Entry-Proofness and Discriminatory Pricing under Adverse Selection*

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Abstract

This paper studies competitive allocations under adverse selection. We first provide a general necessary and sufficient condition for entry on an inactive market to be unprofitable. We then use this result to characterize, for an active market, a unique budget-balanced allocation implemented by a market tariff making additional trades with an entrant unprofitable. Motivated by the recursive structure of this allocation, we finally show that it emerges as the essentially unique equilibrium outcome of a discriminatory ascending auction. These results yield sharp predictions for competitive nonexclusive markets.

Keywords: Adverse Selection, Entry-Proofness, Discriminatory Pricing, Nonexclusive Markets, Ascending Auctions.
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1 Introduction

It has long been recognized that markets subject to adverse selection can unravel to a no-trade equilibrium. As shown by Akerlof (1970), this can occur even when trade would always be mutually beneficial if the quality of the goods traded were commonly known. This failure of the price mechanism has been recently invoked to explain phenomena such as insurance rejections (Hendren (2013)) and to justify public intervention in the presence of liquidity or credit freezes (Philippon and Skreta (2012), Tirole (2012)). In this paper, we build on this insight to provide a new characterization of competitive allocations under adverse selection, based on the standard premise that a perfectly competitive market should be immune to entry.

To this end, we consider a general adverse-selection economy in which little structure is imposed on the buyers' preferences. This setting encompasses insurance economies as well as standard trade environments, with or without wealth effects. The main restrictions are a single-crossing condition and a monotonicity condition on costs implying weak adverse selection, in the sense that buyer types who are more willing to make larger purchases are on average more costly to serve. The corresponding expected costs turn out to play a key role in the description of market outcomes, as in Akerlof (1970).

In this context, entry-proofness provides a tractable and detail-free alternative to the strategic approaches adopted in the literature. We apply this requirement to prove two theorems that respectively pertain to active and inactive markets, depending on whether or not trade opportunities are available on the market. At the core of our approach is a unified treatment of these two cases.

Theorem 1 states a necessary and sufficient condition for entry to be unprofitable on an inactive market, which generalizes the market-unraveling condition first formulated by Akerlof (1970) and recently extended by Hendren (2013) to Rothschild and Stiglitz (1976) insurance economies. The intuition is as follows. Under weak adverse selection, the cost of selling a unit of the good depends on the buyer types who purchase it; on average, the cost is the upper-tail conditional expectation of unit costs, starting from the first type who purchases this unit. Our entry-proofness (EP) condition then simply states that the willingness-to-pay of each type at the no-trade point should not exceed this cost. The necessity of Condition EP for entry-proofness is straightforward and only requires the use of single-contract offers. By contrast, its sufficiency must account for an entrant's ability to offer a menu of contracts; we identify a weak but tight assumption on the buyers' preferences under which entry with a menu of contracts is unprofitable as soon as entry with a single contract is unprofitable.

We next turn to active markets. Rothschild and Stiglitz (1976) have characterized the set of exclusive contracts preventing an entrant from making a profit; we perform a similar analysis for the case of nonexclusive contracts, whereby each buyer can privately trade with several sellers.

It should be stressed from the outset that we assume trade to be nonanonymous: that is, each seller can perfectly monitor all the trades each buyer makes with him, although he can monitor none of the trades this buyer makes with his competitors.[1] Therefore, the notion of nonexclusivity we use in this paper differs from the traditional one, akin to anonymity; in particular, nonanonymity enables price discrimination.

However, price discrimination is harder under nonexclusivity than under exclusivity as a seller does not observe a buyer's aggregate trade: for instance, a buyer may purchase a large quantity by splitting it between several sellers. Under weak adverse selection, this is of concern to the sellers, because buyer types with higher demands are on average more costly to serve. One way to hedge against this risk is for each seller to post a convex tariff that prices successive marginal quantities at an increasing rate.

Our analysis initially abstracts from the determination of individual tariffs and directly imposes three properties on the market tariff obtained from them by aggregation. First, we require that the market tariff be convex, which is true if each seller posts a convex tariff. Second, we require that the market tariff implement a budget-feasible allocation, which must be the case if each seller on the market is to earn a nonnegative expected profit. Third, we require that the market tariff be entry-proof, which is our stability condition for market outcomes. Under nonexclusive contracting, this means that no entrant can make a profit by offering a menu of contracts, given that each buyer is free to combine a contract offered by the entrant with a trade along the market tariff.

The convexity assumption is analytically convenient because it allows us to use Theorem 1 to characterize entry-proof markets tariffs. The basic idea is to factor all the trade opportunities available on the market into the buyers' preferences. Indeed, from an entrant's viewpoint, everything happens as if he were facing an inactive market on which the buyers' preferences for additional trades are represented by indirect utility functions incorporating their optimal trades along the market tariff. The key point is that, when this tariff is convex, these indirect utility functions inherit single-crossing from the primitive utility functions. This, in turn, implies that the entrant faces weak adverse selection, exactly like on an inactive market. As we argue in Section 4, the convexity assumption can nevertheless be

[1]For instance, an insurance company selling life-insurance or annuity contracts can perfectly identify each of its customers, and thus is fully aware of all of their mutual contractual obligations; it does not, however, observe the contracts its customers may enter into with other companies.

significantly relaxed without affecting our results.

Theorem 2 singles out a unique budget-feasible allocation implemented by an entry-proof convex market tariff, and an essentially unique such tariff; existence obtains under very general conditions. This market tariff is typically nonlinear, reflecting the nonanonymity of trade. Specifically, each layer along this tariff is priced at the expected cost of serving the types who optimally choose to trade it, so that the corresponding expected profit is zero; under weak adverse selection, this cost is equal to the upper-tail conditional expectation of unit costs, starting from the marginal type. When the buyers' preferences are linear, subject to a capacity constraint, these properties lead to Akerlof (1970) pricing and to the competitive-equilibrium allocation that maximizes the gains from trade. When the buyers' preferences are strictly convex, these properties lead to a marginal version of Akerlof (1970) pricing and to an allocation generalizing those highlighted, in specific contexts, by Jaynes (1978), Hellwig (1988), and Glosten (1994). We will accordingly refer to the *JHG allocation* and to the *JHG tariff*.

A noticeable feature of the JHG allocation is its recursive structure. On the first layer, the price is the expected cost of serving all types, and the quantity supplied is exactly the demand of the first type at this price. Indeed, supplying less would inefficiently ration demand, while supplying more would entail losses on the excess quantity. On the second layer, the first type is no longer active, and the same reasoning applies: the price is the expected cost of serving all types except the first, and the quantity supplied is exactly the residual demand of the second type at this price—and so on. Overall, the quantity supplied on each layer matches the residual demand of the marginal type, at a price equal to expected cost. In short, the JHG allocation is competitive.

The existence, uniqueness, and competitive features of the JHG allocation are arguably strong arguments in favor of using entry-proofness as a conceptual tool for predicting the outcomes of nonexclusive markets under adverse selection. However, this approach remains silent on how to implement this allocation in a decentralized way, because it does not explain how the JHG tariff comes into existence. It is thus natural to ask whether the JHG allocation and the JHG tariff can be derived as the equilibrium outcome of a game in which strategic sellers compete to serve the buyers' demand.

In this respect, the recursive structure of the JHG allocation suggests a setting in which competition takes place sequentially, layer by layer. To validate this intuition, we model the strategic interactions between sellers as a discriminatory ascending auction. Prices are quoted sequentially, in increasing order, and according to a discrete price grid with a minimum tick size. Each time a new price is quoted, each seller publicly announces the maximum quantity he stands ready to trade with each buyer at this price. Once this process is completed, each buyer selects the quantity she wishes to purchase from each seller at each price, according to her type. As it is optimal to take up the best price offers first, each buyer effectively faces a convex market tariff.

These simple trading rules define a standard extensive-form game with almost-perfect information. Our main results are encapsulated in two theorems. Theorem 3 exhibits a simple equilibrium in which, at each price and in each subgame, the sellers equally share the profitable residual demand. The resulting aggregate equilibrium allocation converges to the JHG allocation when the tick size goes to zero. Theorem 4 then reinforces this result by showing that, modulo a natural refinement, any sequence of aggregate equilibrium allocations converges to the JHG allocation when the tick size goes to zero. Thus the JHG allocation emerges as the essentially unique outcome of competition when each seller can quickly react to his competitors' offers. These positive results invite us to reconsider the role of sequential trading for financial and insurance markets.

Contributions to the Literature

Theorem 1 generalizes results obtained by Akerlof (1970), Glosten (1994), and Mailath and Nöldeke (2008) in the quasilinear case, and by Hendren (2013) in the case of a Rothschild and Stiglitz (1976) economy. Our contribution is to state a general necessary and sufficient condition for an inactive market to be entry-proof, to point out a technical condition on preferences that has been so far overlooked, and to provide a comprehensive yet elementary proof that may be useful for pedagogical purposes.

The unique allocation that survives entry in a nonexclusive market on which supply is described by a convex tariff corresponds to the allocations characterized by Akerlof (1970) in the case of an indivisible good, and by Jaynes (1978), Hellwig (1988), and Glosten (1994) in the case of a divisible good. Beyond extending these results to general preferences, our contribution is to apply our results on inactive markets to active markets, exploiting the idea that a nonexclusive tariff is entry-proof if and only if no additional trades are both incentive-feasible and profitable. This approach allows for a unified treatment of active and inactive markets, and of linear and strictly convex preferences.

Entry-proofness in exclusive markets has been well understood since Rothschild and Stiglitz (1976). The unique candidate is the Riley (1979) allocation, characterized by the absence of cross-subsidies between types and downward-binding local incentive-compatibility constraints. However, this allocation generally fails to be entry-proof when there are many types (Riley (1985)). The main difference with the nonexclusive markets studied in this paper is that the buyers' indirect utility functions induced by an exclusive tariff do not satisfy single-crossing, so that an entrant can engage in cream-skimming without worrying about adverse selection. By contrast, single-crossing is satisfied under nonexclusivity as long as the market tariff is convex; as a result, cream-skimming is impossible, and this explains why an entry-proof market tariff always exists.

Despite the renewed interest for nonexclusive markets under adverse selection, there is no consensus in the literature about the corresponding competitive outcomes. Early work was based on the idea that nonexclusivity is best represented by assuming linear pricing. Applying this idea in the context of insurance, Pauly (1974) generalizes Akerlof (1970) by showing that the equilibrium price of coverage is equal to the average riskiness of consumers, weighted by their demands for coverage.[2] Linear pricing follows from the usual argument that consumers can avoid price discrimination by trading many small contracts with different sellers (Chiappori (2000)). Notice that this requires a form of anonymity, which creates difficulties of its own; for instance, Bisin and Gottardi (1999, 2003) argue that the existence of a competitive equilibrium may require a minimal degree of nonlinear pricing in the form of bid-ask spreads or entry fees.

An alternative route towards the characterization of equilibrium trades has been recently taken by considering competitive-screening games in which sellers simultaneously offer menus of contracts, or nonlinear tariffs, from which a buyer is free to choose according to her private information (Peters (2001), Martimort and Stole (2002)). One of the goals of this oligopolistic approach was to build a strategic model of a discriminatory limit-order book in which market makers place limit orders that are executed in order of price priority.

In this spirit, Biais, Martimort, and Rochet (2000) construct an equilibrium in convex tariffs in a setting where the buyer has strictly convex preferences and the distribution of types is continuous. The equilibrium market tariff is not entry-proof, but it converges to the JHG tariff when the number of sellers grows large. This sounds promising, but Attar, Mariotti, and Salanié (2014, 2019) argue that discretizing the distribution of types leads to a completely different picture: the unique candidate-equilibrium allocation is now the JHG allocation, but it can be supported in equilibrium only in the extreme case where it features a single layer.[3] These discontinuity and existence problems make the equilibrium predictions of competitive-screening games somewhat fragile, as they ultimately hinge on fine modeling details.[4] By contrast, focusing on entry-proof market tariffs leads to a sharp and robust

[2]This formula is popular in the annuity literature (Sheshinski (2008), Hosseini (2015), Rothschild (2015)).

[3]As shown by Attar, Mariotti, and Salanié (2011), this special case notably arises when the buyer's preferences are linear, subject to a capacity constraint.

[4]The findings in Back and Baruch (2013) and Biais, Martimort, and Rochet (2013) illustrate that the

prediction for nonexclusive competitive markets, which may be seen as a natural extension of Akerlof (1970) to the case of a divisible good and general preferences.

The recursive structure of the JHG allocation has motivated us to design an ascending discriminatory auction in which the market tariff is built sequentially. This contrasts with competitive-screening games, which can be interpreted as discriminatory auctions in which sellers simultaneously bid at all prices. The advantage of a sequential auction lies in its transparency, a point emphasized in other contexts by Milgrom (2000) and Ausubel (2004): each seller can directly react at each stage of the auctioning phase to the past supply decisions of his competitors. This allows for a richer set of punishments than in competitive-screening games—in which deviations can only be punished through the buyer's decisions—and this guarantees the existence of an equilibrium. Our contribution is to provide a fully strategic foundation for the JHG allocation, a result that has so far eluded the literature.[5]

An alternative derivation of the JHG allocation is provided by Beaudry and Poitevin (1995), who study a sequential game in which a risk-averse entrepreneur whose project can be of low or high riskiness can repeatedly solicit financing from successive cohorts of uninformed lenders, thereby signaling the type of her project. In comparison, a realistic feature of our auction format is that the set of sellers is fixed throughout the auctioning phase, so that each seller must anticipate the future consequences of his supply decisions at any price. Moreover, signaling plays no role in our analysis, whereas it requires an appropriate selection of lenders' beliefs off the equilibrium path in Beaudry and Poitevin (1995).

