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A DYNKIN GAME ON ASSETS WITH INCOMPLETE INFORMATION ON THE RETURN

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ABSTRACT. This paper studies a 2-players zero-sum Dynkin game arising from pricing an option on an asset whose rate of return is unknown to both players. Using filtering techniques we first reduce the problem to a zero-sum Dynkin game on a bi-dimensional diffusion (X, Y). Then we characterize the existence of a Nash equilibrium in pure strategies in which each player stops at the hitting time of (X, Y) to a set with moving boundary. A detailed description of the stopping sets for the two players is provided along with global C^1 regularity of the value function.

1. INTRODUCTION

Zero-sum optimal stopping games (Dynkin games) have received a lot of attention since the seminal paper by Dynkin [8], see also the classical references [1] and [20]. In particular, these games have found applications in mathematical finance where the arbitrage-free pricing of American options with early cancellation (game options) relies on the computation of the value of a zero-sum game of optimal stopping between the buyer and the seller (see [17],[19]). A common assumption in the financial application of Dynkin games is that the players have complete information about the parameters of the underlying stochastic process. In practice, however, there are many situations in which parameters are difficult to estimate like for instance, the rate of dividend in the Black-Scholes setting or the convenience yield in the pricing of commodity contingent claims.

Motivated by the above considerations, in this paper we study zero-sum optimal stopping games with incomplete information about the return of the underlying asset. We are interested in the existence of the value as well the existence and characterization of Nash equilibria for the game. To tackle these questions, we will focus on a specific setting by considering a game between the buyer (player 1) and the seller (player 2) of a call option written on a stock that evolves according to

$$dS_t = (r - \delta_0 D)S_t dt + \sigma S_t dB_t, \quad S_0 = x > 0, \tag{1.1}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion, $r > 0$ the risk-free rate, $\sigma > 0$ the volatility, $\delta_0 D$ the dividend rate, with $\delta_0 > 0$ and deterministic, and D random. The uncertainty about dividend payments is modeled by a non-negative random variable D which is unobserved by the two players and is independent of B. The random variable D takes the values 0 or 1 with $\mathbb{P}(D = 1) = y$.

In our setting we denote by $\mathcal{F}^S := (\mathcal{F}^S)_{t\geq 0}$ the filtration generated by the observed process S and by $\mathcal{T}^S$ the set of $\mathcal{F}^S$-stopping times. The buyer aims at maximising the discounted expected payoff of early exercise by choosing a stopping time $\tau \in \mathcal{T}^S$ at which to use the option. On the other hand the seller has the right to cancel the contract by paying the call payoff plus a penalty. The seller picks a stopping time $\gamma \in \mathcal{T}^S$ at which cancellation occurs and aims at minimising expected costs.

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In the context of the classical Black-Scholes model (1.1) holds with $\mathbb{P}(D=1)=1$ or $\mathbb{P}(D=1)=0$ (the non-dividend case). Explicit computations have been established by [11] and [28] in the perpetual case. Both papers show that the dividend parameter δ_0 plays an important role for the existence of an equilibrium in the game and this will be the case also in the present work.

In this paper we fix $K>0$ and $\varepsilon_0>0$ and let

$$G_1(x):=(x-K)^+, \qquad G_2(x):=(x-K)^+ + \varepsilon_0 \tag{1.2}$$

be the payoff of immediate stopping for player 1 (the option buyer) and the cost of cancellation for player 2 (the seller), respectively. Then the formulation of our game is the following: on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ the expected discounted payoff of the game is

$$M_{x,y}(\tau,\gamma)=\mathbb{E}[e^{-r\tau}G_1(S_\tau)\mathbb{1}_{\{\tau\le\gamma\}}+e^{-r\gamma}G_2(S_\gamma)\mathbb{1}_{\{\gamma<\tau\}}] \tag{1.3}$$

where $\tau,\gamma\in\mathcal{T}^S$ are the stopping times for the two players described above. The notation $M_{x,y}$ accounts for the dependence of the stopping functional on the initial stock price and on the a priori probability of the event $\{D=1\}$. This notation will be fully justified and explained in Section 2 below.

As usual we define the upper value and the lower value of the stopping game, respectively by

$$\overline{V}(x,y)=\inf_\gamma\sup_\tau M_{x,y}(\tau,\gamma) \text{ and } \quad \underline{V}(x,y)=\sup_\tau\inf_\gamma M_{x,y}(\tau,\gamma). \tag{1.4}$$

When $\underline{V}(x,y)=\overline{V}(x,y)$, the game has a value $V(x,y):=\underline{V}(x,y)=\overline{V}(x,y)$. Moreover, if there exist two stopping times (τ_*,γ_*) such that

$$M_{x,y}(\tau,\gamma_*)\le M_{x,y}(\tau_*,\gamma_*)\le M_{x,y}(\tau_*,\gamma)$$

for all stopping times τ and γ, the pair (τ_*,γ_*) is a saddle point or a *Nash equilibrium* for the optimal stopping game and in that case the game has a value with $V(x,y)=M_{x,y}(\tau_*,\gamma_*)$.

We recall now some results from the existing literature so that we can later discuss the mathematical novelty of our work. The *existence of the value* for optimal stopping games with multi-dimensional Markov processes was proved in [10] using martingale methods and by Bensoussan and Friedman [1] via variational inequalities. These methods require suitable integrability of the payoff processes, i.e., in our notation, the processes $e^{-rt}G_i(S_t)$, $i=1,2$ must be uniformly integrable. When such condition is not fulfilled, the existence of the value was proven in [11] but only for one-dimensional diffusions. Results in [11] rely upon a generalized type of concavity introduced in [9] and brought up to date in [5].

On the other hand sufficient conditions for the *existence of Nash equilibria* in Markovian setting have been studied in [10] and [11]. For a rather general class of Markov processes these conditions include the above mentioned uniform integrability of the payoff processes. In the special case of one-dimensional diffusions weaker integrability may instead be sufficient (see [11], Proposition 4.3).

In our setting we are faced with two main technical difficulties in establishing existence of the value and of a Nash equilibrium: (i) the process S is not Markovian and (ii) it fails to fulfil the condition of uniformly integrability (see Remark 2.1), i.e. for any initial condition $S_0=x\in\mathbb{R}_+$ we have

$$\mathbb{E}\left(\sup_{0\le t<\infty}e^{-rt}G_i(S_t)\right)=+\infty, \quad i=1,2. \tag{1.5}$$

To overcome the first difficulty we rely upon filtering theory and increase the dimension of our state space. Informally we could say that we take into account the progressive update of the players' estimate on D, based on the observation of S. This approach leads us to study a two dimensional Markovian system which we denote by $(X_t, Y_t)_{t\geq 0}$, where (at least formally) $X = S$ and $Y_t = \mathbb{E}[D|\mathcal{F}^S_t]$. On the other hand, to tackle the lack of uniform integrability and prove the existence of the value of the game, we adapt methods developed by Lepeltier-Maingueneau [20] and Ekstrom-Peskir [10].

After we prove existence of the value, we are then in the position to carry out a detailed analysis of the structure of the stopping sets for the two players, i.e. the subsets of the state space in which $V = G_i$, $i = 1, 2$. Denoting $\mathcal{S}_i := \{V = G_i\}$, $i = 1, 2$ we study properties of the boundaries of $\mathcal{S}_1$ and $\mathcal{S}_2$ which we subsequently use to state conditions for the existence of a saddle point (Nash equilibrium). The latter is provided in terms of hitting times to $\mathcal{S}_1$ and $\mathcal{S}_2$.

In our analysis we use two equivalent representations of the two-dimensional dynamics. These are linked to one another by a deterministic transformation. Indeed we first observe that the process (X_t, Y_t) is driven by only one Brownian motion and it is therefore degenerate; then we perform a change of coordinates to obtain a new process (Z_t, Y_t). Here Z_t is deterministic and either increasing or decreasing, depending on the choice of parameters in the problem. Effectively the process Z plays the role of a 'time' process.

We would like to emphasise that the probabilistic study of free boundary problems related to zero-sum Dynkin games on two dimensional diffusions has not received much attention so far. Works in this direction but in a parabolic setting are [7] and [28]. Our analysis here goes beyond results in those papers by showing for example that the value of the game is a globally C^1 function of the state variables (x, y).

The outline of the paper is as follows. In Section 2, we specify the model and provide a Markovian formulation of the zero-sum game. Existence and continuity of the value for the game (1.3) is obtained in Section 3. The geometry of the stopping sets is obtained in Section 4, in the (x, y)-plane, and in Section 5, in the (z, y)-plane (parabolic formulation). Hitting times to those sets are used in Section 6 to prove higher regularity of the value, e.g. its global C^1 regularity. Finally we obtain sufficient conditions for the existence of a saddle point in Section 7. Some technical results are collected in Appendix.

2. Dynamics of the underlying asset

We begin by considering the probability space $\Omega^0 = C([0 + \infty), \mathbb{R}) \times \{0, 1\}$ endowed with a sigma algebra $\mathcal{F}^0$ and the product probability $\mathbb{P}^0_y = \mathbb{W} \otimes \pi(y)$ where $\mathbb{W}$ is the standard Wiener measure and $\pi(y) = (1 - y, y)$. Let us denote $((B_t)_{t\geq 0}, D)$ a canonical element of Ω^0 and let $r > 0$, $\delta_0 > 0$ and $\sigma > 0$ be fixed. Then the stock price with uncertainty on its dividend rate which is described by (1.1) has an explicit expression in terms of the couple (B, D), i.e.

$$S^x_t = xe^{\sigma B_t + (r - \delta_0 D)t - \frac{t\sigma^2}{2}} \tag{2.1}$$

The process S^x is a geometric Brownian motion whose drift parameter depends on the unobservable random variable D. We recall that the latter is independent of the Brownian motion B. As discussed in the introduction a technical difficulty arising in our model is the lack of uniform integrability of the process S^x.

Remark 2.1. *If $y \in (0, 1)$, the process $e^{-rt}S^x_t$ is not uniformly integrable because*

$$\lim_{t\to+\infty} e^{-rt}S^x_t = \lim_{t\to+\infty} xe^{\sigma B_t - \frac{t\sigma^2}{2}}\left(\mathbb{1}_{\{D=0\}} + \mathbb{1}_{\{D=1\}}e^{-\delta_0 t}\right) = 0, \quad \mathbb{P}^0_y - a.s.,$$

whereas

$$\lim_{t\to+\infty}\mathbb{E}^0_y[e^{-rt}S^x_t]=(1-y)x.$$

Hence, by linearity of the payoffs G_i, $i=1,2$ in (1.2) we obtain (1.5).

We aim at giving a rigorous formulation for the game call option (1.3). One way to do it is to replace $\mathbb{P}$ and $(S_t)_{t\geq 0}$ in (1.3) by $\mathbb{P}^0_y$ and $(S^x_t)_{t\geq 0}$ defined above. Letting $\mathcal{F}^S:=(\mathcal{F}^S)_{t\geq 0}$ be the (right-continuous) filtration generated by S^x and denoting $\mathcal{T}^S$ the set of $\mathcal{F}^S$-stopping times, then the optimisation is taken over stopping times $(\tau,\gamma)\in\mathcal{T}^S$. The disadvantage of this formulation is that the dynamics of S^x is not Markovian and therefore for the solution of the problem we cannot rely upon free boundary methods. To overcome this difficulty we want to reduce our problem to a Markovian framework by using filtering techniques.

Define the process $(D^y_t)_{t\geq 0}$ as an $\mathcal{F}^S$-càdlàg version of the martingale $(\mathbb{E}^0_y[D|\overline{\mathcal{F}}^S_t])_{t\geq 0}$ where $\overline{\mathcal{F}}^S:=(\overline{\mathcal{F}}^S)_{t\geq 0}$ denotes the usual augmentation of the filtration $\mathcal{F}^S$ with the $\mathbb{P}^0_y$-null sets (and therefore satisfies the usual assumptions). Notice that $(D^y_t)_{t\geq 0}$ is a bounded martingale that converges almost surely to D. The latter is $\overline{\mathcal{F}}^S_\infty$-measurable because

$$\frac{1}{t}\ln(S^x_t)=\frac{1}{t}\left[\ln(x)+(\sigma B_t-\frac{t\sigma^2}{2})-\delta_0 t\mathbb{1}_{\{D=1\}}\right]\underset{t\to\infty}{\longrightarrow}\frac{\sigma^2}{2}-\delta_0\mathbb{1}_{\{D=1\}}.$$

According to Chapter 9 in Lipster-Shiryaev [21] (see also Chapter 4.2 in Shiryaev [27]), the process (S^x,D^y) is the unique strong solution to the following SDE,

$$\left\{\begin{array}{l} dS^x_t=(r-\delta_0 D^y_t)S^x_t dt+\sigma S^x_t d\widehat{W}_t\\ dD^y_t=-\frac{\delta_0}{\sigma}D^y_t(1-D^y_t)d\widehat{W}_t\end{array}\right. \tag{2.2}$$

where

$$\widehat{W}_t=\frac{1}{\sigma}\left(\int_0^t(S^x_u)^{-1}dS^x_u+\int_0^t(r-\delta_0 D^y_u)du\right)$$

is an $\overline{\mathcal{F}}^S$-adapted Brownian motion under $\mathbb{P}^0_y$. The couple (S^x,D^y) is therefore adapted to the augmentation of the filtration generated by $\widehat{W}$, which we denote by $\overline{\mathcal{F}}^{\widehat{W}}$. This implies in particular $\overline{\mathcal{F}}^S\subseteq\overline{\mathcal{F}}^{\widehat{W}}$ and $\overline{\mathcal{F}}^{\widehat{W}}=\overline{\mathcal{F}}^S$ because $\widehat{W}$ is $\overline{\mathcal{F}}^S$-adapted. Notice also that the process $(D^y_t)_{t\geq 0}$ is adapted to the filtration $\overline{\mathcal{F}}^S$ by construction, so that it is no surprise that the new Brownian motion $\widehat{W}$ is also adapted to $\overline{\mathcal{F}}^S$.

Above we have obtained (S^x,D^y) on the space $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0_y)$ which depends on the probability distribution of the random variable D. We prefer to get rid of such dependence and consider another process $(X_t,Y_t)_{t\geq 0}$, having the same law as $(S^x_t,D^y_t)_{t\geq 0}$, but defined below on a new probability space.

Take a probability space $(\Omega,\mathcal{F},\mathbb{P})$, denote by $W:=(W_t)_{t\geq 0}$ a Brownian motion on this space and by $\mathbb{F}:=(\mathcal{F}_t)_{t\geq 0}$ the augmentation of the filtration that it generates. For $(x,y)\in\mathbb{R}_+\times(0,1)$, let (X,Y) be the unique strong solution of the bi-dimensional SDE

$$\left\{\begin{array}{ll} dX_t=(r-\delta_0 Y_t)X_t dt+\sigma X_t dW_t, & X_0=x,\\ dY_t=-\frac{\delta_0}{\sigma}Y_t(1-Y_t)dW_t, & Y_0=y.\end{array}\right. \tag{2.3}$$

To keep track of the initial point we use the notation $(X^{x,y},Y^y)$ and notice that by standard theory $(t,x,y)\mapsto(X^{x,y}_t,Y^y_t)$ is indeed continuous $\mathbb{P}$-almost surely. Notice also that the second equation is independent of the first one and therefore its solution, Y^y, is independent of x.

Since the processes $\{(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}),(X^{x,y},Y^y)\}$ and $\{(\Omega^0,\mathcal{F}^0,\overline{\mathcal{F}}^{\widehat{W}},\mathbb{P}^0_y),(S^x,D^y)\}$ have the same law then the game option is more conveniently formulated using the former since it is Markovian and the probability measure is independent of y. This will be done in the next section.

Often in what follows we use the notation $\mathbb{P}_{x,y}(\,\cdot\,)=\mathbb{P}(\,\cdot\,|X_0=x,Y_0=y)$ and drop the apex in the couple (X,Y). Before closing the section we notice that for all $t\geq 0$

$$X^{x,y}_t = x\exp\left(\int_0^t (r-\delta_0 Y^y_s-\frac{\sigma^2}{2})\,ds+\sigma W_t\right),\quad \mathbb{P}-\text{a.s.} \tag{2.4}$$

Moreover we recall that since $(\zeta_t)_{t\geq0}:=(e^{-rt}X_t)_{t\geq0}$ is a continuous super-martingale, with last element $\zeta_\infty:=\lim_{t\to\infty}\zeta_t=0$, the optional sampling theorem guarantees (see [15, Thm.1.3.22])

$$\mathbb{E}_x\left[e^{-r\rho}X_\rho|\mathcal{F}_\nu\right]\leq e^{-r\nu}X_\nu,\quad \mathbb{P}_x-\text{a.s.} \tag{2.5}$$

for all stopping times $\rho\geq\nu$.

3. The game and its value

The payoffs G_i, $i=1,2$ in (1.2) are non-decreasing and 1-Lipschitz continuous on $\mathbb{R}_+$ with $0\leq G_1<G_2$. It is also clear that

$$\lim_{t\to\infty}e^{-rt}G_i(X^{x,y}_t)=0,\qquad \mathbb{P}-a.s. \tag{3.1}$$

for any $(x,y)\in\mathbb{R}_+\times(0,1)$, due to the first formula in Remark 2.1. We now recall the formulation of the game expected payoff (1.3) given in the Introduction and notice that, thanks to the equivalence explained in the previous section, we can rewrite it as

$$M_{x,y}(\tau,\gamma)=\mathbb{E}\left[e^{-r\tau}G_1(X^{x,y}_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(X^{x,y}_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\right]. \tag{3.2}$$

The stopping times (τ,γ) are drawn from the set $\mathcal{T}$ of $\mathbb{F}$-stopping times and the dependence of $M_{x,y}(\tau,\gamma)$ on (x,y) is clearly expressed. Thanks to (3.1) on the event $\{\tau\wedge\gamma=+\infty\}$ we simply get a zero payoff for both players.

We recall here that player 1 (the buyer) picks τ in order to maximise (3.2), whereas player 2 (the seller) chooses γ in order to minimise (3.2). The upper value $\overline{V}$ and the lower value $\underline{V}$ of the game are expressed as in (1.4). We spend the rest of this section proving that these functions indeed coincide so that the game has a value V.

We start by proving some regularity result of $\overline{V}$ and $\underline{V}$.

Lemma 3.1. *The functions $\overline{V}$ and $\underline{V}$ are:*

(i) *non-decreasing with respect to (w.r.t.) x and non-increasing w.r.t. y*
(ii) *1-Lipschitz w.r.t. x, uniformly w.r.t. $y\in[0,1]$*
(iii) *locally Lipschitz w. r. t. y, i.e. for $f=\underline{V}$ or $f=\overline{V}$ and a given constant $C>0$ we have*

$$|f(x,y)-f(x,y')|\leq C(1+|x|)|y-y'|,\quad \forall x>0,\forall y,y'\in[0,1].$$

Proof of Lemma 3.1. Without loss of generality, we only provide full details for $\underline{V}$.

[*Proof of (i)*] Let us first prove monotonicity with respect to x. Fix $y\in(0,1)$ and $x\geq x'$, then for any $\varepsilon>0$, there exist a couple $(\tau_\varepsilon,\gamma_\varepsilon)$ such that

$$M_{x,y}(\tau_\varepsilon,\gamma_\varepsilon)\leq\underline{V}(x,y)+\frac{\varepsilon}{2}\quad\text{and}\quad M_{x',y}(\tau_\varepsilon,\gamma_\varepsilon)\geq\underline{V}(x',y)-\frac{\varepsilon}{2}. \tag{3.3}$$

Therefore we also have

$$\begin{aligned}\underline{V}(x,y)-\underline{V}(x',y)\geq& M_{x,y}(\tau_\varepsilon,\gamma_\varepsilon)-M_{x',y}(\tau_\varepsilon,\gamma_\varepsilon)-\varepsilon\\ =&\mathbb{E}\left[e^{-r(\tau_\varepsilon\wedge\gamma_\varepsilon)}((X^{x,y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-K)^+-(X^{x',y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-K)^+)\right]-\varepsilon\\ \geq&-\varepsilon\end{aligned}$$

where the last inequality follows by observing that $X^{x,y}_t\geq X^{x',y}_t$, $\mathbb{P}$-a.s. for $t\geq 0$ thanks to (2.4). Since ε was arbitrary we have $x\mapsto \underline{V}(x,y)$ non-decreasing.

