“Twin Peaks” in Energy Prices:
A Hotelling Model with Pollution and Learning

by

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Abstract

We study how environmental regulation in the form of a cap on aggregate emissions from a fossil fuel (e.g., coal) affects the arrival of a clean substitute (e.g., solar energy). The cost of the substitute decreases with cumulative use because of learning-by-doing. We show that energy prices may initially increase but then decline upon attaining the targeted level of pollution, followed by another cycle of rising and falling prices. The surprising result is that with pollution and learning, the Hotelling model predicts the cyclical behavior of energy prices in the long run. The alternating trends in upward or downward price movements we show may at least partially explain recent empirical findings by Lee, List and Strazicich (2006) that long run resource prices are stationary around deterministic trends with structural breaks in intercept and trend slope. The main implication of our results is that testing for secular price trends as predicted by the textbook Hotelling model may lead to incorrect conclusions regarding the predictive power of the theory of nonrenewable resource economics.

Key words: Dynamic Models, Energy Markets, Environmental Externalities, Global Warming, Technological Change

JEL Classification: *Q12, Q32, Q41*

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1. Introduction

More than 90% of commercial energy today is supplied by the three major fossil fuels, namely coal, oil and natural gas. Each of these resources, in varying degrees, is a major contributor to environmental problems such as global warming. These resources are also nonrenewable. However, there are many clean substitutes such as solar and wind energy which are currently more expensive in the cost of producing a unit of electricity or usable heat. Empirical evidence suggests that the cost of these clean substitutes fall as they begin to acquire more market share (Argote and Epple, 1990).

In this paper, we examine the substitution of a clean energy source for a polluting one in energy production. For example, solar or wind energy are clean but expensive substitutes for coal in electricity generation.[2] The Kyoto Treaty or a similar international agreement sets a carbon concentration target or equivalently, a standard on the stock of pollution emitted from the use of coal.[3,4] We ask: if there is significant learning in the clean substitute, how will that affect energy prices and the process of substitution?

[2] Even when plausible externality costs are accounted for, as pointed out by Borenstein (2008) in a detailed study of the cost of electricity from solar photovoltaics in California.

[3] Since fossil fuels account for 75% of global emissions (rest is deforestation) the effect of an international environmental agreement (e.g., the Kyoto Treaty) can be assumed to be a direct restriction on carbon emissions from the production of energy.

[4] Many empirical studies on the effect of environmental regulation on energy use have concluded that the energy sector that is likely to be most impacted by climate regulation is electricity because it mainly uses coal, the dirtiest of all fuels. Clean substitutes for oil in the transportation sector or natural gas in residential heating (the cleanest of all fossil fuels) are much more expensive than the substitutes available in electricity generation such as hydro and nuclear power. That is, relative to coal, oil and natural gas have strong comparative advantage in their respective uses.

Many studies have looked at the problem of nonrenewable resource extraction as well as learning by doing in new technologies. However, the focus of our paper is on the role played by environmental regulation in this substitution process, an issue not directly examined before.[5] Specifically, we ask how a cap on the stock of pollution or equivalently, a carbon concentration target may affect the switch to the clean substitute.

We characterize this problem by assuming a fairly general cost specification for the nonrenewable resource and the learning technology. Unit extraction costs for coal increase with cumulative depletion. The average cost of solar energy is assumed to decrease with cumulative use but increase with the quantity supplied each period. For example, the unit cost of a solar panel may decline over time, the higher the number of panels built in the past.[6] But at any given time, the unit cost is increasing and convex with respect to the number of units supplied. This is quite realistic because as more and more solar units are brought into the market at any moment, they may have to be deployed in regions that are less favorable to solar energy such as those with lower incidence of solar radiation or in dense urban areas with higher installation costs. In our model, environmental regulation is imposed in the form of a cap on the aggregate stock of pollution.

[5] Except by a few studies described below.

[6] This is consistent with historical evidence which suggests that cheaper production units will be brought online over time, gradually replacing more expensive units. For instance, McDonald and Schrattenholzer (2001) report average cost reductions of 5 to 35% from a doubling of cumulative production in solar and wind energy generation. Duke and Kammen (1999) also find significant reductions in average cost from a rise in the cumulative production of solar panels.

There are several solutions to this model, which we describe in the paper. But the key result arises when the clean technology is used before regulation becomes binding.[7] We show that in the initial period, energy prices rise because of scarcity and impending regulation. As soon as regulation becomes binding and the stock of pollution is at its maximum level, energy prices start falling. Clean energy use increases during this period but emissions cannot increase because of regulatory constraints. However there comes a time when resources are scarce enough that regulation no longer binds, and prices rise again, driven by the scarcity of the fossil fuel until it is no longer economical to mine lower grade deposits. This rise in prices also leads to an increased adoption of clean energy. Finally the polluting fossil fuel becomes too expensive to mine and the clean alternative takes over as the sole supplier of energy and once again, energy prices fall because of learning.

In standard models of Hotelling, the price of a nonrenewable resource rises until a clean substitute is used. If the substitute is available in infinite supply and fixed cost, energy prices rise until this transition and then stay constant. When learning in the backstop is included, energy prices rise until the resource is economically exhausted then fall once substitution to the backstop has taken place (see Oren and Powell, 1985). We show that with both learning and regulation, energy prices may rise and fall successively.

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[7] Empirically, this may be the most plausible case. Carbon regulation is not yet binding in most energy markets and stylized facts suggest that clean substitutes such as solar and wind energy already occupy a small but fast-growing share of the energy market. For example the global market for solar photovoltaics was worth more than 17 billion US dollars in 2007, exhibiting a 62% growth from the previous year (EETimes, 2008). The wind turbine market is even bigger, with revenues of $36 billion in 2007. It accounted for a significant 30% of new power generation in the United States in 2007, which along with Germany and Spain has the highest installed capacity (EWEA, 2008).

This long-run cyclical behavior of energy prices is counter-intuitive and occurs because of the interplay of regulation, scarcity of the fossil fuel and learning in the clean technology. More muted but clear cycles in energy prices are obtained under other scenarios described in the paper, such as when the clean technology is deployed during the period when regulation binds or after it.

A recent review of the empirical significance of the Hotelling model suggests that "its most important empirical implication is that market price must rise over time in real terms, provided that costs are time-invariant" (Livernois, 2008). It also points out that empirical tests of the model have been generally unsuccessful. Our results suggest that in the long run, resource prices may exhibit significant structural variations driven by regulatory policy and market forces, that may result in alternating phases with secular upward or downward price movements. This finding is supported by empirical tests of the Hotelling model over a long time horizon (1870-1990) which employ endogenously determined structural breaks in the data (see Lee, List and Strazicich, 2006). They find that resource prices are stationary around deterministic trends with structural breaks in slope and intercept. In other words, prices may show upward *and* downward trends, these trends broken by the endogenous structural breakpoints. They may not just be rising *or* falling, as predicted by the Hotelling model. Specifically, our results suggest that the same Hotelling model, when subject to regulation and learning effects, may predict alternating bands of rising and falling prices, and not a secular trend as is commonly assumed. In this sense, the results obtained here have a direct bearing on the literature that aims to empirically verify the predictions of the Hotelling model. It suggests that testing for a secular trend in prices may be akin to testing a mis-specified Hotelling model.

Although for convenience, the paper is motivated in terms of coal and a clean substitute such as solar energy, it is equally applicable to other settings, such as the monopoly production of oil by a cartel such as OPEC with a competitive clean technology (e.g., a hydrogen car). The solution predicts that oil prices may rise, followed by a *decline* when emissions become binding. They rise again when regulation ceases to bind, followed by an eventual decline when there is a complete transition to the clean substitute. What is surprising is that energy prices may start decreasing upon attaining the regulated level of emissions.

In reality, there may be many short-run factors (e.g., speculation in commodity markets) that are at play in the determination of energy prices, but these results may at least partly explain the fluctuations in the prices of fossil fuels such as crude oil, natural gas and coal in recent years at a time when there is a general expectation that environmental regulation will bind at some time in the near future. The model then predicts that if say, the Kyoto Treaty imposed a target of 450 ppm (parts per million) of carbon,[8] we would expect

[8] They are currently at about 380 ppm, not counting other greenhouse gases. Although the Kyoto Treaty mandates limits on carbon emissions, its ultimate goal is the stabilization of carbon in the atmosphere, as suggested by numerous studies of the Intergovernmental Panel on Climate Change (e.g., see Gupta, 2007). We model regulation as a limit on the stock of carbon, not on flow. In reality it is some combination of the two, if one considers the short term limits imposed by the Kyoto Treaty on emissions within a five year period and the long term goal of climate stabilization. A limit on the stock allows for arbitrage over time, and implicitly assumes that

prices to rise initially but start decreasing as soon as this constraint becomes binding. When the constraint no longer binds and we fall below the 450 ppm level, energy prices will rise again, and finally fall when we make a complete transition to the clean substitute. The textbook Hotelling model, with learning or pollution regulation, does not predict this cyclical behavior.

