Optimal expectations with complete markets

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(Incomplete version. Do not quote)

July 4, 2007

Abstract

Because agents have anticipatory feelings about future risks, it is optimal for them to manipulate their expectations. As in Brunnermeier and Parker (2003) and Gollier (2004), we examine the trade-off between the costs of erroneous decisions based on these manipulated beliefs and the benefits of anticipating a better future. In this paper, we assume that contingent markets are complete, with applications to portfolio choices, insurance and markets for lotteries. We show that agents will overestimate the probabilities of the good states, a form of optimism. Moreover, this bias in beliefs is approximately independent of the agent's degree of risk aversion.

Keywords: anticipatory feelings, portfolio choice, optimism, optimal expectations.

1 Introduction

Human beings seems to have a natural tendency to believe what makes them happier. Faced by an uncertain future, they tend to discard more easily information that would force them to revise their expectations downwards. This manipulation of beliefs is useful to extract more pleasure from anticipatory feelings about the likelihood of a reward, or to limit the disutility of a stress generated from a risk of loss. Religious beliefs yielding a reward after dead, the optimistic attitude towards lotteries and horsetrack bettings, or the limited efforts to prevent a health risk to occur, could illustrate this phenomenon. The problem is that distorting beliefs has a negative impact on the adequacy of risk management. Overoptimistic agents invest too much in risky assets, purchase too many lottery tickets, and retain too much of insurable risks. There is thus a trade-off in this manipulation of expectations between the desire to improve anticipatory feelings and the willingness to limit the adverse consequences of the inefficient risk management that this manipulation generates.

Brunnermeier and Parker (2005) and Gollier (2004) examined a portfolio choice problem with only one risky asset and one risk-free asset. They showed that, at least when the intensity of anticipatory feelings is not too large, risk-averse investors optimally manipulate their beliefs in an optimistic way, inducing them to purchase too much of the risky asset compared to what would maximize the objective expected utility. An important characteristic of optimal beliefs in this framework is that they set positive probability only to the worst and best possible returns of the risky asset.

In this paper, we consider the same basic framework, but with an alternative decision problem under uncertainty. When there are only two assets but more than two possible states of nature, markets are incomplete. We alternatively assume that contingent markets are complete. This is useful to examine a wide set of applications. For example, in a n-horse race, markets are usually complete in the sense that claims on any bet are priced. Can this model explain why risk-averse agents accept to gamble in spite of the unfair pricing of parimutuel contracts, or the well-documented favorite-longshot bias (Thaler and Ziemba (1988))? In the theory of finance, investors are often assumed to have access to option markets that complete the market for contingent claims. Thus, we may ask how anticipatory feelings influence the willingness to diversify risks, or to purchase portfolio insurance. In the literature of optimal insurance (Gollier (2000)), the policyholder is allowed to select any indemnity schedule contingent to the loss. If the premium is proportional to the actuarial value of the policy, Arrow (1971) has shown that the optimal insurance contract contains a straight deductible. We examine how the existence of anticipatory feelings affects thsi important result.

2 The classical portfolio choice under complete markets

Our model is static, with a decision date $t = 0$ and a consumption date $t = 1$. At date 0, the consumer selects an asset portfolio, but has no utility for consumption. The portfolio is liquidated at date 1, and the agent extracts utility from consuming its market value. The uncertainty at date 0 is characterized by the existence of S possible states of nature indexed by $s = 1, ..., S$ that could occur at date 1. There is an objective probability distribution $\pi = (\pi_1, ..., \pi_S) > 0$ that belongs to the simplex $\mathcal{S} = \{(\pi_1, ..., \pi_S) \in R^S \mid \pi_s \geq 0 \text{ for all } s \text{ and } \Sigma_s \widehat{\pi}_s = 1\}$ in R^S. The consumer is endowed with a state-contingent wealth $(\omega_1, ..., \omega_S) > 0$ at date 1. We assume that, at date 0, there is a complete set of markets for contingent claims. The Arrow-Debreu security associated to state s yields 1 unit of the single consumption good at date $t = 1$ if and only if state s occurs. The price of this asset at date 0 is denoted $p_s > 0$. The price-taking agent exchanges these securities and ends up with portfolio $C = (c_1, ..., c_S)$, where c_s represents the demand for asset s and the consumption level in state s. The portfolio choice problem is subject to the standard budget constraint

$$\sum_{s=1}^{S} p_s c_s = w =_{def} \sum_{s=1}^{S} p_s \omega_s, \tag{1}$$

where w denotes the wealth level of the agent. Among the different feasible portfolios, the agent can decide to purchase a risk-free portfolio with a state-independent demand $c = w/\Sigma_s p_s$ for each of Arrow-Debreu asset. Without loss of generality, we assume that the risk-free rate of the economy is zero, so that $\Sigma_s p_s = 1$. It means that the agent can secure a sure consumption $c = w$ at date $t = 1$.

The consumer has a von Neumann-Morgenstern utility function u that is assumed to be twice differentiable, increasing and strictly concave. We assume that the Inada conditions are satisfied, with $\lim_{c\to 0_+} u'(c) = +\infty$ and $\lim_{c\to\infty} u'(c) = 0$. Let $T(c) = -u'(c)/u''(c)$ denote the absolute risk tolerance of u measured at c. Without anticipatory feelings, the investor selects the portfolio $C(\pi)$ that maximizes the expected utility of final consumption:

$$C(\pi) = \arg \max_{(c_1,...,c_S)} \sum_{s=1}^{S} \pi_s u(c_s) \quad s.t. \quad \sum_{s=1}^{S} p_s c_s = w. \tag{2}$$

This standard Arrow-Debreu portfolio problem has a unique solution with the following first-order condition: for all $s = 1, ..., S$,

$$u'(c_s(\pi)) = \xi \frac{p_s}{\pi_s}. \tag{3}$$

The following well-known properties of the objectively optimal portfolio are easily derived from this condition.[1]

Lemma 1 *The optimal portfolio $C(\pi)$ in the absence of anticipatory feelings satisfies the following properties:*

1. *The demand for the contingent claim associated to state s depends upon s only through the state price per unit of probability $\pi_s = p_s/\pi_s : c_s = \phi(p_s/\pi_s)$;*

2. *This demand is inversely related to the state price per unit of probability: $\pi_{s'} \geq \pi_s$ implies $c_{s'} \leq c_s$;*

3. *The optimal portfolio risk is increasing in the investor's risk tolerance: $|\phi'(\pi)| = T(\phi(\pi))/\pi$;*

4. *Consider two agents respectively with utility functions u_1 and u_2 and with optimal demand functions ϕ_1 and ϕ_2. Agent 1 is more risk-averse than agent u_2 if and only if ϕ_1 crosses ϕ_2 from below, for all price structures of financial markets: $[\forall c : -u_1'(c)/u_1''(c) \leq -u_2'(c)/u_2''(c)] \Longleftrightarrow \forall(p_1, ..., p_S) \in \mathcal{S}$, $\exists \pi_0 \forall \pi : (\phi_1(\pi) - \phi_2(\pi))(\pi - \pi_0) \geq 0$.*

[1] See for example Gollier (2001, chapter 13).

