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“Double auction with interdependent values: incentives and efficiency”

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DOUBLE AUCTION WITH INTERDEPENDENT VALUES: INCENTIVES AND EFFICIENCY

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ABSTRACT. We study a double auction environment where buyers and sellers have interdependent valuations and multi-unit demand and supply. We propose a new mechanism which satisfies ex post incentive compatibility, individual rationality, feasibility, non-wastefulness, and no budget deficit. Moreover, this mechanism is asymptotically efficient in that the trade outcome in the mechanism converges to the efficient level as in a competitive equilibrium as the numbers of the buyers and sellers become large. Our mechanism is the first double auction mechanism with these properties in the interdependent values setting.

JEL Classification Numbers: D44, D47, D82.

Keywords: double auction, interdependent values, multi-unit demand and supply, ex post incentive compatibility, asymptotic efficiency

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1. Introduction

Double auctions are among the most prevalent forms of economic transactions. They also occupy a central place in economic theory, as the microfoundation of the idea of the market in standard microeconomics.

Despite their importance, double auction markets are not easy to organize or analyze. Most common mechanisms quote a price that equates supply and demand and let the objects change hands at that price, but such mechanisms are not always incentive compatible. That is, participants sometimes have incentives to misreport their preferences. The resulting misreporting can lead to inefficiency of equilibrium outcomes. The problem becomes even more difficult once we allow for traders with interdependent values or multi-unit demand or supply, and yet these are common features for many double auction environments.

The goal of our paper is to study whether desirable properties are mutually compatible in double auction markets with interdependent values and multi-unit demand and supply. To address this question, we construct a mechanism that, with an arbitrary number of buyers and sellers, satisfies ex post incentive compatibility and ex post individual rationality. These properties make truth-telling and voluntary participation an ex post equilibrium under this mechanism. Moreover, we show that the mechanism never runs deficit,[1] and has the property that the number of objects sold by the sellers coincides with the number of objects bought by the buyers.

As is the case for other mechanisms studied in the literature, our mechanisms are not fully efficient. In fact, the celebrated impossibility result by Myerson and Satterthwaite (1983) implies that it is impossible to achieve all those properties even under private values and single-unit demand and supply. However, we establish asymptotic efficiency of our mechanism. That is, our second main result shows that the trade outcome in the mechanism converges to the efficient level as in a competitive equilibrium under certain additional conditions, as the number of buyers and sellers go to infinity. This result suggests that the outcome of our mechanism is close to a fully efficient, first-best outcome, at least in large economies. In all, our analysis shows that incentive compatibility and other desirable properties can be achieved while achieving asymptotic efficiency as well.

[1]While our mechanism can run surplus, it is easy to modify the mechanism to satisfy exact ex post budget balance. See footnote 18 for detail.

Our positive result is obtained by constructing a new class of double auction mechanisms, which we call the groupwise-price double-auction mechanisms (or simply, groupwise-price mechanisms). A groupwise-price mechanism divides the entire market into a number of submarkets. Each submarket is composed of a subset of buyers and sellers, and all trades happen between buyers and sellers in the same submarket. For each submarket, we set a reference price for that submarket which is independent of reported types of agents in that submarket. Agents in the submarket trade based on the reference price, although not necessarily at it.[2] We show that these mechanisms satisfy all the aforementioned desiderata such as ex post incentive compatibility and asymptotic efficiency.

Related Literature. Few existing studies have offered a double auction mechanism that is ex post incentive compatible and asymptotically efficient. McAfee (1992) is an important exception, who makes a seminal contribution to this problem. He considers buyers and sellers with private values and single-unit supply and demand. In that setting, he proposes a mechanism that is dominant-strategy incentive compatible (which is equivalent to ex post incentive compatibility with private values) and asymptotically efficient. Our marginal contribution over McAfee (1992) is that we allow for interdependent values and multi-unit demand and supply. Both features are important for most double auction markets in practice. McAfee's mechanism handles neither of these features, and thus our mechanism is based on a different idea. In fact, even in the case with private values and single-unit demand and supply, our mechanism does not reduce to McAfee's.

