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Abstract

What is the effect of future information on today's actions? The answer may help understand, or justify, low investment in the presence of adjustment costs, a preference for holding liquid money, self-insurance or precautionary savings motives, environmental preservation and global warming abatement policies. Within a three-period model, Epstein (1980) showed that the effect of future information depends on a condition on an indirect value function. We provide the necessary and sufficient condition on the model's primitives. Furthermore, we derive a generic ambiguity result, and characterize all model specifications for which the question can be answered without ambiguity. These specifications include all classical models discussed in the literature. The paper also discusses the interpretation of the concept of flexibility in this literature.

Introduction

Economic agents often make decisions under uncertainty, anticipating that future information may reduce uncertainty over time. This paper provides a systematic analysis of the anticipated effect of future information on today's decisions. This analysis is relevant to study the option value to wait for future information,[1] but it can virtually apply to all problems in which decisions are made sequentially. Applications include investment decisions in the presence of adjustment costs, the decision to preserve or to develop an environmental area, portfolio composition decisions with costly portfolio rebalancing, the timing of CO_2 emissions or consumption smoothing over time.

Our framework is standard. It is based on a model of learning with a Bayesian, von Neumann-Morgenstern decision-maker. We consider a three-period model in which the decision-maker first chooses a; then gets some information on the unknown state of nature x; and finally chooses b. The decision-maker has preferences $U(x, a, b)$. Information arrival is modelled using a Blackwell information structure, whose precision can be varied. The question is: how does the optimal first-period decision a change when the information structure is replaced by a more precise one?[2]

There has been an important literature analyzing the effect of learning using this framework. The literature begins with Arrow and Fisher (1974) and Henry (1974) who show that learning always favors more flexible decisions. This classical result is usually coined the "irreversibility effect". It is well-known, however, that Arrow-Fisher-Henry consider a specific problem

[1] This value is also coined the quasi-option value.

[2] The literature uses equivalent terms to characterize a more precise information structure in the sense of Blackwell. These terms include an ealier resolution of uncertainty (Epstein, 1980), an increase in uncertainty (Jones and Ostroy, 1984), learning (Ulph and Ulph, 1997), or a better information structure (Gollier, Jullien and Treich, 2000). In this paper, we will often say that we study the effect of learning, or that of a more precise information.

using additive separable preferences of the form $U(x,a,b) = u(a,x) + v(b,x)$. Epstein (1980) emphasizes this limitation, and considers a general nonseparable model in which today's decision a can *directly* affect the future utility $v(b,x)$. Epstein derives a condition that permits to sign the effect of the precision of information in this model. The subsequent literature, including Demers (1991), Kolstad (1996), Ulph and Ulph (1997), and Gollier, Jullien and Treich (2000), considers nonseparable models and therefore studies the Epstein's condition.[3] Jones and Ostroy (1984) extend the Epstein's condition to non-differentiable problems.

It must be stressed, however, that the Epstein's condition bears on the value function of the second-period problem. More precisely, the condition depends on the convexity in p of the derivative in a of the function $j(a,p) \equiv \max_b \sum_x p(x)U(x,a,b)$, in which $p(x)$ is the probability of state x, and p the associated probability distribution. It is recognised that it is technically difficult to solve the comparative static analysis of learning using this condition. It is also difficult to connect this condition to the primitives of the model U. Papers using the Epstein's condition usually assume simple functional forms. The analysis of the effect of learning thus does not usually provide interpretative properties on preferences and technologies, unlike the literature on the effect of more risk (Rothschild and Stigliz, 1970, Gollier, 2001). Moreover, there are "ambiguous" results in the literature, in the sense that an increase in the precision of information may reduce or increase the first period's decision depending on precise properties of the function U, of prior beliefs, or of the information structures and probability distributions.

This paper first characterizes a condition on U that indicates whether the problem is "ambiguous", that is whether the effect of more precise information can always, or can never, be signed. This result therefore identifies

[3]See also three excellent surveys of the option value literature that base their analysis on the Epstein's condition: Graham-Tomasi (1995), Gollier (2001, Chapter 25) and Fisher and Mäler (2006).

the exact conditions ensuring that there is a connection possible between the Epstein's condition and the primitives of the model. The negative side of this result is that this condition only holds non-generically in the space of functions U. The positive side is that the non-generic class of functions U satisfying our condition is straightforward to characterize. It contains for example all functions U that are linear in the state of nature x, together with other, non-linear specifications, as in a global warming model when the decision-maker is risk-averse with linear tolerance to risk. This condition, that amounts to characterize a separability property, thus tells us when existing results in the literature could never be generalized. In particular, we study when this property is, or is not, satisfied under risk averse preferences.[4]

Furthermore, when this property is satisfied, and thus when the effect of learning is non-ambiguous, we derive an additional condition on U that directly indicates what is the systematic effect of learning. This additional condition is such that it makes the decision-maker's payoff more dependent on the second period decisions. By directly applying this technical condition, we revisit and/or extend all applications (that we are aware of) that have been considered so far the literature. For instance, we show that the irreversibility effect usually does not hold when adjustment costs are "smooth". We also identify the precise conditions on the utility function and on the damage function so that learning increase early CO_2 emissions within a fairly standard model of global warming. Also, this condition allows us to discuss the concept of flexibility, and to provide an interpretation to this concept in terms of the anticipated cost of ex post mistakes. Finally, we use this condition to solve new problems, for instance to examine the effect of learning on the demand of risky assets with adjustment costs.