The first property means that if there are two states with the same price per unit of probability, the demand for the corresponding two contingent assets must be the same. To illustrate, suppose that asset prices as actuarially fair in the sense that $p_s = \pi_s$ for all s. By property 1, investors facing this price structure should purchase a risk-free portfolio with $c_s = w$ for all s. Some risk is acceptable only if prices are not fair. In that case, it is optimal to consume more in states with a smaller state price per unit of probability (property 2). How much risk is desirable depends upon the agent's degree of risk tolerance. The portfolio riskiness can be measured by how the variability of state prices per unit of probability is transfered to the variability of contingent consumption, i.e., by $\phi'(\pi) = -T(c(\pi))/\pi$. Agents with a larger risk tolerance select a consumption plan that is more sensitive to differences in state prices per unit of probability (property 3). Only agents with a zero risk tolerance are willing to purchase a risk free portfolio when assets are not actuarially priced.

Notice moreover that, as observed by Dybvig (1988), any portfolio $C > 0$ satisfying properties 1 and 2 can be rationalized by an increasing and concave utility function u whose derivative would be defined by condition (3). Thus, the expected utility model can be tested only through the comonotonicity of vectors C and $(\pi_1, ..., \pi_S)$. Property 4 tells us that comparative risk aversion can be tested by determining whether the demand functions of the two agents cross only once.

To illustrate, consider an economy with $S = 10$ states of nature. Objectively, these states are equally likely to occur: $\pi_1 = ... = \pi_{10} = 0.1$. The state prices are $p_s = 0.045 + 0.01s$. For example, this means that the Arrow-Debreu security $s = 1$ yields a return of $100(1 - 0.055)/0.055 = 1718\%$ if state 1 occurs, and -100% otherwise. Its expected return equals 81.8%. The Arrow-Debreu security $s = 10$ yields a return of $100(1-0.145)/0.145 = 590\%$ only if state 10 occurs. Its expected return equals -31%. Suppose that the investor's utility function is $u(c) = c^{1-\gamma}/(1-\gamma)$, with constant relative risk aversion. In Figure 1, we depicted the demand for the 10 Arrow-Debreu securities when relative risk aversion is $\gamma = 2$ or $\gamma = 4$. The portfolio selected by the more risk-tolerant agent has a larger expected return (4.6% rather than 2.2%), but the standard deviation of returns is also larger (17% rather than 8%).

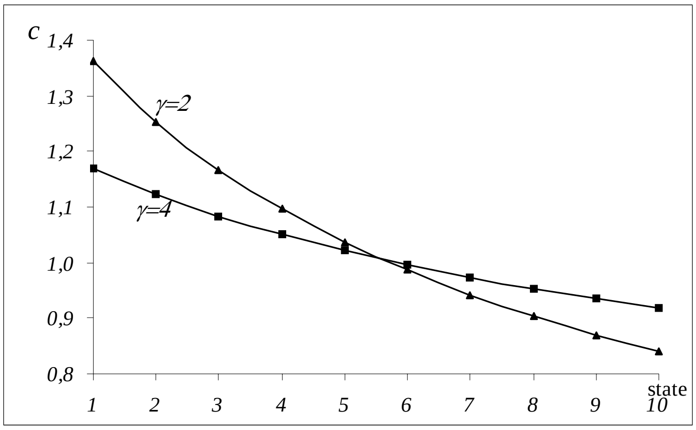

Figure 1: Demand for contingent claims without anticipatory feelings, respectively when relative risk aversion equals 2 (triangles) and 4 (squares).

3 A model of portfolio choice with anticipatory feelings

We now allow investors to use a subjective distribution of states when choosing their portfolio that differs from the objective probability distribution π. At decision date $t = 0$, the beliefs of the consumer are characterized by a subjective probability distribution $\widehat{\pi} = (\widehat{\pi}_1, ..., \widehat{\pi}_S)$ beloging to $\mathcal{S}$. Given these beliefs $\widehat{\pi}$, the consumer selects the portfolio that maximizes his subjective future expected utility on consumption. We obtain the following well-behaved decision problem:

$$S(\widehat{\pi}) = \max_{(c_1,...,c_S)} \quad \sum_{s=1}^{S} \widehat{\pi}_s u(c_s) \quad s.t. \quad \sum_{s=1}^{S} p_s c_s = w. \tag{4}$$

$S(\widehat{\pi})$ measures the felicity at date $t = 0$ generated by anticipatory feelings. The portfolio selected by the agent is denoted $C(\widehat{\pi}) = (c_1(\widehat{\pi}), ..., c_S(\widehat{\pi}))$. It satisfies the following set of necessary and sufficient conditions: for all $s = 1, ..., S$:

$$\widehat{\pi}_s u'(c_s) = \xi p_s, \tag{5}$$

where ξ is the Lagrange multiplier associated to (1).

Because of the potential bias in the subjective beliefs, the objective expected utility of the consumer at date 1 may differ from $S(\widehat{\pi})$. The objective expected utility of a consumer with subjective beliefs $\widehat{\pi}$ equals

$$O(\widehat{\pi}) = \sum_{s=1}^{S} \pi_s u(c_s(\widehat{\pi})). \tag{6}$$

The consumer's objective expected utility depends upon the subjective probability distribution $\widehat{\pi}$ only through the choice of the portfolio allocation induced by $\widehat{\pi}$.

We now specify the lifetime well-being of the consumer with subjective beliefs $\widehat{\pi}$. At date $t = 0$, the consumer savors his subjective future utility, yielding savoring felicity $S(\widehat{\pi})$ at that date. At date $t = 1$, the agent extracts felicity $O(\widehat{\pi})$ from consuming his terminal wealth. His lifetime well-being W is assumed to be a convex combination of his felicity at these two dates:

$$W(\widehat{\pi}) = kS(\widehat{\pi}) + (1-k)O(\widehat{\pi}). \tag{7}$$

As in Gollier (2004), parameter $k \in [0, 1[$ measures the intensity of anticipatory feelings in lifetime utility. When $k = 0$, the consumer has no anticipatory feeling at date 0. When k tends to unity, he extracts felicity just from savoring future consumption flows. Brunnermeier and Parker (2003) consider the special case with $k = 1/2$.

As justified in the introduction, we assume that prior to date $t = 0$, the agent controls his thoughts. He selects the beliefs $\widehat{\pi}$ that maximizes his lifetime well-being:

$$\widehat{\pi}^* = \arg\max_{\widehat{\pi} \in \mathcal{S}} W(\widehat{\pi}). \tag{8}$$

The optimal demand for the risky asset is $C^* = C(\widehat{\pi}^*)$. The main objective of the paper is to compare $\widehat{\pi}^*$ to π, and C^* to $C(\pi)$, the portfolio that is optimal under the objective beliefs.