To prove monotonicity with respect to y we argue in a similar way. We fix $x\in\mathbb{R}_+$ and $y\leq y'$, and for any $\varepsilon>0$ we can find a couple $(\tau_\varepsilon,\gamma_\varepsilon)$ such that

$$\begin{aligned}\underline{V}(x,y)-\underline{V}(x,y')\geq& M_{x,y}(\tau_\varepsilon,\gamma_\varepsilon)-M_{x,y'}(\tau_\varepsilon,\gamma_\varepsilon)-\varepsilon\\ =&\mathbb{E}\left[e^{-r(\tau_\varepsilon\wedge\gamma_\varepsilon)}((X^{x,y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-K)^+-(X^{x,y'}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-K)^+)\right]-\varepsilon\\ \geq&-\varepsilon.\end{aligned}$$

For the last inequality this time we have used the comparison principle for SDEs, which guarantees $Y^y_t\leq Y^{y'}_t$, $\mathbb{P}$-a.s. for $t\geq 0$, and (2.4), which gives $X^{x,y}_t\geq X^{x,y'}_t$, $\mathbb{P}$-a.s. for $t\geq 0$. By arbitrariness of ε we obtain the claim.

[*Proof of (ii)*] As above we fix $y\in(0,1)$ and $x\geq x'$ so that $\underline{V}(x,y)-\underline{V}(x',y)\geq 0$. For any $\varepsilon>0$ we can find a couple $(\tau_\varepsilon,\gamma_\varepsilon)$ such that

$$\begin{aligned}0\leq\underline{V}(x,y)-\underline{V}(x',y)\leq& M_{x,y}(\tau_\varepsilon,\gamma_\varepsilon)-M_{x',y}(\tau_\varepsilon,\gamma_\varepsilon)+\varepsilon\\ \leq&\mathbb{E}\left[e^{-r(\tau_\varepsilon\wedge\gamma_\varepsilon)}|X^{x,y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-X^{x',y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}|\right]+\varepsilon\end{aligned}\tag{3.4}$$

where the second inequality uses the Lipschitz property of the call payoff. From (2.4) we have

$$e^{-r(\tau_\varepsilon\wedge\gamma_\varepsilon)}|X^{x,y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-X^{x',y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}|\leq|x-x'|e^{\sigma W_{\tau_\varepsilon\wedge\gamma_\varepsilon}-\frac{\sigma^2}{2}(\tau_\varepsilon\wedge\gamma_\varepsilon)}.$$

Since $\exp(\sigma W_t-\frac{\sigma^2}{2}t)$, $t\geq 0$ is a positive surmartingale, we deduce that

$$\mathbb{E}[e^{-r(\tau_\varepsilon\wedge\gamma_\varepsilon)}|X^{x,y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}-X^{x',y}_{\tau_\varepsilon\wedge\gamma_\varepsilon}|]\leq|x-x'|$$

and Lipschitz continuity in x follows from (3.4) since $\varepsilon>0$ is arbitrary.

[*Proof of (iii)*] Now we use the equivalence between the couple $(X^{x,y},Y^y)$ on the space $(\Omega,\mathcal{F},\mathbb{P})$ and the couple (S^x,D^y) on the space $(\Omega^0,\mathcal{F}^0,\mathbb{P}^0_y)$ (see explanation in Sec. 2 and (1.3) and (3.2)) to write

$$\begin{aligned}M_{x,y}(\gamma,\tau)=&\,\mathbb{E}^0_y\left[e^{-r\tau}G_1(S^x_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(S^x_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\right]\\ =&\,y\,\mathbb{E}^0_y\left[e^{-r\tau}G_1(S^x_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(S^x_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\middle|D=1\right]\\ &+(1-y)\,\mathbb{E}^0_y\left[e^{-r\tau}G_1(S^x_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(S^x_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\middle|D=0\right]\end{aligned}$$

for any couple $(\tau,\gamma)\in\mathcal{T}^S$. Set

$$S^{1,x}_t=xe^{\sigma B_t+(r-\delta_0-\frac{\sigma^2}{2})t},\qquad S^{0,x}_t=xe^{\sigma B_t+(r-\frac{\sigma^2}{2})t}$$

and notice that conditionally on D, the law of S^x is independent of y, so denoting $\mathbb{E}^W$ the expectation under the Wiener measure $\mathbb{W}$ we get

$$\begin{aligned}M_{x,y}(\gamma,\tau)=&\,y\,\mathbb{E}^W\left[e^{-r\tau}G_1(S^{1,x}_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(S^{1,x}_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\right]\\ &+(1-y)\,\mathbb{E}^W\left[e^{-r\tau}G_1(S^{0,x}_\tau)\mathbb{1}_{\{\tau\leq\gamma\}}+e^{-r\gamma}G_2(S^{0,x}_\gamma)\mathbb{1}_{\{\gamma<\tau\}}\right]\end{aligned}\tag{3.5}$$

Now we use the above representation of the game payoff as follows. Fix $x \in \mathbb{R}_+$ and $y \le y'$, then for any $\varepsilon > 0$ we find $(\tau_\varepsilon, \gamma_\varepsilon) \in \mathcal{T}^S$ such that

$$\begin{aligned} 0 \le& \underline{V}(x,y) - \underline{V}(x,y') \\ \le& M_{x,y}(\gamma_\varepsilon,\tau_\varepsilon) - M_{x,y'}(\gamma_\varepsilon,\tau_\varepsilon) + \varepsilon \\ \le& |y-y'| \Big(\mathbb{E}^W \left[e^{-r\tau_\varepsilon} G_1(S^{1,x}_{\tau_\varepsilon}) \mathbb{1}_{\{\tau_\varepsilon \le \gamma_\varepsilon\}} + e^{-r\gamma_\varepsilon} G_2(S^{1,x}_{\gamma_\varepsilon}) \mathbb{1}_{\{\gamma_\varepsilon < \tau_\varepsilon\}} \right] \\ &\qquad + \mathbb{E}^W \left[e^{-r\tau_\varepsilon} G_1(S^{0,x}_{\tau_\varepsilon}) \mathbb{1}_{\{\tau_\varepsilon \le \gamma_\varepsilon\}} + e^{-r\gamma_\varepsilon} G_2(S^{0,x}_{\gamma_\varepsilon}) \mathbb{1}_{\{\gamma_\varepsilon < \tau_\varepsilon\}} \right] \Big) + \varepsilon. \end{aligned}$$

For any stopping time ρ and for $k = 0, 1$ we have $\mathbb{E}^W[e^{-r\rho} S^{k,x}_\rho] \le x$ as in (2.5). Moreover G_1, G_2 have linear growth so that the Lipschitz property of $\underline{V}(x, \cdot)$ follows. $\square$

Now we can prove the existence of the value for the game. As explained in the introduction, the main difficulty comes from the fact that we are working with a bi-dimensional stopping game with a lack of uniform integrability on the stopping payoff.

Theorem 3.2. *The game with payoff (3.2) has a value $V(x,y) = \underline{V}(x,y) = \overline{V}(x,y)$ for all $(x,y) \in \mathbb{R}_+ \times [0,1]$. Moreover player 2, i.e. the minimiser (seller), has an optimal strategy*

$$\gamma_*(x,y) = \inf\{t \ge 0 \,|\, G_2(X^{x,y}_t) \le V(X^{x,y}_t, Y^{x,y}_t)\} \tag{3.6}$$

and the process

$$e^{-r(t\wedge\gamma_*)} V(X^{x,y}_{t\wedge\gamma_*}, Y^y_{t\wedge\gamma_*}), \quad t \ge 0$$

is a ***closed*** *supermartingale.*

Finally, if we define

$$\tau_*(x,y) = \inf\{t \ge 0 \,|\, V(X^{x,y}_t, Y^y_t) \le G_1(X^{x,y}_t)\}, \tag{3.7}$$

the process

$$e^{-r(t\wedge\tau_*)} V(X^{x,y}_{t\wedge\tau_*}, Y^y_{t\wedge\tau_*}), \quad t \ge 0$$

is a (not necessarily closed) submartingale.

Proof: The proof of Theorem 3.2 is postponed to the Appendix.

Remarks 3.3. *According to Lemma 3.1, the value function is non-increasing with respect to y. Therefore for each $y \in (0,1)$, we have*

$$V(x,y) \ge \lim_{y\to 1} V(x,y) \overset{def}{=} V_1(x),$$

where V_1 is the game value when the dividend rate is δ_0. According to [28], Theorem 2.1, the value function V_1 is strictly positive therefore V is also strictly positive.

4. Properties of the stopping regions

Having established that the game has a value V we can introduce the so-called continuation region

$$\mathcal{C} := \{(x,y) \in \mathbb{R}_+ \times [0,1] \mid G_1(x) < V(x,y) < G_2(x)\} \tag{4.1}$$

and the stopping regions for the two players, i.e.

$$\mathcal{S}_1 = \{(x,y) \in \mathbb{R}_+ \times [0,1] \mid V(x,y) = G_1(x)\}, \tag{4.2}$$

for player 1, and

$$\mathcal{S}_2 = \{(x,y) \in \mathbb{R}_+ \times [0,1] \mid V(x,y) = G_2(x)\}, \tag{4.3}$$

for player 2. It is clear that $\mathcal{C}$ is open and $\mathcal{S}_1$, $\mathcal{S}_2$ are closed, because V is jointly continuous (see Lemma 3.1), and obviously $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$.

These sets are important because, according to theory on zero-sum Dynkin games, the only candidate to be a Nash equilibrium is the pair (γ_*, τ_*) given by (3.6) and (3.7) (see [25]). Under complete information the perpetual game call option has been studied in [11] for $y = 0$ and in [28] for $y = 1$. Those papers analyse the geometry of the continuation and stopping regions and for completeness we account for a summary of their results in appendix. For future reference here we only note that [11] obtain

$$\varepsilon_0 < K \Leftrightarrow \mathcal{S}_2 \cap \{y = 0\} = [K, +\infty). \tag{4.4}$$

In the rest of this section we study the shape of the stopping regions. For that we need to introduce the infinitesimal generator of the two-dimensional diffusion (X, Y), i.e. for any $g \in C^2(\mathbb{R}_+ \times [0, 1])$

$$\begin{aligned}(\mathcal{L}g)(x, y) :=\Big[&(r - \delta_0 y)x\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 g}{\partial x^2} \\ &+ \frac{\delta_0^2}{2\sigma^2}y^2(1-y)^2\frac{\partial^2 g}{\partial y^2} - \delta_0 xy(1-y)\frac{\partial^2 g}{\partial x\partial y}\Big](x, y).\end{aligned} \tag{4.5}$$

Let us also introduce the sets

$$A_1 := \{(x, y) \in (K, +\infty) \times [0, 1] \,|\, (\mathcal{L}G_1 - rG_1)(x, y) > 0\} \tag{4.6}$$

$$A_2 := \{(x, y) \in (K, +\infty) \times [0, 1] \,|\, (\mathcal{L}G_2 - rG_2)(x, y) < 0\} \tag{4.7}$$

and notice that indeed $A_1 = \{(x, y) \,|\, xy < rK/\delta_0 \text{ and } x \geq K\}$ and $A_2 = \{(x, y) \,|\, xy > r(K - \varepsilon_0)/\delta_0 \text{ and } x \geq K\}$. We denote the complements of these sets by A_i^c, $i = 1, 2$ and define $\{x > K\} := (K, +\infty) \times [0, 1]$.

Proposition 4.1. *We have,*

$$\mathcal{S}_1 \subseteq A_1^c \cap \{x > K\} \quad \textit{and} \quad \mathcal{S}_2 \cap \{x > K\} \subseteq A_2^c.$$

Proof. It is sufficient to prove the first inclusion (i.e. for $\mathcal{S}_1$) because arguments for the second one (i.e. for $\mathcal{S}_2$) are analogous.

Because V is strictly positive (Remark 3.3), it is clear that $\mathcal{S}_1 \subset \{x > K\}$. Fix $(x_0, y_0) \in \{x > K\} \cap A_1$, then it is possible to find an open neighbourhood R of (x_0, y_0) such that $R \subset \{x > K\} \cap A_1$, i.e. $(\mathcal{L} - r)G_1 > 0$ on R. Let τ_R be the exit time of (X^{x_0,y_0}, Y^{y_0}) from R and let $\rho := \tau_* \wedge \gamma_* \wedge \tau_R$, then Theorem 3.2 guarantees that

$$e^{-r(t\wedge\rho)}V(X_{t\wedge\rho}, Y_{t\wedge\rho}) \quad \text{is a } \mathbb{P}_{x_0,y_0}\text{-martingale for } t \geq 0.$$

Using this property and Itô's formula we obtain

$$\begin{aligned} V(x, y) &= \mathbb{E}_{x_0,y_0}\left[e^{-r(t\wedge\rho)}V(X_{t\wedge\rho}, Y_{t\wedge\rho})\right] \\ &\geq \mathbb{E}_{x_0,y_0}\left[e^{-r(t\wedge\rho)}G_1(X_{t\wedge\rho})\right] \\ &= G_1(x_0) + \mathbb{E}_{x_0,y_0}\left[\int_0^{t\wedge\rho} e^{-rs}(\mathcal{L}G_1 - rG_1)(X_s, Y_s)\, ds\right] > G_1(x_0), \end{aligned}$$

which implies $(x_0, y_0) \notin \mathcal{S}_1$. □

Our next lemma shows that the stopping region $\mathcal{S}_1$ is up and right-connected while the region $\mathcal{S}_2$ is down and left-connected on $\{x > K\}$.

Lemma 4.2. *The following properties hold*

(i) $(x, y) \in \mathcal{S}_1 \Rightarrow (x, y') \in \mathcal{S}_1$ *for* $y' \geq y$.
(ii) $(x, y) \in \mathcal{S}_2 \Rightarrow (x, y') \in \mathcal{S}_2$ *for* $y' \leq y$.
(iii) $(x, y) \in \mathcal{S}_1 \Rightarrow (x', y) \in \mathcal{S}_1$ *for* $x' \geq x \geq K$.
(iv) $(x, y) \in \mathcal{S}_2 \Rightarrow (x', y) \in \mathcal{S}_2$ *for* $x \geq x' \geq K$.

Proof. The two first properties follow directly from the fact that $y \mapsto V(x,y)$ is non-increasing. To prove (iii) let us fix $(x,y) \in \mathcal{S}_1$ (notice that in particular $x \geq K$). Since $V(x,y)$ is 1-Lipschitz w.r.t. x and non-decreasing (see (i)-(ii) in Lemma 3.1) then for all $x' \geq x$, we have

$$V(x',y) \leq V(x,y) + (x'-x) = V(x,y) + G_1(x') - G_1(x) = G_1(x'), \tag{4.8}$$

where we have used that $G_1(x') - G_1(x) = x' - x$ for $x \geq x' \geq K$, and that $G_1(x) = V(x,y)$ by assumption. Clearly (4.8) implies $(x',y) \in \mathcal{S}_1$ as claimed. Similar arguments give (iv). □

Lemma 4.3. *For $x < K$, $V(x,y) < G_2(x)$. Hence $\mathcal{S}_2 \cap [(0,K) \times (0,1)] = \emptyset$.*

Proof. Notice that $G_2(x) = \varepsilon_0$ for $(x,y) \in (0,K) \times (0,1)$ and therefore $\mathcal{L}G_2 - rG_2 < 0$ on $(0,K) \times (0,1) \subset A_2$. Let $R \subset (0,K) \times (0,1)$ be an open set and fix $(x,y) \in R$. Denote $\rho_R := \inf\{t \geq 0 \mid (X_t, Y_t) \notin R\}$ and let τ_* be defined by (3.7). Notice also that $\tau_* \geq \rho_R$, $\mathbb{P}$-a.s. because player 1 does not stop in $(0,K)$.

Then using Theorem 3.2 and Itô formula we obtain

$$\begin{aligned} V(x,y) \leq& \mathbb{E}\left(e^{-r(t\wedge\rho_R)} V(X^{x,y}_{t\wedge\rho_R}, Y^y_{t\wedge\rho_R})\right) \leq \mathbb{E}\left(e^{-r(t\wedge\rho_R)} G_2(X^{x,y}_{t\wedge\rho_R})\right) \\ &= G_2(x) - r\varepsilon_0 \mathbb{E}\left(\int_0^{t\wedge\rho_R} e^{-rs}\,ds\right) < G_2(x). \end{aligned}$$

□

The next Lemma shows that if the penalty for cancellation does not exceed the strike price, i.e. $\varepsilon_0 < K$, then the stopping region $\mathcal{S}_2$ is non-empty and unbounded.

Lemma 4.4. *If $\varepsilon_0 < K$ then the set $\mathcal{S}_2 \cap [M, +\infty) \times (0,1)$ is non-empty for all $M \geq K$.*

Proof. We argue by contradiction and assume that $\mathcal{S}_2 \cap [M, +\infty) \times (0,1)$ is empty for some $M > K$. Fix $(x,y) \in (M, +\infty) \times (0,1)$ and denote

$$\rho_M(x,y) = \inf\{t \geq 0 \,|\, X^{x,y}_t \leq M\},$$

then clearly $\gamma_* \geq \rho_M$ almost surely. Theorem 3.2 therefore implies that

$$t \mapsto e^{-r(t\wedge\rho_M)} V(X^{x,y}_{t\wedge\rho_M}, Y^{x,y}_{t\wedge\rho_M}) \quad \text{is a supermartingale.}$$

For any stopping time τ we have

$$V(x,y) \geq \mathbb{E}[e^{-r\rho_M} V(M, Y^y_{\rho_M}) \mathbb{1}_{\{\rho_M < \tau\}} + e^{-r\tau} G_1(X^{x,y}_\tau) \mathbb{1}_{\{\tau \leq \rho_M\}}]. \tag{4.9}$$

Using Lipschitz continuity (Lemma 3.1) and (4.4), we also have

$$V(M,y) \geq V(M,0) - C(1+M)y = G_2(M) - C(1+M)y.$$

Plugging the latter into (4.9) to estimate $V(M, Y^y_{\rho_M})$, recalling $x > M$ and using that $(e^{-rt} Y^y_t)_{t\geq 0}$ is a positive, bounded, supermartingale we obtain

$$V(x,y) \geq \mathbb{E}[e^{-r\rho_M} G_2(M) \mathbb{1}_{\{\rho_M < \tau\}} + e^{-r\tau} G_1(X^{x,y}_\tau) \mathbb{1}_{\{\tau \leq \rho_M\}}] - C(1+x)y.$$

Since τ was arbitrary we then have $V(x,y) \geq f_M(x,y) - C(1+x)y$ where

$$f_M(x,y) := \sup_\tau \mathbb{E}[e^{-r\rho_M} G_2(M) \mathbb{1}_{\{\rho_M < \tau\}} + e^{-r\tau} G_1(X^{x,y}_\tau) \mathbb{1}_{\{\tau \leq \rho_M\}}].$$

The same arguments as in the proof of Lemma 3.1 allow us to prove that

$$|f_M(x,y) - f_M(x,y')| \leq C(1+x)|y-y'| \quad \text{for all } y, y' \in [0,1] \text{ and } x \in \mathbb{R}_+.$$

We can now use the above to obtain

$$V(x,y) \geq f_M(x,y) - C(1+x)y \geq f_M(x,0) - 2C(1+x)y. \tag{4.10}$$

Next we want to find a lower bound for $f_M(x,0)$. Notice that for $t \geq 0$

$$Y_t^0 = 0 \quad \text{and} \quad X_t^{x,0} = xe^{\sigma W+(r-\sigma^2/2)t}, \; \mathbb{P}-\text{a.s.}$$

For $n \geq x$, setting

$$\varphi(x) := x^{-\frac{2r}{\sigma^2}}, \quad \psi(x) := x \quad \text{and} \quad \tau_n := \inf\{t \geq 0 \,|\, X^{x,0} \geq n\}$$

we can rely on standard formulae for the Laplace transform of ρ_M and τ_n to obtain

$$\begin{aligned} f_M(x,0) \geq & G_2(M)\mathbb{E}\left[e^{-r\rho_M}\mathbb{1}_{\{\rho_M<\tau_n\}}\right] + G_1(n)\mathbb{E}\left[e^{-r\tau_n}\mathbb{1}_{\{\rho_M\geq\tau_n\}}\right] \\ = & G_2(M)\frac{\psi(x)\varphi(n)-\varphi(x)\psi(n)}{\psi(M)\varphi(n)-\varphi(M)\psi(n)} + G_1(n)\frac{\psi(M)\varphi(x)-\varphi(M)\psi(x)}{\psi(M)\varphi(n)-\varphi(M)\psi(n)}. \end{aligned}$$

Letting $n \to \infty$ it is easy to check that

$$f_M(x,0) \geq x - (K-\varepsilon_0)\left(\frac{M}{x}\right)^{\frac{2r}{\sigma^2}} > G_2(x)$$

where the final inequality uses $x > M$. The latter and (4.10) imply that $V(x,y) > G_2(x)$ for y sufficiently small, and thus a contradiction. □

Thanks to above lemmas we can define boundaries of the stopping regions as follows

$$b_1(y) := \inf\{x \in [0,+\infty) \mid V(x,y) = G_1(x)\}, \tag{4.11}$$

$$b_2(y) := \sup\{x \in [0,+\infty) \mid V(x,y) = G_2(x)\}, \tag{4.12}$$

with the usual convention that $\inf\emptyset = +\infty$ and $\sup\emptyset = 0$. Notice that $\mathcal{S}_2 \cap (\mathbb{R}_+ \times \{y\}) = [K, b_2(y)]$ if $b_2(y) \geq K$ and it is empty otherwise. From Lemma 4.2 and because the sets $\mathcal{S}_i$ are closed, we deduce the next corollary

Corollary 4.5. *The functions b_1 and b_2 are non-increasing on their respective domains. Moreover b_1 is lower semi-continuous (hence right-continuous) whereas b_2 is upper-semi-continuous (hence left-continuous).*

Next we show b_1 exists and is finite on $(0,1)$.