There is a large literature on the role of environmental regulation in generating endogenous technological change, and the importance of considering these incentives in setting policy. Arrow (1962) introduced the notion of learning-by-doing, where cumulative experience rather than the passage of time or directed investment leads to lower marginal production costs. Newell, Jaffe, and Stavins (2002) demonstrate how policies change the long-run cost structure for the firm and drive innovation. Popp (2006), Gerlagh and van der Zwaan (2003), Nordhaus (2002), and Goulder and Matthai (2000) account for the potential for induced technological change in determining optimal climate change policy. Bramoullé and Olson (2005) show how new abatement technologies may be preferred to existing ones because of the dynamic incentives arising from learning-by-doing. Like these papers, the dynamic incentives provided by endogenous technological change are important in our results, but we examine how these interact with resource scarcity and environmental regulation.

Endogenous technological change has also been examined in the context of finite resource extraction. Tahvonen and Salo (2001) characterize the optimal extraction of

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damages are only a function of the stock. It may be important in future work to consider other more realistic but complex regulatory mechanisms, such as a hybrid flow and stock approach.

finite resources when physical capital becomes more productive with use, increasing the marginal productivity of alternative and traditional energy sources over time. Their paper extends the literature that deals with learning in nonrenewable resources in the Hotelling (1931) model. In particular, Dasgupta and Stiglitz (1981) and Dasgupta, Gilbert and Stiglitz (1982) characterize optimal resource extraction rates when probability of the discovery of a substitute technology can be altered through investment. Oren and Powell (1985) develop a model of resource extraction in which the cost of the backstop declines with cumulative production. None of these studies consider environmental regulation, however.

Recent work has also focused on the interplay between resource extraction and environmental regulation. Traditionally, the backstop is available in infinite supply and at constant cost. The entire economy instantaneously shifts from fossil fuels to the alternative energy source.[9] Chakravorty, Roumasset, and Tse (1997) apply the multi-sector extension of Hotelling originally proposed by Nordhaus (1973) to show that a more realistic transition would see clean energy being adopted in those sectors where it is cost-competitive and as traditional energy sources become more expensive, more and more sectors will switch to alternative energy sources. Our model attempts to capture this gradual transition. In each period, clean energy supply is upward-sloping, which captures

[9] Chakravorty, Magné, and Moreaux (2006) examine a similar resource extraction problem when regulation is set to cap the stock of emissions. They investigate how the clean backstop resource may be used jointly with the polluting nonrenewable resource especially when the demand for energy is nonstationary. However, in their model, there is no learning. This no-learning assumption may be restrictive given the long run nature of the energy transition process and the significance of learning effects over an extended time horizon (e.g., decades). They find that energy prices may rise and subsequently fall for a time when the demand for energy declines exogenously. The mechanism we consider in this paper does not depend on exogenous demand shifts but is based on learning in the clean substitute.

not only the fact that a merit order of supplying facilities is present with an increasing cost structure, but also that the marginal product of alternative energy may vary by sector. As energy prices increase, *ceteris paribus*, more of the clean energy is used. This in turn, leads to learning, which lowers the cost of clean energy and thereby makes it cost-competitive over more of the economy.

Section 2 develops the dynamic model of a nonrenewable resource with learning in the clean technology. We discuss the necessary conditions and develop some basic insights. Section 3 characterizes the energy price paths and the sequence of resource use. Section 4 concludes the paper by highlighting policy implications and limitations of the framework considered in this paper.

2. The Model

In this section we develop the Hotelling model with a clean backstop whose unit cost is determined by past experience with this technology. Let the utility or gross surplus from energy consumption be given by $U(q)$ where $q(t)$ is the energy consumed at any time t, which is the sum of the extraction rates for a polluting fossil fuel denoted by $x(t)$ and the consumption of the clean backstop resource denoted by $y(t)$ so that $q = x + y$.[10] For ease of exposition, we denote the fossil fuel as "coal" and the clean backstop as 'solar." The utility function is strictly increasing and concave with respect to q, i.e., $U_q > 0,\ U_{qq} < 0$

[10] We sometimes avoid writing the time argument t to avoid notational clutter.

and satisfies the Inada condition, $\lim_{q \downarrow 0} U_q = \infty$. Denote the marginal surplus by $p(q) \equiv U_q(q)$ and by $d(p)$ the corresponding demand function.

The initial stock of coal is defined by $X_0 > 0$ and the residual stock at any time t is given by $X(t)$, with $\dot{X}(t) = -x(t)$. The unit extraction cost of coal is assumed to be of the form $c(X)$ with $c'(X) < 0$, $c''(X) > 0$ and $\lim_{X \downarrow 0} c(X) = \infty$. Thus $c(X(t))x(t)$ is the total cost of coal at any given time. Coal use leads to pollution. Let there be a 1:1 relationship between coal use and the pollution emitted. We can easily re-work the model by assuming θ units of pollution per unit of coal use, but that will not change any of our results below.[11] The stock of pollution (say carbon) in the atmosphere is given by $Z(t)$ with initial stock $Z(0) = Z_0 > 0$. Its dynamics is given by

$$\dot{Z}(t) = x(t) - f(Z(t)) \qquad (1)$$

where $f(Z)$ is the natural rate of dilution of pollution in the atmosphere. A higher stock of pollution implies a higher rate of dilution, i.e., $f'(Z) > 0$. We model a carbon cap by postulating that the regulator sets a cap on the stock of pollution at $\bar{Z} > Z_0$.[12] Thus at any time t, $Z(t) \le \bar{Z}$.

[11] For a model with two polluting resources with different values of θ (e.g., coal and natural gas in electricity generation), see Chakravorty, Moreaux and Tidball (2008). Even without any learning, the price paths get quite complicated.

[12] This cap may be the outcome of some negotiation process. The alternative way to model the externality may be to explicitly consider damages as a function of the pollution stock Z.

The unit cost of the clean substitute, solar energy decreases with cumulative production denoted by $Y(t)$, i.e., $Y(t) = Y_0 + \int_0^t y(\tau)d\tau$, where $Y(0) = Y_0$ is the cumulative production of solar energy at the initial period. We specify this unit cost as $g(y,Y)$. The total cost of solar energy then becomes $g(y,Y)y$. We assume that for any $y > 0$ and $Y > 0$, the unit cost $g(y,Y)$ is strictly positive. We further assume that $g_y(y,Y) > 0$ with $\lim_{y \downarrow 0} g_y > 0$ and $g_{yy}(y,Y) > 0$.[13] That is, increased use of solar energy raises its unit cost at an increasing rate. The marginal cost of solar energy is positive, i.e., $\frac{d}{dy}[g(y,Y)y] = g + g_y y > 0$. It is also increasing since $\frac{d}{dy}[g + g_y y] = 2g_y + g_{yy}y > 0$. We further assume that $g_Y < 0$ and $g_{YY} > 0$ with $\lim_{y \downarrow 0} |g_Y| > 0$ and $\lim_{y \downarrow 0} g_{YY} > 0$.The average cost declines with experience, at a decreasing rate.

However, the solution is generally determined by the shape of this damage function, which may exhibit a high degree of uncertainty. Our cap is in effect a limit form of a specific damage function – with low damages below the cap and high damages beyond.

[13] Most Hotelling models assume a constant cost ($g_{yy} = 0$) in the backstop resource (e.g., Chakravorty, et al 1997). Constant costs over the quantity supplied will change many of our results. This point is discussed again below.

We also postulate that $g_{yY} = g_{Yy} < 0$. The rate of change of average cost decreases with experience. Hence the marginal cost of solar energy given by $g + g_y y$ is decreasing with experience, i.e., $\frac{d}{dY}[g + g_y y] = g_Y + g_{yY} y < 0$.[14]

When the industry accumulates an infinite level of experience, then the average cost is given by the limit value $\lim_{Y \uparrow \infty} g(y,Y) = \underline{g}(y)$ with $\underline{g}(y) > 0$ and $\underline{g}_y > 0$ and $\underline{g}_{yy} > 0$ with $\lim_{y \downarrow 0} \underline{g}'(y) > 0$ and $\lim_{y \downarrow 0} \underline{g}''(y) > 0$.[15]

Let the social rate of discount be given by r. The problem of the social planner is to determine the optimal portfolio of energy sources over time. The planner faces an externally imposed and perfectly enforceable stabilization scenario which may limit the use of the polluting resource. Specifically, the economy must respect a constraint $Z(t) \le \bar{Z}$ over an infinite horizon. We can now write the optimization program for a constrained social planner as:

$$\underset{x,y}{Maximize} \int_0^\infty \left[U(x+y) - c(X)x - g(y,Y)y\right] e^{-rt} dt \qquad \text{(P)}$$

[14] Although we discuss solar energy as a likely candidate for the clean technology, we avoid being very specific about the nature of the technology in this formulation. For example, solar panels may have a long life (30-40 years), in which case, at any given time, there would be an inherited installed capacity of solar panels, net of obsolescence. This will introduce complications in the model. However, we do not think that the results will change because all the solutions to the model imply increasing solar use over time, so the question of unusable capacity does not arise. Any technology that exhibits a unit cost that increases with the quantity supplied works, provided there are cost reductions due to experience.

