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# “Blow up of the solutions to a linear elliptic system involving schrödinger operators"

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# BLOW UP OF THE SOLUTIONS TO A LINEAR ELLIPTIC SYSTEM INVOLVING SCHRÖDINGER OPERATORS

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**Abstract.** We show how the solutions to a $2 \times 2$ linear system involving Schrödinger operators blow up as the parameter $\mu$ tends to some critical value which is the principal eigenvalue of the system; here the potential is continuous positive with superquadratic growth and the square matrix of the system is with constant coefficients and may have a double eigenvalue.

*Keywords:* Maximum Principle, Antimaximum Principle, Elliptic Equation and Systems, Cooperative and Non-cooperative Systems, Principle Eigenvalue.

*AMS classification:* 35P, 35J10.

## §1. Introduction

We study here the behavior of the solutions to a $2 \times 2$ system (considered in its variational formulation):

$$(S) \qquad LU := (-\Delta + q(x))U = AU + \mu U + F(x) \text{ in } \mathbb{R}^N,$$

$$U(x)_{|x|\to\infty} \to 0$$

where $q$ is a continuous positive potential tending to $+\infty$ at infinity with superquadratic growth; $U$ is a column vector with components $u_1$ and $u_2$ and $A$ is a $2 \times 2$ square matrix with constant coefficients. $F$ is a column vector with components $f_1$ and $f_2$.

Such systems have been intensively studied mainly for $\mu = 0$ and for $A$ with 2 distinct eigenvalues; here we consider also the case of a double eigenvalue. In both cases, we show the blow up of solutions as $\mu$ tends to some critical value $\nu$ which is the principal eigenvalue of System $(S)$. This extends to systems involving Schrödinger operators defined on $\mathbb{R}^N$ earlier results valid for systems involving the classical Laplacian defined on smooth bounded domains with Dirichlet boundary conditions.

This paper is organized as follows: In Section 2 we recall known results for one equation. In Section 3 we consider first the case where $A$ has two different eigenvalues and then we study the case of a double eigenvalue.

## §2. The equation

We shortly recall the case of one equation

$$(E) \qquad Lu := (-\Delta + q(x))u \; = \; \sigma u + f(x) \in \mathbb{R}^N,$$

$$\lim_{|x|\Rightarrow+\infty} u(x) = 0.$$

$\sigma$ is a real parameter.

**Hypotheses**
$(H_q)$ $q$ is a positive continuous potential tending to $+\infty$ at infinity.
$(H_f)$ $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ and $f > 0$ on some subset with positive Lebesgue measure.
It is well knwon that if $(H_q)$ is satisfied, $L$ possesses an infinity of eigenvalues tending to $+\infty$: $0 < \lambda_1 < \lambda_2 \leq \ldots$.

**Notation:** $(\Lambda, \phi)$ Denote by $\Lambda$ the smallest eigenvalue of $L$; it is positive and simple and denote by $\phi$ the associated eigenfunction, positive and with $L^2$-norm $||\phi|| = 1$.

It is classical ([9], [11]) that if $f > 0$ and $\sigma < \Lambda$ the positivity is improved, or in other words, the maximum principle **(MP)** is satisfied:

$$(MP) \qquad f \geq 0, \not\equiv 0 \;\Rightarrow\; u > 0.$$

Lately, for potentials growing fast enough (faster than the harmonic oscillator), another notion has been introduced ([2], [3], [5], [6]) which improves the maximum (or antimaximum principle): the "groundstate positivity" (**GSP**) (resp. " negativity" (**GSN**)) which means that there exists $k > 0$ such that

$$u > k\phi \text{ (GSP) (resp. } u < -k\phi \text{ (GSN))} .$$

We also say shortly "fundamenal positivity" or" negativity", or also "$\phi$-positivity" or "negativity".
The first steps in this direction use a radial potential. Here we consider a small perturbation of a radial one as in [5].