Lemma 4.6. *For all $y \in (0,1)$, $b_1(y) < \infty$.*

Proof. Arguing by contradiction let us assume that there exists $y_0 \in (0,1)$ such that $b_1(y_0) = +\infty$. Then by monotonicity of b_1 ((i) and (iii) of Lemma 4.2) and lower semi continuity, it holds $b_1(y) = +\infty$ on $[0, y_0]$.

Denote $\rho_0 := \inf\{t \geq 0 \mid Y_t \geq y_0\}$. We thus have $\rho_0 \leq \tau_*$, $\mathbb{P}_{x,y}$-a.s. for any starting point (x,y) with $y \in (0,y_0)$. From now on fix $y \in (0,y_0)$. Theorem 3.2 guarantees that

$$t \mapsto e^{-r(t\wedge\rho_0)}V(X^{x,y}_{t\wedge\rho_0}, Y^y_{t\wedge\rho_0}) \quad \text{is a submartingale.}$$

Therefore, using also that $V \leq G_2$, for any $t > 0$ we have

$$V(x,y) \leq \mathbb{E}\left[e^{-r(\rho_0\wedge t)}V(X^{x,y}_{\rho_0\wedge t}, Y^y_{\rho_0\wedge t})\right] \leq \mathbb{E}\left[e^{-r(\rho_0\wedge t)}X^{x,y}_{\rho_0\wedge t}\right] + \varepsilon_0 = \alpha(t,y)x + \varepsilon_0$$

with

$$\alpha(t,y) := \mathbb{E}\left[e^{\sigma W_{\rho_0\wedge t} - \frac{\sigma^2}{2}(\rho_0\wedge t)}e^{-\delta_0\int_0^{\rho_0\wedge t}Y_s^y ds}\right] < \mathbb{E}\left[e^{\sigma W_{\rho_0\wedge t} - \frac{\sigma^2}{2}(\rho_0\wedge t)}\right] = 1.$$

According to the last two expressions above, for fixed $y \in (0,y_0)$, we get

$$\lim_{x\to\infty} x^{-1}V(x,y) = \alpha(t,y) < 1 = \lim_{x\to\infty} x^{-1}G_1(x)$$

which contradicts $V \geq G_1$. □

Lemma 4.7. *If* $\varepsilon_0 < K$ *then*

$$\lim_{y\to 0} b_2(y) = +\infty, \qquad \lim_{y\to 0+} b_1(y) = +\infty.$$

Proof. From Proposition 4.1 and the definition of A_1 and A_2 we have $b_1(y) \geq b_2(y)$ for all $y \in (0,1)$. Since Lemma 4.4 holds, then it must be $\lim_{y\to 0} b_2(y) = +\infty$. The latter also gives $\lim_{y\to 0} b_1(y) = +\infty$. □

From now on, whenever we refer to properties of $\mathcal{S}_2$ and its boundary, we tacitly assume that $\mathcal{S}_2 \cap (\mathbb{R}_+ \times (0,1)) \neq \emptyset$. As proven above this is always true for $\varepsilon_0 < K$. In this context we also denote

$$b_2^K := \sup\{y > 0 \mid (K, y) \in \mathcal{S}_2\}. \tag{4.13}$$

5. A parabolic formulation of the problem

In order to study existence of Nash equilibria and regularity of the value function of the game (beyond continuity) it is useful to introduce a deterministic transformation of the process (X, Y). Such transformation also unveils a parabolic nature of the problem.

Given $(x, y) \in (0, \infty) \times (0, 1)$, let us define $z = \ln(x) + \frac{\sigma^2}{\delta_0} \ln(\frac{y}{1-y})$ and the process Z^z such that $Z_0^z = z$ and:

$$Z_t^z = \ln(X_t^{x,y}) + \frac{\sigma^2}{\delta_0} \ln\left(\frac{Y_t^y}{1 - Y_t^y}\right). \tag{5.1}$$

Then setting

$$k := (r - \tfrac{\sigma^2}{2} - \tfrac{\delta_0}{2}) \tag{5.2}$$

it is not hard to check, by using Itô's formula, that Z^z evolves according to

$$Z_t^z = z + kt, \qquad t \geq 0. \tag{5.3}$$

From (5.1) we observe that $\mathbb{P}$-almost surely

$$X_t^{x,y} = F(Z_t^z, Y_t^y), \qquad t \geq 0 \tag{5.4}$$

with $F : \mathbb{R} \times (0,1) \to \mathbb{R}_+$ defined by

$$F(z, y) = \exp\left(z - \frac{\sigma^2}{\delta_0} \ln\left(\frac{y}{1-y}\right)\right) = e^z \left(\frac{1-y}{y}\right)^{\frac{\sigma^2}{\delta_0}}. \tag{5.5}$$

Notice that F is C^2 on $\mathbb{R} \times (0,1)$. The process Z is indeed deterministic and of bounded variation, hence it plays the role of a "time" process. Whether Z is increasing or decreasing depends on the sign of k. In the rest of the paper we study the case $k \neq 0$ which is *truly* two-dimensional. We leave aside the case $k = 0$ that reduces to a one-dimensional problem parametrised in the variable z.

Remark 5.1. *In the new coordinates it becomes clear that the law of* $(X_t^{x,y}, Y_t^y)$ *is supported on the curve* $\{(F(z + kt, \zeta), \zeta),\, \zeta \in (0, 1)\}$*, which is a set of null Lebesgue measure in* $\mathbb{R}_+ \times [0, 1]$.

We can now look at our game in the new coordinates and consider the functions $H_1, H_2, v : \mathbb{R} \times (0, 1) \to \mathbb{R}_+$ given by

$$v(z, y) := V(F(z, y), y), \quad H_1(z, y) := G_1(F(z, y)), \quad H_2(z, y) := G_2(F(z, y)). \tag{5.6}$$

By construction, we have $H_1 \leq v \leq H_2$ and v is equal to the value of the stopping game

$$v(z, y) = \sup_{\tau\in\mathcal{T}} \inf_{\gamma\in\mathcal{T}} \mathbb{E}\left[e^{-r\tau} H_1(Z_\tau^z, Y_\tau^y)\mathbb{1}_{\{\tau\leq\gamma\}} + e^{-r\gamma} H_2(Z_\gamma^z, Y_\gamma^y)\mathbb{1}_{\{\gamma<\tau\}}\right]$$

For this new parametrization of the game we naturally introduce the continuation and stopping regions

$$\mathcal{C}' := \{(z,y)\in\mathbb{R}\times(0,1)\mid H_1(z,y)<v(z,y)<H_2(z,y)\}$$
$$\mathcal{S}'_1 := \{(z,y)\in\mathbb{R}\times(0,1)\mid v(z,y)=H_1(z,y)\},$$
$$\mathcal{S}'_2 := \{(z,y)\in\mathbb{R}\times(0,1)\mid v(z,y)=H_2(z,y)\}.$$

Using Lemma 3.1 it is immediate to verify that v is locally Lipschitz continuous in $\mathbb{R}\times(0,1)$ so that $\mathcal{C}'$ is open and $\mathcal{S}'_i$, $i=1,2$ are closed. Moreover, it is clear that γ_* and τ_* as in (3.6)-(3.7) are the entry times of (Z,Y) into $\mathcal{S}'_2$ and $\mathcal{S}'_1$, respectively.

The infinitesimal generator associated with (Z,Y) is defined by

$$(\mathcal{G}\,f)(z,y) := k\frac{\partial f}{\partial z}(z,y)+\frac{1}{2}\left(\frac{\delta_0}{\sigma}\right)^2 y^2(1-y)^2\frac{\partial^2 f}{\partial y^2}(z,y), \tag{5.7}$$

for $f\in C^{1,2}(\mathbb{R}\times[0,1])$. One advantage of this formulation is that $\mathcal{G}$ is a parabolic operator and it is non-degenerate on $\mathbb{R}\times(0,1)$, so that the associated Cauchy-Dirichlet problems admit classical solutions under standard assumptions on the boundary conditions.

Since v is continuous then $\{e^{-rt}v(Z_t,Y_t),\ t\le\tau'_{\mathcal{C}}\}$ is a continuous martingale for $\tau'_{\mathcal{C}} := \inf\{t\ge0\mid(Z_t,Y_t)\notin\mathcal{C}'\}$ (the latter follows from Theorem 3.2 and the fact that (X,Y) is linked to (Z,Y) by a deterministic map). We can use results of interior regularity for solutions to parabolic PDEs (see, e.g., [18, Corollary 2.4.3]) and Itô's formula to deduce that any solution f to $(\mathcal{G}f-rf)(z,y)=0$ on $R_\eta=(z_0,z_0+\eta)\times(y_0-\eta,y_0+\eta)\subset\mathcal{C}'$ with $f=v$ on ∂R_η is $C^\infty(R_\eta)$ and coincides with v. Therefore, $v\in C^\infty(\mathcal{C}')$ and thus it satisfies

$$(\mathcal{G}v-rv)(z,y)=0,\qquad \text{for } (z,y)\in\mathcal{C}', \tag{5.8}$$

$$H_1\le v\le H_2,\qquad \text{on } \mathbb{R}\times(0,1), \tag{5.9}$$

$$v|_{\partial\mathcal{S}'_1}=H_1|_{\partial\mathcal{S}'_1}\quad\text{and}\quad v|_{\partial\mathcal{S}'_2}=H_2|_{\partial\mathcal{S}'_2}. \tag{5.10}$$

As a consequence, $V\in C^\infty(\mathcal{C})$ as well.

We denote by R_K the closure in $\mathbb{R}\times(0,1)$ of the set in which $H_1>0$, i.e.

$$R_K = \{(z,y)\in\mathbb{R}\times(0,1)\mid F(z,y)\ge K\}=\{(z,y)\in\mathbb{R}\times(0,1)\mid y\le y_K(z)\} \tag{5.11}$$

where $y_K(z):=e^{\delta_0/\sigma^2 z}/(K^{\delta_0/\sigma^2}+e^{\delta_0/\sigma^2 z})$. According to Proposition 4.1 and Lemma 4.3, the stopping regions $\mathcal{S}'_1$ and $\mathcal{S}'_2$ lie in R_K. Notice that since y_K is increasing, if $(z_0,y_0)\in R_K$ then any pair (z,y_0) belongs to R_K for $z\ge z_0$. Somewhat in analogy with (4.13) we also define

$$z_K := \sup\{z\in\mathbb{R}\mid(z,y_K(z))\in\mathcal{S}'_2\}\ \text{and}\ \overline{y}_K := y_K(z_K)=b_2^K. \tag{5.12}$$

In the new coordinates the sets $\mathcal{S}'_1$ and $\mathcal{S}'_2$ are connected with respect to the z variable, as illustrated in the next lemma.

Lemma 5.2. *Let* $(z_0,y_0)\in R_K$.

(i) $(z_0,y_0)\in\mathcal{S}'_2 \implies (z,y_0)\in\mathcal{S}'_2$ *for all* $z\le z_0$, *such that* $(z,y_0)\in R_K$,
(ii) $(z_0,y_0)\in\mathcal{S}'_1 \implies (z,y_0)\in\mathcal{S}'_1$ *for all* $z\ge z_0$.

Proof. Using that $z\mapsto F(z,y)$ is increasing for each $y\in(0,1)$ it is not difficult to show (by direct comparison) that $z\to v(z,y)$ is also increasing. To prove (i) take $z\le z_0$, then (ii) of Lemma 3.1 implies

$$\begin{aligned}0 \le v(z_0,y_0) - v(z,y_0) &= V(F(z_0,y_0),y_0) - V(F(z,y_0),y_0)\\ &\le F(z_0,y_0) - F(z,y_0) = H_2(z_0,y_0) - H_2(z,y_0).\end{aligned} \tag{5.13}$$

If $(z_0,y_0) \in \mathcal{S}_2'$ then $v(z_0,y_0) - H_2(z_0,y_0) = 0$ yielding $v(z,y_0) - H_2(z,y_0) \ge 0$. With an analogous argument we can prove (ii). □

The stopping sets are not necessarily connected with respect to the y variable and indeed we only have connected sets for some values of dividend rate and volatility of X. In particular in the rest of the paper we make the following standing assumption (unless otherwise specified).

Assumption 5.3. *We assume* $\frac{\sigma^2}{\delta_0} \ge 1$.

For $\frac{\sigma^2}{\delta_0} \ge 1$ the sets $\mathcal{S}_1'$, $\mathcal{S}_2'$ enjoy the next desired property.

Lemma 5.4. *Let* $(z_0,y_0) \in R_K$, *then*

(i) $(z_0,y_0) \in \mathcal{S}_2' \implies (z_0,y) \in \mathcal{S}_2'$ *for all* $y \ge y_0$ *such that* $(z_0,y) \in R_K$,
(ii) $(z_0,y_0) \in \mathcal{S}_1' \implies (z_0,y) \in \mathcal{S}_1'$ *for all* $y \le y_0$.

Moreover it also holds

(iii) $v(z,y') \le v(z,y)$ *for* $y' \ge y > y_K(z)$, $z \in \mathbb{R}$.

Proof. Take $(x_0,y_0) \in R_K$, fix $y \ge y_0$ and let $x_0 = F(z_0,y_0)$. Let $\gamma := \gamma_*(x_0,y)$ be optimal for player 2 in the game started at (x_0,y) and τ an ε-optimal stopping time for player 1 in the game started at (x_0,y_0). Recall also that on $\{\tau \wedge \gamma = +\infty\}$ both players have zero payoff due to (3.1). Then using that $H_2(z,y) - H_2(z,y') = H_1(z,y) - H_1(z,y')$ for all $z \in \mathbb{R}$ and $y,y' \in (0,1)$ we obtain

$$\begin{aligned} v(z_0,y_0) - v(z_0,y) &\\ \le& \mathbb{E}\left[e^{-r\gamma}H_2(Z^{z_0}_\gamma, Y^{y_0}_\gamma)\mathbb{1}_{\{\gamma<\tau\}} + e^{-r\tau}H_1(Z^{z_0}_\tau, Y^{y_0}_\tau)\mathbb{1}_{\{\tau\le\gamma\}}\right]\\ &- \mathbb{E}\left[e^{-r\gamma}H_2(Z^{z_0}_\gamma, Y^{y}_\gamma)\mathbb{1}_{\{\gamma<\tau\}} + e^{-r\tau}H_1(Z^{z_0}_\tau, Y^{y}_\tau)\mathbb{1}_{\{\tau\le\gamma\}}\right] + \varepsilon\\ =& \mathbb{E}\left[e^{-r(\gamma\wedge\tau)}\left(H_1(Z^{z_0}_{\gamma\wedge\tau}, Y^{y_0}_{\gamma\wedge\tau}) - H_1(Z^{z_0}_{\gamma\wedge\tau}, Y^{y}_{\gamma\wedge\tau})\right)\right] + \varepsilon. \end{aligned} \tag{5.14}$$

Now we notice that, since $Y^y_t \ge Y^{y_0}_t$, $\mathbb{P}$-a.s. for all $t \ge 0$ and $y \mapsto F(z,y)$ is decreasing, then

$$H_1(Z^{z_0}_{\gamma\wedge\tau}, Y^{y_0}_{\gamma\wedge\tau}) - H_1(Z^{z_0}_{\gamma\wedge\tau}, Y^{y}_{\gamma\wedge\tau}) \le F(Z^{z_0}_{\gamma\wedge\tau}, Y^{y_0}_{\gamma\wedge\tau}) - F(Z^{z_0}_{\gamma\wedge\tau}, Y^{y}_{\gamma\wedge\tau}) \tag{5.15}$$

and the right-hand side of the inequality is positive. Therefore we can use Fatou's lemma and (5.15) to obtain

$$\begin{aligned} v(z_0,y_0) - v(z_0,y) \le& \mathbb{E}\left[e^{-r(\gamma\wedge\tau)}\left(F(Z^{z_0}_{\gamma\wedge\tau}, Y^{y_0}_{\gamma\wedge\tau}) - F(Z^{z_0}_{\gamma\wedge\tau}, Y^{y}_{\gamma\wedge\tau})\right)\right] + \varepsilon\\ \le& \liminf_{t\to+\infty}\mathbb{E}\left[e^{-r(\gamma\wedge\tau\wedge t)}\left(F(Z^{z_0}_{\gamma\wedge\tau\wedge t}, Y^{y_0}_{\gamma\wedge\tau\wedge t}) - F(Z^{z_0}_{\gamma\wedge\tau\wedge t}, Y^{y}_{\gamma\wedge\tau\wedge t})\right)\right] + \varepsilon \end{aligned} \tag{5.16}$$

Setting $M^\zeta_s = e^{-rs}F(Z^{z_0}_s, Y^\zeta_s)$, for any $\zeta \in (0,1)$ and $s \in [0,t]$, Itô formula gives

$$dM^\zeta_s = -\delta_0 Y^\zeta_s M^\zeta_s dt + \sigma M^\zeta_s dW_s.$$

Hence substituting the above into (5.16) and noticing that $M^\zeta \in L^2([0,t]\times\Omega)$, we can use the optimal sampling theorem to obtain

$$\begin{aligned} v(z_0,y_0) - v(z_0,y) \le& F(z_0,y_0) - F(z_0,y) + \varepsilon\\ &+ \delta_0 \liminf_{t\to\infty}\mathbb{E}\left[\int_0^{\gamma\wedge\tau\wedge t}\left(Y^y_s F(Z^{z_0}_s, Y^y_s) - Y^{y_0}_s F(Z^{z_0}_s, Y^{y_0}_s)\right)dt\right]. \end{aligned} \tag{5.17}$$

Using now that, for $\frac{\sigma^2}{\delta_0} \geq 1$, the map $y \to yF(z,y)$ is non-increasing with

$$\frac{\partial}{\partial y}\left(yF(z,y)\right) = e^z \left(\frac{1-y}{y}\right)^{\frac{\sigma^2}{\delta_0}} \left(1 - \frac{\sigma^2/\delta_0}{(1-y)}\right), \tag{5.18}$$

and recalling once again that $Y^y_\cdot \geq Y^{y_0}_\cdot$, we see that (5.17) implies

$$v(z_0,y_0) - v(z_0,y) \leq F(z_0,y_0) - F(z_0,y) + \varepsilon. \tag{5.19}$$

Since ε is arbitrary then $y \mapsto (v(z,y) - F(z,y))$ is increasing and therefore (i) and (ii) easily follow.

The proof of (iii) follows from the fact that $y \mapsto H_i(z,y)$ is decreasing for $i = 1,2$. □

The next corollary is a simple consequence of Lemma 5.2 and 5.4. We recall that $\mathcal{S}_i \cap R^c_K = \emptyset$ as no player stops for $X_t < K$ (see Remark 3.3 and Lemma 4.3).

Corollary 5.5. *There exists increasing functions $c_1 : \mathbb{R} \to [0,1]$, $c_2 : (-\infty, z_K] \to [0,1]$, with $c_1(\cdot) \leq c_2(\cdot) \leq y_K(\cdot)$ on $(-\infty, z_K]$ and $c_1(\cdot) \leq y_K(\cdot)$ on $(z_K, +\infty)$, such that*

$$\mathcal{S}'_1 = \{(z,y) \in \mathbb{R} \times [0,1] \mid y \leq c_1(z)\}, \tag{5.20}$$

$$\mathcal{S}'_2 = \{(z,y) \in (-\infty, z_K] \times [0,1] \mid y \in [c_2(z), y_K(z)]\}. \tag{5.21}$$

Next we provide continuity of the boundaries b_i and c_i, $i = 1,2$.

