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Sign of the solution to a non-cooperative system

ABSTRACT. Combining the results of a recent paper by Fleckinger-Hernández-de Thélin [14] for a non cooperative 2×2 system with the method of PhD Thesis of M. H. Lécureux we compute the sign of the solutions of a $n \times n$ non-cooperative systems when the parameter varies near the lowest principal eigenvalue of the system.

KEY WORDS. Maximum Principle, Antimaximum Principle, Elliptic Equations and Systems, Non Cooperative Systems, Principal Eigenvalue.

1 Introduction

Many results have been obtained since decades on Maximum Principle and Antimaximum principle for second order elliptic partial differential equations involving *e.g.* Laplacian, p-Laplacian, Schrödinger operator, ... or weighted equations. Then most of these results have been extended to systems.

The maximum principle (studied since centuries) has many applications in various domains as physic, chemistry, biology, ... Usually it shows that for positive data the solutions are positive (positivity is preserved). It is generally valid for a parameter below the "principal" eigenvalue (the smallest one). The Antimaximum principle, introduced in 1979 by Clément and Peletier ([8]), shows that, for one equation, as this parameter goes through this principal eigenvalue, the sign are reversed; this holds only for a small interval. The original proof relies on a decomposition into the groundstate (principal eigenfunction of the operator) and its orthogonal. It is the same idea which has been used in [14] (combined with a bootstrap method) to derive a precise estimate for the validity interval of the Antimaximum principle for one equation. By use of this result, Fleckinger-Hernández-de Thélin ([14]) deduce results on the sign of solution for some 2×2 non-cooperative systems. Indeed many papers have appeared for cooperative systems involving various elliptic operators: ([1], [2], [4], [9], [10],

[11], [12], [13], ...). Concerning non cooperative systems the literature is more restricted ([7], [14], ...).

In this paper we extend the results obtained in [14], valid for 2×2 non-cooperative systems involving Dirichlet Laplacian, to $n \times n$ ones. Recall that a system is said to be "cooperative" if all the terms outside the diagonal of the associated square matrix are positive.

For this aim we combine the precise estimate for the validity interval of the antimaximum principle obtained in [14] with the method used in [15], [1] for systems.

In Section 2 we are concerned with one equation. We first recall the precise estimate for the validity interval for the antimaximum principle ([14]); then we give some related results used in the study of systems.

In Section 3 we first state our main results for a $n \times n$ system (eventually non-cooperative) and then we prove them.

Finally, in Section 4, we compare our results with the ones of [14]. Our method, which uses the matricial calculus and in particular Jordan decomposition, allows us to have a more general point of view, even for a 2×2 system.

2 Results for one equation:

In [14], the authors consider a non-cooperative 2×2 system with constant coefficients. Before studying the system they consider one equation and establish a precise estimate of the validity interval for the antimaximum principle. We recall this result that we use later.

2.1 A precise Antimaximum for the equation [14]

Let Ω be a smooth bounded domain in $I\!\!R^N$. Consider the following Dirichlet boundary value problem

$$-\Delta z = \sigma z + h \text{ in } \Omega\,, \quad z = 0 \text{ on } \partial\Omega, \tag{2.1}$$

where σ is a real parameter.

The associated eigenvalue problem is

$$-\Delta \phi = \lambda \phi \text{ in } \Omega\,, \quad \phi = 0 \text{ on } \partial\Omega. \tag{2.2}$$

As usual, denote by $0 < \lambda_1 < \lambda_2 \leq \ldots$ the eigenvalues of the Dirichlet Laplacian defined on Ω and by ϕ_k a set of orthonormal associated eigenfunctions, with $\phi_1 > 0$.