Independently from our study, an ongoing work by Loertscher and Mezzetti (2014) considers an extension of McAfee (1992) to an environment with multi-unit demand and supply. In a private-values environment, they present a dominant-strategy incentive compatible mechanism and its simple "clock" implementation. A main advantage of our paper compared to theirs is that we allow for interdependent values. On the other hand, they allow for multi-dimensional types while we only allow for one-dimensional types (see the next paragraph for difficulties with multi-dimensional interdependent values known in the literature).

Our paper is part of the literature of mechanism design with interdependent valuations, where many existing studies have found impossibility results. For example, Jehiel and Moldovanu (2001) and Jehiel et al. (2006) demonstrate the difficulties associated with interdependent values and multidimensional signals under the transferable utility setup.

[2]The specific manner that the trading price is determined is important for incentive compatibility. We will defer detail to the main body of the paper, because it is rather complicated and needs formal definitions in order to describe it precisely.

Che, Kim and Kojima (2015) show that, even with single-dimensional signals, similar impossibility results are obtained in a non-transferable utility setup. In our paper, we circumvent those impossibility results by considering an environment where each agent's signal is summarized by a one-dimensional statistic with the standard single-crossing condition (Maskin, 1992; Dasgupta and Maskin, 2000), and the agents have quasi-linear utilities.[3]

Our work was partly inspired by a recent work by Hashimoto (2013).[4] In an object allocation setting, he offers a general procedure to modify a given mechanism into an ex post incentive compatible one, while approximating the original mechanism in large markets. He applies his technique to construct a mechanism that is ex post incentive compatible and asymptotically efficient. While inspired by his work, our result is independent of his. The main difference is that his method presumes that no individual initially owns an object, such as in (one-sided) auction environments. As such, his mechanism does not necessarily guarantee individual rationality if applied to double auction, although individual rationality is crucial in the double auction environment.[5]

Our groupwise-price mechanism defines prices for a subset of agents independently of their own reports, thereby preventing some obvious price-manipulation incentives. Similar ideas are used in several earlier contributions, such as Cordoba and Hammond (1998) and Kovalenkov (2002) for exchange economies, Segal (2003) for optimal pricing, and Baliga and Vohra (2003) for double auction markets.[6] However, as opposed to their private-value settings, with interdependent values, using groupwise prices does not immediately imply that truth-telling is ex post incentive compatible. One issue is that, because an agent's signal can affect the other agents' demands or supplies, even if she cannot affect her price directly, she may have an incentive to manipulate her report to affect the quantities of

[3]Under these assumptions, Maskin (1992), Dasgupta and Maskin (2000), and Perry and Reny (2002) show that exact efficiency can be achieved. However, note that they study *one-sided* auction, and the results do not extend to our double auction setting. In fact, as mentioned below, exact efficiency is not achievable.

[4]Azevedo and Budish (2012) provide mechanisms that are approximately, but not exactly, incentive compatible. The main goal of the current study is different in that we obtain an exactly incentive compatible mechanism, but the basic motivation is similar.

[5]Another paper related to ours is Matsushima (2008). Although he considers double auction with interdependent values like us, his mechanism does not satisfy individual rationality because some agents earn negative payoffs with a small, but positive, probability.

[6]The basic idea of defining personalized or groupwise prices appears to be well-known, and the authors have been unable to locate the first to propose it. Jackson and Manelli (1997) call this type of mechanisms "folk" mechanisms.

trade.[7] We overcome this issue by designing a mechanism such that an agent's report does not affect the demand or supply of the other agents in the same group. Because of these features, our mechanism cannot use the agents' information in a fully efficient way, which necessitates extra care for showing asymptotic efficiency.