4 The first-order condition for optimal expectations

In order to determine the optimal beliefs, it is important to know of a change in them disorts the structure of the portfolio selected by the agent. This question has raised much attention since the work by Rothschild and Stiglitz (1971) who have shown that a mean-preserving spread in the distribution of the return of the risky asset does not necessarily reduce the demand for it.[2] Abel (2002) described two very restrictive changes in distribution that yield an unambiguous increase in the demand for the risky asset for investors with constant relative risk aversion. Because this literature relies on an incomplete market framework with only two assets, these results cannot be used here. Up to our knowledge, we are the first to restate the problem under complete markets. The following proposition characterizes the change in the structure of the optimal portfolio due to a change in beliefs in that case.

Proposition 1 *Consider a change $d\widehat{\pi} = (d\widehat{\pi}_1, ..., d\widehat{\pi}_S)$ in the beliefs of the agent. The demand for the contingent claim associated to state s, $s = 1, ..., S$, is increased by*

$$dc_s = \sum_{t=1}^{S} \frac{\partial c_s}{\partial \widehat{\pi}_t} d\widehat{\pi}_t$$

with

$$\frac{\partial c_s}{\partial \widehat{\pi}_t} = \begin{cases} -\frac{p_t T(c_t)}{\widehat{\pi}_t \sum_{\tau=1}^{S} p_\tau T(c_\tau)} T(c_s) & if \quad t \neq s \\ -\frac{p_s T(c_s)}{\widehat{\pi}_s \sum_{\tau=1}^{S} p_\tau T(c_\tau)} T(c_s) + \frac{T(c_s)}{\widehat{\pi}_s} & if \quad t = s \end{cases} \tag{9}$$

where $T(c) = -u'(c)/u''(c)$ is the absolute risk tolerance of the agent.

The proof of this proposition is skipped. It is easily obtained by totally differentiating the system of equations (1,5) with respect to $\widehat{\pi}$. Observe that, as suggested by the intuition, an increase in the subjective probability $\widehat{\pi}_t$ associated to state t raises the demand for the contingent claim associated to this state, and it reduces the demand for all other contingent claims. These distortions are stronger in states where the risk tolerance of the agent is larger. Again, this is very intuitive: if the agent is very risk-averse ($T = 0$), the risk-free portfolio $c_s = w$ is optimal, and a change in the subjective beliefs has no effect on this portfolio.

We can now turn to the analysis of the choice of expectations. We assume that beliefs are selected to maximize a weighted average of respectively the

[2] Gollier (1995) provides a survey of this literature. He characterizes the changes in beliefs that guarantees that all risk-averse investors reduce their demand for the risky asset.

subjective and objective expected utilities:

$$W(\widehat{\pi}) = \sum_{t=1}^{S} (k\widehat{\pi}_s + (1-k)\pi_s) u(c_s(\widehat{\pi})).$$

The Inada conditions implies that the non-negativity constraints $\widehat{\pi}_s \geq 0$ are never binding. It implies that the first-order condition for program (8) can be written as

$$\frac{\partial W}{\partial \widehat{\pi}_t} = ku(c_t) + (1-k) \sum_{s=1}^{S} \pi_s u'(c_s) \frac{\partial c_s}{\partial \widehat{\pi}_t} - \psi = 0,$$

where ψ is the Lagrange multiplier associated to constraint $\Sigma_s \widehat{\pi}_s = 1$. Substituting $\partial c_s / \partial \widehat{\pi}_t$ by its expression given in (9) allows us to rewrite the above condition as

$$\widehat{\pi}_t \frac{\partial W}{\partial \widehat{\pi}_t} = k\widehat{\pi}_t u(c_t) + (1-k) u'(c_t) T(c_t) \left[\pi_t - \widehat{\pi}_t \frac{\sum_{s=1}^{S} \pi_s u'(c_s) T(c_s)}{\sum_{s=1}^{S} \widehat{\pi}_s u'(c_s) T(c_s)} \right] - \widehat{\pi}_t \psi = 0. \tag{10}$$

This condition must hold for all $t = 1, ..., S$. Taking the sum over all t yields

$$k \sum_{t=1}^{S} \widehat{\pi}_t u(c_t) = \psi. \tag{11}$$

Combining the above two conditions yields the following proposition.

Proposition 2 *The first-order condition for optimal expectations can be written as*

$$H_t(\widehat{\pi}^*) = k\widehat{\pi}_t^* \left(u(c_t) - S(\widehat{\pi}^*) \right) + (1-k) u'(c_t) T(c_t) \left[\pi_t - \widehat{\pi}_t^* \frac{\sum_{s=1}^{S} \pi_s u'(c_s) T(c_s)}{\sum_{s=1}^{S} \widehat{\pi}_s^* u'(c_s) T(c_s)} \right] = 0 \tag{12}$$

for $t = 1, ..., S$ and $c_s = c_s(\widehat{\pi}^*)$ for all s.

This system of equations has usually no analytical solution. A simple consequence of this proposition is presented in the following corollary. It states that, except in the case where the feelings-free agent select the risk-free position, it is always optimal to manipulate beliefs.

Corollary 1 *Suppose that k is positive. Suppose also that asset prices are not objectively actuarially fair in the sense that there exists at least one state t such that p_t is not equal to π_t . It implies that the objective distribution cannot be optimal: $\widehat{\pi}^* \neq \pi$.*

Proof: Suppose by contradiction that $\widehat{\pi}^* = \pi$. It would imply that the second term in the right-hand side of (12) is zero for all t. It would also imply that $c_s = c_s(\pi)$ which may not be state-independent, because state-prices are not objectively fair. But, by the Inada assumption, equation (12) can be satisfied only with $u(c_t) = S(\widehat{\pi}^*)$ for all t. This is possible only if consumption is state-independent, a contradiction. ■

5 The case of small anticipatory feelings

The benchmark case is when there is no anticipatory feelings ($k = 0$). Observe that even without any such feelings, the agent needs to form beliefs ex ante in order to choose his portfolio. Because $W(\widehat{\pi})$ equals $O(\widehat{\pi})$ when $k = 0$, the agent will select the beliefs that induce him to purchase the portfolio $C(\pi)$ that maximizes the objective expected utility. As a striking difference with the model of Brunnermeier and Parker (2003) and Gollier (2004), there is here only one subjective distribution $\widehat{\pi}$ that guarantees the absence of any bias in the portfolio choice. This is the objective distribution π. As shown in Proposition 1, any other subjective beliefs would yield a dominated portfolio. This is compatible with the observation that $\widehat{\pi}^* = \pi$ is a solution of the first-order conditions (12) when $k = 0$. Thus, when $k = 0$, the first-order condition (12) is necessary and sufficient. By a simple continuity argument, this condition is also necessary and sufficient when k is small enough.

