Valuation of natural capital under uncertain substitutability

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Abstract
Natural capital is complex to value notably because of the high uncertainties surrounding the substitutability of its future ecosystem services. We examine a Lucas economy in which a consumption good is produced by combining different inputs, one of them being an ecosystem service that is partially substitutable with other inputs. The growth rate of these inputs and the elasticity of substitution evolve in a stochastic way. We characterize the socially efficient ecological discount rates that should be used to value future ecosystem services at different time horizons. We show that the inverse of the elasticity of substitution can be interpreted as the CCAPM beta of natural capital. We also show that any increase in risk of this beta reduces the ecological discount rate. If our collective beliefs about the elasticity of substitution of ecosystem services are Gaussian, the ecological discount rates go to minus infinity for finite maturities. In that case, a marginal increase in natural capital has an infinite value. We provide a realistic calibration of the model that is coherent with observed asset prices by using the model of extreme events of Barro (2006). The bliss maturity for infinite discount factors is less than 100 years in this calibration.

Keywords: Relative price effect, CCAPM beta, ecological discounting, bioeconomics.

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1. Introduction

The substitutability of scarce environmental goods by manufactured goods is central to any cost-benefit analysis of environmental policies and to the notion of sustainability. In the late 17th century, the French administration expanded oak forests in the perspective of being able to build ships two centuries later to fight the British naval forces, long before realizing that oak would be substituted by steel. More recently, wars have been made to control oil fields before realizing that oil could well be substituted by non-conventional gas reserves and by renewable sources of energy in the near future. Optimistic futurists believe that the need for material goods and disappearing natural capital will be reduced or even eliminated by new technologies. Finally, the debate about the preservation of natural resources and biodiversity is made complex because of the uncertainty about whether these resources could be substituted by other non-natural inputs in the future.

Of course, our attitude towards the preservation of natural capital such as water, biodiversity, fossil fuels, climate or unspoiled natural sites is determined by the way we price it. But most natural assets generate ecological services that will persist for centuries, and deep uncertainties surround the valuation of these services by future generations. This implies that there is no consensus about how to value natural capital. For this reason, this notion remained up to now a metaphor rather than an instrument (Fenichel and Abbott (2014)). Guesnerie (2004), Hoel and Sterner (2007), Sterner and Persson, (2008), Gollier (2010) and Traeger (2011) have stressed the role of the evolution of relative prices in discounting, and thus of the degree of substitutability between different inputs. In a growing economy, the relative scarcity of the non-substitutable non-expendable natural capital will increase over time, thereby raising their relative value for future generations. This relative price effect can be stronger than the discounting effect, so that the same ecosystem service delivered latter can have a larger present monetary value. This is particularly plausible for non-substitutable resources. In this paper, we re-examine this question in a context where the degree of substitutability of natural resources is uncertain.

Suppose that the single consumption good can be produced by combining the constant service of some natural capital with another input, for example some physical capital. Because we take consumption as the numeraire, it can also be interpreted as income. Suppose that the instantaneous monetary value of the ecosystem service has a constant income-elasticity β. In our model, this property is derived from a CES production function with an elasticity of substitution $1/\beta$ between the two inputs, as shown by Ebert (2003). In other words, the elasticity of substitution of natural capital is also the inverse of the consumption-elasticity of the value of the ecosystem services that this natural capital generates. If the elasticity of substitution is small, the consumption-elasticity is large. In a growing economy, this implies a large relative price effect that dampens the effect of discounting. To illustrate, suppose that the physical capital will grow in the future so that consumption is expected to be multiplied by a factor 10 within the next century. If the income-elasticity of the value of ecosystem services is equal to unity, the instantaneous value of ecosystem services will also be multiplied by a factor 10 within the next century. Suppose alternatively that we are unsure about the degree of substitutability, so that the income-elasticity of the value of ecosystem services is either 0 (infinite substitutability) or 2 (weak substitutability) with equal probabilities. In that context, the value of ecosystem services in 100 years will be either stable, or it will be increased by a factor 100. In expectation, the value of ecological services will be increased by a factor 50. Discounted to the present, this shows that the uncertainty affecting the degree of substitution magnifies the value of the natural asset. In this example, the uncertainty surrounding the substitutability of a natural asset raises its social value by a factor 5. In short, the uncertainty surrounding the elasticity of substitution of ecosystem services magnifies the relative price effect in the valuation of natural capital.

However, this simple observation should be reconsidered once one recognizes that the growth rate of the consumption is also uncertain. As is well-known in the Consumption-based Capital Asset Pricing Model (CCAPM), the income-elasticity of the flow of dividends of an asset is also its CCAPM beta, which determines its risk-adjusted discount rate, or expected rate of return, for this asset. An asset with a large income-elasticity concentrates most of its risky benefits in the good states, and provides smaller benefits in the bad states. This justifies penalizing this asset by using a large rate to discount future expected benefits. Under the CCAPM, this risk-adjusted discount rate is linearly increasing in the CCAPM beta, i.e., in β. This means that the uncertainty affecting this substitutability parameter has a discounting effect on top of the relative price effect described above. The main result of this paper is that the relative price effect always dominates the discounting effect, so that any mean-preserving increase in the substitutability parameter β always reduces the ecological discount rate, i.e. the rate at which a sure increase in ecosystem service should be discounted. In other words, the uncertainty surrounding the substitutability of natural capital always increases its value. This is in line with the precautionary principle, a general rule favouring the preservation of natural assets in the face of uncertainty.

These new findings are related to some results in the finance literature. Pastor and Veronesi (2003, 2009) show that the uncertainty affecting the growth rate of dividends of an asset increases its market value. In our model, the uncertainty affecting substitutability translates into an uncertain growth rate of natural dividends. But this risk is correlated with the systematic risk, whereas in Pastor and Veronesi (2003, 2009), an idiosyncratic risk is assumed, so that they are not concerned by the risk-adjusted discounting effect. This paper is also related to the literature on the impact on asset pricing of learning. Collin-Dufresne, Johannes and Lochstoer (2016) examine the case of learning about the trend of economic growth or about the frequency of macroeconomic catastrophes. Jagannathan and Wang (1996), Lettau and Ludvigson (2001) and Adrian and Franzoni (2008) acknowledge that most assets' betas vary stochastically, and that this uncertainty affects asset pricing. This research is also linked to the recent developments aimed at valuing very distant cash flows (Martin (2012), Barro and Misra (2012)). Using our findings, the high price documented by Giglio, Maggiori, and Stroebel (2015) and Giglio, Maggiori, Rao, Stroebel and Weber (2018) for real estate claims maturing in 100 years and more in the United Kingdom and in Singapore could be due to the deep uncertainty affecting the correlation between aggregate consumption and the land rent in the distant future. Our model provides a new explanation for why assets generating benefits with an uncertain degree of substitutability with other goods and services are – or should be -- highly valued.