[15] The limit marginal cost $\underline{g} + \underline{g}_y y$ is increasing, i.e., $\frac{d}{dy}[\underline{g} + \underline{g}_y y] = \underline{g}_y + 2\underline{g}_{yy} y > 0$.

subject to

$$\dot{X}(t) = -x(t) \tag{2}$$

$$\dot{Y}(t) = y(t) \tag{3}$$

$$\dot{Z}(t) = x(t) - f(Z) \tag{4}$$

$$x \geq 0, \quad y \geq 0, \quad \bar{Z} - Z \geq 0\,. \tag{5}$$

The current value Lagrangian is given by

$$L = U(x+y) - c(X)x - g(y,Y)y - \lambda x + \beta y + \mu[x - f(Z)] + \gamma_Z(\bar{Z} - Z) + \gamma_x x + \gamma_y y$$

where $\lambda(t)$, $\mu(t)$ and $\beta(t)$ respectively, are the co-state variables attached to the three equations of motion (2)-(4) and γ_x, γ_y and γ_Z are Lagrange multipliers associated with the Kuhn-Tucker conditions in (5). We obtain the following first order conditions:

$$U_q = c + \lambda - \mu - \gamma_x \tag{6}$$

$$U_q = g + g_y y - \beta - \gamma_y \tag{7}$$

$$\dot{\lambda}(t) = r\lambda + c'(X)x \tag{8}$$

$$\dot{\beta}(t) = r\beta + g_Y y \text{ and} \tag{9}$$

$$\dot{\mu}(t) = (r + f'(Z))\mu + \gamma_Z\,. \tag{10}$$

The complementary slackness conditions are

$$\gamma_x x = 0, \quad \gamma_x \geq 0, \quad x \geq 0 \tag{11}$$

$$\gamma_y y = 0, \quad \gamma_y \geq 0 \quad y \geq 0 \tag{12}$$

$$\gamma_Z(\bar{Z} - Z) = 0, \quad \gamma_Z \geq 0 \quad \bar{Z} - Z \geq 0 \tag{13}$$

and the transversality conditions are given by

$$\lim_{t \uparrow \infty} e^{-\rho t} \lambda(t) X(t) = 0, \tag{14}$$

$$\lim_{t \uparrow \infty} e^{-\rho t} \beta(t) Y(t) = 0 \text{ and} \tag{15}$$

$$\lim_{t \uparrow \infty} e^{-\rho t} \mu(t) Z(t) = 0. \tag{16}$$

The three costate variables capture the dynamic incentives created by resource scarcity, environmental regulation and learning-by-doing. Condition (8) is the usual Hotelling condition for the scarcity rent of the nonrenewable resource $\lambda(t)$ when the unit cost is declining with the residual stock. The variable $\beta(t)$ in (9) represents the shadow price of cumulative production of solar energy. It is the benefit from currently consuming one more unit of solar energy in terms of future reductions in cost. Finally $\mu(t)$ is the shadow cost of the pollution stock and is negative. Its rate of increase over time is given by (10).

Note from (7) that when solar energy is being used ($y(t) > 0$), then $\gamma_y = 0$ so that we must have $p(t) = g + g_y y - \beta$. Recall that the average cost of solar energy is g and the marginal cost is $g + g_y y$. Since $\beta > 0$, this means that price is less than marginal cost and therefore solar energy will need to be subsidized. Moreover if the average cost of solar energy is more than β , then the firm may be making a loss.[16]

From (9), it is clear that the shadow price of the stock of cumulative experience β must increase over time. If over some time interval, there is no solar energy use, i.e., $y(t)=0$, then β exhibits exponential growth, i.e., $\dot{\beta}=r\beta$ and $\beta(t)=\beta_0 e^{rt}$ where $\beta(0)=\beta_0$ is the shadow price of the stock of the clean energy stock at the beginning of the planning horizon. Suppose solar energy were employed in an industry only after some delay, i.e., no solar was used over the interval $[0,t_1]$. Then the right hand side of (2) becomes $g(0,Y_0)-\beta_0 e^{rt}, t\in[0,t_1)$.

Note that if the ceiling does not bind initially but will in the future, then $\gamma_Z=0$ so that (9) yields $\dot{\mu}(t)=(r+f'(Z))\mu$ which gives $\mu(t)=\mu_0 e^{\int_0^t [r+f'(Z(\tau))]d\tau}$ where μ_0 is its initial value at time $t=0$. Thus the shadow price of the pollution stock is strictly decreasing (since $f'(Z)>0$), i.e., its absolute value is strictly increasing. If the ceiling is not binding and will never bind in the future, then $\mu(t)=0$.

3. Price Paths and Resource Use

Because we have a ceiling on the stock of pollution, we have three phases of resource extraction – before, during and after the period when the ceiling is binding. We show that

[16] Some part of the externality may be internalized by firms. Without an explicit micro model, it is difficult to provide further insights on this question.

coal use must be decreasing everywhere except when the ceiling is binding. We also show that solar energy use must increase whenever it is in use. If the ceiling never binds, then the analysis is similar to the post-ceiling phase. In both cases, environmental regulation is no longer relevant, hence $\mu(t)=0$. Coal use will be declining due to Hotelling and solar use will be increasing because of learning.

Before characterizing the solution, let us develop intuition by highlighting some specific cases that may arise.

Before the Ceiling is hit

If the ceiling is not binding yet, then $\bar{Z} > Z(t)$ hence $\gamma_Z = 0$ so that $\dot{\mu} = \mu[r + f'(Z)]$. Suppose there is an initial period when solar energy is not used, and only coal is used. Then $\gamma_x(t) = 0$. Hence from (6), $p = c + \lambda - \mu$. Differentiating with respect to time, using (8) and cancelling terms gives $\dot{p} = r\lambda - \dot{\mu}$. Substituting from (9) yields

$U''(x)\dot{x} = r(\lambda - \mu) - f'(Z)\mu$ so that $\dot{x} = \dfrac{r(\lambda - \mu) - f'(Z)\mu}{U''(x)} < 0$. That is, the use of coal must decrease as in the textbook Hotelling case. This is because of increasing scarcity and the marginal externality cost given by μ which also increases over time.

Now consider the situation when both energy sources are being used simultaneously, i.e., $x(t) > 0$ and $y(t) > 0$. Differentiating (6), we get $\dot{p} = U''(q)\dot{q} = r\lambda - [r + f'(Z)]\mu > 0$ which also implies that $\dot{q} < 0$. The price of energy increases and aggregate energy use decreases. Now we examine what happens to the extraction paths of the two resources.

To do this, we differentiate (7), use (9) and cancel terms to get

$\dot{p} = 2g_y \dot{y} + g_{yy} y\dot{y} + g_{yY} y^2 - r\beta$. Substituting for $\dot{p}$ from above, we have

$\dot{y} = \dfrac{r(\lambda+\beta) - [f'(Z)+r]\mu - g_{yY} y^2}{2g_y + g_{yy} y} > 0$ since the numerator and denominator are both

positive. That is solar energy use must increase under joint use. The use of the nonrenewable resource is given by $\dot{x} = \dot{q} - \dot{y} < 0$. Since aggregate energy use is decreasing and solar use is increasing, coal use must decline over time. The increase in the price of coal because of scarcity drives the increased adoption of solar energy. This effect is compounded by cost reductions achieved by learning.

At the Ceiling

When the economy is at the ceiling $\bar{Z}$ for an interval of time, coal must be in use, since emissions need to be positive and equal natural dilution as shown in (1). So $\dot{Z}(t)$ must be zero, hence we define $\bar{x} = f(\bar{Z})$. This is the constant amount of coal use each period that will keep the stock of pollution at the ceiling. There are two possibilities at the ceiling depending upon whether solar energy is also being used or not. If there is no solar at the ceiling, then $q(t) = x(t) = \bar{x}$ and $p(t) = \bar{p} = U'(\bar{x})$. Coal use as well as the price of energy are constant.

The more interesting case is when solar energy is being used at the ceiling. Note that the same amount of coal must be emitted each time period at the ceiling, otherwise the stock of pollution will not remain at $\bar{Z}$. So $q(t) = \bar{x} + y(t)$ where $y(t) > 0$. This implies that

$\gamma_y = 0$. Then from (7) and following the same procedure as before, we get

$\dot{y} = \dfrac{g_{YY} y^2 - r\beta}{U''(q) - 2g_y - g_{yy} y} > 0$ since both numerator and denominator are negative. That is, solar use must be increasing at the ceiling while coal use will remain constant, by definition. Aggregate energy use will increase. When there is increasing solar energy use, $\dot{p} = U_{qq}\dot{q} = U_{qq}(\dot{x} + \dot{y}) = U_{qq}\dot{y} < 0$. In other words, the price of energy when the pollution ceiling is binding must be decreasing.