Proposition 5.6. *The stopping boundaries b_1, b_2 and c_1, c_2 are continuous.*

Proof. Step 1. First we prove the claim for b_1, b_2. Since the proofs are similar for the two boundaries, we only provide details for b_2. According to Corollary 4.5, the boundary b_2 is left-continuous. To show the right-continuity, we will argue by contradiction.

Assume that there exists $y_0 \in (0,1)$ such that $b_2(y_0+) < b_2(y_0)$ and fix $x_0 \in (b_2(y_0+), b_2(y_0))$. Next define z_0 by $F(z_0,y_0) = x_0$ and notice that since $x_0 < b_2(y_0)$, then $(F(z_0,y_0), y_0) \in \mathcal{S}_2$ and therefore $(z_0,y_0) \in \mathcal{S}'_2$. We take a decreasing sequence $(y_n)_n$ with $y_n \downarrow y_0$ as $n \to \infty$ so that Lemma 5.4 implies that $(z_0, y_n) \in \mathcal{S}'_2$ for all n. Equivalently $x_n = F(z_0,y_n) \leq b_2(y_n)$ so that taking limits and using that F is continuous, we obtain $x_0 = F(z_0,y_0) \leq b(y_0+)$. The latter is a contradiction.

Step 2. Now we show continuity of c_1, c_2. Let us start from c_2 and fix z_0. Take a sequence $z_n \uparrow z_0$ as $n \to \infty$ so that $(z_n, c_2(z_n)) \to (z_0, c_2(z_0-))$, where $c_2(z_0-) \leq c_2(z_0)$ and the limit exists by monotonicity. Since $\mathcal{S}'_2$ is closed we have $(z_0, c_2(z_0-)) \in \mathcal{S}'_2$ and therefore $c_2(z_0-) \geq c_2(z_0)$, hence implying left-continuity.

To prove that c_2 is also right-continuous we use Theorem 3.3 in [6]. Since the latter theorem is not given in our game context we repeat here some arguments for completeness. Let us assume $c_2(z_0) < c_2(z_0+)$ and denote $y_0 := c_2(z_0)$, $y_1 := c_2(z_0+)$ for simplicity. Fix $z_1 > z_0$ such that the open rectangle $\mathcal{R}$ with vertices (z_0,y_0), (z_0,y_1), (z_1,y_1) and (z_1,y_0) is contained in $\mathcal{C}'$. Let $w := v - H_2$ and $\overline{w} := \partial w/\partial y$, then results of interior regularity for solutions to PDEs (see e.g. [18, Corollary 2.4.3]) imply that $\overline{w} \in C^{1,2}(\mathcal{R})$ and, by deriving (5.8) with respect to y, it turns out that

$$\left(k\frac{\partial \overline{w}}{\partial z} + \mathcal{A}\overline{w} - r\overline{w}\right)(z,y) = \delta_0 h(z,y), \quad \text{for } (z,y) \in \mathcal{R}, \tag{5.22}$$

where $h(z,y) := \frac{\partial}{\partial y}(yF(z,y))$ and

$$\mathcal{A} := \tfrac{1}{2}\left(\frac{\delta_0}{\sigma}\right)^2 y^2(1-y)^2 \frac{\partial^2}{\partial y^2} + \left(\frac{\delta_0}{\sigma}\right)^2 y(1-y)(1-2y)\frac{\partial}{\partial y}.$$

Let $\psi \in C_c^\infty(y_0, y_1)$ be positive and such that $\int_{y_0}^{y_1} \psi(y)dy = 1$. Multiply (5.22) by ψ and integrate by parts over (y_0, y_1) to obtain

$$
\begin{aligned}
L_\psi(z) :=& k \int_{y_0}^{y_1} \frac{\partial \overline{w}}{\partial z}(z,y)\psi(y)dy \\
=& \int_{y_0}^{y_1} w(z,y)\frac{\partial}{\partial y}[(r-\mathcal{A}^*)]\psi(y)dy + \delta_0 \int_{y_0}^{y_1} h(z,y)\psi(y)dy,
\end{aligned}
\tag{5.23}
$$

where $\mathcal{A}^*$ is the adjoint operator of $\mathcal{A}$. Now, taking limits as $z \downarrow z_0$, we can use dominated convergence in the right hand side of the above equation and the fact that $w(z_0, \cdot) = 0$ on (y_0, y_1), to find

$$
L_\psi(z_0+) = \lim_{z\downarrow z_0} L_\psi(z) = \delta_0 \int_{y_0}^{y_1} h(z_0,y)\psi(y)dy \leq -\delta_0 \ell. \tag{5.24}
$$

In the final inequality we set $-\ell := \sup_{(y_0,y_1)} h(z_0, y)$ and notice that $\ell > 0$ due to $\sigma^2/\delta_0 \geq 1$ (see (5.18)).

By its definition $L_\psi \in C(z_0, z_1)$ and (5.24) implies that its right limit at z_0 exists and it is strictly negative. Then for some $\delta > 0$, using integration by parts and Fubini's theorem, we have

$$
\begin{aligned}
0 >& \int_{z_0}^{z_0+\delta} L_\psi(z)dz = -k \int_{y_0}^{y_1} \left(\int_{z_0}^{z_0+\delta} \frac{\partial w}{\partial z}(z,y)dz \right) \psi'(y)dy \\
=& -k \int_{y_0}^{y_1} w(z_0+\delta, y)\psi'(y)dy = k \int_{y_0}^{y_1} \frac{\partial w}{\partial y}(z_0+\delta,y)\psi(y)dy \geq 0,
\end{aligned}
\tag{5.25}
$$

where the last inequality follows because $w(z_0 + \delta, \cdot)$ is increasing as shown in the proof of Lemma 5.4. Therefore we reach a contradiction and c_2 must be continuous at z_0. By arbitrariness of z_0 we conclude that c_2 is continuous.

To prove continuity of c_1 we simply refer to [6, Thm. 3.1]. The latter is not obtained in a game context but arguments as above allow a straightforward extension to it. We also notice that in applying that theorem we use that v is locally Lipschitz on $\mathbb{R} \times (0, 1)$. □

6. Regularity across the boundaries

In this section we show that the value function V is indeed C^1 in $\mathbb{R}_+^* \times (0, 1)$. The key to this result is the so-called *regularity* of the optimal boundaries. Roughly speaking this means that the process (X, Y) immediately enters the interior of the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ upon hitting their boundaries $\partial\mathcal{S}_1$ and $\partial\mathcal{S}_2$. Analogous considerations apply to the process (Z, Y) and the sets $\mathcal{S}_1'$, $\mathcal{S}_2'$.

We recall that we work under Assumption 5.3. Let us introduce the hitting times

$$
\hat{\tau}_* := \inf\{t > 0 \,|\, (X_t, Y_t) \in \mathcal{S}_1\} = \inf\{t > 0 \,|\, (Z_t, Y_t) \in \mathcal{S}_1'\} \tag{6.1}
$$

$$
\hat{\gamma}_* := \inf\{t > 0 \,|\, (X_t, Y_t) \in \mathcal{S}_2\} = \inf\{t > 0 \,|\, (Z_t, Y_t) \in \mathcal{S}_2'\}. \tag{6.2}
$$

The next lemma provides a clear statement of the regularity of the optimal boundaries for the diffusions (X, Y) and (Z, Y). Its proof is postponed to the end of the section so that we can move quickly towards the main result, i.e. Proposition 6.4.

Lemma 6.1. *If $(x_0, y_0) \in \partial\mathcal{S}_1$ (resp. $(z_0, y_0) \in \partial\mathcal{S}_1'$) then*

$$
\mathbb{P}_{x_0,y_0}(\hat{\tau}_* > 0) = 0 \qquad (\textit{resp. } \mathbb{P}_{z_0,y_0}(\hat{\tau}_* > 0) = 0). \tag{6.3}
$$

Similarly, if $(x_0, y_0) \in \partial\mathcal{S}_2$ (resp. $(z_0, y_0) \in \partial\mathcal{S}_2'$) then

$$
\mathbb{P}_{x_0,y_0}(\hat{\gamma}_* > 0) = 0 \qquad (\textit{resp. } \mathbb{P}_{z_0,y_0}(\hat{\gamma}_* > 0) = 0). \tag{6.4}
$$

Notice that if $k>0$ (6.4) holds with $(x_0,y_0)\neq(K,b_2^K)$ (resp. $(z_0,y_0)\neq(z_K,\overline{y}_K)$).

Adopting the convention that $[K,b_2(y)]=\emptyset$ for $y>\overline{y}_K$ and $[c_2(z),y_K(z)]=\emptyset$ for $z>z_K$, we can use Corollary 5.5 and write $\mathbb{P}$-a.s.

$$\hat{\tau}_*=\inf\{t>0\,|\,X_t\geq b_1(Y_t)\}=\inf\{t>0\,|\,Y_t\leq c_1(Z_t)\} \tag{6.5}$$

$$\hat{\gamma}_*=\inf\{t>0\,|\,X_t\in[K,b_2(Y_t)]\}=\inf\{t>0\,|\,Y_t\in[c_2(Z_t),y_K(Z_t)]\}. \tag{6.6}$$

To avoid further technicalities we assume that

$$c_2(z)\neq y_K(z)\text{ for }z<z_K\quad(\text{resp. }b_2(y)\neq K\text{ for }y<b_2^K),$$

however all the results of this section can be easily adapted to the case in which $c_2=y_K$ for some z (i.e. $b_2=K$ for some y).

We consider hitting times to the interior of the stopping sets, i.e. we define $\mathbb{P}$-a.s.

$$\check{\tau}:=\inf\{t>0\,|\,X_t>b_1(Y_t)\}=\inf\{t>0\,|\,Y_t<c_1(Z_t)\} \tag{6.7}$$

$$\check{\gamma}:=\inf\{t>0\,|\,X_t\in(K,b_2(Y_t))\}=\inf\{t>0\,|\,Y_t\in(c_2(Z_t),y_K(Z_t))\}. \tag{6.8}$$

Notice that for each line, the second equality follows from the continuity of the optimal boundaries. Precisely, for all $(z,y)\in R_K$, we have the equivalences

$$F(z,y)<b_2(y)\Leftrightarrow y>c_2(z),\ \text{ and }\ F(z,y)>b_1(y)\Leftrightarrow y<c_1(z).$$

We remark that if $c_2=y_K$ on an interval $\mathcal{I}$ then $\check{\gamma}$ should account also for the first *crossing* time of $c_2\,|_{\mathcal{I}}$.

An argument used in [4], Corollary 8 (see eq. (2.39) therein) allows us to obtain the next useful lemma. The proof, originally developed in [4] is given in Appendix B for the reader's convenience.

Lemma 6.2. *For any $(x,y)\in\mathbb{R}\times[0,1]$ we have*

$$\mathbb{P}_{x,y}(\hat{\tau}_*=\check{\tau})=\mathbb{P}_{x,y}(\hat{\gamma}_*=\check{\gamma})=1. \tag{6.9}$$

Equivalently for any $(z,y)\in\mathbb{R}\times(0,1]$ we have

$$\mathbb{P}_{z,y}(\hat{\tau}_*=\check{\tau})=\mathbb{P}_{z,y}(\hat{\gamma}_*=\check{\gamma})=1. \tag{6.10}$$

The above lemma says that the process (X,Y) (or equivalently (Z,Y)), upon hitting the optimal boundaries, will immediately enter the interior of the stopping set. This has the following important consequence

Proposition 6.3. *Let $(x_n,y_n)_n$ be a sequence in $\mathcal{C}$ and let $\hat{\tau}^n_*:=\hat{\tau}_*(x_n,y_n)$ and $\hat{\gamma}^n_*:=\hat{\gamma}_*(x_n,y_n)$ denote the corresponding hitting times for the process (X^{x_n,y_n},Y^{y_n}). It follows that*

(i) *If $(x_n,y_n)\to(x_0,y_0)\in\mathcal{S}_1$ as $n\to+\infty$, then $\hat{\tau}_*(x_n,y_n)\to0$, $\mathbb{P}$-a.s.*
(ii) *If $(x_n,y_n)\to(x_0,y_0)\in\mathcal{S}_2$ as $n\to+\infty$, then $\hat{\gamma}_*(x_n,y_n)\to0$, $\mathbb{P}$-a.s.*

Notice that if $k>0$ the above holds with $(x_0,y_0)\neq(K,b_2^K)$.

Proof. Let us consider (ii) and with no loss of generality let $x_0=b_2(y_0)$ (arguments as below apply also to $x_0=K$). Denote $\check{\gamma}^n:=\check{\gamma}(x_n,y_n)$. Since $\hat{\gamma}^n_*=\check{\gamma}^n$ by Lemma 6.2, it is sufficient to prove that $\check{\gamma}^n\to0$. In particular $\check{\gamma}(x_0,y_0)=0$, $\mathbb{P}$-a.s. by Lemma 6.2 and Lemma 6.1. Hence there exists a set of null measure $\mathcal{N}$ such that $\check{\gamma}(x_0,y_0)=0$ and $(x,y)\to(X^{x,y},Y^y)$ is continuous, for all $\omega\in\Omega\setminus\mathcal{N}$. Fix $\omega\in\Omega\setminus\mathcal{N}$ and an arbitrary $\alpha>0$. We can find $t<\alpha$ such that $X^{x_0,y_0}_t(\omega)<b_2(Y^{y_0}_t(\omega))$. It follows that for all n sufficiently large $X^{x_n,y_n}_t(\omega)<b_2(Y^{y_n}_t(\omega))$ because $(X^{x_n,y_n}_t(\omega),Y^{y_n}_t(\omega))\to(X^{x_0,y_0}_t(\omega),Y^{y_0}_t(\omega))$ and b_2 is continuous. Therefore $\limsup_n\check{\gamma}^n(\omega)<\alpha$. Since α is arbitrary and the argument holds for a.e. ω we obtain (ii).

The proof of (i) follows from an analogous argument. □

Now we can use the result above to obtain continuous differentiability of the value function. In preparation for that we need to recall some results concerning differentiability of the stochastic flow. In particular by [26], Theorem 39, Chapter V.7 we can define the process

$$\text{for all } t \geq 0,\ U_t^y := \frac{\partial Y_t^y}{\partial y} \qquad \mathbb{P}-\text{a.s.} \tag{6.11}$$

which is continuous in both t and y and solves the SDE

$$dU_t^y = -\frac{\delta_0}{\sigma}(1-2Y_t^y)U_t^y dW_t, \qquad U_0^y = 1 \ \mathbb{P}-\text{a.s.} \tag{6.12}$$

Notice that the couple (Y,U) forms a Markov process and that U_t^y is an exponential local martingale. Moreover, since the process Y is bounded, it is not difficult to see that Novikov condition holds and U_t^y is indeed an exponential martingale. Finally we also remark here that (Y,U) is a strong solution of a SDE and notice that, using the explicit representation (2.4), we also have

$$\frac{\partial}{\partial x}X_t^{x,y} = X_t^{1,y} \quad \mathbb{P}-\text{a.s.}.$$

For all $(x,y) \in \mathbb{R}\times[0,1]$ we set

$$u(x,y) := V(x,y) - (x-K), \tag{6.13}$$

(6.14)

and define the process $(P_t)_{t\geq 0}$ as

$$P_t = e^{rt}u(X_t,Y_t) + \int_0^t e^{-rs}(rK - \delta_0 X_s Y_s)ds \qquad \mathbb{P}_{x,y}-\text{a.s.} \tag{6.15}$$

Then from the semi-harmonic characterisation of the value function provided in Theorem 3.2, we obtain for any $T>0$

$$(P_{t\wedge\gamma_*\wedge\tau_*})_{t\leq T} \quad \text{is a } \mathbb{P}_{x,y} \text{ martingale} \tag{6.16}$$

$$(P_{t\wedge\tau_*})_{t\leq T} \quad \text{is a } \mathbb{P}_{x,y} \text{ sub-martingale} \tag{6.17}$$

$$(P_{t\wedge\gamma_*})_{t\leq T} \quad \text{is a } \mathbb{P}_{x,y} \text{ super-martingale.} \tag{6.18}$$

For future reference we also introduce

$$\tau_K(x,y) := \inf\{t\geq 0 \,:\, X_t^{x,y} \leq K\} \tag{6.19}$$

and denote by $\overline{\mathcal{C}}'$ the closure of $\mathcal{C}'$.

Proposition 6.4. *The value function V is C^1 in $\mathbb{R}_+^*\times(0,1)$ (possibly with the exception of the point (K,b_2^K) if $k>0$). Moreover v_{yy} (see (5.6)) is continuous on $\overline{\mathcal{C}}'$ (possibly on $\overline{\mathcal{C}}'\setminus(z_K,\overline{y}_K)$ if $k>0$).*

Proof. The value function is C^1 inside the continuation set $\mathcal{C}$ by simply recalling that $v\in C^1$ in $\mathcal{C}'$ (see the free boundary problem (5.8)–(5.10)). Therefore we only need to prove the C^1 property across the optimal boundaries. We provide full details for the continuity of $u_y := \partial u/\partial y$ as the continuity of $u_x := \partial u/\partial x$ follows analogous arguments up to trivial modifications.

Let us start by looking at points of $\partial\mathcal{S}_1$, i.e. the boundary of the stopping region for the buyer. Let us fix $(x_0,y_0)\in\partial\mathcal{S}_1$ and let us pick (x,y) inside the continuation set $\mathcal{C}\cap\{x>K\}$. Later we will take limits $(x,y)\to(x_0,y_0)$ and use Proposition 6.3.

