On the Foundations of Ex Post Incentive Compatible Mechanisms*

Takuro Yamashita† Shuguang Zhu‡

January, 2021

Abstract

In private-value auction environments, Chung and Ely (2007) establish maxmin and Bayesian foundations for dominant-strategy mechanisms. We first show that similar foundation results for ex post mechanisms hold true even with interdependent values if the interdependence is only *cardinal*. This includes, for example, the one-dimensional environments of Dasgupta and Maskin (2000) and Bergemann and Morris (2009b). Conversely, if the environment exhibits *ordinal* interdependence, which is typically the case with multi-dimensional environments (e.g., a player's private information

*We are grateful to Thomas Mariotti, Renato Gomes, Jiangtao Li, Pierre Boyer, Takakazu Honryo, Raphaël Levy, Andras Niedermayer, Chengsi Wang, Yi-Chun Chen, Tilman Börgers, Takashi Kunimoto, and seminar participants at Toulouse School of Economics, Mannheim (Center for Doctoral Studies in Economics), Hitotsubashi University, and Decentralization Conference (Michigan). Takuro Yamashita also gratefully acknowledges financial support from the European Research Council (Starting Grant #714693).

†Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. takuro.yamashita@tse-fr.eu

‡School of Economics, Shanghai University of Finance and Economics, Shanghai, P. R. China. zhu.shuguang@mail.shufe.edu.cn

comprises a noisy signal of the common value of the auctioned good and an idiosyncratic private-value parameter), then in general, ex post mechanisms do not have foundation. That is, there exists a non-ex-post mechanism that achieves strictly higher expected revenue than the optimal ex post mechanism, regardless of the agents' higher-order beliefs.

1 Introduction

The recent literature on mechanism design provides a series of studies on the robustness of mechanisms, motivated by the idea that a desirable mechanism should not rely too heavily on the agents' common knowledge structure.[1] One approach taken in the literature is to adopt stronger solution concepts that are insensitive to various common knowledge assumptions. For instance, in private-value environments, Segal (2003) studies dominant-strategy incentive compatible sales mechanisms. In interdependent-value environments, Dasgupta and Maskin (2000) study efficient auction rules that are independent of the details under the concept of ex post incentive compatibility.

However, a mechanism that achieves desired outcomes without the agents' common knowledge assumption does not immediately imply dominant-strategy or ex post incentive compatibility. In revenue maximization in private-value auction (under "regularity" conditions), Chung and Ely (2007) fill in this gap by establishing the *maxmin* and *Bayesian* foundation of the optimal dominant-strategy mechanism, in the following sense. Consider a situation where the seller in an auction (principal) only knows a joint distribution of the bidders' (agents) valua-

[1]See, for example, Wilson (1985).

tion profile for the auctioned object, which may be based on data about similar auctions in the past. On the other hand, he does not have reliable information about the bidders' beliefs about each other's value. For example, the bidders may have more or less information than the seller, or may simply have a "wrong" belief (from the seller's point of view) for various reasons. Thus, the seller's objective is to find a mechanism that achieves a good amount of revenue *regardless of the bidders' (higher-order) beliefs.* Note that, in a dominant-strategy mechanism, it is always an equilibrium for each bidder to report his true value, and therefore, it always guarantees the same level of expected revenue. On the other hand, in non-dominant-strategy mechanisms, expected revenue may vary with the bidders' (higher-order) beliefs. In the definition of Chung and Ely (2007), there is a maxmin foundation for a dominant-strategy mechanism if, for any non-dominant-strategy mechanism, there is a possible belief of the seller with which the dominant-strategy mechanism achieves (weakly) higher expected revenue than the non-dominant-strategy mechanism.[2]

In this paper, we examine the existence of such foundations for ex post incentive compatible mechanisms in interdependent-value environments. Our main observation is that the key property that guarantees such foundations is what we call the *cardinal* vs. *ordinal* interdependence. To explain these concepts, imagine an auction problem, where each bidder's willingness-to-pay depends both on his own type and the other bidders' types. If one type of each bidder always has a higher valuation for the good than another type *regardless of the other bidders'*

[2] As a stronger concept, if the same belief can be found for any non-dominant-strategy mechanism with which a dominant-strategy mechanism achieves (weakly) higher expected revenue, then there is a *Bayesian* foundation, because, as long as the seller is Bayesian rational and has that particular belief, he finds it optimal to offer a dominant-strategy mechanism, even though he can also offer any other mechanism.

types (even if each type's valuation itself may vary with the others' types), then we say that the environment exhibits only *cardinal* interdependence. Conversely, if the types cannot be ordered in such a uniform manner with respect to the others' types, then we say that the environment exhibits *ordinal* interdependence.[3]

We first show that, in the environments with only cardinal interdependence, (both maxmin and Bayesian) foundations exist for ex post mechanisms. This includes, for example, private-value environments (in this sense, our result is a generalization of Chung and Ely (2007)), and the one-dimensional environments of Dasgupta and Maskin (2000) and Bergemann and Morris (2009b).

Conversely, if the environment exhibits *ordinal* interdependence, which is typically the case with multi-dimensional environments (e.g., a player's private information comprises a noisy signal of the common value of the auctioned good and an idiosyncratic private-value parameter), then in general, ex post mechanisms do not have foundation. That is, there exists a non-ex-post mechanism that achieves strictly higher expected revenue than the optimal ex post mechanism, regardless of the agents' higher-order beliefs.

Regarding the foundation results, Chen and Li (2018) consider a general class of private-value environments where agents have multi-dimensional payoff types, and show that if the environment satisfies the *uniform-shortest-path-tree* property, then the maxmin (and Bayesian) foundation exists for dominant-strategy mechanisms. This property simply means that, for any allocation rule the principal

[3] These interdependence concepts are obviously related to the "size" of interdependence (e.g., private-value environments are special cases of cardinally interdependent cases). However, they are not necessarily corresponding to each other. For example, if a bidder's valuation in an auction is a sum of a function only of his own type and another function of the others' types, then however large is the second term, the environment never exhibits ordinal interdependence. In this sense, a more appropriate interpretation is that these interdependent concepts are related to the *diversity* of interdependence across types.

desires to implement, the set of binding constraints is invariant. This holds true in the single-good auction environment of Chung and Ely (2007) with regularity, and in this sense, their result generalizes that of Chung and Ely (2007), keeping the private-value assumption. Our work is a complement to Chen and Li (2018) in that we consider interdependent-value environments. For our foundation result (Theorem 1), a similar property to their uniform-shortest-path-tree property holds, which suggests that some of their argument may be applicable even in interdependent-value environments.

Regarding the no-foundation results, there are several papers in the literature that provide examples or a restrictive class of environments in which (various versions of) foundations for dominant-strategy or ex post mechanisms do not exist. For example, for interdependent-value environments, Bergemann and Morris (2005) provide examples in the context of implementation of certain ("non-separable") social choice correspondences, and Jehiel, Meyer-ter Vehn, Moldovanu, and Zame (2006) provide an example for revenue maximization in sequential sales. Chen and Li (2018) also provide an instance of environment where, without their uniform-shortest-path-tree property, there might not exist a foundation for dominant-strategy mechanisms, even in private-value environments. Our work contributes to this line of research by providing a class of environments with a no-foundation result (and sufficient conditions on the primitives in that environment), and the economic intuition based on the cardinal vs. ordinal interdependence.

Other closely related papers include Bergemann and Morris (2005) and Börgers (2017). In interdependent-value environments, Bergemann and Morris (2005) show that any *separable* social choice correspondence that is implementable given any (higher-order) belief structure of the agents must satisfy ex post incentive compatibility. In this sense, they provide another sort of foundation for ex post incentive compatible mechanisms. Their separable social choice correspondence necessarily admits a unique non-monetary allocation for each payoff-type profile, and hence, in general, excludes revenue maximization as the principal's objective. Thus, our work is complementary to theirs in that we consider revenue maximization.

Börgers (2017) criticizes the foundation theorems by constructing a non-dominant-strategy (or more generally, a non-ex-post) mechanism that yields *weakly* higher expected revenue than the optimal dominant-strategy mechanism for any belief structure of the agents, while it yields *strictly* higher expected revenue for some belief structures. Our no-foundation result is stronger in that it provides a *strict* improvement in expected revenue for *any* (higher-order) belief structure, though under stronger conditions on the environment.

One may wonder how the "generic constancy" result by Jehiel, Meyer-ter Vehn, Moldovanu, and Zame (2006) is related (if any) to our foundation or no-foundation results. Recall that, for their result, a crucial assumption is that each agent's relative valuation — the difference in valuation between any two alternatives — is everywhere strictly responsive to one's own signal. This is perhaps a reasonable assumption for example in a voting environment, but typically violated in "private-goods" environments such as in auction (see Bikhchandani (2006) more about this point). In this sense, we think that their result is basically orthogonal to ours.

2 Model

There is a finite set of risk-neutral agents, $\mathcal{I} = \{1, 2, \ldots, I\}$. Agent i's privately-known *payoff type* is $\theta_i \in \Theta_i \subseteq \mathbb{R}^d$, where we assume $|\Theta_i| = N$ for each i as in Chung and Ely (2007). A payoff-type profile is written as $\theta = (\theta_1, \ldots, \theta_I) \in \Theta_1 \times \ldots \times \Theta_I = \Theta$. The principal's (subjective) prior belief for θ is given by $f \in \Delta(\Theta)$, where we assume $f(\theta) > 0$ for all $\theta \in \Theta$.

Each agent i's willingness-to-pay for $q_i \in Q_i \subseteq \mathbb{R}_+$ units of the good is denoted by $v_i(q_i, \theta)$. We assume that $0 \in Q_i$, $|Q_i| < \infty$,[4] $v_i(0, \theta) = 0$, and $v_i(\cdot, \theta)$ is increasing for all θ. Moreover, as a standard single-crossing condition, we assume that:

Assumption 0. For each $\theta_i \neq \theta_i'$, and θ_{-i}, we have either

$$v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) > v_i(q_i, \theta_i', \theta_{-i}) - v_i(q_i', \theta_i', \theta_{-i}), \quad \forall q_i > q_i';$$

or

$$v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) < v_i(q_i, \theta_i', \theta_{-i}) - v_i(q_i', \theta_i', \theta_{-i}), \quad \forall q_i > q_i'.$$

In the first (second) case, we denote $\theta_i \succ_i^{\theta_{-i}} \theta_i'$ ($\theta_i \prec_i^{\theta_{-i}} \theta_i'$, respectively).

Clearly, the payoff environment in this paper includes the standard auction environment where $v_i(q_i, \theta_i, \theta_{-i}) = v_i(\theta_i, \theta_{-i})q_i$. Besides the preferences that are linear in q_i, Assumption 0 also holds for preferences satisfying $v_i(q_i, \theta_i, \theta_{-i}) = g_i(\theta_i, \theta_{-i})h_i(q_i) + H_i(q_i, \theta_{-i})$, where $h_i(q_i)$ is a monotonic non-negative function.

Our assumption throughout the paper is that $\prec_i^{\theta_{-i}}$ is a total ordering over Θ_i for any θ_{-i}, although $\prec_i^{\theta_{-i}}$ can be different from $\prec_i^{\theta_{-i}'}$. To see this, consider an auction environment in which each agent i's payoff-type comprises $(c_i, d_i) \in \Theta_i \subseteq \mathbb{R}^2$, where

[4] As it becomes clearer, the finiteness of Q_i is without loss of generality (though it simplifies the notation), given that Θ is finite and we only consider finite mechanisms (including ex post incentive compatible mechanisms).

c_i denotes a "common-value" component and d_i denotes an idiosyncratic "private-value" component, and his valuation for the good is $\pi_i(c_1, \dots, c_N) + d_i$ for some function π_i strictly increasing in all the arguments.[5] Then, for $(c_i, d_i), (c'_i, d'_i) \in \Theta_i$ such that $c_i < c'_i$ and $d_i > d'_i$, it is possible that, given some c_{-i}, (c_i, d_i) has a higher valuation for the good than (c'_i, d'_i) (i.e., $\pi_i(c_i, c_{-i}) + d_i > \pi_i(c'_i, c_{-i}) + d'_i$), while given another c'_{-i}, (c_i, d_i) has a lower valuation than (c'_i, d'_i). Such environments are said to exhibit *ordinal* interdependence.

Definition 1. We have *ordinal interdependence* if there exists i, θ_{-i}, and θ'_{-i} such that $\prec_i^{\theta_{-i}} \neq \prec_i^{\theta'_{-i}}$.

Paying $p_i \in \mathbb{R}$ to the principal, agent i's final payoff is $v_i(q_i, \theta) - p_i$. The principal's objective is the total revenue, $\sum_i p_i$. The feasible set of $q = (q_1, \dots, q_I)$ is denoted by $Q \subseteq \prod_i Q_i$, where the shape of Q depends on the specific environment of interest. For example, auctions, trading, and public-goods environments are in this class, with (or without) interdependence.

2.1 Type space

The agents' private information includes their own payoff types, their (first-order) beliefs about their payoff types, and their arbitrarily higher-order beliefs. To model this, we introduce type spaces as in Bergemann and Morris (2005).

A ("known-own-payoff-type") type space, denoted by $\mathcal{T} = (T_i, \widehat{\theta}_i, \widehat{\pi}_i)_{i=1}^I$, is a collection of a measurable space of types T_i for each agent i, a measurable function $\widehat{\theta}_i : T_i \to \Theta_i$ that describes the agent's payoff type, and a measurable function $\widehat{\pi}_i : T_i \to \Delta(T_{-i})$ that describes his belief about the others' types. Let $\widehat{\beta}_i(t_i)$

[5] See, for example, Example 5.1 in Jehiel, Meyer-ter Vehn, Moldovanu, and Zame (2006).

denote the belief hierarchy associated with type t_i (i.e., it describes t_i's first-order belief about θ_{-i}, second-order belief, and so on, up to an arbitrary higher order). We say that $\mathcal{T}$ has *no redundant types* if for each i, mapping $t_i \mapsto (\widehat{\theta}_i(t_i), \widehat{\beta}_i(t_i))$ is one-to-one.

In fact, there exists a (compact) *universal type space* $\mathcal{T}^* = (T_i^*, \widehat{\theta}_i^*, \widehat{\pi}_i^*)_{i=1}^I$, such that any type space without redundant types can be embedded into it, in the following sense.[6]

Lemma 1. Let $\mathcal{T}$ be a type space with no redundant types. Then, for each i, there exist subsets $\widehat{T}_i \subset T_i^*$ and bijections $h_i : T_i \to \widehat{T}_i$ such that:

1. $\widehat{\theta}_i^*(h_i(t_i)) = \widehat{\theta}_i(t_i)$ for all $t_i \in T_i$; and
2. $\iint_{t_i \in A, t_{-i} \in B} d\widehat{\pi}_i^*(h(t_i))[h_{-i}(t_{-i})] = \iint_{t_i \in A, t_{-i} \in B} d\widehat{\pi}_i(t_i)[t_{-i}]$ for all $A \subseteq T_i$ and $B \subseteq T_{-i}$.

where $h_{-i}(t_{-i}) = (h_1(t_1), \ldots, h_{i-1}(t_{i-1}), h_{i+1}(t_{i+1}), \ldots, h_I(t_I))$.

In what follows, we directly work with this universal type space.[7] Specifically, let $\mu \in \Delta(T^*)$ represent the principal's prior belief over T^* such that $\mu(\{t | \widehat{\theta}^*(t) = \theta\}) = f(\theta)$ for each θ, that is, the principal's (first-order) belief for θ is given by $f(\theta)$, as assumed above. The other information contained in μ captures the principal's belief over the agents' possible belief structures. Let $\mathcal{M} \subseteq \Delta(T^*)$ represent the set of all such μ.

In some contexts, it may be reasonable to assume that (the principal believes that) the agents do not have extreme (non-full-support) first-order beliefs. For

[6] For constructions of universal type spaces, see Mertens and Zamir (1985) and Brandenburger and Dekel (1993).

[7] The results would not change even if we allow for type spaces with redundant types, but the notation would be more involved.

example, instead of assuming that each agent's belief or knowledge is exogenous, one may be interested in a situation where each agent engages in his own information acquisition (through which his belief is updated), where the information acquisition cost is a linear function of relative entropy (Sims (2003)). Then, it is infinitely costly for each agent to know other agents' payoff types.

Formally, let $\mathcal{M}^{\text{full}} \subset \mathcal{M}$ denote the set of μ such that every agent i has a full-support first-order belief about the other agents. More precisely, for each agent i with type t_i, let $\widehat{\pi}_i^{*1}(t_i) \in \Delta(\Theta_{-i})$ denote his first-order belief, that is, $\widehat{\pi}_i^{*1}(t_i)[\theta_{-i}] = \int_{t_{-i}|\widehat{\theta}^*_{-i}(t_{-i})=\theta_{-i}} d\widehat{\pi}_i^*(t_i)[t_{-i}]$ for each θ_{-i}. Then, $\mathcal{M}^{\text{full}}$ is the set of all $\mu \in \mathcal{M}$ such that $\mu\left(\{t \mid \forall i, \theta_{-i},\ \widehat{\pi}_i^{*1}(t_i)[\theta_{-i}] > 0\}\right) = 1$.

2.2 Mechanism

The principal designs a mechanism, denoted by (M, q, p), where M_i represents a message set for each agent i, $M = M_1 \times \ldots \times M_I$, $q : M \to Q$ is an allocation rule, and $p : M \to \mathbb{R}^I$ is a payment function. Each agent i reports a message $m_i \in M_i$ simultaneously, and then he receives $q_i(m)$ units of the good and pays $p_i(m)$ to the principal. We assume that M_i contains a non-participation message $\emptyset \in M_i$ such that $\big(q_i(\emptyset, m_{-i}), p_i(\emptyset, m_{-i})\big) = (0, 0)$ for any $m_{-i} \in M_{-i}$.

