Multi-self-similar Random Fields and Multivariate Lamperti Transformation

Marc G. Genton*

Department of Statistics, Texas A&M University, College Station, TX 77843-3143, USA.

E-mail: genton@stat.tamu.edu

Olivier Perrin

GREMAQ, Université des Sciences Sociales, 21 Allée de Brienne, 31000 Toulouse, France, and

LERNA, INRA. E-mail: perrin@cict.fr

Murad S. Taqqu

Department of Mathematics and Statistics, Boston University, Boston, MA 02215, USA.

E-mail: murad@math.bu.edu

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Abstract

We define multi-self-similar random fields, that is, random fields that are self-similar component-wise. We characterize them, relate them to stationary random fields using a Lamperti-type transformation and study these stationary fields. We also extend the notion of local stationarity and local stationarity reducibility to random fields. Our work is motivated by applications arising from climatological and environmental sciences. We illustrate these new concepts with the fractional Brownian sheet and the Lévy fractional Brownian random field.

Keywords: Fractional Brownian sheet; Lamperti transformation; Lévy fractional Brownian random field; Local stationarity; Random fields; Reducibility; Self-similarity.

1 Introduction

Let $\{X(t), t \in \mathbb{R}\}$ be a real-valued stochastic process. It is self-similar with index $H > 0$ (H-ss) if for all $a > 0$, the finite-dimensional distributions of $\{X(at), t \in \mathbb{R}\}$ are identical to the finite-dimensional distributions of $\{a^H X(t), t \in \mathbb{R}\}$; see Taqqu (2003) for a thorough review.

The standard fractional Brownian motion $B_H(t)$ is a well-known example of an H-ss process with $0 < H \leq 1$. It is Gaussian, has zero mean stationary increments and covariance function

$$\mathrm{E}(B_H(t)B_H(u)) = \frac{1}{2}(|t|^{2H} + |u|^{2H} - |t-u|^{2H}). \tag{1}$$

A non-degenerate H-ss process cannot be stationary, but there is an important correspondence between self-similar and stationary processes, first presented by Lamperti (1962). Namely, if $\{X(t), t \in \mathbb{R}_+\}$ is H-ss, where $\mathbb{R}_+$ denotes the positive real line, then

$$Y(t) = e^{-tH}X(e^t), \quad t \in \mathbb{R}, \tag{2}$$

is stationary. Indeed, for any $h \in \mathbb{R}$,

$$\begin{aligned} Y(t+h) &= e^{-(t+h)H}X(e^h e^t) \\ &\stackrel{d}{=} e^{-tH}X(e^t) \\ &= Y(t), \end{aligned}$$

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. Conversely, if $\{Y(t), t \in \mathbb{R}\}$ is stationary, then

$$X(t) = t^H Y(\ln(t)), \quad t \in \mathbb{R}_+, \tag{3}$$

is H-ss. Relations (2) and (3) characterize H-ss processes in $\mathbb{R}_+$.

Suppose that Y has finite second moments and denote by $R(v)$ its stationary covariance. Then

$$\begin{aligned} \mathrm{Cov}(X(t), X(u)) &= t^H u^H R(\ln(t) - \ln(u)) && (4) \\ &= e^{H(\ln(t)+\ln(u))} R(\ln(t) - \ln(u)), && (5) \end{aligned}$$

which is a locally stationary reducible covariance according to the following definition of Genton and Perrin (2004).

Definition 1 *The process X has locally stationary reducible (LSR) covariance if*

$$Cov(X(t), X(u)) = R_1\left(\frac{g(t)+g(u)}{2}\right) R(g(t) - g(u)), \tag{6}$$

where R_1 is a nonnegative function, R is a stationary covariance, and g is a bijective deformation of the time index.

The covariance (6) is locally stationary after the time deformation g. Indeed, when R_1 is smooth enough, $\mathrm{Cov}(X(t), X(u)) \simeq R_2(g(t) - g(u))$ when $g(u) \in [g(t) - \frac{\epsilon}{2}, g(t) + \frac{\epsilon}{2}]$ for ϵ small enough.

It follows from (5) that self-similar processes with finite second moments are a subclass of LSR processes with

$$R_1(w) = e^{2Hw} \quad \text{and} \quad g(t) = \ln(t). \tag{7}$$

Note that, in this particular case, R_1 is an exponentially convex covariance (Loève, 1965, p. 414), that is $\sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j R_1(x_i + x_j) \geq 0$ for all finite sets of real coefficients $a_1, \ldots, a_m$ and points $x_1, \ldots, x_m \in \mathbb{R}$. In particular, for the fractional Brownian motion, straightforward computations yield

$$R(v) = \cosh(Hv) - 2^{(2H-1)}(\sinh(|v|/2))^{2H} \tag{8}$$

for the corresponding stationary covariance R, see e.g. Perrin and Senoussi (1999).

In this article, we consider real-valued random fields, that is, stochastic processes $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$, for which the parameter space is the Euclidean space $\mathbb{R}^n$, $n \geq 1$. The classical definition of self-similarity on $\mathbb{R}^n$, see Definition 8.1.1 in Samorodnitsky and Taqqu (1994), is analogous to the one for $\mathbb{R}$, namely,

Definition 2 *A random field $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$ is self-similar with index $H > 0$ (H-ss) if for all $a > 0$, the finite-dimensional distributions of $\{X(a\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$ are identical to the finite-dimensional distributions of $\{a^H X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$.*

Two well-known H-ss random fields indexed by $\mathbb{R}^n$ are the fractional Brownian sheet and the Lévy fractional Brownian random field, which are two different generalizations of the fractional Brownian motion indexed by $\mathbb{R}$.