[4]It is often recognised that the option value exists even under risk-neutrality. Dixit (1992, p.110) for instance says that "the value of waiting has nothing to do with risk-aversion". Although this statement is correct, it may lead to believe that risk-aversion does not affect the option value, and, more generally, does not affect the anticipated effect of information on early decisions. See, for instance, Kolstad (1996). Our paper strongly qualifies this belief.

The plan is the following. Section 1 introduces the model, and the Epstein's result. Section 2 defines and characterizes a particular class of payoff functions. Section 3 offers the main result. Section 4 discusses the irreversibility effect using simple examples. Section 5 and 6 study more applications cases found in the literature, and introduce new ones. Section 7 concludes.

1 The Model

We consider a simple three-period model in which a decision-maker first chooses a; then learns some information y on the unknown state of nature $x \in X$; and finally chooses b. The decision-maker is a Bayesian expected utility maximizer, with preferences $U(x, a, b)$ and prior beliefs p. We assume that b is chosen in $\mathbb{R}^n$ ($n \geq 1$); that U is strictly concave with respect to b; that a is chosen in a closed interval of $\mathbb{R}$; that U can be differentiated as many times as needed with respect to (a, b); that X is finite; and finally that beliefs p verify $p(x) > 0$ for all $x \in X$.[5]

For some prior beliefs p on X, the decision-maker's objective function when choosing a is

$$j(a, p) \equiv \max_b \sum_x p(x) U(x, a, b).$$

Notice that the strict concavity of U ensures the uniqueness of the solution $b(a, p)$ to this problem. In the following, we shall without loss of generality restrict attention to the pairs (a, b) such that $b = b(a, p)$ for some p. Notice also that j is convex in p because it is a maximum of linear functions.

Consider a random variable $\tilde{y}$ whose distribution conditional to x is

[5] One could additionally restrict beliefs to belong to an open, convex subset of the set of distributions on X. The conditions derived below remain unchanged. As a consequence, our results hold also locally, in the neighbourhood of a decision a and for small variations in beliefs.

known.[6] Given the prior beliefs p, a realization y of $\tilde{y}$ makes the decision-maker update his prior p into posterior beliefs q_y. Then the decision-maker's objective function changes from $j(a,p)$ to

$$E_y \max_b \sum_{x \in X} q_y(x) U(x,a,b) = E_y j(a, q_y)$$

Notice that Bayesian updating requires that $p = E_y q_y$. From the convexity of j it immediately follows that

$$E_y j(a, q_y) - j(a,p) \geq 0$$

This difference is the option value associated to a; it is simply the value of future information, conditional on the first-period decision a.[7]

Consider now the more general problem in which it is the precision of future information which is learnt to be increased. A more precise information is defined as a random variable $\tilde{y}'$, such that any decision-maker prefers learning $\tilde{y}'$ to learning $\tilde{y}$. As is well-known, this is equivalent to the requirement that $\tilde{y}$ can be obtained from $\tilde{y}'$ by using a 'garbling machine', which adds a noise uncorrelated with the true state of nature.[8] Another useful result about the comparison of information structures is the following : $\tilde{y}'$ is more precise than $\tilde{y}$ if and only if, for any prior p, the distribution of posteriors $q_{y'}$ forms a mean-preserving spread of the distribution of posteriors q_y, in the

[6] $y|x$ then defines an information structure in the sense of Blackwell; assume for simplicity that y takes a finite number of values, and that $prob(y|x)$ is everywhere strictly positive, so that posterior beliefs $q_y(x) = prob(x|y)$ are also everywhere positive.

[7] This definition may follow from the idea that the option value is a correction term, allowing to take into account the effects of learning (see Hanemann, 1989).