The mechanism together with the universal type space $\mathcal{T}^*$ (subject to the principal's belief μ) constitute an incomplete information game. Let $\sigma^* = (\sigma_i^*)_{i \in \mathcal{I}}$ denote the corresponding equilibrium under certain solution concept, where $\sigma_i^* : T_i^* \to M_i$ is agent i's equilibrium strategy. The expected revenue of mechanism Γ

under the principal's belief μ is given by

$$R_\mu(\Gamma) = \int_{t \in T^*} \sum_i p_i(\sigma^*(t)) d\mu(t).$$

First, we define the Bayesian incentive compatible (hereafter BIC for short) mechanism. By revelation principle for Bayesian equilibrium (with $\mathcal{T}^*$), we restrict attention to direct mechanisms with truth-telling equilibrium.

Definition 2. A *BIC mechanism* is a mechanism $\Gamma = (M, q, p)$ such that, for each i, (i) $M_i = T_i^*$, and (ii) for each $t \in T^*$ and $t_i \neq t_i' \in T_i^*$:

$$\int_{t_{-i}} \Big(v_i(q_i(t), \widehat{\theta}^*(t)) - p_i(t) \Big) d\widehat{\pi}_i^*(t_i)[t_{-i}] \geq 0,$$
$$\int_{t_{-i}} \Big(v_i(q_i(t), \widehat{\theta}^*(t)) - p_i(t) \Big) d\widehat{\pi}_i^*(t_i)[t_{-i}] \geq \int_{t_{-i}} \Big(v_i(q_i(t_i', t_{-i}), \widehat{\theta}^*(t)) - p_i(t_i', t_{-i}) \Big) d\widehat{\pi}_i^*(t_i)[t_{-i}],$$

denoted by BIR_i^t and $BIC_i^{t \to t'}$, respectively.

We now introduce a class of mechanisms with ex post incentive compatibility (an EPIC mechanism for short).

Definition 3. An *EPIC mechanism* is a mechanism $\Gamma = (M, q, p)$ such that, for each i, (i) $M_i = \Theta_i$, and (ii) for each $\theta \in \Theta$ and $\theta_i \neq \theta_i' \in \Theta_i$:

$$\begin{aligned} v_i(q_i(\theta), \theta) - p_i(\theta) &\geq 0, \\ v_i(q_i(\theta), \theta) - p_i(\theta) &\geq v_i(q_i(\theta_i', \theta_{-i}), \theta) - p_i(\theta_i', \theta_{-i}), \end{aligned}$$

denoted by $EPIR_i^{\theta|\theta_{-i}}$ and $EPIC_i^{\theta \to \theta'|\theta_{-i}}$, respectively.

The expected revenue in the truth-telling (ex post) equilibrium in an EPIC mechanism is given by:

$$R_f(\Gamma) = \sum_{\theta} \sum_{i} p_i(\theta) f(\theta).$$

Note that this does not depend on μ, and in this sense, $R_f(\Gamma)$ may be interpreted as a "robustly guaranteed" expected revenue with respect to the agents' beliefs and higher-order beliefs. Let R_f^{EP} denote the maximum expected revenue among all EPIC mechanisms.

Applying the standard argument, the optimal mechanism among all EPIC mechanisms is characterized by the corresponding virtual-value maximization. To explain this, let $F_i(\theta_i \mid \theta_{-i}) = \sum_{\tilde{\theta}_i \preceq_i^{\theta_{-i}} \theta_i} f(\tilde{\theta}_i \mid \theta_{-i})$ denote the cumulative distribution function of i's payoff types conditional on the other agents' payoff-type profile being θ_{-i}.

Agent i's *virtual valuation* at payoff-type profile θ is given by:

$$\gamma_i(q_i, \theta) = v_i(q_i, \theta) - \frac{1 - F_i(\theta_i \mid \theta_{-i})}{f(\theta_i \mid \theta_{-i})} \big(v_i^+(q_i, \theta) - v_i(q_i, \theta)\big),$$

where $v_i^+(q_i, \theta_i, \theta_{-i}) = \min_{\tilde{\theta}_i \succ_i^{\theta_{-i}} \theta_i} v_i(q_i, \tilde{\theta}_i, \theta_{-i})$ whenever the right-hand side is well-defined; otherwise $\gamma_i(q_i, \theta) = v_i(q_i, \theta)$. Because $v_i(0, \theta) = 0$, we have $\gamma_i(0, \theta) = 0$.

The following result is standard, so we omit its proof.

Lemma 2.

$$\begin{aligned} R_f^{EP} \quad = \quad & \max_{q:\Theta \to Q} \quad \sum_{\theta} \sum_{i} \gamma_i(q_i(\theta), \theta) f(\theta) \\ & \text{s.t.} \quad \forall i, \theta_i, \theta_i', \theta_{-i} : \\ & \qquad \theta_i \succ_i^{\theta_{-i}} \theta_i' \;\Rightarrow\; q_i(\theta_i, \theta_{-i}) \geq q_i(\theta_i', \theta_{-i}). \qquad \text{(M)} \end{aligned}$$

We assume that the solution exists in this maximization problem, which we denote by $q^{EP} = (q_i^{EP}(\theta))_{i,\theta}$. The corresponding payment rule is denoted by $p^{EP} = (p_i^{EP}(\theta))_{i,\theta}$.[8]

Let $Q_i^+ = \{q_i \in Q_i \mid \exists \theta \in \Theta \text{ s.t. } q_i = q_i^{EP}(\theta)\}$ for each i (which is a finite subset since Θ has a finite number of elements). Define

$$\eta = \min_i \min_{\substack{\theta_i \neq \theta_i', \theta_{-i}, \\ q_i \neq q_i' \in Q_i^+}} |v_i(q_i, \theta_i, \theta_{-i}) + v_i(q_i', \theta_i', \theta_{-i}) - v_i(q_i', \theta_i, \theta_{-i}) - v_i(q_i, \theta_i', \theta_{-i})|,$$

then we have $\eta > 0$ due to Assumption 0 and the finiteness of $\mathcal{I}$, Θ and $(Q_i^+)_{i \in I}$. In particular, this implies that, by taking $0 = q_i' < q_i \in Q_i^+$, we have

$$|v_i(q_i, \theta_i, \theta_{-i}) - v_i(q_i, \theta_i', \theta_{-i})| \geq \eta$$

for all $\theta_i \neq \theta_i'$ and θ_{-i}.

The following notation is extensively used in the subsequent analysis. For each i and $q_i > 0$, define

$$\Theta_i^*(q_i, \theta_{-i}) \quad = \quad \{\theta_i \in \Theta_i | q_i^{EP}(\theta_i, \theta_{-i}) \geq q_i\}$$

as the set of i's payoff types whose allocation given θ_{-i} is greater than or equal to q_i in the optimal EPIC mechanism. Note that, by monotonicity, if $\theta_i \in \Theta_i^*(q_i, \theta_{-i})$ and

[8] p^{EP} is given as follows. For each i, θ_i and θ_{-i}, (i) if there is no $\theta_i' \prec_i^{\theta_{-i}} \theta_i$, then

$$p_i^{EP}(\theta) = v_i(q_i^{EP}(\theta), \theta);$$

(ii) otherwise, letting $\theta_i' \prec_i^{\theta_{-i}} \theta_i$ be such that no θ_i'' satisfies $\theta_i' \prec_i^{\theta_{-i}} \theta_i'' \prec_i^{\theta_{-i}} \theta_i$,

$$p_i^{EP}(\theta) = v_i(q_i^{EP}(\theta), \theta) - v_i(q_i^{EP}(\theta_i', \theta_{-i}), \theta) + p_i^{EP}(\theta_i', \theta_{-i}).$$

$\theta_i' \succ_i^{\theta_{-i}} \theta_i$, then $\theta_i' \in \Theta_i^*(q_i, \theta_{-i})$. Let $\theta_i^*(q_i, \theta_{-i})$ be the lowest element in $\Theta_i^*(q_i, \theta_{-i})$ with respect to $\prec_i^{\theta_{-i}}$, that is, for any $\theta_i \in \Theta_i^*(q_i, \theta_{-i})$, we have $\theta_i \succ_i^{\theta_{-i}} \theta_i^*(q_i, \theta_{-i})$. This $\theta_i^*(q_i, \theta_{-i})$ is called i's *threshold type with respect to* q_i *given* θ_{-i}. Finally, let

$$\Theta_{-i}^*(q_i, \theta_i) = \{\theta_{-i} \in \Theta_{-i} | \theta_i \in \Theta_i^*(q_i, \theta_{-i})\}$$

denote the set of θ_{-i} with which θ_i is allocated greater than or equal to q_i units in the optimal EPIC mechanism.

2.3 Foundations

For a non-EPIC mechanism, expected revenue may vary with the agents' belief structure, and the principal—who does not know the agents' belief structure—may not want to offer a mechanism if the expected revenue is low for some possible belief structures. Following Chung and Ely (2007), we say that there is a *maxmin foundation* for EPIC mechanisms if, for any non-EPIC mechanism $\Gamma = (M, q, p)$, there exists $\mu \in \mathcal{M}$ such that, for any Bayesian equilibrium σ^*, the expected revenue obtained in the equilibrium is less than R_f^{EP}, that is:

$$\int_{t \in T^*} \sum_i p_i(\sigma^*(t)) d\mu \leq R_f^{EP}.$$

If there exists a single $\mu \in \mathcal{M}$ that achieves the above inequality for all Γ, then we say that there is a *Bayesian foundation* for EPIC mechanisms.[9]

In the context where (the principal believes that) the agents have full-support

[9] These definitions are consistent with the verbal explanations of the corresponding definitions in Chung and Ely (2007). However, in fact, the mathematical definitions of them in Chung and Ely (2007) are slightly different: for example, their mathematical definition of maxmin foundation

first-order beliefs, we replace $\mathcal{M}$ by $\mathcal{M}^{\text{full}}$ in the above definitions, and we say that there is a *strong* maxmin / Bayesian foundation for EPIC mechanisms.

3 Without ordinal interdependence

First, we consider the case where, for each i, θ_{-i}, and θ'_{-i}, $\prec_i^{\theta_{-i}}=\prec_i^{\theta'_{-i}}$. This includes the private-value environment (as in Chung and Ely (2007)) as a special case, but also includes some interdependent-value environments. For example, assume that $\Theta_i \subseteq \mathbb{R}$ and $v_i(q_i, \theta_i, \theta_{-i})$ is an increasing function of θ_i for each given q_i, θ_{-i}. Because i's payoff is affected by θ_{-i}, the environment exhibits interdependence, but it is only *cardinal* interdependence in the sense that a higher value of θ_i corresponds to a higher type with respect to $\prec_i^{\theta_{-i}}$ for any θ_{-i}.

We further assume the following "regularity" condition in the same spirit as in Chung and Ely (2007).

Assumption 1. There exists $\varepsilon > 0$ such that, for any distribution over Θ, $\tilde{f}$, such that $\|\tilde{f} - f\| < \varepsilon$ (in a Euclidean distance), the monotonicity constraints (M) are

says that, for any non-EPIC mechanism $\Gamma = (M, q, p)$,

$$\inf_{\mu \in \mathcal{M}} \left[\max_{\sigma^*:\text{Bayesian equilibrium}} \int_{t \in T} \sum_i p_i(\sigma^*(t)) d\mu \right] \leq R_f^{EP}.$$

To see the difference, let $R(\mu)$ denote the term inside the bracket on the left-hand side (i.e., the expected revenue given μ), and imagine a case where (i) $R(\mu) > R_f^{EP}$ for any μ, while (ii) for any $\varepsilon > 0$, there exists μ such that $R(\mu) - \varepsilon < R_f^{EP}$. That is, the non-EPIC mechanism Γ is a *strict* improvement over the optimal EPIC mechanism, while it is not a *uniform* improvement. The verbal definition of Chung and Ely (2007) (which we follow in this paper) suggests that there is no maxmin foundation, while their mathematical definition says there is. The difference is not innocuous, because the non-EPIC mechanism we propose is indeed such a mechanism.

not binding in the problem of $R^{EP}_{\tilde{f}}$. In particular, this implies

$$R^{EP}_f = \max_{q:\Theta\to Q} \sum_\theta \sum_i \gamma_i(q_i(\theta),\theta) f(\theta).$$

Of course, the conditions on the environment that imply the above assumption can vary with the environment. For example, in an auction environment with $Q = \{q \in \{0,1\}^I | \sum_i q_i \le 1\}$, the regularity assumption is satisfied if, for any $i \in \mathcal{I}$, $j \in \{0, 1, \ldots, I\} \setminus \{i\}$, and θ, we have

$$\gamma_i(\theta) \ge \gamma_j(\theta) \Rightarrow \forall \theta'_i \succ_i \theta_i,\ \gamma_i(\theta'_i, \theta_{-i}) > \gamma_j(\theta'_i, \theta_{-i}), \tag{1}$$

where $\gamma_0(\theta) = 0$. In a *digital-good* environment[10] of Goldberg, Hartline, Karlin, Saks, and Wright (2006) with $Q = \{0,1\}^I$, the regularity assumption is satisfied under the strict monotone hazard rate condition, i.e., for each i and θ, $\frac{1-F_i(\theta_i|\theta_{-i})}{f(\theta_i|\theta_{-i})}$ is decreasing in θ_i. In a multi-unit sales environment as in Mussa and Rosen (1978), the regularity assumption is satisfied under the strict monotone hazard rate condition and concavity of each v_i with respect to q_i.

Remark 1. Chung and Ely (2007) call (1) the single-crossing condition in private-value environments. They show that if $\Theta_i = \{\theta_i^1, \ldots, \theta_i^M\}$ satisfying $\theta_i^m - \theta_i^{m-1} = \gamma > 0$ for each m, then condition-(1) is implied by the strict monotone hazard rate property, together with affiliation in f (which includes independent f as a special case). When there is no ordinal interdependence, we show in Appendix F.1 that

[10]The seller can replicate arbitrarily many copies of the digital goods with negligible costs. Moreover, each copy of the digital goods is completely identical with the original one. Examples of digital goods can be computer software, databases, blueprints, DNA sequences, visual images, music, recipes, ideas and so on. Since the seller is no longer subject to the feasibility constraints in digital-goods auction, she sells to an agent if and only if his virtual value is nonnegative.

condition-(1) is also satisfied in interdependent-value environment if we further impose three mild restrictions on valuation functions: v_i is increasing and concave in each argument, supermodular, and satisfies that an increase in an agent's payoff type has a larger effect on his own valuation function than on any other agent's. An example that satisfies these conditions is that, $v_i(\theta) = \theta_i + \alpha \sum_{j \neq i} \theta_j$ with $0 < \alpha < 1$, for all i.

Generalizing Chung and Ely (2007) (for private-value auction environments), we show that no ordinal interdependence implies the strong maxmin / Bayesian foundations for EPIC mechanisms.

Theorem 1. With Assumption 1 and no ordinal interdependence, EPIC mechanisms have the strong Bayesian (and hence strong maxmin) foundation.

Our proof for Theorem 1 is a direct extension of Chung and Ely (2007) in the private-value setting to the interdependent-value environment. We provide a sketch of the proof here, and the formal proof in the Appendix.

First, we impose the *non-singularity* condition on the payoff-type distribution f, which says that f satisfies certain full-rank conditions, and consider the Bayesian mechanism design problem with a simple type space having a particular belief structure. We show that under such a belief structure, it is without loss of generality to treat all participation constraints and all "adjacent downward" incentive constraints with equality, and ignore all the other constraints. Then we show that the total expected revenue in this Bayesian problem is maximized by the optimal EPIC mechanism.

The next step is to relax the non-singularity assumption by choosing a sequence of non-singular distributions which converge to the given payoff-type distribution.

Since the optimal EPIC mechanisms achieve the highest expected revenue over the sequence of simple type spaces with the particular belief structure, by taking the limit, we show that the Bayesian foundation also exists for any arbitrary payoff-type distribution, as long as Assumption 1 is satisfied.[11]

4 With ordinal interdependence

4.1 No strong foundations

We first illustrate by two examples how ordinal interdependence could undermine the foundations of using the optimal EPIC mechanism.

Example 1. Consider a two-agent digital-goods environment, where $I = 2$, $\Theta_1 = \Theta_2 = \{1, 2\}$, and $Q = \{0, 1\}^2$. We focus on agent 1 because the designer decides allocation rules for each agent separately in digital-goods environments. Table 1 collects payoff-type distribution f, agent 1's valuation and virtual value at each payoff type profile, and the corresponding optimal EPIC allocation for agent 1. For agent 2, assume that $v_2(\theta) = \theta_2 + 1$ for all θ so that the optimal EPIC allocation for him is $(q_2^{EP}(\theta), p_2^{EP}(\theta)) = (1, 2)$ for all θ.

Table 1: Auction environment of Example 1.

$f, v_1, \gamma_1, (q_1^{EP}, p_1^{EP})$	$\theta_2 = 1$	$\theta_2 = 2$
$\theta_1 = 1$	$\frac{1}{6}$, 2, 2, $(1,1)$	$\frac{1}{6}$, 1, -1, $(0,0)$
$\theta_1 = 2$	$\frac{1}{3}$, 1, $\frac{1}{2}$, $(1,1)$	$\frac{1}{3}$, 2, 2, $(1,2)$

We have $\Theta_1^*(q_1, \theta_2) = \{1, 2\}$ if $(q_1, \theta_2) = (1, 1)$ and $\Theta_1^*(q_1, \theta_2) = \{2\}$ if $(q_1, \theta_2) = (1, 2)$. Hence, the threshold payoff type of agent 1 given $\theta_2 = 1$ (i.e., $\theta_1 = 2$) is

[11] In Chung and Ely (2007), they show by example that, without the condition corresponding to Assumption 1, there may not exist a Bayesian foundation.

assigned the goods given $\theta_2 = 2$, but the non-threshold winning payoff type of agent 1 given $\theta_2 = 1$ (i.e., $\theta_1 = 1$) is unassigned given $\theta_2 = 2$. This reversal of the order over agent 1's payoff types is crucial for the no-foundation result.