The paper is organized as follows. Section 2 describes the model. Section 3 analyzes inactive markets. Section 4 extends the analysis to active markets. Section 5 studies the discriminatory ascending auction. Section 6 concludes. The main appendix provides the proofs of Theorems 1–4. The online appendices A–F collect supplementary material.

2 The Economy

Consider a buyer (she) endowed with private information, and whose type can take finitely many values $i = 1, \ldots, I$ with strictly positive probabilities m_i. Type i's preferences are represented by a utility function $u_i(q, t)$ that is continuous and weakly quasiconcave in (q, t) and strictly decreasing in t, with the interpretation that q is the nonnegative quantity of a

equilibrium constructed by Biais, Martimort, and Rochet (2000) only exists under rather stringent joint restrictions on the cost function and the distribution of types.

[5] To be fair, Attar, Mariotti, and Salanié (2019) show that, as the number K of sellers grows large, a standard competitive-screening game admits an ε-equilibrium, with ε of the order of $1/K^2$, that supports the JHG allocation. The results in this paper are significantly stronger in that they rely neither on a notion of approximate equilibrium nor on the consideration of a fictitious competitive limit.

divisible good she purchases and t is the payment she makes in return. Types are ordered according to the weak single-crossing condition (Milgrom and Shannon (1994)), which states that higher types are at least as willing to increase their purchases as lower types are:

$$\text{For all } i < j,\ q < q',\ t, \text{ and } t',\ u_i(q,t) \leq (<)\, u_i(q',t') \text{ implies } u_j(q,t) \leq (<)\, u_j(q',t').$$

For future reference, we also state the slightly stronger, strict single-crossing condition:

$$\text{For all } i < j,\ q < q',\ t, \text{ and } t',\ u_i(q,t) \leq u_i(q',t') \text{ implies } u_j(q,t) < u_j(q',t').$$

To define marginal rates of substitution without assuming differentiability, let $\tau_i(q,t)$ be the supremum of the set of prices p such that

$$u_i(q,t) < \max\{u_i(q+q', t+pq') : q' \geq 0\}.$$

Thus $\tau_i(q,t)$ is the slope of type i's indifference curve at the right of (q,t). Quasiconcavity ensures that $\tau_i(q,t)$ is finite, except possibly when $q = 0$, and that it is nonincreasing along an indifference curve of type i. We additionally make the intuitive assumption that, in the absence of transfers, a positive endowment of q reduces this marginal rate of substitution.

Assumption 1 *For all i and $q > 0$, $\tau_i(q,0) \leq \tau_i(0,0)$.*

Our assumptions on the buyer's preferences hold in a Rothschild and Stiglitz (1976) insurance economy, which is the case studied by Hendren (2013); then i indexes the buyer's riskiness, q is the amount of coverage she purchases, and t is the premium she pays in return. As we illustrate in Appendix C, they also hold under many alternative specifications, allowing for multiple loss levels or various forms of nonexpected utility. Finally, they encompass a broad variety of other applications, such as financial and labor markets. It should be noted that we do not require strict single-crossing nor strict convexity of preferences. This choice is not motivated by an idle desire for generality, but is meant to pave the way for the analysis of active markets provided in Section 4.

Each seller (he) is risk-neutral and thus maximizes his expected profit. All sellers have access to the same linear technology. We denote by $c_i > 0$ the unit cost of serving type i, and by $\bar{c}_i$ the corresponding upper-tail conditional expectation of unit costs,

$$\bar{c}_i \equiv \mathbf{E}[c_j \,|\, j \geq i] = \frac{\sum_{j \geq i} m_j c_j}{\sum_{j \geq i} m_j}.$$

Adverse selection occurs if the unit cost c_i is nondecreasing in i. Here, and unless indicated otherwise, we make the slightly weaker assumption that $\bar{c}_i$ is nondecreasing in i. This weak adverse-selection condition is exactly equivalent to

$$\text{For all } j \leq i,\ c_j \leq \bar{c}_i. \tag{1}$$

A contract between a seller and the buyer specifies a nonnegative quantity and a transfer to be made in return by the buyer.

Our analysis can be extended to the case of multiple buyers by assuming in addition that trade is nonanonymous, contracting is bilateral, and buyers' types are i.i.d. Nonanonymity of trade enables price discrimination by preventing a buyer from making concealed repeat purchases from the same seller. Contracting is bilateral if trade between a seller and a buyer is only contingent on the information reported by the buyer to the seller. Together with the linearity of costs, the independence of types across buyers then implies that the interactions between a seller and each of his potential customers can be studied separately. Finally, if the buyers' types are identically distributed, we can assume, using a symmetry argument, that each seller offers the same contracts to each buyer and that each type of each buyer facing the same choices behaves in the same way. In this way, the multiple-buyer case reduces to a replication of the single-buyer case.

3 Entry-Proofness in Inactive Markets

In this section, we describe the circumstances under which private information impedes trade altogether. We say that a market is *inactive* if only the null contract $(0,0)$ is available, and we say that an inactive market is *entry-proof* if, *for any menu of contracts offered by an entrant, the buyer has a best response such that the entrant earns at most zero expected profit.* Our goal is to characterize the inactive markets that are entry-proof.

Let us first analyze the simple case where the entrant offers a single contract, designed so as to attract some type i. To do so, the entrant can choose some unit price p slightly below $\tau_i(0,0)$. Then, by definition of $\tau_i(0,0)$, there exists a quantity q that strictly attracts type i at this price, that is, $u_i(q,pq) > u_i(0,0)$. As types are ordered according to the weak single-crossing condition, we also have $u_j(q,pq) > u_j(0,0)$ for all $j > i$. Thus any type $j \geq i$ is strictly attracted by the offer (q,pq), and the entrant bears an expected unit cost $\overline{c}_i$ when trading with these types. Finally, some other types $j < i$ may also be attracted, but the weak adverse-selection condition (1) ensures that this can only reduce the entrant's expected unit cost.[6] This simple reasoning shows that the following condition is necessary for entry to be unprofitable.

Condition EP *For each i, $\tau_i(0,0) \leq \overline{c}_i$.*

Notice that Condition EP does not rule out gains from trade, in the usual first-best sense

[6] Alternatively, if we assume strict single-crossing, then we can design (q,pq) so that types $j < i$ are not attracted; as a result, condition (1) is no longer needed for this argument.

of the term; that is, it may well be that $\tau_i(0,0) > c_i$ for some i. The following theorem, a formal proof of which is provided in the main appendix, states that this necessary condition is also sufficient, even when menus of contracts are allowed.

Theorem 1 *An inactive market is entry-proof if and only if Condition EP is satisfied.*

The key to the proof lies in the following remark. Suppose the entrant offers an arbitrary menu of contracts. Under weak single-crossing, a standard monotone-comparative-statics argument implies that the buyer has a best response with nondecreasing quantities; that is, the entrant ends up trading (q_i, t_i) with every type i, with $q_i \leq q_j$ for all $i < j$. Then his expected profit is

$$\sum_i m_i(t_i - c_i q_i),$$

which, using a summation by parts in the spirit of Wilson (1993), we can rewrite as

$$\sum_i \left(\sum_{j \geq i} m_j\right)[t_i - t_{i-1} - \overline{c}_i(q_i - q_{i-1})], \tag{2}$$

where $(q_0, t_0) \equiv (0,0)$. Because $q_i \geq q_{i-1}$ and type i is willing to trade $(q_i - q_{i-1}, t_i - t_{i-1})$ in addition to (q_{i-1}, t_{i-1}), each bracketed term in (2) is at most

$$[\tau_i(q_{i-1}, t_{i-1}) - \overline{c}_i](q_i - q_{i-1})$$

and thus is nonpositive if the marginal rate of substitution is lower than the upper-tail conditional expectation of unit costs,

$$\tau_i(q_{i-1}, t_{i-1}) \leq \overline{c}_i.$$

To show that this holds, recall that, by construction, type $i-1$ prefers her optimal choice (q_{i-1}, t_{i-1}) to the no-trade contract $(0,0)$. Under weak single-crossing, the same property is satisfied by type i, and thus (q_{i-1}, t_{i-1}) lies in the nonnegative orthant, below the indifference curve of type i that goes through the origin. (That we can focus on menus with nonnegative transfers is established in the main appendix.) As illustrated in Figure 1, we can use, in turn, the concavity of the indifference curve of type i, then Assumption 1, and finally Condition EP to obtain the desired inequality:

$$\tau_i(q_{i-1}, t_{i-1}) \leq \tau_i(\underline{q}_i, 0) \leq \tau_i(0,0) \leq \overline{c}_i. \tag{3}$$

This concludes the proof of Theorem 1.

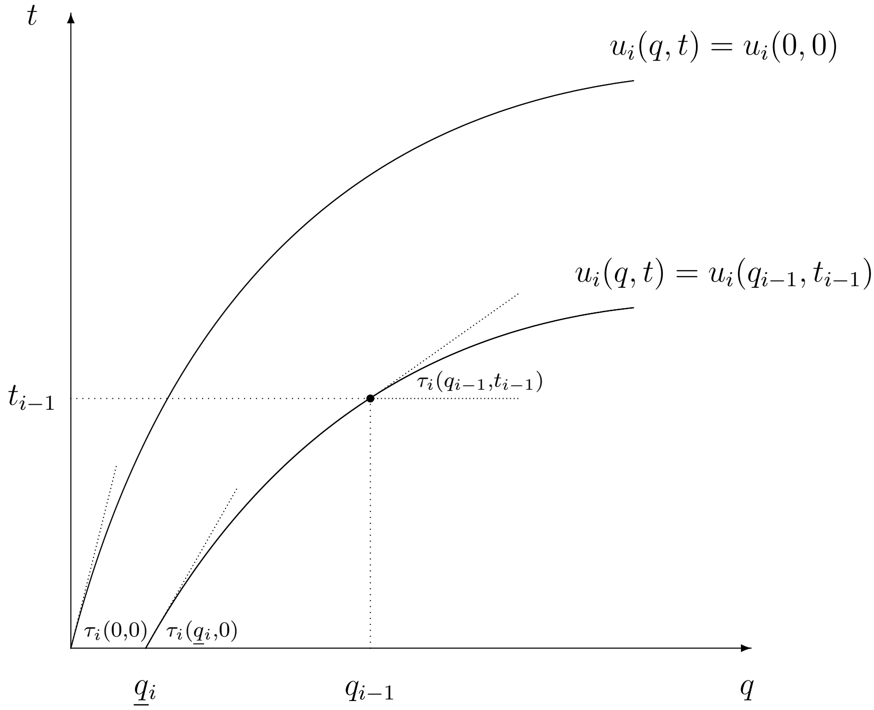

Figure 1: A graphical illustration of (3).

A noticeable feature of this proof is that it does not consider each contract (q_i, t_i) in isolation. Instead, the key role is played by *layers* of the form $(q_i - q_{i-1}, t_i - t_{i-1})$. Under weak single-crossing, the i^{th} layer can be thought of as traded by all types $j \geq i$, and thus has expected unit cost $\bar{c}_i$. Condition EP then states that, at this price, type i is not strictly willing to trade, so that each layer must yield a nonpositive expected profit. By contrast, some of the *contracts* proposed in a menu may yield positive profits. For instance, although the condition $t_1 \leq \bar{c}_1 q_1$ ensures that the expected profit on the first layer (q_1, t_1) is nonpositive, it may well be that $t_1 > c_1 q_1$.

The assumptions of Theorem 1 can be weakened in three directions. First, the finiteness of the type distribution is not crucial: we show in Appendix B that the result holds for an arbitrary type distribution with bounded support over the real line. Second, the convexity of preferences can be relaxed if we reinforce Assumption 1 into $\tau_i(q,t) \leq \tau_i(0,0)$ for all (q,t) such that $t \geq 0$ and $u_i(q,t) \geq u_i(0,0)$. Third, the weak adverse-selection condition (1) is not needed if we assume that types are ordered according to the strict single-crossing condition.[7] Appendix D illustrates this point for a variant of Leland and Pyle's (1977) model of equity issuance in which an entrepreneur faces a background risk.

By contrast, the weak single-crossing condition and the seemingly innocuous Assumption 1 are tight. The key role of the weak single-crossing condition in the proof of Theorem 1 is to ensure that the quantity profile chosen by the buyer in the entrant's menu is nondecreasing in her type. As for Assumption 1, Example 1 in Appendix E shows that, in its absence,

[7] See Footnotes 6 and 24 for the only changes needed in the proof.

entry with a menu of contracts can be profitable even though Condition EP is satisfied—the intuition being that type i's marginal rate of substitution can then take values higher than $\tau_i(0,0)$ in the relevant area illustrated in Figure 1.

Condition EP ensures that there exists a best response for the buyer such that entry on an inactive market is unprofitable. However, the literature often focuses on characterizing *market breakdown*, defined as a situation in which *any menu of contracts that strictly attracts at least some type yields a strictly negative expected profit, even if the buyer's best response is most favorable to the entrant.* Condition EP clearly remains necessary for this stronger concept. We now argue that, under slightly stronger conditions on preferences, it remains also sufficient. The proof of the following result is provided in the main appendix, and Appendix E provides two examples showing that the additional conditions are tight.