The classical definition of self-similarity for $\mathbb{R}^n$ does not seem to be appropriate for the fractional Brownian sheet. Indeed, consider the Euclidean space $\mathbb{R}^2$ for illustration. Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2\}$ be a mean zero standardized fractional Brownian sheet with correlation

$$\mathrm{E}[X(\mathbf{t})X(\mathbf{u})] = \frac{1}{4}\left(|t_1|^{2H_1} + |u_1|^{2H_1} - |t_1 - u_1|^{2H_1}\right)\left(|t_2|^{2H_2} + |u_2|^{2H_2} - |t_2 - u_2|^{2H_2}\right), \tag{9}$$

where $\mathbf{t} = (t_1, t_2)^T$, $\mathbf{u} = (u_1, u_2)^T$, and $0 < H_1 \leq 1$, $0 < H_2 \leq 1$. Here, H_1 reflects the self-similarity in the first dimension, whereas H_2 reflects the self-similarity in the second dimension. According to Definition 2, X is H-ss with $H = H_1 + H_2$, so that this global index H does not reflect the self-similarity component-wise. Consequently, we propose in this article a new definition of self-similarity for random fields, called multi-self-similarity, that addresses the previous issue and includes the classical definition of self-similarity as a particular case.

Our definition of multi-self-similarity allows us to extend the Lamperti characterization to multi-self-similar random fields. We will see that the Lamperti characterization applies to the Lévy fractional Brownian random field in $\mathbb{R}^2$ when polar coordinates are used instead of the usual cartesian coordinates.

From the Lamperti characterization of random fields, we derive a multivariate Lamperti transformation and use it to generalize local stationarity reducibility to random fields. As in the one-dimensional case, multi-self-similar random fields with finite second moments are a subclass of locally stationary random fields. We give the explicit correlation structure of the reduced fractional Brownian sheet in $\mathbb{R}^n$ and the reduced Lévy fractional Brownian random field indexed by $\mathbb{R}^2$.

Our work is motivated by applications arising from climatological and environmental sciences. Indeed, climatologists have recently discovered patterns of self-similarity across temporal and spatial scales in tropical convection. Thus, there is a need for a rigorous and appropriate definition of self-similarity for random fields. Environmental scientists use statistical methods to reduce nonstationary spatial random fields to stationarity using a deformation of the index in $\mathbb{R}^2$ or $\mathbb{R}^3$. Perrin and Senoussi (2000) gave a characterization of these deformations and random fields. The multivariate Lamperti transformation mentioned above is a specific deformation of the spatial index and thus it is of particular interest for applications.

This article is organized as follows. In Section 2, we introduce a generalization of the notion of self-similarity to random fields and derive the associated Lamperti-type transformation. We further extend the notion of local stationarity and local stationarity reducibility to random fields and illustrate our results with the fractional Brownian sheet. We also discuss multi-self-similar random fields with superficial stationary increments. In Section 3, we study the Lévy fractional Brownian random field indexed by $\mathbb{R}^2$ and discuss properties of the associated stationary correlation function.

2 Multi-self-similar random fields

2.1 Multi-self-similarity

We use the following generalization of the notion of self-similarity to real-valued random fields. Let $\mathbb{R}_+^n$ denotes the n cartesian product $\mathbb{R}_+ \times \cdots \times \mathbb{R}_+$.

Definition 3 *A random field* $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ *is multi-self-similar with index* $\mathbf{H} = (H_1, \ldots, H_n)^T \in \mathbb{R}_+^n$ *(*$\mathbf{H}$*-mss) if*

$$\{X(a_1 t_1, \ldots, a_n t_n), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\} \stackrel{d}{=} \{a_1^{H_1} \cdots a_n^{H_n} X(t_1, \ldots, t_n), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}, \tag{10}$$

for all $a_1 > 0, \ldots, a_n > 0$*, where, as usual,* $\stackrel{d}{=}$ *denotes equality of the finite-dimensional distributions.*

Note that Definition 3 depends on the coordinate system $\mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n$ used to parameterize the random field X. If $a_1 = \cdots = a_n = a > 0$ and $H_1 + \cdots + H_n = H > 0$, then Definition 3 reduces to Definition 2, for which the self-similarity index is the same in all dimensions. In contrast, our definition allows for a component-wise self-similarity with possibly different self-similarity indices in each dimension. Based on Definition 3, we extend the Lamperti theorem (Lamperti, 1962) to random fields.

Proposition 1 *If $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n\}$ is $\mathbf{H}$-mss, then*

$$Y(\mathbf{t}) = e^{-\mathbf{t}^T\mathbf{H}} X(e^{t_1}, \ldots, e^{t_n}), \quad \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n, \tag{11}$$

is stationary. Conversely, if $\{Y(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ is stationary, then

$$X(\mathbf{t}) = t_1^{H_1} \cdots t_n^{H_n} Y(\ln(t_1), \ldots, \ln(t_n)), \quad \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n, \tag{12}$$

is $\mathbf{H}$-mss.