Now we consider a modification of the optimal EPIC mechanism, which asks agent 1's first-order belief. More specifically, agent 1 is asked to report his payoff type θ_1 and his belief for $\theta_2 = 1$, that is:

$$y(t_1) := \widehat{\pi}_1^{*1}(t_1)[1] = \int_{t_2|\widehat{\theta}_2^*(t_2)=1} d\widehat{\pi}_1^*(t_1)[t_2].$$

If he reports $\theta_1 = 1$ and first-order belief $y \in [0,1]$, agent 1 obtains the goods by paying $(2 - \cos\tau)$ under $\theta_2 = 1$, but fails to get the goods and still needs to pay $(1 - \sin\tau)$ under $\theta_2 = 2$, where $\tau = \arctan\frac{1-y}{y}$. We keep the optimal EPIC allocations for both agents in the other cases. It is easy to verify that the new mechanism is Bayesian incentive compatible over the universal type space, since every type (of the universal type space) is assigned the optimal choice among the menu of all possible options.[12]

Because we are interested in the strong foundation, assume that (the principal believes that) agent 1 always has a full-support first-order belief, that is, $y \in (0,1)$ with (μ-)probability one. Then, agent 1 with $\theta_1 = 1$ always pays strictly more than 1 regardless of his (full-support) first-order belief and agent 2's true payoff type: if $\theta_2 = 1$, agent 1 pays $2 - \cos\tau$ for some $\tau \in (0, \frac{\pi}{2})$, which is strictly greater than 1; if $\theta_2 = 2$, agent 1 pays $1 - \sin\tau$ for some $\tau \in (0, \frac{\pi}{2})$, which is strictly greater than 0. Therefore, this new mechanism raises strictly higher expected

[12]More precisely, an option for agent 1 is denoted by $\big(q_1(1), p_1(1); q_1(2), p_1(2)\big)$, meaning that agent 1's allocation rule is $q_1(\theta_2)$, and agent 1's payment is $p_1(\theta_2)$, if agent 2 reports $\theta_2 = 1, 2$. Thus, agent 1 chooses from the menu $\big\{(1, 2 - \cos\tau; 0, 1 - \sin\tau) \mid \tau \in [0, \frac{\pi}{2}]\big\} \cup \{(1, 1; 1, 2)\}$.

revenue than the optimal EPIC mechanism, as long as agent 1 has a full-support first-order belief. □

In this example, the reason why we can increase the payment of $\theta_1 = 1$ without violating all constraints is because: in the optimal EPIC mechanism, $EPIC_1^{1\to 2|\theta_2}$ is satisfied with strict inequality at $\theta_2 = 2$, while $EPIR_1^{1|\theta_2}$ is satisfied with strict inequality at $\theta_2 = 1$. This is possible only when we have ordinal interdependence, so that different subset of constraints become binding given different θ_{-i}.[13] Thus, once agent 1's belief puts strictly positive probability on $\theta_2 = 1$ and 2, both $BIC_1^{1\to 2}$ and BIR_1^1 will become strictly slack in the optimal EPIC mechanism, leaving room for payment increase. Similar reasoning applies to the next example.

Example 2. Assume $I = 2$, $\Theta_1 = \{0, 1, 2\}$, $\Theta_2 = \{0, 1\}$ and $Q = \{0, 1\}^2$. Table 2 collects payoff-type distribution f, agent 1's valuation and virtual value at each payoff type profile, and the corresponding optimal EPIC allocation for agent 1. Clearly, agent 1's preference exhibits ordinal interdependence.

Table 2: Auction environment of Example 2.

$f, v_1, \gamma_1, (q_1^{EP}, p_1^{EP})$	$\theta_2 = 0$	$\theta_2 = 1$
$\theta_1 = 0$	$\frac{1}{6}$, 2, 1, $(1, 2)$	$\frac{1}{6}$, 0, -4, $(0, 0)$
$\theta_1 = 1$	$\frac{1}{6}$, 0, -4, $(0, 0)$	$\frac{1}{6}$, 2, 1, $(1, 2)$
$\theta_1 = 2$	$\frac{1}{6}$, 3, 3, $(1, 2)$	$\frac{1}{6}$, 3, 3, $(1, 2)$

We have $\Theta_1^*(q_1, \theta_2) = \{0, 2\}$ if $(q_1, \theta_2) = (1, 0)$ and $\Theta_1^*(q_1, \theta_2) = \{1, 2\}$ if $(q_1, \theta_2) = (1, 1)$. Hence, neither of these two sets is the subset of the other one, which never happens when we don't have ordinal interdependence. Now we construct a new mechanism as follows. When agent 1 reports $\theta_1 = 2$ and first-order

[13]Instead, if there is no ordinal interdependence, we will have the same ranking of payoff types, and thus the same subset of binding constraints, for all θ_{-i} in the optimal EPIC mechanism.

belief y (for $\theta_2 = 0$), agent 1 obtains the goods by paying $(3 - \cos\tau)$ under $\theta_2 = 0$ and obtains the goods by paying $(3 - \sin\tau)$ under $\theta_2 = 1$, where $\tau = \arg\tan\frac{1-y}{y}$. We keep the optimal EPIC mechanism for both agents in the other cases. As in Example 1, we can show that the new mechanism is Bayesian incentive compatible over the universal type space. Since we assume full-support beliefs, that is, $y \in (0,1)$, then the payment from agent 1 is always strictly greater than 2, the optimal EPIC payment rule, under both $\theta_2 = 0$ and $\theta_2 = 1$. Thus, the new mechanism raises strictly higher expected revenue than the optimal EPIC mechanism regardless of the designer's belief, resulting in no maxmin foundation for the EPIC mechanisms.[14] □

The two examples above identify some cases where revenue improvement is possible. The common feature of these two cases is that there exists a type of an agent whose BIC and BIR constraints (given any full-support belief of him about the other agents) are not binding. Motivated by them, we define the concept of *improvability* as follows.

Definition 4 ("Improvability")**.** Revenue from i is improvable with respect to $(\theta_i, \theta_{-i}, \theta'_{-i})$ if there exists q_i and q'_i such that at least one of the following holds:

(i) $\theta_i \in \Theta_i^*(q'_i, \theta'_{-i}) \cap \Theta_i^*(q_i, \theta_{-i})$, and $\theta_i^*(q_i, \theta_{-i}) \notin \Theta_i^*(q'_i, \theta'_{-i})$, and $\theta_i^*(q'_i, \theta'_{-i}) \notin \Theta_i^*(q_i, \theta_{-i})$;

(ii) $\theta_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q'_i, \theta'_{-i})$, and $\theta_i^*(q_i, \theta_{-i}) \in \Theta_i^*(q'_i, \theta'_{-i})$;

(iii) $\theta_i \in \Theta_i^*(q'_i, \theta'_{-i}) \setminus \Theta_i^*(q_i, \theta_{-i})$, and $\theta_i^*(q'_i, \theta'_{-i}) \in \Theta_i^*(q_i, \theta_{-i})$.

[14] In this example, $EPIC_1^{1\to 2|\theta_2}$ is binding at $\theta_2 = 1$, but is strictly slack at $\theta_2 = 2$; meanwhile $EPIC_1^{1\to 3|\theta_2}$ is binding at $\theta_2 = 2$, but is strictly slack at $\theta_2 = 1$. ($EPIR_1^{1|\theta_2}$ always holds with strict inequality, and thus is irrelevant.) Thus, as long as agent 1 has a full-support first-order belief, both $BIC_1^{1\to 2}$ and $BIC_1^{1\to 3}$ are not binding in the optimal EPIC mechanism.

These two examples essentially show that, given the optimal EPIC mechanism, if the revenue from some agent i is improvable with respect to some $(\theta_i, \theta_{-i}, \theta'_{-i})$, then the strong foundation does not exist. We summarize this result in Proposition 1, and provide in Appendix B the formal proof, which directly follows the ideas of Example 1 and 2.

Proposition 1. Under Assumption 0, improvability implies no strong foundation of EPIC mechanisms.

From Definition 4, ordinal interdependence is the necessary condition to have improvability. Particularly, in case (i) we have $\theta_i^*(q_i, \theta_{-i}) \prec_i^{\theta'_{-i}} \theta_i^*(q'_i, \theta'_{-i})$ and $\theta_i^*(q_i, \theta_{-i}) \succ_i^{\theta_{-i}} \theta_i^*(q'_i, \theta'_{-i})$; while in case (ii) we have $\theta_i \succ_i^{\theta_{-i}} \theta_i^*(q_i, \theta_{-i})$ and $\theta_i \prec_i^{\theta'_{-i}} \theta_i^*(q_i, \theta_{-i})$. As a symmetric case for case (ii), we have $\theta_i \succ_i^{\theta'_{-i}} \theta_i^*(q'_i, \theta'_{-i})$ and $\theta_i \prec_i^{\theta_{-i}} \theta_i^*(q'_i, \theta'_{-i})$ in case (iii). A natural question is when ordinal interdependence implies improvability, and hence no strong foundations of EPIC mechanisms. We further assume the following conditions.

Assumption 2 ("Highest Payoff Type")**.** For each i, there exists $\bar{\theta}_i \in \Theta_i$ such that, for each $\theta_i \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$, we have $\bar{\theta}_i \succ_i^{\theta_{-i}} \theta_i$.

Assumption 3. There exist θ_i, θ'_i, θ_{-i} and θ'_{-i} such that $q_i^{EP}(\theta'_i, \theta_{-i}) < q_i^{EP}(\theta_i, \theta_{-i})$ and $q_i^{EP}(\theta_i, \theta'_{-i}) < q_i^{EP}(\theta'_i, \theta'_{-i})$.

The highest-payoff-type assumption is satisfied if Θ is a complete sublattice in $\mathbb{R}^d$, and $v_i(q_i, \theta)$ is increasing in θ. Assumption 3 says we can find at least one pair of agent i's payoff types such that not only his preference over these two payoff types get reversed (that is, $\theta'_i \prec_i^{\theta_{-i}} \theta_i$ and $\theta'_i \succ_i^{\theta'_{-i}} \theta_i$), but also the ranking of the corresponding allocations varies at the same time. Basically Assumption 3 means that ordinal interdependence has an influence on the allocation rule in a nontrivial way.

Assumption 3 is not directly on the primitives, and hence one may wonder if it is easy to satisfy/check the assumption in any given environment. In the next subsection, we obtain sufficient conditions that are more directly on the primitives (or on the objects that easy to compute based on the primitives, such as the agents' virtual values).

Theorem 2. Under Assumptions 0, 2 and 3, EPIC mechanisms have no strong foundation.

Proof. We first show that Assumptions 2 and 3 jointly lead to improvability. Then the theorem follows from Proposition 1.

Let $q_i = q_i^{EP}(\theta_i, \theta_{-i})$ and $q_i' = q_i^{EP}(\theta_i', \theta_{-i}')$. By Assumption 3, we have

$$\begin{cases} \theta_i \in \Theta_i^*(q_i, \theta_{-i}), \quad \theta_i' \notin \Theta_i^*(q_i, \theta_{-i}); \\ \theta_i \notin \Theta_i^*(q_i', \theta_{-i}'), \quad \theta_i' \in \Theta_i^*(q_i', \theta_{-i}'). \end{cases}$$

By Assumption 2, there exists $\bar{\theta}_i \in \Theta_i$ such that, for any other $\theta_i \in \Theta_i$ and $\theta_{-i} \in \Theta_{-i}$, we have $\bar{\theta}_i \succ_i^{\theta_{-i}} \theta_i$. The monotonicity conditions on q^{EP} implies that $q_i^{EP}(\bar{\theta}_i, \theta_{-i}) \geq q_i^{EP}(\theta_i, \theta_{-i})$ for any $\theta_i \neq \bar{\theta}_i$. Thus, $\bar{\theta}_i \in \Theta_i^*(q_i, \theta_{-i})$. Similarly, we have $\bar{\theta}_i \in \Theta_i^*(q_i', \theta_{-i}')$. Thus, if we further have $\theta_i^*(q_i, \theta_{-i}) \notin \Theta_i^*(q_i', \theta_{-i}')$, and $\theta_i^*(q_i', \theta_{-i}') \notin \Theta_i^*(q_i, \theta_{-i})$, then we get improvability-(i).

If we have $\theta_i^*(q_i, \theta_{-i}) \in \Theta_i^*(q_i', \theta_{-i}')$, together with $\theta_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$ we get improvability-(ii). If we have $\theta_i^*(q_i', \theta_{-i}') \in \Theta_i^*(q_i, \theta_{-i})$, together with $\theta_i' \in \Theta_i^*(q_i', \theta_{-i}') \setminus \Theta_i^*(q_i, \theta_{-i})$ we get improvability-(iii). Therefore, improvability is always implied by Assumptions 2 and 3. □

We conclude this subsection with the following two remarks.

Remark 2. Although Theorem 2 assumes that each agent has a full-support first-order belief about the other agents ("$\mathcal{M}^{\text{full}}$"), and hence the result refers to no *strong* foundation, this full-support assumption can be omitted if we use an alternative definition of (no) foundation. Specifically, let us say that the EPIC mechanism has no foundation if there exists an alternative mechanism which (i) generates at least a weakly higher expected revenue given any belief hierarchy of the agents (even including the ones without full-support first-order beliefs), and (ii) generates a strictly higher expected revenue given some of them (see Börgers (2017)). Theorem 2 shows that our proposed mechanism in the proof achieves (ii), but it also achieves (i) as we will see in Section 4.3.

Remark 3. Observe that our improvement over the optimal EPIC mechanism is solely based on the property that each agent has a full-support first-order belief, regardless of whether that belief is "correct" or not. Indeed, Theorem 2 holds true even if we restrict attention to a subset of agents' beliefs which are consistent with a common prior: For example, imagine that the agents' payoff types follow a joint distribution f (i.e., the principal's prior belief), there exists an additional signal space $S = \prod_i S_i$, and a joint distribution over the payoff types and signals $g \in \Delta(\Theta \times S)$ such that $g(\cdot, S) = f(\cdot)$ (i.e., g's marginal on Θ coincides with f). This g is interpreted as a common prior in the sense that each agent's belief about the others is based on Bayes' updated belief given his payoff type θ_i and his signal $s_i \in S_i$. Imagine that the principal evaluates a mechanism based on the worst-case expected revenue among all (S, g) which satisfies the above conditions, and we say that the EPIC mechanism has a strong foundation if for any mechanism, there exists a full-support (S, g) with which the optimal EPIC mechanism is (weakly) better than that mechanism.[15] Then, Theorem 2 immediately shows that, under Assumptions 0, 2, and 3, the EPIC mechanism has no strong foundation.

In this sense, our main logic of improvement is different from the one in Börgers (2017), whose improvement is based on the mutually beneficial side-bets across agents with non-common priors.

4.2 A sufficient condition for Assumption 3

The goal of this subsection is to provide sufficient conditions more directly on the primitives with which Theorem 2 holds. Indeed, the following lemma provides a set of conditions on the virtual values[16] that imply Assumption 3. Together with Assumptions 0 and 2 (which are already directly on the primitives), Theorem 2 holds.

Lemma 3. Assumption 3 is satisfied if:

(i) If $\theta_i \prec_i^{\theta_{-i}} \theta'_i$, then we have $\gamma_i(q_i, \theta_i, \theta_{-i}) \leq \gamma_i(q_i, \theta'_i, \theta_{-i})$ for all q_i;

(ii) There exist i, θ_i, θ'_i, θ_{-i} and θ'_{-i} such that (a) for any q_i, $\gamma_i(q_i, \theta'_i, \theta_{-i})$, $\gamma_i(q_i, \theta_i, \theta'_{-i}) \leq 0$; (b) for some q_i, $\gamma_i(q_i, \theta_i, \theta_{-i}) > 0$; (b') for some q'_i, $\gamma_i(q'_i, \theta'_i, \theta'_{-i}) > 0$; (c) for any $j \neq i$, any $\tilde{\theta}_i$ and any q_j, we have $\gamma_j(q_j, \tilde{\theta}_i, \theta_{-i}) \leq 0$ and $\gamma_j(q_j, \tilde{\theta}_i, \theta'_{-i}) \leq 0$;

(iii) Q is a lower set (i.e., if $q \in Q$ and $q' \leq q$, then $q' \in Q$); and

(iv) for any i, θ, $\gamma_i(q_i, \theta)$ is strictly quasi-concave in q_i.

[15] Du (2018) and Brooks and Du (2019) studies the worst-case optimal auction mechanisms in this sense in (pure-)common-value environments.

[16] Recall that the virtual values are straightforwardly computed based on the primitives.