Hypothesis 1 *Assume $h \in L^q$, $q > N$ if $N \geq 2$ and $q = 2$ if $N = 1$.*

Hypothesis 2 *Assume $h^1 := \int h\phi_1 > 0$.*

Writing

$$h = h^1\phi_1 + h^\perp \tag{2.3}$$

where $\int_\Omega h^\perp \phi_1 = 0$ one has:

Lemma 2.1 [14] *We assume $\lambda_1 < \sigma \leq \Lambda < \lambda_2$ and $h \in L^q$, $q > N \geq 2$. We suppose that there exists a constant C_1 depending only on Ω, q, and Λ such that z satisfying (2.1) is such that*

$$\|z\|_{L^2} \leq C_1\|h\|_{L^2}. \tag{2.4}$$

Then there exist constants C_2 and C_3, depending only on Ω, q and Λ such that

$$\|z\|_{C^1} \leq C_2\|h\|_{L^q} \text{ and } \|z\|_{L^q} \leq C_3\|h\|_{L^q}. \tag{2.5}$$

Remark 2.1 The same result holds for $\Lambda < \sigma < \lambda_1$ where Λ is any given constant $< \lambda_1$, with the same proof.

Remark 2.2 Inequality (2.4) cannot hold, for all $\lambda_1 < \sigma \leq \Lambda$, unless h is orthogonal to ϕ_1.

Theorem 1 [14]*: Assume Hypotheses 1 and 2; fix Λ such that $\lambda_1 < \sigma \leq \Lambda < \lambda_2$. There exists a constant K depending only on Ω, Λ and q such that, for $\lambda_1 < \sigma < \lambda_1 + \delta(h)$ with*

$$\delta(h) = \frac{Kh^1}{\|h^\perp\|_{L^q}}, \tag{2.6}$$

the solution z to (2.1) satisfies the antimaximum principle, that is

$$z < 0 \text{ in } \Omega; \quad \partial z/\partial\nu > 0 \text{ on } \partial\Omega, \tag{2.7}$$

where $\partial/\partial\nu$ denotes the outward normal derivative.

2.2 Other remarks for one equation

Consider again Equation (2.1). For $\sigma \neq \lambda_k$, z solution to (2.1) is

$$z \;=\; z^1\phi_1 + z^\perp \;=\; \frac{h^1}{\lambda_1 - \sigma}\phi_1 \;+\; z^\perp, \tag{2.8}$$

with $z^\perp$ satisfying

$$-\,\Delta z^\perp \;=\; \sigma z^\perp + h^\perp \;\text{ in } \Omega\,; \quad z^\perp = 0 \text{ on } \partial\Omega. \tag{2.9}$$

In the next section, our proofs will use the following result.

Lemma 2.2 *We assume Hypothesis 1 and $\sigma < \lambda_1$. Then $z^\perp$ (and its first derivatives) is bounded: There exits a positive constant C_0, independent of σ such that*

$$\|z^\perp\|_{\mathcal{C}^1} \leq C_0 \|h\|_{L^q}. \tag{2.10}$$

Moreover, if $\sigma < \Lambda < \lambda_1$, where Λ is some given constant $< \lambda_1$, z is bounded and there exits a positive constant C_0', independent of σ such that

$$\|z\|_{\mathcal{C}^1} \leq C_0' \|h\|_{L^q}. \tag{2.11}$$

Proof: This is a simple consequence of the variational characterization of λ_2:

$$\lambda_2 \int_\Omega |z^\perp|^2 \leq \int_\Omega |\nabla z^\perp|^2 = \sigma \int_\Omega |z^\perp|^2 + \int_\Omega z^\perp h^\perp \leq \lambda_1 \int_\Omega |z^\perp|^2 + \int_\Omega z^\perp h^\perp.$$

By Cauchy-Schwarz we deduce

$$\|z^\perp\|_{L^2} \leq \frac{1}{\lambda_2 - \lambda_1} \|h^\perp\|_{L^2}. \tag{2.12}$$

This does not depend on $\sigma < \lambda_1$.

Then one can deduce (2.10), that is $z^\perp$ (and its derivatives) is bounded. This can be found *e.g.* in [6] (for $\sigma < \lambda_1$ and $\lambda_1 - \sigma$ small enough) or it can be derived exactly as in [14] (where the case $\sigma > \lambda_1$ and $\sigma - \lambda_1$ small enough is considered).