Our paper is part of the extensive literature on double auctions. Existing studies have shown that behavior under Bayesian equilibria converges to truth-telling as the number of traders increases (and the outcome achieves asymptotic, though not exact, efficiency) in a broad class of double auction mechanisms. Important contributions in this tradition include Gresik and Satterthwaite (1989), Rustichini, Satterthwaite and Williams (1994), Fudenberg, Mobius and Szeidl (2007), and Cripps and Swinkels (2006) for the private values case, and Reny and Perry (2006) for the interdependent values case.[8] The main difference between this line of research and ours is that these papers study Bayesian equilibrium behavior of the participants in mechanisms that are *not* necessarily ex post incentive compatible. Our motivation is to design a mechanism that is ex post incentive compatible, which makes truthtelling a best response irrespective of the participants' beliefs about others' signals. As such, we believe that our paper complements the existing studies of double auctions.

More broadly, our asymptotic analysis can be situated in a long tradition of economic theory on large-market properties of mechanisms. In large exchange economies, Roberts and Postlewaite (1976) demonstrate that the Walrasian mechanism is difficult to manipulate under some conditions. Jackson (1992), Jackson and Manelli (1997), and Andreyanov and Sadzik (2016) investigate exchange economies from asymptotic perspectives as well. More recently, Roth and Peranson (1999), Immorlica and Mahdian (2005), Kojima and Pathak (2009), Lee (2011), and Ashlagi, Kanoria and Leshno (2013) show that the deferred acceptance algorithm due to Gale and Shapley (1962) becomes increasingly hard to manipulate in large markets. In the object allocation setting without transfers, asymptotic incentive compatibility and asymptotic efficiency of various mechanisms have been established by Kojima and Manea (2010), Che and Kojima (2010), Liu and Pycia (2011), and Azevedo and Budish (2012). Our paper identifies another case in which both incentive compatibility and efficiency become achievable in large economies, reinforcing the insights from these existing studies.

[7]To satisfy the feasibility constraint, we use a rationing rule (through an auction mechanism of Ausubel (1999)). Hence, manipulation of quantities by an agent (without affecting prices) is a relevant concern.

[8]See also Kazumori (2013) who shows that, under interdependent values, every trembling hand perfect equilibrium asymptotically approximates ex post price taking behavior.

2. Model

There are a set of buyers B and a set of sellers S. Let $n_B \in \mathbb{N}$ be the number of buyers, and $n_S \in \mathbb{N}$ be the number of sellers. There is one type of indivisible object, as well as divisible money.

Each agent can buy or sell at most $m \in \mathbb{N}$ units of the object. Each agent $i \in B \cup S$ is endowed with a signal, which we refer to as her **type**, $t_i \in [0,1]$. Type t_i is agent i's private information. Let $t = (t_i)_{i \in B \cup S}$ denote the profile of types. Given type profile t, each agent i's value profile is $(v_i^\ell(t))_{\ell=1}^m$. For each buyer b and index $\ell \in \{1, \dots, m\}$, $v_b^\ell(t) \in [0,1]$ is b's valuation for the ℓ-th unit of the object. For each seller s and each $\ell \in \{1, \dots, m\}$, $v_s^\ell(t) \in [0,1]$ is the cost of giving up the ℓ-th unit of the object for seller s. We assume that each agent has a quasi-linear utility function. More precisely, for each buyer b, her payoff from consuming $\ell \in \{1, \dots, m\}$ units of the object and paying money $\tau \in \mathbb{R}$ is given by

$$\sum_{\ell'=1}^{\ell} v_b^{\ell'}(t) - \tau.$$

For each seller s, her payoff from giving up $\ell \in \{1, \dots, m\}$ units of the object and receiving money $\tau \in \mathbb{R}$ is given by

$$\tau - \sum_{\ell'=1}^{\ell} v_s^{\ell'}(t).$$

For each i and ℓ, we assume that $v_i^\ell(\cdot)$ is continuous, non-decreasing in each argument, and strictly increasing in t_i. For each ℓ, we assume $v_b^\ell(t) > v_b^{\ell+1}(t)$ for all $b \in B$ and $v_s^\ell(t) < v_s^{\ell+1}(t)$ for all $s \in S$. That is, buyers have diminishing marginal utility and sellers have increasing marginal cost. We also impose a single-crossing condition. More specifically, for each $i, j \in S \cup B$, $\ell, \ell' \in \{1, \dots, m\}$, and $t = (t_i, t_{-i})$, if $v_i^\ell(t) \geq v_j^{\ell'}(t)$, then for any $t_i' > t_i$, $v_i^\ell(t_i', t_{-i}) > v_j^{\ell'}(t_i', t_{-i})$. For normalization, we assume that the highest possible valuation is 1 and the lowest possible valuation is 0 (across buyers and sellers, and across units of the object).