[8] One manner to obtain a more precise information is to consider that the decision-maker faces a given experiment $\tilde{y}|x$, and to reduce his confidence in his prior beliefs. Then the decision-maker will revise his beliefs differently, by giving more weight to the new information and less weight to his prior beliefs. Jones and Ostroy (1984) argue that this leads to a new structure of posteriors corresponding to a more precise information. Hence switching to a more precise experiment may be interpreted as introducing more uncertainty in the decision-maker's prior beliefs.

space of posteriors (Marschak and Miyazawa, 1968). In other words, one can find (hypothetical) conditional probabilities $prob(y'|y)$ such that

$$\forall\ y \qquad q_y = \sum_{y'} prob(y'|y) q_{y'}.$$

Once more, switching from $\tilde{y}$ to $\tilde{y}'$ can only benefit the decision-maker, whatever his first-period decision a : by convexity of j in p we have

$$E_y j(a, q_y) \leq E_y prob(y'|y) j(a, q_{y'}) = E_{y'} j(a, q_{y'})$$

The same reasoning can now be applied to check whether the change from $\tilde{y}$ to $\tilde{y}'$ increases the first-period decision a. Since the optimal decision obtains by maximizing $E_y j(a, q_y)$, this will be the case if[9]

$$E_y j_a(a, q_y) \leq E_{y'} j_a(a, q_{y'}).$$

This result forms Epstein condition :

Proposition 1 *(Epstein, 1980) An increase in precision of future information increases the optimal first period decision if and only if $j_a(a, p)$ is convex in p.*

This condition has been applied in the literature to various simple specifications for the function U. Because the condition bears on an indirect value function and not on U directly, it is not easy to check. The first aim of this paper is to replace this condition bearing on j by an equivalent condition bearing on the primitive U.

2 Payoff Functions Admitting an Invariant

The analysis would be easy if the optimal second-period choice $b(a, p)$ did not depend on a. Notice, however, that one can proceed to a change of variable on b, without any impact on the choice of a. Hence the definition:

[9]Subscripts denote partial derivatives.

Definition 1 *Let $b(a,p)$ denote the unique solution to*

$$\max_b \sum_x p(x)U(x,a,b)$$

The payoff function U admits an invariant if and only if there exists a change of variable $B = f(a,b)$ such that :

i) for any a, $b \neq b' \Rightarrow f(a,b) \neq f(a,b')$

ii) for any p, $f(a,b(a,p))$ does not depend on a.

Therefore, if U admits an invariant f, then f^{-1} exists and one can define

$$V(x,a,B) = U(x,a,f^{-1}(B,a))$$

Then the optimal second-period decision for a decision-maker endowed with preferences V and beliefs p does not depend on a : it is

$$B(p) \equiv f(a,b(a,p))$$

Equivalently, we get $b(a,p) = f^{-1}(B(p),a)$, so that $b(a,p)$ depends on p only through a statistic $B(p)$. Since the change of variable must be one-to-one, the constraint is that this statistic $B(p)$ must have the same dimension as b.

This class of functions may seem restrictive, but it contains for example all functions U that are linear in x. Indeed consider

$$U(x,a,b) = u(a,b) + v(a,b) \cdot x$$

where $\cdot$ denotes a scalar product. The decision-maker's choice $b(a,p)$ is characterized by

$$u_b(a,b) + v_b(a,b)E_p x = 0$$

so that $b(a,p)$ depends on p only through the statistic $E_p x$. The problem thus admits the invariant

$$f(a,b) = -[v_b(a,b)]^{-1}u_b(a,b)$$

which is well-defined because we have assumed U strictly concave. In the case when both b and x are one-dimensional, this invariant is the marginal rate of substitution between the risk-free part u and the exposure to risk v; for given beliefs the decision-maker always make the same trade-off between these two elements.

Other examples of functions U admitting an invariant are discussed in the next sections. The following result gives a simple property for these functions:

Suppose that U admits an invariant. Then for any a and b, there exists a vector $d(a, b)$ and a matrix $M(a, b)$ such that

$$\forall\ x \qquad U_{ab}(x, a, b) + U_{bb}(x, a, b)d(a, b) = M(a, b)U_b(x, a, b) \tag{1}$$

Moreover, if this property holds, then

$$\forall\ a, p \qquad d(a, b(a, p)) = \frac{\partial b}{\partial a}(a, p).$$

The restriction in (1) is that it must be valid for all $x \in X$. On the other hand, (1) only depends on U_b, so that this property is unaffected if one adds an arbitrary function $w(a, x)$ to U.

The following Corollary may also be useful when $U(x, a, b)$ is differentiable.

Corollary 1 *When b and x are real numbers and $U(x, a, b)$ is differentiable, (1) is equivalent to*

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial}{\partial a}\frac{\partial}{\partial x}\log U_b}{\frac{\partial}{\partial b}\frac{\partial}{\partial x}\log U_b}\right) = 0$$

In the following sections, we will mostly use (1) which allows for multi-dimensional b and arbitrary x.

3 Main Result

Our main result is the following :

Proposition 2 *If an increase in precision of future information increases the optimal first-period decision, then U verifies (1).*

Conversely, if U verifies (1), and if moreover the matrix $M+d_b$ is positive (resp. negative) semi-definite for any a and b, then an increase in precision of future information increases (resp. decreases) the optimal first-period decision.

Finally, if b is one-dimensional the condition on the sign of $M+d_b$ is also necessary.