Denote by $\tau_*=\tau_*(x,y)$ the first entry time of $(X^{x,y},Y^y)$ into $\mathcal{S}_1$ and by $\gamma_\varepsilon=\gamma_*(x,y+\varepsilon)$ the first entry time of $(X^{x,y+\varepsilon},Y^{y+\varepsilon})$ into $\mathcal{S}_2$ for some $\varepsilon>0$. From (i) of Lemma 3.1 and (6.13) we know that $u(x,y+\varepsilon)-u(x,y)\le 0$ since V is decreasing in y. In order to find a lower bound for $u(x,y+\varepsilon)-u(x,y)$ we want to use the semi-harmonic property of $(P_t)_{t\ge0}$. For that we introduce the stopping time $\lambda_\varepsilon:=\tau_*\wedge\gamma_\varepsilon\wedge\tau^\varepsilon_K\wedge T$ where $T>0$ is fixed and $\tau^\varepsilon_K=\tau_K(x,y+\varepsilon)$. Notice that since $X^{x,y+\varepsilon}\le X^{x,y}$ (see (2.4)) then $\tau_K(x,y+\varepsilon)\le\tau_K(x,y)$. Now, using (6.17) and (6.18) we obtain

$$
\begin{aligned}
(6.20)\qquad & u(x,y+\varepsilon)-u(x,y)\\
\ge&\mathbb{E}\Big[e^{-r\lambda_\varepsilon}u(X^{x,y+\varepsilon}_{\lambda_\varepsilon},Y^{y+\varepsilon}_{\lambda_\varepsilon})+\int_0^{\lambda_\varepsilon}e^{-rt}(rK-\delta_0X^{x,y+\varepsilon}_tY^{y+\varepsilon}_t)dt\Big]\\
&-\mathbb{E}\Big[e^{-r\lambda_\varepsilon}u(X^{x,y}_{\lambda_\varepsilon},Y^{y}_{\lambda_\varepsilon})+\int_0^{\lambda_\varepsilon}e^{-rt}(rK-\delta_0X^{x,y}_tY^{y}_t)dt\Big].
\end{aligned}
$$

Notice that $0\le u\le\varepsilon$ on $[K,+\infty)\times[0,1]$ and

$$
\begin{aligned}
\tau_*\le\gamma_\varepsilon\wedge\tau^\varepsilon_K\wedge T &\implies u(X^{x,y}_{\lambda_\varepsilon},Y^{y}_{\lambda_\varepsilon})=0\le u(X^{x,y+\varepsilon}_{\lambda_\varepsilon},Y^{y+\varepsilon}_{\lambda_\varepsilon})\\
\gamma_\varepsilon\le\tau_*\wedge\tau^\varepsilon_K\wedge T &\implies u(X^{x,y+\varepsilon}_{\lambda_\varepsilon},Y^{y+\varepsilon}_{\lambda_\varepsilon})=\varepsilon\ge u(X^{x,y}_{\lambda_\varepsilon},Y^{y}_{\lambda_\varepsilon})
\end{aligned}
$$

so that

$$
(6.21)\qquad \text{on }\{\tau_*\wedge\gamma_\varepsilon\le\tau^\varepsilon_K\wedge T\}\text{ we have }u(X^{x,y+\varepsilon}_{\lambda_\varepsilon},Y^{y+\varepsilon}_{\lambda_\varepsilon})\ge u(X^{x,y}_{\lambda_\varepsilon},Y^{y}_{\lambda_\varepsilon}).
$$

Using this fact in (6.20) we get

$$
\begin{aligned}
&u(x,y+\varepsilon)-u(x,y)\\
\ge&\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\wedge\gamma_\varepsilon\}}e^{-r(\tau^\varepsilon_K\wedge T)}\Big(u(X^{x,y+\varepsilon}_{\tau^\varepsilon_K\wedge T},Y^{y+\varepsilon}_{\tau^\varepsilon_K\wedge T})-u(X^{x,y}_{\tau^\varepsilon_K\wedge T},Y^{y}_{\tau^\varepsilon_K\wedge T})\Big)\Big]\\
&+\delta_0\mathbb{E}\Big[\int_0^{\lambda_\varepsilon}e^{-rt}(X^{x,y}_tY^y_t-X^{x,y+\varepsilon}_tY^{y+\varepsilon}_t)dt\Big].
\end{aligned}
$$

Now we use that $X^{x,y+\varepsilon}\le X^{x,y}$ (see (2.4)) and that $x\mapsto u(x,y)$ is non-increasing (as shown in the proof of (iii) and (iv) of Lemma 4.2). Therefore from the right-hand side of the above inequality we easily get

$$
\begin{aligned}
(6.22)\qquad & u(x,y+\varepsilon)-u(x,y)\\
\ge&\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\wedge\gamma_\varepsilon\}}e^{-r(\tau^\varepsilon_K\wedge T)}\Big(u(X^{x,y}_{\tau^\varepsilon_K\wedge T},Y^{y+\varepsilon}_{\tau^\varepsilon_K\wedge T})-u(X^{x,y}_{\tau^\varepsilon_K\wedge T},Y^{y}_{\tau^\varepsilon_K\wedge T})\Big)\Big]\\
&+\delta_0\mathbb{E}\Big[\int_0^{\lambda_\varepsilon}e^{-rt}X^{x,y}_t(Y^y_t-Y^{y+\varepsilon}_t)dt\Big].
\end{aligned}
$$

Lower bounds can be provided for both terms on the right-hand side of the above expression. For the first term we recall (iii) of Lemma 3.1 and get

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\wedge\gamma_\varepsilon\}}e^{-r(\tau^\varepsilon_K\wedge T)}\Big(u(X^{x,y}_{\tau^\varepsilon_K\wedge T},Y^{y+\varepsilon}_{\tau^\varepsilon_K\wedge T})-u(X^{x,y}_{\tau^\varepsilon_K\wedge T},Y^{y}_{\tau^\varepsilon_K\wedge T})\Big)\Big]\\
&\geq -C\,\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\wedge\gamma_\varepsilon\}}e^{-r(\tau^\varepsilon_K\wedge T)}(1+X^{x,y}_{\tau^\varepsilon_K\wedge T})\Big(Y^{y+\varepsilon}_{\tau^\varepsilon_K\wedge T}-Y^{y}_{\tau^\varepsilon_K\wedge T}\Big)\Big]\\
&= -\varepsilon\,C\,\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\wedge\gamma_\varepsilon\}}e^{-r(\tau^\varepsilon_K\wedge T)}(1+X^{x,y}_{\tau^\varepsilon_K\wedge T})\Delta Y^{\varepsilon}_{\tau^\varepsilon_K\wedge T}\Big]\\
&\geq -\varepsilon\,C\,\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\}}e^{-r(\tau^\varepsilon_K\wedge T)}(1+X^{x,y}_{\tau^\varepsilon_K\wedge T})\Delta Y^{\varepsilon}_{\tau^\varepsilon_K\wedge T}\Big]
\end{aligned}\tag{6.23}$$

where $\Delta Y^\varepsilon_s := \frac{1}{\varepsilon}(Y^{y+\varepsilon}_t - Y^y_t)$, and the final inequality follows by observing that $\{\tau_*\wedge\gamma_\varepsilon > \tau^\varepsilon_K\wedge T\}\subseteq\{\tau_* > \tau^\varepsilon_K\wedge T\}$ and that the quantity under expectation is positive. For the integral term in (6.22) we argue in a similar way and obtain

$$\begin{aligned}
\mathbb{E}\Big[\int_0^{\lambda_\varepsilon} e^{-rt}X^{x,y}_t(Y^y_t-Y^{y+\varepsilon}_t)dt\Big] &= -\varepsilon\,\mathbb{E}\Big[\int_0^{\lambda_\varepsilon} e^{-rt}X^{x,y}_t\Delta Y^\varepsilon_t dt\Big]\\
&\geq -\varepsilon\,\mathbb{E}\Big[\int_0^{\tau_*\wedge\tau^\varepsilon_K\wedge T} e^{-rt}X^{x,y}_t\Delta Y^\varepsilon_t dt\Big]
\end{aligned}\tag{6.24}$$

Collecting (6.22), (6.23) and (6.24) we find

$$\begin{aligned}
&\frac{u(x,y+\varepsilon)-u(x,y)}{\varepsilon}\\
&\geq -C\,\mathbb{E}\Big[\mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\}}e^{-r(\tau^\varepsilon_K\wedge T)}(1+X^{x,y}_{\tau^\varepsilon_K\wedge T})\Delta Y^{\varepsilon}_{\tau^\varepsilon_K\wedge T}\Big]-\mathbb{E}\Big[\int_0^{\tau_*\wedge\tau^\varepsilon_K\wedge T} e^{-rt}X^{x,y}_t\Delta Y^\varepsilon_t dt\Big]
\end{aligned}$$

and we now aim at taking limits as $\varepsilon\to 0$. In order to apply dominated convergence, it is sufficient to prove that the family of random variables $(X^{x,y}_\tau,\Delta Y^\varepsilon_\tau)$ is uniformly bounded in L^4 when τ ranges through all $[0,T]$ valued stopping times and $\varepsilon\in(0,1-y)$. Indeed, by Cauchy-Shwarz inequality, this will imply that $X^{x,y}_\tau\cdot\Delta Y^\varepsilon_\tau$ is bounded in L^2 uniformly with respect to ε and τ.

The bound for X follows directly from the explicit expression (2.4). Note then that ΔY^ε is an exponential martingale. Indeed, denoting $H^\varepsilon_t = -\frac{\delta_0}{\sigma}(1-Y^y_t-Y^{y+\varepsilon}_t)$, we have

$$\Delta Y^\varepsilon_t = \exp\left[\int_0^t H^\varepsilon_s dW_s - \frac{1}{2}\int_0^t (H^\varepsilon_s)^2 ds\right].$$

It follows that

$$\begin{aligned}
(\Delta Y^\varepsilon_t)^4 &= \exp\left[4\int_0^t H^\varepsilon_s dW_s - 4\frac{1}{2}\int_0^t (H^\varepsilon_s)^2 ds\right]\\
&= \exp\left[6\int_0^t (H^\varepsilon_s)^2 ds\right]\exp\left[\int_0^t 4H^\varepsilon_s dW_s - \frac{1}{2}\int_0^t (4H^\varepsilon_s)^2 ds\right]
\end{aligned}$$

Since H^ε is uniformly bounded by $\frac{\delta_0}{\sigma}$, the second term in the above expression is a martingale, and we deduce that for any stopping time τ taking values in $[0,T]$

$$\mathbb{E}[(\Delta Y^\varepsilon)^4_\tau] \leq \exp(6T\frac{\delta_0^2}{\sigma^2}).$$

Using that $\Delta Y^\varepsilon_t \to U^y_t$ almost surely for all $t \ge 0$, as $\varepsilon \to 0$, we conclude

$$0 \ge u_y(x,y) \ge -C\,\mathbb{E}\Big[\mathbb{1}_{\{\tau_K\wedge T<\tau_*\}}e^{-r(\tau_K\wedge T)}(1+X^{x,y}_{\tau_K\wedge T})U^y_{\tau_K\wedge T}\Big] - \mathbb{E}\Big[\int_0^{\tau_*\wedge\tau_K\wedge T} e^{-rt}X^{x,y}_t U^y_t dt\Big]. \tag{6.25}$$

In the above estimate we have used that

$$\tau^\varepsilon_K \uparrow \tau_K \text{ and } \mathbb{1}_{\{\tau^\varepsilon_K\wedge T<\tau_*\}} \to \mathbb{1}_{\{\tau_K\wedge T<\tau_*\}} \qquad \text{as } \varepsilon \to 0, \tag{6.26}$$

which follows from the continuity of $(x,y) \to X^{x,y}$ and the fact that $\mathbb{P}(\tau_* = \tau_K) = 0$ (see Proposition 4.1).

Notice that the above estimates also imply that U^y_τ is bounded in L^4 and $X^{x,y}_\tau \cdot U^y_\tau$ is bounded in L^2, uniformly with respect to stopping times $\tau \in [0,T]$ and $(x,y) \in [K, x_0+1]\times(0,1)$.

It remains to take limits as $(x,y) \to (x_0,y_0)$ with $(x,y) \in \mathcal{C}$. By continuity of the sample paths $\tau_*(x,y) = \hat\tau_*(x,y)$ for $(x,y) \in \mathcal{C}$. We use (i) of Proposition 6.3, dominated convergence and (6.25) (along with the fact that $\mathbb{P}_{x_0,y_0}(\tau_K > 0) = 1$) to obtain

$$\lim_{(x,y)\to(x_0,y_0)} u_y(x,y) = 0. \tag{6.27}$$

The latter implies continuity of u_y at $\partial\mathcal{S}_1$.

To prove that u_y is also continuous across $\partial\mathcal{S}_2$ we need to argue in a slightly different way. Fix $(x_0,y_0) \in \partial\mathcal{S}_2$ with $y_0 < b^K_2$ and pick $(x,y) \in \mathcal{C}$. With no loss of generality we consider $x_0 = b_2(y_0)$ as the proof requires minor changes for $x_0 = K$. We set $\gamma_* = \gamma_*(x,y)$ the first entry time of $(X^{x,y}, Y^y)$ into $\mathcal{S}_2$ and denote by $\tau_\varepsilon = \tau_*(x, y-\varepsilon)$ the first entry time of $(X^{x,y-\varepsilon}, Y^{y-\varepsilon})$ into $\mathcal{S}_1$ for some $\varepsilon > 0$. Then we define $\eta_\varepsilon := \tau_\varepsilon \wedge \gamma_* \wedge \tau_K \wedge T$ for some $T > 0$. Again we recall that $\tau_K = \tau_K(x,y) \le \tau_K(x, y-\varepsilon)$.

We know that $u(x,y) - u(x,y-\varepsilon) \le 0$ from (i) of Lemma 3.1 and (6.13). In order to find a lower bound we use (6.17) and (6.18) and get

$$\begin{aligned} &u(x,y) - u(x,y-\varepsilon)\\ \ge&\mathbb{E}\Big[e^{-r\eta_\varepsilon}u(X^{x,y}_{\eta_\varepsilon}, Y^y_{\eta_\varepsilon}) + \int_0^{\eta_\varepsilon} e^{-rt}(rK - \delta_0 X^{x,y}_t Y^y_t)dt\Big]\\ &-\mathbb{E}\Big[e^{-r\eta_\varepsilon}u(X^{x,y-\varepsilon}_{\eta_\varepsilon}, Y^{y-\varepsilon}_{\eta_\varepsilon}) + \int_0^{\eta_\varepsilon} e^{-rt}(rK - \delta_0 X^{x,y-\varepsilon}_t Y^{y-\varepsilon}_t)dt\Big]. \end{aligned}$$

From this point onwards we can repeat the arguments used above up to trivial modifications. These allow us to conclude that u_y is continuous across $\partial\mathcal{S}_2$ with the possible exception of (K, b^K_2), because Proposition 6.3 does not hold at that point if $k > 0$.

As already mentioned, analogous arguments allow to prove that u_x is also continuous everywhere with the possible exception of (K, b^K_2). It follows that $V \in C^1$ on $(\mathbb{R}_+\times(0,1)) \setminus (K, b^K_2)$ and $v \in C^1$ on $(\mathbb{R}\times(0,1)) \setminus (z_K, \overline{y}_K)$ (see (5.6)). The latter and (5.8) imply that v_{yy} is continuous on $\overline{\mathcal{C}}' \setminus (z_K, \overline{y}_K)$ as claimed.

□

It remains to prove Lemma 6.1 and for that it is convenient to change variables to the coordinate system (z,y). We denote $w(z,y) = u(F(z,y),y)$, $w_z := \partial w/\partial z$, $w_y := \partial w/\partial y$ and $w_{yy} := \partial^2 w/\partial y^2$. In these variables τ_K from (6.19) reads

$$\tau_K(z,y) = \inf\{t \ge 0 \,:\, F(Z^z_t, Y^y_t) \le K\}.$$

Notice that for $k > 0$ the boundary c_1 is increasing and the stopping set $\mathcal{S}'_1$ lies below it. Hence (6.3) is a consequence of standard arguments involving the law of iterated logarithm. Showing (6.4) for $k > 0$ is instead more difficult because c_2 is also increasing but $\mathcal{S}_2'$ lies above the boundary. A symmetric situation occurs for $k < 0$.

In what follows we first show that the classical smooth-fit condition holds and then prove that under our assumptions this implies Lemma 6.1. In the next lemma we only consider smooth-fit in those cases when the monotonicity of the boundary does not allow a direct proof of (6.3) or (6.4) based on the law of iterated logarithm.

Lemma 6.5. *If $(z_0, y_0) \in \partial\mathcal{S}_1'$ and $k < 0$, then $w_y(z_0, y_0+) = 0$. Analogously if $(z_0, y_0) \in \partial\mathcal{S}_2'$, with $z_0 < z_K$ and $y_0 = c_2(z_0)$, and $k > 0$ then $w_y(z_0, y_0-) = 0$. Finally, if $(z_0, y_0) \in \partial\mathcal{S}_2'$, with $y_0 = y_K(z_0)$ and $k < 0$ then $v_y(z_0, y_0+) = 0$.*

Proof. We carry out the proof under the assumption of $k > 0$ (see (5.2)). This induces no loss in generality as symmetric arguments hold for $k < 0$.

Let $(z_0, y_0) \in \partial\mathcal{S}_2'$ with $y_0 = c_2(z_0)$. Notice that for $y \in (y_0, y_K(z_0))$ we have $w_y(z_0, y) = 0$. Also we know from the proof of (i) in Lemma 5.4 that $w_y \geq 0$ locally at (z_0, y_0). We argue by contradiction and assume $w_y(z_0, y_0-) \geq \lambda_0 > 0$. The latter limit exists because w_z is locally bounded (see (5.13)) and $|w_{yy}| \leq c|w_z|$ in $\mathcal{C}'$ due to (5.8), for a suitable $c > 0$.

Fix $\varepsilon > 0$, consider the open rectangle $R_\varepsilon := (z_0, z_0 + \varepsilon) \times (y_0 - \varepsilon, y_0 + \varepsilon)$ and let $\rho_\varepsilon = \inf\{t \geq 0 : (Z_t^{z_0}, Y_t^{y_0}) \notin R_\varepsilon\}$. With no loss of generality we assume $\rho_\varepsilon \leq \tau_K \wedge \tau_*$ and from (6.17) we obtain

(6.28)
$$w(z_0, y_0) \leq \mathbb{E}\left[e^{-r(t\wedge\rho_\varepsilon)} w(Z^{z_0}_{t\wedge\rho_\varepsilon}, Y^{y_0}_{t\wedge\rho_\varepsilon}) + \int_0^{t\wedge\rho_\varepsilon} e^{-rs}(rK - \delta_0 Y_s^{y_0} F(Z_s^{z_0}, Y_s^{y_0}))ds\right].$$

Since $w(\,\cdot\,, y)$ is decreasing (see (5.13)) and R_ε is bounded we can find a constant $C_\varepsilon > 0$ depending on R_ε and such that

$$w(z_0, y_0) \leq \mathbb{E}\left[e^{-r(t\wedge\rho_\varepsilon)} w(z_0, Y^{y_0}_{t\wedge\rho_\varepsilon}) + C_\varepsilon(t \wedge \rho_\varepsilon)\right]. \tag{6.29}$$

Recalling that $w_{yy}(z_0, \cdot)$ is bounded on $[y_0 - \varepsilon, y_0 + \varepsilon] \setminus \{y_0\}$, we can apply Itô-Tanaka formula to get

$$\begin{aligned} w(z_0, y_0) \leq & w(z_0, y_0) + \mathbb{E}\left[\int_0^{t\wedge\rho_\varepsilon} e^{-rs} \frac{\delta_0^2}{2\sigma^2} \left[Y_s^{y_0}(1 - Y_s^{y_0})\right]^2 w_{yy}(z_0, Y_s^{y_0}) \mathbb{1}_{\{Y_s^{y_0} \neq y_0\}} ds\right] \\ & + \mathbb{E}\left[\frac{1}{2}\int_0^{t\wedge\rho_\varepsilon} e^{-rs}(w_y(z_0, y_0+) - w_y(z_0, y_0-)) dL_s^{y_0}(Y^{y_0}) + C_\varepsilon(t \wedge \rho_\varepsilon)\right]. \end{aligned} \tag{6.30}$$

Boundedness of $w_{yy}(z_0, \cdot)$ and the assumption $w_y(z_0, y_0-) \geq \lambda_0$ give

$$0 \leq -\frac{1}{2}\lambda_0 \mathbb{E}\left[\int_0^{t\wedge\rho_\varepsilon} e^{-rs} dL_s^{y_0}(Y^{y_0})\right] + C_\varepsilon' \mathbb{E}\left[(t \wedge \rho_\varepsilon)\right] \tag{6.31}$$

for some positive $C_\varepsilon' > 0$. For $0 < p < 1$, Burkholder-Davis-Gundy inequality and some algebra give

$$\begin{aligned} \mathbb{E}\left[\int_0^{t\wedge\rho_\varepsilon} e^{-rs} dL_s^{y_0}(Y^{y_0})\right] \geq & \mathbb{E}\left[e^{-rt} L^{y_0}_{t\wedge\rho_\varepsilon}(Y^{y_0})\right] \\ = & e^{-rt}\mathbb{E}\left[|Y^{y_0}_{t\wedge\rho_\varepsilon} - y_0|\right] \geq \frac{e^{-rt}}{(2\varepsilon)^p}\mathbb{E}\left[|Y^{y_0}_{t\wedge\rho_\varepsilon} - y_0|^{1+p}\right] \\ \geq & \frac{e^{-rt}}{(2\varepsilon)^p} c_p \mathbb{E}\left[\langle Y^{y_0}\rangle^{\frac{1+p}{2}}_{t\wedge\rho_\varepsilon}\right] \geq \frac{e^{-rt}}{(2\varepsilon)^p} c_{p,\varepsilon} \mathbb{E}\left[(t \wedge \rho_\varepsilon)^{\frac{1+p}{2}}\right], \end{aligned} \tag{6.32}$$

with $c_{p,\varepsilon} > 0$ depending on p and ε. Plugging the latter inside (6.31) and letting $t \to 0$ we reach a contradiction. Therefore it must be $w_y(z_0, y_0-) = 0$.

The proof is entirely analogous for $(z_0, y_0) \in \partial\mathcal{S}'_1$ and $k < 0$. It is also worth noticing that for $(z_0, y_0) \in \partial\mathcal{S}'_2$ with $y_0 = y_K(z_0)$, the smooth-fit condition amounts to $v_y(z_0, y_0+) = 0$ because the stopping payoff is ε_0. Using (iii) in Lemma 5.4 and arguments similar to those above we can prove that $v_y(z_0, y_0+) = 0$ holds. □

Proof of Lemma 6.1. Here we only consider the case $k > 0$ but the same results hold for $k < 0$ and these can be proven by symmetric arguments.

The proof of (6.3), which we omit for brevity, is a straightforward consequence of the fact that c_1 is increasing and Y_t is non-degenerate away from 0 and 1, so that the law of iterated logarithm can be applied. The same rationale allows to prove that (6.4) holds for $y_0 = y_K(z_0)$ for $z_0 < z_K$.