Corollary 1 *Suppose that the buyer's preferences are strictly convex and that types are ordered according to the strict single-crossing condition. Then there is market breakdown if and only if Condition EP is satisfied.*

Mailath and Nöldeke (2008) obtain a related result for an economy in which the buyer has quadratic quasilinear preferences, as in Glosten (1989), Biais, Martimort, and Rochet (2000, 2013), and Back and Baruch (2013). Hendren (2013) studies a Rothschild and Stiglitz (1976) insurance economy, and his Theorem 1 is the analogue of Corollary 1 in this particular setting. As emphasized by the author, an implication of Condition EP is that the highest-risk type I must not be willing to purchase coverage at the actuarially fair rate c_I. Given that her preferences have an expected-utility representation, this is possible only if type I incurs a loss with certainty. In that case, type I's preferences are no longer strictly convex, and the above result becomes that all types except perhaps type I must be excluded from trade, as in Akerlof's (1970) classic example of market breakdown.

4 Entry-Proofness in Active Markets

We now turn to *active* markets, on which nonnull contracts are available. In line with Rothschild and Stiglitz (1976), our goal is to characterize when entry in such a market is unprofitable, given the contracts available; in contrast with them, we suppose that the buyer can trade with several sellers. That is, trade is *nonexclusive.*

To this end, the proper object of study is the *market tariff*, which describes the frontier of the set of aggregate trades that can be achieved by trading on the market. Hence the market tariff specifies, for any nonnegative q, the minimum aggregate transfer $T(q)$ required to purchase an aggregate quantity q, with $T(q) \equiv \infty$ if this is impossible; notice that we obviously have $T(0) = 0$.

The key restriction we impose in this section is that the market tariff be *convex*. A case in point is when each seller k posts a convex tariff t^k such that $t^k(0) = 0$. As pointed out in the Introduction, one reason to do so is to hedge against the risk of attracting high-cost types buying large quantities.[8] Then the market tariff $T(q) \equiv \min\{\sum_k t^k(q^k) : \sum_k q^k = q\}$, which incorporates the possibility of trading with several sellers on the market, is indeed a convex function of the aggregate quantity q.[9]

We throughout assume that the domain of the market tariff T is a compact interval with lower bound 0. Every type i selects q_i so as to maximize $u_i(q, T(q))$. We then say that the allocation $(q_i, T(q_i))_{i=1}^I$ is *implemented* by the tariff T; this allocation is *budget-feasible* if

$$\sum_i m_i[T(q_i) - c_i q_i] \geq 0. \tag{4}$$

We assume that types are ordered according to the strict single-crossing condition, so that the optimal quantities q_i are nondecreasing in i.

Now, suppose an entrant can propose additional trades to the buyer, in the form of a menu of contracts. We say that the tariff T is *entry-proof* if, *for any menu of contracts offered by an entrant, the buyer has a best response such that the entrant earns at most zero expected profit, given that the buyer is free to combine any contract offered by the entrant with a trade along the tariff* T. The last clause of this definition is crucial, and captures the nonexclusivity of trade. Our goal is to characterize the set of budget-feasible allocations that are implemented by entry-proof convex market tariffs.

Let us first observe that, from the entrant's viewpoint, everything happens as if he were facing modified types with indirect utility functions

$$u_i^T(q', t') \equiv \max\{u_i(q + q', T(q) + t') : q\}, \tag{5}$$

reflecting that the buyer is free to combine any contract (q', t') offered by the entrant with a trade $(q, T(q))$ along the tariff T.[10] In particular, $u_i^T(0, 0)$ represents type i's utility when she only trades on the market and not with the entrant, and thus defines the relevant individual-rationality constraint for type i from the entrant's viewpoint.

[8] In the literature, convex tariffs are often used to model collections of limit orders placed by strategic market makers and executed in order of price priority by an informed insider (Biais, Martimort, and Rochet (2000, 2013), Back and Baruch (2013), Attar, Mariotti, and Salanié (2019), Baruch and Glosten (2019)).

[9] This is because T is the *infimal convolution* of the convex tariffs t^k (Rockafellar (1970, Theorem 5.4)).

[10] Clearly, q in (5) should belong to the domain of T. The admissible set for q may also vary continuously in (q', t'), as, for instance, when the consumption set of the buyer is bounded. To simplify notation, we do not explicitly mention such admissibility constraints in the maximization problems considered in this section.

Because the tariff T is continuous over a compact domain, the maximum in (5) is attained and $u_i^T(q',t')$ is continuous in (q',t').[11] Moreover, because the tariff T is convex and the primitive utility functions $u_i(q,t)$ are weakly quasiconcave in (q,t) and strictly decreasing in t, the indirect utility functions $u_i^T(q',t')$ are weakly quasiconcave in (q',t') and strictly decreasing in t'. As a result, we can define the marginal rates of substitution $\tau_i^T(q',t')$ associated to them exactly as we did in Section 2 for the primitive utility functions. Finally, because the primitive types are ordered according to the strict single-crossing condition, the modified types are ordered according to the weak single-crossing condition.[12]

Thus, to apply Theorem 1, there only remains to ensure that Assumption 1 holds for the indirect marginal rates of substitution $\tau_i^T(q',0)$. A convenient way to proceed is to require that each type's family of primitive indifference curves satisfy a slightly stronger fanning-out condition than in Assumption 1.

Assumption 2 *For all i and t, $\tau_i(q,t)$ is nonincreasing in q.*

That is, a higher quantity traded reduces the buyer's willingness-to-pay. As we illustrate in Appendix C, this assumption is satisfied by a large variety of preference relations. The following result is established in Appendix A.

Lemma 1 *If Assumption 2 holds for the primitive marginal rates of substitution $\tau_i(q,t)$, then Assumption 1 holds for the indirect marginal rates of substitution $\tau_i^T(q',0)$.*

We can now deduce from Theorem 1 that the tariff T is entry-proof if and only if

$$\text{For each } i,\ \tau_i^T(0,0) \leq \bar{c}_i. \tag{6}$$

To see what this abstract condition entails for the tariff T and the allocation $(q_i, T(q_i))_{i=1}^I$ it implements, recall from (5) that $\tau_i^T(0,0)$ is the supremum of the set of prices p such that

$$u_i(q_i, T(q_i)) = u_i^T(0,0) < \max\{u_i^T(q', pq') : q'\} = \max\{u_i(q+q', T(q)+pq') : q, q'\}.$$

Thus, according to (6), we have

$$\text{For each } i,\ u_i(q_i, T(q_i)) \geq \max\{u_i(q+q', T(q)+\bar{c}_i q') : q, q'\}. \tag{7}$$

Fixing $q_0 \equiv 0$ and applying (7) to $q \in [q_{i-1}, q_i]$ and $q' = q_i - q$ yields

$$\text{For all } i \text{ and } q \in [q_{i-1}, q_i],\ T(q_i) \leq T(q) + \bar{c}_i(q_i - q). \tag{8}$$

[11] This follows from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)).

[12] This follows from Attar, Mariotti, and Salanié (2019, Supplementary Appendix, Proof of Lemma 1).

In particular, for $q = q_{i-1}$, we have

$$T(q_i) \leq T(q_{i-1}) + \overline{c}_i(q_i - q_{i-1}). \tag{9}$$

Now, rewriting the expected profit (4) as in (2) and imposing that the allocation $(q_i, T(q_i))_{i=1}^I$ be budget-feasible, we have

$$\sum_i \left(\sum_{j \geq i} m_j \right) [T(q_i) - T(q_{i-1}) - \overline{c}_i(q_i - q_{i-1})] \geq 0.$$

The only possibility is thus that the inequalities (9) hold as equalities,

$$\text{For each } i, \; T(q_i) = T(q_{i-1}) + \overline{c}_i(q_i - q_{i-1}), \tag{10}$$

which, in turn, implies, according to (7),

$$\text{For each } i, \; u_i(q_i, T(q_i)) = \max\{u_i(q_{i-1} + q', T(q_{i-1}) + \overline{c}_i q') : q'\}. \tag{11}$$

Finally, because T is convex and satisfies both (8) and (10), it must be that T is affine with slope $\overline{c}_i$ over the interval $[q_{i-1}, q_i]$. The following theorem, a formal proof of which is provided in the main appendix, summarizes this discussion and states that the necessary conditions (10)–(11) for entry-proofness are also sufficient.

Theorem 2 *An allocation* $(q_i, T(q_i))_{i=1}^I$ *is budget-feasible and is implemented by an entry-proof convex market tariff* T *with domain* $[0, q_I]$ *if and only if they jointly satisfy the following recursive system:*

(i) $(q_0, T(q_0)) \equiv (0, 0)$.

(ii) For each i, $q_i - q_{i-1} \in \arg\max\{u_i(q_{i-1} + q', T(q_{i-1}) + \overline{c}_i q') : q'\}$.

(iii) For each i, *if* $q_{i-1} < q_i$, *then* T *is affine with slope* $\overline{c}_i$ *over the interval* $[q_{i-1}, q_i]$.

In particular, any such allocation is exactly budget-balanced.

Let us first comment on each item of this result. First, it is natural to focus on tariffs defined up to the maximum quantity q_I; one can build other entry-proof tariffs by suitably prolonging T beyond this point, but this is in no way needed. Next, (i) is merely a convention. Finally, (ii)–(iii) are substantial, and indicate how to recursively build a complete family of quantities, as well as a tariff that by construction is convex, because the upper-tail conditional expectation of unit costs is nondecreasing in the buyer's type.[13]

[13] Thus the weak adverse-selection condition (1), which is not needed for Theorem 1 under strict single-crossing, is essential for Theorem 2.

Existence of an entry-proof convex market tariff obtains as soon as each maximization problem in (ii) admits a solution. This is, for instance, ensured by the following Inada condition, which states that demand is finite when the price is positive:

$$\text{For all } i,\ (q,t),\ \text{and } p>0,\ \arg\max\{u_i(q+q',t+pq'):q'\}<\infty. \tag{12}$$

Therefore, under nonexclusivity, budget-feasibility and entry-proofness are not conflicting requirements, in contrast with the pervasive nonexistence problems arising under exclusivity (Rothschild and Stiglitz (1976)).[14]

Uniqueness of an entry-proof convex market tariff also follows if the solution to each maximization problem in (ii) is unique. This is the case if the buyer's preferences are strictly convex. If they are only weakly convex, multiple solutions may appear if the marginal rate of substitution of some type i equals $\overline{c}_i$ over a whole interval of quantities, but this is clearly a nongeneric phenomenon.

Theorem 2 thus characterizes an essentially unique allocation. Following Attar, Mariotti, and Salanié (2014, 2019), we label this allocation, which was originally introduced in different contexts by Jaynes (1978), Hellwig (1988), and Glosten (1994), the *JHG allocation*, and we denote it by $(q_i^*,t_i^*)_{i=1}^I$. Similarly, the *JHG tariff* consists of a sequence of layers with unit prices $\overline{c}_i$, and features an upward kink at any quantity $q_i^*\in(0,q_I^*)$ such that $q_{i+1}^*>q_i^*$ and $\overline{c}_{i+1}>\overline{c}_i$.[15] The JHG allocation is exactly budget-balanced, because each marginal quantity is priced at the expected cost of serving the types who purchase it. This property can be interpreted as a marginal version of Akerlof (1970) pricing.

The JHG allocation and the JHG tariff are illustrated in Figure 2 in the case of three types with strictly convex preferences.

We can also apply Theorem 2 to preferences that are only weakly convex. Consider, for instance, linear utility functions $u_i(q,t)\equiv v_iq-t$, subject to a capacity constraint $q\in[0,1]$; such linear preferences generalize those in Akerlof (1970) to a divisible good, and strict single-crossing requires that v_i be strictly increasing in i. Each problem in (ii) admits a unique solution if $v_i\neq\overline{c}_i$ for all i, and we then have two possibilities:

1. If $v_i<\overline{c}_i$ for all i, then, according to (ii), every quantity q_i must be zero. Moreover,

[14] Our approach does not apply when competition is exclusive. The reason is that the buyer's indirect utility functions no longer satisfy single-crossing: by offering a cream-skimming contract, the entrant can attract any type i without attracting types $j>i$, which allows him to target type i without worrying about adverse selection. Thus the pervasive nonexistence problems arising under exclusivity are arguably not due to private information or entry-proofness per se, but rather to this violation of single-crossing—or, to put it more provocatively, to the fact that the exclusive model does not capture the full extent of adverse selection.

[15] In line with the literature cited in Footnote 8, this sequence of layers can be interpreted as a family of limit orders with maximum quantities $q_i^*-q_{i-1}^*$ and unit prices $\overline{c}_i$.