Finally we write $z = z_1\phi_1 + z^\perp$ and deduce (2.11).

Remark 2.3 Note that in (2.8), since $h^1 > 0$, $\frac{h^1}{\lambda_1 - \sigma} \to +\infty$ as $\sigma \to \lambda_1$, $\sigma < \lambda_1$.

3 Results for a $n \times n$ system:

We consider now a $n \times n$ (eventually non-cooperative) system defined on Ω a smooth bounded domain in $I\!R^N$:

$$-\Delta U = AU + \mu U + F \text{ in } \Omega\,,\ U = 0 \text{ on } \partial\Omega, \tag{S}$$

where F is a column vector with components f_i, $1 \leq i \leq n$. Matrix A is not necessarily cooperative, that means that its terms outside the diagonal are not necessarily positive. First we introduce some notations concerning matrices. Then, with these notations we can state our results and prove them.

3.1 The matrix of the system and and the eigenvalues

Hypothesis 3 *A is a $n \times n$ matrix which has constant coefficients and has only real eigenvalues. Moreover, the largest one which is denoted by ξ_1 is positive and algebrically and geometrically simple. The associated eigenvectors X_1 has only non zero components.*

Of course some of the other eigenvalues can be equal. Therefore we write them in decreasing order

$$\xi_1 > \xi_2 \geq \ldots \geq \xi_n. \tag{3.13}$$

The eigenvalues of $A = (a_{ij})_{1 \leq i,j \leq n}$, denoted , ξ_1, ξ_2,. . . , ξ_n , are the roots of the associated characteristic polynomial

$$p_A(\xi) = det(\xi I_n - A) = \prod(\xi - \xi_k), \tag{3.14}$$

where I_n is the $n \times n$ identity matrix.

Remark 3.1 By above, $\xi > \xi_1 \Rightarrow p_A(\xi) > 0$.

Denote by $X_1 \ldots X_n$ the eigenvectors associated respectively to eigenvalue $\xi_1, \ldots, \xi_n$.

Jordan decomposition Matrix A can be expressed as $A = PJP^{-1}$, where $P = (p_{ij})$ is the change of basis matrix of A and J is the Jordan canonical form (lower triangular matrix) associated with A. The diagonal entries of J are the ordered eigenvalues of A and $p_A(\xi) = p_J(\xi)$.

Notation : In the following, set

$$U = P\tilde{U} \Leftrightarrow \tilde{U} = P^{-1}U, \; F = P\widetilde{F} \Leftrightarrow \tilde{F} = P^{-1}F. \tag{3.15}$$

Here $\widetilde{U}$ and $\widetilde{F}$ are column vectors with components $\widetilde{u}_i$ and $\widetilde{f}_i$.

Eigenvalues of the system: μ is an eigenvalue of the system if there exists a non zero solution U to

$$-\Delta U = AU + \mu U \text{ in } \Omega \,, \; U = 0 \text{ on } \partial\Omega. \tag{S_0}$$

We also say that μ is a "principal eigenvalue" of System (S) if it is an eigenvalue with components of the associated eigenvector which does not change sign. (Note that the components do not change sign but are not necessarily positive as claimed in [14]).

Then $\phi_j X_k$ is an eigenvector associated to eigenvalue

$$\mu_{jk} = \lambda_j - \xi_k. \tag{3.16}$$

3.2 Results for $|\mu - \mu_{11}| \to 0$

We study here the sign of the component of U as $\mu \to \mu_{11} = \lambda_1 - \xi_1$.

For this purpose we use the methods in [15] or [1] combined with [14]. Note that by (3.13), $\mu_{11} < \mu_{1k} = \lambda_1 - \xi_k$, for all $2 \le k \le n$.