Proof. Let $\theta_1, \ldots, \theta_p$ be real numbers. If $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n\}$ is $\mathbf{H}$-mss, then for any $\mathbf{t}_1, \ldots, \mathbf{t}_p \in \mathbb{R}^n$, with $\mathbf{t}_j = (t_{j1}, \ldots, t_{jn})^T$, and $\mathbf{h} = (h_1, \ldots, h_n)^T \in \mathbb{R}^n$,

$$\begin{aligned}
\sum_{j=1}^p \theta_j Y(\mathbf{t}_j + \mathbf{h}) &= \sum_{j=1}^p \theta_j e^{-(t_{j1}H_1 + \cdots + t_{jn}H_n)} e^{-(h_1H_1 + \cdots + h_nH_n)} X(e^{h_1}e^{t_{j1}}, \ldots, e^{h_n}e^{t_{jn}}) \\
&\stackrel{d}{=} \sum_{j=1}^p \theta_j e^{-(t_{j1}H_1 + \cdots + t_{jn}H_n)} X(e^{t_{j1}}, \ldots, e^{t_{jn}}) \\
&= \sum_{j=1}^p \theta_j Y(\mathbf{t}_j),
\end{aligned}$$

proving that $\{Y(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ is stationary.

Conversely, if $\{Y(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ is stationary, then for any $\mathbf{t}_1, \ldots, \mathbf{t}_p \in \mathbb{R}_+^n$ and $a_1 > 0, \ldots, a_n > 0$,

$$\begin{aligned}
\sum_{j=1}^p \theta_j X(a_1t_{j1}, \ldots, a_nt_{jn}) &= \sum_{j=1}^p \theta_j a_1^{H_1} \cdots a_n^{H_n} t_{j1}^{H_1} \cdots t_{jn}^{H_n} Y(\ln(a_1) + \ln(t_{j1}), \ldots, \ln(a_n) + \ln(t_{jn})) \\
&\stackrel{d}{=} \sum_{j=1}^p \theta_j a_1^{H_1} \cdots a_n^{H_n} t_{j1}^{H_1} \cdots t_{jn}^{H_n} Y(\ln(t_{j1}), \ldots, \ln(t_{jn})) \\
&= \sum_{j=1}^p \theta_j a_1^{H_1} \cdots a_n^{H_n} X(\mathbf{t}_j),
\end{aligned}$$

proving that $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n\}$ is $\mathbf{H}$-mss. □

2.2 Locally stationary random fields

Let $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}_+^n\}$ be an $\mathbf{H}$-mss random field with finite second moments. According to Equation (12) in Proposition 1, the covariance of X can be written as:

$$\begin{aligned} \mathrm{E}(X(\mathbf{t})X(\mathbf{u})) &= \prod_{i=1}^{n} \left[e^{2H_i \frac{\ln(t_i)+\ln(u_i)}{2}} \right] R\left(\ln(t_1) - \ln(u_1), \ldots, \ln(t_n) - \ln(u_n)\right), \\ &= R_1\left(\frac{\mathbf{g}(\mathbf{t}) + \mathbf{g}(\mathbf{u})}{2}\right) R\left(\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{u})\right), \end{aligned} \tag{13}$$

where

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}}, \ \mathbf{g}(\mathbf{t}) = (\ln(t_1), \ldots, \ln(t_n))^T \text{ and } R \text{ is a stationary covariance.} \tag{14}$$

In light of this remark, we extend the notions of local stationarity of Silverman (1957) and local stationarity reducibility of Genton and Perrin (2004) to random fields.

Definition 4 *A random field $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ with finite second moments is locally stationary (LS) if its covariance function c can be written in the form:*

$$c(\mathbf{t}, \mathbf{u}) = R_1\left(\frac{\mathbf{t} + \mathbf{u}}{2}\right) R\left(\mathbf{t} - \mathbf{u}\right), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n, \tag{15}$$

where R_1 is a nonnegative function and R is a stationary covariance.

Definition 5 *A random field $\{X(\mathbf{t}), \mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n\}$ with finite second moments is locally stationary reducible (LSR) if its covariance function c can be written in the form:*

$$c(\mathbf{t}, \mathbf{u}) = R_1\left(\frac{\mathbf{g}(\mathbf{t}) + \mathbf{g}(\mathbf{u})}{2}\right) R\left(\mathbf{g}(\mathbf{t}) - \mathbf{g}(\mathbf{u})\right), \quad \mathbf{t}, \mathbf{u} \in \mathbb{R}^n, \tag{16}$$

where R_1 is a nonnegative function, R is a stationary covariance and $\mathbf{g}$ is a bijective deformation of the index space $\mathbb{R}^n$. If X is Gaussian with zero mean, then $X(\mathbf{t}) \overset{d}{=} Y(\mathbf{g}(\mathbf{t}))$, where Y is an LS random field. We call Y the reduced random field.

Therefore, multi-self-similar random fields with finite second moments are a subclass of LSR random fields. In this particular case, the deformation $\mathbf{g}$ does not depend on the index $\mathbf{H}$.

Note also that $R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}}$ is an exponentially convex covariance. Ehm et al. (2003) establish a bijection between exponentially convex functions and entire positive definite functions that provides parametric covariance models R_1 for LS random fields. However, it is not necessary that R_1 be exponentially convex in Definitions 4 and 5.