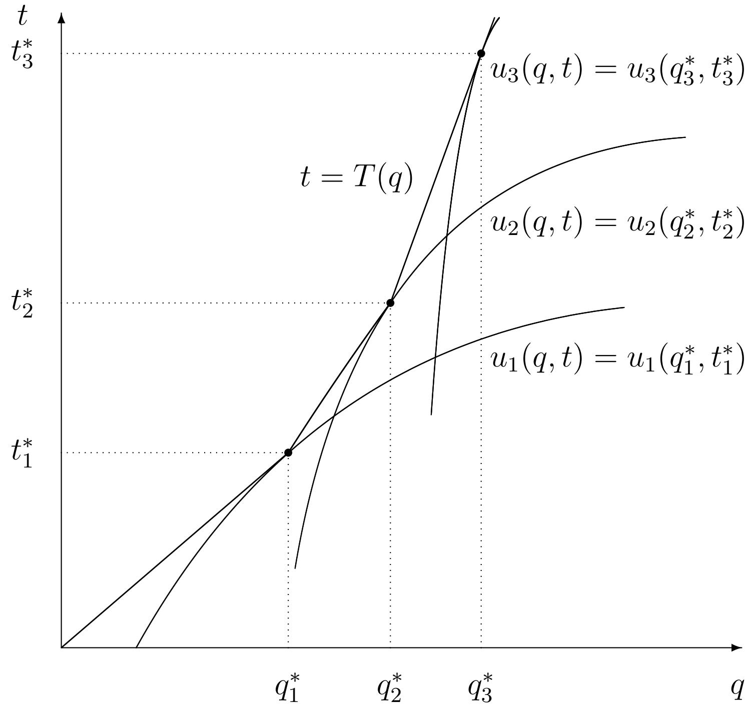

Figure 2: The JHG allocation and the JHG tariff for $I = 3$.

according to (iii), the essentially unique entry-proof convex market tariff is only defined at zero, with $T(0) = 0$, and the market is inactive.

2. Suppose, alternatively, that $v_i > \overline{c}_i$ for some i, and let i^* be the lowest such i. Then, according to (ii), i^* trades up to capacity at unit price $\overline{c}_{i^*}$. By strict single-crossing, so do types $i > i^*$, while types $i < i^*$ do not trade at all. Moreover, according to (iii), the unique entry-proof convex market tariff is linear, with $T(q) = \overline{c}_{i^*}q$ for all $q \in [0, 1]$.

Thus, generically, the JHG allocation features a single layer when the buyer's preferences are linear, and corresponds to competitive-equilibrium allocation in Akerlof (1970) that maximizes the gains from trade.

The property that the modified types be ordered according to the weak single-crossing condition has played a key role in our analysis. This property itself resulted from the two assumptions that the primitive types be ordered according to the strict single-crossing condition and that the market tariff be convex. Because this second assumption effectively constraints market outcomes, it is natural to ask to which extent it can be relaxed. In this respect, it should first be noted that convexity of the market tariff can be dispensed with altogether if the buyer has linear preferences, as above, or if there are only two buyer types.[16] Second, and more generally, the JHG allocation turns out to be the only budget-feasible allocation implemented by an entry-proof market tariff that is first convex and then concave; the details of the argument are provided in Appendix F. Thus entry-proofness per se selects

[16]This follows from Attar, Mariotti, and Salanié (2011, 2020).

a convex tariff in a large class of admissible tariffs, allowing for quantity discounts.

5 A Discriminatory Ascending Auction

5.1 Motivation

Our results so far illustrate the power of the entry-proofness requirement, which selects a unique candidate for aggregate market outcomes. However, by design, the approach we have followed, which imposes properties directly on the market tariff, eschews the question of how this tariff can be derived as the equilibrium outcome of a game in which strategic sellers compete to serve the buyer's demand. That is, it does not explain how the sellers' tariffs, from which the market tariff is ultimately obtained by aggregation, are determined in a decentralized way. This question is especially pressing under nonexclusivity, because the buyer is free to combine contracts issued by different sellers.[17]

The recursive construction of the JHG allocation suggests that it be implemented via a dynamic process whereby trade first takes place at a low price until sellers stop serving the demand at this price, after which the price moves up, making sellers willing to supply additional quantities—and so on, until demand vanishes. We formalize this intuition by designing a discriminatory auction in which prices are quoted in ascending order. Each time a new price is quoted, each seller publicly announces the maximum quantity he stands ready to trade at this price. Once this auctioning phase is completed, the buyer decides which quantities to purchase from which sellers in a nonexclusive way.

This auction departs from the Walrasian *tâtonnement* process in that, at each price, sellers cannot withdraw the quantities they supplied at lower prices. Because the buyer in the end optimally selects the best price offers first, everything happens from her perspective as if the sellers were posting convex tariffs; the fact that they do so sequentially is immaterial to her. As a consequence, the auction discovers a convex market tariff instead of converging to a single equilibrium price.

5.2 Timing and Assumptions

We throughout postulate a discrete price grid. This is first for the sake of realism; for instance, prices quoted on financial markets come in multiples of a minimum tick size. Second, an ascending auction with a discrete price grid can be modeled as a standard extensive-form game, allowing us to avoid the conceptual difficulties raised by continuous-time games (Simon and Stinchcombe (1989)). For a tick size $\Delta > 0$, we thus fix a price

[17] Under exclusivity, if an entry-proof tariff exists, there exists an equilibrium in which each seller posts it.

grid $\{0, \Delta, 2\Delta, \dots\}$; to simplify the analysis, we assume that the upper-tail conditional expectations of unit costs $\overline{c}_i$ all belong to that grid.

The game unfolds in two phases.

In a first phase, the auctioneer quotes the prices in the grid in ascending order. When a new price p is quoted, $K \geq 2$ sellers simultaneously and publicly announce the maximum quantities $s^k(p) \geq 0$, $k = 1, \dots, K$, they stand ready to trade at this price. The auctioneer then goes on to the next price $p + \Delta$, and this process is repeated until all prices have been quoted.[18] Once this first phase is over, we can build a market tariff by aggregating the quantities successively supplied, as follows. Let $s(p) \equiv \sum_k s^k(p)$ be the aggregate supply at price p and $S(p) \equiv \sum_{p' \leq p} s(p)$ be the aggregate supply at prices lower than or equal to p. Then the market tariff T is defined recursively by $T(0) \equiv 0$ and

$$\text{For each } q \in [S(p - \Delta), S(p)],\ T(q) \equiv T(S(p - \Delta)) + p[q - S(p - \Delta)].$$

By construction, T is convex.

In a second phase, the buyer learns her type, and decides which quantities to buy from which sellers. In the aggregate, she purchases a quantity q in exchange for a transfer $T(q)$. Therefore, the price p of the last purchased unit is the left-derivative $\partial^- T(q)$ of T at q. The revenue earned by every seller k at any inframarginal price $p' < p$ is $p' s^k(p')$, because it is in the buyer's interest to exhaust supply at any such price. The aggregate revenues earned by the sellers at price p are $p[q - S(p - \Delta)]$. If $q < S(p)$, the buyer is indifferent to the manner she allocates this revenue among the sellers; her equilibrium strategy will specify how she breaks these ties. Overall, each seller's expected profit is the expected sum of his revenues at all prices, minus the expected cost of sales.

To simplify the exposition, we assume that every type i has quasilinear, strictly convex, and differentiable preferences satisfying the Inada condition (12), so that her demand $D_i(p)$ at any price $p > 0$ is single-valued, finite, and continuous and strictly decreasing in p as long as it is strictly positive. In particular, $D_i(p)$ goes to zero when p goes to ∞. Finally, to avoid nongeneric cases, we slightly strengthen the strict single-crossing condition by requiring that $D_i(p)$ be strictly increasing in i for each $p > 0$ as long as it is strictly positive.

We denote by Γ the corresponding extensive-form game with almost-perfect information. Our equilibrium concept is pure-strategy subgame-perfect Nash equilibrium. The remainder of this section provides our characterization results.

[18]We do not need to specify a stopping rule for this phase, because our game is formally well-defined even with infinitely many prices in the grid. In practice, one may end this phase when the aggregate supply exceeds the highest possible demand at the current price. See also Footnote 19 for an alternative timing.

5.3 A Simple Equilibrium

In our equilibrium construction, the sellers' supply decisions at any history in the first phase of Γ only depend on the current price p and on the aggregate quantity Q^- supplied at prices $p' < p$. We call (p, Q^-) the current *state* of the game, which starts in state $(0, 0)$. In any state (p, Q^-), every type j has a residual demand $[D_j(p) - Q^-]^+$, where $[x]^+$ is the positive part of x. Under strict single-crossing, any quantity purchased at price p by some type i is also purchased by types $j > i$. Thus, in any state (p, Q^-) such that $\overline{c}_i < p \le \overline{c}_{i+1}$, maximizing aggregate expected profits exactly requires serving the residual demand $[D_i(p) - Q^-]^+$ of type i, which we call the *profitable residual demand* in state (p, Q^-). The following theorem, a formal proof of which is provided in the main appendix, exhibits an equilibrium of Γ in which sellers equally share this profitable residual demand in any state.

Theorem 3 *There exists an equilibrium of* Γ *in which, in any state* (p, Q^-),

(i) If $p \le \overline{c}_1$, *each seller supplies a zero quantity.*

(ii) If $\overline{c}_1 < p \le \overline{c}_I$, *each seller supplies an equal share of the profitable residual demand.*

(iii) If $p > \overline{c}_I$, *each seller supplies an infinite quantity.*

These strategies induce the following equilibrium outcome. As soon as the price reaches $\overline{c}_1 + \Delta$, the sellers collectively serve the demand $D_1(\overline{c}_1 + \Delta)$ of type 1, thereby satiating her demand; this quantity will also be purchased by types $i > 1$. Then, as soon as the price reaches $\overline{c}_2 + \Delta$, the sellers collectively serve the residual demand $[D_2(\overline{c}_2 + \Delta) - D_1(\overline{c}_1 + \Delta)]^+$ of type 2, thereby satiating her demand; this quantity will also be purchased by types $i > 2$. This process is repeated until the price reaches $\overline{c}_I + \Delta$, at which point the sellers flood the market by supplying an infinite quantity. It is readily checked that the resulting aggregate equilibrium allocation converges to the JHG allocation when Δ goes to zero. We will establish a general version of this result in the next section.

The proof of Theorem 3 relies on three arguments.

First, the game effectively stops when the price $\overline{c}_I + \Delta$ is reached. This allows us to apply the one-shot deviation property in our analysis of the sellers' deviations.

Second, a seller may try to increase his market share $1/K$ in state (p, Q^-) by increasing his supply. Given his competitors' equilibrium strategies, however, all profitable types at price p, that is, all types i such that $p > \overline{c}_i$, can choose to ignore this deviation and carry on trading the same quantity with each seller. Hence the deviating seller will only succeed at selling more to unprofitable types, which lowers his expected profit at price p. Moreover, because unprofitable types trade more at this price, their residual demands at higher prices will also be reduced. Evaluating the overall impact on the continuation path, we show that such upward deviations cannot be profitable.

Third, a seller may try to reduce his supply in state (p, Q^-) by an amount δ, so that some profitable type i is now rationed in this state. Her residual demand at the next price $p + \Delta$ thus increases, say, for simplicity, by exactly δ. From the deviating seller's viewpoint, the problem is that the main part of this increase will go to his competitors. Indeed, following their equilibrium strategies, they will collectively react by increasing their supply at price $p + \Delta$ by $(K - 1)\delta/K$, leaving only δ/K to him. Hence, instead of selling δ at price p, the deviating seller ends up selling only δ/K at price $p + \Delta$, which is less profitable as $p > \Delta$ and $K \geq 2$. Though this intuition is simple, the proof is more involved; indeed, as demand is elastic, the reduction in supply at price p does not translate into an equivalent increase of the residual demand at price $p+\Delta$. Evaluating again the overall impact on the continuation path, we show that such downward deviations cannot be profitable.

Key to this existence result is the sequentiality of the ascending auction, which allows each seller to condition his supply at each price on his competitors' past supply decisions. The only constraint is subgame-perfection, but this constraint is mild as punishments take the form of profitable increases in supply. By contrast, simultaneous models of nonexclusive competition under adverse selection generally conclude to the nonexistence of equilibrium when preferences are strictly convex (Attar, Mariotti, and Salanié (2014, 2019)). Indeed, in such games, the natural candidate for equilibrium is similar to the one described in Theorem 3: each seller supplies a share of the profitable residual demand at each price and, therefore, is indispensable for serving that demand. The difference is that, in a simultaneous game, a seller can reduce his supply at a given price without triggering a reaction by his competitors. Indeed, the only available device to block such a deviation consists in the buyer sending appropriate reports to the nondeviating sellers, translating into different choices in the menus or tariffs they offer. Such reports, however, have to be sequentially rational from the buyer's viewpoint, which considerably restricts the set of available punishments. By contrast, in our equilibrium construction for the discriminatory ascending auction, the main thrust of punishments is borne by the sellers themselves, leaving for the buyer only the task of breaking ties at the expense of the deviating seller.

5.4 Convergence of Equilibrium Allocations

We now show that the JHG allocation uniquely emerges as the limit of equilibrium allocations when the tick size goes to zero, generalizing an insight of Theorem 3. To establish this result, we focus on equilibrium strategies for the buyer that satisfy a minimal robustness property. Recall that, in equilibrium, every type i accepts all offers up to some price p_i. At this last price, the sellers' aggregate supply $s(p_i)$ may exceed her residual demand, so that type i can allocate it in different ways among the sellers; although she is indifferent between all such allocations, her choice may matter to the sellers. We say that an equilibrium of Γ is *robust to irrelevant offers* if any type i's allocation of trades at price p_i does not depend on offers made at prices $p > p_i$. Intuitively, we do not allow the buyer to punish a seller for deviating at a price that is irrelevant to her as she is not willing to trade at this price. The equilibrium constructed in Theorem 3 satisfies this refinement.[19] The following theorem, a formal proof of which is provided in the main appendix, encapsulates our convergence result.