2.1. Mechanisms and Desirable Properties.

A (double auction) **mechanism** is a pair of functions $\varphi = (\zeta, \tau)$ from the set of type profiles to the sets of object allocations and transfers. More specifically, for each type profile $t = (t_i)_{i \in S \cup B}$ and agent $i \in B \cup S$, $\zeta_i(t) \in \{1, \dots, m\}$ is the number of objects that i trades (so, $\zeta_i(t)$ is the number of objects received if i is a buyer and the number of objects sold if i is a seller), and $\tau_i(t)$ is the transfer for i (so, $\tau_i(t)$ is the money that i pays if i is a buyer, and the payment that i receives if i is a seller).

In the remainder of this section, we introduce desirable properties of mechanisms. The main goal of our study is to construct a mechanism that satisfies these properties, which we will do in the rest of the paper.

First, we introduce our central incentive compatibility concept. A mechanism $\varphi = (\zeta, \tau)$ is **ex post incentive compatible** if, for each t, we have

$$\sum_{\ell=1}^{\zeta_b(t)} v_b^\ell(t) - \tau_b(t) \geq \sum_{\ell=1}^{\zeta_b(\tilde{t}_b, t_{-b})} v_b^\ell(t) - \tau_b(\tilde{t}_b, t_{-b}), \text{ for each } b \in B \text{ and } \tilde{t}_b \in [0,1], \text{and}$$

$$\tau_s(t) - \sum_{\ell=1}^{\zeta_s(t)} v_s^\ell(t) \geq \tau_s(\tilde{t}_s, t_{-s}) - \sum_{\ell=1}^{\zeta_s(\tilde{t}_s, t_{-s})} v_s^\ell(t), \text{ for each } s \in S \text{ and } \tilde{t}_s \in [0,1].$$

This condition requires that, given that every other agent reports her true type, reporting the true type is a best response *even in the ex post sense*, i.e., it is a best response even after all true types are revealed to the agent. This property provides certain robust incentives to report true types (see Bergemann and Morris (2005) for instance).[9]

A mechanism $\varphi = (\zeta, \tau)$ is **ex post individually rational** (or individually rational for short) if, for each t, we have

$$\sum_{\ell=1}^{\zeta_b(t)} v_b^\ell(t) - \tau_b(t) \geq 0 \text{ for each } b \in B, \text{and}$$

$$\tau_s(t) - \sum_{\ell=1}^{\zeta_s(t)} v_s^\ell(t) \geq 0 \text{ for each } s \in S.$$

This is a standard condition in the literature and important for voluntary participation.

A mechanism $\varphi = (\zeta, \tau)$ is **feasible** if, for each t, we have

$$\sum_{b \in B} \zeta_b(t) \leq \sum_{s \in S} \zeta_s(t).$$

This condition requires that the number of the objects that are sold is weakly larger than the number of the objects that are bought. This ensures that the trade is feasible, as the set of the objects sold by the sellers offers enough supply to satisfy the demand by the buyers who are prescribed to buy the objects.

A mechanism $\varphi = (\zeta, \tau)$ is **non-wasteful** if, for each t, we have

$$\sum_{b \in B} \zeta_b(t) \geq \sum_{s \in S} \zeta_s(t).$$

This condition requires that the mechanism never wastes an object by buying up objects from sellers while not assigning all of these objects to buyers.

[9] See also Wilson (1987).

A mechanism $\varphi = (\zeta, \tau)$ runs **no ex post budget deficit** (or no budget deficit for short) if, for each t, we have

$$\sum_{b\in B} \tau_b(t) \geq \sum_{s\in S} \tau_s(t).$$

This condition ensures that the auction organizer never runs deficit. We regard this condition as important for the sustainability of a mechanism.