2.3 Fractional Brownian sheet

Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}_+^n\}$ be a mean zero standard fractional Brownian sheet with covariance

$$\mathrm{E}[X(\mathbf{t})X(\mathbf{u})] = \frac{1}{2^n}\prod_{i=1}^{n}\left(t_i^{2H_i} + u_i^{2H_i} - |t_i - u_i|^{2H_i}\right), \tag{17}$$

where $\mathbf{t} = (t_1, \ldots, t_n)^T$, $\mathbf{u} = (u_1, \ldots, u_n)^T$, and $0 < H_i \le 1$, $i = 1, \ldots, n$. Then it follows from Definition 3 that X is $\mathbf{H}$-mss with $\mathbf{H} = (H_1, \ldots, H_n)^T$. From Proposition 1, we obtain that:

$$X(\mathbf{t}) = t_1^{H_1}\cdots t_n^{H_n} Y(\ln(t_1), \ldots, \ln(t_n)),$$

where $Y(\mathbf{t})$ is a mean zero Gaussian stationary process with covariance $\mathrm{E}[Y(\mathbf{t})Y(\mathbf{t}+\mathbf{v})] = R(\mathbf{v})$, with

$$R(\mathbf{v}) = \prod_{i=1}^{n}\left(\cosh(H_i v_i) - 2^{(2H_i-1)}(\sinh(|v_i|/2))^{2H_i}\right). \tag{18}$$

If we restrict the process X to one dimension, for example setting $\mathbf{t} = (t, c, \ldots, c)^T$, where $c > 0$, then $Z(t) = X(\mathbf{t})$, $t > 0$, is the classical fractional Brownian motion indexed by $\mathbb{R}$, which is H-ss with index $H = H_1$. On the other hand, with Definition 2, $Z(t)$ is H-ss with index $H = H_1 + \cdots + H_n$, which does not reflect the self-similarity component-wise.

It follows from Definition 5 and Relation (14) that fractional Brownian sheets are LSR random fields with

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}},\ \mathbf{g}(\mathbf{t}) = (\ln(t_1), \ldots, \ln(t_n))^T \text{ and } R \text{ given by (18).}$$

2.4 Multi-self-similar random fields with superficial stationary increments

The following result is a generalization of Lemma 7.2.1 in Samorodnitsky and Taqqu (1994).

Proposition 2 *Suppose that $X = \{X(\mathbf{t}), \mathbf{t} = (t_1, t_2)^T \in \mathbb{R}^2\}$ is a non-constant $\mathbf{H}$-mss random field with finite second moments and stationary superficial increments of the form:*

$$S_{\mathbf{h}}(\mathbf{t}) = X(t_1 + h_1, t_2 + h_2) - X(t_1 + h_1, t_2) - X(t_1, t_2 + h_2) + X(t_1, t_2),$$

for all $\mathbf{h} = (h_1, h_2) \in \mathbb{R}^2$. Then $X(\mathbf{0}) = X(t_1, 0) = X(0, t_2) = 0$ almost surely, $H_1 \le 1$, $H_2 \le 1$ and the variance of $X_{\mathbf{h}}(\mathbf{t})$ is:

$$Var(S_{\mathbf{h}}(\mathbf{t})) = |h_1|^{2H_1}|h_2|^{2H_2}\,Var(X(\mathbf{1})). \tag{19}$$

Moreover, for all $\mathbf{t} \in \mathbb{R}^2$, if $H_1 < 1$ or $H_2 < 1$, then $E(X(\mathbf{t})) = 0$. If $H_1 = H_2 = 1$, then $X(\mathbf{t}) = t_1 t_2 X(\mathbf{1})$ almost surely, where $\mathbf{1} = (1, 1)^T$.

Proof. The **H**-mss property gives $X(\mathbf{0}) \stackrel{d}{=} a_1^{H_1} a_2^{H_2} X(\mathbf{0})$ so that $X(\mathbf{0}) = 0$ almost surely. For the same reason we also get $X(t_1, 0) = X(0, t_2) = 0$ almost surely. From $S_{\mathbf{1}}(0,0) \stackrel{d}{=} S_{\mathbf{1}}(-1,-1)$ we obtain that $X(1,1) \stackrel{d}{=} X(-1,-1)$ and from $S_{(-1,1)}(0,0) \stackrel{d}{=} S_{(-1,1)}(1,-1) \stackrel{d}{=} S_{(-1,1)}(1,0)$ we obtain that $X(-1,1) \stackrel{d}{=} X(1,-1)) \stackrel{d}{=} -X(1,1)$. Moreover

$$\mathrm{E}(|X(2,1)|) \leq \mathrm{E}(|X(2,1) - X(1,1)|) + \mathrm{E}(|X(1,1)|). \tag{20}$$

Because $X(\mathbf{0}) = X(t_1, 0) = X(0, t_2) = 0$ almost surely, we have $X_{(1,1)}(1,0) \stackrel{d}{=} X(2,1) - X(1,1)$ so that (20) becomes

$$2^{H_1}\mathrm{E}(|X(1,1)|) \leq 2\mathrm{E}(|X(1,1)|),$$

by using stationarity of the superficial increments and multi-self-similarity. Let us now prove that $\mathrm{E}(|X(1,1)|) \neq 0$ so that the last inequality imply $H_1 \leq 1$. If $X(1,1) = 0$ almost surely then $X(1,-1) = X(-1,1)) = X(-1,-1) = 0$ almost surely and by multi-self-similarity X is identically vanishing which contradicts our assumption. Similar arguments give $H_2 \leq 1$.

