Propositional Opinion Diffusion

Umberto Grandi
IRIT, University of Toulouse
France
umberto.grandi@irit.fr

Emiliano Lorini
IRIT, University of Toulouse
France
emiliano.lorini@irit.fr

Laurent Perrussel
IRIT, University of Toulouse
France
laurent.perrussel@irit.fr

ABSTRACT

We present a formal model of opinion diffusion and formation which combines notions from social network analysis together with concepts and techniques from judgment aggregation and merging. The model allows us to study the propagation of individual opinions, represented in the form of yes/no answers to a set of multiple binary issues, in a multiagent system linked by an influence network. The process is iterative with discrete time. We are interested in characterizing properties of the network structure which guarantee convergence of the iterative process for every initial configuration of the agents' opinions, and in developing tractable algorithms for computing the set of opinions at convergence.

Keywords

Formal models of agency, social choice theory, trust and reputation, social networks, reasoning in agent-based systems

1. INTRODUCTION

Studies of social influence in social sciences and in social psychology have emphasized the role of interpersonal processes in how people construe and form their perceptions, judgments, and impressions (e.g., [1, 11, 34]). According to this literature, social influence consists in forming an opinion on the basis of the opinions expressed by the other individuals in the society. Social influence is one of the basic mechanisms driving the diffusion of opinions in human societies: certain agents influence other agents in the society to acquire a certain view who, in turn, influence other agents in the society to acquire the same view, and so on.

The present paper intends to provide a formal model of opinion diffusion and formation in social networks which combines concepts from the theory of social networks [22] together with concepts and techniques from judgment aggregation [32] and merging [27]. With the term 'opinion' we refer to the public expression of

a view by an agent about some issue(s). In this sense, the term *opinion* does not refer to an agent's mental attitude but rather to the *expression* of a mental attitude of the agent.

Our model of opinion diffusion and formation can be summarized as follows. We start from a given population of agents. Each agent in the population is identified with a number of opinions on a given set of binary issues. The issues about which agents have to form their judgments may be dependent: an agent's opinion about a given issue may depend on the agent's opinion about other issues. Dependence between issues is expressed via an integrity constraint which is nothing but a formula of propositional logic in our context.

Consider for instance the following example, inspired from the literature on computational social choice [4]. Four agents, Bob, Ann, Jesse and Mary, need to take a decision on whether to build a swimming pool (S) or a tennis court (T) in the common area of the residence where they live. In other words, they have to form opinions about whether S and/or T have to be case. Issues S and T are dependent insofar as the common area of the residence is so small that a swimming pool and a tennis court cannot be built at the same time. Specifically, the following integrity constraint is assumed: $\neg(T \wedge S)$. Let us suppose that, at the beginning of the interaction process (time t_0), both Ann and Bob have a positive opinion about S and a negative opinion about T. That is, both Ann and Bob express their preference for the construction of a swimming pool rather than for a tennis court. On the contrary, both Mary and Jesse have a negative opinion about S and a positive opinion about T. That is, both Mary and Jesse express their preferences for a tennis court rather than a swimming pool.

Moreover, each agent i identifies a set of neighbours that contains all agents in the population who can influence i's opinions or, conversely, all agents in the population who are trusted by agent i. This induces an influence (viz. trust) network and an edge from agent j to agent i indicates that j can influence i's opinions (viz. i trusts agent j). For example, suppose that Ann trusts both Mary and Jesse while she does not trust Bob: only Mary and Jesse can influence her opinions. At the opposite Bob does not trust anybody apart from Ann.

At each stage in the interaction process, the agents in the population aggregate the current opinions of their neighbours to form their new opinions. Different aggregation criteria can be used such as unanimity (e.g., change your current opinion if all your neighbours have a different opinion) or majority (e.g., change your current opinion if the majority of your neighbours have a different opinion). For example, if the aggregation criterion used by the agents in the population is either unanimity or majority then, at time t_1, Ann will start to have a negative opinion about S as she is influenced by Jesse and Mary's opinions at time t_0 about this issue. At time t_2 Ann will maintain her opinion, while Bob will change his

opinion and start to have a negative opinion about S by taking into account Ann's negative opinion about S at time t_1.

Through a formal formal model of opinion diffusion and formation, our aim is to: (i) study the properties of the network structure which guarantee convergence of the iterative process for every initial configuration of individual views, (ii) exhibit the set of opinions at convergence, and (iii) provide algorithms for computing this result. Examples of properties of the network structure are acyclicity of the underlying graph or whether the cycles in the underlying graph are disjoint or not.

The paper is organized as follows. Section 2 discusses related work. Section 3 presents the basic definitions of our model of opinion diffusion in networks. Section 4 presents general results about convergence of opinions in specific classes of networks. These results are independent from the aggregation procedure used by the agents to form new opinions on the basis of those of their neighbours in the network. Sections 5 and 6 focus on on two variants of our model in which agents are assumed to use specific aggregation procedures, namely the unanimity rule and the majority rule. Section 7 concludes by discussing perspectives for future research.

2. RELATED WORK

Our work stems from two research agendas: the development of formal models of influencing power, mainly developed in the game-theoretic literature, and the study of opinion diffusion and formation based on influence mechanisms, mainly developed in the literature on social sciences and social network analysis.

Influencing power.

Formal models of influencing power developed in social sciences are focused on measuring how much an agent in a social network can influence the opinions of the other agents in the network. A formal model of influencing power was introduced in the social sciences more than fifty years ago by Isbell [21] and, more recently, starting from the work of Hoede and de Bakker [19], a series of papers developed a first formal model of influencing power in situations of *binary decisions* [14, 15]. In this model, a set of individuals make a yes/no decision on a single issue, and an influence function determines the dynamics from a profile of individual opinions $x \in \{0, 1\}^n$ to the next state. The key concept is the *followers function*, which specifies for each coalition of individuals the set of individuals that follow their (unanimous) decision.

Opinion diffusion.

There have been several attempts in the literature to build models of opinion diffusion in social networks that are based on influence mechanisms. The aim of these models is to study the evolution of the opinion distribution and, in particular, the conditions under which the opinions of the agents in a society tend to converge to consensus in the long run. Two families of models have been studied, namely discrete models and continuous models.