Theorem 4 *For each $n \in \mathbb{N}$, fix an equilibrium robust to irrelevant offers of the ascending auction Γ_n with tick size $\Delta_n \equiv \Delta/2^n$. Then the resulting sequence of aggregate equilibrium allocations converges to the JHG allocation.*

This result confirms the prominent role played by the JHG allocation under adverse selection and nonexclusive competition. Although the proof of Theorem 4 is involved, its logic follows from a generalized Bertrand argument that we sketch here, before turning to technical difficulties. To this end, let us hypothetically place ourselves in the limiting case $\Delta = 0$. Recall that, in any state (p, Q^-), type i's residual demand is $[D_i(p) - Q^-]^+$. When the aggregate supply in this state is s, sellers collectively earn

$$B(p, Q^-, s) \equiv \sum_i m_i(p - c_i) \min\{[D_i(p) - Q^-]^+, s\}.$$

Now, suppose, by way of contradiction, that there exists a state (p, Q^-) reached on the equilibrium path such that

$$B^*(p, Q^-) \equiv \max\{B(p, Q^-, s) : s \geq 0\} > 0. \tag{13}$$

Because the highest price at which trade takes place turns out to be bounded along the sequence of equilibria under consideration, let us, for the sake of the argument, focus on the highest price p satisfying (13). At even higher prices, B^* is at most zero. Notice, however, that aggregate continuation profits beyond p must be nonnegative; otherwise, some seller

[19] A simple way to justify this refinement is to consider a slightly different timing for the game, as follows. At any price p quoted by the auctioneer, the sellers announce their supplies $s^k(p)$, and the buyer immediately reacts by choosing which quantities to purchase from which sellers. The game stops at price p if the buyer purchases less than the aggregate supply $s(p)$ at this price; otherwise, the auctioneer goes on to the next price $p + \Delta$. It is easy to check that, in any perfect Bayesian equilibrium of this game, every type i optimally selects quantities by accepting all offers below some threshold p_i. At that threshold price, the game stops, so that the allocation of trades cannot depend on offers that will never be made.

would find it profitable to stop making offers, without jeopardizing his profits at lower prices as the equilibrium is robust to irrelevant offers. Because aggregate continuation profits are an integral of aggregate expected profits B at all prices $p' > p$, each lying by construction below B^*, one must have $B = B^* = 0$ at any such price.

Now, at price p, sellers collectively earn at most $B^*(p, Q^-) > 0$. As a result, each seller is tempted to appropriate these entire aggregate expected profits. The classical Bertrand undercutting deviation consists in making a well-chosen offer at a price $p' < p$ arbitrarily close to p at which the aggregate supply of his competitors is zero; such a price always exists if $\Delta = 0$. The deviating seller is then certain to attract in priority the relevant types, and to secure himself an expected profit at price p' arbitrarily close to $B^*(p, Q^-)$. Because continuation profits beyond p are zero and profits at lower prices are unaffected, this deviation is thus profitable, a contradiction. Therefore, we can conclude that B^* is at most zero in every state reached on the equilibrium path, and thus, reasoning as above, that $B = B^* = 0$ in any such state. Finally, we show that this property actually characterizes the JHG allocation and the JHG tariff, which concludes the proof of Theorem 4.

The technical difficulties in this reasoning should not be overlooked. First, we need to establish a convergence result for supply functions and market tariffs when the tick size goes to zero. To do so, we rely on Helly's selection theorem. Second, it may well be that, in the limit, there exists no highest price p such that (13) holds. This requires a careful limiting argument. Third, with a discrete price grid, the choice of the price $p' < p$ used for undercutting is more delicate, because the nondeviating sellers may also supply positive quantities at any such price, making priority difficult to achieve. Fortunately, when the tick size goes to zero, the number of available prices just below p grows without bounds. This guarantees that the aggregate supply of nondeviating sellers becomes negligible in a left-neighborhood of p, which validates the informal argument given above.

6 Concluding Remarks

In this paper, we have provided a unified perspective on entry-proofness under adverse selection, which is relevant both for inactive markets and for active markets on which buyers cannot be prevented from making additional trades with an entrant. These two scenarios turn out to be intimately linked: indeed, the second one reduces to the first one when the buyers' utilities are modified to incorporate their optimal trades along the market tariff. Our existence and uniqueness results suggest that entry-proofness is a simple and powerful way to characterize the competitive outcomes of nonexclusive markets.

The JHG allocation and the JHG tariff that implements it emerge as the extension of Akerlof (1970) pricing to a rich class of preferences. The JHG allocation can be decomposed into successive layers, each of them priced at the expected cost of serving those types who trade it. This particular structure has motivated us to design a discriminatory ascending auction. In contrast with the simultaneous competitive-screening games so far studied in the literature, which generally conclude to the nonexistence of equilibrium, this sequential auction essentially uniquely implements the JHG allocation. Beyond making a theoretical point, this result offers a useful complement to studies that advocate a transformation of continuous markets into batch auctions, so as to avoid inefficiencies linked to high-frequency trading (Budish, Cramton, and Shin (2015)).

Although an empirical illustration is beyond the scope of this paper, our results suggest new avenues for empirical work. In the context of insurance, nonexclusive markets have been so far investigated through the lens of exclusive-competition models, exploiting the observation that, under adverse selection, there should be a positive correlation between the coverage purchased by a consumer and her risk (Chiappori and Salanié (2000)).[20] An alternative approach building on our analysis would be to exploit price and cost data to compare the price of successive layers of insurance to their average cost, as measured by the empirical loss frequency of the consumers who trade them. This approach would extend that proposed by Einav, Finkelstein, and Cullen (2010) to richer environments where firms offer insurance tariffs and consumers can combine different levels of coverage from different firms, and that proposed by Hendren (2013) to the case of active markets. Estimates of upper-tail conditional expectations of unit costs should arguably be a key variable for future tests of adverse selection in nonexclusive insurance markets.

Finally, it is fair to acknowledge a limitation of our analysis. Following a time-honored tradition initiated by Akerlof (1970), Pauly (1974), and Rothschild and Stiglitz (1976), we have assumed that the buyers' private information is one-dimensional and that their types are ordered according to a single-crossing condition; moreover, we have assumed, in our analysis of active markets, that higher buyer types are on average more costly to serve, implying a weak form of adverse selection.[21] These restrictions stand in contrast with the important role of multidimensional private information documented in the recent empirical literature.[22] There are in comparison few theoretical analyses of this question, and they

[20]See, for instance, Cawley and Philipson's (1999) on life insurance, Finkelstein and Poterba (2004) on annuities, and Finkelstein and McGarry (2006) on long-term care.

[21]Recall that this last assumption is not needed for the analysis of inactive markets.

[22]See, for instance, Finkelstein and McGarry (2006), Fang, Keane, and Silverman (2008), and Einav, Finkelstein, and Schrimpf (2010) for empirical analyses of insurance markets on which consumers differ both in terms of riskiness and of risk-aversion.

have so far focused on the case of exclusive markets;[23] the main conceptual hurdle is that the single-crossing condition is typically no longer satisfied. An important challenge for future research is thus to understand the impact of multidimensional private information on the functioning of nonexclusive markets. Our hope is that the general methodology developed in this paper will prove useful to this end.

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Appendix

Proof of Theorem 1. The proof consists of three steps.

Step 1 We first formulate the entrant's problem. According to the revelation and taxation principles, there is no loss of generality in letting the entrant offer a menu of contracts $\{(q_1, t_1), \ldots, (q_I, t_I)\}$ that is incentive-compatible:

$$\text{For all } i \text{ and } j,\ u_i(q_i, t_i) \geq u_i(q_j, t_j),$$

and individually rational:

$$\text{For each } i,\ u_i(q_i, t_i) \geq u_i(0, 0).$$

We claim that, for any such menu, the buyer has a best response with quantities that are nondecreasing in her type. Indeed, if i optimally trades (q_i, t_i) and $j > i$ optimally trades (q_j, t_j), then it must be that $u_i(q_i, t_i) \geq u_i(q_j, t_j)$ and $u_j(q_j, t_j) \geq u_j(q_i, t_i)$. Now, suppose that $q_i > q_j$. Because $i < j$, applying weak single-crossing to the first inequality yields $u_j(q_i, t_i) \geq u_j(q_j, t_j)$, which, along with the second inequality, implies $u_j(q_i, t_i) = u_j(q_j, t_j)$. Hence type j could optimally trade (q_i, t_i) as well.[24] The same reasoning applies to any such pair (i, j) for which quantities are strictly decreasing, which proves the claim.

Because we want entry to be profitable no matter the buyer's best response, we are thus allowed to add the monotonicity constraint that quantities q_i be nondecreasing in i to the entrant's profit-maximization problem. We can also relax this problem by focusing on the downward local constraints, that is, the downward local incentive-compatibility constraints of types $i > 1$ and the individual-rationality constraint of type $i = 1$. The entrant's expected profit is thus bounded above by

$$\max\left\{\sum_i m_i(t_i - c_i q_i) : q_i \text{ is nondecreasing in } i \text{ and } u_i(q_i, t_i) \geq u_i(q_{i-1}, t_{i-1}) \text{ for all } i\right\},$$

where $(q_0, t_0) \equiv (0, 0)$. We call $\mathcal{P}$ this relaxed problem.

Step 2 We now prove that we can focus in $\mathcal{P}$ on menus with nonnegative transfers. Indeed, suppose that a menu $\{(q_1, t_1), \ldots, (q_I, t_I)\}$ satisfies all the constraints in $\mathcal{P}$, and is such that at least one type makes a strictly negative payment. Let i be the lowest such type. Then we can build a new menu by assigning (q_{i-1}, t_{i-1}) to both types $i - 1$ and i. We claim that this new menu satisfies all the constraints in $\mathcal{P}$. First, because the original menu displays nondecreasing quantities, so does the new menu. Second, the downward local constraint for type i is now an identity. Third, the downward local constraint for type $i + 1$, if such type exists, now writes as $u_{i+1}(q_{i+1}, t_{i+1}) \geq u_{i+1}(q_{i-1}, t_{i-1})$, which follows from observing that the

[24] Assuming strict single-crossing would enable us to reach a contradiction at this point, so that any best response of the buyer would feature nondecreasing quantities.

initial menu satisfies $u_{i+1}(q_{i+1}, t_{i+1}) \geq u_{i+1}(q_i, t_i)$, $q_i \geq q_{i-1}$, and $u_i(q_i, t_i) \geq u_i(q_{i-1}, t_{i-1})$, and from applying weak single-crossing to the last inequality. This proves the claim. The resulting variation in expected profit is, up to multiplication by m_i,

$$(t_{i-1} - c_i q_{i-1}) - (t_i - c_i q_i) = t_{i-1} - t_i + c_i(q_i - q_{i-1}),$$

which is strictly positive as $t_{i-1} \geq 0 > t_i$ by construction and $q_i \geq q_{i-1}$. It follows that the initial menu cannot be solution to $\mathcal{P}$. The entrant's expected profit is thus bounded above by the value of the problem $\mathcal{P}_+$ obtained by adding to $\mathcal{P}$ the constraints $t_i \geq 0$ for all i.

Step 3 Fix a menu $\{(q_1, t_1), \ldots, (q_I, t_I)\}$ that satisfies all the constraints in $\mathcal{P}_+$ and, for any type i, consider the trade (q_{i-1}, t_{i-1}). For $i = 1$, we clearly have $u_i(q_{i-1}, t_{i-1}) \geq u_i(0, 0)$ as $(q_0, t_0) = (0, 0)$. For $i > 1$, we know that type $i - 1$ weakly prefers (q_{i-1}, t_{i-1}) to $(0, 0)$. By weak single-crossing, so does type i. Thus, in any case, we have $u_i(q_{i-1}, t_{i-1}) \geq u_i(0, 0)$. Because $t_{i-1} \geq 0$, this shows that the indifference curve of type i going through (q_{i-1}, t_{i-1}) must cross the q-axis at some point $(\underline{q}_i, 0)$, with $\underline{q}_i \in [0, q_{i-1}]$. The argument in the text then shows that $\sum_i m_i(t_i - c_i q_i) \leq 0$. Hence the result. ■

Proof of Corollary 1. According to Footnote 24, strict single-crossing implies that any best response of the buyer features nondecreasing quantities. Suppose, by way of contradiction, that the entrant trades, so that $q_i > q_{i-1}$ for some type i. Because any such type's preferences are strictly convex and $u_i(q_i, t_i) \geq u_i(q_{i-1}, t_{i-1})$, the inequalities (3) now imply

$$t_i - t_{i-1} - \overline{c}_i(q_i - q_{i-1}) < 0.$$

Thus the expected profit (2) is strictly negative, a contradiction. Hence the result. ■

Proof of Theorem 2. The necessity part is shown in the text. Assume now that (i)–(iii) hold. According to (iii), T is defined over $[0, q_I]$, and it is convex because $\overline{c}_i$ is nondecreasing in i. The proof consists of two steps.

Step 1 We first check that T implements the quantities q_i, in the sense that, for each i, q_i maximizes $u_i(q, T(q))$ with respect to q. This is easily shown by induction. First, according to (ii), type 1 optimally chooses q_1 when facing the tariff $T_1(q) \equiv \overline{c}_1 q$. Because $T_1 \leq T$ and $T_1(q_1) = T(q_1)$, it follows that q_1 is indeed an optimal choice for type 1 when facing T. Next, suppose that type $i - 1$ optimally chooses q_{i-1} when facing T. By weak single-crossing, for type i we can then focus on quantities $q \geq q_{i-1}$. According to (ii), type i optimally chooses q_i when facing the tariff T_i that coincides with T for quantities $q \leq q_{i-1}$ and has slope $\overline{c}_i$ beyond q_{i-1}. Because $T_i \leq T$ and $T_i(q_i) = T(q_i)$, it follows that q_i is indeed an optimal choice of type i when facing T. This concludes the induction step.