The proof is given in Appendix F.2. Condition-(i) is a standard monotonicity condition on virtual values, which guarantees that the seller never sells to an agent with negative virtual value. Condition-(ii) requires that ordinal interdependence affect the virtual values in a particular way. More specifically, $\theta_{-i}, \theta'_{-i}$ are $-i$'s types such that they themselves are not assigned (condition (c)), but important for i in the sense that i's virtual value can have opposite signs depending on θ_{-i} or θ'_{-i}. For example, one may imagine that $\theta_{-i}, \theta'_{-i}$ include important "common value" information (and hence i's virtual value crucially depends on it), but $-i$ have very low "private value" or "idiosyncratic" shocks so that their willingness to pay are small (and hence have negative virtual values). Condition-(iii) is satisfied in many environments of private-good assignment, such as in single-unit or multi-unit auction, bilateral trading, partnership dissolution, and so on. Condition-(iv) is trivially satisfied if v_i is linear in q_i (e.g., auction, bilateral trade). It is also typically satisfied in many multi-unit environment, where the condition, together with $\gamma_i(0, \theta) = 0$, essentially says: If $\gamma_i(q_i, \theta) > 0$ and $0 < q'_i < q_i$, then $\gamma_i(q'_i, \theta) > 0$. We provide a more concrete example of the conditions in the above lemma in a single-unit auction context.

Example 3. Consider a single-object auction with two agents and two states. Each agent i's payoff type is $\theta_i = (c_i, d_i^{(1)}, d_i^{(2)})$, and his valuation is $v_i = c_j d_i^{(1)} + (1 - c_j) d_i^{(2)}$, which depends on his own private-value component and agent j's common-value component.[17] For simplicity, assume that for $i = 1, 2$, we have

[17]Because v_i does not depend on c_i, any two payoff types with the same private-value component will induce the same valuation, thus Assumption 0 is violated in this example. However, by allowing each agent's valuation v_i to slightly depend on one's own common-value component c_i, Example 3 can easily satisfy Assumption 0.

It is worth noting that the modified mechanism in Proposition 1 also works in the current version of Example 3, even though Assumption 0 is violated. To see this, fixed arbitrary d_i and

$c_i \in \{\frac{1}{3}, \frac{2}{3}\}$, $d_i = (d_i^{(1)}, d_i^{(2)}) \in \{0,1\} \times \{0,1\}$ satisfying $(c_i, d_i^{(1)}, d_i^{(2)})$ are mutually independent and uniformly distributed. Given $c_j = \frac{2}{3}$, agent i's valuations for $d_i = (0,0), (0,1), (1,0), (1,1)$ are 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, respectively; and the corresponding virtual values are -1, $-\frac{1}{3}$, $\frac{1}{3}$ and 1. While given $c_j = \frac{1}{3}$, agent i's valuations for $d_i = (0,0), (0,1), (1,0), (1,1)$ are 0, $\frac{2}{3}$, $\frac{1}{3}$ and 1, respectively; and the corresponding virtual values are -1, $\frac{1}{3}$, $-\frac{1}{3}$ and 1. Immediately, the monotonicity condition on virtual values (condition-(i) of Lemma 3) is satisfied, and that $d_i = (0,0)$ is the worst private-value component for both agents (condition-(ii-c) of Lemma 3). Moreover, fixed any c_i, let $\bar{\theta}_i = (c_i, (1,1))$, $\theta_i = (c_i, (0,1))$, $\theta_i' = (c_i, (1,0))$, $\theta_j = (\frac{1}{3}, (0,0))$ and $\theta_j' = (\frac{2}{3}, (0,0))$, and we have $\gamma_i(\bar{\theta}_i, \theta_j) > \gamma_i(\theta_i, \theta_j) > 0 > \gamma_i(\theta_i', \theta_j)$, $\gamma_i(\bar{\theta}_i, \theta_j') > \gamma_i(\theta_i', \theta_j') > 0 > \gamma_i(\theta_i, \theta_j')$ (condition-(ii-a,b,b') of Lemma 3). By Lemma 3, we have $q_i^{EP}(\bar{\theta}_i, \theta_j) = q_i^{EP}(\bar{\theta}_i, \theta_j') = q_i^{EP}(\theta_i, \theta_j) = q_i^{EP}(\theta_i', \theta_j') = 1$, $q_i^{EP}(\theta_i, \theta_j') = q_i^{EP}(\theta_i', \theta_j) = 0$. Thus, revenue from agent i is improvable with respect to $(\bar{\theta}_i, \theta_j, \theta_j')$, and hence there is no strong foundation. □

4.3 No foundations

Next, we study if EPIC mechanisms have the (not necessarily strong) foundation. The following example suggests that the same mechanism as above does not generally work, if the agents have non-full-support first-order beliefs.

Example 4. In the new mechanism proposed in Example 1, if we allow for non-full-support beliefs, there exists a situation where agent 1 always correctly predicts agent 2's payoff types. Formally, let $C = \{t \in T^* | \widehat{\theta}^*(t) = (1,1), \widehat{\pi}_1^{*1}(t_1)[1] = 1\}$,

any $c_i \neq c_i'$, we have $v_i(q_i, \theta_i, \theta_{-i}) = v_i(q_i, \theta_i', \theta_{-i})$ for all q_i and all θ_{-i} where $\theta_i = (c_i, d_i)$ and $\theta_i' = (c_i', d_i)$. Due to incentive compatibility constraints, we only need to set the allocations (as well as the payments) for agent i at (θ_i, θ_{-i}) and (θ_i', θ_{-i}) to be the same.

$C' = \{t \in T^* | \widehat{\theta}^*(t) = (1,2), \widehat{\pi}_1^{*1}(t_1)[2] = 1\}$, and consider μ such that $\mu(C) = f(1,1)$ and $\mu(C') = f(1,2)$. Because the optimal choice for agent 1 is $\tau^* = 0$ (or reporting $y = 1$ as his belief for $\theta_2 = 1$) if $t \in C$, and $\tau^* = \frac{\pi}{2}$ (or reporting $y = 0$) if $t \in C'$, the equilibrium payments in the new mechanism coincide with those in the optimal EPIC mechanism. Thus, without the full-support belief assumption, the new mechanism in Example 1 only *weakly* improves the expected revenue.

Now we further modify the mechanism as follows. Unless agent 1 reports $\theta_1 = 1$ and $y = 0$, the allocation is the same as the previous mechanism proposed in Example 1. If agent 1 reports $\theta_1 = 1$ and $y = 0$, then the following events happen: agent 1 does not buy the good for any θ_2, he pays $M(> 3)$ if $\theta_2 = 1$ (i.e., when his belief turns out to be "wrong"), and the principal offers price 3 for agent 2 (so that agent 2 buys only if $\theta_2 = 2$, i.e., when agent 1's belief turns out to be "right"), instead of price 2. As before, the new mechanism is Bayesian incentive compatible on the universal type space $\mathcal{T}^*$.[18]

This new mechanism achieves a weakly higher expected revenue than in the optimal EPIC mechanism. First, this weak improvement is obvious unless $\theta_1 = 1$ and $y = 0$. If $\theta_1 = 1$ and $y = 0$, the principal earns $M > 3$ from agent 1 if $\theta_2 = 1$ (while the optimal EPIC mechanism yields total revenue 3), and earns 3 from agent 2 if $\theta_2 = 2$ (while the optimal EPIC mechanism yields total revenue 2).

To show a strict improvement in expected revenue for any $\mu \in \mathcal{M}$, consider the case where $\theta_1 = 1$ and $\theta_2 = 2$. Because $f(1,2) > 0$, it suffices to show that, for

[18]For agent 1, the only change is that his payment increases when agent 1 reports $\theta_1 = 1$ and $y = 0$, and agent 2 reports $\theta_2 = 1$. Thus, any other type of agent 1 won't pretend to have $\theta_1 = 1$ and $y = 0$; meanwhile, this change has no effect on agent 1 with $\theta_1 = 1$ and $y = 0$, since he deems the probability of having $\theta_2 = 1$ to be zero. As for agent 2, he is always offered a posted-price mechanism (depending on agent 1's report only), then his Bayesian incentive compatible constraints are satisfied.

any $y \in [0,1]$ reported by agent 1, the new mechanism achieves a strictly higher revenue than 2, the revenue in the optimal EPIC mechanism. First, as we see above, if $y = 0$ is reported, then the new mechanism yields 3 (from agent 2), and hence there is a strict improvement. If $y > 0$, then agent 2 pays 2, and agent 1 pays $1 - \sin(\arctan \frac{1-y}{y}) > 0$, and hence, there is again a strict improvement.

Notice that the key for strict improvement is to use agent 1's belief to modify the price for agent 2. If agent 1 is correct, such modification is profitable for the principal. Otherwise, the principal collects a "fine" from agent 1, which is also profitable. □

As suggested in the example, if an agent always correctly predicts the other agents' payoff types, we can use this agent's prediction to raise additional revenue from the other agents (and to fine him if his prediction turns out to be wrong in order for the principal to "hedge", as in the example above). Because this means that we need to be able to change an agent's allocation without changing the others' – more precisely, we reduce the allocation at the threshold payoff type in order to charge higher prices for non-threshold winning types – we assume that the feasible allocation set Q is a lower set, that is, if q is in Q and $q' \leq q$, then q' is also in Q. Obviously, the standard auction belongs to this class of environments.

In addition, even if an agent correctly predicts the occurrence of some θ_{-i} (or its non-occurrence), such information does not necessarily make the principal earn strictly more revenue from the other agents (for example, imagine that any $j(\neq i)$'s virtual valuation is negative given θ_{-i}). Thus, we need a stronger version of the improvability.

Definition 5. We have the *strong improvability* if there exist $i, j, \theta_i, \theta_j, q_j, \theta_{-ij}$

such that $\theta_j \in \Theta^*_j(q_j, \theta_i, \theta_{-ij})$, and that revenue from i is improvable with respect to $\big(\theta_i, (\theta_j, \theta_{-ij}), (\theta^*_j(q_j, \theta_i, \theta_{-ij}), \theta_{-ij})\big)$.

Roughly, the strong improvability implies that, if agent i with θ_i correctly predicts that $-i$'s payoff types are not θ'_{-i}, then (given θ_{-ij}) the principal can know that j's type is not a threshold type for some q_j. Such information enables the principal to earn higher expected revenue from j.

Proposition 2. Under Assumption 0, if Q is a lower set, then strong improvability implies no foundation of EPIC mechanisms.

The formal proof is given in Appendix C, which directly follows from the idea of Example 4. The main difference from Proposition 1 is that, when the agent does not have a full-support first-order belief, the principal extracts more surplus from other agents by modifying their allocations. This imposes a stronger requirement on the feasible allocation, so that makes Proposition 2 not applicable to some environments. For example, in the single-object auction, strong improvability requires that agent j is assigned the object at both $(\theta_i, \theta_j, \theta_{-ij})$ and $(\theta_i, \theta^*_j(1, \theta_{-j}), \theta_{-ij})$, which means agent i cannot get the good at either payoff type profile, contradicting the fact that strong improvability also means that agent i should win the object under at least one payoff type profile.

However, other environments such as a multiple-unit auction where each agent has a unit demand would satisfy the strong improvability. In the previous auction with common-value and private-value components, we assume that there are two objects and more than two agents. To get strong improvability, first choose $c_{-i} = (c_j, c_{-ij})$, $c'_{-i} = (c'_j, c_{-ij})$ satisfying $\gamma_i(d_i, c_{-i}) \leq 0 < \gamma_i(d'_i, c_{-i})$, $\gamma_i(d_i, c'_{-i}) > 0 \geq \gamma_i(d'_i, c'_{-i})$; then let any agent $k \neq i, j$ have some worst private-value component, $\underline{d}_k$, which induces negative virtual value, so that he will not get the object in the optimal EPIC mechanism; finally choose c_i, d_j and d'_j such that the smaller one of $\{\gamma_j(d_j, c_i, c_{-ij}), \gamma_j(d'_j, c_i, c_{-ij})\}$ is just above 0, and thus becomes the threshold payoff type under (c_i, c_{-ij}). Through a similar argument with Lemma 3, we can show that revenue from i is improvable with respect to $(c_i, \tilde{d}_i)$, $\big((c_j, d_j), (c_{-ij}, \underline{d}_{-ij})\big)$ and $\big((c'_j, d'_j), (c_{-ij}, \underline{d}_{-ij})\big)$, where $\tilde{d}_i$ depends on which of the three cases in Definition 4 (Improvability) we actually have.

5 Necessary and sufficient condition

A natural question is, under which additional conditions, the ordinal interdependence implies the strong improvability, so that EPIC mechanisms do not have the foundation if and only if we do not have the ordinal interdependence. A sufficient condition is the following richness condition on Q.

Assumption 4. For each i, θ_i, and θ_{-i}, we have $q_i^{EP}(\theta_i, \theta_{-i}) > 0$, and for each $\theta'_i \neq \theta_i$, we have $q_i^{EP}(\theta_i, \theta_{-i}) \neq q_i^{EP}(\theta'_i, \theta_{-i})$.

A representative example is a monopoly problem with multiple buyers and multiple units of trading.[19] Assume that each agent i's payoff is given by

$$v_i(\theta_i, \theta_{-i}, q_i, p_i) = u_i(\theta_i, \theta_{-i})q_i - q_i^2 - p_i,$$

where $q_i (\geq 0)$ is the quantity assigned to agent i, and p_i is agent i's payment to the principal. Assume that there is no feasibility constraint on q. The principal

[19] See Mussa and Rosen (1978) and Segal (2003) (or their straightforward generalizations) for such environments, although they focus on private-value environments.

maximizes the expected revenue. We can see that an agent's valuation is concave in q_i. The optimal EPIC mechanism is given by:

$$\begin{aligned}
\max_{q,p} \quad & \sum_{\theta} \sum_{i} f(\theta) p_i(\theta) \\
s.t. \quad & \forall i, \theta_i, \theta_i', \theta_{-i} : \quad q_i(\theta) \geq 0; \\
& u_i(\theta_i, \theta_{-i}) q_i(\theta_i, \theta_{-i}) - q_i^2(\theta_i, \theta_{-i}) - p_i(\theta_i, \theta_{-i}) \\
& \geq \max \big\{0, \quad u_i(\theta_i, \theta_{-i}) q_i(\theta_i', \theta_{-i}) - q_i^2(\theta_i', \theta_{-i}) - p_i(\theta_i', \theta_{-i})\big\}.
\end{aligned}$$

By the standard argument, the problem is equivalent to:

$$\begin{aligned}
\max_{q,p} \quad & \sum_{\theta} \sum_{i} f(\theta) \big[\gamma_i(\theta) q_i(\theta) - q_i^2(\theta)\big] \\
s.t. \quad & \forall i, \theta_i, \theta_{-i}, \theta_i' \prec_i^{\theta_{-i}} \theta_i : \quad q_i(\theta) \geq 0; \quad q_i(\theta_i, \theta_{-i}) \geq q_i(\theta_i', \theta_{-i}).
\end{aligned}$$

The virtual value γ_i is given by:

$$\gamma_i(\theta_i, \theta_{-i}) := u_i(\theta_i, \theta_{-i}) - \frac{1 - F_i(\theta_i \mid \theta_{-i})}{f(\theta_i \mid \theta_{-i})} \Big(u_i^+(\theta_i, \theta_{-i}) - u_i(\theta_i, \theta_{-i})\Big),$$

where

$$u_i^+(\theta_i, \theta_{-i}) := \min_{\tilde{\theta}_i \succ_i^{\theta_{-i}} \theta_i} u_i(\tilde{\theta}_i, \theta_{-i}).$$

If we assume (i) for all i and θ, we have $\gamma_i(\theta) > 0$, and (ii) $\gamma_i(\theta_i', \theta_{-i}) < \gamma_i(\theta_i, \theta_{-i})$ whenever $\theta_i' \prec_i^{\theta_{-i}} \theta_i$, then the optimal EPIC mechanism, given by $q_i^{EP}(\theta) = \frac{\gamma_i(\theta)}{2}$, satisfies Assumption 4.

Theorem 3. Under Assumptions 0, 1 and 4, if Q is a lower set, then EPIC mechanisms have the foundation if and only if we do not have ordinal interdependence.

Remark 4. In some contexts, the assumption that each agent is always assigned a positive quantity may be considered as a restrictive assumption. However, we adopt this assumption because it is relatively transparent, and it simplifies the proof of the theorem. Weaker conditions may suffice at the cost of less transparent statement and more complicated argument.

More specifically, imagine an environment where, given some θ_{-i}, agent i is assigned $q_i^{EP}(\theta) = 0$ for some subset of θ_i, while for any θ_i, θ_i' with $q_i^{EP}(\theta_i, \theta_{-i})$, $q_i^{EP}(\theta_i', \theta_{-i}) > 0$, we have $q_i^{EP}(\theta_i, \theta_{-i}) \neq q_i^{EP}(\theta_i', \theta_{-i})$. Assume that there exists a subset of payoff types, $\tilde{\Theta}_i$ for each i, such that those types are always assigned some non-zero quantities. Then, one can show that EPIC mechanisms do not have the foundation if ordinal interdependence occurs for those always-positive-quantity type profiles.

6 "Unimprovable" mechanisms?

Given our result that EPIC mechanisms could be improved when the environment exhibits significant interdependence, it seems natural to ask: Then, which mechanism(s) is (are) "unimprovable", that is, a mechanism for which no other mechanism can achieve a higher expected revenue *regardless of* the principal's belief μ (strictly at least for some μ)?

To simplify the analysis, we consider a "reduced form" of the multi-agent environment, where we focus on a particular agent with privately-known payoff type θ, and the other agents' payoff type profile θ_{-i} is represented by a state variable $\omega \in \Omega$ which becomes publicly known after the agent's report. Thus, the agent's *type* is a pair (θ, β), where θ is his payoff-type and $\beta \in \Delta(\Omega)$ represents his belief about ω. Let $T = \Theta \times \Delta(\Omega)$ denote the agent's type space.

We assume that the allocation is denoted by $(q, p) \in [0, 1] \times \mathbb{R}$, where q represents the probability of selling a single good, and p represents the monetary transfer from the agent to the principal. The principal's payoff is the revenue, p, and the agent's payoff given (q, p) is denoted by $v(\omega, \theta)q - p$. It is without loss of generality to focus on the class of *direct* mechanisms $\Gamma = (M, q, p)$ where the agent participates in the mechanism and reports (θ, β) truthfully.