The necessity part of the Proposition shows that in general one can only answer the question of the impact of more information on today's optimal policy in non-generic cases; indeed (1) has to be satisfied. The proof shows that this ambiguity is also local: it still holds in neighborhood of a decision a, and only small variations of beliefs are allowed.

At this stage, we can offer some intuition on the sufficiency part of Proposition 2. Recall that if U admits an invariant f, and if moreover f verifies differentiability assumptions, then one can work as well with

$$V(x,a,B) = U(x,a,f^{-1}(B,a))$$

and the optimal second-period decision $B(p)$ does not depend on a. Therefore (1) reduces to

$$V_{aB}(x,a,B) = M(a,B)V_B(x,a,B)$$

In the case when B is one-dimensional, a necessary and sufficient condition for learning to favour higher decisions a is that the real number $M(a,b)$ be positive. From the above equality, this means that a higher a increases the absolute value of V_B. Thus the decision-maker's payoff becomes more dependent on B, for each state of nature x.

To go further, define the ex-post cost from choosing B as

$$[\max_{B'} V(x,a,B')] - V(x,a,B)$$

This term can be interpreted as the ex post cost of a 'mistake' (as checked after the realization of x) induced by the choice of B. Our result says that decisions that are favoured by more learning are those that increase these costs. Intuitively, a decision-maker who knows he will be better informed cares less about these mistakes, and can handle relatively more easily an increase in their costs.

It is useful to relate this intuition to the one given in Jones and Ostroy (1984) about the concept of flexibility. Jones and Ostroy suggest that a decision is more flexible if it is favoured by the future arrival of a more precise information. The idea is that tomorrow a better informed decision-maker will make better choices. Hence the anticipation of more information should favour decisions which are more 'flexible', in the intuitive sense of allowing for a wider range of tomorrow's decisions at a lower cost. One manner to give a precise content to this last sentence is thus to characterize decisions which are favoured by an increase in the precision of future information. This is what Jones and Ostroy do, and what we do as well. Yet, contrary to the intuition given by Jones and Ostroy, the formulation above suggests that the decisions favoured by learning *reduce* the range of tomorrow's decisions that are accessible at a given cost – and thus appear less 'flexible', and not more.

Finally, we provide a result that complements Corollary 1.

Corollary 2 *Assume that b and x are real numbers, that $U(x, a, b)$ is differentiable and that*

$$g(a, b) \equiv \frac{\frac{\partial}{\partial a}\frac{\partial}{\partial x}\log U_b}{\frac{\partial}{\partial b}\frac{\partial}{\partial x}\log U_b}$$

is independent from x. An increase in the precision of future information increases the optimal first-period decision if and only if

$$\frac{1}{U_b(x, a, b)}\frac{\partial}{\partial b}(U_a(x, a, b) - g(a, b)U_b(x, a, b))$$

is positive.

The following sections study the applications found in the literature. In particular, we show that the class of functions that admit an invariant contains many simple specifications for which it is easy to sign $M + d_b$.[10]

4 The Irreversibility Effect

As we said in introduction, the literature on the irreversibility effect traditionally considers additive separable preferences of the form[11]

$$U(x, a, b) = u(a, x) + v(b, x)$$

Following Proposition 2, these preferences imply $M + d_b = 0$. Learning thus does not matter because changes in b do not affect the payoffs from a: $U_{ab} = 0$ implies that (1) is verified with $M = d = 0$.

However, in the literature on the irreversibility effect, learning has an effect because the today's decision affects the future decision set. Typically, there is an irreversibility constraint. Consider for example the case when a firm chooses an investment a, and can adjust it only upward. The irreversibility constraint writes

$$b \geq a$$

It is then easily shown[12] that *in the presence of an irreversibility constraint, an increase in precision of future information reduces the optimal choice of a.* This classical result always holds precisely because U is additive separable.

[10]In the following sections, we will present several applications. We will thus need to introduce several economic models. Nevertheless, we will only present a very brief introduction to each of these models. Instead, we will focus on the properties of these models that are important for the analysis of the effect of learning. For a more accurate presentation of the various models, we recommend to read their initial presentation in the references that will be systematically given.

[11]See, e.g., Arrow and Fisher (1974), Henry (1974), Freixas and Laffont (1984), Fisher and Hanemann (1987) and Hanemann (1989).

[12]Using Epstein's condition, one easily find that $j_a(a, p) = \sum p(x)u_a(a, x) + \min(\sum p(x)v_b(a, x), 0)$, and is thus concave in p. Similar proofs can be found in Epstein (1980).