After the Ceiling Period

There are three possibilities in the post-ceiling phase – only coal is used, both resources are used and only solar is used. Now the regulation phase is no longer binding hence there is no pollution stock externality. So the shadow price of the stock carbon μ is zero.[17]

If only coal is used over some period in the post ceiling phase, then from (6) we have $p = c(X) + \lambda$ which upon differentiation yields $\dot{p} = c'(X)\dot{X} + \dot{\lambda} = r\lambda$ using (8) and cancelling terms. Thus the price of energy is increasing and energy use is decreasing, i.e., $\dot{q} = \dot{x} < 0$. This phase is essentially a pure Hotelling path with a rising extraction cost function. This cannot go on forever because at some point, $c(X) + \lambda$ will catch up with the declining price of solar, given by $g(0, Y_0) - \beta$ (see equation (7)).

[17] It can be shown that the ceiling can be hit at most over one time interval, since coal which causes the pollution gets scarce over time.

If both coal and solar are used simultaneously, then (6) continues to hold as $p = c(X) + \lambda$.

Time differentiating gives $\dot{p} = r\lambda > 0$. Energy prices must rise and aggregate consumption must decline, $\dot{q} < 0$. As before, we can use the necessary conditions to determine the rate of change of solar energy use with respect to time,

as $\dot{y} = \dfrac{r(\lambda + \beta) - g_{yY} y^2}{2g_y + g_{yy} y} > 0$, which is a special case of the pre-ceiling result with $\mu(t) = 0$. Since solar use is increasing and aggregate energy use is decreasing, then coal use must decline, $\dot{x} = \dot{q} - \dot{y} < 0$.

When only solar energy is being used, $\gamma_y = 0$ and using (7), we have

$\dot{y} = \dfrac{g_{YY} y^2 - r\beta}{U''(q) - 2g_y - g_{yy} y} > 0$. This is the same result as in the ceiling with joint use, except that instead of $x = \overline{x}$, now $x = 0$ since no coal is being used. Then $\dot{p} = U''(y)\dot{y} < 0$. The price of energy must fall over time.

As solar energy continues to be used, the cumulative experience in solar energy $Y(t)$ increases with time. What is the limiting value of Y? We have $\lim_{t \uparrow \infty} Y(t) = \infty$ and the average cost of solar energy approaching the limit cost $\underline{g}(y)$. In the limit the supply of solar energy is given by $p = U'(y) = \underline{g}(y) + \underline{g}'(y)y$. Since $U'(y)$ is downward sloping and the marginal cost of solar energy is upward sloping, the solution is unique. Let us define this limit quantity as $\bar{y}$.[18]

The Ceiling is Never Active

It is important to observe that the ceiling may not be binding at all, in which case the solution is a pure Hotelling model with learning by doing in the backstop resource. This may happen for instance, if learning effects induce significant substitution of coal by solar, so that the stock of pollution never builds up to the regulated level. If the solution to the model implies a path for the pollution stock $Z(t)$ that never equals the regulated cap $\bar{Z}$ for a non-degenerate interval of time, then the ceiling is redundant. In that case, we may have two possible outcomes depending on whether solar energy is used right from the beginning of the planning horizon or not.

When solar energy is used from the beginning, the resource use profile is shown in Fig.1. During the initial period of joint use, $\dot{p}(t) > 0$, $\dot{q}(t) < 0$, $\dot{x}(t) < 0$ and $\dot{y}(t) > 0$. Finally at

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[18] There exists a finite time t_x at which the use of the nonrenewable resource is terminated provided that the cost of the clean substitute declines sufficiently due to cumulative experience.. Then since $\lim_{X \uparrow 0} c(X) = \infty$, the terminal condition that determines when no more of the coal is used is given by $c(X(t_x)) = p(t_x)$. As pointed out by Salant, Eswaran and Lewis (1983), the extraction period of coal may be infinite even if the inverse demand function is bounded from above, i.e., $p(0) < \infty$. In our case, the inverse demand is not bounded because of the Inada condition. But the residual demand of coal is bounded and decreasing over time. Therefore coal extraction over an infinite period cannot be excluded *a priori*. However, suppose the initial extraction cost of coal is very low relative to $\bar{p}$, the price at the ceiling and this extraction cost does not increase significantly over a large part of the coal stock. Suppose also that the initial cost of solar is very high, but the limit price $\underline{g}(y)$ is lower than the initial extraction cost of coal and that this limit price is low over a large range of values of the solar extraction rate. If with given demand parameters, it takes a long time for solar energy to come "close" to the limit price, then he price of energy cannot increase forever. The necessary conditions for an infinite period of coal extraction will not be satisfied. This is the setting we consider in this paper.

time t_x, coal is no longer used, and all energy is supplied by solar. Then the price of energy decreases and solar use increases because of learning. Note that the price of energy rises at first then declines. In the figure, $x(0)$ may be more or less than $\bar{y}$. Similarly, there is no *a priori* ordering of $q(0)$ and $\bar{y}$.

When the ceiling never binds but solar energy is not deployed from the beginning, we may have three phases, as shown in Fig. 2. In the first phase $[0, t_y]$ only coal is exploited. The price of energy keeps increasing while the true cost of solar energy given by (7), i.e., $g(0, Y_0) - \beta$ keeps decreasing. The gap between them shrinks. Finally at time t_y in the second phase solar energy becomes economical and both resources are exploited at the same time. The price of energy keeps increasing and peaks at time t_x when coal gets exhausted. Beyond this time, the price of energy keeps falling until infinity. Solar energy use increases and approaches an upper bound in the limit.

There must always be an intermediate phase in which both resources are used. This is because the marginal cost of solar energy increases with the volume supplied at any instant, i.e., $\frac{d}{dy}[g + g_y y] = 2g_y + g_{yy} y > 0$. Suppose that there is no joint use. That is, before time t_x, there is no solar energy use and beginning at time t_x, solar supplies the whole industry. Then at t_x, there is a jump of solar energy consumption from zero to $y = q(t_x) = \lim_{t \uparrow t_x} x(t) > 0$. That is at time t_x, by condition (7), we must have $p(t_x) = g(q(t_x), Y_0) + g'(q(t_x), Y_0) q(t_x) - \beta(t_x)$. Now consider the fact that $p(t)$ is continuous and increasing for $t < t_x$, $\dot{\beta} = r\beta$ by (9) hence $\beta(t) = \beta_0 e^{rt}$ for $t < t_x$ and $g(y, Y_0) + g'(y, Y_0)y$ is strictly increasing in y. Then for $t = t_x - \varepsilon$ where $\varepsilon > 0$ is sufficiently small, there is a level of solar energy use y_ε, $0 < y_\varepsilon < q(t_x)$, which satisfies condition (2) with $\gamma_y = 0$ and $p(t) = g(y_\varepsilon, Y_0) + g'(y_\varepsilon, Y_0)y_\varepsilon - \beta(t)$. But this violates the fact that since solar energy use is zero before time t_x, we must have $p(t) \le g(0, Y_0) - \beta(t) - \gamma_y$ at time $t = t_x - \varepsilon$ with $\gamma_y \ge 0$. Similar arguments hold in each of the solutions described below.

A Binding Ceiling

When the ceiling binds over a non-zero interval, then solar energy may become economical before, during or after the ceiling. Consider the last case first.

Solar Arrives after the Ceiling

The price path and resource extraction is shown in Fig. 3. Initially emissions are higher than natural dilution, i.e., $\dot{x}(t) > f(Z)$ so the stock of pollution $Z(t)$ increases as given by (1). However because of rising resource prices, coal extraction and emissions decline and the higher stock of pollution means increased dilution since $f'(Z) > 0$. Finally at time $t = \underline{t_c}$, emissions exactly equal natural dilution. In the following phase, coal use is constant, and the stock of pollution stays at the regulated maximum $\bar{Z}$. Emissions equal $\bar{x}$ and energy consumption is constant. At the end of this phase, $\mu(t) = 0$ and is always zero beyond.

From time $\overline{t_c}$ only the nonrenewable resource is used again, and the solution is pure Hotelling since there is no environmental regulation. Emissions fall below $\bar{x}$ and the stock of pollution begins to fall as resource prices rise due to Hotelling. All this time, as shown in Fig.3, the price of the clean alternative continues to decline but not enough to be economical. Finally at time t_y we have $p(t_y) = g(0, Y_0) - \beta_0 e^{rt_y}$. Both resources are now exploited, although coal use declines while solar use increases, driven partly by increasing energy prices and by learning effects.