**The potential** $q$ We define first a class $\mathcal{P}$ of radial potentials:

$$\mathcal{P} := \{Q \in C(\mathbb{R}_+, (0, \infty))/\ \exists R_0 > 0, Q' > 0\, a.e.\, on\, [R_0, \infty), \int_{R_0}^{\infty} Q(r)^{-1/2} < \infty\}. \tag{1}$$

The last inequality holds if $Q$ is growing sufficiently fast ($> r^2$). Now we give results of GSP or GSN for a potential $q$ which is a small perturbation of $Q$; we assume:
$(H'_q)$ $q$ satisfies $(H_q)$ and there exists two functions $Q_1$ and $Q_2$ in $\mathcal{P}$, and two positive constants $R_0$ and $C_0$ such that

$$Q_1(|x|) \leq q(x) \leq Q_2(|x|) \leq C_0 Q_1(|x|),\ \forall x \in \mathbb{R}^N, \tag{2}$$

$$\int_{R_0}^{\infty} (Q_2(s) - Q_1(s)) \int_{R_0}^{s} exp( - \int_{r}^{s} [Q_1(t)^{1/2} + Q_2(t)^{1/2}]dt) drds < \infty. \tag{3}$$

Denoting by $\Phi_1$ (resp. $\Phi_2$) the groundstate of $L_1 := -\Delta + Q_1$ (resp. $L_2 = -\Delta + Q_2$), Corollary 3.3 in [5] says that all these groundstates are "comparable" that is there exists constants $0 < k_1 \leq k_2 \leq \infty$ such that $k_1\phi \leq \Phi_1, \Phi_2 \leq k_2\phi$.

**Theorem 1.** *(GSP) ([5]) If $(H_q')$ and $(H_f)$ are satisfied, then, for $\sigma < \Lambda$, there is a unique solution $u$ to $(E)$ which is positive, and there exists a constant $c > 0$, such that*

$$u > c\phi. \tag{4}$$

*Moreover, if also $f \leq C\phi$ with some constant $C > 0$, then*

$$u \leq \frac{C}{\Lambda - \sigma}\phi. \tag{5}$$

*Remark* 1. This holds also if we only assume $f \in L^2$ and $f^1 := \int f\phi > 0$

**The space $\mathcal{X}$** : It is convenient for several results to introduce the space of "groundstate bounded functions":

$$\mathcal{X} := \{h \in L^2(\mathbb{R}^N) : \ h/\phi \in L^\infty(\mathbb{R}^N)\}, \tag{6}$$

equipped with the norm $||h||_{\mathcal{X}} = ess \sup_{\mathbb{R}^n}(|h|/\phi)$.

**Hypothesis** $(H_f')$ We consider now functions $f$ which are such that

$(H_f')$: $f \in \mathcal{X}$ and $f^1 := \int f\phi > 0$.

For a potential satisfying $(H_q')$ and a function $f \in \mathcal{X}$, there is also a result of "groundstate negativity" **(GSN)** for $(E)$; it is an extension of the antimaximum principle, introduced by Clément and Peletier in 1978 ([8]) for the Laplacian when the parameter $\sigma$ crosses $\Lambda$.

**Theorem 2.** *(GSN) ([5] ) Assume $(H_q')$ and $(H_f')$ are satisfied; then there exists $\delta(f) > 0$ and a positive constant $c' > 0$ such that for all $\sigma \in (\Lambda, \Lambda + \delta)$,*

$$u \leq -c'\phi. \tag{7}$$

**Theorem 3.** *Assume $(H_q')$ and $(H_f')$ are satisfied. Then there exists $\delta > 0$, independant of $\sigma$, such that for $\Lambda - \delta < \sigma < \Lambda$ there exists positive constants $k'$ and $K'$, depending on $f$ and $\delta$ such that*

$$0 < \frac{k'}{\Lambda - \sigma}\phi < u < \frac{K'}{\Lambda - \sigma}\phi. \tag{8}$$

*If $\Lambda < \sigma < \Lambda + \delta$, there exists positive constants $k$" and $K$", depending on $f$ and $\delta$ such that*

$$\frac{k"}{\Lambda - \sigma}\phi < u < \frac{K"}{\Lambda - \sigma}\phi < 0. \tag{9}$$

This result extends earlier one in [10] and a close result is Theorem 2.03 in [7]. It shows in particular that $u \in \mathcal{X}$ and $|u| \to \infty$ as $|\nu - \mu| \to 0$.