To prove (6.4) with $y_0 = c_2(z_0)$ and $z_0 < z_K$ let us argue by contradiction and assume that $(x_0, y_0) \in \partial\mathcal{S}_2 \cap \{x > K\}$ is not regular or equivalently $(z_0, y_0) \in \partial\mathcal{S}'_2 \cap R_K$ is not regular (with $F(z_0, y_0) = x_0$), i.e. (6.4) does not hold. Pick $y < y_0$ and $\varepsilon > 0$ such that $y + \varepsilon < y_0$. Denote $\hat\gamma_\varepsilon = \hat\gamma_*(z_0, y+\varepsilon)$, $\hat\tau = \hat\tau_*(z_0, y)$, $\hat\gamma = \hat\gamma_*(z_0, y)$, $\tau_K^\varepsilon = \tau_K(z_0, y+\varepsilon)$. Notice that $\tau_K(z_0, y) \geq \tau_K(z_0, y+\varepsilon)$, then from (6.17) and (6.18) and setting $\lambda_\varepsilon := \hat\tau \wedge \hat\gamma_\varepsilon \wedge \tau_K^\varepsilon \wedge T$ we obtain

$$
\begin{aligned}
w(z_0, y+\varepsilon) - w(z_0, y) \geq & \mathbb{E}\left[\int_0^{\lambda_\varepsilon} e^{-rt}\delta_0\left(Y_t^y F(Z_t^{z_0}, Y_t^y) - Y_t^{y+\varepsilon} F(Z_t^{z_0}, Y_t^{y+\varepsilon})\right)dt\right] \\
& + \mathbb{E}\left[e^{-r\lambda_\varepsilon}\left(w(Z_{\lambda_\varepsilon}^{z_0}, Y_{\lambda_\varepsilon}^{y+\varepsilon}) - w(Z_{\lambda_\varepsilon}^{z_0}, Y_{\lambda_\varepsilon}^{y})\right)\right] \\
\geq & \mathbb{E}\left[\int_0^{\lambda_\varepsilon} e^{-rt}\delta_0\left(Y_t^y F(Z_t^{z_0}, Y_t^y) - Y_t^{y+\varepsilon} F(Z_t^{z_0}, Y_t^{y+\varepsilon})\right)dt\right]
\end{aligned} \tag{6.33}
$$

where in the last inequality we have used that $y \mapsto w(z, y)$ is non-decreasing as shown in the proof of Lemma 5.4. Recall that $\frac{\partial}{\partial y}(yF(z, y))$ is strictly negative (see (5.18)) so that almost surely and for all $\varepsilon > 0$ we have

$$
\int_0^{\lambda_\varepsilon} e^{-rt}\delta_0\left(Y_t^y F(Z_t^{z_0}, Y_t^y) - Y_t^{y+\varepsilon} F(Z_t^{z_0}, Y_t^{y+\varepsilon})\right)dt \geq 0.
$$

As in the proof of Proposition 6.4 we have $\tau_K^\varepsilon \uparrow \tau_K$ as $\varepsilon \to 0$. Moreover $\hat\gamma_\varepsilon$ increases[1] as $\varepsilon \to 0$, hence $\hat\gamma_- := \lim_{\varepsilon\to 0}\hat\gamma_\varepsilon \leq \hat\gamma$, $\mathbb{P}$-a.s. To prove the reverse inequality we fix $\omega \in \Omega$ and pick $\delta > 0$ such that $\hat\gamma(\omega) > \delta$. Then in particular we have

$$
\inf_{0\leq t\leq\delta}(c_2(Z_t^{z_0}) - Y_t^y)(\omega) \geq c_\delta(\omega) > 0 \tag{6.34}
$$

for some c_δ. Recall that $(t, y) \mapsto U_t^y(\omega)$ is continuous, hence bounded on $[0, \delta] \times [0, 1]$ by a constant $c'_\delta(\omega) > 0$. Using that $|Y_t^{y+\varepsilon} - Y_t^y|(\omega) \leq c'_\delta(\omega)\cdot\varepsilon$ we find

$$
\inf_{0\leq t\leq\delta}(c_2(Z_t^{z_0}) - Y_t^{y+\varepsilon})(\omega) \geq c_\delta(\omega) - c'_\delta(\omega)\cdot\varepsilon
$$

from (6.34). This implies that for all ε sufficiently small $\hat\gamma_\varepsilon(\omega) > \delta$. Since δ was arbitrary we conclude $\lim_{\varepsilon\to 0}\hat\gamma_\varepsilon(\omega) = \hat\gamma(\omega)$. The argument holds for a.e. $\omega \in \Omega$ hence we obtain

$$
\lim_{\varepsilon\to 0}\hat\gamma_\varepsilon = \hat\gamma, \qquad \mathbb{P}-\text{a.s.}
$$

Convergence of $\hat\gamma_\varepsilon$ and τ_K^ε imply

$$
\lim_{\varepsilon\to 0}\lambda_\varepsilon = \hat\tau \wedge \hat\gamma \wedge \tau_K \wedge T, \qquad \mathbb{P}-\text{a.s.}
$$

[1] Notice that, due to the geometry of $\mathcal{S}'_2$, (Z^{z_0}, Y^{y_0}) can only enter $\mathcal{S}'_2$ by hitting c_2.

Dividing (6.33) by ε and taking limits as $\varepsilon \to 0$, we may use Fatou's theorem and the expression (5.18) for $\frac{\partial}{\partial y}(yF(z,y))$ to obtain

$$w_y(z_0,y) \geq -\delta_0 \mathbb{E}\left[\int_0^{T\wedge\hat{\tau}\wedge\hat{\gamma}\wedge\tau_K} e^{-rt} U_t^y F(Z_t^{z_0}, Y_t^y)\left(\frac{1-Y_t^y-\frac{\sigma^2}{\delta}}{1-Y_t^y}\right) dt\right]. \tag{6.35}$$

Now we let $y \uparrow y_0$ and use that $\mathbb{P}$-a.s. the following limits hold

$$\hat{\gamma}_*(z_0,y) \downarrow \hat{\gamma}_*^+(z_0,y_0) \geq \hat{\gamma}_*(z_0,y_0),$$

$$\hat{\tau}_*(z_0,y) \uparrow \hat{\tau}_*(z_0,y_0) \quad \text{and} \quad \tau_K(z_0,y) \downarrow \tau_K(z_0,y_0).$$

In particular we notice that for the convergence of $\hat{\tau}_*$ we can use the same arguments as those used above for the convergence of $\hat{\gamma}_\varepsilon$. Clearly

$$\mathbb{P}(\tau_K(z_0,y_0) > 0) = \mathbb{P}(\hat{\tau}_*(z_0,y_0) > 0) = 1,$$

and by assumption, $\mathbb{P}(\hat{\gamma}_*(z_0,y_0) > 0) > 0$. Using again Fatou's lemma, taking limits in (6.35) the stopping time $\theta(z_0,y) := (\hat{\tau} \wedge \hat{\gamma} \wedge \tau_K)(z_0,y)$ converges to a stopping time $\theta(z_0,y_0) > 0$, $\mathbb{P}$-a.s. Hence $w_y(z_0,y_0-) > 0$, which contradicts the smooth-fit principle proven in Lemma 6.5. In conclusion (z_0,y_0) must be regular for $\mathcal{S}_2'$, i.e. (6.4) holds. ☐

7. Existence of a Nash equilibrium

Building on the results of the previous sections, we can prove the existence of a Nash equilibrium for our game with incomplete information. We recall here that the two main difficulties for such existence arise from the lack of uniform integrability of the stopping payoffs and the fact that the problem is bi-dimensional. In the rest of this section we make the next standing assumption.

Assumption 7.1. *We assume* $\frac{\sigma^2}{\delta_0} > 1$.

The next result will allow us to circumvent the lack of uniform integrability and it shows that the boundary c_1 of $\mathcal{S}_1'$ is always strictly positive.

Lemma 7.2. *For every* $z \in \mathbb{R}$ *we have* $c_1(z) > 0$.

Proof. Arguing by contradiction we assume that there exists $z_0 \in \mathbb{R}$ such that $c_1(z_0) = 0$. Hence

$$(z_0,y) \notin \mathcal{S}_1' \text{ and } (F(z_0,y),y) \notin \mathcal{S}_1 \qquad \text{for all } y \in (0,1). \tag{7.1}$$

Since $F(\,\cdot\,,y)$ is increasing, properties of $\mathcal{S}_1$ studied in Section 4 imply that for fixed $h > 0$ we may define a strip

$$C_0(h) := \{(z,y) \in \mathbb{R}\times(0,1) \mid F(z_0-h,y) \leq x \leq F(z_0,y)\}$$

and $C_0(h) \cap \mathcal{S}_1 = \emptyset$.

In particular if we pick $y \in (0,1)$ and $x = F(z_0-h,y)$ then, assuming without loss of generality that $k > 0$ (see (5.2)), we have $\tau_* \geq h$, $\mathbb{P}_{x,y}$-a.s. The latter follows by the fact that for all $t \in [0,h]$ the couple $(X_t^{x,y}, Y_t^y)$ lies in $C_0(h)$ because its joint distribution is supported along a curve $\{(F(z_0-h+kt,\zeta),\zeta),\ \zeta \in (0,1)\}$ (see Remark 5.1). Notice that for $k < 0$ and with $x = F(z_0-h,y)$, monotonicity of $F(\,\cdot\,,\,y)$ and (7.1) imply $\tau_* = +\infty$ $\mathbb{P}_{x,y}$-a.s.).

Theorem 3.2 gives

$$\begin{aligned} V(x,y) &\leq \mathbb{E}_{x,y}\left[e^{-r(h\wedge\tau_*)}V(X_{h\wedge\tau_*},Y_{h\wedge\tau_*})\right] = \mathbb{E}_{x,y}\left[e^{-rh}V(X_h,Y_h)\right] \\ &\leq \mathbb{E}_{x,y}[e^{-rh}X_h] = F(z_0-h,y)\mathbb{E}_y[e^{-\delta_0\int_0^h Y_t\,dt}M_h], \end{aligned} \tag{7.2}$$

where $M_h = \exp(\sigma W_h - \frac{\sigma^2}{2}h)$. We aim at showing that for y sufficiently close to zero we get

$$F(z_0-h,y)\mathbb{E}_y\left[e^{-\delta_0\int_0^h Y_t\,dt}M_h\right] \le G_1(F(z_0-h,y)) = F(z_0-h,y)-K$$

or equivalently

$$\Theta(y) := F(z_0-h,y)\left(1-\mathbb{E}_y\left[e^{-\delta_0\int_0^h Y_t\,dt}M_h\right]\right) \ge K. \tag{7.3}$$

The latter and (7.2) lead to $V(x,y)\le x-K$, hence a contradiction.

Defining the probability measure $\mathbb{P}^{(\sigma)}$ by

$$\frac{d\mathbb{P}^{(\sigma)}_y}{d\mathbb{P}_y} = M_h$$

by Girsanov's theorem we have that $W^{(\sigma)}_t = W_t - \sigma t$, $t\ge 0$ is a Brownian motion under $\mathbb{P}^{(\sigma)}_y$. Moreover under the new measure Y evolves according to

$$Y_t = y - \frac{\delta_0}{\sigma}\int_0^t Y_s(1-Y_s)dW^{(\sigma)}_s - \delta_0\int_0^t Y_s(1-Y_s)ds.$$

From the above dynamics it follows immediately that $\mathbb{E}^{(\sigma)}_y(Y_t)\le y$ for all $t\ge 0$ and

$$\mathbb{E}^{(\sigma)}_y(Y_t) \ge y - \delta_0\int_0^t \mathbb{E}^{(\sigma)}_y(Y_s)\,ds \ge y(1-\delta_0 t). \tag{7.4}$$

Using the inequality $1-e^{-u}\ge u-\frac{u^2}{2}$ valid for $u\ge 0$, we have

$$\begin{aligned}\Theta(y) &= F(z_0-h,y)\mathbb{E}_y\left[\left(1-e^{-\delta_0\int_0^h Y_t\,dt}\right)M_h\right]\\ &\ge F(z_0-h,y)\left(\delta_0\mathbb{E}^{(\sigma)}_y\left[\int_0^h Y_t\,dt\right]-\frac{\delta_0^2}{2}\mathbb{E}^{(\sigma)}_y\left[\int_0^h Y_t\,dt\right]^2\right)\\ &\ge F(z_0-h,y)\left(y\delta_0\int_0^h(1-\delta_0 t)\,dt-\frac{\delta_0^2 h}{2}\mathbb{E}^{(\sigma)}_y\left[\int_0^h (Y_t)^2\,dt\right]\right),\end{aligned}$$

where for the last inequality we used (7.4) and Cauchy-Schwarz inequality. We aim at showing that

$$\mathbb{E}^{(\sigma)}_y\left[\int_0^h (Y_t)^2\,dt\right] \le hy^2. \tag{7.5}$$

To see this, we observe that Y^2 is a supermartingale under the probability measure $\mathbb{P}^{(\sigma)}$. Indeed, applying Itô's formula we get

$$d(Y^y_t)^2 = -2\frac{\delta_0}{\sigma}(Y^y_t)^2(1-Y^y_t)dW^{(\sigma)}_t - \delta_0(Y^y_t)^2(1-Y^y_t)dt + \frac{\delta_0^2}{\sigma^2}(Y^y_t)^2(1-Y^y_t)^2dt,$$

and the drift part of the SDE is non-positive because $\frac{\sigma^2}{\delta_0}>1$. Thus (7.5) holds as claimed.

Finally we obtain

$$\Theta(y) \ge \delta_0 h\, y\, F(z_0-h,y)\left(1-\frac{\delta_0 h}{2}(1+y)\right). \tag{7.6}$$

Recalling that $y\in[0,1]$, for h sufficiently small we have $1>\frac{\delta_0 h}{2}(1+C\,y)$. Moreover when $\sigma^2/\delta_0>1$ it is immediate to check that $y\,F(z_0-h,y)\to+\infty$ as $y\to 0$ (see (5.5)). In conclusion the right-hand side in (7.6) diverges, yielding the desired contradiction. □

We can now prove existence of a saddle point for our game.

Proposition 7.3. *If $k > 0$ the pair (γ_*, τ_*) defined in Theorem 3.2 is a saddle point.*

Proof. Since Theorem 3.2 guarantees the optimality of γ_*, i.e.

$$V(x,y) \geq M_{x,y}(\tau, \gamma_*), \quad \text{for all } \tau \in \mathcal{T},$$

it remains to prove the optimality of τ_*, that is

$$V(x,y) \leq M_{x,y}(\tau_*, \gamma), \quad \text{for all } \gamma \in \mathcal{T},$$

Let $z \in \mathbb{R}$ be fixed and set $x = F(z,y)$. Invoking Theorem 3.2 and observing that for any fixed $t > 0$ and γ,

$$V(X_{\tau_*}, Y_{\tau_*})\mathbb{1}_{\{\tau_* \leq t\wedge\gamma\}} = G_1(X_{\tau_*})\mathbb{1}_{\{\tau_* \leq t\wedge\gamma\}}$$

we obtain

$$\begin{aligned} V(x,y) \leq & \mathbb{E}_{x,y}\left[e^{-r(t\wedge\tau_*\wedge\gamma)}V(X_{t\wedge\tau_*\wedge\gamma}, Y_{t\wedge\tau_*\wedge\gamma})\right] \\ \leq & \mathbb{E}_{x,y}\left[e^{-r\tau_*}G_1(X_{\tau_*})\mathbb{1}_{\{\tau_* \leq t\wedge\gamma\}} + e^{-r\gamma}G_2(X_\gamma)\mathbb{1}_{\{\gamma < t\wedge\tau_*\}}\right] \\ & + \mathbb{E}_{x,y}\left[e^{-rt}V(X_t, Y_t)\mathbb{1}_{\{t\leq\tau_*\wedge\gamma\}}\right] \end{aligned}$$

for any stopping time γ.

We now prove that the last term of the expression above converges to zero as $t \to +\infty$. Notice first that c_1 is increasing (see Corollary 5.5) and therefore $\zeta \mapsto (b_1 \circ c_1)(\zeta)$ is decreasing due to Corollary 4.5. For $t \leq \tau_*$ we have $X^x_t = F(Z^z_t, Y^y_t) \leq b_1(Y_t)$, which implies that $Y^y_t \geq c_1(Z^z_t) \geq c_1(z)$ after the change of variables. Then $b_1(Y_t) \leq (b_1 \circ c_1)(z)$ and we have the uniform bound $X^x_t \leq (b_1 \circ c_1)(z) =: a_z$ for $t \leq \tau_*$. Notice that $c_1(z) > 0$ thanks to Lemma 7.2, so that we also have $a_z < +\infty$.

Using such bound we get

$$\mathbb{E}_{x,y}\left[e^{-rt}V(X_t, Y_t)\mathbb{1}_{\{t\leq\tau_*\wedge\gamma\}}\right] \leq G_2(a_z)\, e^{-rt} \to 0, \quad \text{as } t \to 0.$$

Next, the monotone convergence theorem yields

$$\begin{aligned} V(x,y) \leq & \lim_{t\to+\infty}\mathbb{E}_{x,y}\left[e^{-r\tau_*}G_1(X_{\tau_*})\mathbb{1}_{\{\tau_* \leq t\wedge\gamma\}} + e^{-r\gamma}G_2(X_\gamma)\mathbb{1}_{\{\gamma < t\wedge\tau_*\}}\right] \\ = & \mathbb{E}_{x,y}\left[e^{-r\tau_*}G_1(X_{\tau_*})\mathbb{1}_{\{\tau_* \leq \gamma\}} + e^{-r\gamma_*}G_2(X_\gamma)\mathbb{1}_{\{\gamma < \tau_*\}}\right] \\ = & M_{x,y}(\tau_*, \gamma), \end{aligned}$$

that is, τ_* is optimal for the buyer. $\square$

Let us now analyze the case $k < 0$, for which we prove existence of a Nash equilibrium under stronger assumptions on the parameters. We start with an auxiliary lemma, which will require the following assumption (recall also that $\sigma^2/\delta_0 > 1$ by Assumption 7.1).

Assumption 7.4. *We take r such that*

$$\frac{\delta_0}{\sigma^2}\left(\frac{\delta_0 + \sigma^2}{2}\right) < r < \left(\frac{\delta_0 + \sigma^2}{2}\right). \tag{7.7}$$

Notice that (7.7) indeed implies $k < 0$.

Lemma 7.5. *Under Assumption 7.4 it holds that:*

$$\lim_{t\to\infty} e^{-rt}F(z + kt, c_1(z + kt)) = 0, \qquad \forall z \in \mathbb{R}.$$

Proof. First note that

$$e^{-rt}F(z + kt, c_1(z + kt)) = e^{z+(k-r)t}\left(\frac{1 - c_1(z + kt)}{c_1(z + kt)}\right)^{\frac{\sigma^2}{\delta_0}}.$$

Then recall that $c_1(\cdot) \le y_K(\cdot)$ (see (5.11)) and since $k < 0$ then $c_1(z+kt) \to 0$ as $t \to +\infty$. It is therefore sufficient to prove that as $t \to \infty$

$$\frac{1}{c_1(z+kt)} \le c\, e^{\alpha t}, \tag{7.8}$$

for some constants $c > 0$ and $\alpha < \frac{\delta_0}{\sigma^2}(r-k)$.