This paper is structured as follows. In Section 2, we present a simple asset pricing model using a CES production function with two inputs. We specify this two-input CCAPM model by assuming a bivariate geometric Brownian motion for the growth of consumption and ecosystem services with constant relative risk aversion. We characterize the impact of uncertainty for the substitutability parameter in Section 4. In Section 5, we provide an analytical solution for the ecological discount rate when the distribution of the substitutability parameter is Gaussian. Because the CCAPM specification implies the well-known asset pricing puzzles, this model would generate pricing rules for natural capital that are inconsistent with the price of assets on financial markets. In order to resolve this serious limitation of the model, we follow an approach proposed by Barro (2006, 2009).

In Section 6, we extend the CCAPM model by introducing the possibility of infrequent extreme macroeconomic catastrophes, and we obtain an analytical solution for the ecological discount rates that are coherent with market pricing. This model is calibrated to illustrate our results.

2. *The model*

Consider an economy with a representative agent consuming at discrete dates $t = 0,1,2,...$. At any date *t*, the agent consumes c_t units of the single consumption good, which is the numeraire. We consider the standard utilitarian social welfare function W with

$$W = \sum_{\tau=0} e^{-\delta\tau} Eu(c_\tau), \tag{1}$$

where δ is the rate of pure preference for the present. The expectation operator in this equation is relative to the information set available at date 0. We assume that $u(c) = c^{1-\gamma} / (1-\gamma)$ with a non-negative relative risk aversion γ, and that W exists and is finite.

We assume that the consumption good is produced by combining two inputs. The first input, available in quantity y_t at date *t*, is some aggregate economic capital (physical capital, labor, scientific knowledge,…). The second input, available in quantity x_t at date *t*, assembles various ecosystem services that are exogenously generated from natural capital. Following Guesnerie (2004), Hoel and Sterner (2007), Sterner and Persson, (2008), Gollier (2010), Traeger (2011) and Barro and Misra (2016), we assume a CES production function:

$$c_t = c(x_t, y_t, t) = \left[\alpha x_t^{1-\beta_t} + (1-\alpha) y_t^{1-\beta_t}\right]^{\frac{1}{1-\beta_t}}. \tag{2}$$

Parameter $\alpha \in [0,1]$ measures the weight of the services of the natural capital under scrutiny in the aggregate good consumed by the representative agent. Parameter $\beta_t \in \mathbb{R}$ is the inverse of the elasticity of substitution between the economic capital and the natural capital at date t in the production function. When β tends to unity, we get as a limit case the Cobb-Douglas specification with $c = x^\alpha y^{1-\alpha}$.

Let F_t denote the willingness-to-pay (or value) of one unit of ecosystem services at date *t*, i.e., the marginal rate of substitution between the ecosystem service and consumption at that date. We have that

$$F_t = \frac{\partial W / \partial x_t}{\partial W / \partial c_t} = \alpha \left(\frac{c_t}{x_t} \right)^{\beta_t} . \tag{3}$$

This equation shows that the income-elasticity of the willingness-to-pay for the ecosystem service is equal to β, the inverse of the elasticity of substitution. This has already been demonstrated by Ebert (3003). We contemplate an action today that will increase the ecosystem services x_t at some arbitrary date *t* by $\Delta = 1$. In order to implement a standard cost-benefit analysis of this action, we characterize P, which denotes the equivalent increase in present consumption that has the same welfare effect than the action itself. This value is given by the marginal rate of substitution between future ecological services and current consumption:

$$P = \frac{\partial W / \partial x_t}{\partial W / \partial c_0} = E\left[\frac{\partial W / \partial c_t}{\partial W / \partial c_0} \times \frac{\partial W / \partial x_t}{\partial W / \partial c_t} \right] = E\left[m_t F_t \right], \tag{4}$$

where

$$m_t = e^{-\delta t} \left(\frac{c_t}{c_0} \right)^{-\gamma} \tag{5}$$

is the standard pricing kernel of the CCAPM. Random variable m_t is the price at date 0 of the Arrow-Debreu security that delivers one unit of state-contingent consumption at date *t*. In general, it is statistically related to the future value F_t of ecological services.

There are two approaches to interpret the pricing equation (4). The more useful for our purpose is based on the method of ecological discounting proposed by Malinvaud (1953) and pioneered by Hoel and Sterner (2007). It consists in measuring the immediate increase in ecological services that has the same value as one more unit of ecological services at date *t*, and then to translate this ecological discount value in monetary terms using the exchange rate F_0:

$$P = F_0 \, e^{-\rho_t t}. \tag{6}$$

The present value *P* of future ecosystem services is equal to its current value F_0 multiplied by an ecological discount factor $\exp(-\rho_t t)$, which is the marginal rate of substitution between future and current ecosystem services. The ecological discount rate ρ_t is defined as follows:

$$\rho_t = -t^{-1} \ln\left(E\left[m_t \frac{F_t}{F_0} \right] \right). \tag{7}$$

The ecological discount rate ρ_t is useful to characterize how the valuation of natural capital is related to the uncertainties surrounding growth and the degree of substitutability of natural capital. A more traditional method is obtained by representing price *P* by discounting the expected future value of ecological services at a risk-adjusted monetary rate r_t:

$$P = e^{-r_t t} \, E\left[F_t\right] \tag{8}$$

where

$$r_t = -t^{-1} \ln \frac{E\left[m_t F_t\right]}{E\left[F_t\right]} \tag{9}$$

is the (risk-adjusted) monetary discount rate. Equation (9) is the standard pricing equation of the consumption based pricing model (Rubinstein (1976), Lucas (1978)). Of course, the monetary discount rate r_t and the ecological discount rate ρ_t are interrelated. In fact, we have that

$$\rho_t = r_t - f_t, \tag{10}$$

where f_t is the growth rate of the expected value of ecological services:

$$f_t = t^{-1} \ln \frac{EF_t}{F_0}. \tag{11}$$

Equation (10) is central to the literature on ecological discounting, as it emphasizes the role of the evolution of relative prices characterized by f_t. In particular, the ecological discount rate ρ_t will be negative if the relative price effect f_t is larger than the discounting effect r_t. The intensity of the price effect depends upon the elasticity of substitution. For example, in the case of perfect substitutability ($\beta = 0$), the value F_0 of ecological services is a constant α, so that f_t vanishes. Moreover, if we also assume that consumption grows at a sure rate g_c, the above equations immediately yield the well-known Ramsey rule $\rho = r = \delta + \gamma g_c$.