**Proof:** Decompose $u$ and $f$ on $\phi$ and its orthogonal:

$$u = u^1\phi + u^\perp \ ; \ f = f^1\phi + f^\perp. \tag{10}$$

We derive from $(E)$: $Lu = \sigma u + f$ that

$$Lu^\perp = \sigma u^\perp + f^\perp \tag{11}$$

$$Lu^1\phi = \Lambda u^1\phi = \sigma u^1\phi + f^1\phi. \tag{12}$$

We notice that, since $q$ is smooth, so is $u$. Also, since $f \in \mathcal{X}$, $f^\perp$, $u$ and $u^\perp$ are also in $\mathcal{X}$ and hence are bounded. Choose $\sigma < \Lambda$ and assume $(H'_f)$. We derive from Equation (11) (by [4]Thm 3.2) that : $\|u^\perp\|_{\mathcal{X}} < K_1$. Therefore $|u^\perp|$ is bounded by some $cste.\phi > 0$.
From Equation (12) we derive

$$u^1 = \frac{f^1}{(\Lambda - \sigma)} \;\rightarrow\; \pm\infty \, as \, (\Lambda - \sigma) \rightarrow 0. \tag{13}$$

Take $\delta$ small enough and $\sigma \in (\Lambda - \delta, \Lambda)$. Since $u = u^1\phi + u^\perp$, then

$$0 \;<\; \frac{K'}{\Lambda - \sigma}\phi \;<\; u \;<\; \frac{K"}{\Lambda - \sigma}\phi.$$

For $\sigma > \Lambda$. we do exactly the same, except that the signs are changed for $u^1$ in (13).

## §3. A $2 \times 2$ Linear system

Consider now a linear system with constant coefficients.

$$(S) \qquad LU = AU + \mu U + F(x) \; in \; \mathbb{R}^N.$$

As above, $L := -\Delta + q$ where the potential $q$ satisfies $(H'_q)$, and where $\mu$ is a real parameter. $L$ can be detailed as 2 equations:

$$(S) \qquad \left\{ \begin{array}{lcl} Lu_1 & = & au_1 + bu_2 + \mu u_1 + f_1(x) \\ Lu_2 & = & cu_1 + du_2 + \mu u_2 + f_2(x) \end{array} \right. \quad in \; \mathbb{R}^N, .$$

$$u_1(x), u_2(x)_{|x|\rightarrow\infty} \rightarrow 0.$$

Assume

$$(H_A) \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } b > 0 \, \text{and} \, D := (a-d)^2 + 4bc \geq 0.$$

Note that $b > 0$ does not play any role since we can always change the order of the equations. The eigenvalues of $A$ are

$$\xi_1 = \frac{a + d + \sqrt{D}}{2} \geq \xi_2 = \frac{a + d - \sqrt{D}}{2}.$$

As far as we know, all the previous studies suppose that the largest eigenvalue $\xi_1$ is simple (i.e. $D = (a-d)^2 + 4bc > 0$). Here we also study, in the second subsection, the case of a double eigenvalue $\xi_1 = \xi_2$, that is $D = 0$; this implies necessarily $bc < 0$ and necessarily the matrix is not cooperative.

## 3.1. Case $\xi_1 > \xi_2$

This is the classical case where $\xi_1$ is simple. Set $\xi_1 > \xi_2$. The eigenvectors are

$$X_k = \begin{pmatrix} b \\ \xi_k - a \end{pmatrix},$$

As above, denote by $(\Lambda, \phi)$, $\quad \phi > 0$, the principal eigenpair of the operator $L = (-\Delta + q(x))$. It is easy to see that

$$L(X_k\phi) - AXk\phi = (\Lambda - \xi_k)X_k\phi,\ k = 1, 2$$

Set $X := X_1$. Hence

$$\nu = \Lambda - \xi_1 \tag{14}$$

is the principal eigenvalue of $(S)$ with associated eigenvector $X\phi$. Note that the components of $X\phi$ do not change sign, but, in the case of a non cooperative matrix they are not necessarily both positive. We prove:

**Theorem 4.** *Assume $(H_q')$, $b > 0$ and $D > 0$. Assume also that $f_1$ and $f_2$ are in $\mathcal{X}$ and*

$$(a - \xi_2)f_1^1 + bf_2^1 > 0. \tag{15}$$

*Then, there exists $\delta > 0$, independant of $\mu$, such that if $\nu - \delta < \mu < \nu$, there exists a positive constant $\gamma$ depending only on $F$ and Matrix $A$ such that*

*For cooperative systems*

$$c > 0 \ \Rightarrow\ u_1, u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0, \tag{16}$$

*For non-cooperative systems*

$$d > a \ \Rightarrow\ u_1, u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0, \tag{17}$$

$$a > d \ \Rightarrow\ u_1, -u_2 \geq \frac{\gamma}{\nu - \mu}\phi > 0. \tag{18}$$

*If $\nu < \mu < \nu + \delta$, the sign are reversed.*

*Remark* 2. It is noticeable that for all these cases, $|u_1|$, $|u_2| \to +\infty$ as $|\nu - \mu| \to 0$.

These results extend Theorem 4.2 in [2].

**Proof:** As in [1], we use $J$ the associated Jordan matrix (which in this case is diagonal) and $P$ the change of basis matrix which are such that

$$A = PJP^{-1}.$$

Here

$$P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}, \quad P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix}. \tag{19}$$

$$J = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}.$$

Denoting $\tilde{U} = P^{-1}U$ and $\tilde{F} = P^{-1}F$, we derive from System $(S)$ (after multiplication by $P^{-1}$ to the left):

$$L\tilde{U} = J\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

Since $J$ is diagonal we have two independant equations:

$$L\tilde{u}_k = (\xi_k + \mu)\tilde{u}_k + \tilde{f}_k,\ k = 1\, or\, 2. \tag{20}$$

The projection on $\phi$ and on its orthogonal for $k = 1$ and 2 gives

$$\tilde{u}_k = (\tilde{u}_k)^1\,\phi + \tilde{u}_k^\perp, \quad \tilde{f}_k = (\tilde{f}_k)^1\,\phi + \tilde{f}_k^\perp;$$

hence

$$L(\tilde{u}_k)^1\,\phi = \Lambda(\tilde{u}_k)^1\,\phi = \xi_k(\tilde{u}_k)^1\,\phi + \mu(\tilde{u}_k)^1\,\phi + (\tilde{f}_k)^1\phi, \tag{21}$$

$$L\tilde{u}_k^\perp = \xi_k\tilde{u}_k^\perp + \mu\tilde{u}_k^\perp + \tilde{f}_k^\perp. \tag{22}$$

If both $f_k$ are in $\mathcal{X}$, $f_k/\phi$ are bounded and hence both $\tilde{f}_k^\perp/\phi$ are bounded. Therefore, by (22) both $\tilde{u}_k^\perp/\phi$ are also bounded since the smallest eigenvalue for $L$ acting on $\phi^\perp$ is $\lambda_2 \neq< \Lambda$).
We derive from (21) that

$$(\tilde{u}_k)^1 \;=\; \frac{(\tilde{f}_k)^1}{\Lambda - \xi_k - \mu}.$$

Consider again Equation (21) for $k = 2$; obviously, $(\tilde{u}_2)^1$ stays bounded as $\mu \to \nu = \Lambda - \xi_1 < (\neq)\Lambda - \xi_2$ and therefore $\tilde{u}_2/\phi$ stays bounded.
For $k = 1$, $(\tilde{u}_1)^1 = \frac{(\tilde{f}_1)^1}{\nu-\mu} \to \infty$ as $\mu \to \nu = \Lambda - \xi_1$, where $(\tilde{f}_1)^1 = \frac{1}{\xi_1-\xi_2}((a - \xi_2)f_1^1 + bf_2^1) > 0$; this is the condition (15) which appears in Theorem 4. Then, we simply apply Theorem 3 to (20) for $k = 1$ and deduce that there existes $\delta > 0$, such that, for $|\Lambda - \xi_1 - \mu| = |\nu - \mu| < \delta$, there exists a positive constant $C > 0$ such that

$$\mu < \nu \;\Rightarrow\; \tilde{u}_1 \geq \frac{C}{\nu - \mu}\phi > 0;\ \mu > \nu \;\Rightarrow\; \tilde{u}_1 \leq \frac{C}{\nu - \mu}\phi < 0.$$