Define $\lambda_a = \inf\{t > 0 \,|\, Y_t \le a\}$. Let $z \in \mathbb{R}$, $y > c_1(z)$ and $x = F(z,y)$. Note that since $z \to c_1(z)$ is non-decreasing and $k < 0$, we have $\tau_* \ge \lambda_{c_1(z)}$ $\mathbb{P}_{x,y}$-almost surely. Therefore, for all $t \ge 0$ Theorem 3.2 gives

(7.9)
$$\begin{aligned} V(x,y) &\le \mathbb{E}_{x,y}\left[e^{-r(t\wedge\lambda_{c_1(z)})} V\left(X_{t\wedge\lambda_{c_1(z)}}, Y_{t\wedge\lambda_{c_1(z)}}\right)\right] \\ &= \mathbb{E}_{x,y}\left[e^{-rt} V\left(X_t, Y_t\right) \mathbb{1}_{\{t<\lambda_{c_1(z)}\}} + e^{-r\lambda_{c_1(z)}} V\left(X_{\lambda_{c_1(z)}}, c_1(z)\right) \mathbb{1}_{\{\lambda_{c_1(z)} \le t\}}\right]. \end{aligned}$$

On the event $\{t < \lambda_{c_1(z)}\}$ we have $X_t \le b_1(Y_t)$ and $Y_t \ge c_1(z)$, $\mathbb{P}_{x,y}$-a.s., so that $X_t \le (b_1 \circ c_1)(z) =: a_z < +\infty$, $\mathbb{P}_{x,y}$-a.s. (as in the proof of Proposition 7.3). The latter implies

$$e^{-rt} V(X_t, Y_t) \mathbb{1}_{\{t<\lambda_{c_1(z)}\}} \le e^{-rt} X_t \mathbb{1}_{\{t<\lambda_{c_1(z)}\}} \le e^{-rt} a_z \mathbb{1}_{\{t<\lambda_{c_1(z)}\}}$$

and hence

$$\lim_{t\to 0} \mathbb{E}_{x,y}\left[e^{-rt} V(X_t, Y_t) \mathbb{1}_{\{t<\lambda_{c_1(z)}\}}\right] = 0, \quad \mathbb{P}_{x,y}\text{-a.s.}$$

Taking limits in (7.9) as $t \to \infty$ and using monotone convergence, we deduce that

$$\begin{aligned} V(x,y) &\le \mathbb{E}_{x,y}\left[e^{-r\lambda_{c_1(z)}} V\left(X_{\lambda_{c_1(z)}}, c_1(z)\right) \mathbb{1}_{\{\lambda_{c_1(z)}<\infty\}}\right] \\ &\le \mathbb{E}_{x,y}\left[e^{-r\lambda_{c_1(z)}} X_{\lambda_{c_1(z)}} \mathbb{1}_{\{\lambda_{c_1(z)}<\infty\}}\right] \\ &= \mathbb{E}_{y}\left[e^{-r\lambda_{c_1(z)}} F(z + k\lambda_{c_1(z)}, c_1(z)) \mathbb{1}_{\{\lambda_{c_1(z)}<\infty\}}\right] \\ &= e^z \left(\frac{1-c_1(z)}{c_1(z)}\right)^{\frac{\sigma^2}{\delta_0}} \mathbb{E}_{y}\left[e^{(-r+k)\lambda_{c_1(z)}} \mathbb{1}_{\{\lambda_{c_1(z)}<\infty\}}\right] \end{aligned}$$

In order to compute the Laplace trasform of $\lambda_{c_1(z)}$ we need to recall the fundamental solutions of $\mathcal{L}_Y f - (r-k)f = 0$, where $\mathcal{L}_Y$ denotes the infinitesimal generator of the diffusion Y. Letting ψ be the unique positive, increasing solution and ϕ the unique positive, decreasing one, we have

$$\psi(y) = y^{\beta}(1-y)^{1-\beta} \quad \text{and} \quad \phi(y) = y^{1-\beta}(1-y)^{\beta}$$

where $\beta = \frac{\sigma^2}{\delta_0} + 1$ is the largest solution of

$$\beta(\beta-1) = \frac{2\sigma^2(r-k)}{\delta_0^2} = \frac{\sigma^2}{\delta_0}\left(\frac{\sigma^2}{\delta_0} + 1\right).$$

In terms of ϕ the Laplace transform of $\lambda_{c_1(z)}$ reads (recall that $y > c_1(z)$)

$$\mathbb{E}_{y}\left[e^{(-r+k)\lambda_{c_1(z)}} \mathbb{1}_{\{\lambda_{c_1(z)}<\infty\}}\right] = \frac{y^{1-\beta}(1-y)^{\beta}}{c_1(z)^{1-\beta}(1-c_1(z))^{\beta}}.$$

In conclusion, for any $z \in \mathbb{R}$ and $y > c_1(z)$, taking $x = F(z,y)$ we have (recall (5.6))

(7.10)
$$\begin{aligned} v(z,y) &= V(x,y) \\ &\le e^z \left(\frac{1-c_1(z)}{c_1(z)}\right)^{\frac{\sigma^2}{\delta_0}} \frac{y^{1-\beta}(1-y)^{\beta}}{c_1(z)^{1-\beta}(1-c_1(z))^{\beta}} = \frac{(1-y)}{(1-c_1(z))} F(z,y). \end{aligned}$$

Now we fix $z \in \mathbb{R}$ and pick $a > 1$ such that $a\,c_1(z) < 1$. Since $v(z+kt, a\,c_1(z+kt)) \geq F(z+kt, a\,c_1(z+kt)) - K$ for all $t \geq 0$ we can use the latter and (7.10), replacing (z, y) therein by $\big(z+kt, a\,c_1(z+kt)\big)$, to estimate

$$-K \leq (v - F)(z+kt, a\,c_1(z+kt)) \leq -\frac{(a-1)c_1(z+kt)}{1 - c_1(z+kt)} F(z+kt, a\,c_1(z+kt)).$$

Simple algebra gives

$$\frac{1}{c_1(z+kt)} \leq c\, e^{\alpha t}$$

for some constant $c > 0$ depending on z, K and a, and with $\alpha = -k\delta_0/(\sigma^2 - \delta_0)$. Now Assumption 7.4 implies that $\frac{-k\delta_0}{\sigma^2 - \delta_0} < \frac{\delta_0}{\sigma^2}(r-k)$ as required in (7.8). □

Proposition 7.6. *Under Assumption 7.4 the pair (τ_*, γ_*) is a saddle point.*

Proof. As in Proposition 7.3, we only have to prove the optimality of τ_* and we argue in a similar way. Let $z \in \mathbb{R}$ be fixed and set $x = F(z, y)$, then as in the proof of Proposition 7.3 we find

$$\begin{aligned} V(x,y) &\leq \mathbb{E}_{x,y}\left[e^{-r(t\wedge\tau_*\wedge\gamma)} V(X_{t\wedge\tau_*\wedge\gamma}, Y_{t\wedge\tau_*\wedge\gamma})\right] \\ &= \mathbb{E}_{x,y}\left[e^{-r\tau_*} G_1(X_{\tau_*}) \mathbb{1}_{\{\tau_* \leq t\wedge\gamma\}} + e^{-r\gamma} G_2(X_\gamma, Y_\gamma) \mathbb{1}_{\{\gamma < t\wedge\tau_*\}}\right] \\ &\quad + \mathbb{E}_{x,y}\left[e^{-rt} V(X_t, Y_t) \mathbb{1}_{\{t \leq \tau_*\wedge\gamma\}}\right] \end{aligned} \tag{7.11}$$

for any stopping time γ and any t. Under $\mathbb{P}_{x,y}$ we have $X_t = F(z+kt, Y_t)$ and for $t < \tau_*$ we have $Y_t \geq c_1(z+kt)$, which implies $X_t \leq F(z+kt, c_1(z+kt))$. The latter gives

$$\mathbb{E}_{x,y}\left[e^{-rt} V(X_t, Y_t) \mathbb{1}_{\{t \leq \tau_*\wedge\gamma\}}\right] \leq e^{-rt} F(z+kt, c_1(z+kt)),$$

which goes to zero according to Lemma 7.5. Then taking limits as $t \to \infty$ in (7.11) and using also monotone convergence we conclude the proof. □

Appendix A. Proof of Theorem 3.2

The main idea of the proof is to approximate our game by a sequence of games with bounded stopping payoffs indexed by $n \in \mathbb{N}$. For each approximating problem we can apply the results of [10] regarding existence of the value and of a saddlepoint. Eventually we pass to the limit as $n \to \infty$ to obtain the existence of the value for the game with unbounded payoffs.

For $n \geq 1$ let us define the functions $G_i^{(n)}(x) = G_i(x \wedge n)$, $i = 1, 2$. Next for $\tau, \gamma \in \mathcal{T}$ let us introduce the the associated payoff

$$M_{x,y}^{(n)}(\tau, \gamma) = \mathbb{E}_{x,y}[e^{-r\tau} G_1^{(n)}(X_\tau) \mathbb{1}_{\{\tau \leq \gamma\}} + e^{-r\gamma} G_2^{(n)}(X_\gamma) \mathbb{1}_{\{\gamma < \tau\}}]. \tag{A.1}$$

According to Theorem 2.1. in [10], the game with payoff (A.1) has a value, i.e.

$$V^{(n)}(x,y) = \sup_\tau \inf_\gamma M_{x,y}^{(n)}(\tau, \gamma) = \inf_\gamma \sup_\tau M_{x,y}^{(n)}(\tau, \gamma).$$

Moreover, the stopping times

$$\tau_n = \inf\{t \geq 0 \,|\, V^{(n)}(X_t, Y_t) = G_1^{(n)}(X_t)\},$$

$$\gamma_n = \inf\{t \geq 0 \,|\, V^{(n)}(X_t, Y_t) = G_2^{(n)}(X_t)\},$$

form a Nash equilibrium. Since

$$G_1^{(n)}(x) \leq V^{(n)}(x,y) \leq \sup_\tau M_{x,y}^{(n)}(\tau, +\infty) \leq (n-K)^+$$

and $G_1^{(n)}(x) = (n-K)^+$ for $x \geq n$, then $V^{(n)}(x,y) = G_1^{(n)}(x)$ for $(x,y) \in [n,+\infty)\times[0,1]$. The latter implies that

$$\{(x,y) \mid V^{(n)}(x,y) = G_2^{(n)}(x,y)\} \subset [0,n] \times [0,1] = \{(x,y) \mid G_2(x,y) = G_2^{(n)}(x,y)\}$$

and therefore

$$\gamma_n = \inf\{t \geq 0 \,|\, V^{(n)}(X_t,Y_t) = G_2(X_t)\}.$$

Concerning the value of the approximating game, it is easy to check that Lemma 3.1 holds for $V^{(n)}$ with the same proof. Moreover the sequence $M^{(n)}_{x,y}(\tau,\gamma)$ is non-decreasing in n and it is bounded from above by $M_{x,y}(\tau,\gamma)$. Hence the sequence $(V^{(n)})_{n\geq 1}$ is non-decreasing in n with

$$V^{(n)}(x,y) \leq \underline{V}(x,y) \leq \overline{V}(x,y) \quad \text{and} \quad V^{\infty}(x,y) := \lim_{n\to\infty} V^{(n)}(x,y) \tag{A.2}$$

for all $(x,y) \in \mathbb{R}_+ \times [0,1]$. In particular $V^{(n)} \leq \overline{V}$ implies

$$\gamma_n \geq \gamma_* := \inf\{t \geq 0 \,|\, \overline{V}(X_t,Y_t) = G_2(X_t)\} \quad \text{for all } n \geq 1.$$

Since $V^{(n)}$ is non-decreasing in n then γ_n is non-increasing and we set $\gamma_\infty := \lim_{n\to\infty}\gamma_n$.

Now we aim at showing that $V^\infty \geq \overline{V}$ so that (A.2) implies $\underline{V} = \overline{V}$ and therefore the value exists and it coincides with V^∞. For all $\tau \in \mathcal{T}$, we have

$$\begin{aligned} M^{(n)}_{x,y}(\tau,\gamma_n) =& \mathbb{E}_{x,y}\left[e^{-r\tau}G_1^{(n)}(X_\tau)\mathbb{1}_{\{\tau\leq\gamma_n\}} + e^{-r\gamma_n}G_2^{(n)}(X_{\gamma_n})\mathbb{1}_{\{\gamma_n<\tau\}}\right] \\ =& \mathbb{E}_{x,y}\left[e^{-r\tau}G_1(X_\tau)\mathbb{1}_{\{\tau\leq\gamma_n\}} + e^{-r\gamma_n}G_2(X_{\gamma_n})\mathbb{1}_{\{\gamma_n<\tau\}}\right] \\ & - \mathbb{E}_{x,y}\left[e^{-r\tau}(X_\tau - n)\mathbb{1}_{\{X_\tau\geq n\}}\mathbb{1}_{\{\tau\leq\gamma_n\}}\right]. \end{aligned}$$

Observe that

$$0 \leq \mathbb{E}_{x,y}[e^{-r\tau}(X_\tau - n)\mathbb{1}_{\{X_\tau\geq n\}}\mathbb{1}_{\{\tau\leq\gamma_n\}}] \leq \mathbb{E}_{x,y}[e^{-r\tau}X_\tau\mathbb{1}_{\{X_\tau\geq n\}}] \tag{A.3}$$

and recall that $\mathbb{E}_{x,y}[e^{-r\tau}X_\tau] \leq x$ by (2.5). Using dominated convergence in (A.3) we obtain

$$\lim_{n\to\infty}\mathbb{E}_{x,y}[e^{-r\tau}(X_\tau - n)\mathbb{1}_{\{X_\tau\geq n\}}\mathbb{1}_{\{\tau\leq\gamma_n\}}] = 0.$$

On the other hand, Fatou's Lemma implies

$$\begin{aligned} &\liminf_{n\to\infty} \mathbb{E}_{x,y}\left[e^{-r\tau}G_1(X_\tau)\mathbb{1}_{\{\tau\leq\gamma_n\}} + e^{-r\gamma_n}G_2(X_{\gamma_n})\mathbb{1}_{\{\gamma_n<\tau\}}\right] \\ &\quad \geq \mathbb{E}_{x,y}\left[e^{-r\tau}G_1(X_\tau)\mathbb{1}_{\{\tau\leq\gamma_\infty\}} + e^{-r\gamma_\infty}G_2(X_{\gamma_\infty})\mathbb{1}_{\{\gamma_\infty<\tau\}}\right] = M_{x,y}(\tau,\gamma_\infty). \end{aligned}$$

Collecting the above limits we deduce that

$$\liminf_{n\to\infty} M^{(n)}_{x,y}(\tau,\gamma_n) \geq M_{x,y}(\tau,\gamma_\infty). \tag{A.4}$$

Now, for $\varepsilon > 0$, let τ_ε be such that

$$M_{x,y}(\tau_\varepsilon,\gamma_\infty) \geq \sup_\tau M_{x,y}(\tau,\gamma_\infty) - \varepsilon.$$

Using optimality of γ_n in the approximating problem, and (A.4) we obtain

$$\begin{aligned} V^\infty(x,y) = \lim_{n\to\infty}V^{(n)}(x,y) = \lim_{n\to\infty}\sup_\tau M^{(n)}_{x,y}(\tau,\gamma_n) \geq \liminf_{n\to\infty} M^{(n)}_{x,y}(\tau_\varepsilon,\gamma_n) \\ \geq M_{x,y}(\tau_\varepsilon,\gamma_\infty) \geq \sup_\tau M_{x,y}(\tau,\gamma_\infty) - \varepsilon \geq \overline{V}(x,y) - \varepsilon. \end{aligned}$$

Finally, letting $\varepsilon \to 0$ and recalling (A.2), we obtain

$$\underline{V}(x,y) \geq V^\infty(x,y) \geq \overline{V}(x,y),$$

and hence the existence of the value $V := V^\infty$. As a byproduct we also obtain that γ_∞ is optimal for player 2, that is

$$V(x,y) = \sup_\tau M_{x,y}(\tau, \gamma_\infty).$$

Next we want to prove optimality of γ_* and super/sub-martingale properties of V. For all n and any $\tau \in \mathcal{T}$ we have (see [10, Thm. 2.1.])

$$\begin{aligned} V^{(n)}(x,y) \geq& \mathbb{E}_{x,y}[e^{-r(\tau\wedge\gamma_n)} V^{(n)}(X_{\tau\wedge\gamma_n}, Y_{\tau\wedge\gamma_n})] \\ =& \mathbb{E}_{x,y}[e^{-r\tau} V^{(n)}(X_\tau, Y_\tau) \mathbb{1}_{\{\tau\leq\gamma_n\}}] + \mathbb{E}_{x,y}[e^{-r\gamma_n} G_2(X_{\gamma_n}) \mathbb{1}_{\{\tau>\gamma_n\}}] \\ \geq& \mathbb{E}_{x,y}[e^{-r\tau} V^{(n)}(X_\tau, Y_\tau) \mathbb{1}_{\{\tau\leq\gamma_n\}}] + \mathbb{E}_{x,y}[e^{-r\gamma_n} V(X_{\gamma_n}, Y_{\gamma_n}) \mathbb{1}_{\{\tau>\gamma_n\}}] \\ =& \mathbb{E}_{x,y}[e^{-r(\tau\wedge\gamma_n)} V(X_{\tau\wedge\gamma_n}, Y_{\tau\wedge\gamma_n})] \\ &+ \mathbb{E}_{x,y}[e^{-r\tau}(V^{(n)}(X_\tau, Y_\tau) - V(X_\tau, Y_\tau)) \mathbb{1}_{\{\tau\leq\gamma_n\}}] \end{aligned}$$

where in the second inequality we used that $G_2 \geq V$. Now we take limits as $n \to \infty$. Recalling that $V^{(n)} \leq V$, that $0 \leq V(x,y) \leq x + \varepsilon_0$ and that $e^{-r\tau} X_\tau$ is integrable, the second term in the last expression above converges to zero by dominated convergence. Moreover, Fatou's Lemma yields,

$$V(x,y) \geq \mathbb{E}_{x,y}[e^{-r(\tau\wedge\gamma_\infty)} V(X_{\tau\wedge\gamma_\infty}, Y_{\tau\wedge\gamma_\infty})]. \tag{A.5}$$

Since $\tau \in \mathcal{T}$ was arbitrary the process $e^{-r(t\wedge\gamma_\infty)} V(X_{t\wedge\gamma_\infty}, Y_{t\wedge\gamma_\infty})$, $t \geq 0$ is a super-martingale. Noticing that $\gamma_\infty \geq \gamma_*$ and choosing $\tau = \rho \wedge \gamma_*$ in (A.5) for some $\rho \in \mathcal{T}$, we see that also the process $e^{-r(t\wedge\gamma_*)} V(X_{t\wedge\gamma_*}, Y_{t\wedge\gamma_*})$, $t \geq 0$ is a super-martingale as claimed. As it is a non-negative super-martingale, Fatou's lemma gives

$$V(x,y) \geq \liminf_{t\to\infty} \mathbb{E}_{x,y}[e^{-r(t\wedge\gamma_*)} V(X_{t\wedge\gamma_*}, Y_{t\wedge\gamma_*})] \geq \mathbb{E}_{x,y}[e^{-r\gamma_*} V(X_{\gamma_*}, Y_{\gamma_*})]$$

hence the super-martingale is closed.

Finally, we prove that γ_* is optimal for the seller, i.e. player 2. We have

$$\begin{aligned} V(x,y) &\geq \mathbb{E}_{x,y}[e^{-r(\tau\wedge\gamma_*)} V(X_{\tau\wedge\gamma_*}, Y_{\tau\wedge\gamma_*})] \\ &= \mathbb{E}_{x,y}[e^{-r\tau} V(X_\tau, Y_\tau) \mathbb{1}_{\{\tau\leq\gamma_*\}}] + \mathbb{E}_{x,y}[e^{-r\gamma_*} G_2(X_{\gamma_*}) \mathbb{1}_{\{\tau>\gamma_*\}}] \\ &\geq \mathbb{E}_{x,y}[e^{-r\tau} G_1(X_\tau) \mathbb{1}_{\{\tau\leq\gamma_*\}}] + \mathbb{E}_{x,y}[e^{-r\gamma_*} G_2(X_{\gamma_*}) \mathbb{1}_{\{\tau>\gamma_*\}}] \\ &= M_{x,y}(\tau, \gamma_*) \end{aligned}$$

Taking the supremum over τ gives the optimality of strategy $\gamma*$ for player 2.

It remains to prove the sub-martingale property. Let us denote

$$\mathcal{S}_1^{(n)} = \big\{(x,y) \in \mathbb{R}_+ \times [0,1] \mid V^{(n)}(x,y) = G_1^{(n)}(x)\big\}, \tag{A.6}$$

the stopping region of player 1. Notice that an analogous set can be defined relatively to V and G_1 (see (4.2)). In Section 4 properties of $\mathcal{S}_1$ are proven in Lemma 4.2 by using continuity and monotonicity of V. The same methodology can be applied to $V^{(n)}$ to prove analogous properties for $\mathcal{S}_1^{(n)}$. To be precise it is worth noticing that (4.8) holds for $V^{(n)}$ provided that $x \leq x' \leq n$ therein. The rest of (iii) in Lemma 4.2 follows by recalling that $[n, +\infty) \times [0,1] \subseteq \mathcal{S}_1$. The analogy holds with Corollary 4.5 as well. In particular there exists a non-increasing lower-semi-continuous map $b_1^{(n)} : [0,1] \to \mathbb{R}_+$ such that for $x \geq K$ it holds $(x,y) \in \mathcal{S}_1^{(n)} \Leftrightarrow x \geq b_1^{(n)}(y)$.