For the variance of $S_{\mathbf{h}}(\mathbf{t})$, we have:

$$\begin{aligned} \mathrm{Var}(S_{\mathbf{h}}(\mathbf{t})) &= \mathrm{Var}(S_{\mathbf{h}}(\mathbf{0})) \\ &= |h_1|^{2H_1}|h_2|^{2H_2}\mathrm{Var}(X(\mathbf{1})), \end{aligned}$$

by using stationarity of the superficial increments, multi-self-similarity and the relations

$$X(1,1) \stackrel{d}{=} X(-1,-1) \ \text{ and } \ X(-1,1) \stackrel{d}{=} X(1,-1) \stackrel{d}{=} -X(1,1).$$

Suppose $H_1 < 1$ or $H_2 < 1$. Since $\mathrm{E}(S_{\mathbf{1}}(0,0)) = \mathrm{E}(S_{\mathbf{1}}(1,0))$, we obtain $\mathrm{E}(X(\mathbf{1})) = \mathrm{E}(X(2,1)) - \mathrm{E}(X(1,1)) = (2^{H_1} - 1)\mathrm{E}(X(\mathbf{1}))$, so that $\mathrm{E}(X(\mathbf{1})) = 0$, and hence $\mathrm{E}(X(\mathbf{t})) = 0$. When $H_1 = H_2 = 1$, we have from the **H**-mss property:

$$\begin{aligned} \mathrm{E}\left[(X(\mathbf{t}) - t_1t_2X(\mathbf{1}))^2\right] &= \mathrm{E}(X^2(\mathbf{t})) - 2t_1t_2\mathrm{E}(X(\mathbf{t})X(\mathbf{1})) + t_1^2t_2^2\mathrm{E}(X^2(\mathbf{1})) \\ &= (t_1^2t_2^2 - 2t_1^2t_2^2 + t_1^2t_2^2)\mathrm{E}(X^2(\mathbf{1})) \\ &= 0, \end{aligned}$$

so that $X(\mathbf{t}) = t_1t_2X(\mathbf{1})$ almost surely.

□

This can be generalized to random fields indexed by $\mathbb{R}^n$, $n \geq 3$.

2.5 Another illustrative example

Let $X = \{X(\mathbf{t}) = B_{1/2}(t_1^{2H_1} \cdots t_n^{2H_n}), \mathbf{t} \in \mathbb{R}_+^n\}$ where $\mathbf{t} = (t_1, \ldots, t_n)^T$, $0 < H_i \leq 1$, $i = 1, \ldots, n$ and $B_{1/2}$ is the standard Brownian motion indexed by $\mathbb{R}_+$, that is, a standard fractional

Brownian motion with $H = 1/2$. The random field X is $\mathbf{H}$-mss but does not have superficial stationary increments as defined in Proposition 2 above, unlike the fractional Brownian sheet.

From (6), (7) and (8), we get the covariance function of $B_{1/2}$, for $t, u > 0$

$$\mathrm{E}[B_{1/2}(t)B_{1/2}(u)] = \exp\left(\frac{\ln(t)+\ln(u)}{2}\right)\exp\left(-\frac{|\ln(t)-\ln(u)|}{2}\right). \tag{21}$$

Therefore, we deduce from (21) the covariance function of X, for $\mathbf{t}, \mathbf{u} \in \mathbb{R}_+^n$

$$\mathrm{E}[X(\mathbf{t})X(\mathbf{u})] = R_1\left(\frac{\mathbf{g}(\mathbf{t})+\mathbf{g}(\mathbf{u})}{2}\right) R\left(\mathbf{g}(\mathbf{t})-\mathbf{g}(\mathbf{u})\right), \tag{22}$$

where

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}},\; R(\mathbf{v}) = \exp(-|\mathbf{H}^T\mathbf{v}|), \text{and } \mathbf{g}(\mathbf{t}) = (\ln(t_1), \ldots, \ln(t_n))^T.$$

Hence X is LSR.

3 Lévy fractional Brownian random fields indexed by $\mathbb{R}^2$

The following theorem describes the stationary field associated with the Lévy fractional Brownian random field. Observe that the arguments of the stationary field involve polar coordinates.

3.1 Lamperti characterization

Theorem 1 *Let $X = \{X(\mathbf{t}), \mathbf{t} = (t_1, t_2)^T \in \mathbb{R}^2\}$ be a mean zero Lévy fractional Brownian random field with covariance*

$$E[X(\mathbf{t})X(\mathbf{u})] = \frac{1}{2}(\|\mathbf{t}\|^{2H} + \|\mathbf{u}\|^{2H} - \|\mathbf{t}-\mathbf{u}\|^{2H}), \tag{23}$$

where $0 < H \leq 1$ and $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^2$. Then

$$X(\mathbf{t}) \stackrel{d}{=} \rho_{\mathbf{t}}^H Y(\ln(\rho_{\mathbf{t}}), \theta_{\mathbf{t}}), \tag{24}$$

with $\rho_{\mathbf{t}} = \sqrt{t_1^2 + t_2^2}$, $\theta_{\mathbf{t}} = \arctan(t_2/t_1) + k\pi$, $k \in \mathbb{Z}$, and where $Y(\mathbf{t})$ is a mean zero Gaussian stationary process with correlation $E[Y(\mathbf{t})Y(\mathbf{t}+\mathbf{v})] = R(\mathbf{v})$, given by

$$R(\mathbf{v}) = \frac{1}{2}\left(e^{v_1 H} + e^{-v_1 H} - (e^{v_1} + e^{-v_1} - 2\cos(v_2))^H\right). \tag{25}$$