In this section, we explore the properties of the optimal beliefs in the neighborhood of $k = 0$. To any degree k of anticipatory feelings, there is an optimal subjective distribution $\widehat{\pi}^*(k)$ and an optimal portfolio allocation $C^*(k) = C(\widehat{\pi}^*(k))$. By determining $d\widehat{\pi}^*/dk$ at $k = 0$, we will be able to evaluate the optimal beliefs when k is positive by using the following Taylor approximation:

$$\widehat{\pi}^*(k) = \pi + k \left. \frac{d\widehat{\pi}^*}{dk} \right|_{k=0} + o(k^2).$$

In addition to allowing us to derive simple properties of optimal beliefs, the assumption of a small anticipatory feelings is made here because of the difficulty to check the second-order condition otherwise. It is indeed noteworthy that the first-order condition (12) is not sufficient for optimal expectations, because the lifetime utility function W is generally not concave in the vector $\widehat{\pi}$ of subjective beliefs. Notice in particular that the subjective expected utility S is a convex function of $\widehat{\pi}$, since by definition (4) it is the upper envelope of linear functions of $\widehat{\pi}$. On the contrary, we have shown above that the objective expected utility function $O(\widehat{\pi})$ is well-behaved with a single maximum at $\widehat{\pi} = \pi$. By a standard continuity argument, condition (12) is necessary and sufficient when k is positive but small. It needs not be sufficient for larger intensities of anticipatory feelings.

In order to determine the derivative of $\widehat{\pi}_t^*$ with respect to k, we totally differentiate $H_t(\widehat{\pi}^*)$ defined by (12) with respect to k. Doing this at $k = 0$ makes life much easier because we know that $\widehat{\pi}^*(0) = \pi$. It implies that the bracketted term in (12) vanishes. Thus, we get

$$\left.\frac{dH_t(\widehat{\pi}^*)}{dk}\right|_{k=0} = \widehat{\pi}_t^* \left[u_t - O(\pi)\right] - u_t' T_t \frac{d}{dk}\left.\widehat{\pi}_t^* \frac{\sum_{s=1}^{S} \pi_s u'(c_s) T(c_s)}{\sum_{s=1}^{S} \widehat{\pi}_s^* u'(c_s) T(c_s)}\right|_{k=0} = 0,$$

where $u_t = u(c_t(\pi))$, $u_t' = u'(c_t(\pi))$ and $T_t = T(c_t(\pi))$. Using again that $\widehat{\pi}^*(0) = \pi$, this condition can be rewritten as

$$\left.\frac{1}{\widehat{\pi}_t^*}\frac{d\widehat{\pi}_t^*}{dk}\right|_{k=0} = \frac{u_t - O(\pi)}{u_t' T_t} + \frac{\sum_{s=1}^{S} \left.\frac{d\widehat{\pi}_s^*}{dk}\right|_{k=0} u_s' T_s}{\sum_{s=1}^{S} \widehat{\pi}_s^* u_s' T_s}. \tag{13}$$

This must be true for all $t = 1, ..., S$. This system of S equations has the following unique solution:

$$\left.\frac{1}{\widehat{\pi}_t^*}\frac{d\widehat{\pi}_t^*}{dk}\right|_{k=0} = \frac{u_t - \sum_{s=1}^{S} \pi_s u_s}{u_t' T_t} - \sum_{\tau=1}^{S} \pi_\tau \frac{u_\tau - \sum_{s=1}^{S} \pi_s u_s}{u_\tau' T_\tau}. \tag{14}$$

Turning to the impact of anticipatory feelings on the optimal portfolio, we have that

$$\left.\frac{dc_t^*}{dk}\right|_{k=0} = \sum_{s=1}^{S} \frac{\partial c_t}{\partial \widehat{\pi}_s} \left.\frac{d\widehat{\pi}_s^*}{dk}\right|_{k=0}.$$

Using conditions (9) and (14), this simplifies to

$$\left.\frac{dc_t^*}{dk}\right|_{k=0} = \frac{u_t - \sum_{s=1}^{S} \pi_s u_s}{u_t'}. \tag{15}$$

Finally, we derive from this equation that

$$\left.\frac{dO(\widehat{\pi}^*)}{dk}\right|_{k=0} = 0 \quad \text{and} \quad \left.\frac{dS(\widehat{\pi}^*)}{dk}\right|_{k=0} = cov\left(u, \frac{u - Eu}{u'T}\right).$$

We summarize the characteristics of the impact of marginal anticipatory feelings in the next proposition.

Proposition 3 *The optimal subjective beliefs are a function $\widehat{\pi}^*(k)$ of the intensity k of anticipatory feelings. When $k = 0$, it is optimal not to manipulate beliefs: $\widehat{\pi}^*(0) = \pi$. The optimal portfolio is $(c_1(\pi), ..., c_S(\pi)) = C(\pi)$ in that case. Everything else unchanged, the marginal sensitivities of the optimal portfolio and the optimal beliefs to anticipatory feelings are respectively equal to*

$$\left.\frac{dc_t^*}{dk}\right|_{k=0} = \Delta_t =_{def} \frac{u_t - \sum_{s=1}^{S} \pi_s u_s}{u_t'}, \tag{16}$$

and

$$\left.\frac{1}{\widehat{\pi}_t^*}\frac{d\widehat{\pi}_t^*}{dk}\right|_{k=0} = \frac{\Delta_t}{T_t} - \eta, \quad with\ \eta = \sum_{s=1}^{S} \pi_s \frac{\Delta_s}{T_s}, \tag{17}$$

for all $t = 1, ..., S$, where functions are evaluated for the optimal objective consumption plan $C(\pi)$. At the margin, the objective expected utility is not affected by these distortions, whereas the increase in the subjective expected utility is proportional to the objective covariance between u_t and Δ_t/T_t.

At the margin, the objective expected utility of the decision maker is not affected by the optimal manipulation of beliefs induced by anticipatory feelings. In other words, anticipatory feelings have only a second order effect on the objective welfare. On the contrary, anticipatory feelings have in general a first order effect on the asset allocation, the optimal subjective distribution of returns and the subjective expected utility. The only exception is when objective asset prices are actuarially fair: $p_s = \pi_s$ for all $s = 1, ..., S$. In that case, the feelings-free agent would fully insure against risk: $c_s(\pi) = w$ and $u_s = u(w)$ for all s. Because it implies that $\Delta_s = 0$ for all s, we conclude from the above proposition that full insurance remains the optimal portfolio strategy for agents with a limited intensity of anticipatory feelings. More generally, we obtain the following corollary.