Without loss of generality, we hereafter assume that $x_0 = c_0 = 1$.

3. *Valuing natural capital when the elasticity of substitution is known*

The expectation operator appearing in the previous section is related to three sources of uncertainties. The growth of consumption and the evolution of the ecosystem services are both uncertain. We assume that (x_t, c_t) follows a discrete version of a bivariate geometric Brownian motion. Let $g_{xt} = \ln x_{t+1} / x_t$ and $g_{ct} = \ln c_{t+1} / c_t$ denote the per-period growth rate of respectively the ecosystem services and the numeraire good. We assume that the pair $(g_{xt}, g_{ct})\big|_{t=0,1,\ldots}$ follows a stationary random walk with (g_x, g_c) being normally distributed with mean (μ_x, μ_c), variance (σ_x^2, σ_c^2) and covariance $k\sigma_x\sigma_c$, where $k \in [-1,1]$ denotes the coefficient of correlation between the two growth rates. In this section, we assume that the elasticity of substitution is known, but we will relax this assumption later in this paper.

Let us define function $\chi(t,z) = \ln E\exp(tz)$, which is the Cumulant-Generating Function (CGF) of random variable z. Most results presented in this paper are derived from the following Lemma, which provides some well-known properties of CGF functions (see Billingsley (1995)).

Lemma 1 : *If it exists, the CGF function* $\chi(t,z) = \ln E\exp(tz)$ *has the following properties:*

i. $\chi(t,z) = \sum_{n=1}^{\infty} \kappa_n(z) t^n / n!$ *where* $\kappa_n(z)$ *is the nth cumulant of random variable z.*

ii. A well-known special case is when z is $N(\mu, \sigma^2)$*, so that* $\chi(t,z) = t\mu + 0.5t^2\sigma^2$.

iii. $t^{-1}\chi(t,z)$ *is increasing in t, from Ez to the supremum of the support of z when t goes from zero to infinity.*

If β_t takes value β with certainty, equation (7) implies that

$$\begin{aligned}\rho_t &= \delta - t^{-1}\ln E\left[\exp\left(-\beta \ln x_t + (\beta-\gamma)\ln c_t\right)\right] \\ &= \delta - t^{-1}\ln E\left[\exp\sum_{\tau=0}^{t-1}\left(-\beta g_{x\tau} + (\beta-\gamma) g_{c\tau}\right)\right] \\ &= \delta - t^{-1}\ln\left(E\exp\left(-\beta g_x + (\beta-\gamma)g_c\right)\right)^t \\ &= \delta - \ln\left(E\exp\left(-\beta g_x + (\beta-\gamma)g_c\right)\right) = \delta - \chi\left(1, -\beta g_x + (\beta-\gamma)g_c\right).\end{aligned} \tag{12}$$

The third equality comes from the fact that $(g_{xt}, g_{ct})\big|_{t=0,1,...}$ follows a stationary random walk. Now, observe that $z = -\beta g_x + (\beta-\gamma)g_c$ is normally distributed. Applying property *ii* of Lemma 1 allows us to rewrite the above equation as $\rho_t = \rho(\beta_t)$ with

$$\rho(\beta) = r_f - a\beta - 0.5b\beta^2, \tag{13}$$

with

$$r_f = \delta + \gamma\mu_c - 0.5\gamma^2\sigma_c^2, \tag{14}$$

$$a = \mu_c - \mu_x - \gamma\sigma_c\left(\sigma_c - k\sigma_x\right), \tag{15}$$

$$b = Var(g_c - g_x) = \sigma_c^2 + \sigma_x^2 - 2k\sigma_x\sigma_c \geq 0. \tag{16}$$

Equation (13) generalizes the characterization of the ecological discount rate obtained by Traeger (2011) when there is no uncertainty ($\sigma_c = \sigma_x = 0$). In that case, we obtain that

$$\rho(\beta) = \delta + \gamma\mu_c - \beta\left(\mu_c - \mu_x\right). \tag{17}$$

Traeger refers to term $\gamma\mu_c$ as the "overall growth effect". In a growing economy, investing for the future increases intergenerational inequalities, and is therefore socially undesirable. The discount rate (net of the rate of pure preference for the present) can then be interpreted as the minimum rate of return necessary to compensate this adverse effect of investing. Traeger also refer to term $\beta\left(\mu_c - \mu_x\right)$ as the "real substitutability effect", but is a relative price effect. When consumption grows faster than ecoservices, the willingness-to-pay for them increases with time at a rate equals to the product of the income-elasticity β by $\mu_c - \mu_x$. This reduces the ecological discount rate by that amount.

When the growth of consumption and ecoservices are uncertain, the easiest way to interpret equations (13)-(16) is to rely on equation (10), which states that the ecological discount rate is equal to the difference between the monetary discount rate and the growth rate of the expected value of ecological services. Using Lemma 1 and equations (9) and (11), we obtain the following characterization when $\beta_t = \beta$ is certain:

$$r_t = r_f + \beta\gamma\sigma_c(\sigma_c - k\sigma_x) \tag{18}$$

$$f_t = (\mu_c - \mu_x)\beta + 0.5b\beta^2. \tag{19}$$

Let us start with the case $\beta = 0$, which means that the value of ecosystem services is a known constant, thereby eliminating the relative price effect ($f_t = 0$). In that case, the ecological and monetary discount rates simplify to r_f, so that r_f can be interpreted as the risk-free interest rate. Equation (14) tells us that it corresponds to the Ramsey rule adapted to an uncertain growth rate of consumption. The last term in this equation reduces the interest rate to induce more precautionary investments in natural capital. In fact, equation (14) is the well-known extended Ramsey rule (Cochrane (2001)).