If $|\mu - \nu|$ small enough

$$(\tilde{u}_1)^1 \geq \frac{K}{\nu - \mu}\ if\ \mu < \nu\,;\ \tilde{u}_1^1 \leq -\frac{K}{\nu - \mu}\ if\ \mu > \nu$$

where $K$ is a positive constant depending only on $F$ and $A$.
Now, it follows from $U = P\tilde{U}$, that

$$u_1 = b(\tilde{u}_1 + \tilde{u}_2),\ u_2 = (\xi_1 - a)\tilde{u}_1 + (\xi_2 - a)\tilde{u}_2.$$

As $\nu - \mu \to 0$, since $\tilde{u}_2/\phi$ stays bounded, $u_1$ behaves as $b(\tilde{u}_1)^1\phi > 0$ ; $u_2$ behaves as $(\xi_1 - a)(\tilde{u}_1)^1\phi$.
Therefore 3 cases appear according to matrix $A$:

If $A$ is cooperative $(b > 0, c > 0)$, then $\xi_2 < a < \xi_1$ so that $(\xi_1 - a) > 0$ and $u_2 > 0$.

If $A$ is non-cooperative with $b > 0, c < 0$ , $d > a$, then $a < \xi_2 < \xi_1 \;\Rightarrow\; (\xi_1 - a) > 0$, $u_2 > 0$.

If $A$ is non-cooperative with $b > 0, c < 0, a > d$, then $\xi_2 < \xi_1 < a \;\Rightarrow\; (\xi_1 - a) < 0$, $u_2 < 0$.

*Remark* 3. Indeed, we always assume that $b > 0$, hence $u_1 > 0$ for $\nu - \mu > 0$ small enough.

**Behavior of the solution near the eigenvalue** $\nu' := \Lambda - \xi_2$.

Obviously, $\nu' := \Lambda - \xi_2$ is also an eigenvalue of the system with associated eigenvector $X_2\phi$.

**Theorem 5.** *Assume* $(H_q')$, $b > 0$, $D > 0$ *and* $\nu' < \lambda_2$. *Assume also that* $f_1$ *and* $f_2$ *are in* $\mathcal{X}$ *and* $(\xi_1 - a)f_1^1 - bf_2^1 > 0$ *is satified. Then, for* $0 < \nu' - \mu$ *small enough, there exists a positive constant* $\gamma'$ *depending only on F and Matrix A such that*
*For cooperative systems,* ($c > 0$)*, then*

$$u_1, -u_2 \geq \frac{\gamma'}{\nu' - \mu}\phi > 0,$$

*For non-cooperative systems* ($c < 0$)*, then*

$$d > a \Rightarrow u_1, u_2 \geq \frac{\gamma'}{\nu' - \mu}\phi > 0. \tag{23}$$

$$a > d \Rightarrow u_1, -u_2 \geq \frac{\gamma'}{\nu - \mu}\phi > 0. \tag{24}$$

*If* $0 < \mu - \nu'$ *small enough, the sign are reversed.*

**Proof** The proof is exactly the same as for Theorem 4 except that we derive from (21) that $(\tilde{u}_1)^1$ stays bounded and $(\tilde{u}_2)^1 = \frac{(\tilde{f}_2)^1}{\nu' - \mu} \to \infty$ as $\nu' - \mu \to 0$. This holds also since $\mu + \xi_2 <$ $mu + \xi_1 < \Lambda < \lambda_2$. Now $u_1$ behaves as $b(\tilde{u}_2)$ and $u_2$ as $(\xi_2 - a)(\tilde{u}_2)$, and the result follows.