At time t_x, coal extraction is complete with $p(t_x) = c(X(t_x))$ implying that the scarcity rent of coal $\lambda(t_x)$ is zero. Solar energy becomes the sole supplier of energy. A degenerate case may occur if $p(\overline{t_c}) = g(0, Y_0) - \beta_0 e^{r\overline{t_c}}$, i.e, solar energy arrives exactly when the ceiling ceases to be binding. In this case we will not get a period with only coal use after the ceiling. This may happen if coal resources are limited or learning effects are higher in the solar technology.

Solar Arrives at the Ceiling

When the price of energy is high relative to the cost of solar energy, it is possible that solar energy may become economical when the economy is still at the regulated level of pollution. In this case, over the time interval $[t_y, \overline{t_c}]$ the use of solar energy must rise because of learning (see Fig.4). Since coal use must be constant, the aggregate consumption of energy rises, which in turn implies that the price of energy declines during this period of joint use at the ceiling. That is, $q(t) = \bar{x}(t) + y(t)$, so that

$\dot{q}(t) = \dot{y}(t) > 0$ and $\dot{p}(t) = U''(q)\dot{q}(t) < 0$. As before the shadow price of the pollution stock is zero at the end of the ceiling. The remaining path is similar to that shown in Fig. 3.

Here the price of energy fluctuates, initially increasing and staying at a peak, then declining and again increasing and decreasing over another cycle.

Solar Arrives before the Ceiling - The Price of Energy Peaks Twice

Perhaps the most interesting cases are those in which the solar energy arrives before the regulation binds. This may happen if the cost of solar is low relative to the price of energy or regulation is less stringent. Here we obtain two “peaks” in energy prices - one when the regulation begins and another when coal gets exhausted. There are two corresponding troughs in energy consumption, shown in Fig. 5.

Coal is used initially until solar becomes economical at a price below the ceiling, i.e., $p(t_y) = g(0, Y_0) - \beta(t_y) < \bar{p}$. Both resources are exploited and the price of energy continues to rise. However, solar energy use expands while coal use declines although the emissions generated are higher than the dilution rate, so that the stock of pollution continues to build towards $\bar{Z}$. Finally the ceiling becomes effective at time $\underline{t_c}$. The remaining path is similar to the previous case when solar arrives at the ceiling.

We can broadly summarize the main results as follows:

Proposition: *When a polluting nonrenewable resource has a clean substitute which exhibits learning by doing, the price path is single-peaked if regulation does not bind. Under binding environmental regulation, the price path is again single-peaked if the substitute arrives after regulation ceases to bind. Cyclical price trends are observed when the substitute arrives during the period when regulation binds. Finally, prices exhibit twin peaks when the substitute arrives before binding regulation.*

To understand the reason behind the cyclical behavior of energy prices, it is important to note that energy prices rise because of scarcity of coal resources. In the textbook Hotelling model, the price of a nonrenewable resource rises due to scarcity until it is substituted by the backstop. If there was learning by doing in the backstop, then the arrival of the backstop will trigger a fall in energy prices, thus the overall price path is a single peak. What happens in the present model is that when the use of coal is restricted because of regulation, solar energy use must increase because of learning. Thus aggregate energy use increases, which means the energy price must decline for some time. However, when coal reserves get sufficiently depleted, then there is not enough to coal to continue emitting at the ceiling, so the stock of pollution is off the regulated level. The textbook Hotelling model takes over. Coal use declines, and its price rises again. This leads to the second peak in the price path. In fact, coal gets exhausted precisely at the second peak, beyond which prices must again fall due to learning in the clean technology.

Under what circumstances will the twin peaks in prices arise? They are likely when solar energy is relatively cheap, learning effects are significant and the environmental cap $\bar{Z}$ is set at a relatively high level. For example, if solar is expensive and/or learning effects are small, it is more likely to become economical during or after the ceiling is attained, in which case the price cycles will not arise. If the environmental cap is relatively strict (low $\bar{Z}$) then again solar may arrive after the period when regulation is binding. The abundance and pollution intensity of coal may have the same effect. The more abundant the coal, the lower its price and higher the emissions. This is likely to postpone the onset of the clean energy because *ceteris paribus,* it becomes more expensive relative to abundant coal. The ceiling will be achieved earlier with abundance in the fossil fuel since pollution will be higher. Other parameters not considered in the paper such as the pollution intensity of the nonrenewable resource will have similar effects. A higher pollution intensity of coal is likely to hasten the arrival of the ceiling relative to deployment of solar energy.

There is yet another case in which a similar cyclical price path is achieved, when solar energy arrives at the beginning of the planning horizon, at time t_0. This essentially implies that we will not have the zone with exclusive coal use at the beginning, and the rest of the extraction sequence remains the same.

The assumption of increasing unit cost of solar energy each period is not only realistic from an empirical point of view, as we have discussed earlier, but also important for our results. If the unit cost were constant ($g_{yy} = 0$) then there may be jumps in the supply of solar energy. The periods of joint use of the two resources may not occur, except when at the ceiling. The “twin peaks” result may not hold. More precise assumptions on solar energy supply may be necessary before a complete characterization can be done, which is beyond the scope of this paper.

4. Concluding Remarks

We consider the effect of environmental regulation on substitution of a polluting fossil fuel by a clean backstop technology, when the latter exhibits learning by doing. We show that the price of energy may exhibit cyclical trends driven by scarcity of the nonrenewable resource, the effect of regulation and cost reductions in the clean technology. Such price cycles do not emerge in traditional models of energy use.

The textbook model of Hotelling with a clean backstop suggests that energy prices will rise until an instantaneous transition to the clean backstop. The model with learning in the backstop suggests in general that energy prices rise until economic exhaustion and then fall when the renewable is used. In our model prices rise, and then fall under regulation, then rise again with scarcity and finally fall with learning.

In particular, what may be surprising is that energy prices may start declining the moment environmental regulation begins to bind. But once regulation is no longer effective, prices will rise, and finally fall upon transition to the clean substitute. The scarcity of a nonrenewable resource has a positive effect, since it drives the increase in the price of energy, which in turn accelerates the process of substitution and learning in the clean alternative.

This non-monotonic behavior of prices may explain, at least partly, some of the cyclical trends we observe in the long run movement of world energy prices. For example, Lee, List and Strazicich (2006) examine prices for eleven nonrenewable resources during the period 1870-1990. They conduct unit root tests with two structural breaks endogenously determined by the data. They find evidence that resource prices are stationary with structural breaks in slope and intercept.

This implies that cycles that are a common feature in price movements of commodities such as crude oil may not be entirely due to macroeconomic phenomena or business cycle effects. The emergence of new energy sources and new substitution possibilities over time (Rhodes, 2007) may lead to repeated cycles in energy prices in the same manner that we have demonstrated in this paper. In order to better understand how the model proposed in this paper may explain some of these observed long run price cycles, one would need to extend this framework to explicitly consider substitution possibilities among nonrenewable resources. For instance, both coal and natural gas are used to supply electricity. Without recognizing these important features of the energy market, testing the textbook Hotelling model that predicts secular price movements may lead to incorrect conclusions regarding the predictive power of the theory of nonrenewable resources.

For instance, our results suggest that the price of the nonrenewable resource may rise initially, fall and then rise again until economic exhaustion. However *energy* prices exhibit the twin peaks discussed earlier. So in a long run model with many nonrenewable resources, it is likely that there may be multiple peaks. As prices rise because of scarcity, a new substitute may be deployed. This new resource may initially exhibit significant learning effects so that prices may fall. Ultimately, scarcity dominates the learning effect and prices rise, only to be substituted by yet another source of energy. Of course, for all this to happen, there may need to be some degree of uncertainty in terms of the availability of these new resources. Otherwise extraction could be ordered by cost and price movements will be secular. Moreover, it may not be important to have environmental regulation as in the present framework. Any increase in the cost of energy production say due to scarcity or a rise in transportation costs may be sufficient to generate price peaks but a precise model is left for future work.

There are other limitations of the proposed framework that could be modeled. One is the growth in energy demand. It may be interesting to see how substitution of the clean energy may occur under exogenous demand growth, as is currently occurring in the global economy. The second issue is uncertainty which could not only be in the form of environmental regulation, but also the stocks of the fossil fuel and the cost reductions that could be achieved through an increase in market share of the clean fuel. *Ex-ante,* it is not clear how these uncertainties will affect the substitution process. Yet another issue completely ignored here is the endogeneity of research and development and the associated market structure – the nature of the fossil fuel industry and the clean alternative and how their interaction may affect the process of technology adoption. A fuller specification of the research process is needed to determine the precise effects of regulation.

An interesting issue that could be handled only with a calibrated model is the effect of the stringency of regulation on resource use. If the target is lax, or the cost of extraction is high or learning-by-doing is significant, then we are likely to obtain a single peak in prices and regulation will not bind. A stricter target implies that the fossil fuel and the clean energy co-exist for a period of time with the twin peaks in prices. A lower target raises the externality cost of the polluting fuel but also accelerates the substitution process. However, it is not clear *ex-ante* whether the twin peaks are higher than the single peak.