When natural and physical capitals are imperfectly substitutable, the uncertain growth of consumption and natural capital implies that the value of future ecological services is uncertain. This uncertainty has two effects on discount rates. First, notice that this value is correlated with aggregate consumption. The CCAPM tells us that in that case, a risk premium must be added to the interest rate r_f to determine the risk-adjusted monetary discount rate r_t. This risk premium is given by the last term in equation (18). From the CCAPM, we know that it is positive if $\ln(F_t) = \ln(\alpha) + \beta\ln(c_t) - \beta\ln(x_t)$ and $\ln(c_t)$ are positively correlated. This is the case when $\beta(\sigma_c - k\sigma_x)$ is positive. On the contrary, when $\beta(\sigma_c - k\sigma_x)$ is negative, the value of natural capital and consumption are negatively correlated, and the monetary discount rate r_t is smaller than the risk-free rate r_f. This is because investing in natural capital hedges the macroeconomic risk in this economy. When consumption and ecological growth rates are independent ($k=0$), β can be interpreted as the CCAPM beta of natural capital, i.e., the elasticity of the value of natural capital to changes in aggregate consumption. When these growth rates are positively correlated, this elasticity is reduced.

But when the two types of capital are imperfectly substitutable ($\beta \neq 0$), the growth rate of the expected value of ecological services is not zero. This yields a second effect of growth uncertainty on the ecological discount rate. This relative price effect introduces a wedge between the two discount rates, as stated by equation (10): The difference between the ecological discount rate and the monetary discount rate is equal to the growth rate f_t of EF_t, which is a quadratic function of β (equation (19)). This quadratic term will be a key element in our analysis of the effect of the uncertainty affecting β_t.

Observe that parameter *b* is positive, so that the ecological discount rate is a hump-shaped quadratic function of β, with a maximum at $\beta = -a/b$. The ecological discount rate and thus the value of natural capital are non-monotone in the elasticity of substitution of the ecosystem services. To understand this result, remember that the substitutability parameter β can be interpreted as the CCAPM beta of natural capital. This implies that the monetary discount rate is linearly increasing in β, as shown by equation (18). Remember also from equation (10) that the ecological discount rate is the difference between the monetary discount rate and the growth rate f_t of the monetary value F_t of ecological service. From equation (19), this relative price effect is a quadratic convex function of β. Thus, the ecological discount rate is a quadratic concave function of β. As already explained in Dietz, Gollier and Kessler (2018), a project with a large β is not necessarily a bad news for its social valuation. It is true that a large β implies that the project's social benefit is positively correlated with aggregate consumption. It increases the macroeconomic risk, and should be penalized for this by adding a risk premium in the monetary discount rate. But a large β also means a smaller elasticity of substitution. In a growing economy, the monetary value F_t of the ecological services will thus grows faster. It is therefore possible that a larger β increases the monetary value faster than it reduces the discount factor, with a positive net effect for the present value of the service. This happens when β is larger than *–a/b*.

4. Uncertain substitutability, ecological discounting and value

We hereafter consider that β_t is a random variable whose distribution characterizes our current beliefs about the degree of substitutability of the services provided by the natural capital at date *t*. Using the law of iterated expectations, we have that

$$e^{-\rho_t t} = E\left[e^{-\rho(\beta_t)t}\right], \tag{20}$$

where function $\rho(.)$ is characterized by equation (13). When β_t is uncertain, the ecological discount factor equals the expectation of the ecological discount factor conditional on β_t. This is summarized in the following proposition.

Proposition 1: *Let β_t denote the inverse of the degree of substitutability of the ecosystem services. The ecological discount rate ρ_t associated with time horizon t is given by the following equation:*

$$\rho_t = r_f - t^{-1}\chi\left(t, a\beta_t + 0.5b\beta_t^2\right), \tag{21}$$

where r_f, a and b are three scalars defined respectively in equations (14), (15) *and* (16).

In the remainder of this section, we use equation (21) to derive some properties of the impact of the uncertainty affecting the degree of substitutability on the ecological discount rate and on the value of natural capital. It is a direct consequence of the fact that the conditional discount rate $\rho(\beta)$ is non-monotone in β that first-order stochastic changes in the distribution of β_t have an intrinsically ambiguous effect on the ecological discount rate, and therefore on the value of natural capital. However, we show in Proposition 2 that a mean-preserving spread in β_t always raises the value of natural capital. This is the main result of this paper.

Proposition 2: *The ecological discount rate* ρ_t *is reduced by any mean preserving spread of* β_t. *In this sense, the uncertainty affecting the degree of substitutability of ecosystem services always raises the economic value of the natural capital that generates them.*

Proof: We can rewrite equation (21) as follows:

$$\rho_t = r_f - t^{-1} \ln Eh(\beta_t), \tag{22}$$

where $h(\beta) = \exp\left(t\left(a\beta + 0.5b\beta^2\right)\right)$. Observe that because b is non-negative, function h compounds two convex functions, so is convex. By Jensen's inequality, $Eh(\beta_t)$ is increased by any mean-preserving spread of β_t. This concludes the proof. □

There are two reasons why an increase in risk about β_t reduces the ecological discount rate. The first reason can be identified in the special case of short maturities. Indeed, we know from property *iii* of Lemma 1 that the limit of $t^{-1}\chi(t, a\beta + 0.5b\beta^2)$ when *t* vanishes is equal to the expectation of $a\beta + 0.5b\beta^2$. This implies that

$$\lim_{t \to 0_+} \rho_t = r_f - aE\beta_{0_+} - 0.5bE\beta_{0_+}^2 \le r_f - aE\beta_{0_+} - 0.5b\left(E\beta_{0_+}\right)^2. \tag{23}$$

If β_{0_+} is uncertain, the ecological discount rate for short maturities is reduced by $0.5bVar\left(\beta_{0_+}\right)$. The second technical reason of the negative impact of the uncertain β on the ecological discount rate is due to the fact that $t^{-1}\chi(t, a\beta + 0.5b\beta^2)$ is increasing in *t* when β is uncertain. This is reminiscent of an argument made by Weitzman (1998, 2001) who claimed that the uncertainty affecting future discount rates should induce us to use a certainty equivalent discount rate smaller than its expected value.

In Proposition 3, we show that the term structure of ecological discount rates is decreasing when the uncertainty affecting β_t is increasing with maturity, in the sense of Rothschild and Stiglitz (1970).