In this section, we show that this result strongly depends on the modelling of irreversibility, through a constraint. This constraint implies that there is an infinite cost to choose b lower than a, and a zero cost otherwise. Suppose instead that one considers instead "smooth" adjustment costs $c(a,b)$, where c is strictly convex in b. We write

$$U(x,a,b) = u(a,x) + v(b,x) - c(a,b) \tag{2}$$

and we can apply Propositions 2 and 2. In particular, equation (1) becomes

$$v_{bb}d - Mv_b = dc_{bb} + c_{ab} - Mc_b$$

Because the right-hand-side of this equation is independent from x, so must be the left-hand-side. It is easily checked that the associated restriction on v is that one can write

$$v_b(b,x) = A(b)K(x) + L(b)$$

Hence an increase in the precision of information does not necessarily cause a systematic change in a; to be able conclude one has to impose strong restrictions on v. After this step, one can directly get the values of M and d. However, because the expressions are complex, and $M + d_b$ still has an ambiguous sign.[13]

The rest of this section revisits standard examples initially analyzed in literature on the irreversibility effect, using an irreversibility constraint. We underline that the systematic effect of learning in favor of flexibility may disappear once the irreversibility constraint is replaced by smooth adjustment costs.

[13]In particular, if $A(b) > 0$, then $M + d_b$ is positive if and only if $c_{ab}/(\frac{\partial}{\partial b}\frac{L-c_b}{A})$ is increasing with b. Concavity of U only implies that the denominator is negative, so that this property is in no way guaranteed.

4.1 The Arrow-Fisher example

Arrow and Fisher (1974) considers a model similar to the following one. A forest of unit size whose ecological value x is uncertain. The risk-neutral decision-maker can cut a share a in the first period, and a share b in the second period. He or she can sell timber at a price p_0 and p_1 in each period. The irreversibility constraint is

$$0 \leq b \leq 1 - a$$

which states that one can neither replant, nor cut more than unit size of the forest.

We consider a variant version of the Arrow-Fisher example with smooth adjustment cost. It is based on the following preferences

$$U(x, a, b) = x(1 - a - b) + p_0 a + p_1 b - c(a, b)$$

Replacing into (1), we get

$$M(a, b) = 0 \qquad\qquad d(a, b) = -\frac{c_{ab}}{c_{bb}}(a, b)$$

so that the effect of learning on a depends on whether d increases with b. Hence, even in this simple setting the effects of learning depend on third derivatives of the adjustment costs. One can for instance specify these costs to set

$$c(a, b) = C_1(b) + C_2(1 - a - b)$$

which may be interpreted as a "smooth" version of the constraint $0 \leq b \leq 1 - a$. We then get

$$d(a, b) = \frac{-1}{C_1''(b)/C_2''(1 - a - b) + 1}$$

which is decreasing with b if both $C_1^{'''}$ and $C_2^{'''}$ are negative.

4.2 The Jones-Ostroy model for holding money

Jones and Ostroy (1984) give an example which offers a rationale for holding money. The idea is that an investor first chooses a portfolio a, then observes some information on future portfolio returns x, and reallocates the initial portfolio a into portfolio b. What is not invested in the portfolio is interpreted as holding liquid money.

Our framework may be useful to revisit the Jones and Ostroy (1984)'s example. Consider

$$U(x, a, b) = u(a, x) + b \cdot x - c(b - a)$$

where function c is associated to the transaction cost from switching from a to b. By setting $b' = b - a$, we can rewrite

$$\tilde{U}(x, a, b') = u(a, x) + a \cdot x + b' \cdot x - c(b')$$

which is additive separable. *Hence learning has absolutely no impact, even though arbitrary transaction costs are introduced.*

In fact, the rationale for holding money originates in another feature: in the Jones and Ostroy (1984)'s model, financial titles other than money can only be owned in bounded quantities : $b_i \in [0, 1]$ for all $i > 1$ (that is, other than money). The lower bound corresponds to the absence of short sales; the upper bound is justified by some indivisibilities. This creates constraints on b':

$$-a_i \leq b'_i \leq 1 - a_i$$

so that holding money naturally appears as more profitable. These constraints would not play any role in the absence of learning, since then the investor would choose $b = a$, or equivalently $b' = 0$.[14]

[14] One may alternatively consider that these bounds are an extreme form of concavity of returns: so that we could write $U(x, a, b) = u(a, x) + v(b, x) - c(b - a)$. But we are then back to the ambiguity underlined when studying (2): to conclude one has to assume that v is linear in some transform of x.

4.3 Adjusting the capital stock over time

We consider a three-period version of the model in Demers (1991). A firm first invests a, then additionally invests b. We write

$$U(x,a,b) = R(a+b,x) - wb$$

where R is the final revenue (a first period revenue $R_1(a,x)$ could be included without change), and w is the investment cost. Clearly we can define the final capital stock as $b' = a+b$, and get

$$\tilde{U}(x,a,b') = R(b',x) - wb' + wa$$

The problem is additive separable, so that *learning has absolutely no impact on the choice on* a.[15] This suggests that the irreversibility constraint of the form

$$b' \geq a$$

plays a major role in the Demers (1991)'s study. Replacing this constraint by a smooth adjustment cost function of the form $c(b-a)$ would lead to the same type of ambiguity that we discussed before.