Discrete models assume that agents can only have binary (two-state) opinions such as yes-no, for-against. Some of these models are concerned with the problem of studying the conditions under which the opinion of a minority of agents tends to maintain itself (e.g., [30, 23]). Granovetter [17] and Schelling [36] were among the first to propose discrete models of opinion diffusion in the area of social sciences. These are called thresholds models, as they assume that each agent in the population is identified with a given numerical value which characterizes the number or proportion of neighbours who must have a certain opinion before the agent adopts it. A generalisation of thresholds models, called the linear thresh-

old model, has been extensively used in recent years in studying diffusion models (see, e.g., [25, 24]). In the linear threshold model each edge in the social network has a non-negative weight attached which characterizes the strength of the influence relation. Another example of model of opinion diffusion in networks is the so called voter model. In this model the propagation of opinions follows a random process in which at each step an individual on the network chooses the opinion of a random neighbour [5, 20]. The label propagation algorithm used in the related setting of community detection and formation, is a similar model where every individual follows the most approved opinion in her neighbourhood [35]. This latter setting coincides with the framework studied in this paper with the *plurality rule with random tie-breaking* as individual aggregation function.

Continuous models assume that agents may have continuous opinions about a given issue, indeed a numerical value. For example, in DeGroot's model [6] (also known as the Lehrer-Wagner model [31]) agents update their opinions depending on the weight they attach to the others' opinions, where this weight can be viewed as a rough approximation of a notion of trust. In particular, in De-Groot's model it is assumed that every agent i attaches to the opinion of another agent j a given weight $k_{i,j}$ such that the higher the value of $k_{i,j}$, the higher the influence of j's opinion on i's opinion. In Chatterjee and Seneta's model [3] it is assumed that the weight an agent attaches to her own opinions can vary over time, while in DeGroot's model this weight as well as the weights an agent attaches to the others' opinions are kept constant over time. In the bounded confidence (BC) model [28, 18, 8], the agents have continuous opinions and the agents actually influence each other only if the distance between their opinions is below a threshold. The BC model can be seen as a non-linear version of previous continuous opinion models [12, 13]. A variant of the BC model is the Relative Agreement Model (RA) [7] in which, during interactions, the agents are assumed to influence both each other's uncertainties and each other's opinions. Related models (e.g., [2]) consider vectors of binary traits rather than continuous opinions as a better approximation of the diffusion of culture in a social network.

The present paper.

The present paper combines the preceding two lines of research on influencing power and opinion diffusion in social networks. As in existing models of influencing power, our work aims at understanding the basic influence mechanisms which are responsible for the dynamics of opinions in a social network. As in existing models of opinion diffusion, we are interested in the convergence of the iterated opinion propagation process. However, we do not focus on the characterization of situations converging at consensus. We also consider situations in which the opinions of the agents in the network converge without being necessarily aligned.

Hereafter, we commit to the discrete model perspective on opinions, as we want to consider their qualitative dimension. However, out setting generalizes existing models of opinion diffusion under two important aspects. First, we do not limit opinions to two values but rather to a list of binary values; opinion are expressed on a set of possibly correlated issues. We represent the interplay between issues by means of a constraint expressed in propositional logic grounding our perspective on opinion diffusion on concepts and techniques issued from judgment aggregation and merging. Second, the proposed model does not commit to any specific aggregation criterion by means of which an agent aggregates the opinions of her neighbours in the network. This is the main innovation and key difference with the existing models of opinion diffusion discussed above.

3. BASIC DEFINITIONS

In this section we present our model for the propagation of individuals' opinions on an influence network. We represent individual opinions as yes/no answers to a set of possibly correlated questions, a setting that has been shown to be general enough to model a variety of individual expressions such as preferences, judgments and approval sets [16]. We model the influence network as a directed graph, and we define an iterative propagation process with discrete time, with individuals updating their opinions from those of their neighbours by making use of an aggregation procedure.

3.1 Individual opinions and influence network

Let $\mathcal{I} = \{p_1, \ldots, p_m\}$ be a finite set of questions or issues. Each individual in a finite set $\mathcal{N} = \{1, \ldots, n\}$ expresses an *opinion* $B_i \in \{0, 1\}^{\mathcal{I}}$ in the form of a yes/no evaluation over each issue in $\mathcal{I}$. We often call B_i a ballot, borrowing the terminology from voting theory. Observe that in defining opinions we do not tolerate uncertainty. Hence B_i should not be considered as agent i's belief but rather as the *public* expression of i's view about the issues in $\mathcal{I}$. Examples of opinions are expressions of value judgments (i.e., whether a state of affairs should be considered good or bad), expressions of preferences (i.e., whether object a is preferred to object b) or desires (i.e., whether a state of affairs is desirable or not) and, finally, expressions of choices (i.e., whether an action should be performed or not).

Let $\mathcal{L}_{PS}$ be the language of propositional logic over atoms $\{p_1, \ldots, p_m\}$. An *integrity constraint* IC is any formula in $\mathcal{L}_{PS}$, and it is used to define a set of feasible individual opinions consisting of its satisfying assignments $\mathcal{X} = \mathrm{Mod}(\mathrm{IC}) \subseteq \{0, 1\}^{\mathcal{I}}$. For instance, if we are interested in modelling opinions as expressions of individual preferences over a set of items $A = \{a, b, c \ldots\}$, then we may use a set of issues $\mathcal{I}_A = \{p_{ab} \mid a, b \in A\}$, and an integrity constraint that enforces the required properties of a preference relation. For instance, reflexivity can be represented by the formula $\bigwedge_{a \in A} p_{aa}$, forcing each individual to accept issue p_{aa}; transitivity corresponds to the conjunction of formulas $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for all distinct $a, b, c \in A$. We assume that each individual opinion is feasible, i.e., $B_i \in \mathcal{X}$ for all $i \in \mathcal{N}$. Thus, we consider IC as a common rationality assumption shared by all individuals.

A generalisation of the present model would allow for different agents in the population to have different integrity constraints. That is, according to this generalisation, every agent $i \in \mathcal{N}$ has her own integrity constraint $\mathrm{IC}_i \in \mathcal{L}_{PS}$ and agent i's set of feasible opinions is defined as $\mathcal{X}_i = \mathrm{Mod}(\mathrm{IC}_i) \subseteq \{0, 1\}^{\mathcal{I}}$. Feasibility of ballots is obtained by imposing that $B_i \in \mathcal{X}_i$ for all $i \in \mathcal{N}$. To keep the model simpler, here we prefer to assume that the integrity constraint is the same for all agents. It is worth noting that most results presented in the paper easily extend to this general case.