A stronger condition than no budget deficit is of some interest. A mechanism $\varphi = (\zeta, \tau)$ is **ex post budget-balanced** (or budget-balanced for short) if, for each t, we have

$$\sum_{b\in B} \tau_b(t) = \sum_{s\in S} \tau_s(t).$$

While sometimes assumed in the literature, we do not regard budget balance to be indispensable as far as the mechanism runs no budget deficit. For that reason, our group-wise price mechanism never runs a budget deficit, but can run budget surplus, violating the exact budget balance. However, we will later show that it is straightforward to modify our groupwise-price mechanism into a mechanism that satisfies the exact budget balance. See Section 4.1 for detail.

As the first-best benchmark, we consider a (complete information) competitive equilibrium. Formally, a mechanism $\varphi = (\zeta, \tau)$ is said to be a **competitive mechanism** if it satisfies the following condition: For any type profile t,

$$\sum_{b\in B} \zeta_b(t) = \sum_{s\in S} \zeta_s(t),$$

and there exists $p(t)$ such that

(1) For each buyer $b \in B$,

$$\zeta_b(t) \in \arg\max_{\ell\in\{0,\ldots,m\}} \sum_{\ell'=1}^{\ell} v_b^{\ell'}(t) - p(t)\ell,$$

(2) For each seller $s \in S$,

$$\zeta_s(t) \in \arg\max_{\ell\in\{0,\ldots,m\}} p(t)\ell - \sum_{\ell'=1}^{\ell} v_s^{\ell'}(t),$$

(3) $\tau_i(t) = p(t)\zeta_i(t)$ for every agent $i \in B \cup S$.

In other words, a competitive mechanism lets each agent buy or sell optimally given price $p(t)$ where the price $p(t)$ balances demand and supply.[10] From the definition it is

[10]It is straightforward to see that there exists a competitive mechanism.

obvious that any competitive mechanism satisfies individual rationality, feasibility, non-wastefulness, budget balance and, perhaps most importantly, efficiency. The main drawback of a competitive mechanism is that it does not satisfy ex post incentive compatibility (it fails even weaker conditions such as Bayesian incentive compatibility). In fact, there exists no mechanism that satisfies all of these desirable properties including efficiency in the exact sense (Myerson and Satterthwaite, 1983).[11] The main goal of this paper is to offer a mechanism that achieves an efficiency level arbitrarily close to a competitive mechanism, while satisfying ex post incentive compatibility and the other desirable properties.

2.2. **The Groupwise-Price Double Auction Mechanism.** The class of double auction mechanisms we examine is called the **groupwise-price double auction mechanisms**, or the **groupwise-price mechanisms** for short. A groupwise-price mechanism is defined as follows (because its formal definition is somewhat complicated, we provide an informal description following the formal definition).

(1) Let B be (possibly randomly) partitioned into K sets $B_1, \ldots, B_K$ of equal size. Let S be (possibly randomly) partitioned into K sets $S_1, \ldots, S_K$ of equal size.[12]
(2) Each agent i simultaneously reports type t_i (not necessarily truthfully).

Now, for each $k \in \{1, \ldots, K\}$, the **submarket** k is composed of the set of agents $B_k \cup S_k$. The trading procedure in this submarket is described as follows.

(1) Let $p_k = p_k((t_i)_{i \notin (B_k \cup S_k)})$ be a real number that depends on $(t_i)_{i \notin (B_k \cup S_k)}$ while not on $(t_i)_{i \in (B_k \cup S_k)}$. We call it the **reference price** for the submarket k in the sense explained below.[13]
(2) Let $\underline{t} = ((t_i)_{i \notin S_k}, (0)_{i \in S_k})$ and $\overline{t} = ((t_i)_{i \notin B_k}, (1)_{i \in B_k})$.
(3) Define

$$\begin{aligned} B_k^*(\underline{t}) &= \{(b, \ell) \in B_k \times \{1, \ldots, m\} | v_b^\ell(\underline{t}) \geq p_k\}, \\ S_k^*(\overline{t}) &= \{(s, \ell) \in S_k \times \{1, \ldots, m\} | v_s^\ell(\overline{t}) < p_k\}. \end{aligned}$$

[11] In the setting with multiple buyers and sellers, Williams (1999) finds conditions for the existence of a mechanism that satisfies these desirable properties. His conditions are in general not satisfied in out setting.