The following example suggests that we need to make a further restriction on the principal's belief μ; otherwise would not be a well-defined question to find unimprovable mechanisms.

Example 5. Let $\Omega = \{0, 1\}$ and $\Theta = \{1\}$, and the agent's valuation is $v(\omega, \theta) = \theta(= 1)$. That is, ω does not change his value, and in this sense, it is a *private-value* environment. Furthermore, the agent has no payoff-relevant private information. The optimal EPIC mechanism fully extract the valuation: $q^{EP}(\omega, \theta) = p^{EP}(\omega, \theta) = \theta(= 1)$.

This mechanism may seem to be "obviously optimal". However, it is (unboundedly) improvable by the following mechanism $\Gamma = (T, q, p)$: $q(\omega, \theta, \beta) = 1$ for all ω, θ, β; $p(\omega, \theta, \beta) = 1$ if $\beta(\omega) > 0$; and $p(\omega, \theta, \beta) = P(> 0)$ if $\beta(\omega) = 0$. It is obvious that truth-telling of $\theta(= 1)$ and β is optimal for the agent, which implies that, for any μ, the revenue is at least 1. Moreover, for μ with $\mu(\{\omega, \theta, \beta | \beta(\omega) = 0\}) > 0$ (i.e., the principal believes that, with a positive probability, the agent is "completely wrong"), the principal's expected revenue is strictly greater than 1.

Actually, P can be any number greater than 1, and increasing P always implies further improvement, and hence, no "unimprovable" mechanism exists. □

The above example suggests that the question of unimprovable mechanism can only make sense by (further) restricting our attention on the class of μ which does not assign a positive probability that the agent is "completely wrong".

Formally, we say that μ is *strongly admissible* if it is in $\mathcal{M}$ and it does not assign a positive probability for the agent being completely wrong:

$$\mu(\{\omega, \theta, \beta | \beta(\omega) = 0\}) = 0, \ \forall \omega, \theta.$$

Then, based on Börgers (2017), we say that a (direct) mechanism $\Gamma = (T, q, p)$ is *dominated* (instead of "improved") by another (direct) mechanism $\Gamma' = (T, q, p')$ if (i) for any strongly admissible μ, the expected revenue in Γ' is weakly higher than that in Γ, and (ii) there exists a strongly admissible μ with which the expected revenue in Γ' is strictly higher. A mechanism is *undominated* (or "unimprovable") if it is not dominated.[20] Applying this definition, the optimal EPIC mechanism where the agent always pays 1 is undominated in Example 5.

Characterizing undominated mechanisms in the general environment is an interesting question but beyond the scope of the paper, so left for future research. Nevertheless, we believe that our idea of mechanism construction in the previous sections could be useful for this ambitious question. In order to illustrate this point, in what follows, we characterize *all* the mechanisms that (i) dominate the optimal EPIC mechanism and (ii) are undominated, in the context of Example 2.

[20] Börgers (2017) shows that the dominant-strategy mechanism of Chung and Ely (2007) is dominated in private-value environments with three or more agents. Characterizing the class of undominated mechanisms seems an open question.

6.1 Example 2 revisited

We consider the following class of mechanisms $\Gamma = (T, q, p)$: $q(\cdot) = q^{EP}(\cdot)$; $p(\omega, \theta, \beta) = p^{EP}(\omega, \theta)$ if $\theta \in \{0, 1\}$; and,

$$\big(p(0, 2, \beta), p(1, 2, \beta)\big) = \big(a(y), b(y)\big) \quad \text{if} \quad \beta(1) = y \in [0, 1].$$

It is a class of mechanisms in the sense that $(a(y), b(y))_{y \in [0,1]}$ are the free parameters. We denote this class by $\mathbf{\Gamma}^*$. In particular, the non-EPIC mechanism constructed in Example 2 (that dominates the optimal EPIC mechanism) is in this class.

Let $\mathbf{\Gamma}^U$ denote the set of all mechanisms that are undominated and that dominate the optimal EPIC mechanism.

Proposition 3. Γ is in $\mathbf{\Gamma}^U$ if and only if it is in $\mathbf{\Gamma}^*$ with $(a(y), b(y))_{y \in [0,1]}$ satisfying (i) $(a(y), b(y))_{y \in [0,1]}$ (as a set of points on $\mathbb{R}^2$) lies on a continuous, convex, non-increasing curve that connects $(2, 3)$ and $(3, 2)$; and (ii) either $\lim_{y \uparrow 1} a(y) = 3$ or $\lim_{y \downarrow 0} b(y) = 3$ holds (or both).

In particular, the result says that the non-EPIC mechanism constructed in Example 2 (that dominates the optimal EPIC mechanism) is undominated. That is, if any other mechanism achieve a strictly higher expected revenue for some (strongly admissible) μ than that non-EPIC mechanism, such a mechanism necessarily achieves a strictly *lower* revenue for another (strongly admissible) μ. In this sense, that non-EPIC mechanism cannot be further improved.

7 Conclusion

If the environment exhibits only cardinal interdependence (and certain regularity conditions), then there exist the maxmin and Bayesian foundations for EPIC mechanisms, in the sense of Chung and Ely (2007). If the environment exhibits ordinal interdependence, (and certain additional conditions), then such a foundation may not exist.

In interdependent-value environments, Yamashita (2015) provides an alternative solution concept (that is, *incentive compatibility in value revelation*), which is also robust to the agents' belief structure in a related sense and useful in the implementation of social choice correspondences in undominated strategies. It may be interesting to investigate similar sorts of foundation results for this alternative solution concept.

A Proof of Theorem 1

Because $\succ_i^{\theta_{-i}} = \succ_i^{\theta'_{-i}}$ for all i, θ_{-i}, and θ'_{-i}, we denote this ordering by $\succ_i$ with no superscript. Also, let $\Theta_i = \{\theta_i^1, \ldots, \theta_i^N\}$ (where $N = |\Theta_i|$) so that $\theta_i^n \prec_i \theta_i^{n+1}$ for all $n = 1, \ldots, N-1$.

Consider the simple type space $\widehat{\mathcal{T}}^f = (T_i, \widehat{\theta}_i, \widehat{\pi}_i)_{i=1}^I$ with $T_i = \Theta_i$ and the agents' beliefs defined by $\widehat{\pi}_i(\theta_i^n)[\theta_{-i}] = \big(\sum_{\theta'_{-i} \in \Theta_{-i}} G_i(\theta_i^n, \theta'_{-i})\big)^{-1} G_i(\theta_i^n, \theta_{-i})$ for all $\theta_{-i} \in \Theta_{-i}$, where $G_i(\theta_i^n, \theta_{-i}) = \sum_{k=n}^N f(\theta_i^k, \theta_{-i})$. By convention, $G_i(\theta_i^{N+1}, \theta_{-i}) = 0$.

The optimal Bayesian mechanism given this simple type space achieves:

$$
\begin{aligned}
V(f) = & \max_{(q,p):\Theta \to Q \times \mathbb{R}^I} \sum_{\theta \in \Theta} f(\theta) \sum_{i \in I} p_i(\theta) \\
& s.t. \quad \forall i \in \mathcal{I}, \forall n, l \in \{1, \ldots, N\}, \forall \theta \in \Theta : \\
& \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i})\big) \geq 0, \qquad (BIR_i^n) \\
& \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^n, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^n, \theta_{-i})\big) \\
& \geq \sum_{\theta_{-i} \in \Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}] \big(v_i(q_i(\theta_i^l, \theta_{-i}), \theta_i^n, \theta_{-i}) - p_i(\theta_i^l, \theta_{-i})\big). \qquad (BIC_i^{n \to l})
\end{aligned}
$$

Because the identity function $\widehat{\theta}_i$ is one-to-one, by Lemma 1, $\widehat{\mathcal{T}}^f$ can be embedded in the universal type space $\mathcal{T}^*$ through a bijection h such that $t_i^n = h_i(\theta_i^n)$. Thus, $V(f)$ provides an upper bound for the best expected revenue given the universal type space $\mathcal{T}^*$ (and the principal's belief $\mu^* \in \mathcal{M}$ such that $\mu^*(h(\widehat{\theta}^{-1}(\theta))) = f(\theta)$). Therefore, in order to show the Bayesian foundation for EPIC mechanisms given f, it suffices to show that $V(f) \leq R_f^{EP}$.

We first prove the claim by imposing the non-singularity condition on f, which assumes that $\Omega_i = (f(\theta_i^1,\cdot),\ldots,f(\theta_i^N,\cdot))^\intercal$ has rank N for each i, where $f(\theta_i^n,\cdot) = (f(\theta_i^1,\theta_{-i}))_{\theta_{-i}\in\Theta_{-i}}$ is a $(I-1)N$-dimensional vector.

Lemma 4. In the solution of $V(f)$, $(BIC_i^{n\to n-1})$ holds with equality for all i and $n \neq 1$, and (BIR_i^n) holds with equality for all i and n.

The lemma implies that, for all i and n:

$$\begin{aligned}
&\sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i(\theta_i^n,\theta_{-i})\big)\\
&= \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^{n-1},\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i(\theta_i^{n-1},\theta_{-i})\big) = 0,
\end{aligned}$$

or equivalently:

$$\begin{aligned}
&\sum_{\theta_{-i}\in\Theta_{-i}} \Big(\sum_{\theta'_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta'_{-i})\Big)^{-1} G_i(\theta_i^n,\theta_{-i})\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i(\theta_i^n,\theta_{-i})\big) = 0,\\
&\sum_{\theta_{-i}\in\Theta_{-i}} \Big(\sum_{\theta'_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta'_{-i})\Big)^{-1} G_i(\theta_i^n,\theta_{-i})\big(v_i(q_i(\theta_i^{n-1},\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i(\theta_i^{n-1},\theta_{-i})\big) = 0.
\end{aligned}$$

This implies:

$$\begin{aligned}
\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i}) v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) &= \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i}) p_i(\theta_i^n,\theta_{-i}),\\
\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i}) v_i(q_i(\theta_i^{n-1},\theta_{-i}),\theta_i^n,\theta_{-i}) &= \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i}) p_i(\theta_i^{n-1},\theta_{-i})\big),
\end{aligned}$$

and therefore, the objective becomes:

$$
\begin{aligned}
&\sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\sum_{\theta_{-i}\in\Theta_{-i}} f(\theta_i^n,\theta_{-i})p_i(\theta_i^n,\theta_{-i})\\
=&\sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\sum_{\theta_{-i}\in\Theta_{-i}} \big(G_i(\theta_i^n,\theta_{-i})-G_i(\theta_i^{n+1},\theta_{-i})\big)p_i(\theta_i^n,\theta_{-i})\\
=&\sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\left(\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})p_i(\theta_i^n,\theta_{-i})-\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^{n+1},\theta_{-i})p_i(\theta_i^n,\theta_{-i})\right)\\
=&\sum_{i\in\mathcal{I}}\sum_{n=1}^{N}\sum_{\theta_{-i}\in\Theta_{-i}} \big(G_i(\theta_i^n,\theta_{-i})v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i})-G_i(\theta_i^{n+1},\theta_{-i})v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^{n+1},\theta_{-i})\big)\\
=&\sum_{i\in\mathcal{I}}\sum_{\theta\in\Theta} f(\theta)\gamma_i(q_i,\theta).
\end{aligned}
$$

Therefore, under Assumption 1, we have $V(f)=R_f^{EP}$.

Proof of Lemma 4. We first show that each upward incentive constraint, $(BIC_i^{n\to l})$ with $n<l$, can be ignored without loss. Let $\Pi_i=\big(\widehat{\pi}_i(\theta_i^1),\dots,\widehat{\pi}_i(\theta_i^N)\big)^\intercal$ denote the matrix of agent i's beliefs, where each $\widehat{\pi}_i(\theta_i^n)$ is a $(I-1)N$-dimensional vector. Then:

$$
\Pi_i=\begin{pmatrix}\kappa_i^1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & \kappa_i^N\end{pmatrix}_{N\times N}\begin{pmatrix}1 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 1\end{pmatrix}_{N\times N}\Omega,
$$

where $\kappa_i^n=\big(\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^n,\theta_{-i})\big)^{-1}$, and hence Π_i has a rank N. Thus, there exists $\lambda\in\mathbb{R}^{(I-1)N}$ such that:

$$
\Pi_i\lambda=(1,\dots,1,\underbrace{0}_{l\text{-th element}},\dots,0)^\intercal.
$$

If we add λ to $p_i(\theta_i^l, \cdot)$, each $BIC_i^{n\to l}$ with $n < l$ is relaxed, while no other (BIC) and (BIR) constraints are affected. Moreover, from $\widehat{\pi}_i(\theta_i^l) \cdot \lambda = 0$ and $\widehat{\pi}_i(\theta_i^{l+1}) \cdot \lambda = 0$, we obtain:

$$\sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^l, \theta_{-i})\lambda(\theta_{-i}) = 0, \qquad \sum_{\theta_{-i}\in\Theta_{-i}} G_i(\theta_i^{l+1}, \theta_{-i})\lambda(\theta_{-i}) = 0,$$

which implies that $\sum_{\theta_{-i}\in\Theta_{-i}} f(\theta_i^l, \theta_{-i})\lambda(\theta_{-i}) = 0$, that is, the principal's expected revenue is also unaffected.

Next, we show that for any mechanism (q, p) satisfying the remaining constraints, there exists a mechanism (q', p') which satisfies not only the remaining constraints, but also (BIR_i^n) for $n = 1, \dots, N$ and ($BIC_i^{n\to n-1}$) for $n = 2, \dots, N$ with equality, and raises at least as high expected revenue as (q, p).

Given any such mechanism (q, p), if ($BIC_i^{n\to n-1}$) is satisfied with strict inequality for some i and n, then let $\beta_i^{n\to n-1}$ be the amount of the slackness of this constraint ($BIC_i^{n\to n-1}$). Let Π_i' be the matrix generated by substituting the n-th row of Π_i with the vector $f(\theta^{n-1}, \cdot)$. That is:

$$\Pi_i' = \begin{pmatrix} \kappa_i^1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \kappa_i^{n-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \kappa_i^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \kappa_i^N \end{pmatrix} \begin{pmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \Omega,$$

and hence, Π_i' has a rank N. Thus, there exists $\lambda \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i'\lambda = (0, \dots, 0, \underbrace{1}_{n\text{-th element}}, 0, \dots, 0)^\intercal.$$

Because $\widehat{\pi}_i(\theta_i^{n-1}) \cdot \lambda = 0$ and $f(\theta^{n-1}, \cdot) \cdot \lambda = 1$, we have:

$$\widehat{\pi}_i(\theta_i^n) \cdot \lambda = \frac{\kappa_i^n}{\kappa_i^{n-1}} \widehat{\pi}_i(\theta_i^{n-1}) \cdot \lambda - \kappa_i^n f(\theta^{n-1}, \cdot) \cdot \lambda < 0,$$

and thus, $\varepsilon = -\beta_i^{n \to n-1} / (\widehat{\pi}_i(\theta_i^n) \cdot \lambda) > 0$. If we add $\varepsilon\lambda$ to $p_i(\theta_i^{n-1}, \cdot)$, then all the constraints for types θ_i^l with $l \neq n$ are unaffected because $\widehat{\pi}_i(\theta_i^l) \cdot \lambda = 0$ for all $l \neq n$, and for type θ_i^n only constraint $(BIC_i^{n \to n-1})$ is changed, which holds with equality under the new payment rule. Because $f(\theta^{n-1}, \cdot) \cdot (\varepsilon\lambda) = \varepsilon > 0$, the expected revenue increases under the new payment rule.

Similarly, if (BIR_i^n) is satisfied with strict inequality for some i and n, then let β_i^n be the amount of the slackness of this constraint (BIR_i^n). Because Π_i has a rank N, there exists $\lambda \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i \lambda = (\beta_i^1, \dots, \beta_i^N)^\intercal \geq 0.$$

Adding λ to each $p_i(\theta_i^n, \cdot)$ does not affect any (BIC) constraint, while all the participation constraints are satisfied with equality in the new mechanism. The change in the total expected revenue is:

$$
\begin{aligned}
\sum_{n=1}^{N} \sum_{\theta_{-i} \in \Theta_{-i}} f(\theta_i^n, \theta_{-i}) \lambda(\theta_{-i}) &= \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) \sum_{n=1}^{N} f(\theta_i^n, \theta_{-i}) \\
&= \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) G_i(\theta_i^1, \theta_{-i}) \\
&= \frac{1}{\kappa_i^1} \sum_{\theta_{-i} \in \Theta_{-i}} \lambda(\theta_{-i}) \widehat{\pi}_i(\theta_i^1)[\theta_{-i}] \\
&= \beta_i^1,
\end{aligned}
$$

which is non-negative. □

Next, we consider the case where f is singular, that is, for some i, Ω_i has a rank strictly less than N. Consider a sequence of distributions over Θ, $\{f_r\}_{r=1}^{\infty}$, such that each f_r is full-support and $f_r \to f$ (in the standard Euclidean distance).[21] By Assumption 1, without loss of generality, we assume that the monotonicity constraints (M) are not binding in the problem of $R_{f_r}^{EP}$.

We prove the following continuity lemma.

Lemma 5. For each $\varepsilon > 0$, there exists $r_\varepsilon \in \mathbb{N}$ such that, for any $r \geq r_\varepsilon$, $R_{f_r}^{EP} \leq R_f^{EP} + \varepsilon$ and $V(f_r) \geq V(f) - \varepsilon$.

Proof of Lemma 5. For the first inequality, recall that

$$
R_f^{EP} = \sum_{\theta} f(\theta) \cdot \max_{q(\theta)} \sum_{i} \gamma_i\big(q_i(\theta), \theta\big),
$$

which is obviously continuous in f.