References

Argote, Linda and Dennis Epple (1990), "Learning Curves in Manufacturing," *Science* 247, 920.

Arrow, Kenneth J. (1962). "The Economic Implications of Learning by Doing," *Review of Economic Studies* 29: 155-73.

Borenstein, Severin (2008). "The Market Value and Cost of Solar Photovoltaic Electricity Production," CSEM Working Paper 176, Berkeley: University of California Energy Institute.

Bramoullé, Yann and Lars Olson (2005), "Allocation of Pollution Abatement under Learning by Doing," *Journal of Public Economics* 89(9), 1935-60.

Chakravorty, Ujjayant, James Roumasset and KinPing Tse (1997), "Endogenous Substitution of Energy Resources and Global Warming," *Journal of Political Economy* 105(6), 1201-1233.

Chakravorty, Ujjayant, Bertrand Magné, and Michel Moreaux (2006), "A Hotelling Model with a Ceiling on the Stock of Pollution," *Journal of Economic Dynamics and Control* 30(12), 2875-2904.

Chakravorty, Ujjayant, Michel Moreaux and Mabel Tidball (2008), "Ordering the Extraction of Polluting Nonrenewable Resources," *American Economic Review* 98(3), 1128-44.

Dasgupta, Partha, Richard Gilbert and Joseph Stiglitz (1982), "Invention and Innovation under Alternative Market Structures: The Case of Natural Resources," *Review of Economic Studies* 49(4), 567-82.

Dasgupta, Partha and Joseph Stiglitz (1981), "Uncertainty, Industrial Structure and the Speed of R&D," *Bell Journal of Economics* 11(1), 1-28.

Duke, R.D. and D.M. Kammen (1999), "The Economics of Energy Market Transformation Initiatives," *Energy Journal* 20(4), 15-64.

EETimes (2007), "Solar Market Grew 62% in 2007," by Mark LaPedus, March 18.

EWEA (2008), "US, China and Spain Lead World Wind Power Market in 2007," February 15, European Wind Energy Association.

Goulder, Lawrence and K. Mathai (2000), "Optimal CO2 abatement in the presence of induced technological change," *Journal of Environmental Economics and Management* 39(1), 1-38.

Gerlagh Reyer, and B.C.C. van der Zwaan (2003), "Gross World Product and Consumption in a Global Warming Model with Endogenous Technological Change," *Resource and Energy Economics*, 25, 35-57.

Gupta, S., et al, (2007), "Policies, Instruments and Co-operative Arrangements", in *Climate Change 2007: Mitigation. Contribution of Working Group III to the Fourth Assessment Report of the Intergovernmental Panel on Climate Change*, Cambridge University Press, Cambridge, U.K. and New York, USA.

Hotelling, Harold, (1931), "The Economics of Exhaustible Resources," *Journal of Political Economy* 39(2), 137-75.

Lee, Junsoo, John A. List and Mark Strazicich (2006), Non-renewable Resource Prices: Deterministic or Stochastic Trends?" *Journal of Environmental Economics and Management* 51, 354-70.

Livernois, John, (2008), "The Empirical Significance of the Hotelling Rule," *Review of Environmental Economics and Policy* 2008, 1-20.

McDonald A., and L. Schattenholzer (2001), "Learning Rates for Energy Technologies", *Energy Policy* 29:255-261.

Newell, Richard G, Adam B. Jaffe, and Robert N. Stavins. 2002 , "The Induced Innovation Hypothesis and Energy-Saving Technological Change," *Quarterly Journal of Economics*. 114(3): 941-975.

Nordhaus, William (1973), "The Allocation of Energy Resources," *Brookings Papers on Economic Activity,* 3, 529-70.

Nordhaus, William (2002), "Modeling Induced Innovation in Climate-Change Policy," in *Induced Innovation and Climate Change: Collected Essays*, ed. Arnulf Grubler, Nebojsa Nakicenovic, and William Nordhaus, Resources for the Future Press, Washington, D.C.

Oren, Shmuel and Stephen Powell (1985), "Optimal Supply of a Depletable Resource with a Backstop Technology: Heal's Theorem Revisited," *Operations Research* 33(2), 277-92.

Popp, David, "ENTICE-BR: The Effects of Backstop Technology R&D on Climate Policy Models," *Energy Economics,* March 2006, 28(2), 188-222.

Rhodes, Richard (2007), "Energy Transitions: A Curious History," Stanford University: Center for International Security and Cooperation.

Salant, Stephen, Mukesh Eswaran and Tracy Lewis (1983), "The Length of Optimal Extraction Programs when Depletion affects Extraction Costs," *Journal of Economic Theory* 31, 364-74.

Tahvonen, Olli and Seppo Salo (2001), “Economic Growth and Transitions between Renewables and Nonrenewable Energy Resources,” *European Economic Review,* 1379-98.

Appendix

Determination of the optimal sequence when the ceiling constraint does not bind

Here we only consider the case illustrated in Fig.1 in which solar energy arrives at the beginning of the planning horizon and the ceiling constraint is not binding. The other cases are similar and therefore are not examined separately. We show how the optimal paths for $x(t), X(t), y(t), Y(t), \mu(t)$ and $\beta(t)$ are determined in the first phase when both resources are used, together with the time t_x at which coal use is terminated. The equations determining the above six variables are:

$$\dot{X}(t) = -x(t)$$

$$\dot{Y}(t) = y(t)$$

$$U_q = c + \lambda$$

$$U_q = g + g_y y - \beta$$

$$\dot{\lambda}(t) = r\lambda + c'(X)x\text{, and}$$

$$\dot{\beta}(t) = r\beta + g_Y y\,.$$

We need to have six conditions that determine the six dimensional vector $(x, X, y, Y, \lambda, \beta)$ and another condition that determines the time t_x when the first phase ends. These are:

$$X(0) = X_0$$

$$Y(0)=Y_0$$

$$U'(x(0)+y(0))=c(X_0)+\lambda_0$$

$$U'(x(0)+y(0))=g(y(0),Y_0)+g_y(y(0),Y_0)y(0)-\beta_0;\ \ Y_0 \text{ given}$$

$$U'(x(t_x)+y(t_x))=c\left(X_0-\int_0^{t_x}x(t)dt\right) \text{ which implies } \lambda(t_x)=0\,.$$

$$U'(x(t_x)+y(t_x))=g\left(y(t),Y_0+\int_0^{t_x}y(t)dt\right)+g_Y\left(y(t_x),Y_0+\int_0^{t_x}y(t)dt\right)y(t_x)-\beta(t_x);$$

$$x(t_x)=0$$

Equivalently, the system of four first order differential equations in X, Y, λ and β can be solved. They are as follows:

$$U'(-\dot{X}(t)+\dot{Y}(t))=c(X(t))+\lambda(t)$$

$$U'(-\dot{X}(t)+\dot{Y}(t))=g(\dot{Y}(t),Y(t))+g_Y(\dot{Y}(t),Y(t))\dot{Y}(t)-\beta(t)$$

$$\dot{\lambda}(t)=r\lambda(t)-c'(X(t))\dot{X}(t))$$

$$\dot{\beta}(t)=r\beta(t)+g_Y(\dot{Y}(t),Y(t))\dot{Y}(t)$$

We need four points in the (X,Y,λ,β) space to determine the solution. These are:

$X(0)=X_0$ given

$Y(0)=Y_0$ given

$$U'(-\dot{X}(t_x)+\dot{Y}(t_x)) = c\left(X_0 + \int_0^{t_x} \dot{X}(t)dx\right) \text{implies } \lambda(t_x) = 0$$

$$U'(-\dot{X}(t_x)+\dot{Y}(t_x)) = g(\left(\dot{Y}(t_x), Y_0 + \int_0^{t_x} \dot{Y}(t)dx\right) + g_Y\left(\dot{Y}(t_x), Y_0 + \int_0^{t_x} \dot{Y}(t)dx\right)\dot{Y}(t_x) - \beta(t_x)$$

Furthermore, at $t = t_x$, we must have $\dot{X}(t_x) = 0$, which determines t_x.