Individuals are connected by an *influence network* represented in the form of a directed graph $E \subseteq \mathcal{N} \times \mathcal{N}$, with the interpretation that $(i, j) \in E$ if and only if agent j is influenced by agent i. We also refer to E as the influence graph and to individuals in $\mathcal{N}$ as the nodes of the graph. Note that the influence network is directed, hence $(j, i) \in E$ represents the fact that j influences i.

3.2 The iterative process

Define the influence set $Inf(i) = \{j \in N \mid (j, i) \in E\}$ of agent i to be the set of influencers of i. We model opinion diffusion as a discrete time process: at time $t \in \mathbb{N}$, $t \neq 0$ each individual updates her opinion by aggregating the opinion of all agents in her influence set at time $t - 1$. This process is guided by an *aggregation procedure* F_i for each individual i, that computes how an

agent changes her opinions starting from those of her influencers. Given an integrity constraint IC that defines a set of feasible opinions $\mathcal{X}$, a (collectively rational) aggregation procedure is a class of functions $F : \mathcal{X}^n \rightarrow \mathcal{X}$ for each $n \in \mathbb{N}$. Notable examples are the majority rule, that accepts an issue if and only if there is a majority of individuals accepting it (see Section 6 for a formal definition), or distance-based procedures that merge the opinion of the influencers using a suitable minimisation process.

Let therefore $B_i^t \in \mathcal{X}$ be the opinion of agent i at time t, and $\boldsymbol{B}^t = (B_1^t, \ldots, B_n^t)$ the associated profile. The iterative process is defined as follows:

$$B_i^t = \begin{cases} B_i^{t-1} & \text{if } Inf(i) = \emptyset \\ F_i(\boldsymbol{B}_{Inf(i)}^{t-1}) & \text{otherwise} \end{cases}$$

where $\boldsymbol{B}_{Inf(i)}^{t-1}$ is profile $\boldsymbol{B}^{t-1}$ restricted to the set $Inf(i)$ of influencers of agent i. We call this process *propositional opinion diffusion (POD)*. If $F_i = F$ for all $i \in \mathcal{N}$, i.e., all individuals use the same aggregation procedure, we call the process *uniform-POD*.

EXAMPLE 1. *Let us go back to a variant of the example discussed in the introduction. Ann, Bob and Jesse need to take a decision on two mutually exclusive actions S and T, corresponding to building a swimming pool or a tennis court in the common area of the residence where they live. The set of possible opinions is thus represented as the set of models of the propositional formula* $\mathrm{IC} = \neg(S \wedge T)$, *i.e.,* $\mathcal{X} = Mod(\mathrm{IC}) = \{01, 10, 00\}$, *representing the three possibilities of performing only the first action, only the second action, or neither actions. Let us consider a more complex influence network connecting the three individuals in question. Bob is influenced by both Ann and Jesse, and also takes into consideration his current opinion (he is a compromising agent). Jesse is influenced by Ann, and does not take into consideration her current opinion (she is a conformist agent). Finally, Ann is influenced by herself only (she is stubborn). The influence network is depicted in Figure 1.*
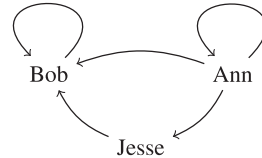
Figure 1: A simple influence network.

Let us start from a profile of initial opinions $\boldsymbol{B}^0 = (01, 00, 10)$, corresponding respectively to Ann, Bob, and Jesse's opinion. Assume moreover that each agent update their opinions using the majority rule, i.e., they change their opinion if and only if all influencers agree. At the first step, Bob does not change his mind since Ann and Jesse disagree, Ann is only influenced by herself, hence her opinion remain stable, and Jesse moves to 01 since she is influenced by Ann. Hence, $\boldsymbol{B}^1 = (01, 00, 01)$. At the second step Ann and Jesse do not update their opinions, but Bob changes to 01 since a majority of its influencers agree on 01 – all influencers in this particular case. Thus, the opinion diffusion process converges to a consensus state $\boldsymbol{B}^* = (01, 01, 01)$, corresponding to building a tennis court and not building a swimming pool.

In this paper we are interested in characterising classes of networks on which convergence of *POD* is guaranteed. To do so, we first formally define the notion of convergence:

DEFINITION 1. *Given a class of graphs $\mathcal{E} \subseteq 2^{E^2}$, we say that POD converges on $\mathcal{E}$ if for all graphs $E \in \mathcal{E}$ and for all profiles of initial opinions $\boldsymbol{B}^0 \in X^{\mathcal{N}}$ there is a convergence time $\bar{t} \in \mathbb{N}$ such that $\boldsymbol{B}^t = \boldsymbol{B}^{\bar{t}}$ for all $t \geq \bar{t}$.*

Our notion of convergence is centered on the notion of graph property, e.g., acyclic or tree-shaped graphs, rather than on the set of initial individual opinions, on which we do not assume to have any prior knowledge.

3.3 Aggregation procedures

There is a vast number of aggregation procedures that have been introduced in the literature on social choice theory and on judgment aggregation in particular. Classic examples are the unanimous rule seen in Example 1, the majority rule which will be studied in Section 6, and distance-based rules, which select the closest opinion to that of the influencers making use of a suitable notion of distance. In this paper the role of an aggregation procedure is that of modeling the opinion update process brought about by the influence structure. Therefore, we discard situations of *negative influence*, in which individuals change their opinion to the opposite opinion of their influencers. Similar situations can be avoided by assuming some axiomatic properties on the aggregation procedure. A first property is the following:

Ballot-Monotonicity: for all profiles $\boldsymbol{B} = (B_1, \ldots, B_n)$, if $F(\boldsymbol{B}) = B^*$ then for any $1 \leq i \leq n$ we have that $F(\boldsymbol{B}_{-i}, B^*) = B^*$.

Where $\boldsymbol{B}_{-i}$ consists of profile $\boldsymbol{B}$ without ballot B_i. Ballot-monotonicity implies that the result of aggregation in a given profile should not change if one of the agents gives additional support to the winning opinion. This is a very weak form of monotonicity and is satisfied by most known aggregation procedures. Observe that, when there is only one single agent, any ballot-monotonic aggregation rule boils down to a simple copying of the influencer's opinion, i.e., $F(B) = B$ for any $B \in \mathcal{X}$ when $n = 1$.

Another property that we shall consider is that of unanimity:

Unanimity: for all profiles $\boldsymbol{B} = (B_1, \ldots, B_n)$, if $B_i = B$ for all $1 \leq i \leq n$ then $F(\boldsymbol{B}) = B$.