Proposition 3: *Suppose that, for all* $t' > t$, $\beta_{t'}$ *is a mean-preserving spread of* β_t*. Under this condition, the term structure of ecological discount rates* ρ_t *is decreasing.*

Proof: Consider any pair (t, t') such that t' is larger than *t*. Because $t^{-1}\chi(t,z)$ is increasing in *t* by property *iii* of Lemma 1, equation (21) implies that

$$\begin{aligned}\rho_{t'} &= r_f - t'^{-1}\chi\left(t', a\beta_{t'} + 0.5b\beta_{t'}^2\right) \\ &\leq r_f - t^{-1}\chi\left(t, a\beta_{t'} + 0.5b\beta_{t'}^2\right) = r_f - t^{-1}\ln Eh(\beta_{t'}),\end{aligned} \tag{24}$$

where $h(\beta) = \exp\left(t\left(a\beta + 0.5b\beta^2\right)\right)$ is convex in β. By Jensen's inequality, it implies that $Eh(\beta_{t'}) \geq Eh(\beta_t)$. Combining these two results implies that

$$\rho_{t'} \leq r_f - t^{-1}\ln Eh(\beta_{t'}) \leq r_f - t^{-1}\ln Eh(\beta_t) = \rho_t. \tag{25}$$

This concludes the proof. □

The decreasing nature of the term structure of the ecological discount rate says something important about the intrinsic value of natural capital. In a world in which the degree of substitutability of the ecosystem services is uncertain, natural capital is particularly valuable if it can deliver ecological benefits in the distant future. An economic intuition of this central result of this paper can be derived from two observations. First, from equation (20), we know that the ecological discount rate equals the expectation of the ecological discount factors conditional on β_t. Second, the discount factor $\exp(-\rho(\beta)t)$ is a convex function of β, and the degree of convexity of this function is increasing in *t*. These two observations implies that the term structure of the ecological discount rates must be decreasing when the distribution of β_t becomes more dispersed for longer maturities. Again, this result is in line with Weitzman (1998, 2001) who showed that the term structure of the discount rates must be decreasing when the rate at which sure benefits must be discounted in the future is uncertain.

Because of the presence of a term in β^2 is $\rho(\beta)$, there is usually no analytical solution to $\chi(t, a\beta + 0.5b\beta^2)$. However, if one knows the first few cumulants of random variable $a\beta_t + 0.5b\beta_t^2$, one can approximate equation (21) by using property *i* of lemma 1:

$$\rho_t = r_f - \sum_{n=1}^{\infty} \kappa_n \left(a\beta_t + 0.5b\beta_t^2 \right) \frac{t^{n-1}}{n!}. \tag{26}$$

For example, if the distribution of β_t is independent of *t* in the neighborhood of $t = 0_+$, then the above equation implies that

$$\lim_{t \to 0_+} \frac{\partial \rho_t}{\partial t} = -0.5 Var\left(a\beta_{0_+} + 0.5b\beta_{0_+}^2 \right). \tag{27}$$

One can finally use property *iii* of Lemma 1 to determine the asymptotic value of the ecological discount rate. Suppose that the support of the distribution of β_t is bounded when *t* tends to infinity, with $\lim_{t\to\infty} \text{supp}\, \beta_t = [\beta_{\min}, \beta_{\max}]$. We know that $t^{-1}\chi(t,x)$ converges to the supremum of the support of $a\beta_t + 0.5b\beta_t^2$. This implies that the ecological discount rates asymptotically tend to

$$\lim_{t\to\infty} \rho_t = \begin{cases} \rho(\beta_{\min}) = r_f - a\beta_{\min} - 0.5b\beta_{\min}^2 & \text{if} \quad \beta^* \le -a/b \\ \rho(\beta_{\max}) = r_f - a\beta_{\max} - 0.5b\beta_{\max}^2 & \text{if} \quad \beta^* > -a/b, \end{cases} \tag{28}$$

where $\beta^* = 0.5(\beta_{\min} + \beta_{\max})$ is the center of the support of β_∞.

5. The Gaussian case

We now examine the special case in which the beliefs about β_t can be represented by a normal distribution. The proof of the following result is relegated to the Appendix.

Proposition 4 : *Suppose that random variable* β_t *is normally distributed with mean* μ_{β_t} *and standard deviation* σ_{β_t}. *Then, as long as* $b\sigma_{\beta_t}^2 t < 1$,

$$\rho_t = r_f + \frac{1}{2t}\ln\left(1 - b\sigma_{\beta_t}^2 t\right) - \frac{a\mu_{\beta_t} + 0.5b\mu_{\beta_t}^2 + 0.5a^2\sigma_{\beta_t}^2 t}{1 - b\sigma_{\beta_t}^2 t} \qquad (29)$$

When $b\sigma_{\beta_t}^2 t$ tends to 1, ρ_t tends to $-\infty$.

This proposition provides an analytical solution for the ecological discount rate at all maturities *t* such that $b\sigma_{\beta_t}^2 t$ is smaller than unity. It is unbounded below for all other maturities. This means that any natural capital that delivers a positive service in time horizons *t* such that $b\sigma_{\beta_t}^2 t > 1$ has an infinite value. Technically, this is because the conditional ecological discount rate $\rho(\beta_t)$ is a quadratic function of β_t. This fattens the tails of the distribution of the future benefits of ecological services. This result is in the spirit of Weitzman (2007), who shows how introducing uncertainty on the variance of future consumption growth can generate an unbounded negative interest rate.

An immediate application of Proposition 4 is when parameter β is constant over time, but unknown at date 0. The term structure of ecological discount rates is defined in this case by equation (29) with $\mu_{\beta_t} = \mu_\beta$ and $\sigma_{\beta_t}^2 = \sigma_\beta^2$. The true value of β will be learned over time by observing the correlation between consumption and the value of ecological services. The analysis of the impact of learning on the future term structure of ecological discount rates would necessitate the quantification of the posterior distributions of β. However, because we focus our analysis on the current term structure of discount rates, this analysis can be skipped.

In Corollary 1, we illustrate Proposition 4 in another special case in which β_t evolves stochastically from the current β_0 by following an arithmetic Brownian motion. This is a direct application of Proposition 4 with $\mu_{\beta_t} = \beta_0 + \mu_\beta t$ and $\sigma_{\beta_t}^2 = t\sigma_\beta^2$.

Corollary 1: *Suppose that β_t follows an arithmetic Brownian motion with drift μ_β and volatility σ_β. This implies that the term structure of ecological discount rates exists for all maturities $t < T = (b\sigma_\beta^2)^{-1/2}$, with*

$$\rho_t = r_f + \frac{1}{2t}\ln\left(1 - b\sigma_\beta^2 t^2\right) - \frac{a\left(\beta_0 + \mu_\beta t\right) + 0.5b\left(\beta_0 + \mu_\beta t\right)^2 + 0.5a^2\sigma_\beta^2 t^2}{1 - b\sigma_\beta^2 t^2}. \tag{30}$$

This term structure tends to $-\infty$ for maturities tending to T.

