h-type sellers after cheating is the equilibrium value at the entry level, denoted by $V^{\dagger}_h(\mu_1)$, rather than $V^{\dagger}_h(0)$, the optimality condition of h-type sellers is more involved to verify than when restarting is impossible.

However, in the proof of Proposition 1 of this Online Appendix, we showed that if (1) holds then the slope of V^*_h is larger than the slope of V^*_ℓ at all level of reputation and consequently, the sorting condition (2) holds. The same reasoning applies to the current case to verify that

$$V^{\dagger}_h(\mu) - V^{\dagger}_h(\mu_1) > V^{\dagger}_{v_1}(\mu) - V^{\dagger}_{v_1}(\mu_1) \quad \forall \mu > \mu_1.$$

Therefore, if (1) holds then it constitutes an equilibrium for ℓ-type sellers to announce according to $y^{\dagger}_{v_1}(\mu)$ and for h-type sellers to always announce honestly for $\mu \geq \mu_1$, provided that $\delta(V^{\dagger}_h(1) - V^{\dagger}_h(\mu_1)) \geq 1$.[8]

Lastly, to show that $\delta(V^{\dagger}_h(1) - V^{\dagger}_h(\mu_1)) \geq 1$, as $\mu_1 < \gamma$ and $y^{\dagger}_{v_1} > \hat{y}$ we obtain

$$V^{\dagger}_h(\mu_1) < h p_G\big(\gamma, \frac{h-\ell}{1-\ell}\big) + \delta h V^{\dagger}_h(\pi_{Gg}(\gamma)) + \delta\,(1-h)\,V^{\dagger}_h(1).$$

As $p_G(\mu, \frac{h-\ell}{1-\ell}) = \frac{\mu(h-\ell)+\ell}{h}$, by extending the formula recursively we get

$$\begin{aligned} V^{\dagger}_h(\mu_1) &< \sum_{t=0}^{\infty} \delta^t h^t \left(\pi^t_{Gg}(\gamma)\,(h-\ell) + \ell\right) + \frac{\delta\,(1-h)}{1-\delta h}\frac{h}{1-\delta} \\ &= \sum_{t=0}^{\infty} \delta^t h^t \left(\frac{\gamma h^{t+1} + (1-\gamma)\,\ell^{t+1}}{\gamma h^t + (1-\gamma)\,\ell^t}\right) + \frac{\delta\,(1-h)}{1-\delta h}\frac{h}{1-\delta} \end{aligned}$$

where the equality follows from $\pi^t_{Gg}(\gamma) = \frac{\gamma h^t}{\gamma h^t + (1-\gamma)\ell^t}$. Then, as $V^{\dagger}_h(1) = h/(1-\delta)$,

$$\begin{aligned} \delta\left(V^{\dagger}_h(1) - V^{\dagger}_h(\mu_1)\right) &> \delta\left(\frac{h}{1-\delta h} - \sum_{t=0}^{\infty} \delta^t h^t \left(\frac{\gamma h^{t+1} + (1-\gamma)\,\ell^{t+1}}{\gamma h^t + (1-\gamma)\,\ell^t}\right)\right) \\ &= \delta\left(\sum_{t=0}^{\infty} \delta^t h^t \left(h - \frac{\gamma h^{t+1} + (1-\gamma)\,\ell^{t+1}}{\gamma h^t + (1-\gamma)\,\ell^t}\right)\right) \\ &= \delta\left(\sum_{t=0}^{\infty} \delta^t h^t\,(1-\gamma)\,\ell^t \left(\frac{h-\ell}{\gamma h^t + (1-\gamma)\,\ell^t}\right)\right). \end{aligned}$$

[8] The proof is omitted because it is the same as the proof of Proposition 1 of this online Appendix with obvious changes, such as v_1 and $\bar{\mu}^{\dagger}$ in place of $V^*_\ell(0)$ and $\bar{\mu}$, respectively.

Taking the limit as $\delta \to 1$,

$$\begin{aligned}
V_h^\dagger(1) - V_h^\dagger(\mu_1) &\geq \sum_{t=0}^{\infty}(1-\gamma)\ell^t\left(\frac{h-\ell}{\gamma+(1-\gamma)(\ell/h)^t}\right) \\
&> (h-\ell)\left[1-\gamma+\sum_{1}^{\infty}\frac{(1-\gamma)\ell^t}{\gamma+(1-\gamma)(\ell/h)^t}\right] \\
&> (h-\ell)\left[1-\gamma+\frac{(1-\gamma)\ell}{(1-\ell)\big(\gamma+(1-\gamma)(\ell/h)\big)}\right] \\
&\to (h-\ell)\Big(1+\frac{h}{1-\ell}\Big) \quad \text{as} \quad \gamma \to 0.
\end{aligned} \tag{49}$$

Therefore, $\delta(V_h^\dagger(1) - V_h^\dagger(\mu_1)) \geq 1$ holds for sufficiently large δ if γ is sufficiently small, so long as $h > h^\dagger(\ell)$ where $h^\dagger(\ell) \in (\ell, 1)$ is a solution to $(h-\ell)\big(1+\frac{h}{1-\ell}\big) = 1$. This completes the proof of existence of an equilibrium as stated in Theorem 3. Note that the lower bound of h above which an honest equilibrium exists decreases as the sum (49) is expanded beyond $t = 1$.