A unanimous aggregation procedure copies the opinion of the collectivity if there is no disagreement.

In both definitions the quantification over profiles was left deliberately open, and has to be intended to be restricted to the domain of the aggregation procedure F. This ambiguity should not create problems in the technical results that follow, where aggregation procedures are defined on all admissible ballots and for as many input ballots as there are influencers in the network.

Aggregation procedures may or may not satisfy a given set of axiomatic properties. For example, both the unanimity rule used in Example 1 and the majority rule sketched in the introduction satisfy both ballot-monotonicity and unanimity. Research in social choice theory has often focused on characterising unfeasible sets of axiomatic properties [32]. In the sequel we will explicitly mention when any such property is being assumed on the aggregation procedures defining the opinion diffusion process.

4. THE GENERAL MODEL

In this section we prove convergence of *POD* on different classes of graphs and we provide an algorithm for computing the opinions at convergence for the class of directed acyclic networks with loops. We will not focus on particular aggregation procedures to be used by individuals, but we will rather constrain their choice with the use of axiomatic properties.

4.1 Convergence results

Let us first introduce some useful notation. A *directed acyclic graph (DAG)* is a directed graph that does not contain cycles. A *DAG with loops* is a DAG where we allow only for cycles of size one, i.e., edges of the type (i, i). Loops are very important in an opinion diffusion process, since they indicate whether an agent takes her current opinion into consideration when updating. Let a *source* of a graph E be a node such that $Inf(i) \subseteq \{i\}$, and let $diam(E)$ be the diameter of a graph E, i.e., the length of the longest path between a source and any of the nodes.

THEOREM 2. *If F_i satisfies ballot-monotonicity for all $i \in \mathcal{N}$, then POD converges on the class of DAG with loops after at most $diam(E) + 1$ number of steps.*

PROOF. Let $i \in \mathcal{N}$ be a node and $d(i)$ be the maximal distance from i to a source node, i.e., the length of the longest simple path from i to any of the sources. As the influence graph E is acyclic, $d(i)$ is finite and is bounded by the diameter of E. We now prove by induction on $d(i)$ that *POD* converges for a node i after exactly $d(i)$ steps.

Let $d(i) = 0$, i.e., i is a source. If $Inf(i) = \emptyset$, then the *POD* procedure simply copies i's opinion, hence $B_i^t = B_i^0$ for all $t \geq 1$. If $Inf(i) = \{i\}$, then, as observed in Section 3, by a consequence of ballot-monotonicity we obtain that $B_i^1 = F_i(B_i^0) = B_i^0$. The same holds for all $t \geq 1$.

Assume now that all nodes j such that $d(j) = k$ have converged to a stable opinion at step $k + 1$. Let i be an individual such that $d(i) = k + 1$. Observe first that, since the graph is acyclic, $Inf(i) \setminus \{i\}$ contains only nodes that are at distance at most k from a source, hence by inductive hypothesis we can assume they have reached a stable opinion B_j^*. If $i \notin Inf(i)$ then $B_i^t = F_i(\boldsymbol{B}_{Inf(i)}^*)$ for all $t \geq k + 2$, since all opinions of i's influencers are stable from step $k + 1$ onwards, hence showing convergence of B_i^t from $t = k + 2$. If instead $i \in Inf(i)$, then $B_i^{k+2} = F_i(\boldsymbol{B}_{Inf(i) \setminus \{i\}}^*, B_i^{k+1})$, and $B_i^{k+3} = F_i(\boldsymbol{B}_{Inf(i) \setminus \{i\}}^*, B_i^{k+2}) = B_i^{k+2}$, the last equality obtained by ballot-monotonicity. Hence $B_i^t = B_i^{k+2}$ for all $t \geq k + 2$, obtaining the desired bound on the number of steps to reach convergence. $\square$

The assumption of ballot-monotonicity in Theorem 2, albeit very weak, is necessary. As a counterexample, consider an anti-majority rule on a single issue, which dictates to each individual to take the opposite opinion to that of the majority of her influencers. In this case, any source with $Inf(i) = \{i\}$ will constantly change opinion from 0 to 1, undermining the convergence on the whole network.

The assumption of acyclicity is also crucial to obtain convergence, as shown by the following theorem. Let a circle be a graph composed of a single cycle, and let us work under uniformity assumption, i.e., $F_i = F$ for all $i \in \mathcal{N}$.

THEOREM 3. *If F is not ballot-monotonic, then uniform-POD does not converge on the class of all graphs.*

PROOF SKETCH. We show that, if F is not ballot-monotonic, then we can construct an influence network that is a circle $C_m = \{(1, 2), (2, 3), \ldots, (m - 1, 1)\}$ of length m, and an initial vector of opinions $\boldsymbol{B}^0$ on which *POD* does not converge. Recall that the action of F on a circle describes how to update one's own opinion from the opinion of a single influencer, i.e., how to obtain B_j^t from B_{j-1}^{t-1}. The aggregator has therefore the form $F : \mathcal{X} \to \mathcal{X}$. Since $\mathcal{X}$ is finite and F is not ballot-monotonic, which in this case is equivalent to say that there exists a B such that $F(B) \neq B$, we can construct a cycle of opinions $B_1 \ldots B_k$ such that $F(B_i) = B_{i+1}$

for all $i \leq k-1$ and $F(B_k) = B_1$. Now take any cycle C_m of length $m \neq k$, and profile of initial opinions $\boldsymbol{B}^0 = (B_1, \ldots, B_1)$. *POD* does not converge as the cycle of individual opinions will continue rotating in the circle of individuals. $\square$

Another class of networks on which we can show convergence for *uniform-POD* is that of complete graphs, i.e., graphs $E = \mathcal{N} \times \mathcal{N}$ where every individual is connected to each other.

Theorem 4. *If F is unanimous, then uniform-POD converges on the class of complete graphs.*

Proof. In a complete graph we have that $Inf(i) = \mathcal{N}$ for all $i \in \mathcal{N}$. Hence, $B_i^1 = F(\boldsymbol{B}^0)$ for all $i \in \mathcal{N}$, making the opinion profile at step one unanimous, i.e., $\boldsymbol{B}^1 = (F(\boldsymbol{B}^0), \ldots, F(\boldsymbol{B}^0))$. Therefore, by assumption of unanimity of F, we obtain that $B_i^t = F(\boldsymbol{B}^1) = F(\boldsymbol{B}^0)$ for all $t > 1$. $\square$

Theorems 3 and 4 show an interesting dicotomy in the convergence of *uniform-POD*: on the one hand, if individuals form a single cycle we lose the convergence result of Theorem 2, while on the other hand convergence is guaranteed if individuals form all possible influence links composing a complete graph. In Section 6 we are able to obtain a convergence result between these two extremes, albeit by focusing on the majority rule as the common aggregation procedure.