Corollary 2 *Suppose that k is small and that there exist two states (t, t') with the same state price per unit of objective probability: $p_t/\pi_t = p_{t'}/\pi_{t'}$. The introduction of a small intensity of anticipatory feelings affects the demand for the two contingent claims together with their subjective log probabilities in the same way. It implies that agents with a small intensity of anticipatory feelings purchase the same quantity of the two corresponding Arrow-Debreu securities: $c_t^* = c_{t'}^*$.*

Proof: From (3), $p_t/\pi_t = p_{t'}/\pi_{t'}$ implies that $c_t = c_{t'} = c$. It implies in turn that $\Delta_t = \Delta_{t'} = \Delta$ and

$$c_s^* = c + k\Delta + o(k^2)$$

for $s = t$ or t'. This concludes the proof. ■

Notice that we also obtain from (17) that the percentage increase in $\widehat{\pi}_t^*$ and $\widehat{\pi}_{t'}^*$ from respectively π_t and $\pi_{t'}$ are equal at the margin. This corollary shows that property 1 in Lemma 1 is robust to the introduction of marginal anticipatory feelings. It has several applications. For example, it means that all risks that can be diversified at an objectively fair price will be diversified away in individual portfolios. Another application concerns the theory of optimal insurance. The main result (Arrow (1971)) in this theory states that, if transaction costs are proportional to the indemnity, the optimal insurance contract is full insurance above a straight deductible. The above corollary shows that this result is robust to the introduction of a small intensity of anticipatory feelings.

Suppose alternatively that the objectively optimal portfolio $C(\pi)$ is risky, because asset prices are not objectively fair. We hereafter determine how the riskiness of the selected portfolio is affected by anticipatory feelings. Let $\widehat{c}$ denote the objective certainty equivalent of the consumption plan $C(\pi)$ that is optimal under the objective beliefs. The following result is directly derived from Proposition 3.

Corollary 3 *Suppose that k is small and define the objective certainty equivalent $\widehat{c}$ by $u(\widehat{c}) = \Sigma_s \pi_s u(c_s(\pi))$. Define $\widehat{\pi}$ as the state price per unit of probability such that $\widehat{c} = \phi(\widehat{\pi})$. The optimal portfolio strategy has the following two properties:*

1. *Anticipatory feelings raise the demand for all Arrow-Debreu securities whose demand is larger* $\widehat{c}$*, and they reduce the demand for all Arrow-Debreu securities whose demand is less than* $\widehat{c}$*. In this sense, they raise the riskiness of the portfolio selected by the investor.*

2. *The effect of anticipatory feelings on the demand for the contingent claim associated to state* t *is approximately proportional to the product of the investor's intensities of anticipatory feelings and of risk tolerance:*

$$c_t^* - c_t(\pi) = kT(\widehat{c})\frac{\widehat{\pi} - \pi_t}{\widehat{\pi}} + o(k^2) + o((\widehat{\pi} - \pi_t)^2). \qquad (18)$$

Proof: Define $\Delta(c) = (u(c) - u(\widehat{c}))/u'(c)$. Observe that $\Delta'(\widehat{c}) = 1$, which implies that function Δ satisfies the following single-crossing property: $(c - \widehat{c})\Delta(c) \geq 0$. Combining this observation with (16) proves the single crossing property 1. Define function g such that $g(\pi) = \Delta(\phi(\pi))$. Because $\phi(\widehat{\pi}) = \widehat{c}$, $\Delta'(\widehat{c}) = 1$ and $\phi'(\pi) = -T(c)/\pi$ as stated in Lemma 1, we have that $g'(\widehat{\pi}) = -T(\widehat{\pi})/\widehat{\pi}$. Because $g(\widehat{\pi}) = 0$, we have that

$$\Delta_t = g(\pi_t) = T(\widehat{\pi})\frac{\widehat{\pi} - \pi_t}{\widehat{\pi}} + o((\widehat{\pi} - \pi_t)^2).$$

Combining this with condition (16) yields (18). ■

This single-crossing property 1 is a central result of this paper, since it shows that optimal expectations yield an excess of portfolio risk compared to what would be optimal based on the objective state probability distribution. The optimal manipulation of beliefs affects the structure of the selected portfolio exactly as an exogeneous reduction in risk aversion. Both changes yield a single-crossing change in asset demand. To illustrate this result, consider again the numerical example presented at the end of section 2, with $S = 10$ states and constant relative risk aversion $\gamma = 4$. The squares in Figure 2 correpond to the optimal portfolio without any anticipatory feelings, whereas the triangles describe the optimal portfolio when $k = 0.2$. The increased demand for contingent claims for low states and the corresponding reduction for high states yield a riskier portfolio. Compared to the objectively optimal portfolio, the objective expected return has been increased from 2.2% to 3.2%, and the objective standard deviation from 8.3% to 12%.

Property 2 in the above corollary tells us that more risk-tolerant agents have a demand for contingent claims that is more sensitive to the intensity

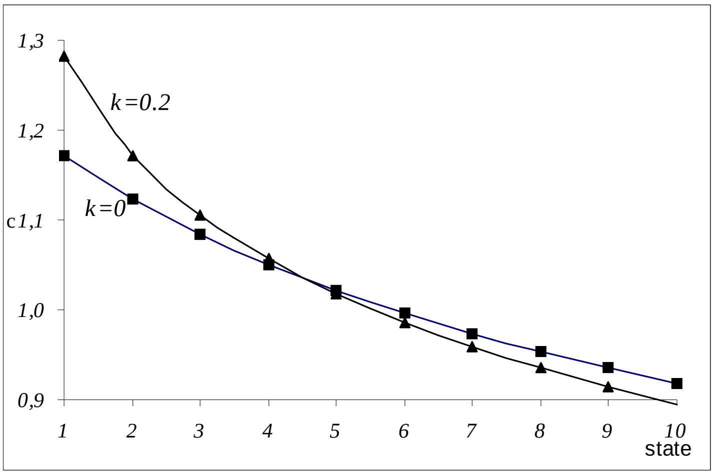

Figure 2: Effect of anticipatory feelings on the asset allocation, when relative risk aversion γ equals 4 and $k = 0$ (squares) or $k = 0.2$ (triangles).

of anticipatory feelings. More precisely, if agents 1 and 2 are such that agent 1 has twice the degree of risk tolerance of agent 2 and half of her intensity of anticipatory feelings, both agents will manipulate their beliefs in such a way to change their demand for Arrow-Debreu securities in the same way. This result holds only when the riskiness of the objectively optimal portfolio is small, i.e. when $c_s(\pi) - \widehat{c}$ are small. Indeed, this result has been obtained by a linear approximation for $\Delta(c) \simeq c - \widehat{c}$. Notice however that Δ is locally convex in c at $\widehat{c}$ since $\Delta''(\widehat{c}) = 1/T(\widehat{c}) > 0$. This means that the effect of anticipatory feelings on the demand for contingent claim s is larger than $k\,(c_s(\pi) - \widehat{c})$. In consequence, this effect is magnified for states where the demand is already large. This is apparent in Figure 2.