In the second phase, only solar is used. The corresponding differential equations that need to be solved for $Y(t)$ and $\beta(t)$ are

$$U'(\dot{Y}(t)) = g(\dot{Y}(t), Y(t)) + g_Y(\dot{Y}(t), Y(t))\dot{Y}(t) - \beta(t) \text{ and}$$

$$\dot{\beta}(t) = r\beta(t) + g_Y(\dot{Y}(t), Y(t))\dot{Y}(t)$$

together with the initial conditions $Y(t_x)$ and $\beta(t_x)$ given by the first phase. ▪

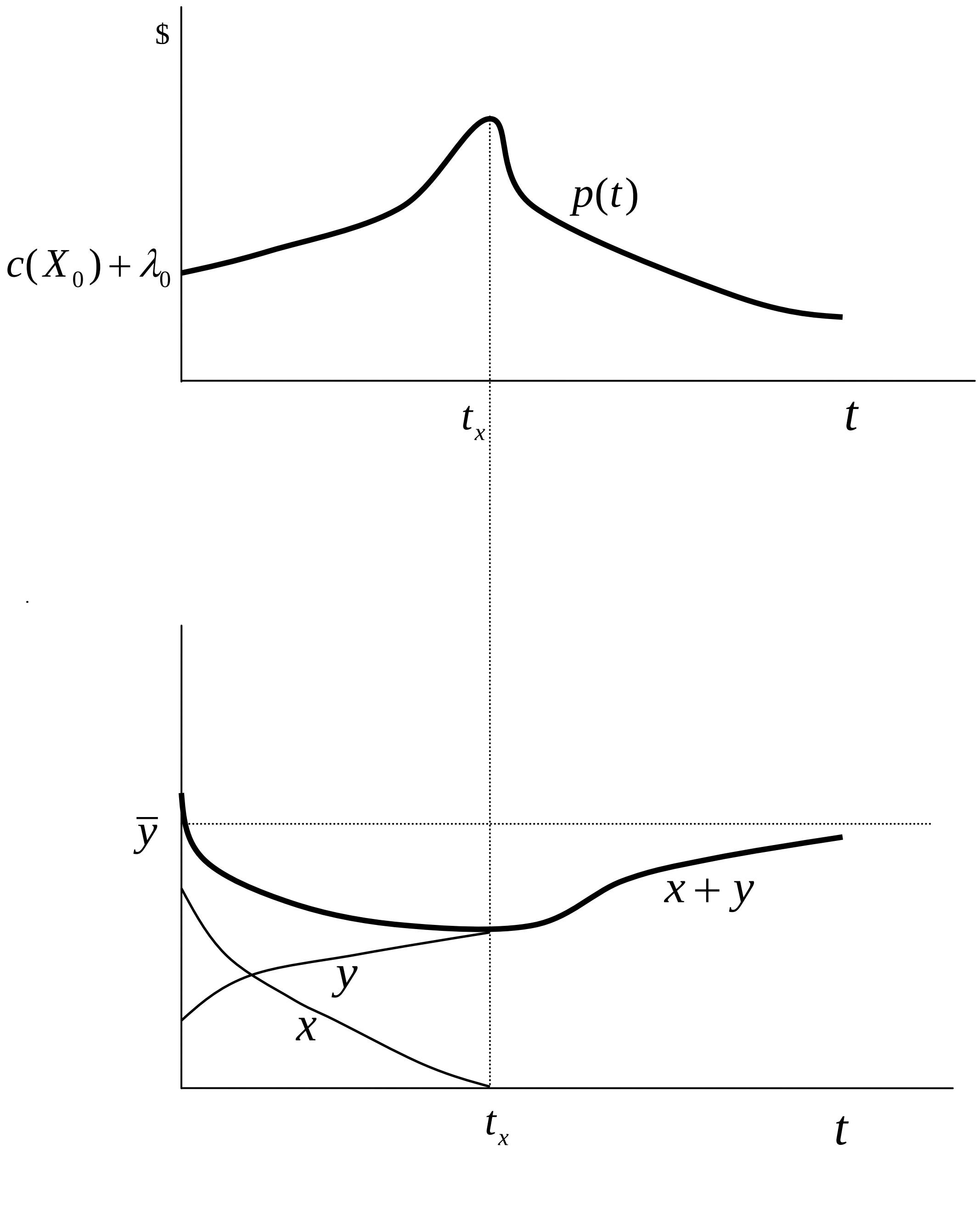