Conversely, if $Y(\mathbf{t})$, $\mathbf{t} = (t_1, t_2)^T \in \mathbb{R}^2$, is a mean zero Gaussian stationary process with correlation $R(\mathbf{v})$ given by (25), then $Y(\mathbf{t})$ can be expressed as:

$$Y(\mathbf{t}) \stackrel{d}{=} e^{-t_1 H} X(e^{t_1}\cos(t_2), e^{t_1}\sin(t_2)), \tag{26}$$

where $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^2\}$ is a mean zero Lévy fractional Brownian random field.

Proof. The variance of X is $\mathrm{Var}(X(\mathbf{t})) = \|\mathbf{t}\|^{2H}$ and therefore its correlation is given by:

$$r(\mathbf{t},\mathbf{u}) = \frac{1}{2}\left(\frac{\|\mathbf{t}\|^H}{\|\mathbf{u}\|^H} + \frac{\|\mathbf{u}\|^H}{\|\mathbf{t}\|^H} - \frac{\|\mathbf{t}-\mathbf{u}\|^{2H}}{\|\mathbf{t}\|^H\|\mathbf{u}\|^H}\right).$$

We have:

$$\begin{aligned}\frac{\|\mathbf{t}\|^H}{\|\mathbf{u}\|^H} + \frac{\|\mathbf{u}\|^H}{\|\mathbf{t}\|^H} &= \left(e^{\ln\frac{\|\mathbf{t}\|}{\|\mathbf{u}\|}}\right)^H + \left(e^{-\ln\frac{\|\mathbf{t}\|}{\|\mathbf{u}\|}}\right)^H \\ &= e^{v_1 H} + e^{-v_1 H},\end{aligned}$$

where $v_1 = \ln\|\mathbf{t}\| - \ln\|\mathbf{u}\| = \ln(\rho_{\mathbf{t}}) - \ln(\rho_{\mathbf{u}})$, and:

$$\begin{aligned}\frac{\|\mathbf{t}-\mathbf{u}\|^{2H}}{\|\mathbf{t}\|^H\|\mathbf{u}\|^H} &= \left(\frac{\|\mathbf{t}\|^2 + \|\mathbf{u}\|^2 - 2\|\mathbf{t}\|\|\mathbf{u}\|\cos(\theta_{\mathbf{t}} - \theta_{\mathbf{u}})}{\|\mathbf{t}\|\|\mathbf{u}\|}\right)^H \\ &= \left(\frac{\|\mathbf{t}\|}{\|\mathbf{u}\|} + \frac{\|\mathbf{u}\|}{\|\mathbf{t}\|} - 2\cos(\theta_{\mathbf{t}} - \theta_{\mathbf{u}})\right)^H \\ &= \left(e^{\ln\frac{\|\mathbf{t}\|}{\|\mathbf{u}\|}} + e^{-\ln\frac{\|\mathbf{t}\|}{\|\mathbf{u}\|}} - 2\cos(\theta_{\mathbf{t}} - \theta_{\mathbf{u}})\right)^H \\ &= \left(e^{v_1} + e^{-v_1} - 2\cos(v_2)\right)^H,\end{aligned}$$

where $v_2 = \theta_{\mathbf{t}} - \theta_{\mathbf{u}}$. Thus, we have:

$$\begin{aligned}r(\mathbf{t},\mathbf{u}) &= \frac{1}{2}\left(e^{v_1 H} + e^{-v_1 H} - \left(e^{v_1} + e^{-v_1} - 2\cos(v_2)\right)^H\right) \\ &= R(v_1, v_2) \\ &= R(\ln(\rho_{\mathbf{t}}) - \ln(\rho_{\mathbf{u}}), \theta_{\mathbf{t}} - \theta_{\mathbf{u}}),\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{E}(X(\mathbf{t})X(\mathbf{u})) &= \rho_{\mathbf{t}}^H \rho_{\mathbf{u}}^H R(\ln(\rho_{\mathbf{t}}) - \ln(\rho_{\mathbf{u}}), \theta_{\mathbf{t}} - \theta_{\mathbf{u}}) \\ &= \rho_{\mathbf{t}}^H \rho_{\mathbf{u}}^H \mathrm{E}(Y(\ln(\rho_{\mathbf{t}}), \theta_{\mathbf{t}}) Y(\ln(\rho_{\mathbf{u}}), \theta_{\mathbf{u}})).\end{aligned}$$