When the inverse of the elasticity of substitution follows a Brownian motion, the term structure of ecological discount rates is decreasing and tends to $-\infty$ when the maturity tends to $T = (b\sigma_\beta^2)^{-1/2}$. For example, if natural capital delivers a sure flow of services, we have that $\sigma_x = 0$ and $b = \sigma_c^2$, which implies in turn that $T = 1/\sigma_c\sigma_\beta$. The volatility of the growth rate of aggregate consumption over the last century in the western world has been between 2% and 4% per year. If we assume that the volatility of the growth rate of β is also between 2% and 4%, we find that the bliss maturity T is somewhere between 625 and 2500 years. If the natural capital delivers a positive service above this bliss maturity *T*, it has an unbounded economic value, at the margin.

6. A credible calibration

Up to know, we assumed for simplicity that the growth rates of consumption and ecological services are governed by a Brownian process. We know from three decades of research on asset pricing that when this assumption is combined with constant relative risk aversion, this leads to shocking puzzles, such as the equity premium puzzle and the risk-free rate puzzle. One possible solution to these puzzles has been proposed by Barro (2006, 2009). Barro introduced the possibility that consumption growth be subject of very infrequent disasters. This is modelled by assuming that the annual growth rate of consumption is i.i.d. with a probability distribution being a mixture of two normal distributions. Most often, the annual growth rate is extracted from a “business-as-usual” urn calibrated from historical data. But in rare instances, the annual growth rate is extracted from a “catastrophe” urn, with a very negative mean and a large standard deviation. The possibility of catastrophes can explain why, over the last century or so, the observed real interest rates have been so small, and the equity premium have been so large.

In the spirit of Barro, we assume in this section that the pair of random variables (g_{xt}, g_{ct}) exibits no serial correlation and is sensitive to the state of nature denoted *i*, with *i=1,…,n*. The state of nature occurring at date *t* stipulates from which of the *n* urns the pair (g_{xt}, g_{ct}) is extracted at date *t*. The probability $p_i > 0$ of occurrence of state *i* is time-independent, with $\Sigma_i p_i = 1$. Following Martin (2012), we assume that (g_{xt}, g_{ct}) conditional on state *i* is normally distributed with mean (μ_x, μ_i), variance (σ_x^2, σ_i^2) and covariance $k_i \sigma_x \sigma_i$, where $k_i \in [-1,1]$ denotes the coefficient of correlation between the two growth rates conditional on state *i*. As stated in this notation, we assume for simplicity that the trend and volatility of the growth of ecological services are state-independent.

Following the same line of developments as in sections 3 and 4, it is easy to show that equation (21) has a simple generalization in this new framework. It yields the following pricing equation:

$$\rho_t = -t^{-1} \ln\left(E\left[\left(\sum_{i=1}^{n} p_i \exp\left(-r_{fi} + a_i \beta_t + \frac{1}{2} b_i \beta_t^2 \right) \right)^t \right] \right) \tag{31}$$

with

$$r_{fi} = \delta + \gamma \mu_i - 0.5 \gamma^2 \sigma_i^2, \tag{32}$$

$$a_i = \mu_i - \mu_x - \gamma \sigma_i \left(\sigma_i - k_i \sigma_x \right), \tag{33}$$

and

$$b_i = Var(g_i - g_x) = \sigma_i^2 + \sigma_x^2 - 2 k_i \sigma_x \sigma_i \geq 0. \tag{34}$$

If β_t is Gaussian, one can use the same trick (Lemma 2 in the Appendix) to derive an analytical solution for the expectation operators appearing in the right-hand side of equation (31). To keep the analysis simple, let us assume that there are only two possible states (*n*=2). In that case, this equation can be rewritten as follows:

$$\rho_t = -t^{-1} \ln\left(\sum_{j=0}^{t} \frac{t!\, p_1^{t-j} p_2^j}{j!(t-j)!} E\left[\exp\left(\left((t-j)a_1 + j a_2\right)\beta_t + \left((t-j)b_1 + j b_2\right)\frac{\beta_t^2}{2} - (t-j) r_{f1} - j r_{f2} \right) \right] \right),$$

or, equivalently, using Lemma 2, assuming that $\max(b_1, b_2)\sigma_{\beta_t}^2 t$ is smaller than unity,

$$\rho_t = -t^{-1} \ln\left(\sum_{j=0}^{t} \frac{t!\, p_1^{t-j} p_2^j \exp\left(R_{tj}\right)}{j!(t-j)!\sqrt{1-\left((t-j)b_1 + jb_2\right)\sigma_{\beta_t}^2}} \right), \tag{35}$$

with

$$R_{tj} = \frac{\left((t-j)a_1 + ja_2\right)\mu_{\beta_t} + \left((t-j)a_1 + ja_2\right)^2 \frac{\sigma_{\beta_t}^2}{2} + \left((t-j)b_1 + jb_2\right)\frac{\mu_{\beta_t}^2}{2}}{1 - b\sigma_z^2} - (t-j)r_{f1} - jr_{f2}. \tag{36}$$

Equation (35) provides a simple analytical method to compute the term structure of discount rates. We calibrate this equation by using the parameter values that are described in Table 1. We assume that the ecological services are constant through time. In the business-as-usual state, the trend of consumption growth is 2% and the volatility of consumption is 4%, in line with the U.S. data during the last century. In the catastrophe state, in the spirit of Martin (2012), we assume an expected reduction of consumption of 30%, and a standard deviation of this reduction of 25%. The coefficient of relative risk aversion is 3 and the probability of the catastrophe state is 1.7% per year, as in Barro (2006).

Parameter	Value	Interpretation
δ	0	Rate of pure preference for the present
γ	3	Relative risk aversion
μ_1	2%	Trend of consumption growth in the BAU state
σ_1	4%	Volatility of consumption growth in the BAU state
p_2	1.7%	Probability of catastrophe
μ_2	-30%	Trend of consumption growth in the catastrophe state
σ_2	25%	Volatility of consumption in the catastrophe state
μ_x	0%	Trend of growth of ecological services
σ_x	0%	Volatility of growth of ecological services
$k_1 = k_2$	0	Coefficient of correlation between consumption and ecological services

Table 1: Calibration of the model with one Business-As-Usual (BAU) state and one catastrophe state.