4.2 Computing opinion diffusion in a DAG

The proof of Theorem 2 suggests a polynomial algorithm for computing the result of *POD* at convergence, provided that the opinion update process defined by the individual aggregation procedures can also be performed in polynomial time. Our algorithm, together with the observation that checking acyclicity of a graph can be done in polynomial time, shows that *POD* on directed acyclic graphs with loops is a tractable problem.

Input: A DAG with loops E over $\mathcal{N} = \{1, \ldots, n\}$, an initial opinion vector $\boldsymbol{B}^0 = (B_1^0, \ldots, B_n^0)$
Output: Final opinion vector $\boldsymbol{B}^* = (B_1^*, \ldots, B_n^*)$
for $i \in \mathcal{N}$ **do**
$\quad$| $curr\_opinion\_i = B_i^0$;
end
$stable = \{$sources of $E\}$;
while $stable \neq \mathcal{N}$ **do**
$\quad$| **for** $i \notin stable$ **do**
$\quad\quad$| | $curr\_opinion\_i = POD\_update\_i$;
$\quad\quad$| | $stable = stable \cup next(stable)$;
$\quad$| **end**
end
return $(curr\_opinion\_1, \cdots, curr\_opinion\_n)$;

Algorithm 1: *POD* computation on DAG with loops.

With the help of Algorithm 1, we are now in a position to show that the opinions at convergence of *POD* can be computed in polynomial time as long as the aggregation procedure is also polynomial. Recall that an aggregation procedure is polynomial to compute if the result of $F(\boldsymbol{B})$ can be computed in polynomial time on all profiles.

Corollary 5. *If E is acyclic and F is ballot-monotonic and polynomial to compute then the opinion profile $\boldsymbol{B}^*$ at convergence of POD can be computed in polynomial time.*

Proof. By the proof of Theorem 2 we know that *POD* converges in a number of steps equal to the maximal distance between a source and any node (i.e., the *diameter* of the graph) plus one additional step. Algorithm 1 can hence be used for computing the result of *POD*. The algorithm uses a subprocedure called "*POD*_update_i" for computing the updated opinion of an individual using aggregation procedure F_i, a subprocedure which by assumption can be performed in polynomial time. The subprocedure "next" simply computes the nodes at distance one from a given set of nodes (not considering loops), and guarantees termination of the while loop in a number of steps equal the diameter of the graph, plus an additional step for loops. $\square$

Algorithm 1 propagates individual opinions from the sources of the network to the furthest nodes, making use of subprocedure *POD*_update_i at most $k \times |\mathcal{N}|$ times, where k is the diameter of E (the worst case being an influence graph that is a transitive, complete and reflexive ordering of the individuals). It is easy to see that if E is a DAG, hence E does not contain loops, the number of calls to *POD*_update_i is bounded by $|\mathcal{N}|$, i.e., each F_i needs to be computed only once.

Most interesting aggregation procedures known from the literature are unfortunately super-polynomial to compute [26, 10, 29]. Algorithm 1 anyway shows that the number of times that the aggregation function needs to be computed is bounded by the diameter of the influence graph and by the number of individuals forming an instance of *POD*. Hence, convergence is tractable under specific conditions on the graph and the aggregation procedures.

5. THE UNANIMOUS CASE

In this section, we show a simple convergence result that identifies sufficient conditions for convergence of unanimous opinion diffusion over possibly cyclic networks.

Consider the case of individuals that change their opinions only if influencers all share the same opinion, as was the case in the introductory example. Call *U-POD* the propositional diffusion model where each F_i is the unanimity rule. More precisely:

$$B_i^t = \begin{cases} B_i^{t-1} & \text{if } Inf(i) = \emptyset \\ B & \text{if } B_j = B \text{ for all } j \in Inf(i) \\ B_i^{t-1} & \text{otherwise} \end{cases}$$

Even if unanimous update may seem restrictive, it is anyway an interesting and tractable example of opinion diffusion. In the following, we consider that all agents exclude themselves from their influence sets, hence restricting to graph without loops. Two cycles are called *vertex-disjoint* if they have no internal vertex in common.

Theorem 6. *Let E be an influence network without loops such that all cycles contained in E are vertex-disjoint and, for each cycle in E, there exists $i \in \mathcal{N}$ belonging to the cycle such that $|Inf(i)| \geqslant 2$. Then U-POD converges on E after at most $|\mathcal{N}|$ steps.*

Proof. Let us first show convergence for the case of a network containing a single cycle and satisfying the property in the statement. Let therefore $C = \{m_1, m_2, \ldots, m_k\}$ be the nodes in the single cycle, such that $(m_j, m_{j+1}) \in E$ for all $j \leq k$ and $(m_k, m_1) \in E$, and let m_i be a node on the cycle with at least one influencer that is external to the cycle, i.e., $|Inf(m_i)| \geq 2$ (recall that E has no loops). Let us make the assumption that all external influencers of nodes in the circle have reached a stable opinion (cf. Theorem 2). We distinguish two cases. First, if the influencers of m_i that are external to the cycle have non-unanimous opinions then

m_i will never change its opinion, since we are using the unanimity rule. If instead all its external influencers agree on a given opinion B, and $B_{m_i}^0 \neq B$, then m_i can update its opinion only in case m_{i-1} also agrees with B at a certain point in time. Observe however that once $B_{m_i}^t = B$ then $B_{m_i}^{t'} = B$ for all $t' > t$, that is, once m_i updates to the unanimous opinion of its external influencers then it is not anymore possible to change it. We have shown that the existence of an individual with at least one external influencer on the cycle is sufficient to guarantee the convergence of her own individual opinion. This fact implies the convergence of all other individual opinions on the cycle, since the stability of m_i's opinion does not allow for cyclic behaviour in the opinion dynamic. Note that, in the worst case, we need to update the individual opinions on the cycle a number of steps equal to the size of C.