The effect of anticipatory feelings on subjective probabilities is quantified by equation (17). It tells us that anticipatory feelings yields a direct increase in $\log \widehat{\pi}_t^*$ by Δ_t/T_t, for $t = 1, ..., S$, together with an indirect reduction by $\Sigma_s \pi_s \Delta_s / T_s$ to guarantee that this changes preserve the identity $\Sigma_s \widehat{\pi}_s^* = 1$. Of course, the direct effect measured by Δ/T has the single-crossing property: it is positive when c is larger than $\widehat{c}$, and it is negative otherwise. The existence of anticipatory feelings induces the agent to raise the subjective probabilities associated to all states where consumption is larger than $\widehat{c}$, and to reduce the subjective probabilities of the other states. This can be defined as a concept of optimism. However, these changes in subjective probabilities do not necessarily sum up to zero. The second term denoted η in the right-hand side of (17) takes care of this, but it can potentially break down the single-crossing property for relative changes in probabilities. This is due to the fact that $h(c) = \Delta(c)/T(c)$ is in general not increasing in c. It is increasing only locally around $c = \widehat{c}$, an observation that yields optimal optimism for small risks, i.e., when the state prices per unit of objective probability are in a small neighborhood of their objective mean $\widehat{\pi} = 1$.

In order to guarantee that anticipatory feelings yield an increase in the probability of the good states and a reduction in the probability of the bad states, we would need that $h(c) = \Delta(c)/T(c)$ be increasing in c. In spite of the fact that it is not true in general, it is true when the utility function exhibits constant relative risk aversion, i.e., when $u(c) = c^{1-\gamma}/(1-\gamma)$. Indeed, in that special case, we have that

$$h'(c) = \frac{\partial}{\partial c}\left(\frac{u(c) - \sum_{s=1}^{S} \pi_s u_s}{u'(c)T(c)}\right) = \gamma c^{\gamma-2} \sum_{s=1}^{S} \pi_s c_s^{1-\gamma},$$

which is positive for all c.

Corollary 4 *Suppose that k is small. The optimal subjective beliefs have the following two properties:*

1. *The effect of anticipatory feelings on the subjective log probability associated to state t is approximately proportional to the agent's intensity of anticipatory feelings. At the margin, it is independent of the agent's degree of risk tolerance:*

$$\log \widehat{\pi}_t^* - \log \pi_t = k\frac{\widehat{\pi} - \pi_t}{\widehat{\pi}} + o(k^2) + o((\widehat{\pi} - \pi_t)^2). \tag{19}$$

2. *Suppose that $\widehat{\pi} - \pi_t$, $t = 1, ..., S$, are small, or that relative risk aversion is constant. Anticipatory feelings raise the subjective probability of states where the excess return of their portfolio is positive, and they reduce the subjective probability of states where this return is negative. In this sense, anticipatory feelings raises the agent's optimism.*

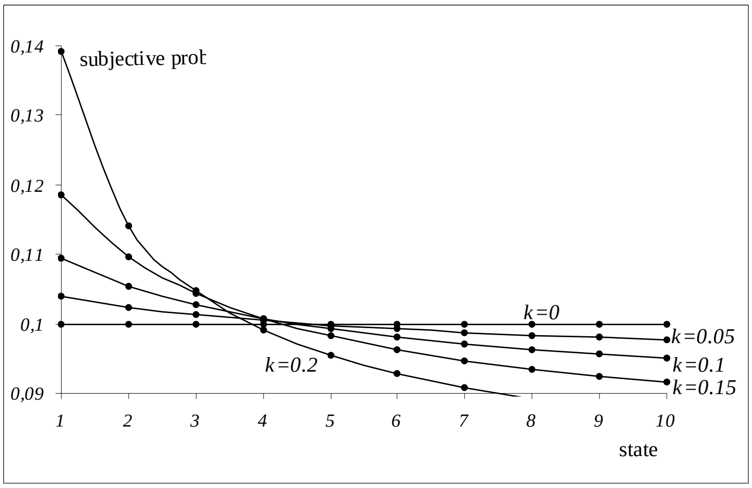

Figure 3: The optimal subjective probability of the $S = 10$ states when $\gamma = 4$, for different intensities of anticipatory feelings.

Proof: See the Appendix.

As suggested by the intuition, anticipatory feelings induce optimism. In order to feel better, investors overestimate the probabilities of states in which their selected portfolio has a positive return. To illustrate, let us go back once again to the numerical example introduced in section 2. In this example, the portfolio that is optimal for a feelings-free agent with constant relative risk aversion $\gamma = 4$ has an excess return that is positive for the low price states $s = 1$ to 4, and has a negative return for the high price states $s = 6$ to 10. In Figure 3, we draw the optimal subjective beliefs when k is between 0 and 0.20. We see that, as predicted by the above corollary, the agent with $k = 0.05$ overestimates the probabilities of states $s \in \{1, 2, 3, 4\}$, and he underestimates the probabilities of states $s \in \{6, 7, 8, 9, 10\}$. The optimal optimism is reinforced for investors with more powerful anticipatory feelings.

Property 2 in Corollary 4 tells us that the induced optimism is not much sensitive to the risk tolerance of the investor, as shown in Figure 4 when $k = 0.2$. For all states except $s = 1$, the difference between the optimal subjective state probabilities of the investor with $\gamma = 2$ and and the one with $\gamma = 6$ does not exceed 0.3%. However, these two agents differ much

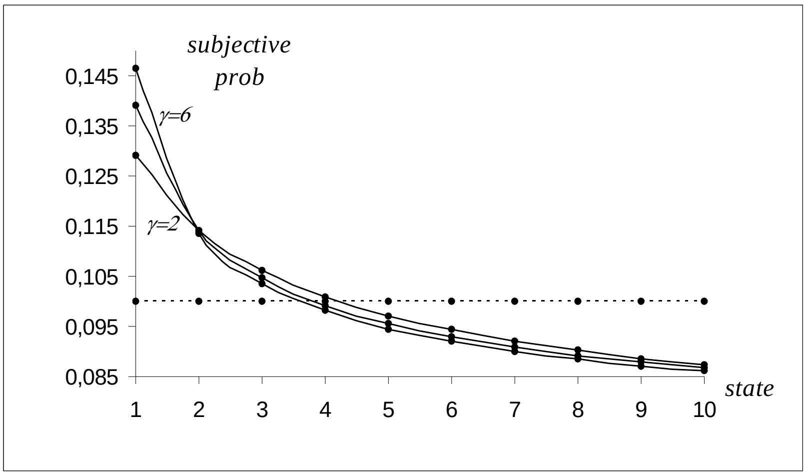

Figure 4: The optimal subjective probability of the $S = 10$ states when $k = 0.2$, for different degrees of relative risk aversion ($\gamma = 2, 4$ or 6).

about the subjective probability of the best state, with $\widehat{\pi}_1^* = 12.9\%$ or $\widehat{\pi}_1^* = 14.6\%$ respectively for $\gamma = 2$ and $\gamma = 6$.