## 3.2. Case $\xi_1 = \xi_2$

Consider now the case where the coefficients of the matrix $A$ satisfy $b > 0$ and

$$D := (a - d)^2 + 4bc = 0.$$

Of course this implies $bc < 0$ and since $b > 0$ , then $c < 0$: only for non-cooperative systems a double root can appear. Now $\xi_1 = \xi_2 = \xi = \frac{a+d}{2}$. The proof of Theorem 4 is no more valid since *e.g.* in (19) there is a factor of the form $\frac{1}{\xi_1 - \xi_2}$. Moreover Matrix $J$ is triangular and the system in $\tilde{U}$ is no more decoupled. We prove here

**Theorem 6.** *Assume* $(H_q')$ *and* $b > 0, c < 0$ *with* $(a - d)^2 + 4bc = 0$*; assume also that* $f_1, f_2$ *are in* $\mathcal{X}$ *and :*

$$\frac{(a - d)}{2} f_1^1 + bf_2^1 > 0. \tag{25}$$

*If* $\nu - \delta < \mu < \nu + \delta$ ($\nu = \Lambda - \xi$)*,* $\delta$ *small enough, there exists a positive constant* $\gamma$ *such that*

*if* $a > d$ $\qquad u_1 \geq \frac{\gamma}{|\nu - \mu|}\phi, \quad u_2 \leq -\frac{\gamma}{|\nu - \mu|}\phi.$

*if* $d > a$ $\qquad u_1 \geq \frac{\gamma}{|\nu - \mu|}\phi, \quad u_2 \geq \frac{\gamma}{|\nu - \mu|}\phi.$

*Remark* 4. We notice that $u_1$ is always positive whatever the sign of $d-a$ or of $\nu-\mu$. Also $u_2$ keeps the same sign for $\mu$ going over $\nu$. Things work as having 2 eigenvalues $\xi_1$ and $\xi_2$ with $\xi_1 - \xi_2 \to 0$. The functions $u_1$ and $u_2$ change sign twice (as $\mu$ goes over $\nu$ and $\nu'$) and finally they keep the same sign.

*Remark* 5. Note that the condition $\frac{(a-d)}{2} f_1^1 + b f_2^1 > 0$ in Theorem 6 is the same than in Theorem 4: $(\xi_2 - a) f_1^1 < b f_2^1$, since in Theorem 6, $\xi_2 = \xi = \frac{a+d}{2}$.

**Proof** The eigenvector associated to eigenvalue $\xi$ is

$$X = \begin{pmatrix} b \\ \frac{d-a}{2} \end{pmatrix}.$$

The vector $X\phi$ is thus an eigenvector for $L - A$,

$$L(X\phi) - AX\phi = (\Lambda - \xi)X\phi = \nu X\phi.$$

We will need to use two different decompositions of the matrix $A$. For the decomposition 1 we choose

$$P_1 = \begin{pmatrix} b & \frac{2b}{a-d} \\ \frac{d-a}{2} & 0 \end{pmatrix}, \quad P_1^{-1} = \frac{1}{b} \begin{pmatrix} 0 & -\frac{2b}{a-d} \\ \frac{a-d}{2} & b \end{pmatrix}.$$

So the associated triangular matrix $J_1$ is

$$J_1 = P_1^{-1} A P_1 = \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}.$$

As above, setting $\tilde{U} = P_1^{-1} U$ and $\tilde{F} = P_1^{-1} F$, we derive from System $(S)$

$$L\tilde{U} = J_1 \tilde{U} + \mu \tilde{U} + \tilde{F}.$$

We do not have anymore a decoupled system but

$$\begin{cases} L\tilde{u}_1 &= (\xi+\mu)\tilde{u}_1 + \tilde{u}_2 + \tilde{f}_1 \\ L\tilde{u}_2 &= \qquad\qquad\quad + (\xi+\mu)\tilde{u}_2 + \tilde{f}_2; \end{cases} \tag{26}$$

here $\tilde{f}_1 = \frac{-2}{a-d} f_2$ and $\tilde{f}_2 = \frac{(a-d)}{2b} f_1^1 + f_2^1$ are in $\mathcal{X}$ and $\tilde{f}_2 > 0$ by (25).