Let us now consider the general case. Let $C_1, \ldots, C_s$ be a clustering of the network E such that each C_j is either one of the vertex-disjoint cycles in E or a single vertex. Assume moreover that the order is consistent with the graph, i.e., if $(i_j, i_k) \in E$ and $i_j \in C_j$ and $i_k \in C_k$ for distinct j and k, then $j < k$. Obtaining such an ordering is possible since all cycles are vertex-disjoint. By induction on this ordering we now prove that all nodes reach convergence. First, observe that C_1 must be a source node, hence it converges in one step as in the proof of Theorem 2. Otherwise, C_1 would be a circle that has no external influencer, contradicting the hypothesis that at least one node in the circle has more than two influencers. Assume now that all nodes until C_k reach convergence after $|C_1| + \cdots + |C_k|$ steps, and consider C_{k+1}. Observe that all influencers of nodes in C_{k+1} belong to $C_1 \cup \cdots \cup C_k$, since all cycles need to be disjoint, and hence by inductive hypothesis have reached convergence. Therefore, either C_{k+1} is a single node, and then we can apply *U-POD* once to obtain convergence, or C_{k+1} is a cycle with stable influencers, in which case we know by the first part of the proof that it converges after $2|C_{k+1}|$ steps. Since $|C_1| + \cdots + |C_s| = |\mathcal{N}|$ we obtain the desired statement. □

Theorem 6 shows that if cycles in the influence network are vertex-disjoint, then a minimal condition suffices to ensure convergence of unanimous *POD*. The result at convergence can then be computed in polynomial time by simply computing opinion diffusion a sufficient number of steps, i.e., a number of steps equal to the number of nodes in the network.

6. THE MAJORITARIAN CASE

In this section we study the diffusion of opinions over a network where each individual follows the opinion of the majority of her influencers (possibly including herself). We are able to strenghten Theorem 2 obtaining convergence on a larger class of graphs, and we provide a closed form to compute the set of individual opinions at convergence from the initial opinions of the network sources.

6.1 Convergence results

On multi-issue domains the majority rule does not always result in admissible outcomes, i.e., it is possible that all individuals have admissible opinions while the result of the majority is not admissible; this is a well known result in judgment aggregation and merging. To avoid similar problems, in this paper we restrict to the case of a single issue. Our results generalise immediately to all cases in which the majority rule constitutes a collectively rational aggregator; that is, to all domains defined by integrity constraints that are equivalent to a formula in 2-CNF [16].

Let p be a single issue. In this case, individuals have initial opinions $B_i \in \{0, 1\}$, forming an initial profile $\boldsymbol{B}^0 \in \{0, 1\}^n$. Given a profile $\boldsymbol{B}$, let $n_{\boldsymbol{B}}^1$ be the number of individuals that accept the issue

in profile $\boldsymbol{B}$ and $n_{\boldsymbol{B}}^0$ the number of those that reject it. Formally, the majority rule $maj(\boldsymbol{B}) = 1$ if $n_{\boldsymbol{B}}^1 > n_{\boldsymbol{B}}^0$, $maj(\boldsymbol{B}) = 0$ if $n_{\boldsymbol{B}}^0 > n_{\boldsymbol{B}}^1$ and $maj(\boldsymbol{B}) = \{0, 1\}$ in case $n_{\boldsymbol{B}}^1 = n_{\boldsymbol{B}}^0$. Technically, the above-defined majority rule constitutes a non-resolute aggregation procedure, since it can output a set of admissible opinions rather than always a single one. This explains the following setting which takes care of possible ties:

$$B_i^t = \begin{cases} B_i^{t-1} & \text{if } Inf(i) = \emptyset \\ B_i^{t-1} & \text{if } |maj(\boldsymbol{B}_{Inf(i)}^{t-1})| = 2 \\ maj(\boldsymbol{B}_{Inf(i)}^{t-1}) & \text{otherwise} \end{cases}$$

Under these assumptions, an agent changes her opinion at time t only if she observes a strict majority of opposite opinions at time $t-1$ among her influencers. We call this process *majoritarian propositional opinion diffusion (maj-POD)*.

We are interested in strenghtening Theorem 2, obtaining convergence on a larger class of networks. The following theorem shows that convergence can be guaranteed even if the graph contains cycles:

THEOREM 7. *Let E be an influence network such that all cycles contained in E are vertex-disjoint and, if a node i belongs to a cycle, then $|Inf(i)|$ is of even cardinality, then maj-POD converges on E after at most $|\mathcal{N}|$ steps.*

PROOF. Let us first show that if E contains a single cycle and satisfies the properties in the statement, then *maj-POD* converges. Let therefore $C = \{m_1, m_2, \ldots, m_k\}$ be the nodes in the single cycle, such that $(m_j, m_{j+1}) \in E$ for all $j \leq k$ and $(m_k, m_1) \in E$. Let $ExtInf(m_j) = Inf(m_j) \setminus (m_{j-1})$ be the set of influencers of m_j different than m_{j-1}, and possibly including herself. Consider any node $m_j \in C$ at time 1. Since $Inf(m_j)$ has even cardinality, we can distinguish two cases:

(a) there is strong majority among $ExtInf(m_j)$ either supporting or against the issue, i.e., either $n_{\boldsymbol{B}^1}^1 \geq n_{\boldsymbol{B}^1}^0 + 2$ or $n_{\boldsymbol{B}^t}^0 \geq n_{\boldsymbol{B}^t}^1 + 2$. In this case the opinion of the internal influencer m_{j-1} is irrelevant: at step 1 agent m_j will update her opinion to that of the majority of $ExtInf(m_j)$ and her opinion will not change for the rest of the process.

(b) there is a simple majority among external influencers for either accepting or rejecting the issue, i.e., there is one more individual supporting than rejecting the issue or viceversa. We distinguish two sub-cases:

(b1) if m_{j-1} agrees with the majority of external influencers, then m_j updates her opinion accordingly and will not change it anymore. To see this, observe that after step 1 the opinion of m_{j-1} becomes irrelevant, since in the worst case m_j will observe a tie in the influencers opinions and will not change her view.

(b2) if m_{j-1} does not agree with the majority, m_j does not change her opinion since she observes a tie.

Hence, we have shown that once a node on the cycle updates her opinion to that of the majority of her sources, it is then not possible to revert it. Therefore, in at most k steps the opinions on the cycle stabilise, where k is the size of the cycle. The worst case is the influence network depicted in Figure 2, where in exactly k steps the systems converges to consensus on 1.