6 Asset prices with anticipatory feelings

In this section, we examine an asset pricing model which is based on the portfolio choice problem presented above. We assume that the economy is composed of a large number N of identical consumers, indexed $i = 1, ..., N,$, all with the same utility function u, the same intensity of anticipatory feelings k and the same endowment of contingent claims $(\omega_1, ..., \omega_S)$. A competitive equilibrium in this economy is a set $\{(C^{i*}, \widehat{\pi}^{i*})_{i=1,...,N}, (p_1, ..., p_S)\}$ such that (1) for each i, given asset prices p and beliefs $\widehat{\pi}^{i*}$, agent $i's$ portfolio C^{i*} maximizes his subjective expected utility subject to his budget constraint; (2) beliefs $\widehat{\pi}^{i*}$ maximizes agent's i lifetime utility W^i; and (3) the S markets for contingent claims clear.

As in the previous section, we assume that k is small enough, so that, for any price vector $(p_1, ..., p_S) > 0$, the beliefs selection problem has a unique solution, which implies that the equilibrium must be symmetric. Therefore, we hereafter drop the index i from our notation. The symmetry of the equilibrium implies that the market-clearing condition is written as $c_t^* = \omega_t$ for all t. The optimality of this portfolio (condition (1)) requires in turn that

$$\widehat{\pi}_t^* u'(\omega_t) = \xi p_t, \tag{20}$$

for $t = 1, ..., S$. To keep our notation simple, we replace our normalizing assumption that prices sum up to unity by the alternative condition $\xi = 1$. Finally, the optimality of beliefs $\widehat{\pi}^*$ requires the S equations (12) to be satisfied. Using the market-clearing conditions, this is written as

$$k\widehat{\pi}_t^* \left[u(\omega_t) - \sum_{s=1}^{S} \widehat{\pi}_s^* u(\omega_s) \right] + (1-k)u'(\omega_t)T(\omega_t) \left[\pi_t - \widehat{\pi}_t^* \frac{\sum_{s=1}^{S} \pi_s u'(\omega_s)T(\omega_s)}{\sum_{s=1}^{S} \widehat{\pi}_s^* u'(\omega_s)T(\omega_s)} \right] = 0 \tag{21}$$

for $t = 1, ..., S$. Thus, finding the equilibrium asset prices requires first to solve this system of S equations for the S unknowns $(\widehat{\pi}_1^*, ..., \widehat{\pi}_S^*)$. In a second step, we use equation (20) to derive $(p_1, ..., p_S)$. The difficulty is that (21) is a system of polynomial equations of degree 3, yielding at most three possible equilibria. As in the previous section, we simplify this problem by limiting the analysis to small intensities of anticipatory feelings. From Proposition 3, we know that equation (21) implies that

$$\widehat{\pi}_t^* = \pi_t + k\pi_t \left[\frac{\Delta_t}{T_t} - \eta \right] + o(k^2), \tag{22}$$

where Δ_t is evaluated at ω_t. This yields the following result.

Proposition 4 *Suppose that the intensity of anticipatory feelings is small. Then, there exists a unique competitive equilibrium. It is symmetric. The equilibrium price vector satisfies the following condition: for* $t = 1, ..., S$, $p_t/\pi_t = b(\omega_t)$ *with*

$$b(\omega) = u'(\omega) \left\{ 1 + k \left[\frac{u(\omega) - u(\widehat{\omega})}{u'(\omega)T(\omega)} - \eta \right] \right\} + o(k^2), \tag{23}$$

where $\widehat{\omega}$ *is the certainty equivalent consumption defined by* $u(\widehat{\omega}) = \sum_{s=1}^{S} \pi_s u(\omega_s)$, *and* $\eta = \sum_{s=1}^{S} \pi_s (u(\omega_s) - u(\widehat{\omega}))/u'(\omega_s)T(\omega_s)$. *If the macroeconomic risk is* *small or if relative risk aversion is constant, anticipatory feelings raise (resp. reduce) the price of states where aggregate wealth is larger (resp. smaller) than some threshold $\overline{\omega}$ defined by $u(\overline{\omega}) - u(\widehat{\omega}) = \eta u'(\overline{\omega})T(\overline{\omega})$.*

The result specific to constant relative risk aversion comes from the fact that the fraction in the right-hand side is increasing in ω in that special case.

In order to discuss this proposition, observe first that the equilibrium state prices per unit of objective probability depend upon the state only through the aggregate wealth ω of the corresponding state: $p_t/\pi_t = b(\omega_t)$. This implies in particular that at equilibrium, diversifiable risks are actuarially priced. This property of the equilibrium is a consequence of the observation made earlier that investors diversify diversifiable risks in their portfolio. If there is no aggregate risk, equilibrium prices are objectively actuarially fair.

When there is an aggregate risk, the above proposition tells us that anticipatory feelings distort equilibrium prices by raising the price of good states, and by reducing the price of bad state. This is an intuitive consequence of the optimal optimism, which increases the demand for contingent claims associated to the good states. The theshold wealth level $\overline{\omega}$ corresponding to the state for which anticipatory feelings do not affect the equilibrium price is characterized in Proposition 4. The simplest case is when consumers have logarithmic utility functions. In that case, it is easy to check that $\eta = 0$, which implies $\overline{\omega} = \widehat{\omega}$. But η is not zero in general. Because it can be checked that η is positive when relative risk aversion is larger than unity,[3] the theshold $\overline{\omega}$ is larger than $\widehat{\omega}$ in that case.

The most important property of the price kernel p/π in the classical case is that it is monotone decreasing in ω. This property implies there is a positive equity premium on financial markets. Observe that the price kernel $b(\omega)$ needs not be decreasing in our model, because the bracketted term in the right-hand side of (23) is increasing in ω.

7 The case of large anticipatory feelings

Because by construction $S(\widehat{\pi})$ is convex in $\widehat{\pi}$, the problem of maximizing $W(\widehat{\pi}) = kS(\widehat{\pi}) + (1-k)O(\widehat{\pi})$ is well-behaved only when k is small. In this

[3] More generally, η is positive (resp. negative) when $u'''u'/u''^2$ is smaller (resp. larger) than 2.