• If $\xi + \mu < \Lambda$ (that is $\mu < \nu$), by Theorem 3 applied to the second equation, there exists a constant $K > 0$, such that $\tilde{u}_2 > \frac{K}{\nu-\mu}\phi$. Hence, for $\nu - \mu$ small enough for any $\tilde{f}_1 \in \mathcal{X}$, $\tilde{u}_2 + \tilde{f}_1 > 0$ and is in $X$; then again Theorem 3 applied to the first equation implies that there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.

For $a > d$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_1 \tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 + \frac{2b}{a-d}\tilde{u}_2 > \frac{\gamma}{\nu-\mu}\phi \\ u_2 = \frac{d-a}{2}\tilde{u}_1 < -\frac{\gamma}{\nu-\mu}\phi \end{cases}$$

• If $\mu > \nu$ we have reversed sign for $\tilde{u}_2$. Hence, for $\mu - \nu$ small enough for any $\tilde{f}_1 \in \mathcal{X}$, $\tilde{u}_2 + \tilde{f}_1 < 0$ and is in $X$; then again Theorem 3 for the first equation implies that there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\mu-\nu}\phi$.

For $d > a$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_1\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 + \frac{2b}{a-d}\tilde{u}_2 > \frac{\gamma}{\mu-\nu}\phi \\ u_2 = \frac{d-a}{2}\tilde{u}_1 > \frac{\gamma}{\mu-\nu}\phi \end{cases}$$

For the remaining cases, we need to use an other decomposition of Matrix $A$. For the decomposition 2 we choose

$$P_2 = \begin{pmatrix} b & 0 \\ \frac{d-a}{2} & 1 \end{pmatrix}, \quad P_2^{-1} = \frac{1}{b}\begin{pmatrix} 1 & 0 \\ \frac{a-d}{2} & b \end{pmatrix}.$$

So the associated triangular matrix $J_2$ is

$$J_2 = P_2^{-1}AP_2 = \begin{pmatrix} \xi & 1 \\ 0 & \xi \end{pmatrix}.$$

As above, setting $\tilde{U} = P_2^{-1}U$ and $\tilde{F} = P_2^{-1}F$, we derive from System $(S)$ the same system with the same function $\tilde{f}_2 = \dfrac{(a-d)}{2b}f_1 + f_2$ :

$$\begin{cases} L\tilde{u}_1 = (\xi+\mu)\tilde{u}_1 + \tilde{u}_2 + \tilde{f}_1 \\ L\tilde{u}_2 = \quad + (\xi+\mu)\tilde{u}_2 + \tilde{f}_2 \end{cases} \tag{27}$$

• If $\xi + \mu < \Lambda$ (that is $\mu < \nu$), since $\frac{(a-d)}{2b}f_1^1 + f_2^1 > 0$, we get (exactly as for decomposition 1) that there exists a constant $K > 0$, such that $\tilde{u}_2 > \frac{K}{\nu-\mu}\phi$ and there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.
For $d > a$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_2\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 > \frac{\gamma}{\nu-\mu}\phi \\ u_2 = \frac{d-a}{2}\tilde{u}_1 + \tilde{u}_2 > \frac{\gamma}{\nu-\mu}\phi \end{cases}$$

• If $\mu > \nu$ we have reversed sign for $\tilde{u}_2$. Hence, there exists a constant $K' > 0$, such that $\tilde{u}_1 > \frac{K'}{\nu-\mu}\phi$.
For $a > d$, we can conclude that there exists a constant $\gamma > 0$,

$$U = P_2\tilde{U} = \begin{cases} u_1 = b\tilde{u}_1 > \frac{\gamma}{\mu-\nu}\phi \\ u_2 = \frac{d-a}{2}\tilde{u}_1 + \tilde{u}_2 < -\frac{\gamma}{\mu-\nu}\phi \end{cases}$$

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