Step 2 To conclude the proof, we only need to check that (6) holds for the tariff T defined by (i)–(iii). For each i, $\tau_i^T(0,0)$ is the supremum of the prices p such that

$$u_i(q_i, T(q_i)) < \max\{u_i(q+q', T(q)+pq') : q, q'\} \equiv U_i^T(p). \tag{14}$$

Let us compute $U_i^T(\bar{c}_i)$. Because $\partial^- T(q) \leq \bar{c}_i$ for $q < q_{i-1}$ and $\partial^- T(q) \geq \bar{c}_i$ for $q > q_{i-1}$, there exists for $p = \bar{c}_i$ a solution to the maximization problem in (14) such that $q = q_{i-1}$. It then follows from (ii) that $U_i^T(\bar{c}_i) = u_i(q_i, T(q_i))$. Thus (14) does not hold for $p = \bar{c}_i$, which implies $\tau_i^T(0,0) \leq \bar{c}_i$ as $U_i^T(p)$ is nonincreasing in p. Hence the result. ■

Proof of Theorem 3. We throughout set $\min \emptyset \equiv \infty$ and $\sum_{j<0} = \sum_{j>I} \equiv 0$. The proof consists of two steps.

Step 1 We first compute each seller's continuation profit in state (p, Q^-), which is the sum of all the expected profits he earns at prices $p' \geq p$ by trading with every type i such that $D_i(p) > Q^-$. In any of her best responses, any such type purchases Q^- at prices $p' < p$.

Case 1: $p > \bar{c}_I$ According to (iii), for any value of Q^-, each seller supplies an infinite quantity in state (p, Q^-). The best response we select for every type i is to equally split her residual demand in state (p, Q^-) between the sellers. Because each type can purchase her demand at price p, she makes no purchases at prices $p' > p$. Thus each seller's continuation profit in state (p, Q^-) is

$$\sum_i m_i(p - c_i)\,\frac{[D_i(p) - Q^-]^+}{K}. \tag{15}$$

Case 2: $\bar{c}_i < p \leq \bar{c}_{i+1}$ According to (ii), for any value of Q^-, each seller supplies an equal share of type i's residual demand in state (p, Q^-). The best response we select for every type $j < i$ is to equally split her residual demand in state (p, Q^-) between the sellers. Next, by strict single-crossing, every type $j \geq i$ purchases $[D_i(p) - Q^-]^+/K$ from each seller at price p. Finally, each seller earns a continuation profit, which is the sum of all his expected profits at prices $p' > p$. Because type i can purchase her demand at price p, she makes no purchases at prices $p' > p$. To characterize the types who make purchases in excess of $\max\{D_i(p), Q^-\}$, we rank the demands $\mathcal{D}_j \equiv D_j(\bar{c}_j + \Delta)$ according to the following recursive definition.

Definition 1 *Let* $r(1) \equiv 1$ *and, for each* ι, *let* $r(\iota+1) \equiv \min\{j : j > r(\iota)$ *and* $\mathcal{D}_j > \mathcal{D}_{r(\iota)}\}$; *finally, let* $\bar{\iota} \equiv \max\{\iota : r(\iota) < \infty\}$ *and* $\mathcal{D}_\infty \equiv \infty$.

Now, let $\iota_i(p, Q^-) \equiv \min\{\iota : r(\iota) > i \text{ and } \mathcal{D}_{r(\iota)} > \max\{D_i(p), Q^-\}\}$. According to (ii), $\bar{c}_{r(\iota_i(p,Q^-))} + \Delta$ is the first price at which a quantity in excess of $\max\{D_i(p), Q^-\}$ is supplied, and $r(\iota_i(p,Q^-))$ is the first type willing to purchase some of it. By strict single-crossing, every type $j \geq r(\iota_i(p,Q^-))$ purchases $(\mathcal{D}_{r(\iota_i(p,Q^-))} - \max\{D_i(p),Q^-\})/K$ from each seller at price $\overline{c}_{r(\iota_i(p,Q^-))} + \Delta$, so that the expected margin on these trades is Δ. Next, every type $j \geq r(\iota_i(p,Q^-)+1)$ purchases $(\mathcal{D}_{r(\iota_i(p,Q^-)+1)} - \mathcal{D}_{r(\iota_i(p,Q^-))})/K$ from each seller at price $\overline{c}_{r(\iota_i(p,Q^-)+1)} + \Delta$—and so on. Thus each seller's continuation profit in state (p,Q^-) is

$$\begin{aligned}\sum_{j<i} m_j(p-c_j)\,\frac{[D_j(p)-Q^-]^+}{K}\\ +\left(\sum_{j\geq i} m_j\right)(p-\overline{c}_i)\,\frac{[D_i(p)-Q^-]^+}{K}\\ +\left(\sum_{j\geq r(\iota_i(p,Q^-))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_i(p,Q^-))}-\max\{D_i(p),Q^-\}}{K}\\ +\sum_{\iota=\iota_i(p,Q^-)+1}^{\overline{\iota}}\left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)}-\mathcal{D}_{r(\iota-1)}}{K}.\end{aligned} \tag{16}$$

Case 3: $p \leq \overline{c}_1$ Let $\iota_0(Q^-) \equiv \min\{\iota : \mathcal{D}_{r(\iota)} > Q^-\}$. According to (i)–(ii), $\overline{c}_{r(\iota_0(Q^-))} + \Delta$ is the first price at which a quantity in excess of Q^- is supplied, and $r(\iota_0(Q^-))$ is the first type willing to purchase some of it. Thus each seller's continuation profit in state (p,Q^-) is

$$\left(\sum_{j\geq r(\iota_0(Q^-))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_0(Q^-))}-Q^-}{K}+\sum_{\iota=\iota_0(Q^-)+1}^{\overline{\iota}}\left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)}-\mathcal{D}_{r(\iota-1)}}{K}.$$

Step 2 We now check that no seller can strictly increase his continuation profit by deviating from the candidate-equilibrium strategy. As the buyer's decisions to purchase from each seller at prices $p' < p$ do not depend on the offers he makes at prices $p' \geq p$, this implies that no deviation is profitable.

Case 1: $p > \overline{c}_I$ According to (iii), for any value of Q^-, each seller supplies an infinite quantity in state (p,Q^-). If a seller deviates to a finite supply s, then the best response we select for every type i is to purchase $\min\{s,[D_i(p)-Q^-]^+/K\}$ from him at price p. Because $p > \overline{c}_I \geq c_i$ for all i, (15) implies that the deviating seller cannot thereby strictly increase his continuation profit. Thus no seller has an incentive to deviate, and the game ends in state $(p+\Delta,\infty)$. In particular, whatever the sellers' decisions at prices $p' \leq \overline{c}_I$, the highest price at which trade can take place is $\overline{c}_I + \Delta$. This allows us to apply the one-shot deviation property at prices $p' \leq \overline{c}_I$.

Case 2: $\overline{c}_i < p \leq \overline{c}_{i+1}$ According to (ii), for any value of Q^-, each seller supplies an equal share of type i's residual demand in state (p,Q^-). If a seller deviates and supplies s, the aggregate supply at prices $p' \leq p$ becomes

$$S(p, Q^-, s) \equiv Q^- + \frac{K-1}{K}\,[D_i(p) - Q^-]^+ + s. \tag{17}$$

We consider two types of deviations in turn.

Downward Deviations If $D_i(p) > Q^-$, a seller can deviate to $s < [D_i(p) - Q^-]/K$. We compute his continuation profit from doing so by using the one-shot deviation property. First, the best response we select for every type $j < i$ is to purchase $\min\{s, [D_j(p) - Q^-]^+/K\}$ from him at price p. Next, type i is rationed at price p because the aggregate supply at prices $p' \leq p$ is $S(p, Q^-, s) < D_i(p)$ by (17). Hence, by strict single-crossing, every type $j \geq i$ purchases s from the deviating seller at price p. Finally, the deviating seller earns a continuation profit, which is the sum of all his expected profits at prices $p' > p$ and can be computed as in (16), with p replaced by $p + \Delta$ and Q^- replaced by $S(p, Q^-, s)$. Thus each seller's continuation profit from deviating to $s < [D_i(p) - Q^-]/K$ in state (p, Q^-) and returning to equilibrium play afterwards is

$$\begin{aligned}
&\sum_{j<i} m_j \left[(p - c_j)\min\left\{s, \frac{[D_j(p) - Q^-]^+}{K}\right\} + (p + \Delta - c_j)\,\frac{[D_j(p+\Delta) - S(p, Q^-, s)]^+}{K}\right] \\
&\qquad + \left(\sum_{j\geq i} m_j\right)(p - \bar{c}_i)s \\
&\qquad + \left(\sum_{j\geq i} m_j\right)(p + \Delta - \bar{c}_i)\,\frac{[D_i(p+\Delta) - S(p, Q^-, s)]^+}{K} \\
&\qquad + \left(\sum_{j\geq r(\iota_i(p+\Delta, S(p,Q^-,s)))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_i(p+\Delta, S(p,Q^-,s)))} - \max\{D_i(p+\Delta), S(p, Q^-, s)\}}{K} \\
&\qquad + \sum_{\iota=\iota_i(p+\Delta, S(p,Q^-,s))+1}^{\bar{\iota}} \left(\sum_{j\geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}.
\end{aligned} \tag{18}$$

To compare this to (16), we use the definition (17) of $S(p, Q^-, s)$. As $D_j(p+\Delta) > S(p, Q^-, s)$ implies $D_j(p) > Q^-$, we first obtain that the coefficient of s in each term of the first sum in (18), when different from zero, is at least

$$(p - c_j)\left(1 - \frac{1}{K}\right) - \frac{\Delta}{K} \geq \left(1 - \frac{2}{K}\right)\Delta \geq 0$$

because $p \geq \bar{c}_i + \Delta \geq c_j + \Delta$ for $j < i$, and $K \geq 2$. Similarly, by distinguishing whether $D_i(p + \Delta)$ is higher or lower than $S(p, Q^-, s)$, we obtain that the coefficient of s in the next three terms in (18) is at least

$$\left(\sum_{j\geq i} m_j\right)\left[(p - \bar{c}_i)\left(1 - \frac{1}{K}\right) - \frac{\Delta}{K}\right] \geq \left(\sum_{j\geq i} m_j\right)\left(1 - \frac{2}{K}\right)\Delta \geq 0$$

because $p \geq \bar{c}_i + \Delta$ and $K \geq 2$. Hence supplying $s' \in (s, [D_i(p) - Q^-]/K]$ instead of s never decreases the deviating seller's continuation profit as long as $\iota_i(p + \Delta, S(p, Q^-, s'))$ remains constant. Eventually, however, this index may jump up, in which case the last sum in (18) jumps down. When s' is close to but below the value at which such a jump occurs, then $\max\{D_i(p+\Delta), S(p, Q^-, s')\} = S(p, Q^-, s')$ becomes close to $\mathcal{D}_{r(\iota_i(p+\Delta, S(p,Q^-,s')))}$, and hence the third and fourth terms in (18) vanish while the second term in (18) becomes close to

$$\left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i)\left\{\mathcal{D}_{r(\iota_i(p+\Delta,S(p,Q^-,s')))} - Q^- - \frac{K-1}{K}\,[D_i(p) - Q^-]\right\}.$$

As the first sum in (18) is at most equal to the first sum in (16), this reasoning shows that all we need to prove is that $\pi \geq \pi(\hat{\iota})$ for all $\hat{\iota} = \iota_i(p+\Delta, S(p, Q^-, s)), \ldots, \iota_i(p, Q^-) - 1$, where

$$\pi \equiv \left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i)\,\frac{D_i(p) - Q^-}{K} + \left(\sum_{j \geq r(\iota_i(p,Q^-))} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota_i(p,Q^-))} - D_i(p)}{K} + \sum_{\iota=\iota_i(p,Q^-)+1}^{\bar{\iota}}\left(\sum_{j \geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}$$

and, for any such $\hat{\iota}$,

$$\pi(\hat{\iota}) \equiv \left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i)\left\{\mathcal{D}_{r(\hat{\iota})} - Q^- - \frac{K-1}{K}\,[D_i(p) - Q^-]\right\} + \sum_{\iota=\hat{\iota}+1}^{\bar{\iota}}\left(\sum_{j \geq r(\iota)} m_j\right)\Delta\,\frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota-1)}}{K}.$$

For each $\hat{\iota} = \iota_i(p + \Delta, S(p, Q^-, s)) + 1, \ldots, \iota_i(p, Q^-) - 1$, we have

$$\pi(\hat{\iota}) - \pi(\hat{\iota} - 1) = \left[\left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i) - \left(\sum_{j \geq r(\hat{\iota})} m_j\right)\frac{\Delta}{K}\right](\mathcal{D}_{r(\hat{\iota})} - \mathcal{D}_{r(\hat{\iota}-1)}),$$

which is strictly positive because $r(\hat{\iota}) > i$, $p \geq \bar{c}_i + \Delta$, $K \geq 2$, and $\mathcal{D}_{r(\hat{\iota})} > \mathcal{D}_{r(\hat{\iota}-1)}$. Hence, to conclude, we only need to check that $\pi \geq \pi(\iota_i(p, Q^-) - 1)$. We have

$$\pi - \pi(\iota_i(p, Q^-) - 1) = \left[\left(\sum_{j \geq i} m_j\right)(p - \bar{c}_i) - \left(\sum_{j \geq r(\iota_i(p,Q^-))} m_j\right)\frac{\Delta}{K}\right][D_i(p) - \mathcal{D}_{r(\iota_i(p,Q^-)-1)}],$$

which is nonnegative because $r(\iota_i(p, Q^-)) > i$, $p \geq \bar{c}_i + \Delta$, $K \geq 2$, and $D_i(p) \geq \mathcal{D}_{r(\iota_i(p,Q^-)-1)}$ by definition of $\iota_i(p, Q^-)$. This concludes the proof that no deviation to $s < [D_i(p) - Q^-]/K$ can increase a seller's continuation profit in state (p, Q^-).