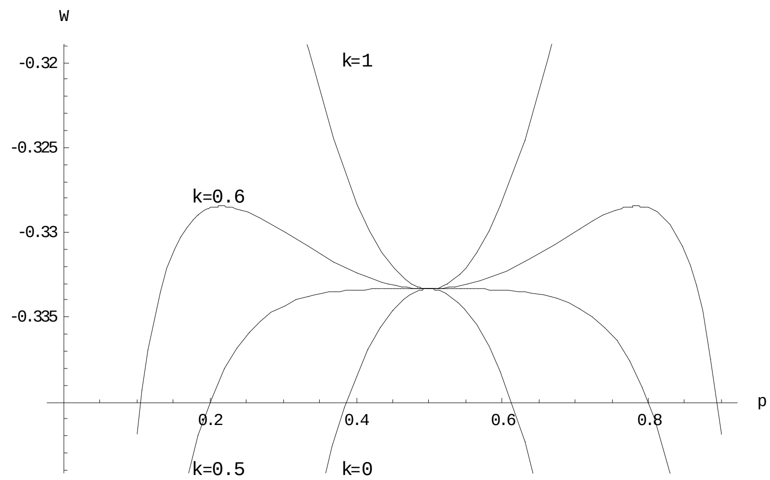

Figure 5: The lifetime well-being as a function of the subjective probability of the high return, for various intensities k of anticipatory feelings. Parameter values: $\gamma = 4$, $\Pi_1 = \Pi_2 = q_1 = q_2 = 1/2$.

section, we illustrate some features of the optimal assets allocation when k is not small. Let us consider the case of constant relative risk aversion $\gamma = 4$ and $w = 1$. We also assume that there are two states that are objectively equally likely: $\pi_1 = \pi_2 = 1/2$. Finally, we assume that the prices of the two contingent claims are actuarially fair: $p_1 = p_2 = 1/2$. From Corollary 2, when k is small enough, consumers fully insure risks and do not manipulate their beliefs. In Figure 5, we have drawn the lifetime well-being W as a function of the subjective probability $\widehat{\pi}$ of state 1, for various values of k. When k is smaller than or equal to 1/2, W is globally single-peaked and the optimal subjective probability is $\widehat{\pi}_1^* = \widehat{\pi}_2^* = 1/2$, implying that investing only in the riskfree asset is optimal as stated in Corollary 2.

When k is in $]1/2, 1[$, function W exhibits a concave-convex-concave shape, with two symmetric optimal beliefs. The optimal subjective probability that is larger than one-half is first constant and then increasing in k, as seen in Figure 6. Notice that the existence of two symmetric optima shows that providing zero-sum gambling opportunities can be helpful to improve welfare in an homogeneous economy of risk-averse agents. Consider an

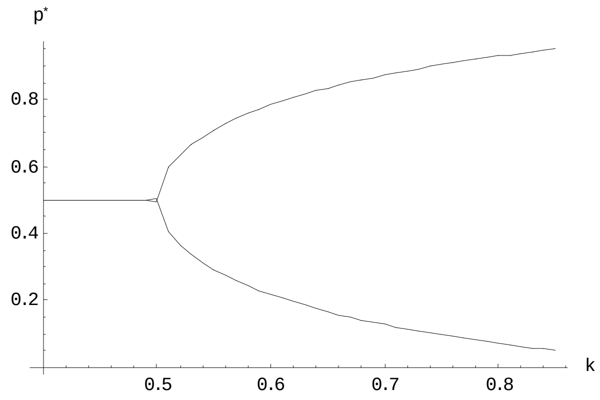

Figure 6: The optimal subjective probability p^* as a function of the intensity k of anticipatory feelings. Parameter values: $\gamma = 4$, $\Pi_1 = \Pi_2 = q_1 = q_2 = 1/2$.

economy with no aggregate risk. Suppose that agents with constant relative risk aversion $\gamma = 4$ and with an intensity $k = 0.6$ of anticipatory feelings are considering playing Heads-or-Tail game with a fair coin. In this economy, there is a competitive equilibrium with fair prices $p_1 = p_2 = 1/2$ where each agent puts 15.96% of initial wealth at stake by betting on either Heads or Tail. Half of the population bets on Heads, believing that the probability of Heads equals $\widehat{\pi}^* = 78,4\%$. The other half bets on Tail, believing that the probability of Tail is $1 - \widehat{\pi}^*$.

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Appendix: Proof of Corollary 4

We prove property 1. When $\widehat{\pi} - \pi_s$, $s = 1, ..., S$ are small, property 2 is a direct consequence of property 1. Property 2 with constant relative risk aversion is immediate from (17) and the fact that Δ/T is increasing. Consider a vector $\Theta = (\theta_1, ..., \theta_S)$ and a family of state prices indexed by ε, with $p_s = \pi_s(1 + \varepsilon\theta_s)$, or $\pi_s = 1 + \varepsilon\theta_s$, for all s. In order to preserve our assumption that the state prices sum up to unity, we impose that $\Sigma_s \pi_s \theta_s = 0$. Let $c_s(\varepsilon)$ and $\widehat{c}(\varepsilon)$ denote the optimal asset allocation and the certainty equivalent consumption as a function of ε. Define $\widehat{\pi}(\varepsilon)$ accordingly. Because this corollary focuses on situations where the heterogeneity of π_s is small, we examine the properties of these functions around $\varepsilon = 0$. Observe that when $\varepsilon = 0$, $c_s(0) = \widehat{c}(0) = w$ and $\widehat{\pi}(0) = 1$. Totally differentiating the first-order condition $u'(c_s) = \xi(1 + \varepsilon\theta_s)$ and using the budget constraint $\Sigma_s \pi_s \pi_s c_s = w$ yields

$$c_s'(0) = -T(w)\theta_s$$

for all s. Totally differentiating condition $u(\widehat{c}) = \Sigma_s \pi_s u(c_s)$ yields in turn that

$$\widehat{c}'(0) = \frac{\sum_{s=1}^{S} \pi_s u'(c_s(0)) c_s'(0)}{u'(\widehat{c}(0))} = -\frac{u'(w)T(w)\sum_{s=1}^{S} \pi_s \theta_s}{u'(w)} = 0.$$

It implies that $\widehat{\pi}'(0) = 0$ and that $\widehat{\pi}(\varepsilon) = 1 + o(\varepsilon^2)$.

Let us define function h_s as $h_s(\varepsilon) = [u(c_s(\varepsilon)) - u(\widehat{c}(\varepsilon))]/u'(c_s(\varepsilon))T(c_s(\varepsilon))$. From the computations above, it is easy to check that $h_s(0) = 0$ for all s, and

$$h_s'(0) = -\theta_s$$

for all s. In consequence, we have that

$$\left. \frac{d}{d\varepsilon} \sum_{s=1}^{S} \pi_s h_s(\varepsilon) \right|_{\varepsilon=0} = -\sum_{s=1}^{S} \pi_s \theta_s = 0.$$

It implies that equation (17) can be rewritten as

$$\begin{aligned} \left. \frac{1}{\widehat{\pi}_t^*} \frac{d\widehat{\pi}_t^*}{dk} \right|_{k=0} &= h_t(\varepsilon) - \textstyle\sum_{s=1}^{S} \pi_s h_s(\varepsilon) \\ &= \varepsilon\, [-\theta_t] + o(\varepsilon^2) \\ &= \frac{\widehat{\pi} - \pi_t}{\widehat{\pi}} + o((\widehat{\pi} - \pi_t)^2). \end{aligned}$$

Equation (19) is a simple rewriting of this condition. It implies the single crossing property 1 for $\log \widehat{\pi}_t^*$ with respect to $\log \pi_t$. ■