In this economy, putting $\beta_t \equiv 0$ in equation (31) implies that the equilibrium interest rate is equal to

$$r_f = -\ln\left(\left(\sum_{i=1}^{2} p_i \exp\left(-r_{fi}\right)\right)\right) = 1.22\%. \tag{37}$$

Similarly, we can examine how this model would predict the price of a claim on aggregate consumption. Because x_t is constant, equation (3) tells us that selecting $\beta_t \equiv 1$ points to an asset generating this consumption claim. Using equations (10) and (31), we obtain that the equilibrium aggregate risk premium is equal to

$$\begin{aligned} \pi_t &= (\rho_t + f_t) - r_f \\ &= -\ln\left(\left(\sum_{i=1}^{2} p_i \exp\left(-r_{fi} + a_i + \frac{1}{2}b_i\right)\right)\right) + \ln\left(\sum_{i=1}^{2} p_i \exp\left(\mu_i - \mu_x + \frac{1}{2}b_i\right)\right) - r_f \\ &= 2.18\%. \end{aligned} \tag{38}$$

These equilibrium returns are in line with observed asset prices, which suggests that this calibration provides asset prices that are coherent with market prices.[2]

Consider now a natural capital that is partially substitutable to physical capital. We first assume that β, the inverse of the elasticity of substitution, is constant but unknown. Our collective beliefs about β is represented by a normal distribution with a mean $\mu_\beta = 1$ and a standard deviation $\sigma_\beta = 0.5$. Figure 1 describes the term structure of discount rates to be used today to value the ecological services generated by this natural capital at different maturities. These discount rates are remarkably constant around 1.62% for maturities below 41 years, but they plunges abruptly for longer maturities. For example, the ecological services delivered in 45 and 50 years should be discounted at rates $\rho_{45} = -8.66\%$ and $\rho_{50} = -182.64\%$, respectively. They tend to minus infinity when for maturities tending to $T = (\sigma_\beta \sigma_2)^{-2} = 64$ years. The large standard deviation $\sigma_2 = 1/4$ of

[2] Suppose alternatively that there is no catastrophe (p_2=0). In that case, the equilibrium interest rate would be equal to 5.28%, whereas the equilibrium aggregate risk premium would be equal to 0.48%.

the growth rate of consumption in the catastrophe state is the key reason for why this bliss maturity T is so short.

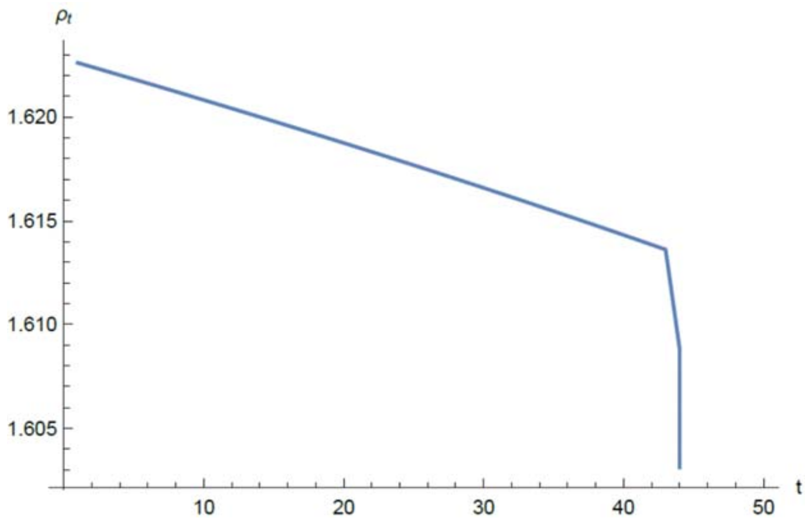

Figure 1: Term structure of ecological discount rates for the calibration described in Table 1, assuming $\mu_\beta = 1$ and $\sigma_\beta = 0.5$.

In the remainder of this paper, we show how the ecological discount rate is affected by the substitutability parameters μ_β and σ_β. We first examine the role of the expected beta. We have shown and explained in Section 3 that when the substitutability parameter β is certain, the ecological discount rate ρ is a quadratic function of β with a minimum at $-a/b$. It is the consequence of the fact that the monetary discount rate is linearly increasing in the CCAPM β, and that the expected growth rate of the willingness-to-pay is quadratic in β. As shown in Figure 2, this property of a hump-shaped relationship between the ecological discount rate and the expected β is preserved when β is uncertain.

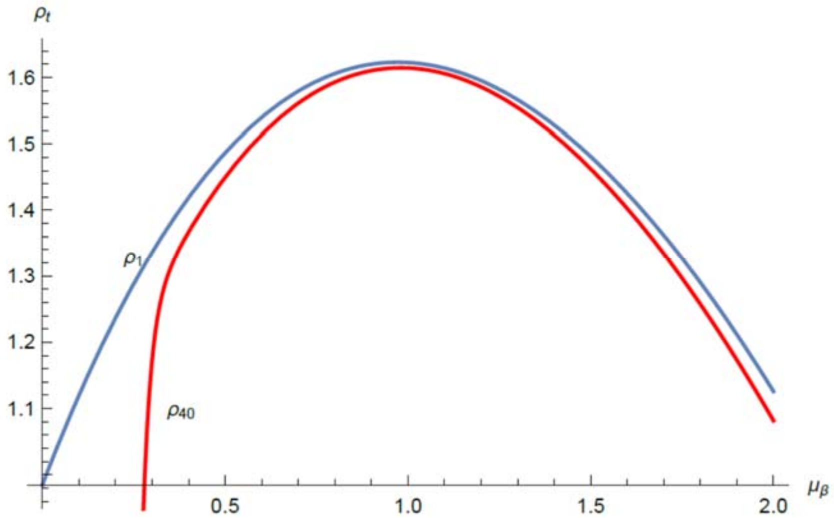

Figure 2: Ecological discount rates at maturities t=1 year and t=40 years as a function of the expected β, under the calibration described in Table 1 and assuming $\sigma_\beta = 0.5$.

In Figure 3, we illustrate the central message of this paper that the uncertainty affecting the elasticity of substitution between natural and physical capital tends to reduce the ecological discount rate, and thus to raise the value of natural capital today. Under the calibration described in Table 1, we consider different types of natural capital, all with the same expected beta ($\mu_\beta = 1$), but with different degree of uncertainty. For short maturities, increasing the standard deviation of random variable β from 0 to 1 reduces the ecological discount rate from 1.76% to 1.18%. For a 40-year maturity, the effect of increasing uncertainty is more dramatic because an increase in σ_β reduces the bliss maturity $T = (\sigma_\beta \sigma_2)^{-2}$.