Hypothesis 4 *F is with components $f_i \in L^q$, $q > N > 2$, $q = 2$ if $N = 1$, $1 \le i \le n$; moreover we assume that the first component $\tilde{f}_1$ of $\tilde{F} = P^{-1}F$ is $\ge 0, \not\equiv 0$.*

Theorem 2 *Assume Hypothesis 3 and 4. Assume also $\mu < \mu_{11}$. Then, there exists $\delta > 0$ independant of μ, such that for $\mu_{11} - \delta < \mu < \mu_{11}$, the components u_i of the solution U have the sign of p_{i1} and the outside normal derivatives $\frac{\partial u_i}{\partial \nu}$ have the sign of $-p_{i1}$.*

Theorem 3 *Assume Hypothesis 3 and 4 are satisfied; then, there exists $\delta > 0$ independant of μ such that for $\mu_{11} < \mu < \mu_{11} + \delta$ the components u_i of the solution U have the sign of $-p_{i1}$ and their outgoing normal derivatives have opposite sign.*

Remark 3.2 The results of Theorems 2 and 3 are still valid if we assume only $\int_\Omega \tilde{f}_1 \phi_1 > 0$ instead of $\tilde{f}_1 \ge 0 \not\equiv 0$.

3.3 Proofs

We start with the proof of Theorem 2 where $\mu < \mu_{11}$; assume Hypotheses 3 and 4.

3.3.1 Step 1: An equivalent system

We follow [15] or [1]. As above set $U = P\tilde{U}$ and $F = P\tilde{F}$.

Starting from

$$-\Delta U = AU + \mu U + F,$$

multiplying by P^{-1}, we obtain

$$-\Delta \tilde{U} = J\tilde{U} + \mu\tilde{U} + \tilde{F}.$$

Note that everywhere we have the homogeneous Dirichlet boundary conditions, but we do not write them for simplicity.

The Jordan matrix J has p Jordan blocks J_i $(1 \le i \le p \le n)$ which are $k_i \times k_i$ matrices of the form

$$J_i = \begin{pmatrix} \xi_i & 0 & \cdots & 0 \\ 1 & \xi_i & 0 & \cdots \\ \ddots & \ddots & \vdots & \\ 0 & \cdots 1 & \xi_i & 0 \\ 0 & \cdots & 1 & \xi_i \end{pmatrix}.$$

By Hypothesis 3, the first block is 1×1 : $J_1 = (\xi_1)$. Hence we obtain the first equation

$$-\Delta\widetilde{u_1} = \xi_1\widetilde{u_1} + \mu\widetilde{u_1} + \tilde{f}_1. \tag{3.17}$$

Since $\tilde{f}_1 \geq 0$, $\not\equiv 0$, $\xi_1 + \mu < \lambda_1$ and by Hypothesis 4, $\tilde{f}_1 \in L^2$, we have the maximum principle and

$$\widetilde{u_1} > 0 \ on\ \Omega.\ \frac{\widetilde{u_1}}{\partial\nu}|_{\partial\Omega} < 0. \tag{3.18}$$

Then we consider the second Jordan blocks J_2 which is a $k_2 \times k_2$ matrix with first line

$$\xi_2,\ 0,\ 0, \ldots$$

The first equation of this second block is

$$-\Delta\tilde{u}_2 = \xi_2\tilde{u}_2 + \mu\tilde{u}_2 + \tilde{f}_2.$$

Since $\mu < \mu_{11} = \lambda_1 - \xi_1 < \lambda_1 - \xi_2 \leq \lambda_1 - \xi_k$, $k \geq 2$. Hence, by Lemma 2.2, $\widetilde{u_2}$ stays bounded as $\mu \to \mu_{11}$. and this holds for all the $\widetilde{u_k}$, $k > 1$. By induction $\widetilde{u_k}$ is bounded for all k.

3.3.2 Step 2: End of the proof of Theorem 2

Now we go back to the functions u_i: $U = P\tilde{U} = (u_i)$ implies that for each u_i, $1 \leq i \leq n$, we have

$$u_i = p_{i1}\widetilde{u_1} + \sum_{j=2}^{n} p_{ij}\widetilde{u_j}. \tag{3.19}$$

The last term in (3.19) stays bounded according to Lemma 2.2; indeed $\sum_{j=2}^{n} p_{ij}\widetilde{u_j}$ is bounded by a constant which does not depend on μ.