For the second inequality, let (q, p) be a solution to the problem of $V(f)$.

[21] We can always find such a sequence because the set of all non-singular distributions is a dense subset of the set of all distributions over Θ.

In the following, for each r, we construct another mechanism (q, p^r) (note that we keep the same q), so that it satisfies all the constraints of the problem of $V(f_r)$, namely:

$$\sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i^r(\theta_i^n,\theta_{-i})\big) \geq 0, \qquad (BIR_i^n(r))$$

$$\sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i^r(\theta_i^n,\theta_{-i})\big)$$
$$\geq \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^l,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i^r(\theta_i^l,\theta_{-i})\big). \qquad (BIC_i^{n\to l}(r))$$

Let:

$$S_i^n(r) = \max\left\{0, \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(p_i(\theta_i^n,\theta_{-i}) - v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i})\big)\right\},$$

denote the size of violation of $(BIR_i^n(r))$ by p. If we consider a modified payment rule p' so that $p_i'(\theta_i^n,\cdot) = p_i(\theta_i^n,\cdot) - S_i^n(r)\mathbf{1}$, then this new payment rule satisfies the participation constraints, but may not satisfy the incentive compatibility constraints. Thus, let:

$$L_i^{n\to l}(r) = \max\Big\{ 0, \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^l,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i'(\theta_i^l,\theta_{-i})\big)$$
$$- \sum_{\theta_{-i}\in\Theta_{-i}} \widehat{\pi}_i^r(\theta_i^n)[\theta_{-i}]\big(v_i(q_i(\theta_i^n,\theta_{-i}),\theta_i^n,\theta_{-i}) - p_i'(\theta_i^n,\theta_{-i})\big)\Big\},$$

denote the size of violation of $(BIC_i^{n\to l}(r))$ by p'. As in the first part of the proof, the matrix of agent i's belief in the simple type space $\widehat{\mathcal{T}}^{f_r}$, $\Pi_i^r = \big(\widehat{\pi}_i^r(\theta_i^1),\ldots,\widehat{\pi}_i^r(\theta_i^N)\big)^\intercal$, has a rank N, and hence, there exists $\lambda_i^1(r), \ldots, \lambda_i^N(r) \in \mathbb{R}^{(I-1)N}$ such that:

$$\Pi_i^r\left(\lambda_i^1(r), \ldots, \lambda_i^N(r)\right) = \left(L_i^{n \to l}(r)\right)_{N \times N},$$

which we denote by $\mathbf{L}_r$. Or equivalently:

$$\mathbf{L}_r = \underbrace{\begin{pmatrix} \kappa_i^1(r) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \kappa_i^N(r) \end{pmatrix}_{N \times N}}_{\triangleq \mathbf{K}_r} \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}_{N \times N}}_{\triangleq \mathbf{A}} \Omega_r\left(\lambda_i^1(r), \ldots, \lambda_i^N(r)\right).$$

Define $p_i^r(\theta_i^n, \cdot) = p_i(\theta_i^n, \cdot) - S_i^n(r)\mathbf{1} + \lambda_i^n(r)$. Then, together with q, it satisfies all the constraints of the problem of $V(f_r)$.

We complete the proof by showing that $\sum_\theta \sum_i (p_i^r(\theta) - p_i(\theta)) f_r(\theta) \to 0$ as $r \to \infty$. Because it is obvious that $S_i^n(r) \to 0$, it suffices to show that:

$$\sum_{n=1}^{N} f_r(\theta_i^n, \cdot) \cdot \lambda_i^n(r) \to 0.$$

Indeed:

$$\sum_{n=1}^{N} f_r(\theta_i^n, \cdot) \cdot \lambda_i^n(r) = tr\big(\mathbf{A}^{-1}\mathbf{K}_r^{-1}\mathbf{L}_r\big) \to 0,$$

as $r \to \infty$, because $\mathbf{L}_r \to 0$. □

Finally, contrarily to the original claim, suppose that $V(f) > R_f^{EP}$, and let $\varepsilon \in (0, \frac{V(f) - R_f^{EP}}{2})$. Then, there exists r_ε such that:

$$V(f_r) - R_{f_r}^{EP} \geq V(f) - R_f^{EP} - 2\varepsilon > 0,$$

which contradicts the first part of this proof.

B Proof of Proposition 1

We show that, for each of these cases, there exists a mechanism that yields a strictly higher expected revenue than the optimal EPIC mechanism.

Case (i): $\theta_i^*(q_i, \theta_{-i}) \notin \Theta_i^*(q_i', \theta_{-i}')$, $\theta_i^*(q_i', \theta_{-i}') \notin \Theta_i^*(q_i, \theta_{-i})$, $\theta_i \in \Theta_i^*(q_i', \theta_{-i}') \cap \Theta_i^*(q_i, \theta_{-i})$.

Consider a new mechanism (M, q^*, p^*) such that $M_i = \Theta_i \times [0,1]$, $M_j = \Theta_j$ for $j \neq i$, and for each $((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) \in M$,

$$\begin{aligned} q^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) &= q^{EP}(\tilde{\theta}), \\ p_j^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) &= p_j^{EP}(\tilde{\theta}), \ \forall j \neq i, \end{aligned}$$

and for p_i^*, we set $p_i^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) = p_i^{EP}(\tilde{\theta})$ unless $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \cap \Theta_i^*(q_i', \theta_{-i}')$ and $\tilde{\theta}_{-i} \in \{\theta_{-i}, \theta_{-i}'\}$; and for each $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \cap \Theta_i^*(q_i', \theta_{-i}')$, we set

$$\begin{aligned} p_i^*((\tilde{\theta}_i, x), \theta_{-i}) &= p_i^{EP}(\tilde{\theta}_i, \theta_{-i}) + \eta(1-x), \\ p_i^*((\tilde{\theta}_i, x), \theta_{-i}') &= p_i^{EP}(\tilde{\theta}_i, \theta_{-i}') + \eta\psi(x), \end{aligned}$$

where $\psi(x) = 1 - \sqrt{1-x^2}$.

Intuitively, $x \in [0,1]$ is related to agent i's first-order belief over θ_{-i} and θ_{-i}' (more precisely, their likelihood ratio). Indeed, if agent i reports his payoff type θ_i truthfully, his optimal choice of x is given by $x^*(\beta, \beta') = \sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$, where β is

i's first-order belief for θ_{-i} and β' is i's first-order belief for θ'_{-i}. Note that, given any $\mu \in \mathcal{M}^{\text{full}}$, agent i chooses $x \in (0,1)$ with probability one.

It is then obvious that, if the agents report their payoff types truthfully (and agent i chooses x optimally), then this new mechanism yields a strictly higher expected revenue than the optimal EPIC mechanism.

For any agent $j \neq i$, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR.

We show the incentive compatibility of agent i with $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \cap \Theta_i^*(q'_i, \theta'_{-i})$ (for the other payoff types, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR). First, obviously, any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \cap \Theta_i^*(q'_i, \theta'_{-i})$ is not profitable. Second, any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q'_i, \theta'_{-i})$ is not profitable either, because, letting β and β' be his first-order beliefs for θ_{-i} and θ'_{-i} respectively, the expected gain by deviation is at most[22]

$$\beta[\eta(1-x^*(\beta,\beta'))]+\beta'[-\eta+\eta\psi(x^*(\beta,\beta'))]\le 0.$$

As shown in Footnote 22, the key step is to prove that in the optimal EPIC mechanism, certain deviations would lead to a payoff loss of at least η for the agent. This is where the $-\eta$ term in the second bracket comes from.

Similarly, we can show that any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i', \theta_{-i}') \setminus \Theta_i^*(q_i, \theta_{-i})$ and $\hat{\theta}_i \notin \Theta_i^*(q_i', \theta_{-i}') \cup \Theta_i^*(q_i, \theta_{-i})$ is not profitable either.

Case (ii): $\theta_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$ and $\theta_i^*(q_i, \theta_{-i}) \in \Theta_i^*(q_i', \theta_{-i}')$.

Consider a new mechanism (M, q^*, p^*) such that $M_i = \Theta_i \times [0,1]$, $M_j = \Theta_j$ for

[22] Particularly, the expected gain by deviation is

$$\begin{aligned}
&\beta\cdot\left[v_i(q_i^{EP}(\hat{\theta}_i,\theta_{-i}),\tilde{\theta}_i,\theta_{-i})-p_i^{EP}(\hat{\theta}_i,\theta_{-i})\right]+\beta'\cdot\left[v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})-p_i^{EP}(\hat{\theta}_i,\theta'_{-i})\right]\\
&\quad-\beta\cdot\left[v_i(q_i^{EP}(\tilde{\theta}_i,\theta_{-i}),\tilde{\theta}_i,\theta_{-i})-p_i^{EP}(\tilde{\theta}_i,\theta_{-i})-\eta\cdot(1-x^*(\beta,\beta'))\right]\\
&\quad-\beta'\cdot\left[v_i(q_i^{EP}(\tilde{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})-p_i^{EP}(\tilde{\theta}_i,\theta'_{-i})-\eta\cdot\psi(x^*(\beta,\beta'))\right]\\
\le&\beta\cdot\left[\eta\cdot(1-x^*(\beta,\beta'))\right]+\beta'\cdot\left[\eta\cdot\psi(x^*(\beta,\beta'))\right]+\beta\cdot 0\\
&\quad-\beta'\cdot\left[v_i(q_i^{EP}(\tilde{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})-p_i^{EP}(\tilde{\theta}_i,\theta'_{-i})-v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})+p_i^{EP}(\hat{\theta}_i,\theta'_{-i})\right].
\end{aligned}$$

Without loss of generality, we assume that $q_i' \in Q_i^+$, so that $q_i^{EP}\big(\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i}\big)=q_i'$; otherwise we can replace q_i' by $q_i^{EP}\big(\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i}\big)$. Because $q_i^{EP}(\tilde{\theta}_i,\theta'_{-i}) \ge q_i' > q_i^{EP}(\hat{\theta}_i,\theta'_{-i})$, $\tilde{\theta}_i \succ_i^{\theta'_{-i}} \theta_i^*(q_i',\theta'_{-i}) \succ_i^{\theta'_{-i}} \hat{\theta}_i$, then the terms in the last bracket satisfy

$$\begin{aligned}
&v_i(q_i^{EP}(\tilde{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})-p_i^{EP}(\tilde{\theta}_i,\theta'_{-i})-v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})+p_i^{EP}(\hat{\theta}_i,\theta'_{-i})\\
\ge&v_i(q_i',\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i})-p_i^{EP}(\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i})-v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i})+p_i^{EP}(\hat{\theta}_i,\theta'_{-i})\\
&+v_i(q_i',\tilde{\theta}_i,\theta'_{-i})-v_i(q_i',\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i})-v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\tilde{\theta}_i,\theta'_{-i})+v_i(q_i^{EP}(\hat{\theta}_i,\theta'_{-i}),\theta_i^*(q_i',\theta'_{-i}),\theta'_{-i})\\
\ge&0+\eta.
\end{aligned}$$

$j \neq i$, and for each $((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) \in M$,

$$\begin{aligned} q^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) &= q^{EP}(\tilde{\theta}), \\ p_j^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) &= p_j^{EP}(\tilde{\theta}), \ \forall j \neq i, \end{aligned}$$

and for p_i^*, we set $p_i^*((\tilde{\theta}_i, x), \tilde{\theta}_{-i}) = p_i^{EP}(\tilde{\theta})$ unless $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$ and $\tilde{\theta}_{-i} \in \{\theta_{-i}, \theta_{-i}'\}$; and for each $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$, we set

$$\begin{aligned} p_i^*((\tilde{\theta}_i, x), \theta_{-i}) &= p_i^{EP}(\tilde{\theta}_i, \theta_{-i}) + \eta(1 - x), \\ p_i^*((\tilde{\theta}_i, x), \theta_{-i}') &= p_i^{EP}(\tilde{\theta}_i, \theta_{-i}') + \eta\psi(x), \end{aligned}$$

where $\psi(x) = 1 - \sqrt{1 - x^2}$.

Again, $x \in [0, 1]$ is related to agent i's first-order belief over θ_{-i} and θ_{-i}'. Indeed, if agent i reports his payoff type θ_i truthfully, his optimal choice of x is given by $x^*(\beta, \beta') = \sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$, where β is i's first-order belief for θ_{-i} and β' is i's first-order belief for θ_{-i}'. Note that, given any $\mu \in \mathcal{M}^{\text{full}}$, agent i chooses $x \in (0, 1)$ with probability one.

It is obvious that, if the agents report their payoff types truthfully (and agent i chooses x optimally), then this new mechanism yields a strictly higher expected revenue than the optimal EPIC mechanism.

For any agent $j \neq i$, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR.

We show the incentive compatibility of agent i with $\tilde{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$ (for the other payoff types, the new mechanism is outcome-equivalent to the optimal EPIC mechanism, and hence satisfies EPIC and EPIR). First, obviously, any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \setminus \Theta_i^*(q_i', \theta_{-i}')$ is not profitable. Second, any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i, \theta_{-i}) \cap \Theta_i^*(q_i', \theta_{-i}')$ is not profitable either, because, letting β and β' be his first-order beliefs for θ_{-i} and θ_{-i}' respectively, the expected gain by deviation is at most

$$\beta[\eta(1 - x^*(\beta, \beta'))] + \beta'[-\eta + \eta\psi(x^*(\beta, \beta'))] \leq 0.$$

Similarly, we can show that any deviation to $\hat{\theta}_i \in \Theta_i^*(q_i', \theta_{-i}') \setminus \Theta_i^*(q_i, \theta_{-i})$ and $\hat{\theta}_i \notin \Theta_i^*(q_i', \theta_{-i}') \cup \Theta_i^*(q_i, \theta_{-i})$ is not profitable either.

Case (iii) is symmetry to case (ii), thus we omit its proof. In conclusion, EPIC mechanisms do not have the strong foundation.

C Proof of Proposition 2

Assume that $(i, \theta_i, \theta_j, q_j, \tilde{\theta}_{-ij})$ satisfies the definition of strong improvability. We use the same mechanism as in Proposition 1, except that the allocations for agent i and j change in case agent i reports θ_i and $x = 1$. Recall that, given his truthfully reporting θ_i, agent i's optimal choice of x is $\sqrt{\frac{(\beta/\beta')^2}{1+(\beta/\beta')^2}}$ where β, β' are his first-order beliefs for θ_{-i}, θ_{-i}', respectively, with $\theta_{-i} = (\theta_j, \tilde{\theta}_{-ij})$ and $\theta_{-i}' = (\theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})$; $x = 1$ means that he predicts that j does not have a threshold type for q_j given $(\theta_i, \tilde{\theta}_{-ij})$. The allocations from agents i and j are then modified as follows (and all the other parts of the mechanism are the same as before):

$$
\begin{aligned}
q_j^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= q_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij}),\\
p_j^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= p_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij}),\\
p_j^{**}((\theta_i,1),\theta_j,\tilde{\theta}_{-ij}) &= p_j^*(\theta_i,\theta_j,\tilde{\theta}_{-ij})+\eta, \qquad \forall \theta_j \succ_j^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})\\
p_i^{**}((\theta_i,1),\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij}),\tilde{\theta}_{-ij}) &= M,
\end{aligned}
$$

where $\hat{\theta}_j$ is j's payoff type that is just below $\theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ with respect to $\prec_j^{\theta_i,\tilde{\theta}_{-ij}}$, and $M>0$ is sufficiently large.

Observe that the modified mechanism satisfies all the constraints. First, except for agents i and j, the allocations are the same as in the previous mechanism. For agent i, large fine M is irrelevant unless he assigns zero probability for θ'_{-i} (because $x=1$ is not optimal for him); on the other hand, if he assigns zero probability for θ'_{-i}, then this large fine is also payoff-irrelevant for him. Finally, for agent j, we only need to check his incentive if i reports $(\theta_i,1)$ and $-ij$ report $\tilde{\theta}_{ij}$: in such a case, j with payoff type $\tilde{\theta}_j \precsim_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ has no incentive of misreporting, because their on-path payoffs would be the same as in the original mechanism, while the other types' payments are higher than in the original mechanism. For $\tilde{\theta}_j \succ_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$, his payoff by deviation is at most[23]

$$
\begin{aligned}
& v_j(q_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij}),\theta_i,\tilde{\theta}_j,\tilde{\theta}_{-ij}) - p_j^*((\theta_i,1),\hat{\theta}_j,\tilde{\theta}_{-ij})\\
\leq\ & v_j(q_j^*((\theta_i,1),\tilde{\theta}_j,\tilde{\theta}_{-ij}),\theta_i,\tilde{\theta}_j,\tilde{\theta}_{-ij}) - p_j^*((\theta_i,1),\tilde{\theta}_j,\tilde{\theta}_{-ij}) - \eta,
\end{aligned}
$$

[23]By the same argument as in Footnote 22, pretending to have $\theta_j \preccurlyeq_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ for agent j with $\tilde{\theta}_j \succ_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$ would lead to a payoff loss of at least η. Moreover, he would not misreport a different $\tilde{\theta}'_j \succ_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})$, because the increase in payment for $\{\theta_j \mid \theta_j \succ_i^{\theta_i,\tilde{\theta}_{-ij}} \theta_j^*(q_j,\theta_i,\tilde{\theta}_{-ij})\}$ is uniform.

but the right-hand side is precisely his on-path payoff. The individual rationality constraints can be checked similarly.