Upward Deviations A seller can deviate to $s > [D_i(p) - Q^-]^+/K$. We compute his continuation profit from doing so by using the one-shot deviation property. First, the best response we select for every type $j \leq i$ is to purchase $[D_j(p) - Q^-]^+/K$ from him at price p. Next, the best response we select for every type $j > i$ is to purchase

$$\min \left\{ s, [D_j(p) - Q^-]^+ - \frac{K-1}{K} [D_i(p) - Q^-]^+ \right\} \geq \frac{1}{K} [D_i(p) - Q^-]^+ \tag{19}$$

from him at price p, reflecting that any such type can first purchase $[D_i(p) - Q^-]^+/K$ from each of the nondeviating sellers and then purchase any additional quantity she is willing to purchase at price p from the deviating seller, within the limit s. Finally, the deviating seller earns a continuation profit, which can be computed as in the case of downward deviations. Thus each seller's continuation profit from deviating to $s > [D_i(p) - Q^-]^+/K$ in state (p, Q^-) and returning to equilibrium play afterwards is

$$\begin{aligned} &\sum_{j<i} m_j (p - c_j) \frac{[D_j(p) - Q^-]^+}{K} \\ &+ \sum_{j \geq i} m_j (p - c_j) \min \left\{ s, [D_j(p) - Q^-]^+ - \frac{K-1}{K} [D_i(p) - Q^-]^+ \right\} \\ &+ \left(\sum_{j \geq r(\iota_i(p, S(p, Q^-, s)))} m_j \right) \Delta \frac{\mathcal{D}_{r(\iota_i(p, S(p, Q^-, s)))} - S(p, Q^-, s)}{K} \\ &+ \sum_{\iota = \iota_i(p, S(p, Q^-, s)) + 1}^{\bar{\iota}} \left(\sum_{j \geq r(\iota)} m_j \right) \Delta \frac{\mathcal{D}_{r(\iota)} - \mathcal{D}_{r(\iota - 1)}}{K}. \end{aligned} \tag{20}$$

The first sum in (20) is the same as in (16). Next, using a summation by parts and (19), we obtain that the second sum in (20) is of the form

$$\sum_{j \geq i} \left(\sum_{k \geq j} m_k \right) (p - \bar{c}_j)(q_j - q_{j-1})$$

for nondecreasing quantities $(q_j)_{j=i-1}^I$ such that $q_{i-1} \equiv 0$ and $q_i \equiv [D_i(p) - Q^-]^+/K$. Because $p \leq \bar{c}_j$ for all $j > i$, this sum is at most equal to its first term corresponding to $j = i$, which itself is equal to the second term in (16). Finally, $S(p, Q^-, s) > \max\{D_i(p), Q^-\}$ and $\iota_i(p, S(p, Q^-, s)) \geq \iota_i(p, Q^-)$ imply that the last two terms of (20) are at most equal to the last two terms of (16). This concludes the proof that no deviation to $s > [D_i(p) - Q^-]^+/K$ can increase a seller's continuation profit in state (p, Q^-).

Case 3: $p \leq \bar{c}_1$ According to (i), for any value of Q^-, each seller supplies a zero quantity at price p. Thus no downward deviation is feasible. The proof that no upward deviation can increase a seller's continuation profit in state (p, Q^-) is similar to that provided in Case 2 and is thus omitted. Hence the result. ■

Proof of Theorem 4. Every type i's preferences can be represented by $U_i(q) - t$ for some strictly concave utility function U_i that is differentiable over $\mathbb{R}_{++}$. The Inada condition (12), which is here equivalent to $\lim_{q \to \infty} U_i'(q) \leq 0$, ensures that $D_i(p) < \infty$ except perhaps for $p = 0$. We will often use the property that, when facing a convex market tariff, each type optimally purchases the sellers' aggregate supply until her demand is satisfied at some price or, equivalently, until the price exceeds her willingness-to-pay. Therefore, if type i trades at price p, then she overall purchases at most $D_i(p)$; if she at least purchases $q > 0$, then she is not willing to trade at prices $p > U_i'(q)$.

We first dispose of the case where the JHG allocation $(q_i^*, t_i^*)_{i=1}^I$ is degenerate, that is, $q_I^* = 0$. Then, by Theorems 1–2, Condition EP is satisfied. As the buyer's preferences are strictly convex and types are ordered according to the strict single-crossing condition, it follows from Corollary 1 that there is market breakdown. Thus no trade takes place in any equilibrium of any game Γ_n, and each equilibrium implements the degenerate JHG allocation. Hence the result.

From now on, we assume that the JHG allocation is nondegenerate, that is, $q_I^* > 0$. By Theorem 2, this amounts to assuming that there exists some i such that $U_i'(0) > \bar{c}_i$ or, equivalently, that $D_i(\bar{c}_i) > 0$.

Our first task is to show that we can put uniform bounds on equilibrium prices and quantities. The proof of the following lemma—and of all the intermediary results used in the proof of Theorem 4—is provided in Appendix A.

Lemma 2 *There exist a finite price $\overline{p}$ and finite quantities $\overline{q} > \underline{q} > 0$ such that, for n high enough, in any equilibrium of Γ_n type I is not willing to trade at prices strictly higher than $\overline{p}$ and purchases an aggregate quantity in $[\underline{q}, \overline{q}]$.*

Thanks to this result, we can in what follows consider that the auction ends when price $\overline{p}$ is reached, that supply functions are defined over $[0, \overline{p}]$ and bounded above by $\overline{q}$, and that tariffs are defined over $[0, \overline{q}]$. This makes no difference for the quantities chosen by the buyer on the equilibrium path, and the profitability of the deviation we shall soon consider does not depend on the values of these functions at higher arguments. Hence, for n high enough and for any equilibrium of Γ_n, there exists a finite highest price $p_{i,n}$ at which type i trades on the equilibrium path; we set $p_{i,n} \equiv U_i'(0)$ if type i does not trade. By strict single-crossing, $p_{1,n} \leq p_{2,n} \leq \ldots \leq p_{I,n} < \overline{p}$. Let $q_{i,n}$ be the aggregate quantity purchased by type i on the equilibrium path. By strict single-crossing again, $q_{1,n} \leq q_{2,n} \leq \ldots \leq q_{I,n} \leq \overline{q}$.

From now on, we fix a sequence of equilibria of $(\Gamma_n)_{n \in \mathbb{N}}$ that are robust to irrelevant offers. For each n, the following objects are defined on the equilibrium path:

- $s_n^k(p)$, seller k's supply at price p;
- $s_n(p) \equiv \sum_k s_n^k(p)$, the aggregate supply at price p;
- $s_n^{-k}(p) \equiv s_n(p) - s_n^k(p)$, the aggregate supply of sellers other than k at price p;
- $S_n(p) \equiv \sum_{p' \leq p} s_n(p')$, the aggregate supply at prices lower than or equal to p;
- $\pi_n^k(p)$, seller k's expected profit at price p;
- $\gamma_n^k(p) \equiv \sum_{p' \geq p} \pi_n^k(p')$, seller k's continuation profit at price p.

In any equilibrium of Γ_n that is robust to irrelevant offers, each seller anticipates that deviating at prices $p' \geq p$ will not affect the buyer's decisions at prices $p' < p$. We can thus focus on continuation profits as in the proof of Theorem 3. This, in particular, implies that $\gamma_n^k(p) \geq 0$ for all k and p; otherwise, seller k could strictly increase his expected profit by withdrawing his offers at prices $p' \geq p$.

To formulate our convergence result, we extend the supply functions $(S_n)_{n \in \mathbb{N}}$ to the whole of $[0, \overline{p}]$ by letting

$$\text{For all } n \text{ and } p,\ S_n(p) \equiv S_n(\Delta_n \lfloor p/\Delta_n \rfloor),$$

where $\lfloor p/\Delta_n \rfloor$ is the integer part of p/Δ_n. By construction, for each n, the function S_n is nondecreasing and right-continuous; moreover, for n high enough, $S_n(p) \in [0, \overline{q}]$ for all $p \in [0, \overline{p}]$. Therefore, by Helly's selection theorem (Billingsley (1995, Theorem 25.9)), there exists a nondecreasing right-continuous function S_∞ and a subsequence of $(S_n)_{n \in \mathbb{N}}$ that converges pointwise to S_∞ over $[0, \overline{p}]$ at the continuity points of S_∞. In what follows, and with no loss of generality, we take this subsequence to be the original sequence $(S_n)_{n \in \mathbb{N}}$. The marginal tariffs associated to S_n and S_∞ are their generalized inverses

$$\text{For each } q \in [0, \overline{q}],\ t_n(q) \equiv \inf\{p : q \leq S_n(p)\} \text{ and } t_\infty(q) \equiv \inf\{p : q \leq S_\infty(p)\},$$

with $\inf \emptyset \equiv \overline{p}$; they are nondecreasing and left-continuous. It follows from the proof of Skorokhod's representation theorem (Billingsley (1995, Theorem 25.6)) that the sequence $(t_n)_{n \in \mathbb{N}}$ converges pointwise to t_∞ at the continuity points of t_∞, that is, everywhere over $[0, \overline{q}]$ except at countably many points. Letting T_n and T_∞ be the convex tariffs obtained by integrating the marginal tariffs t_n and t_∞, we then have

$$\sup_{q \in [0, \overline{q}]} |T_n(q) - T_\infty(q)| = \sup_{q \in [0, \overline{q}]} \left| \int_0^q [t_n(x) - t_\infty(x)] \, \mathrm{d}x \right| \leq \int_0^{\overline{q}} |t_n(x) - t_\infty(x)| \, \mathrm{d}x,$$

which converges to zero by the bounded convergence theorem as the functions $(t_n)_{n \in \mathbb{N}}$ are uniformly bounded by $\overline{p}$ and converge pointwise to t_∞ except at countably many points. Thus the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T_∞. This implies that the graph of T_∞ is the closed limit of the graph of T_n when n goes to ∞ (Aliprantis and Border (2006, Definition 3.80)). As a result, and because every type i has strictly convex preferences, we can conclude from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)) that $q_{i,\infty} \equiv \lim_{n\to\infty} q_{i,n}$ is well-defined and is the unique optimal choice of type i against the limit tariff T_∞. By Lemma 2, we have $q_{I,\infty} \geq \underline{q} > 0$.

With these preliminaries at hand, we turn to our main argument. As $\lim_{n\to\infty} q_{i,n} = q_{i,\infty}$ and the sequence $(T_n)_{n\in\mathbb{N}}$ converges uniformly to T_∞, $\lim_{n\to\infty} T_n(q_{i,n}) = T_\infty(q_{i,\infty})$. Our goal is to show that $(q_{i,\infty}, T_\infty(q_{i,\infty}))_{i=1}^I$ is the JHG allocation. We will rely on the following characterization of the JHG allocation, which is of independent interest.

Lemma 3 *The allocation implemented by a convex tariff T is the JHG allocation if and only if it is budget-feasible and*

$$\textit{For all } p \textit{ and } s,\ B(p,s) \equiv \sum_i m_i(p-c_i)\min\{[D_i(p)-S(p^-)]^+, s\} \leq 0, \tag{21}$$

where S is the supply function associated to T and $S(p^-) \equiv \lim_{p'\uparrow p} S(p')$.

We denote by B_n and B_∞ the functions B in (21) obtained for $S = S_n$ and $S = S_\infty$, respectively. A key observation is that the functions π_n^k, B_n, and s_n are related as follows:

$$\text{For each } p,\ \sum_k \pi_n^k(p) = B_n(p, s_n(p)). \tag{22}$$

The allocation $(q_{i,\infty}, T_\infty(q_{i,\infty}))_{i=1}^I$ is budget-balanced as it is the limit of the equilibrium allocations $(q_{i,n}, T_n(q_{i,n}))_{i=1}^I$. Hence, by Lemma 3, it coincides with the JHG allocation if and only if (21) holds for B_∞. Thus suppose, by way of contradiction, that there exists some p such that $B_\infty^*(p) \equiv \max\{B_\infty(p,s) : s \geq 0\} > 0$, and let $\hat{p}_\infty$ the supremum of such p. Our next result gathers useful properties related to this threshold.

Lemma 4 *The following holds:*

(i) $\hat{p}_\infty \leq \overline{p}$.