By Remark 2.3, $\widetilde{u_1} \to +\infty$ as $\mu \to \lambda_1 - \xi_1$. Hence, each u_i has the same sign than p_{i1} (the first coefficient of the $i - th$ line in matrix P which is also the i-th coefficient of the first eigenvector X_1) for $\lambda_1 - \xi_1 - \mu > 0$ small enough. Analogously, $\frac{\partial u_i}{\partial\nu}$ behaves as $p_{i1}\frac{\partial\tilde{u}_i}{\partial\nu}$ which has the sign of $-p_{i1}$.

It is noticeable that only $\widetilde{u_1}$ plays a role!! □

3.4 Proof of Theorem 3 ($\mu > \mu_{11}$)

Now $\mu_{11} < \mu < \mu_{11} + \epsilon$ where $\epsilon \leq min\{\xi_1 - \xi_2, \lambda_2 - \lambda_1\}$ and $f_i \in L^q, q > N$. We proceed as above but deduce immediately that for $\mu - \mu_{11}$ small enough ($\mu - \mu_{11} < \delta_1 := \delta(\widetilde{f_1}) < \frac{K\widetilde{f}_1^1}{\|f_1^\perp\|_{L^q}}$) defined in [14], Theorem 1), $\widetilde{u}_1 < 0$ by the antimaximum principle. From now on choose

$$\mu - \mu_{11} < \delta, \text{ with } \delta < min\{\epsilon, \delta_1\}. \tag{3.20}$$

For the other equations, by Lemma 2.1, $\widetilde{u}_k > 0$ is bounded as above.

We consider now U. We notice that $F = P\widetilde{F}$ which can also be written $f_i = \sum_{k=1}^n p_{ik}\widetilde{f}_k$ implies $f_i^{\perp} = \sum_{k=1}^n p_{ik}\widetilde{f}_k^{\perp}$. With the same argument as above, the components u_i of the solution U have the sign of $-p_{i1}$ for $\mu - \mu_{11}$ sufficiently small ($\mu - \mu_{11} < \delta$). The normal derivatives of the u_i are of opposite sign. □

4 Annex: The 2 × 2 non-cooperative system

We apply now our results to the 2×2 system, considered in [14]. Consider the 2×2 non-cooperative system depending on a real parameter μ

$$-\Delta U = AU + \mu U + F \text{ in } \Omega\,,\ U = 0 \text{ on } \partial\Omega, \tag{S}$$

which can also be written as

$$-\Delta u \;=\; au \;+\; bv \;+\; \mu u \;+\; f \;\text{ in } \Omega, \tag{S_1}$$

$$-\Delta v \;=\; cu \;+\; dv \;+\; \mu v \;+\; g \;\text{ in } \Omega, \tag{S_2}$$

$$u = v = 0 \text{ on } \partial\Omega. \tag{S_3}$$

Hypothesis 5 *Assume* $b > 0\,, c < 0,$ *and* $D := (a-d)^2 + 4bc > 0.$

Here System (S) has (at least) two principal eigenvalues μ_1^- and μ_1^+ where

$$\mu_1^- \;:=\; \lambda_1 - \xi_1 \;<\; \mu_1^+ \;:=\; \lambda_1 - \xi_2, \tag{4.21}$$

where ξ_1 and ξ_2. are the eigenvalues of Matrix A and we choose $\xi_1 > \xi_2$.