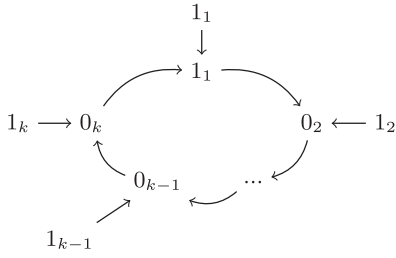The general case of vertex-disjoint cycles can be treated in an analogous way to the proof of Theorem 6. □

Figure 2: *Maj-POD* **converges in** k **steps.**

Theorem 7 shows convergence of *maj-POD* when all cycles present in the graph are vertex-disjoints, and when each node that belongs to a graph has an even number of influencers, i.e., it has at least one external influencer and an odd number of them if there are more. The graph in Figure 2 shows an example where each vertex on the graph has exactly two influencers, one internal and one external. Theorem 7 cannot be strengthened easily. Figure 3 shows a network and an initial opinion vector where *maj-POD* does not converge. To see this, observe that at each step any of the two internal nodes observes a tie in the opinions of their external sources, and will hence copy the opinion of the other individual in the cycle.
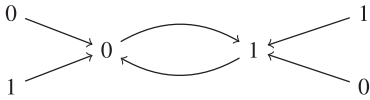

Figure 3: *Maj-POD* **does not converge.**

Theorem 7 guarantees that the result of *maj-POD* at convergence can be computed in polynomial time:

COROLLARY 8. *The result of maj-POD at convergence $\boldsymbol{B}^*$ can be computed in time $O(n^2 m)$, where $n = |\mathcal{N}|$ and $m = |E|$.*

PROOF. Theorem 7 guarantees convergence of *maj-POD* in $|\mathcal{N}| = n$ steps. Hence, we can compute the opinion profile $\boldsymbol{B}^*$ at convergence in an iterative fashion: first we compute B_i^2 for all $i \in \mathcal{N}$ using the majority update, then all B_i^3 for $i \in \mathcal{N}$, until we reach B_i^n in a total of n^2 steps. Each step is a simple application of the majority rule among at most $|E|$ edges, resulting in a bound of $O(n^2 m)$. $\square$

6.2 From dynamic to static computation of opinion diffusion

In this section we focus on a decomposition feature specific to the majority rule, in order to provide faster algorithms for the computation of *maj-POD*. To do so, we express opinions at convergence as a sort of linear combination of the initial opinions of the network sources, defining in passing a novel measure of the (indirect) influence of a source agent on the rest of the graph.

We first need some additional notation. Let E be the network relating the individuals. A *path* on E is a sequence of distinct nodes $i_1, \ldots, i_m$ such that $(i_j, i_{j+1}) \in E$ for all $1 \leq j \leq k - 1$. Given a path $g = i_1, \ldots, i_m$, we say that $k \in g$ if there exists a j such that an $i_j = k$. We denote with $P(i_1, i_2)$ the set of paths connecting node i_1 to node i_2. Let $deg(i) = |Inf(i)|$ be the *indegree* of a node i, i.e. the number of her influencers. Finally, let us indicate with

$C^{-1}(i)$ the set of ancestors of a node i, i.e., all nodes that have a path connecting them to i.

In what follows we restrict to networks that do not allow for ties between the influencers of any given node, guaranteeing the existence of a strict majority at any step of opinion update. Hence, we call a network *resolute* if $deg(i)$ is either zero or it is an odd number for every $i \in \mathcal{N}$. We can finally state the following characterisation of individuals' opinions at convergence of *maj-POD* in terms of the initial opinions of the sources:

THEOREM 9. *Let E be a resolute DAG, and let $\boldsymbol{B}^*$ be the opinion profile at convergence of maj-POD . The following holds:*

$$B_i^* = maj(\alpha(s_1, i)B_{s_1}^0, \ldots, \alpha(s_m, i)B_{s_m}^0) \qquad (1)$$

Where $s_1, \ldots, s_m$ are the sources of E, αB stands for α copies of B, and $\alpha(s_j, i)$ is computed as follows:

$$\alpha(s_j, i) = \sum_{g \in P(s_j, i)} \prod_{\substack{k \notin g \\ k \in C^{-1}(i)}} \widehat{deg}(k) \qquad (2)$$

where $\widehat{deg}(k) = 1$ if k is a source node, and $\widehat{deg}(k) = deg(k)$ otherwise, $\alpha(s_j, i) = 0$ if s_j is not connected to i, and we assume the empty product to be equal to 1. Finally, we set $B_i^* = B_i^0$ if $Inf(i) = \emptyset$, i.e., if i is already a source node.

PROOF. Since the network E is resolute, the majority rule will never result in a tie. Hence, the definition of *Maj-POD* simplifies to $B_i^* = maj(\boldsymbol{B}_{Inf(i)}^*)$. Observe that if k is an odd integer, then:

$$maj(B_0, maj(B_1, \ldots, B_k)) = maj(kB_0, B_1, \ldots, B_k) \qquad (3)$$

where kB_0 stands for k copies of ballot B_0. To see this, observe that the majority rule gives equal weights to each of the ballots that are being aggregated, and if one is itself a majority of k other ballots then the weights should be redistributed to reflect this number. Let us therefore go back to the computation of B_i^*. For each $j \in Inf(i)$ that is not a source, B_j^* is also defined in terms of the majority rule as $B_j^* = maj(\boldsymbol{B}_{Inf(j)}^*)$. Now, since the network is resolute, each node has an odd number of influencers and hence we can iterate equation (3) from i to the sources of E obtaining that $B_i^* = maj(\alpha(s_1, i)B_{s_1}^*, \ldots, \alpha(s_m, i)B_{s_m}^*)$. Every source that is not connected to i will not appear in the formulation, hence $\alpha(s_j, i) = 0$. Since sources do not update their opinions, i.e., $B_{s_j}^* = B_{s_j}^0$, we obtain equation (1).