5 Impact of Learning under Risk-neutrality

This section focusses on cases in which U is linear in x, and thus avoids difficulties associated with risk-aversion:

$$U(x,a,b) = u(a,b) + v(a,b) \cdot x$$

This model can be interpreted as a choice of risk-exposure v by a risk-neutral decision-maker. Consistent with what we said above, this model admits an invariant. Indeed (1) reduces to the system of equations

$$u_{ab} + u_{bb}d = Mu_b$$

[15]This result holds true regardless of the dimensionality of a and b, or whether w is in fact part of the unknown x. Allowing for capital depreciation between period 1 and 2 would not change the result either.

$$v_{ab} + v_{bb}d = Mv_b$$

which permits to find d and M, and so to be conclusive about the effect of learning. This is illustrated by the following examples.

5.1 Production choices

We consider a model similar to that of Epstein (1980, section 7). A firm chooses initially a capital stock a. Then the firm observes the output price x, and chooses another input b (labour), so that its production is $F(a, b)$, where F is a concave production function. The idea is that labour can be adjusted in the short-term, while capital is given. Profits are

$$U(x, a, b) = xF(a, b) - wb$$

where w is the input price.

We would like to know whether the fact that x will be learnt in the future modifies the choice of a. Solving (1) yields the system of equations

$$F_{ab} + F_{bb}d = MF_b$$

$$0 = Mw$$

so that we get

$$M = 0 \qquad\qquad d = -F_{bb}^{-1}F_{ab}$$

According to Proposition 2 the impact of learning on a depends on whether d is increasing or decreasing with b. Such a property may or may not be verified, depending on the third derivative in the production function, as first acknowledged by Epstein (1980).

Nevertheless, we fully characterize here this critical property, and thus we can directly conclude for some standard production functions. For instance, with a Cobb-Douglas production function $F(a, b) = Aa^r b^s$, we have $d = br(a(1-s))^{-1}$ which is increasing in b under r and s in $[0, 1]$. With a constant elasticity of substitution production function $F(a, b) = A(ra^s + (1-r)b^s)^{1/s}$, we have $d = b(a)^{-1}$ which is also increasing in b. Thus learning increases capital stock for these two standard production functions.

5.2 Self-protection

We consider a model in which capital b can be protected by an investment a which costs $c(a)$. We assume that capital b can be adjusted after having obtained some information on the probability of accident x. We write

$$U(x, a, b) = v(b) - c(a) - b(1 - a)x$$

in which $v(b)$ could be interpreted as the profitability that is derived from capital b in the economy. This model is similar to that of Kousky, Luttmer and Zeckhauser (2007), although that they do not study the effect of learning on self-protection in their paper.

Thus (1) writes

$$x + v''(b)d = M(v'(b) - (1 - a)x)$$

whose solution is

$$M = -\frac{1}{1-a} \qquad\qquad d = \frac{1}{1-a}T(b)$$

where $T = -v'/v''$ is a measure of the non-linearity of the profitability function. Hence the effect of learning on the level self-protection a depends on the sign of $T'(b) - 1$. Notice that this quantity is exactly zero for $v(b) = \log b$.

5.3 Global warming and emissions

Ulph and Ulph (1997) analyze the effect of learning on the optimal climate policy.[16] They use a microeconomic stock pollutant model:

$$U(x, a, b) = F(a) + G(b) - xD(a + b)$$

[16] See also Kolstad (1996).

where the concave functions F and G denote respectively the first- and second-period utility derived from emissions of pollution a and b, and where the convex damage function $D(a+b)$ is affected by the multiplicative shock x, and depends on the stock of pollution $a+b$.[17] Ulph and Ulph (1997) indicate that with quadratic functions for F, G and D, learning increases the first-period emissions a.

According to Proposition 2, we get that the effect in fact depends on the sign of

$$[2\frac{-G''}{G'} - \frac{G'''}{-G''}] + [2\frac{D''}{D'} - \frac{D'''}{D''}]$$

In each bracket the first term is positive, so that we can confirm the Ulph and Ulph (1997)'s result for quadratic functions. But the two other terms may modify this result. For example this expression is zero (resp. negative) with well-chosen logarithmic (resp. square root) G and D functions.

6 Impact of Learning under Risk-aversion

We now examine some situations involving a risk-averse decision-maker. These problems are not linear in x anymore, and can only be solved for some specific von Neumann-Morgenstern utility functions. In the global warming example, the problem can be solved only if risk tolerance is linear. In the cake-eating problem prudence must be constant. This observation may explain, in part, why economists have not studied the relationship between the option value and risk-aversion.

6.1 Global warming and emissions (continued)

In Gollier, Jullien and Treich (2000), a and b are first- and second-period emissions, and x is the unknown unit damage. The payoff function may be written

$$U(x, a, b) = u(a) + v(b - x(a+b))$$

[17] Introducing a decay factor would not change the results.