The main theorems in [14] are:

Theorem 4 ([14]) *Assume Hypothesis 5,* $\mu_1^- < \mu < \mu_1^+$ *and* $d < a$. *Assume also*

$$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0,\ f, g \in L^q,\ q > N \,if\, N \geq 2\,;\ q = 2\,if\, N = 1.$$

Then there exists $\delta > 0$*, independent of* μ*, such that* $\mu < \mu_1^- + \delta$ *implies*

$$u < 0,\ v > 0 \;in\; \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} < 0 \;on\; \partial\Omega.$$

Theorem 5 ([14]) *Assume Hypothesis 5,* $\mu_1^- < \mu < \mu_1^+$ *and* $a < d$. *Assume also*

$$f \leq 0,\ g \geq 0,\ f, g \not\equiv 0\,,\ f, g \in L^q,\ q > N \,if\, N \geq 2\,;\ q = 2\,if\, N = 1.$$

Then there exists $\delta > 0$*, independent of* μ*, such that i* $\mu < \mu_1^- + \delta$ *implies*

$$u < 0,\ v < 0 \;in\; \Omega;\ \frac{\partial u}{\partial \nu} > 0,\ \frac{\partial v}{\partial \nu} > 0 \;on\; \partial\Omega.$$

Theorem 6 ([14]) *Assume Hypothesis 5 and $a < d$. Assume also that the parameter μ satisfies: $\mu < \mu_1^-$, and*

$$f \geq 0,\ g \geq 0,\ f, g \not\equiv 0,\ f, g \in L^2.$$

Assume also $t^*g - f \geq 0$, $t^*g - f \not\equiv 0$ with

$$t^* = \frac{d - a + \sqrt{D}}{-2c}.$$

Then

$$u > 0,\ v > 0 \text{ in } \Omega;\ \frac{\partial u}{\partial \nu} < 0,\ \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega.$$

The matrix A is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with eigenvalues $\xi_2 = \frac{a+d-\sqrt{D}}{2} < \xi_1 = \frac{a+d+\sqrt{D}}{2}$ where $D = (a-d)^2 + 4bc > 0$. The eigenvectors are

$$X_k = \begin{pmatrix} b \\ \xi_k - a \end{pmatrix},\quad P = \begin{pmatrix} b & b \\ \xi_1 - a & \xi_2 - a \end{pmatrix}.$$

Note that the characteristic polynomial is $\mathcal{P}(s) = (a - s)(d - s) - bc$. Since $\mathcal{P}(a) = \mathcal{P}(d) = -bc > 0$, a and d are outside $[\xi_2, \xi_1]$.

For $d > a$ both $p_{i1} > 0$ and for $d < a$ $p_{11} > 0$, $p_{21} < 0$.

$$P^{-1} = \frac{1}{b(\xi_1 - \xi_2)} \begin{pmatrix} a - \xi_2 & b \\ \xi_1 - a & -b \end{pmatrix}.$$

$$\tilde{f}_1 = \frac{1}{b(\xi_1 - \xi_2)}[(a - \xi_2)f + bg]. \tag{4.22}$$

In Theorem 2 of [14] $d < a$, $f, g \geq 0$ so that $\tilde{f}_1 > 0$ and u has the sign of $-p_{11} = -b < 0$; v has the sign of $-p_{21} = a - \xi_1 > 0$.

In Theorem 3 of [14] $d > a$, $f \leq 0$ and $g \geq 0$ implies $\tilde{f}_1 > 0$. So that u has the sign of $-p_{11} = -b < 0$; v has the sign of $-p_{12} = a - \xi_2 < 0$.

Finally the hypothesis $\tilde{f}_1 \geq 0$ is sufficient for having the sign of the solutions and the maximum principle holds (all $u_i > 0$) iff $p_{i1} > 0$.

Our results can conclude for other cases; *e.g*, as in Theorem 2, $d < a$, $f \geq 0$, but now $g < 0$ with $\tilde{f}_1 = \frac{1}{b(\xi_1 - \xi_2)}[(a - \xi_2)f + bg] > 0$.

Analogously, in Theorem 4, $f, g \geq 0$ and $\tilde{f}_1 > 0$ implies for having $u, v > 0$ that necessarily $\xi_2 - a > 0$ so that $a < d$. But again we can conclude for the sign in other cases (*e.g.* $a > d$) if only $\tilde{f}_1 > 0$, (which is precisely the added condition in Theorem 4). □

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