□

According to Definition 2, X defined by (23) and (24) is H-ss with $0 < H \leq 1$. A natural question is whether X is also $\mathbf{H}$-mss? The answer is no in cartesian coordinates, but yes in polar coordinates. Indeed, rewriting (24) as

$$Z(\rho_{\mathbf{t}}, \theta_{\mathbf{t}}) = X(\mathbf{t}) \overset{d}{=} \rho_{\mathbf{t}}^{H_1} (e^{\theta_{\mathbf{t}}})^{H_2} Y(\ln(\rho_{\mathbf{t}}), \ln(e^{\theta_{\mathbf{t}}})), \tag{27}$$

with $\mathbf{H} = (H_1, H_2)^T = (H, 0)^T$, we conclude from Proposition 1 that X is $\mathbf{H}$-mss with respect to the polar coordinates $(\rho_{\mathbf{t}}, \theta_{\mathbf{t}})$. Thus, from Definition 5, Lévy fractional Brownian random fields indexed by $\mathbb{R}^2$ are LSR random fields with

$$R_1(\mathbf{w}) = e^{2\mathbf{H}^T\mathbf{w}},\ \mathbf{H} = (H_1, H_2)^T = (H, 0)^T,\ \mathbf{g}(\mathbf{t}) = (\ln(t_1), t_2)^T,\ \text{and } R(\mathbf{v}) \text{ given by (25).}$$

Note that $R(\mathbf{v})$ for $H = 1/2$ was already given by Perrin and Senoussi (2000). Also by setting $v_1 = v$ and $v_2 = 0$ in (25), we recover Relation (8).

It is still an open problem to find the appropriate coordinate system to show that Lévy fractional Brownian random fields indexed by $\mathbb{R}^n$, $n \geq 3$, are $\mathbf{H}$-mss.

3.2 Properties of the stationary correlation function

Consider the stationary correlation function $R(\mathbf{v})$, $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, given by (25) associated with the Lévy fractional Brownian random field indexed by $\mathbb{R}^2$. $R(\mathbf{v})$ is fully symmetric (see Gneiting, 2002), i.e.

$$R(v_1, v_2) = R(-v_1, v_2) = R(v_1, -v_2) = R(-v_1, -v_2). \tag{28}$$

It is sufficient to focus on $R(v_1, v_2)$, $(v_1, v_2) \in \mathbb{R}_+ \times [0, 2\pi)$. Figure 1 depicts the graph of $R(\mathbf{v})$ for $H = 1/4, 1/2, 3/4, 1$ and shows the behavior of that correlation function. It can be checked that for $0 < H \leq \frac{1}{2}$, $0 \leq R(\mathbf{v}) \leq 1$, for all $\mathbf{v} \in \mathbb{R}^2$, and that for $\frac{1}{2} < H \leq 1$, $-1 \leq R(\mathbf{v}) \leq 1$, for all $\mathbf{v} \in \mathbb{R}^2$. Figure 2 depicts the graph of $R(v_1, v_2)$ for $H = 0, 0.1, \ldots, 1$ and: $v_2 = 0$ (top), $v_2 = \pi/2$ (middle), and $v_2 = \pi$ (bottom). In particular,

$$\lim_{v_1 \to +\infty} R(\mathbf{v}) = \begin{cases} 0 & \text{for } 0 < H < 1, \\ \cos(v_2) & \text{for } H = 1. \end{cases} \tag{29}$$

More precisely, the asymptotic behavior of $R(\mathbf{v})$ as $v_1 \to +\infty$ is given by

$$R(\mathbf{v}) \sim \begin{cases} \frac{1}{2} e^{-v_1 H} & \text{for } 0 < H \leq \frac{1}{2}, \\ H e^{-v_1(1-H)} \cos(v_2) & \text{for } \frac{1}{2} < H \leq 1. \end{cases} \tag{30}$$

It is interesting to note that, unlike the Lévy fractional Brownian random field X, the corresponding reduced process Y in (26) has a short-range dependence structure for $0 < H < 1$. The correlation function $R(\mathbf{v})$ is infinitely differentiable except at $v_1 = v_2 = 0$, which corresponds to $\mathbf{t} = \mathbf{u}$ in (23).