Replacing into (1) yields

$$-x(1-x)v''^2 dv'' = M(1-x)v'$$

which does not depend on u. Define the tolerance to risk $T = -v'/v''$, and divide by $(1-x)v''$ to get

$$-x + (1-x)d = -MT$$

The left-hand-side of this equation is linear in x, and so must be the right-hand-side. This implies that T must be linear, a particular restriction on the tolerance to risk of v. Such functions are said to display an hyperbolic absolute risk aversion; classical examples are the quadratic, exponential, logarithmic, and power utility functions. This result corresponds to Proposition 2 in Gollier, Jullien and Treich (2000).

So let us set

$$T(R) = \alpha + \gamma R$$

Replacing in the above equality at $R = b - (a+b)x$, one gets a system of two equations

$$d + M(\alpha + \gamma b) = 0$$

$$1 + d + \gamma M(a+b) = 0$$

whose solution is

$$M = \frac{1}{\alpha - \gamma a} \qquad d = -\frac{\alpha + \gamma b}{\alpha - \gamma a}$$

Finally

$$M + d_b = \frac{1-\gamma}{\alpha - \gamma a}$$

Since $T(a) = \alpha - \gamma a$ is a risk-tolerance and is thus positive, we get that an increase in the precision of information increases optimal emissions today if $\gamma < 1$. This result corresponds to Proposition 1 in Gollier, Jullien and Treich (2000). This result can be shown to extend to the case of multi-dimensional b.

6.2 Eating a cake with unknown size

Eeckhoudt, Gollier and Treich (2005) consider a standard cake-eating model in which the cake has an unknown size. They study the effect of learning about the size of the cake on the initial consumption. The model writes

$$U(x, a, b) = u(a) + v(b) + w(x - a - b)$$

Therefore, (1) becomes

$$w'' + (v'' + w'')d = M(w' - v')$$

so that the only part that depends on x is $dw'' - Mw'$, and setting $M = d = 0$ yields a contradiction. Therefore one must have

$$P^w = -w^{'''}/w'' = \frac{M}{d}$$

in which P^w is the absolute prudence associated to the function w. We now know that it is a constant, and cannot depend on the last period consumption. This gives us an expression for w, that we can use to solve completely the problem. The formulae that we obtain are omitted here. We mention that when w has a constant absolute risk aversion, then the impact of more information depends on the sign of

$$P^w[2A^v(b) - P^v(b) + P^w]$$

where $A^v = -v''/v'$ is the absolute risk-aversion of v, and P^v is its absolute prudence. When w is quadratic, the effect depends only on whether v displays prudence or not.

These results suggest that the effect of learning is usually ambiguous in the cake-eating problem, unless the utility functions are quadratic or exponential. Eeckhoudt, Gollier and Treich (2005), in contrast, obtain some general results because they compare two extreme cases, that is, no learning and perfect learning.

7 Conclusion

This paper has studied the classical question of the effect of learning in a three-period model. It has provided a theorem that characterizes the class of models for which this question can be answered without ambiguity, and another that permits to check the sign of this impact.

Several of our results indicate that ambiguity is the rule. This is the case for generic choices of the payoff function. This is also the case in simple additive problems, when an irreversibility constraint is replaced by smooth adjustment costs. Hence the well-known irreversibility effect is not robust to such changes, apart in the most simple case. Even in problems that are non-ambiguous, the sign of the impact depends on precise properties of the payoff function that are not easy to sign. A consequence is that in simulation exercises one should be cautious when choosing a particular parametric specification for these functions. Another consequence is that risk-aversion is shown to be instrumental for the analysis of the effect of learning.

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Appendix

Proof of Proposition 2 : notice first that $b(a, p)$ is characterized by

$$\sum_x p(x) U_b(x, a, b(a, p)) = 0$$

so that we can write for any a, b, p

$$\sum_x p(x) U_b(x, a, b) = 0 \Rightarrow \sum_x p(x)[U_{ab}(x, a, b) + U_{bb}(x, a, b) \frac{\partial b}{\partial a}(a, p)] = 0 \quad (3)$$

If the problem admits an invariant f, then for any a and p we have

$$B(p) = f(a, b(a, p))$$

Suppose that for some a, p, q we have $b(a, p) = b(a, q)$. Then $B(p) = B(q)$, so that for any a' one has $f(a', b(a', p)) = f(a', b(a', q))$. Because f is one-to-one, we obtain that $b(a', p) = b(a', q)$ for any a'.