Finally, we show that this modified mechanism achieves a strictly higher expected revenue than the original mechanism. First, observe that it does not yield a lower payoff given any payoff-type profile. It is obvious except when the payoff-type profile is $(\theta_i, \theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})$ and agent i chooses $x = 1$; if this is the realized payoff-type profile, and agent i reports $x = 1$, agent i pays a large fine M. Therefore, the principal would be better off by setting M large enough.

Consider a payoff-type profile $(\theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij})$ such that $\tilde{\theta}_j \succ_i^{\theta_i, \tilde{\theta}_{-ij}} \theta_j^*(q_j, \theta_i, \tilde{\theta}_{-ij})$. Due to the full-support assumption on the principal's prior belief for θ, $(\theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij})$ occurs with strictly positive probability; that is, the μ-measure of subset $\{t \in T^* \mid \widehat{\theta}^*(t) = (\theta_i, \tilde{\theta}_j, \tilde{\theta}_{-ij})\}$ is strictly positive. If agent i chooses $x < 1$ (at least with a positive probability), then i pays $\eta(1 - x)(> 0)$ more than in the original mechanism, and hence, strict improvement is achieved. If agent i chooses $x = 1$ (with probability one), then the principal increases j's payment by η as explained above, and thus, again strict improvement is achieved.

D Proof of Theorem 3

It suffices to show that ordinal interdependence implies strong improvability.

By ordinal interdependence, there exist $i, \theta_{-i}, \theta'_{-i}$ such that $\prec_i^{\theta_{-i}} \neq \prec_i^{\theta'_{-i}}$. We first observe the following lemma.

Lemma 6. Ordinal interdependence implies that there exist $j \neq i$, θ_j, θ'_j, and $\tilde{\theta}_{-ij}$ such that $\prec_i^{\theta_j, \tilde{\theta}_{-ij}} \neq \prec_i^{\theta'_j, \tilde{\theta}_{-ij}}$.

Proof of Lemma 6. Let $i = 1$ without loss of generality, and for each $n = 1, \dots, I$, let $\theta^n_{-1} = ((\theta'_j)^n_{j=2}, (\theta_j)^I_{j=n+1})$. Note that $\theta^1_{-1} = \theta_{-1}$ and $\theta^I_{-1} = \theta'_{-1}$.

If $\prec_1^{\theta^{n-1}_{-1}} = \prec_1^{\theta^n_{-1}}$ for all $n = 2, \dots, I$, then we have $\theta^1_{-1} = \theta^I_{-1}$, contradicting that $\prec_1^{\theta_{-1}} \neq \prec_1^{\theta'_{-1}}$. Therefore, there exists $n \in \{2, \dots, I\}$ such that $\prec_1^{\theta^{n-1}_{-1}} \neq \prec_1^{\theta^n_{-1}}$. We complete the proof of the lemma by setting $j = n$ and $\tilde{\theta}_{-1j} = ((\theta'_k)^{n-1}_{k=2}, (\theta_k)^I_{k=n+1})$.

□

By the lemma, there exists θ_i, θ'_i such that $\theta_i \succ_i^{(\theta_j, \tilde{\theta}_{-ij})} \theta'_i$ and $\theta'_i \succ_i^{(\theta'_j, \tilde{\theta}_{-ij})} \theta_i$. Letting $q_i = q_i^{EP}(\theta'_i, \theta_j, \tilde{\theta}_{-ij})$ and $q'_i = q_i^{EP}(\theta'_i, \theta'_j, \tilde{\theta}_{-ij})$, by Assumption 4, we have $\theta'_i = \theta^*_i(q'_i, \theta'_j, \tilde{\theta}_{-ij}) = \theta^*_i(q_i, \theta_j, \tilde{\theta}_{-ij})$. It follows that $\theta_i \in \Theta^*_i(q_i, \theta_j, \tilde{\theta}_{-ij}) \setminus \Theta^*_i(q'_i, \theta'_j, \tilde{\theta}_{-ij})$ and $\theta^*_i(q_i, \theta_j, \tilde{\theta}_{-ij}) \in \Theta^*_i(q'_i, \theta'_j, \tilde{\theta}_{-ij})$. Then, revenue from agent i is improvable with respect to $\big(\theta_i, (\theta_j, \tilde{\theta}_{-ij}), (\theta'_j, \tilde{\theta}_{-ij})\big)$.

If $\theta'_j \prec_j^{\theta_i, \tilde{\theta}_{-ij}} \theta_j$, then choose $q_j = q_j^{EP}(\theta'_j, \theta_i, \tilde{\theta}_{-ij})$. By Assumption 4, we have $\theta'_j = \theta^*_j(q_j, \theta_i, \tilde{\theta}_{-ij})$ and $\theta_j \in \Theta^*_j(q_j, \theta_i, \tilde{\theta}_{-ij}) \setminus \{\theta^*_j(q_j, \theta_i, \tilde{\theta}_{-ij})\}$. Thus, revenue from i is improvable with respect to $\big(\theta_i, (\theta_j, \tilde{\theta}_{-ij}), (\theta^*_j(q_j, \theta_i, \tilde{\theta}_{-ij}), \tilde{\theta}_{-ij})\big)$, which establishes the strong improvability. If $\theta'_j \succ_j^{\theta_i, \tilde{\theta}_{-ij}} \theta_j$, let $q_j = q_j^{EP}(\theta_j, \theta_i, \tilde{\theta}_{-ij})$, then by a symmetry argument we have strong improvability.

E Proof of Proposition 3

The proof comprises several lemmas.

Lemma 7. If $\Gamma = (T, q, p)$ dominates the optimal EPIC mechanism, then $p(\omega, \theta, \beta) \geq p^{EP}(\omega, \theta)$ for all (ω, θ, β).

Proof. First, we allow for μ that does not satisfy our strong admissibility (in the way described below), and prove the statement. Then, we provide a continuity argument showing that, even with strong admissibility, the statement goes through.

Let β^ω be the degenerate belief which puts probability 1 on having ω. Define μ^{EP} such that $\mu^{EP}\big(\{\omega\} \times \{\theta\} \times \{\beta^\omega\}\big) = f(\omega, \theta)$ for all ω, θ.

Because the agent knows the realization of ω when reporting to the principal, the optimal BIC mechanism for μ^{EP} is the optimal EPIC mechanism. Since $R_{\mu^{EP}}(\Gamma) \geq R_f^{EP}$, we have $(q, p)(\omega, \theta, \beta^\omega) = (q^{EP}, p^{EP})(\omega, \theta)$ for all ω, θ.

Suppose that there exists some $\tilde{\omega}, \tilde{\theta}, \tilde{\beta}$ such that $p(\tilde{\omega}, \tilde{\theta}, \tilde{\beta}) < p^{EP}(\tilde{\omega}, \tilde{\theta})$. Consider a joint belief $\hat{\mu}$ satisfying:

(i) $\hat{\mu}\big(\{\omega\} \times \{\theta\} \times \{\beta^\omega\}\big) = f(\omega, \theta)$ for all $(\omega, \theta) \neq (\tilde{\omega}, \tilde{\theta})$;

(ii) $\hat{\mu}\big(\{\tilde{\omega}\} \times \{\tilde{\theta}\} \times \{\tilde{\beta}\}\big) = f(\tilde{\omega}, \tilde{\theta})$.

This $\hat{\mu}$ does not satisfy our strong admissibility, and we take care of this point at the end of the proof.

Given $\hat{\mu}$, we have

$$\begin{aligned} R_{\hat{\mu}}(\Gamma) &= \sum_{(\omega,\theta)\neq(\tilde{\omega},\tilde{\theta})} f(\omega, \theta)p^{EP}(\omega, \theta) + f(\tilde{\omega}, \tilde{\theta})p(\tilde{\omega}, \tilde{\theta}, \tilde{\beta}) \\ &< \sum_{(\omega,\theta)\neq(\tilde{\omega},\tilde{\theta})} f(\omega, \theta)p^{EP}(\omega, \theta) + f(\tilde{\omega}, \tilde{\theta})p^{EP}(\tilde{\omega}, \tilde{\theta}) \\ &= R_f^{EP}. \end{aligned}$$

Now, recall that the above $\hat{\mu}$ does not satisfy our strong admissibility. However, it is easy to construct a sequence $\{\mu^k\}_{k\in\mathbb{N}}$, where each μ^k is strongly admissible and converges to $\hat{\mu}$ (in a weak-* topology), and along the sequence the expected revenue also converges. Therefore, it contradicts that Γ dominates the optimal EPIC mechanism. $\square$

This lemma implies that, if Γ is in $\mathbf{\Gamma}^U$, then it must be in $\mathbf{\Gamma}^*$. The next lemma shows that, based on a similar logic, if a mechanism that dominates the optimal EPIC mechanism is further dominated by an alternative mechanism, this alternative mechanism must have a pointwise (weakly-)higher payment than that mechanism.

Lemma 8. Mechanism $\Gamma = (T, q, p)$ that dominates the optimal EPIC mechanism is dominated by $\hat{\Gamma} = (T, \hat{q}, \hat{p})$ if and only if $p(\omega, \theta, \beta) \geq \hat{p}(\omega, \theta, \beta)$ for all (ω, θ, β).

Next, we obtain further properties of Γ if it is in $\mathbf{\Gamma}^U$.

Lemma 9. If Γ is in $\mathbf{\Gamma}^U$, then it is in $\mathbf{\Gamma}^*$ with $(a(y), b(y))_{y\in[0,1]}$ satisfying (i) $(a(y), b(y))_{y\in[0,1]}$ (as a set of points on $\mathbb{R}^2$) lies on a continuous, convex, non-increasing curve that connects $(2,3)$ and $(3,2)$; and (ii) either $\lim_{y\uparrow 1} a(y) = 3$ or $\lim_{y\downarrow 0} b(y) = 3$ holds (or both).

Proof. We first show (i). By Lemma 7 and the IR constraints, we obtain:

$$\big(q(\omega, \theta, \beta), p(\omega, \theta, \beta)\big) = (q^{EP}(\omega, \theta), p^{EP}(\omega, \theta)),$$

for $(\omega, \theta) = (0,0), (0,1), (1,0), (1,1)$ and any $\beta \in \Delta(\Omega)$; and

$$q(0, 2, \beta) = q(1, 2, \beta) = 1, \quad p(0, 2, \beta) \geq 2, \quad p(1, 2, \beta) \geq 2,$$

for any $\beta \in \Delta(\Omega)$. Recall our notation: $a(y) = p(0, 2, \beta)$, and $b(y) = p(1, 2, \beta)$ for $y = \beta(1)$.

Due to the IC constraint that type $(\theta = 2, y = 0)$ won't pretend to be $(\theta = 0, y = 0)$, we have $a(0) = 2$; similarly we have $b(1) = 2$. For any $y \in (0, 1)$, we must have $a(y) \leq 3$; otherwise reporting $\theta = 1$ is a profitable deviation. Similarly we have $b(y) \leq 3$ for any $y \in (0, 1)$. Notice that

$$(1 - y)(3 - a(y)) + y(3 - b(y)) \geq \max\{1 - y, y\}$$

for all $y \in (0, 1)$. Thus, given any $\epsilon > 0$, we have

$$a(y) - 2 \leq \frac{3 - yb(y) - (1 - y)}{1 - y} - 2 \leq \frac{3 - 2y - (1 - y)}{1 - y} - 2 = \frac{y}{1 - y} < \epsilon,$$

for all $y < \frac{\epsilon}{\epsilon+1}$. Therefore, we have $\lim_{y\downarrow 0} a(y) = 2$. Similarly, we have $\lim_{y\uparrow 1} b(y) = 2$. It follows that $a(y) + b(y) \leq 5$ for any $y \in (0, 1)$; otherwise either $(a(\epsilon), b(\epsilon))$ or $(a(1 - \epsilon), b(1 - \epsilon))$ would be a profitable deviation for sufficiently small $\epsilon (> 0)$. On the other hand, by assumption on μ, the value of $a(1)$ (or $b(0)$) is irrelevant, as long as type $\theta = 2$ with $y \neq 1$ (or $y \neq 0$) won't pretend to be $y = 1$ (or $y = 0$). Thus, without loss of generality, we choose $a(1) = b(0) = 3$.

For each $y \in [0, 1]$, define

$$Y(y) = \big\{(a, b) \in [2, 3]^2 \mid (1 - y)a(y) + yb(y) \leq (1 - y)a + yb\big\},$$

which is convex and compact, and contains $\{(a(y), b(y)) \mid y \in [0, 1]\}$. Define $Y = \bigcap_{y\in[0,1]} Y(y)$, and then Y is also convex and compact. Let ∂Y be the boundary of Y which is below the line $a + b = 5$, that is,

$$\partial Y = \big\{(a, b) \in Y \mid \forall\epsilon > 0, \exists\epsilon, \epsilon_2 \in (0, \epsilon), \text{ s.t. } (a - \epsilon, b - \epsilon_2) \notin Y\big\}.$$

Then, ∂Y is a continuous convex curve connecting the two points $(2,3)$ and $(3,2)$.

∂Y is related with $(a(y), b(y))$ in the following way. On one hand, $(a(y), b(y)) \in \partial Y$ for all $y \in [0,1]$; otherwise there exists $\epsilon, \epsilon_2 > 0$ such that $(a-\epsilon, b-\epsilon_2) \in Y \subseteq Y(y)$, violating the definition of $Y(y)$. On the other hand, for any $(a,b) \in \partial Y$ satisfying $(1-y')a+y'b < (1-y')a'+y'b'$ for some y' and any other $(a',b') \in \partial Y$, we must have $(a,b) = (a(y'), b(y'))$. To prove this, suppose that $(a,b) \neq (a(y'), b(y'))$, then we have $(1-y')a+y'b < (1-y')a(y')+y'b(y')$, which means that $(a,b) \notin Y(y')$. It follows that $(a,b) \notin Y$, contradicting $(a,b) \in \partial Y$.

Putting a on the horizontal axis and b on the vertical axis, we define the function $b(a)$:

$$b(a) = \begin{cases} \lim_{y \to 0} b(y) & \text{if } a = 2 \\ b' \text{ such that } (a,b') \in \partial Y & \text{if } 2 < a \leq 3. \end{cases}$$

Immediately, we have $\partial Y = \{(a,b) \mid b = b(a)\} \cup \{(2,b) \mid b(a) < b \leq 3\}$. Moreover, $b = b(a)$ is convex, continuous, and non-increasing (otherwise it would violate $b(y) \geq 2$ for all y).

Now we show (ii). Suppose we have $\bar{b} = \lim_{y \to 0} b(y) < 3$ and $\bar{a} = \lim_{y \to 1} a(y) < 3$, then define $\bar{y}$ such that $(1-\bar{y})(3-\bar{a}) = \bar{y}(3-\bar{b})$. Consider the following payment rule for $\theta = 2$:

$$\big(\tilde{a}(y), \tilde{b}(y)\big) = \begin{cases} \big(a(y), b(y) + 3 - \bar{b}\big) & \text{if } 0 \leq y \leq \bar{y} \\ \big(a(y) + 3 - \bar{a}, b(y)\big) & \text{if } \bar{y} < y \leq 1. \end{cases}$$

Obviously, $\theta = 0, 1$ won't pretend to be $\theta = 2$. Also, $\theta = 2$ with $y, y' \in [0, \bar{y}]$ (or $y, y' \in (\bar{y}, 1]$) won't pretend to be each other, because the changes in payment are the same. Pick $y \in [0, \bar{y}]$ and $y' \in (\bar{y}, 1]$, since we have

$$\begin{aligned}(1-y)\tilde{a}(y) + y\tilde{b}(y) &= (1-y)a(y) + y\big(b(y) + 3 - \bar{b}\big) \\ &= (1-y)a(y) + yb(y) + y(3-\bar{b}) \\ &\leq (1-y)a(y') + yb(y') + (1-y)(3-\bar{a}) \\ &= (1-y)\big(a(y') + 3 - \bar{a}\big) + yb(y') \\ &= (1-y)\tilde{a}(y') + y\tilde{b}(y'),\end{aligned}$$

then the expected payment is lower by telling the truth than misreporting. By a similar argument, agent with y' won't pretend to have y, either. Agent $\theta = 2$ won't report $\theta = 1$ because

$$\begin{aligned}(1-y)(3-\tilde{a}(y)) + y(3-\tilde{b}(y)) &\geq (1-y)(3-\tilde{a}(1)) + y(3-\tilde{b}(1)) \\ &= (1-y)\big(3 - a(1) - 3 + \bar{a}\big) + y(3-b(1)) \\ &= (1-y)\cdot 0 + y(3 - p(1,1)).\end{aligned}$$

Similarly, $\theta = 2$ won't report $\theta = 0$. Clearly, the new mechanism achieves weakly higher expected revenue than Γ for any μ, and strictly higher expected revenue when $\mu\big(\{0\} \times \{2\} \times \{y \mid \bar{y} < y < 1\}\big) > 0$. □

Now we show the converse, completing the proof of the proposition.

Lemma 10. Suppose that Γ is in $\mathbf{\Gamma}^*$ with $(a(y), b(y))_{y\in[0,1]}$ satisfying (i) $(a(y), b(y))_{y\in[0,1]}$ (as a set of points on $\mathbb{R}^2$) lies on a continuous, convex, non-increasing curve that connects $(2, 3)$ and $(3, 2)$; and (ii) either $\lim_{y\downarrow 1} a(y) = 3$ or $\lim_{y\downarrow 0} b(y) = 3$ holds (or both). Then, Γ is in $\mathbf{\Gamma}^U$.