[12] If $|B|$ is not a multiple of K, then find a largest integer z such that $|B| \geq zK$, exclude $|B| - zK$ buyers, and redefine B in the description of the mechanism as the set of the remaining buyers (and apply a similar redefinition to S as well). By modifying the mechanism in this way, all the results in the paper hold. In the rest of the paper, we assume $|B|$ and $|S|$ are multiples of K without loss.

[13] The choice of the reference prices are crucial for asymptotic efficiency, as we will show in a subsequent section, while all other results hold for an arbitrary choice of the reference prices.

(4) Order the elements in $B_k^*(\underline{t})$ in a decreasing manner in terms of the associated valuations $v_b^\ell(\underline{t})$, and the elements in $S_k^*(\overline{t})$ in an increasing manner in terms of the associated valuations $v_s^\ell(\overline{t})$.[14] Let $x = \min\{|B_k^*(\underline{t})|, |S_k^*(\overline{t})|\}$, and let $B_k^{**}(\underline{t})$ be the set of the first x elements of $B_k^*(\underline{t})$, and $S_k^{**}(\overline{t})$ be the set of first x elements of $S_k^*(\overline{t})$ (so $B_k^{**}(\underline{t}) = B_k^*(\underline{t})$ if $|B_k^*(\underline{t})| \leq |S_k^*(\overline{t})|$, and similarly, $S_k^{**}(\overline{t}) = S_k^*(\overline{t})$ otherwise).

(5) For each $b \in B_k$ and ℓ, define

$$\tilde{t}_b^\ell(\underline{t}_{-b}) = \inf\{\tilde{t}_b | (b, \ell) \in B_k^{**}(\tilde{t}_b, \underline{t}_{-b})\},$$

if the set in the right hand side of this equation is nonempty, and let $\tilde{t}_b^\ell(\underline{t}_{-b}) = 1$ otherwise. Similarly, for each $s \in S_k$ and ℓ, define

$$\tilde{t}_s^\ell(\overline{t}_{-s}) = \sup\{\tilde{t}_s | (s, \ell) \in S_k^{**}(\tilde{t}_s, \overline{t}_{-s})\},$$

if the set in the right hand side of this equation is nonempty, and let $\tilde{t}_s^\ell(\overline{t}_{-s}) = 0$ otherwise.

Note that, by the single-crossing condition, once $(b, \ell) \in B_k^{**}(t_b', \underline{t}_{-b})$ for some t_b', then for any $t_b'' > t_b'$, $(b, \ell) \in B_k^{**}(t_b'', \underline{t}_{-b})$. Also, note that $\tilde{t}_b^\ell(\underline{t}_{-b})$ is non-decreasing in ℓ. Similar properties hold for the sellers.

(6) Buyer b receives the ℓ-th unit of the object if and only if $(b, \ell) \in B_k^{**}(\underline{t})$, and pays the price $v_b^\ell(\tilde{t}_b^\ell(\underline{t}_{-b}), t_{-b})$ for that unit.[15] Seller s sells the ℓ-th unit of the object if and only if $(s, \ell) \in S_k^{**}(\overline{t})$, and receives the price $v_s^\ell(\tilde{t}_s^\ell(\overline{t}_{-s}), t_{-s})$ for that unit.

As is clear in the description, the key parameters of each groupwise-price mechanism comprise the number of submarkets (groups), K, and the reference price for each submarket k, $p_k((t_i)_{i \notin (B_k \cup S_k)})$. In Section 3 we show that all the desirable properties except for asymptotic efficiency hold true for any choice of K and $\{p_k(\cdot)\}_{k=1}^K$. In Section 4, we show asymptotic efficiency for a specific choice of them.