Hence we have shown

$$b(a, p) = b(a, q) \Rightarrow \frac{\partial b}{\partial a}(a, p) = \frac{\partial b}{\partial a}(a, q) \quad (4)$$

Therefore $\frac{\partial b}{\partial a}(a, p)$ can depend on b only through $b(a, p)$. There thus exists d such that

$$\forall\ a, p \qquad d(a, b(a, p)) = \frac{\partial b}{\partial a}(a, p)$$

Replacing in (3) we get, for any a, b, p

$$\sum_x p(x) U_b(x, a, b) = 0 \Rightarrow \sum_x p(x)[U_{ab}(x, a, b) + U_{bb}(x, a, b) d(a, b)] = 0 \quad (5)$$

This implies that there exists a matrix $M(a, b)$ such that (1) holds.[18]

[18] There is a slight complication here, as p is not any vector of $\mathbb{R}^X$: all its components must be strictly positive, and they must sum to one. Since we have restricted attention to pairs $(a,\ b)$ such that $b = b(a, p)$ for some p, these restrictions turn out to not matter at all. Proof omitted for brievity.

Conversely, if (1) holds, then summing over x at $b = b(a,p)$ yields

$$\sum_x p(x)(U_{ab} + U_{bb}d) = \sum_x p(x)MU_b = M\sum_x p(x)U_b = 0.$$

We thus have shown (5). Because by strict concavity of U the hessian matrix

$$H(a,p) \equiv \sum_x p(x)U_{bb}(x,a,b(a,p))) \tag{6}$$

is negative definite, comparing (3) and (5) we get the second statement in the Proposition.■

Proof of Proposition 2 : For given beliefs p and q, and $r \in [0,1]$, define

$$J(a,r) = j(a,(1-r)p + rq)$$

Notice that Epstein's condition is equivalent to J_a being convex in r. Suppose this is the case. Compute

$$J_{rr}(a,r) = [\sum(q(x) - p(x))U_b(x,a,b(a,(1-r)p + rq))].\frac{\partial}{\partial r}b(a,(1-r)p + rq)$$

From the first-order condition

$$\sum_x [(1-r)p(x) + rq(x)]U_b(x,a,b(a,(1-r)p + rq)) = 0 \tag{7}$$

we get the following two expressions

$$\frac{\partial}{\partial r}b(a,(1-r)p+rq) = -H^{-1}(a,(1-r)p+rq)\sum(q(x)-p(x))U_b(x,a,b(a,(1-r)p+rq)). \tag{8}$$

$$\sum(q(x)-p(x))U_b(x,a,b(a,(1-r)p+rq)) = -\frac{1}{r}\sum p(x)U_b(x,a,b(a,(1-r)p+rq))$$

where the hessian matrix H is defined as in (6). Replacing yields

$$\begin{aligned} J_{rr}(a,r) &= -\frac{1}{r^2}[\sum p(x)U_b(x,a,b(a,(1-r)p + rq))]'[H(a,(1-r)p + rq)]^{-1} \\ &\quad [\sum p(x)U_b(x,a,b(a,(1-r)p + rq))]. \end{aligned}$$

where the prime stands for transposition. This expression must be increasing with a, for any p, q, r. This implies that

$$Z(a,p,q) \equiv [\sum p(x)U_b(x,a,b(a,q))]'[H(a,q)]^{-1}[\sum p(x)U_b(x,a,b(a,q))]$$

is decreasing with a, for any p and q.

Suppose that $b(a,p) = b(a,q)$ at some (a, p, q). Then not only $Z(a,p,q) = 0$, but also $Z_a(a,p,q) = 0$ because all terms in the derivative vanish. Since Z is decreasing in a and $Z_a = 0$, Z_a attains a maximum at a and thus $Z_{aa} = 0$. Computing this second derivative, all terms but one vanish, so that

$$[\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q))]'[H(a,q)]^{-1}[\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q))] = 0$$

Since the hessian matrix is negative definite, this implies that

$$\frac{\partial}{\partial a}\sum p(x)U_b(x,a,b(a,q)) = 0$$

We have thus shown (4), which from the preceeding proof is sufficient for proving (1). This concludes the proof of necessity.

Conversely, suppose that U verifies (1), and compute

$$J_{ar}(a,r) = \sum[q(x) - p(x)][U_a(x,a,b(a,(1-r)p + rq)) \\ +U_b(x,a,b(a,(1-r)p+rq))d(a,b(a,(1-r)p+rq))]$$

which depends on r only through $b(a,(1-r)p+rq)$. Differentiating we get

$$J_{arr}(a,r) = \sum(q(x) - p(x))[U_{ab} + U_{bb}d + d_bU_b].\frac{\partial}{\partial r}b(a,(1-r)p+rq)$$

and using (1) we obtain

$$J_{arr}(a,r) = [\sum(q(x) - p(x))U_b]'[M + d_b]'\frac{\partial}{\partial r}b(a,(1-r)p+rq).$$

where the prime stands for transposition. Using (8) we get

$$J_{arr}(a,r) = -[\sum(q(x) - p(x))U_b]'[M + d_b]'H^{-1}[\sum(q(x) - p(x))U_b].$$

This shows the result, from the fact that the hessian matrix H is definite negative.■