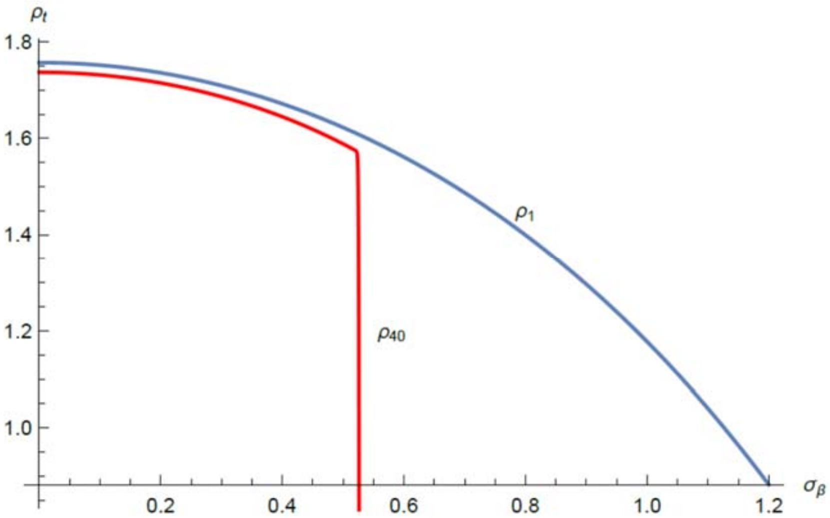

Figure 3: Ecological discount rates at maturities t=1 year and t=40 years as a function of the standard deviation of β, under the calibration described in Table 1 and assuming $\mu_\beta = 1$.

7. Conclusion

The uncertainty affecting the substitutability of ecosystem services in the future is an important source of complexity to estimate the economic value of natural capital. We have shown in this paper that taking account of this uncertainty may indeed have a crucial importance, in particular if this natural capital is expected to deliver services in the distant future. This uncertainty makes the investment in natural capital risky, because the monetary benefits generated by it become uncertain even when its ecological benefit is certain. The main result of this paper is that the economic value of natural capital is always increased by the uncertainty surrounding the elasticity of substitution.

We have also shown that, because of this uncertainty, the rate at which ecosystem services must be discounted is always decreasing with the time horizon at which they materialize. In the Gaussian case, any marginal increment of ecosystem service has an infinite economic value if it is delivered in a time horizon that is larger than some "bliss maturity". If infrequent catastrophic macroeconomic events are introduced into the model as suggested by Barro (2006), this bliss maturity could be measured in decades rather than in centuries.

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Appendix : Proof of Proposition 4

We first prove the following Lemma.

Lemma 2: *Suppose that random variable* z *is normally distributed with mean* μ_z *and standard deviation* σ_z. *Consider any pair* $(a,b)\in\mathbb{R}^2$ *such that* $b<\sigma_z^{-2}$. *Then, we have that*

$$E\exp(az+0.5bz^2)=\left(1-b\sigma_z^2\right)^{-1/2}\exp\left(\frac{a\mu_z+0.5a^2\sigma_z^2+0.5b\mu_z^2}{1-b\sigma_z^2}\right). \quad (39)$$

Proof: We have that

$$E\exp(az+0.5bz^2)=\frac{1}{\sigma_z\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(az+0.5bz^2-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right)dz. \quad (40)$$

After rearranging terms in the integrant, this is equivalent to

$$E\exp(az+0.5bz^2)=\frac{\exp\left(-\frac{\mu_z^2}{2\sigma_z^2}-y\right)}{\sigma_z/\hat{\sigma}}\left[\frac{1}{\hat{\sigma}\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(z-\hat{\mu}\right)^2}{2\hat{\sigma}^2}\right)dz\right], \quad (41)$$

with

$$y=\frac{\left(a+(\mu_z/\sigma_z^2)\right)^2}{2b-(2/\sigma_z^2)},$$

$$\hat{\mu}=-\frac{a+(\mu_z/\sigma_z^2)}{b-(1/\sigma_z^2)},$$

and

$$-\frac{1}{\hat{\sigma}^2}=b-\frac{1}{\sigma_z^2}.$$

Notice that $\hat{\sigma}$ exists only if we assume that $b < 1/\sigma_z^2$. Notice also that the bracketed term in equation (41) is the integral of the density function of the normal distribution with mean $\hat{\mu}$ and variance $\hat{\sigma}^2$. This must be equal to unity. This equation can thus be rewritten as

$$\begin{aligned} E\exp(az+0.5bz^2) &= \frac{\hat{\sigma}}{\sigma_z}\exp\left(-\frac{\mu_z^2}{2\sigma_z^2}-\frac{\left(a+(\mu_z/\sigma_z^2)\right)^2}{2b-(2/\sigma_z^2)}\right) \\ &= \left(1-b\sigma_z^2\right)^{-1/2}\exp\left(\frac{a\mu_z+0.5a^2\sigma_z^2+0.5b\mu_z^2}{1-b\sigma_z^2}\right). \end{aligned} \tag{42}$$

This concludes the proof of Lemma 2. □

Equation (29) in Proposition 4 is a direct consequence of applying Lemma 2 to equation (21). It remains to prove that ρ_t tends to $-\infty$ when $b\sigma_{\beta_t}^2 t$ tends to 1 from below. Observe in equation (29) that the second term in the right-hand side of the equality tends to $-\infty$. We would be done if the numerator of the last term is positive when $b\sigma_{\beta_t}^2 t$ tends to 1. It is equal to

$$num(a) = 0.5b\mu_{\beta_t}^2 + a\mu_{\beta_t} + 0.5a^2\sigma_{\beta_t}^2 t = 0.5\frac{\mu_{\beta_t}^2}{\sigma_{\beta_t}^2 t} + a\mu_{\beta_t} + 0.5a^2\sigma_{\beta_t}^2 t, \tag{43}$$

which is a quadratic function of a, with a minimum at $a^* = -\mu_{\beta_t}^2/\sigma_{\beta_t}^2 t$. This implies that this numerator is always larger than

$$num(a) \geq num(a^*) = 0.5\frac{\mu_{\beta_t}^2}{\sigma_{\beta_t}^2 t} - \frac{\mu_{\beta_t}^2}{\sigma_{\beta_t}^2 t} + 0.5\frac{\mu_{\beta_t}^2}{\sigma_{\beta_t}^2 t} = 0. \tag{44}$$

Thus, the numerator of the last term of equation (29) is positive, and ρ_t tends to $-\infty$ when $b\sigma_{\beta_t}^2 t$ tends to 1. □