3.2 Proof that $y^\dagger(\mu) > y^*(\mu)$.

Observe from $\Delta_{v_1} < \Delta$ that $\bar{\mu}_{v_1}^\dagger < \bar{\mu}$. It is immediate from the definition of $\bar{\mu}_{v_1}^\dagger$ that $y_{v_1}^\dagger(\mu) = 1$ for all $\mu \geq \bar{\mu}_{v_1}^\dagger$. Hence, we consider $\mu < \bar{\mu}_{v_1}^\dagger < \bar{\mu}$ below.

It is straightforward to extend the relevant arguments in the proof of Proposition 1 of the main article, to verify that $y_{v_1}^\dagger$ is continuous and $y_{v_1}^\dagger(\mu) \in (0,1)$ for $\mu < \bar{\mu}_{v_1}^\dagger$. To reach a contradiction, suppose $y_{v_1}^\dagger(\mu') = y^*(\mu')$ for some $\mu' < \bar{\mu}_{v_1}^\dagger$ and $y_{v_1}^\dagger(\mu) > y^*(\mu)$ for all $\mu \in (\mu', \bar{\mu})$. Then,

$$\begin{aligned}
\delta\big(V_\ell^*(\pi_{Bb}(\mu', y^*(\mu'))) - V_\ell^*(0)\big) &= p_G(\mu', y^*(\mu')) \\
&= p_G(\mu', y_{v_1}^\dagger(\mu')) = \delta\big(V_{v_1}^\dagger(\pi_{Bb}(\mu', y_{v_1}^\dagger(\mu'))) - v_1\big)
\end{aligned}$$

and thus,

$$V_\ell^*(\tilde{\mu}) - V_\ell^*(0) = V_{v_1}^\dagger(\tilde{\mu}) - v_1 \quad \text{where} \quad \tilde{\mu} := \pi_{Bb}(\mu', y^*(\mu')) > \mu' \tag{50}$$

and the inequality is from Lemma 5 of Appendix A.3. Furthermore, since

$$V_\ell^*(\tilde{\mu}) = p_G(\tilde{\mu}, y^*(\tilde{\mu})) + \delta\big(\ell V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)V_\ell^*(0)\big) \quad \text{and} \tag{51}$$

$$V_{v_1}^\dagger(\tilde{\mu}) = p_G(\tilde{\mu}, y_{v_1}^\dagger(\tilde{\mu})) + \delta\big(\ell V_{v_1}^\dagger(\pi_{Gg}^*(\tilde{\mu})) + (1-\ell)v_1\big) \tag{52}$$

while $p_G(\tilde{\mu}, y^*(\tilde{\mu})) \geq p_G(\tilde{\mu}, y_{v_1}^\dagger(\tilde{\mu}))$, (50)–(52) would imply

$$\delta\ell((V_\ell^*(\pi_{Gg}^*(\tilde{\mu})) - V_\ell^*(0)) - (V_{v_1}^\dagger(\pi_{Gg}^*(\tilde{\mu})) - v_1)) \leq (\delta - 1)(v_1 - V_\ell^*(0)) < 0. \tag{53}$$

Since $V_\ell^*(1) - V_\ell^*(0) = \Delta > \Delta_{v_1} = V_{v_1}^\dagger(1) - v_1$, there must exist $\mu'' \in [\tilde{\mu}, 1)$ such that $V_\ell^*(\mu'') - V_\ell^*(0) \leq V_{v_1}^\dagger(\mu'') - v_1$ and $V_\ell^*(\mu) - V_\ell^*(0) > V_{v_1}^\dagger(\mu) - v_1$ for all $\mu > \mu''$. However, since $p_G(\mu'', y^*(\mu'')) \geq p_G(\mu'', y_{v_1}^\dagger(\mu''))$ and $\pi_{Gg}^*(\mu'') > \mu''$, (51) and (52) evaluated at $\mu = \mu''$ imply that $V_\ell^*(\mu'') - \delta V_\ell^*(0) > V_{v_1}^\dagger(\mu'') - \delta v_1$ and consequently, $V_\ell^*(\mu'') - V_\ell^*(0) > V_{v_1}^\dagger(\mu'') - v_1$, contradicting the definition of μ''. This completes the proof.