Observe that if $(x,y) \in \mathcal{S}_1^{(n+1)}$ is such that $x < n$, we have

$$V^{(n)}(x,y) \leq V^{(n+1)}(x,y) = G_1^{(n+1)}(x) = G_1(x) = G_1^{(n)}(x),$$

which implies $(x,y) \in \mathcal{S}_1^{(n)}$. Together with the fact that $[n,\infty)\times[0,1] \subset \mathcal{S}_1^{(n)}$, this implies that $\mathcal{S}_1^{(n+1)} \subset \mathcal{S}_1^{(n)}$. By the same arguments, we prove that $\mathcal{S}_1^{(n)} \subset \mathcal{S}_1$. We deduce that the sequence $b_1^{(n)}$ is non-decreasing and that τ_n is a non-decreasing sequence of stopping times such that $\tau_\infty := \lim_n \tau_n \le \tau_*$. Moreover, if $(x,y) \in \mathbb{R}_+ \times [0,1)$ is such that $x < b_1(y)$, then $V(x,y) > G_1(x)$ and, for sufficiently large n, we have $V^{(n)}(x,y) > G_1(x) = G_1^{(n)}(x)$, implying that $x < b_1^{(n)}(y)$. We deduce that $b_1^{(n)}$ converges to b_1 on $[0,1)$ pointwise.

Now, we prove that $\tau_\infty = \tau_*$. Since $\tau_\infty \le \tau_*$, it is sufficient to show that the equality holds $\mathbb{P}_{x,y}$-almost surely on $\{\tau_\infty < \infty\}$. For $(x,y) \in \mathcal{S}_1$ the claim is trivial. Fix $(x,y) \notin \mathcal{S}_1$ and $\omega \in \{\tau_\infty < \infty\}$. Since the sequence $(b_1^{(n)})_{n\in\mathbb{N}}$ is increasing then for fixed $m \in \mathbb{N}$ and any $n \ge m$ we have $b_1^{(n)}(Y_{\tau_n}(\omega)) \ge b_1^{(m)}(Y_{\tau_n}(\omega))$. The latter implies

$$\liminf_{n\to\infty} b_1^{(n)}(Y_{\tau_n}(\omega)) \ge b_1^{(m)}(Y_{\tau_\infty}(\omega))$$

by using that $Y_{\tau_n}(\omega) \to Y_{\tau_\infty}(\omega)$ as well. Taking the supremum over m in the right-hand side of the above expression and recalling that $b_1^{(m)} \uparrow b_1$ pointwise we conclude

$$\liminf_{n\to\infty} b_1^{(n)}(Y_{\tau_n}(\omega)) \ge b_1(Y_{\tau_\infty}(\omega)). \tag{A.7}$$

Since $X_{\tau_n} \ge b_1(X_{\tau_n})$, $\mathbb{P}_{x,y}$-a.s. for all $n \in \mathbb{N}$, using continuity of paths and (A.7), we also find

$$X_{\tau_\infty} = \lim_{n\to\infty} X_{\tau_n} \ge \liminf_{n\to\infty} b_1^{(n)}(Y_{\tau_n}) \ge b_1(Y_{\tau_\infty}) \quad \mathbb{P}_{x,y}\text{-a.s.}$$

which implies $\tau_\infty \ge \tau_*$, $\mathbb{P}_{x,y}$-a.s. as requested.

Finally we notice that the process $e^{-r(t\wedge\tau_n)}V^{(n)}(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n})$ is a sub-martingale for all n. Since $\tau_n \le \tau_{n+p}$ for all $n,p \ge 0$, we deduce that

$$V^{(n+p)}(x,y) \le \mathbb{E}_{x,y}[e^{-r(t\wedge\tau_n)}V^{(n+p)}(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n})].$$

Letting $p \to \infty$, monotone convergence implies that

$$V(x,y) \le \mathbb{E}_{x,y}[e^{-r(t\wedge\tau_n)}V(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n})]$$

for all n. Taking $n \to \infty$ and recalling that $e^{-r(t\wedge\tau_n)}V(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}) \le \sup_{s\in[0,t]} e^{-rs}X_s \in L^1(\mathbb{P}_{x,y})$, bounded convergence implies

$$V(x,y) \le \mathbb{E}_{x,y}[e^{-r(t\wedge\tau^*)}V(X_{t\wedge\tau^*}, Y_{t\wedge\tau^*})].$$

The above result and the Markov property imply that $e^{-r(t\wedge\tau^*)}V(X_{t\wedge\tau^*}, Y_{t\wedge\tau^*})$ is a sub-martingale as claimed.

Appendix B. Proof of Lemma 6.2

The proof is more easily carried out considering the boundaries c_1 and c_2 rather than b_1 and b_2. However we incur no loss of generality thanks to the equivalence of the problem formulation with respect to the coordinates (x,y) and (z,y). We provide a full argument for $k > 0$ but a completely symmetric proof holds for $k < 0$.

Since c_1 is increasing and Y is non-degenerate at all points of $(0,1)$, the law of iterated logarithm implies that $\hat{\tau}_* = \check{\tau}$, $\mathbb{P}$-a.s. Similarly if (Z,Y) hits the line $y_K(\cdot)$ from above then it will immediately cross it downwards.

For the same result relative to the boundary c_2 we repeat the steps in [4, Cor. 8]. In particular let us introduce some notation

$$\hat{\gamma}_\varepsilon := \inf\{t > 0 \,|\, Y_t \geq c_2(Z_t) + \varepsilon\}, \quad \hat{\gamma}^\delta_\varepsilon := \inf\{t > \delta \,|\, Y_t \geq c_2(Z_t) + \varepsilon\} \tag{B.1}$$

$$\check{\gamma}_\varepsilon := \inf\{t > 0 \,|\, Y_t > c_2(Z_t) + \varepsilon\}, \quad \check{\gamma}^\delta_\varepsilon := \inf\{t > \delta \,|\, Y_t > c_2(Z_t) + \varepsilon\} \tag{B.2}$$

so that $\hat{\gamma}_* = \hat{\gamma}_0$ and $\check{\gamma} = \check{\gamma}_0$. We have $\hat{\gamma}_+ := \lim_{\varepsilon\to 0}\hat{\gamma}_\varepsilon = \check{\gamma}$ and

$$\hat{\gamma}^\delta_0 \leq \hat{\gamma}^\delta_+ := \lim_{\varepsilon\to 0}\hat{\gamma}^\delta_\varepsilon = \check{\gamma}^\delta_0.$$

Assume that for any $(z, y) \in R_K$ we have

$$\mathbb{P}_{z,y}(\check{\gamma}^\delta_0 > t) \leq \mathbb{P}_{z,y}(\hat{\gamma}^\delta_0 > t) \tag{B.3}$$

so that $\check{\gamma}^\delta_0 = \hat{\gamma}^\delta_0$, $\mathbb{P}_{z,y}$-a.s. Then

$$\check{\gamma} = \lim_{\varepsilon\to 0}\hat{\gamma}_\varepsilon = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\hat{\gamma}^\delta_\varepsilon = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\hat{\gamma}^\delta_\varepsilon = \lim_{\delta\to 0}\check{\gamma}^\delta_0 = \lim_{\delta\to 0}\hat{\gamma}^\delta_0 = \hat{\gamma}_0 = \hat{\gamma}_*$$

where the last limit is easily verified by definition of $\hat{\gamma}^\delta_0$ and we could swap the limits because $\hat{\gamma}^\delta_\varepsilon$ is increasing in both δ and ε.

Now it remains to verify (B.3). We start by noticing that any interval of the form (δ, t) may be decomposed into the union of countably many intervals over which c_2 is either strictly increasing or flat. Consider the latter, i.e. let $\mathcal{I} \subset \mathbb{R}$ be an interval such that $c_2(\zeta) = y_0$ for $\zeta \in \mathcal{I}$ and a fixed $y_0 \in (0, 1)$. Fix also $(z, y) \in R_K$, then it is immediate to check that on the event $\{\hat{\gamma}_* \in \mathcal{I}\}$ one has $\hat{\gamma}_* = \check{\gamma}$, $\mathbb{P}_{z,y}$-a.s., because Y immediately crosses y_0 after reaching it. This in particular implies that

$$\mathbb{P}_{z,y}\left(Y_s \leq c_2(Z_s),\, \forall s \in \mathcal{I}\right) = \mathbb{P}_{z,y}\left(Y_s < c_2(Z_s),\, \forall s \in \mathcal{I}\right). \tag{B.4}$$

Next we fix $h_0 \in (0, \delta/2)$ so that for $h \in (0, h_0)$ we have $c_2(Z_s) \leq c_2(Z_{s+h})$, $\mathbb{P}_{z,y}$-a.s., because c_2 and Z are increasing. Moreover *the inequality is strict* whenever c_2 is strictly increasing. Hence, the latter consideration and (B.4) imply

$$\begin{aligned}
\mathbb{P}_{z,y}(\check{\gamma}^\delta_0 > t) =& \mathbb{P}_{z,y}\left(Y_s \leq c_2(Z_s),\, \forall s \in (\delta, t]\right)\\
\leq& \mathbb{P}_{z,y}\left(Y_s < c_2(Z_{s+h}),\, \forall s \in (\delta, t]\right)\\
=& \mathbb{P}_{z,y}\left(Y_{r-h} < c_2(Z_r),\, \forall r \in (\delta + h, t + h]\right)\\
\leq& \mathbb{P}_{z,y}\left(Y_{r-h} < c_2(Z_r),\, \forall r \in (\delta + h_0, t]\right)\\
\leq& \mathbb{P}_{y}\left(Y_{r-h} < c_2(z + kr),\, \forall r \in (\delta + h_0, t]\right),
\end{aligned}$$

where in the last expression we have expressed Z explicitly so that it can be treated effectively as a 'time' variable.

We now denote by p_Y and m_Y the probability transition density and the speed measure of Y, respectively. Then by using the Markov property of Y we obtain

$$\begin{aligned}
\mathbb{P}_{z,y}(\check{\gamma}^\delta_0 > t) \leq& \mathbb{P}_{y}\left(Y_{r-h} < c_2(z + kr),\, \forall r \in (\delta + h_0, t]\right)\\
=& \mathbb{E}_y\left[\mathbb{P}_{Y_{\delta/2-h}}\left(Y_{r-\delta/2} < c_2(z + kr),\, \forall r \in (\delta + h_0, t]\right)\right]\\
=& \int_0^1 p_Y(\delta/2 - h, y, \xi)\mathbb{P}_\xi\left(Y_{r-\delta/2} < c_2(z + kr),\, \forall r \in (\delta + h_0, t]\right) m_Y(d\xi).
\end{aligned}$$

Scheffé's theorem (see page 224 in [3]) guarantees that

$$\lim_{h\to 0}\int_0^1 |p_Y(\delta/2 - h, y, \xi) - p_Y(\delta/2, y, \xi)|m_Y(d\xi) = 0$$

thus implying that taking limits as $h \to 0$ we obtain

$$\begin{aligned}\mathbb{P}_{z,y}(\check{\gamma}_0^\delta > t) \le & \int_0^1 p_Y(\delta/2, y, \xi)\mathbb{P}_\xi\left(Y_{r-\delta/2} < c_2(z+kr),\, \forall r \in (\delta+h_0, t]\right) m_Y(d\xi)\\ =&\mathbb{P}_y\left(Y_r < c_2(z+kr),\, \forall r \in (\delta+h_0, t]\right) = \mathbb{P}_{z,y}(\hat{\gamma}_0^{\delta+h_0} > t).\end{aligned}$$

Letting now $h_0 \to 0$ we find (B.3) as claimed, because it is easy to verify that $\hat{\gamma}_0^{\delta+h_0} \downarrow \hat{\gamma}_0^\delta$.

Appendix C. Game with complete information: summary of results

In this appendix we provide a short summary of existing results concerning the stopping regions in the game call option problem with perfect information, i.e. when y is either 0 or 1. The material below is based on results contained in [28], for $y = 1$, and [11], for $y = 0$.

We recall $M_{x,y}(\tau, \gamma)$ as in (1.3) and emphasise that here $y = \{0, 1\}$. Denote by V_∞ the value of the optimal stopping problem for the buyer when there is no possible seller's cancellation (i.e. when $\gamma = +\infty$):

$$V_\infty(x, y) := \sup_\tau M_{x,y}(\tau, +\infty)$$

and by V_K the value of the problem when $\gamma = \gamma_K$, i.e. the hitting time of $\{K\}$ by X:

$$V_K(x, y) := \sup_\tau M_{x,y}(\tau, \gamma_K).$$

We also define the critical dividend levels $\delta_1 < \delta_2$ by

$$\delta_1 := \inf\{\delta > 0 \,|\, \lim_{x\downarrow K} \frac{V_K(x,1) - \varepsilon}{x - K} \le 1\} \quad \text{and} \quad \delta_2 := \inf\{\delta > 0 \,|\, V_\infty(K, 1) < \varepsilon\}.$$

For the process X we recall that the fundamental solutions of

$$\tfrac{\sigma^2}{2} x^2 f''(x) + (r - \delta) x f'(x) - r f(x) = 0, \quad x > 0$$

are $\psi(x) = x^{\lambda_1}$ and $\phi(x) = x^{\lambda_2}$, with ψ increasing (notice that $\lambda_1 > 1$) and ϕ decreasing, and where $\lambda_2 < \lambda_1$ solve

$$\tfrac{\sigma^2}{2}\lambda^2 + (r - \delta_0 - \frac{\sigma^2}{2})\lambda - r = 0.$$

The next Proposition summarises results of [28] and [11].

Proposition C.1. *The following four cases hold*

- ***Case 1:*** *If $\varepsilon_0 \ge K$ we have*
  - $\mathcal{S}_2 \cap \{y = 0\} = \mathcal{S}_2 \cap \{y = 1\} = \emptyset$,
  - $\mathcal{S}_1 \cap \{y = 0\} = \emptyset$ *and* $\mathcal{S}_1 \cap \{y = 1\} = [\frac{\lambda_1}{\lambda_1 - 1}K, +\infty)$.
- ***Case 2:*** *If $\varepsilon_0 < K$ and $\delta_0 \ge \delta_2$ we have*
  - $\mathcal{S}_2 \cap \{y = 0\} = [K, +\infty)$ *and* $\mathcal{S}_2 \cap \{y = 1\} = \emptyset$.
  - $\mathcal{S}_1 \cap \{y = 0\} = \emptyset$ *and* $\mathcal{S}_1 \cap \{y = 1\} = [\frac{\lambda_1}{\lambda_1 - 1}K, +\infty)$.
- ***Case 3:*** *If $\varepsilon_0 < K$ and $\delta_1 \le \delta_0 < \delta_2$ we have*
  - $\mathcal{S}_2 \cap \{y = 0\} = [K, +\infty)$ *and* $\mathcal{S}_2 \cap \{y = 1\} = \{K\}$.
  - $\mathcal{S}_1 \cap \{y = 0\} = \emptyset$ *and* $\mathcal{S}_1 \cap \{y = 1\}| = [\alpha_0, +\infty)$, *where α_0 is the unique solution of*

$$\left(\frac{\alpha_0 - K}{\alpha_0}\lambda_1 - 1\right)\alpha_0^{\lambda_1 - \lambda_2} - \left(\frac{\alpha_0 - K}{\alpha_0}\lambda_2 - 1\right)K^{\lambda_1 - \lambda_2} = \frac{\varepsilon_0}{K}(\lambda_1 - \lambda_2)\alpha_0^{\lambda_1 - 1}K^{1 - \lambda_2}.$$

- ***Case 4:*** *If $\varepsilon_0 < K$ and $0 < \delta_0 < \delta_1$ we have*
  - $\mathcal{S}_2 \cap \{y = 0\} = [K, +\infty)$ *and* $\mathcal{S}_2 \cap \{y = 1\} = [K, \beta_1]$

– $\mathcal{S}_1 \cap \{y=0\} = \emptyset$ *and* $\mathcal{S}_1 \cap \{y=1\} = [\alpha_1, +\infty)$ *where* (α_1, β_1) *is the unique solution of the system of equations*

$$\begin{cases} \left(\frac{\alpha_1 - K}{\alpha_1}\lambda_1 - 1\right)\alpha_1^{\lambda_1-\lambda_2} - \left(\frac{\alpha_1 - K}{\alpha_1}\lambda_2 - 1\right)\beta_1^{\lambda_1-\lambda_2} = \frac{\beta_1 - K + \varepsilon_0}{\beta_1}(\lambda_1-\lambda_2)\alpha_1^{\lambda_1-1}\beta_1^{1-\lambda_2} \\ \left(\frac{\beta_1 - K+\varepsilon_0}{\beta_1}\lambda_1 - 1\right)\beta_1^{\lambda_1-\lambda_2} - \left(\frac{\beta_1 - K+\varepsilon_0}{\beta_1}\lambda_2 - 1\right)\alpha_1^{\lambda_1-\lambda_2} = \frac{\alpha_1 - K}{\alpha_1}(\lambda_1-\lambda_2)\beta_1^{\lambda_1-1}\alpha_1^{1-\lambda_2} \end{cases}$$

References

[1] Bensoussan, A., and Friedman, A. , *Nonlinear variational inequalities and differential games with stopping times*. J. Funct. Anal. 16, 305-352, 1974.
[2] Bismut, J.-M. and Skalli, B., *Temps d'arrêt optimal, théorie générale des processus et processus de Markov*. Probability Theory and Related Fields, **39**, pp. 301–313, 1977.
[3] Billingsley, P., *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968
[4] Cox, A.M.G. and Peskir, G., *Embedding laws in diffusions by functions of time*. Ann. Probab. 43(5), 2481-2510, 2015.
[5] Dayanik, S., and Karatzas, I. *On the optimal stopping problem for one-dimensional diffusions.* Stochastic Process. Appl., 107(2), 173-212, 2003.
[6] De Angelis, T. *A note on the continuity of free boundaries in finite-horizon optimal stopping problems for one-dimensional diffusions.* SIAM J. Control Optim., 53 (1), 167–184, 2015.
[7] De Angelis, T. and Ferrari, G. *A stochastic partially reversible investment problem on a finite time horizon: free boundary analysis*. Stochastic Process. Appl., 124, 4080-4119, 2014.
[8] Dynkin, E. B. , *Game variant of a problem of optimal stopping*. Soviet Math. Dokl. 10, 270-274, 1969.
[9] Dynkin, E. B. and Yushkevich, A. A., Markov Processes: Theorems and Problems. Plenum Press, New York, 1969.
[10] Ekström, E. and Peskir, G., *Optimal stopping games for Markov processes*, SIAM Journal on Control and Optimization, **47**, pp. 684–702, 2008.
[11] Ekström, E. and Villeneuve, S., *On the value of optimal stopping games*, The Annals of Applied Probability, **16**, pp. 1576–1596, 2006.
[12] El Karoui, N., *Les aspects probabilistes du contrôle stochastique*, Ecole d?Eté de Probabilités de Saint-Flour IX-1979, pp. 73–238, 1981, Springer.
[13] Friedman, A. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, New Jersey, 1964.
[14] J. Jacod and A.N. Shiryaev, *Limit theorems for stochastic processes, Second Edition*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 2003.
[15] Karatzas, I. and Shreve, S.E., *Brownian motion and stochastic calculus*, Springer-Verlag New York, 1988.
[16] Karatzas, I. and Shreve, S., *Methods of mathematical finance. Applications of Mathematics (New York), 39*. Springer-Verlag, New York, 1998.
[17] Kifer, Y. *Game options*. Finance Stoch. 4(4), 443-463, 2000.
[18] Krylov N.V. *Lectures on Elliptic and Parabolic Equations in Sobolev Spaces*, Graduate studies in Mathematics, AMS Providence, 2008.
[19] Kyprianou, A. E. *Some calculations for Israeli options*. Finance Stoch. 8, 73-86, 2004.
[20] Lepeltier, J.P. and Maingueneau, M.A., *Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski*, Stochastics, **13**, pp. 25–44, 1984.
[21] R.S. Lipster and A.N. Shiryaev, *Statistics of random Processes. I: General theory. Second edition.*, Springer-Verlag, 2001.
[22] Maingueneau, M.A., *Temps d'arrêt optimaux et théorie générale*, Séminaire de Probabilités XII, pp. 457–467, 1978.
[23] Mertens, J.-F., *Théorie des processus stochastiques généraux applications aux surmartingales*, Probability Theory and Related Fields, **22**, pp. 45–68, 1972.
[24] Peskir, G. and Shiryaev, A., *Optimal stopping and free-boundary problems*, 2006, Birkhäuser Basel.
[25] Peskir, G., *Optimal stopping games and Nash equilibrium*, Theory of Probability & Its Applications, **53**, pp. 558-571, 2009.
[26] Protter, P.E., *Stochastic integration and differential equations* - Second Edition, Springer-Verlag, Berlin Heidelberg, 2004.
[27] Shiryaev, A. *Optimal Stopping Rules*, Springer-Verlag, 1978.
[28] S.C.P. Yam, S.P. Yung, and W. Zhou, *Game call options revisited*, Mathematical Finance, 2012.

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