Fig. 1. Ceiling does not Bind: Solar is Economical from the Beginning

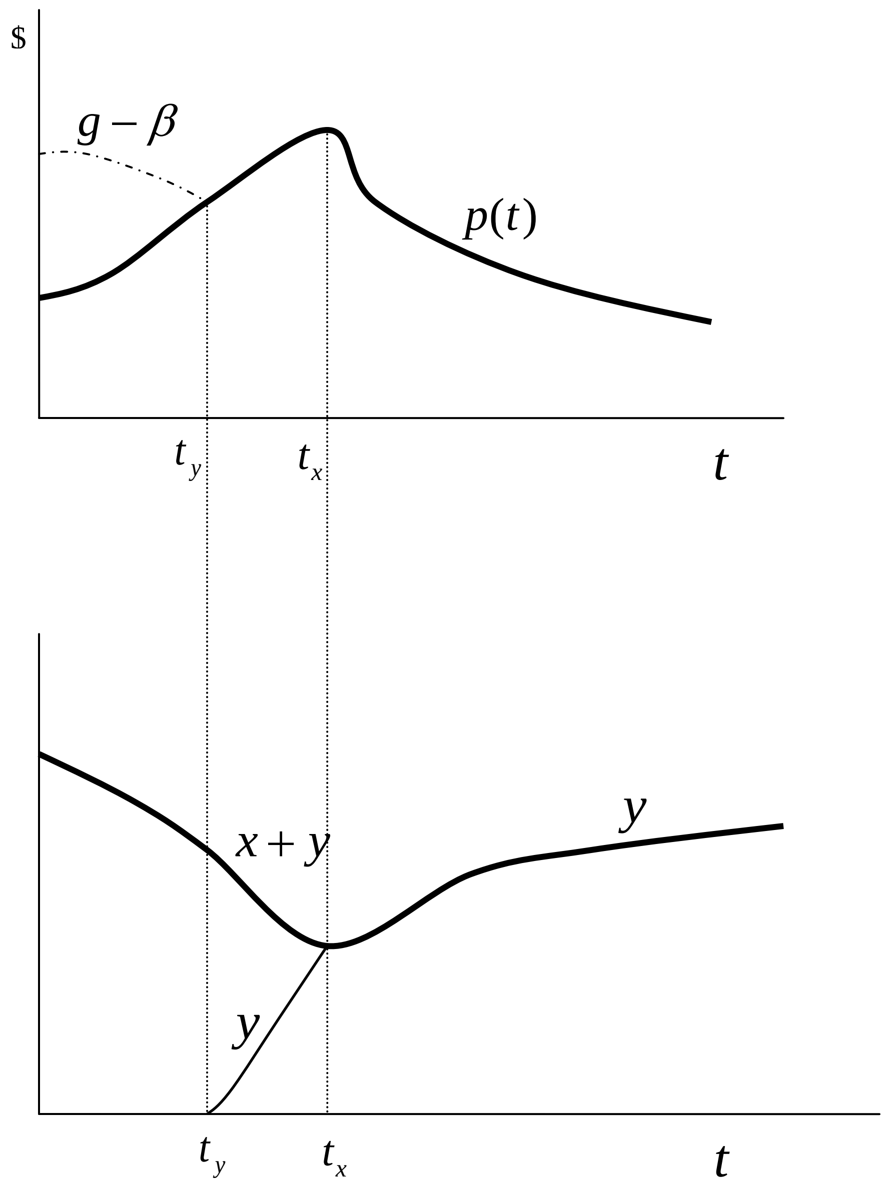

Fig. 2. Ceiling does not Bind: Only Coal is used at the Beginning

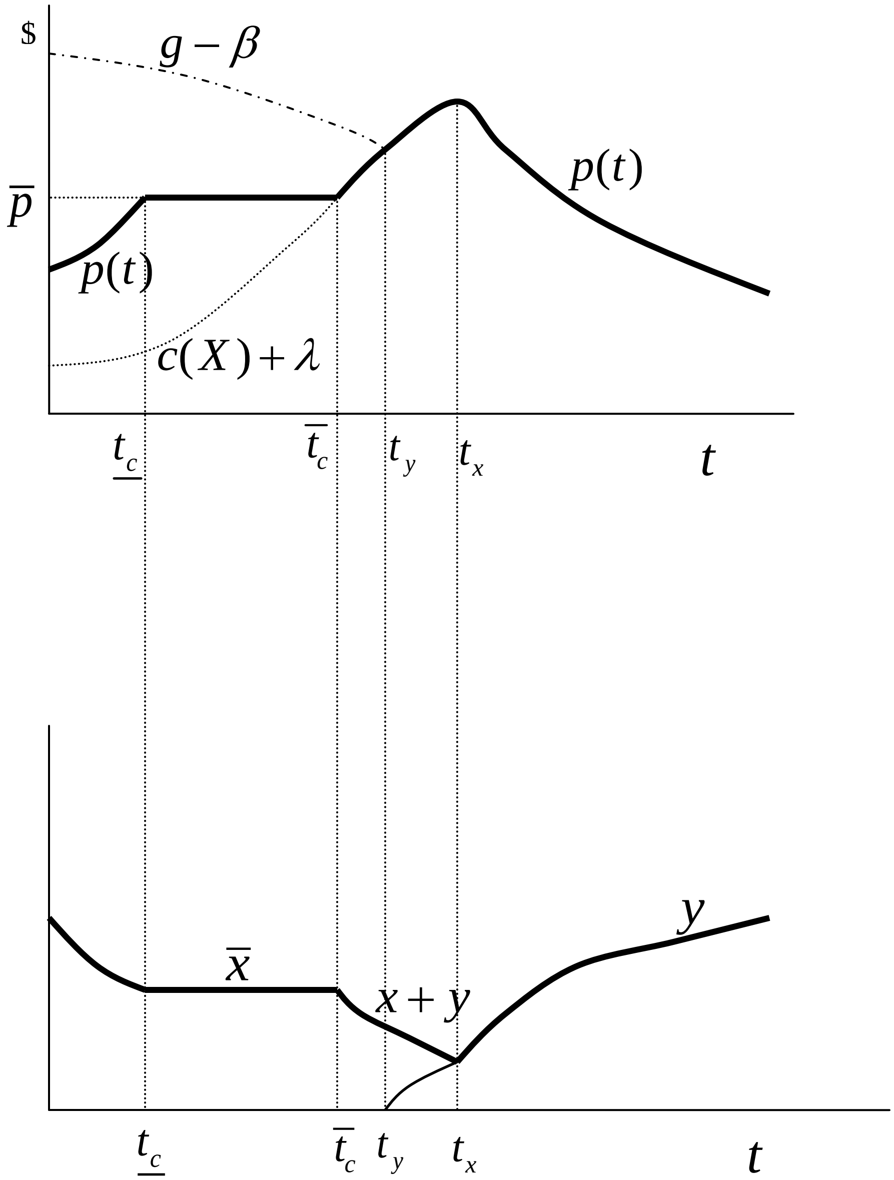

Fig. 3. Solar Energy arrives after the Ceiling Period

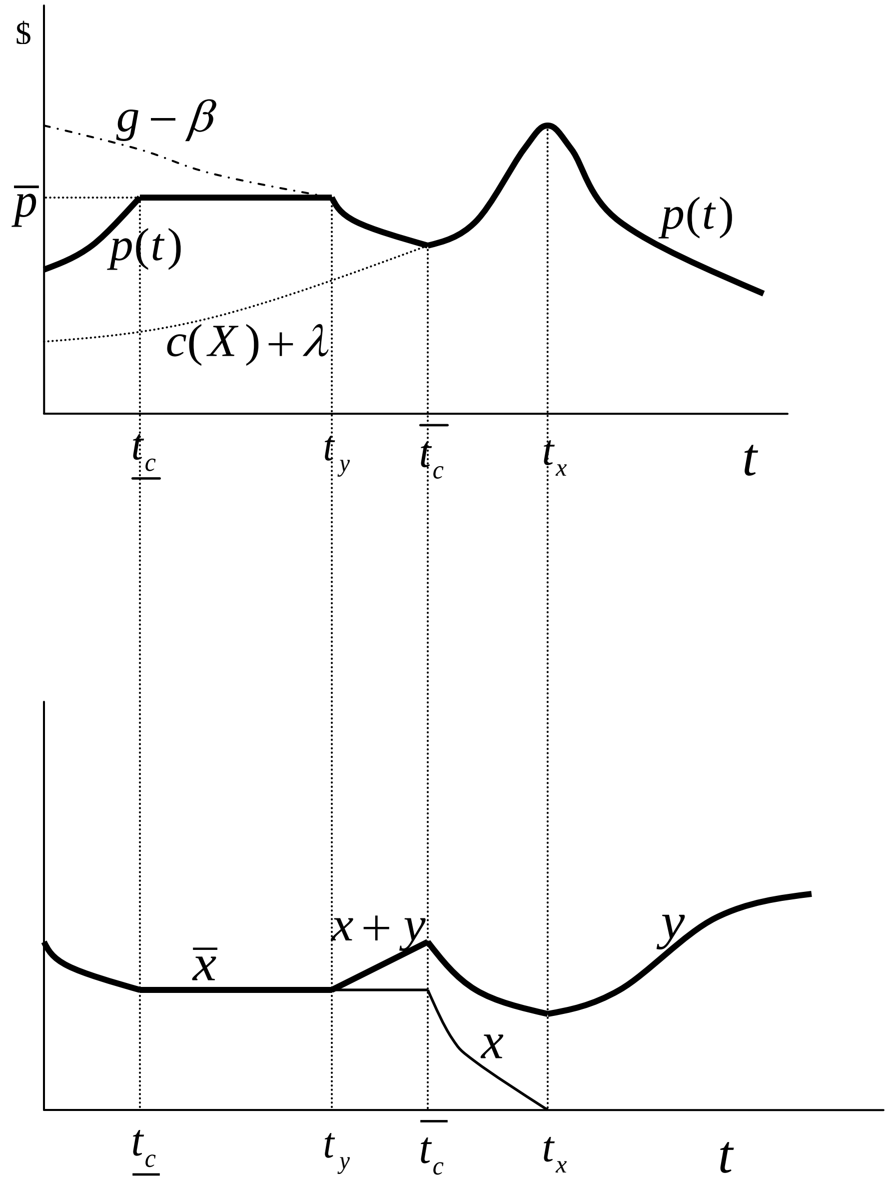

Fig. 4. Solar Energy arrives during the Ceiling Period

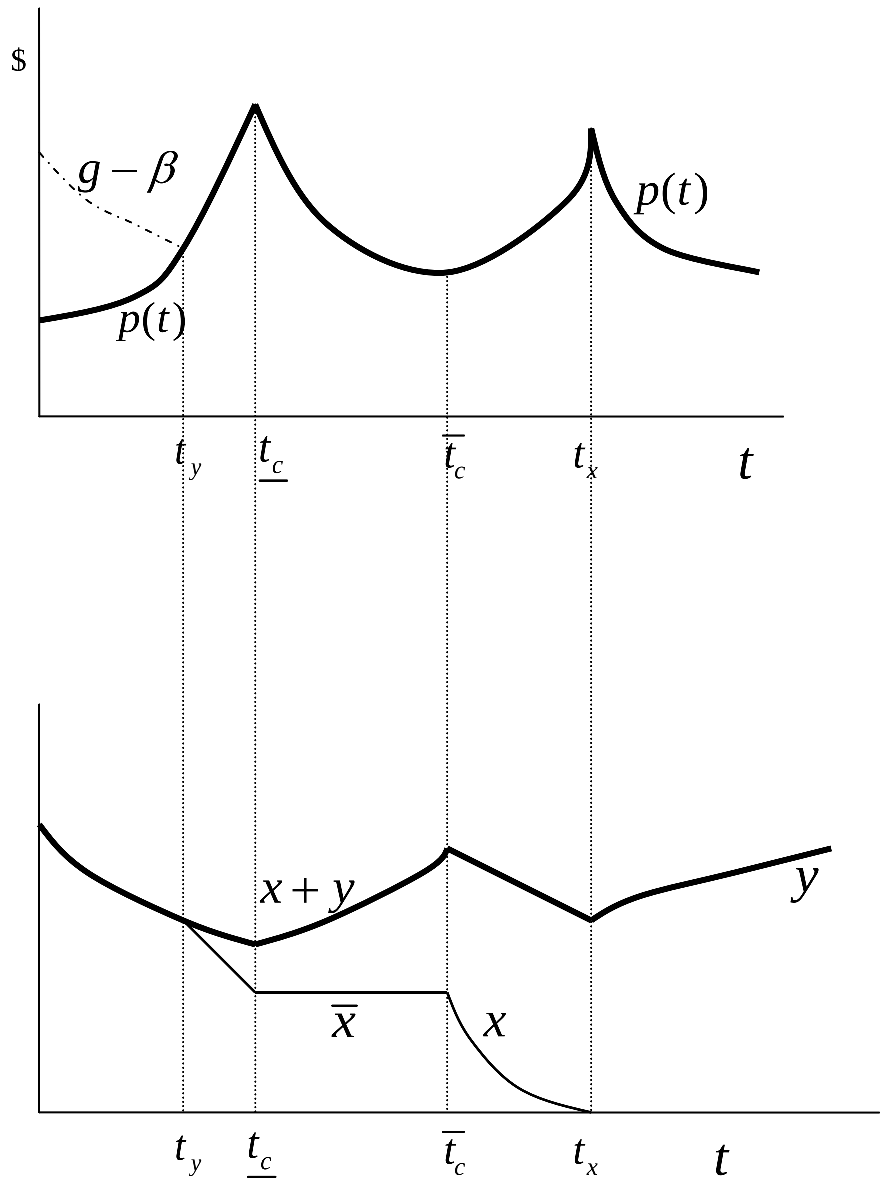

Fig. 5. Solar Energy comes before the Ceiling