Let us now show how to compute the multiplicative coefficients $\alpha(s_j, i)$. First, consider the case of E being a polytree, i.e., a directed graph such that its undirected version is a tree. If E is a polytree then there is a unique path relating any two nodes on the graph, i.e., $|P(i_1, i_2)| = 1$ for any $i_1, i_2 \in \mathcal{N}$. Therefore, equation (2) boils down to $\alpha(s_j, i) = \prod_{k \notin g} deg(k)$ where g is the unique path relating the source s_j to node i. Consider now i's opinion at convergence $B_i^* = maj(\boldsymbol{B}_{Inf(i)}^*)$. We can expand further this expression by observing that each B_j for $j \in Inf(j)$ that is not a source can be written as the result of the majority over its influencers, and then simplify this expression using (3). Observe moreover that the multiplicative coefficient in (3) are equal to the indegrees $deg(j)$ of influencers $j \in Inf(i)$. Let us now use once more the assumption that E is a polytree. Since every source connected to node i reaches i from one and only one influencer, say $\bar{j}$, the overall multiplicative coefficient at step one is equal to $\prod_{j \in Inf(i), j \neq \bar{j}} \widehat{deg}(j)$. We can now repeat the same process one level further, and so on until we reach the sources, obtaining the expression (2) for the case of polytrees.

Let us now consider the general case of an acyclic resolute network without loops. Under these assumptions there are several paths that connect a given source s_j to node i, hence any source s_j is connected to i through one or several direct influencers of i. Hence, each source s_j may occur more than once in the unraveling of B_i^* obtained using (3). However, the multiplicative coefficient of each path is computed as in the previous part of the proof. Since the majority rule is anonymous we can permute the aggregated ballots in order to sum these multiplicative coefficients, each one obtained following a different path that connects a source to i, obtaining equation (2) for the general case. $\square$

We are now able to refine the bound given by Corollary 8 on the computation of opinions at convergence for *maj-POD* :

COROLLARY 10. *If E is a resolute DAG, then B_i^* can be computed in time $O(k(n+m))$, where k is the number of sources of E, $n = |\mathcal{N}|$ and $m = |E|$.*

PROOF SKETCH. First, the indegree of each node in the graph can be computed in time $O(n+m)$. Sources can be identified from the indegree table – recall that the graph is a DAG – and all simple paths that connects a source s with node i can be computed with a depth-first search in time $O(n+m)$. It is then sufficient to use equation (2) to compute the multiplicative coefficient $\alpha(s,i)$ and use equation (1) to compute the result of *maj-POD* at convergence for agent i. $\square$

When loops are present, or when the network is not resolute, the situation is more complex and neither a reduction in the style of Theorem 9 nor a faster computation may be possible. Observe however that loops can be neglected by making use of a suitable notion of graph transformation. More precisely, let E be a graph with loops. Given $i \in \mathcal{N}$ and time $t \in \mathbb{N}$, we can construct a graph E' without loops, which contains a copy of E and is such that $B_i^t = B_{i'}^t$. The graph transformation works as follows: the opinion of a node i with a loop at time t is equivalent to that of a node i' without loops that has exactly the same influencers with the same opinions, but has also a copy of itself at time $t-1$ as external influencer. We can then substitute the loop around i with a copy of all parent nodes of i (possibly containing loops) and an extra copy of itself (containing a loop) at time $t-1$. This defines a recursive procedure that is guaranteed to end after at most t steps, and that can be applied to eliminate each loop in the network. Hence, if we are able to prove convergence for graphs without loops, then the same is guaranteed for graphs with loops. However, the size of the equivalent graph without loops may be prohibitively large.

The multiplicative coefficient $\alpha(s,i)$ introduced in Theorem 9 can be considered as a precise measure of the (indirect) influence of a source s on a node i. As observed in Section 2, influence measures have already been introduced in the literature on formal models of influence. In particular, Grabisch and Rusinovska [15] focus on situations of binary choice such as those considered in this section. However, without the guarantee of convergence, their measure of influence focuses on one step changes in individual opinions, counting how often a given individual changes her opinion according to the unanimous opinion of a given coalition. Exploring potential connections between these two measures of influence constitutes a prominent direction for future work.

Theorem 9 opens several other interesting problems concerning the characterisation and the computation of opinions at convergence of *maj-POD* . For instance, we may be interested in computing the ratio of sources needed to obtain consensus on a given network structure, or study strategic reasoning aspects such as bribery and control to enforce specific patterns of opinions to form at convergence or at a given point in time.

7. CONCLUSIONS

In this paper we presented a formal model of opinion propagation on networks, based on the notion of aggregation procedure. Our work differs from classical models of opinion diffusion and formation as we commit to a fully qualitative view of opinions and we consider each individual opinion formation process as a (possibly different) aggregation procedure.

Our results show that individuals' opinions reach a convergence state on directed acyclic graphs, even when self loops are allowed. For two specific cases, namely that of the unanimity rule and the majority rule, we presented sufficient conditions to guarantee convergence on general networks, provided that there is no interplay among the influence cycles that may be present. For all cases under question we devised tractable algorithms for the computation of opinions at convergence. We also showed that in the specific case of majoritarian opinion diffusion it is possible to reduce the computation of the opinions at convergence to a suitable combination of the initial opinions of the sources.

Amongst other directions, in future work we aim to relax two of the main hypothesis of this paper: the completeness of individual opinions and the tractability of aggregation procedures:

Uncertain opinions Each B_i could be viewed as a subset of the models of IC. In this case, opinion formation is closer to merging and all classical merging procedures could be relevant to our context [27]. Numerous issues then needs to be considered. First, distance-based procedures [26] will become central for defining the impact of influence. Second, as aggregation becomes more complex, tractability may become an issue. The computational complexity of belief merging is no longer polynomial and closed forms such as the one shown in Theorem 9 may be hard to obtain. A first step in this direction may use recently proposed approximations of distance-based procedures in binary aggregation [9].

Opinion aggregator Several other aggregation procedures may be considered in order to take into account more sophisticated phenomena. Fist, in the current paper we have assumed only one network for building an opinion on multiple issues. An immediate extension is to consider multiple influence graphs: one for each issue or subset of issues. Once we consider at the same time correlated issues and different influence networks for different issues, new aggregation procedures need to be defined and novel convergence results be studied. Second, our influence network is qualitative and thus does not handle hierarchy between influencers: a selfish agent may give at first a high importance to her own opinion. This hierarchy can be viewed as bridge between qualitative and quantitative models: If an agent has little influence on a second one, then her opinion should also be weighted in the aggregation procedure. Recent work on trust-based belief change [33] can provide a promising starting point in this direction: it shows how an agent revises her epistemic state with respect to some public announcement and weight of trust, the strength of trust propagating to the strength of belief.

In conclusion, in this paper we combined research in social network analysis and judgment aggregation and merging, obtaining a number of tractable models for opinion dynamics. Our initial results explored a variety of problems posed by the model, and opened several directions for both empirical research and theoretical exploration of the problem of opinion diffusion on networks.

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