(ii) $B_\infty^*(p) > 0$ *if and only if there exists some i such that $p > \overline{c}_i$ and $D_i(p) > S_\infty(p^-)$.*

(iii) The highest i satisfying the property in (ii) is equal to a constant $\hat{\imath}_\infty$ for all p in an open left-neighborhood $\mathcal{V}$ of $\hat{p}_\infty$ and

$$\textit{For each } p \in \mathcal{V},\ D_{\hat{\imath}_\infty}(p) - S_\infty(p^-) \in \arg\max\{B_\infty(p,s) : s \geq 0\}. \tag{23}$$

Using Lemma 4 along with the definition of B_∞ and the left-continuity of the mapping $p \mapsto S_\infty(p^-)$, we can select p_0 arbitrarily close to $\hat{p}_\infty$ such that: (1) $B^*_\infty(p_0) > 0$; (2) $p_0 \in \mathcal{V}$; (3) p_0 is a continuity point of S_∞; (4) p_0 is a multiple of Δ_n for n high enough. Any seller k can then deviate in Γ_n when price p_0 is quoted by supplying

$$\hat{s}^k_n \equiv [D_{\hat{\imath}_\infty}(p_0) - S_n(p_0^-) - s^{-k}_n(p_0)]^+$$

at price p_0 and nothing afterwards. Because $p_0 \in \mathcal{V}$, we have $D_{\hat{\imath}_\infty}(p_0) > S_\infty(p_0^-)$. Moreover, because p_0 is a continuity point of S_∞, we have $S_\infty(p_0^-) = S_\infty(p_0) = \lim_{n\to\infty} S_n(p_0)$. Finally, for each n, $S_n(p_0) \geq S_n(p_0^-) + s^{-k}_n(p_0)$ by definition. Thus, for n high enough, $\hat{s}^k_n$ is strictly positive, and this deviation is nontrivial.

How do the different types react to this deviation, and what is the impact on seller k's continuation profit at price p_0? Observe first that trading with any type $i \leq \hat{\imath}_\infty$ at price p_0 is always profitable as $c_i \leq \bar{c}_{\hat{\imath}_\infty} < p_0$. Thus, from seller k's perspective, any such type will at worst first exhaust his competitors' supply $s^{-k}_n(p_0)$ at price p_0 before purchasing anything from him. That is, her residual demand for the quantity $\hat{s}^k_n$ supplied by seller k at price p_0 is $[D_i(p_0) - S_n(p_0^-) - s^{-k}_n(p_0)]^+$. In particular, type $\hat{\imath}_\infty$ has a unique best response at price p_0 that involves purchasing $\hat{s}^k_n$ from seller k. By strict single-crossing, this a fortiori holds for types $i > \hat{\imath}_\infty$. Therefore, we can conclude that every seller k's continuation profit $\gamma^k_n(p_0)$ at price p_0 is at least $A_n(s^{-k}_n(p_0))$, where

$$\text{For each } s,\ A_n(s) \equiv \sum_i m_i(p_0 - c_i)[\min\{D_i(p_0), D_{\hat{\imath}_\infty}(p_0)\} - S_n(p_0^-) - s]^+.$$

We now aggregate these profits. Because $A_n(s)$ is convex in s, we have

$$\sum_k A_n(s^{-k}_n(p_0)) \geq K A_n\left(\frac{1}{K}\sum_k s^{-k}_n(p_0)\right) = K A_n\left(\frac{K-1}{K}\, s_n(p_0)\right)$$

by Jensen's inequality. As p_0 is a continuity point of S_∞, $\lim_{n\to\infty} S_n(p_0^-) = S_\infty(p_0) = S_\infty(p_0^-)$ and $\lim_{n\to\infty} s_n(p_0) = 0$. Thus

$$\begin{aligned}
\lim_{n\to\infty} A_n\left(\frac{K-1}{K}\, s_n(p_0)\right) &= \sum_i m_i(p_0 - c_i)[\min\{D_i(p_0), D_{\hat{\imath}_\infty}(p_0)\} - S_\infty(p_0^-)]^+ \\
&= \sum_i m_i(p_0 - c_i)\min\{[D_i(p_0) - S_\infty(p_0^-)]^+, D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0^-)\} \\
&= B_\infty(p_0, D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0^-)) \\
&= B^*_\infty(p_0),
\end{aligned}$$

where the fourth equality follows from (23). Hence the aggregate equilibrium continuation profits at p_0 satisfy

$$\liminf_{n\to\infty} \sum_k \gamma^k_n(p_0) \geq K B^*_\infty(p_0) > 0. \tag{24}$$

Our goal in the remainder of the proof consists in deriving an upper bound on aggregate continuation profits that contradicts (24) for an appropriate choice of p_0.

To this end, we first provide an alternative expression for those profits. For each n, summing over the multiples $p \geq p_0$ of Δ_n and taking advantage of Lemma 2 and (22) yields

$$\begin{aligned}
\sum_k \gamma_n^k(p_0) &= \sum_{p \geq p_0} \sum_k \pi_n^k(p) \\
&= \sum_{\overline{p} \geq p \geq p_0} B_n(p, s_n(p)) \\
&= \sum_{\overline{p} \geq p \geq p_0} \sum_i m_i(p - c_i) \min\{[D_i(p) - S_n(p^-)]^+, S_n(p) - S_n(p^-)\} \qquad (25) \\
&= \sum_{\overline{p} \geq p \geq p_0} \sum_i m_i(p - c_i)[\min\{D_i(p), S_n(p)\} - S_n(p^-)]^+ \\
&= \sum_i \int_{[p_0, \overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,n}(\mathrm{d}p),
\end{aligned}$$

where $\boldsymbol{\sigma}_{i,n}$ is the measure with finite support defined by

$$\text{For each } p \in [0, \overline{p}],\ \boldsymbol{\sigma}_{i,n}(\{p\}) \equiv [\min\{D_i(p), S_n(p)\} - S_n(p^-)]^+. \qquad (26)$$

This is a Borel measure over $[0, \overline{p}]$ of at most mass $\overline{q}$. As is customary, let us endow the space of such measures with the weak* topology generated by all continuous real-valued functions. The following lemma then characterizes the weak* limit of the sequence $(\boldsymbol{\sigma}_{i,n})_{n \in \mathbb{N}}$.

Lemma 5 *Let $\overline{p}_{i,\infty} \equiv \inf\{p \in [0, \overline{p}] : S_\infty(p) \geq D_i(p)\}$. Then the unique measure $\boldsymbol{\sigma}_{i,\infty}$ over the Borel sets of $[0, \overline{p}]$ such that*

$$\textit{For each } p \in [0, \overline{p}],\ \boldsymbol{\sigma}_{i,\infty}([0, p]) \equiv \min\{S_\infty(p), D_i(\overline{p}_{i,\infty})\} \qquad (27)$$

is the weak limit of the sequence $(\boldsymbol{\sigma}_{i,n})_{n \in \mathbb{N}}$.*

Because p_0 is a continuity point of S_∞ and hence not an atom of $\boldsymbol{\sigma}_{i,\infty}$ for all i, it follows from (25) and Lemma 5 that

$$\lim_{n \to \infty} \sum_k \gamma_n^k(p_0) = \sum_i \int_{[p_0, \overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p). \qquad (28)$$

The idea is now to cut this integral into two pieces. The following lemma reflects the intuitive idea that there are no profits to be earned at prices $p > \hat{p}_\infty$ as $B_\infty^*(p) = 0$ for any such p.

Lemma 6 *If $p_1 > \hat{p}_\infty$, then*

$$\sum_i \int_{(p_1, \overline{p}]} m_i(p - c_i)\, \boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p) \leq 0.$$

Fix some $p_1 > \hat{p}_\infty$. Lemma 6 together with (24) and (28) implies

$$\begin{aligned} KB^*_\infty(p_0) &\leq \sum_i \int_{[p_0,p_1]} m_i(p-c_i)\,\boldsymbol{\sigma}_{i,\infty}(\mathrm{d}p) \\ &\leq \sum_i m_i(p_1-c_i)\,\boldsymbol{\sigma}_{i,\infty}([p_0,p_1]) \\ &= \sum_i m_i(p_1-c_i)\min\{[D_i(\overline{p}_{i,\infty})-S_\infty(p_0)]^+, S_\infty(p_1)-S_\infty(p_0)\}, \end{aligned}$$

where the equality follows from (27) along with the continuity of S_∞ at p_0. Because $p_1 > \hat{p}_\infty$ is arbitrary and S_∞ is right-continuous, it follows that

$$\sum_i m_i(\hat{p}_\infty-c_i)\min\{[D_i(\overline{p}_{i,\infty})-S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty)-S_\infty(p_0)\} \geq KB^*_\infty(p_0),$$

where p_0 can be arbitrarily close to $\hat{p}_\infty$. We now prove that this inequality leads to a contradiction, which completes the proof of Theorem 4. Observe that, because $B_\infty(p,s)$ is left-continuous in p, we have $\lim_{p_0\uparrow\hat{p}_\infty} B^*_\infty(p_0) = B^*_\infty(\hat{p}_\infty)$. We distinguish two cases.

Case 1 Suppose first that $B^*_\infty(\hat{p}_\infty) > 0$. Letting p_0 converge to $\hat{p}_\infty$ from below, we have

$$\sum_i m_i(\hat{p}_\infty-c_i)\min\{[D_i(\overline{p}_{i,\infty})-S_\infty(\hat{p}_\infty^-)]^+, S_\infty(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)\} \geq KB^*_\infty(\hat{p}_\infty). \tag{29}$$

If $\overline{p}_{i,\infty} = \hat{p}_\infty$, then it is obvious that each minimum in the left-hand side equals

$$\min\{[D_i(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)]^+, S_\infty(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)\}.$$

The same equality actually holds in all other cases. Indeed, if $\overline{p}_{i,\infty} < \hat{p}_\infty$, then $S_\infty(\hat{p}_\infty^-) \geq S_\infty(\overline{p}_{i,\infty}) \geq D_i(\overline{p}_{i,\infty}) > D_i(\hat{p}_\infty)$, and both minima are equal to zero, while, if $\overline{p}_{i,\infty} > \hat{p}_\infty$, then $D_i(\hat{p}_\infty) > D_i(\overline{p}_{i,\infty}) \geq S_\infty(\overline{p}_{i,\infty}^-) \geq S_\infty(\hat{p}_\infty) \geq S_\infty(\hat{p}_\infty^-)$, and both minima are equal to $S_\infty(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)$. It follows that the left-hand side of (29) equals

$$\begin{aligned} &\sum_i m_i(\hat{p}_\infty-c_i)\min\{[D_i(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)]^+, S_\infty(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)\} \\ &= B_\infty(\hat{p}_\infty, S_\infty(\hat{p}_\infty)-S_\infty(\hat{p}_\infty^-)) \\ &\leq B^*_\infty(\hat{p}_\infty), \end{aligned}$$

which contradicts (29) as $K \geq 2$. This case is thus impossible.

Case 2 Suppose next that $B^*_\infty(\hat{p}_\infty) = 0$. Then, proceeding as in Lemma 4, we obtain $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = S_\infty(\hat{p}_\infty^-)$. We have, for p_0 arbitrarily close to $\hat{p}_\infty$,

$$R(p_0) \equiv \frac{\sum_i m_i(\hat{p}_\infty-c_i)\min\{[D_i(\overline{p}_{i,\infty})-S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty)-S_\infty(p_0)\}}{\sum_i m_i(p_0-c_i)\min\{[D_i(p_0)-S_\infty(p_0)]^+, D_{\hat{\imath}_\infty}(p_0)-S_\infty(p_0)\}} \geq K, \tag{30}$$

using (23) along with the continuity of S_∞ at p_0. It follows from $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = S_\infty(\hat{p}_\infty^-)$ that $\overline{p}_{\hat{\imath}_\infty,\infty} = \hat{p}_\infty$ and $\overline{p}_{i,\infty} < \hat{p}_\infty$ for all $i < \hat{\imath}_\infty$. Hence, for any such i, $D_i(p_0) \leq D_i(\overline{p}_{i,\infty}) < S_\infty(p_0)$ for p_0 close enough to $\hat{p}_\infty$. This, in turn, implies that, for any such p_0, the denominator of $R(p_0)$ is equal to

$$\left(\sum_{i \geq \hat{\imath}_\infty} m_i\right)(p_0 - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0)], \tag{31}$$

while the numerator of $R(p_0)$ is equal to

$$\sum_{i \geq \hat{\imath}_\infty} m_i(\hat{p}_\infty - c_i) \min\left\{[D_i(\overline{p}_{i,\infty}) - S_\infty(p_0)]^+, S_\infty(\hat{p}_\infty) - S_\infty(p_0)\right\}.$$

Because $\overline{c}_i \geq \hat{p}_\infty$ for all $i > \hat{\imath}_\infty$ by definition of $\hat{\imath}_\infty$ and $D_{\hat{\imath}_\infty}(\hat{p}_\infty) = D_{\hat{\imath}_\infty}(\overline{p}_{\hat{\imath}_\infty,\infty}) \leq S_\infty(\overline{p}_{\hat{\imath}_\infty,\infty}) = S_\infty(\hat{p}_\infty)$, the numerator of $R(p_0)$ is bounded above by

$$\left(\sum_{i \geq \hat{\imath}_\infty} m_i\right)(\hat{p}_\infty - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(\hat{p}_\infty) - S_\infty(p_0)]. \tag{32}$$

Combining (30)–(32), we obtain

$$\frac{(\hat{p}_\infty - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(\hat{p}_\infty) - S_\infty(p_0)]}{(p_0 - \overline{c}_{\hat{\imath}_\infty})[D_{\hat{\imath}_\infty}(p_0) - S_\infty(p_0)]} \geq R(p_0) \geq K,$$

a contradiction as $K \geq 2$ and p_0 can be arbitrarily close to $\hat{p}_\infty$. Hence the result. ■