Proof. Suppose there exists Γ' that dominates Γ. Assume that the payment rules for $\theta = 2$ in Γ and Γ' are $(a(y), b(y))$ and $(a'(y), b'(y))$, respectively. First, we show that the two corresponding curves, denoted by $b = b(a)$ and $b' = b'(a')$, have no interior common point: If they do, then we can find (s, t) with $s \in (2, 3)$ such that, either right derivatives or left derivatives of the two curves at (s, t) are different; otherwise $b = b(a)$ and $b' = b'(a')$ would be the same. Without loss of generality, assume that the left derivatives, given by

$$k = \partial_{-} b(s) := \lim_{a \to s^{-}} \frac{b(a) - b(s)}{a - s}, \quad k' = \partial_{-} b'(s) := \lim_{a' \to s^{-}} \frac{b(a') - b(s)}{a' - s},$$

satisfies $(-\infty <)\, k < k' \,(< 0)$. Then, pick $y \in (k, k')$, and we have

$$a'(y) < s \leq a(y), \quad b'(y) > t \geq b(y).$$

By Lemma 8, Γ and Γ' cannot dominate each other. Thus, $b = b(a)$ and $b' = b'(a')$ can only have common point in $\{(s, t) \mid s = 2 \text{ or } t = 2\}$. Without loss of generality, we assume that $\lim_{y \to 0} b(y) = \lim_{y \to 0} b'(y) = 3$. Additionally, we must have $b'(a) > b(a)$ for all $a \in (2, 3)$; otherwise Γ' cannot dominate Γ.

Fixed $\bar{a} \in (2, 3)$, since $\lim_{y \to 0} \big(b(y) - b'(y)\big) = 0$, we can find $\underline{a} \in (0, \bar{a})$ such that $b'(\underline{a}) - b(\underline{a}) = \frac{1}{2}\big(b'(\bar{a}) - b(\bar{a})\big)$. By Lemma 9, both $b = b(a)$ and $b' = b'(a')$ have derivatives almost everywhere on $[\underline{a}, \bar{a}]$ (as convex functions, they are absolutely continuous). Suppose $\partial_- b(s) \geq \partial_- b'(s)$ for all $s \in [\underline{a}, \bar{a}]$. Then we have

$$\begin{aligned} b'(\bar{a}) - b(\bar{a}) &= b'(\underline{a}) + \int_{\underline{a}}^{\bar{a}} \partial_- b'(s) ds - b(\underline{a}) - \int_{\underline{a}}^{\bar{a}} \partial_- b(s) ds \\ &= b'(\underline{a}) - b(\underline{a}) + \int_{\underline{a}}^{\bar{a}} \Big(\partial_- b'(s) - \partial_- b(s) \Big) ds \\ &\leq b'(\underline{a}) - b(\underline{a}) < b'(\bar{a}) - b(\bar{a}), \end{aligned}$$

which forms a contradiction. Thus, there exists some $s \in [\underline{a}, \bar{a}]$ such that $\partial_- b(s) < \partial_- b'(s)$. Picking y such that $\partial_- b(s) < y < \partial_- b'(s)$, we have

$$b'(y) > b'(s) > b(s) \geq b(y), \quad a'(y) < s \leq a(y).$$

By Lemma 8, Γ' cannot dominate Γ. □

F Miscellany

F.1 A sufficient condition for Assumption 1

When there is no ordinal interdependence, $\prec_i$ does not depend on θ_{-i} for all i. As in Chung and Ely (2007), we consider the auction environment where $\Theta_i = \{\theta_i^1, \ldots, \theta_i^M\}$ satisfying $\theta_i^m - \theta_i^{m-1} = \gamma > 0$ for each m, and agent i's valuation given payoff type profile θ is $v_i(\theta)$. The following lemma characterizes a class of interdependent-value environments satisfying Assumption 1.

Lemma 11. Under the following conditions,

(1) $f(\theta)$ is affiliated: for all θ and θ', we have $f(\theta \vee \theta') f(\theta \wedge \theta') \geq f(\theta) f(\theta')$;

(2) $f(\theta)$ has increasing hazard rate: $\frac{f(\theta_i|\theta_{-i})}{1-F_i(\theta_i|\theta_{-i})}$ is increasing on θ_i;

(3) $v_i(\theta)$ is increasing and concave: $\frac{dv_i(\theta_i,\theta_{-i})}{d\theta_i} > 0$, $\frac{d^2 v_i(\theta_i,\theta_{-i})}{d(\theta_i)^2} \leq 0$;

(4) $v_i(\theta)$ is supermodular: $\forall \theta, \theta'$, we have $v_i(\theta \vee \theta') + v_i(\theta \wedge \theta') \geq v_i(\theta) + v_i(\theta')$;

(5) $\forall \theta_i < \theta_i'$, $\forall j \neq i$, we have $v_i(\theta_i', \theta_{-i}) - v_i(\theta_i, \theta_{-i}) \geq v_j(\theta_i', \theta_{-i}) - v_j(\theta_i, \theta_{-i})$,

virtual valuation satisfies the regularity conditions: $\forall \theta$, $\forall i \in \{1, \ldots I\}$, $\forall j \in \{0, 1, \ldots I\} \setminus \{i\}$, we have

$$\gamma_i(\theta) \geq \gamma_j(\theta) \Longrightarrow \gamma_i(\hat{\theta}_i, \theta_{-i}) > \gamma_j(\hat{\theta}_i, \theta_{-i}), \quad \forall \hat{\theta}_i > \theta_i,$$

where $\gamma_0(\cdot) \equiv 0$, and

$$\gamma_i(\theta_i^m, \theta_{-i}) := v_i(\theta_i^m, \theta_{-i}) - \frac{1 - F_i(\theta_i^m \mid \theta_{-i})}{f(\theta_i^m \mid \theta_{-i})}\Big(v_i(\theta_i^{m+1}, \theta_{-i}) - v_i(\theta_i^m, \theta_{-i})\Big).$$

Proof. Condition (1) implies: $\forall \theta_i < \theta_i'$, $\forall \theta_j < \theta_j'$, $\forall \theta_{-ij}$, we have

$$\frac{f(\theta_i', \theta_j', \theta_{-ij})}{f(\theta_i', \theta_j, \theta_{-ij})} \geq \frac{f(\theta_i, \theta_j', \theta_{-ij})}{f(\theta_i, \theta_j, \theta_{-ij})}.$$

Condition (4) implies: $\forall \theta_i < \theta_i'$, $\forall \theta_j < \theta_j'$, $\forall \theta_{-ij}$, we have

$$v_i(\theta_i', \theta_j', \theta_{-ij}) - v_i(\theta_i, \theta_j', \theta_{-ij}) \geq v_i(\theta_i', \theta_j, \theta_{-ij}) - v_i(\theta_i, \theta_j, \theta_{-ij}).$$

Write $\theta_i^+ := \theta_i + \gamma$. For any $\hat{\theta}_i > \theta_i$, we have

$$
\begin{aligned}
&\big(\gamma_i(\hat{\theta}_i,\theta_{-i}) - \gamma_j(\hat{\theta}_i,\theta_{-i})\big) - \big(\gamma_i(\theta) - \gamma_j(\theta)\big) \\
=&\Big[v_i(\hat{\theta}_i,\theta_{-i}) - \frac{1-F_i(\hat{\theta}_i \mid \theta_{-i})}{f(\hat{\theta}_i \mid \theta_{-i})} \underbrace{\Big(v_i(\hat{\theta}_i^+,\theta_{-i}) - v_i(\hat{\theta}_i,\theta_{-i})\Big)}_{\le v_i(\theta_i^+,\theta_{-i}) - v_i(\theta_i,\theta_{-i})}\Big] \\
&- \Big[v_j(\hat{\theta}_i,\theta_{-i}) - \frac{1-F_j(\theta_j \mid \hat{\theta}_i,\theta_{-ij})}{f(\theta_j \mid \hat{\theta}_i,\theta_{-ij})} \underbrace{\Big(v_j(\theta_j^+,\hat{\theta}_i,\theta_{-ij}) - v_j(\theta_j,\hat{\theta}_i,\theta_{-ij})\Big)}_{\ge v_j(\theta_j^+,\theta_i,\theta_{-ij}) - v_j(\theta_j,\theta_i,\theta_{-ij})}\Big] \\
&- \Big[v_i(\theta_i,\theta_{-i}) - \frac{1-F_i(\theta_i \mid \theta_{-i})}{f(\theta_i \mid \theta_{-i})}\Big(v_i(\theta_i^+,\theta_{-i}) - v_i(\theta_i,\theta_{-i})\Big)\Big] \\
&+ \Big[v_j(\theta_i,\theta_{-i}) - \frac{1-F_j(\theta_j \mid \theta_{-j})}{f(\theta_j \mid \theta_{-j})}\Big(v_j(\theta_j^+,\theta_i,\theta_{-ij}) - v_j(\theta_j,\theta_i,\theta_{-ij})\Big)\Big] \\
\ge&\big(v_i(\hat{\theta}_i,\theta_{-i}) - v_i(\theta_i,\theta_{-i})\big) - \big(v_j(\hat{\theta}_i,\theta_{-i}) - v_j(\theta_i,\theta_{-i})\big) \\
&+ \Big[\frac{1-F_i(\theta_i \mid \theta_{-i})}{f(\theta_i \mid \theta_{-i})} - \frac{1-F_i(\hat{\theta}_i \mid \theta_{-i})}{f(\hat{\theta}_i \mid \theta_{-i})}\Big]\Big(v_i(\theta_i^+,\theta_{-i}) - v_i(\theta_i,\theta_{-i})\Big) \\
&+ \sum_{\tilde{\theta}_j > \theta_j} \Big[\frac{f(\tilde{\theta}_j,\hat{\theta}_i,\theta_{-ij})}{f(\hat{\theta}_i,\theta_{-i})} - \frac{f(\tilde{\theta}_j,\theta_i,\theta_{-ij})}{f(\theta_i,\theta_{-i})}\Big]\Big(v_j(\theta_j^+,\theta_i,\theta_{-ij}) - v_j(\theta_j,\theta_i,\theta_{-ij})\Big) \\
\ge&0.
\end{aligned}
$$

Thus, $\gamma_i(\hat{\theta}_i,\theta_{-i}) - \gamma_j(\hat{\theta}_i,\theta_{-i}) \ge 0$ as long as $\gamma_i(\theta) - \gamma_j(\theta) \ge 0$. □

F.2 Proof of Lemma 3

Let $\big(q_i^{EP}(\theta), p_i^{EP}(\theta)\big)_{i,\theta}$ denote the optimal EPIC mechanism. First, we prove that any agent with non-positive virtual value should not be assigned the good in the optimal EPIC mechanism.

Lemma 12. Under condition-(i) in Lemma 3, if $\gamma_i(q_i,\theta) \le 0$ for any q_i, then $q_i^{EP}(\theta) = 0$.

Proof of Lemma 12. Due to condition-(i), we can find the type $\theta_i^-(\theta_{-i})$ such that $\gamma_i(q_i,\theta_i,\theta_{-i}) \leq 0$ for all q_i if and only if $\theta_i \preccurlyeq_i^{\theta_{-i}} \theta_i^-(\theta_{-i})$. Consider a relaxed problem where for all $\theta_i \preccurlyeq_i^{\theta_{-i}} \theta_i^-(\theta_{-i})$, we ignore agent i's monotonicity constraints involving (θ_i,θ_{-i}), and replace the feasibility constraints at (θ_i,θ_{-i}) by $(0,(q_j(\theta_i,\theta_{-i})_{j\neq i}) \in Q$. Immediately, the solution to this relaxed problem satisfies $q_i^{Re}(\theta_i,\theta_{-i}) = 0$ for all $\theta_i \preccurlyeq_i^{\theta_{-i}} \theta_i^-(\theta_{-i})$. Notice that $q_i^{Re}(\theta_i,\theta_{-i}) = 0 \leq q_i^{Re}(\theta_i',\theta_{-i})$ for all $\theta_i' \succ_i^{\theta_{-i}} \theta_i^-(\theta_{-i})$, and $(q_i^{Re}(\theta_i,\theta_{-i}),(q_j^{Re}(\theta_i,\theta_{-i}))_{j\neq i}) \in Q$. Then q^{Re} satisfies all the constraints in the original problem. Thus, $q_i^{EP}(\theta_i,\theta_{-i}) = 0$ for all $\theta_i \preccurlyeq_i^{\theta_{-i}} \theta_i^-(\theta_{-i})$. □

Now we can prove Lemma 3. Let θ_i, θ_i', θ_{-i} and θ_{-i}' be the payoff types satisfying condition-(ii). For θ_{-i}, by condition-(i) we can find the type $\theta_i^+(\theta_{-i})$ such that $\gamma_i(q_i,\tilde{\theta}_i,\theta_{-i}) > 0$ for some q_i if and only if $\tilde{\theta}_i \succcurlyeq_i^{\theta_{-i}} \theta_i^+(\theta_{-i})$. By condition-(ii-c), $\gamma_j(q_j,\tilde{\theta}_i,\theta_{-i}) \leq 0$ for all q_j and $\tilde{\theta}_i$. Then by Lemma 12 we have $q_j^{Re}(\tilde{\theta}_i,\theta_{-i}) = 0$ for all $j \neq i$.

It suffices to show that $q_i^{EP}(\theta_i,\theta_{-i}) > 0$.[24]. Suppose not. Then, by monotonicity, there exists $\hat{\theta}_i \succ_i^{\theta_{-i}} \theta_i$ such that $q_i^{EP}(\tilde{\theta}_i,\theta_{-i}) = 0$ if $\tilde{\theta}_i \prec_i^{\theta_{-i}} \hat{\theta}_i$, and $q_i^{EP}(\tilde{\theta}_i,\theta_{-i}) > 0$ if $\tilde{\theta}_i \succcurlyeq_i^{\theta_{-i}} \hat{\theta}_i$.

Let q_i^+ be the smallest $q_i > 0$ such that $(q_i,0,\ldots,0) \in Q$. By condition-(iv), we have $\gamma_i(q_i^+,\theta_i^+(\theta_{-i}),\theta_{-i}) > 0$.[25] Because $q_i^+ \leq q_i^{EP}(\hat{\theta}_i,\theta_{-i})$, we can modify the allocation rule so that $q_i^{EP}(\tilde{\theta}_i,\theta_{-i}) = q_i^+$ if $\theta_i^+(\theta_{-i}) \preccurlyeq_i^{\theta_{-i}} \tilde{\theta}_i \prec_i^{\theta_{-i}} \hat{\theta}_i$ (and the other parts of q^{EP} stays the same as before) without violating any constraint, and hence, this modified mechanism is an EPIC mechanism. However, it is strictly better than the original q^{EP}, contradicting that q^{EP} is an optimal EPIC mechanism.

[24] $q_i^{EP}(\theta_i',\theta_{-i}') > 0$ can be similarly shown by a symmetric argument

[25] Because $\gamma_i(q_i,\theta_i^+(\theta_{-i}),\theta_{-i}) > 0$ for some q_i, we only need to consider the case $q_i > q_i^+$. Since $\gamma_i(0,\theta_i^+(\theta_{-i}),\theta_{-i}) = 0$, by strict quasi-concavity, we have $\gamma_i(q_i^+,\theta_i^+(\theta_{-i}),\theta_{-i}) > 0$.

In conclusion, θ_i, θ'_i, θ_{-i} and θ'_{-i} satisfy the requirement in Assumption 3.

References

Bergemann, D., and S. Morris (2005): "Robust mechanism design," *Econometrica*, 73(6), 1771–1813.

——— (2009b): "Robust implementation in direct mechanisms," *The Review of Economic Studies*, 76(4), 1175–1204.

Bikhchandani, S. (2006): "Ex post implementation in environments with private goods," *Theoretical Economics*, 1(3), 369–393.

Börgers, T. (2017): "(No) Foundations of dominant-strategy mechanisms: a comment on Chung and Ely (2007)," *Review of Economic Design*, 21(2), 73–82.

Brandenburger, A., and E. Dekel (1993): "Hierarchies of Beliefs and Common Knowledge," *Journal of Economic Theory*, 59, 189–198.

Brooks, B. A., and S. Du (2019): "Optimal auction design with common values: An informationally-robust approach," *Available at SSRN 3137227*.

Chen, Y.-C., and J. Li (2018): "Revisiting the foundations of dominant-strategy mechanisms," *Journal of Economic Theory*, 178, 294–317.

Chung, K.-S., and J. C. Ely (2007): "Foundations of dominant-strategy mechanisms," *The Review of Economic Studies*, 74(2), 447–476.

Dasgupta, P., and E. Maskin (2000): "Efficient auctions," *Quarterly Journal of Economics*, 115(2), 341–388.

Du, S. (2018): "Robust mechanisms under common valuation," *Econometrica*, 86(5), 1569–1588.

Goldberg, A. V., J. D. Hartline, A. R. Karlin, M. Saks, and A. Wright (2006): "Competitive auctions," *Games and Economic Behavior*, 55(2), 242–269.

Jehiel, P., M. Meyer-ter Vehn, B. Moldovanu, and W. R. Zame (2006): "The limits of ex post implementation," *Econometrica*, 74(3), 585–610.

Mertens, J.-F., and S. Zamir (1985): "Formulation of Bayesian Analysis for Games with Incomplete Information," *International Journal of Game Theory*, 14, 1–29.

Mussa, M., and S. Rosen (1978): "Monopoly and Product Quality," *Journal of Economic Theory*, 18(2), 301–317.

Segal, I. (2003): "Optimal pricing mechanisms with unknown demand," *The American economic review*, 93(3), 509–529.

Sims, C. A. (2003): "Implications of rational inattention," *Journal of Monetary Economics*, 50(3), 665–690.

Wilson, R. (1985): "Incentive efficiency of double auctions," *Econometrica: Journal of the Econometric Society*, pp. 1101–1115.

Yamashita, T. (2015): "Implementation in Weakly Undominated Strategies: Optimality of Second-Price Auction and Posted-Price Mechanism," *Review of Economic Studies*, 82(3), 1223–1246.