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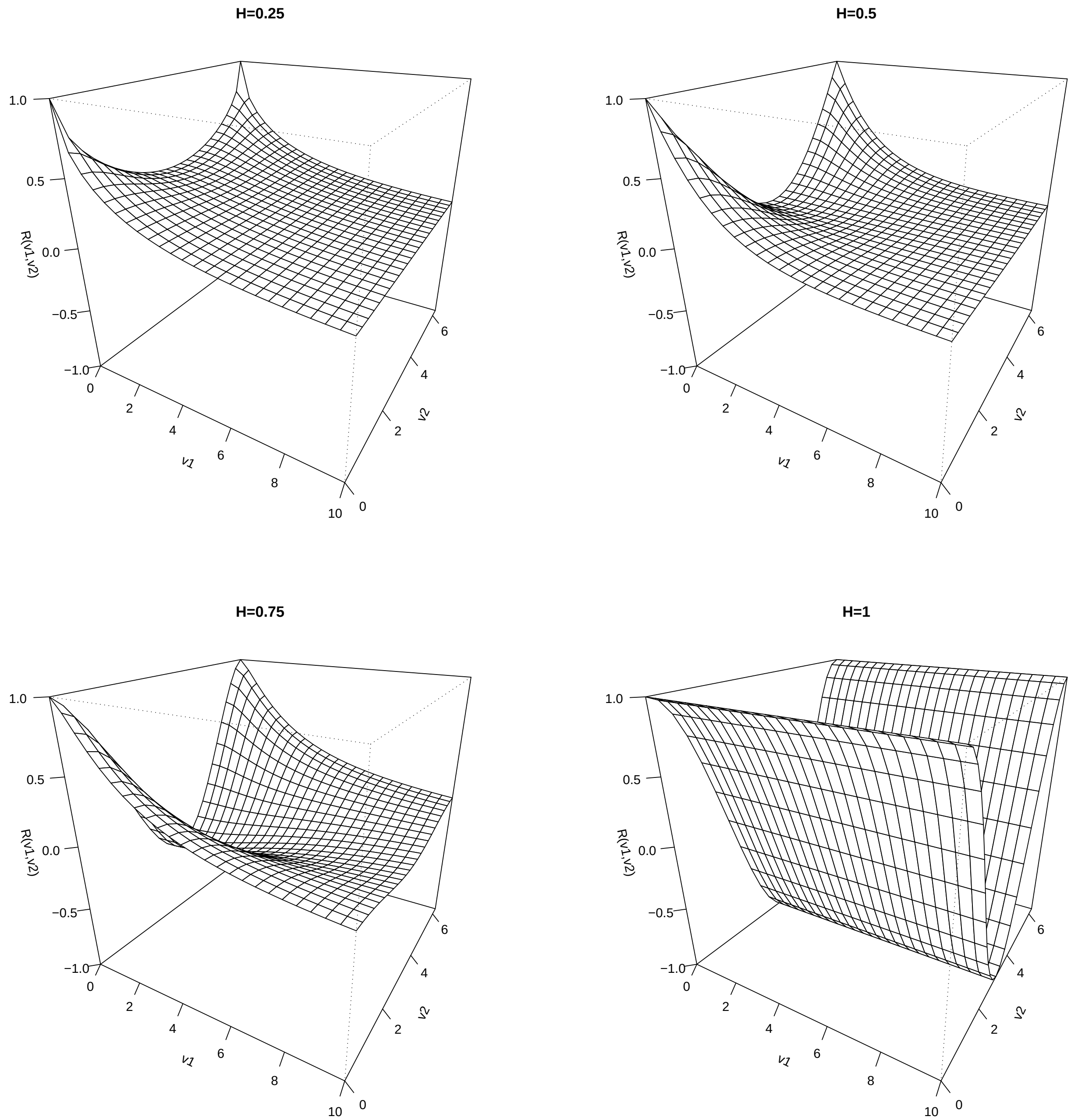

Figure 1: Graph of $R(\mathbf{v})$ for $H = 1/4, 1/2, 3/4, 1$.

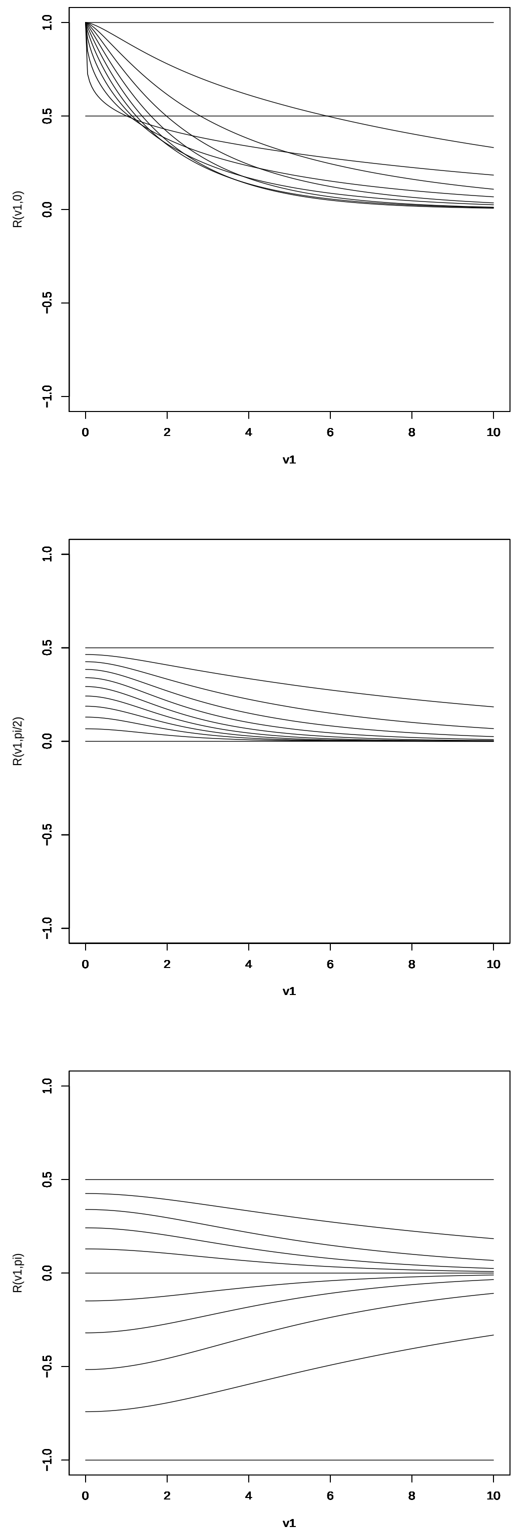

Figure 2: Graph of $R(v_1, v_2)$ for $H = 0, 0.1, \ldots, 1$ and: $v_2 = 0$ (top); $v_2 = \pi/2$ (middle); $v_2 = \pi$ (bottom).