While the formal definition of this mechanism is somewhat involved, the basic idea is simple: divide the market into a number of submarkets (groups), and use group-specific prices (hence the name "groupwise-price mechanism"). Each submarket is composed of a subset of buyers and sellers, and all trades happen only between buyers and sellers in the same submarket. For each submarket, we set a reference price for that submarket

[14]When the valuations of multiple agents are identical, we order them in some fixed order (where that order is independent of reported types).

[15]Note that, if b trades the ℓ-th unit, then she necessarily trades the ℓ'-th unit for every $\ell' < \ell$ as well. If b receives ℓ units of the object in total, then her payment is $\sum_{\ell'=1}^{\ell} v_b^{\ell'}(\tilde{t}_b^{\ell'}(\underline{t}_{-b}), t_{-b})$. A similar comment applies to sellers.

independently of reported types of agents in that submarket. In this way, we can prevent some obvious price-manipulation incentives.

However, because the reference price in submarket k does not use any information about the agents' types in that submarket, it is possible that the reference price does not "clear" the demand and supply of that submarket. For example, the reference price may be so high that the number of units the sellers in S_k want to sell is greater than the number of units the buyers in B_k want to buy. Then, the mechanism runs a generalized VCG auction (Ausubel, 1999) separately for each side of the market to satisfy feasibility and non-wastefulness.[16]

Another incentive issue we need to overcome is specific to interdependent-value double auction environments. Because a seller's type report can affect buyers' willingness to pay, the seller may have an incentive to overreport her type so that the buyers in the same submarket would buy more. Similarly, a buyer may have an incentive to underreport her type. Our mechanism eliminates such an incentive by defining B_k^* independent of any report by the sellers in submarket k, and similarly, defining S_k^* independent of any report by the buyers in submarket k.

By tailoring the detail in this manner, the mechanism satisfies a number of desirable properties (namely, ex post incentive compatibility, individual rationality, feasibility, non-wastefulness, and no budget deficit), but at a cost of efficiency through the following two channels. First, because the reference price in submarket k does not use any information about the agents' types in that submarket, some efficiency-enhancing trades *within* the submarket may be prevented. In Section 4, we will address this problem by increasing the number of submarkets and thus diminishing the effect of lost information for

[16]To run a generalized VCG auction as in Ausubel (1999), the designer must know the valuation function of each agent i, $v_i^\ell(\cdot)$, for each ℓ. This may be considered to be too demanding as the designer's prior knowledge. Dasgupta and Maskin (2000) and Perry and Reny (2002) study implementation of efficient allocations in auction environments as an equilibrium of a game whose form does not depend on the functional forms of the agents' valuation functions ("detail-free" mechanisms, in the spirit of Wilson (1987)). For example, Dasgupta and Maskin (2000) consider an auction mechanism where each bidder names not only a single bid but the entire valuation functions given his signal. Perry and Reny (2002) consider an auction mechanism where (at most two rounds of) a second-price auction is run for each pair of bidders. These mechanisms are detail-free in the sense that their auction mechanisms do not require the designer's knowledge about the bidders' valuation functions. We use a generalized VCG auction as in Ausubel (1999) for its simple description, even though it is not detail-free. However, given that our model satisfies both single-crossing preferences as in Dasgupta and Maskin (2000) and decreasing-marginal-values as in Perry and Reny (2002), we conjecture that similar detail-free mechanisms may work too. We leave this question for future research.

each submarket. Second, because we divide the market into (many) submarkets, some efficiency-enhancing trades *across* different submarkets may be prevented. Our technical contribution for analyzing efficiency is to find that there is an appropriate growth rate of the number of submarkets that balances out this tradeoff.

To illustrate how the mechanism works, we consider the following example.

Example 1. Let $m = 2$. In a submarket k, there are one seller s and two buyers $b = b_1, b_2$, each with two units of supply and demand. The type of seller s is $t_s = 0.2$, buyer b_2's type is $t_{b_2} = 0.3$, and $t_i = 0$ for any $i \notin B_k, S_k$. The values of seller s are $v_s^\ell(t) = 5t_s = 1$ for $\ell = 1, 2$.[17] For each buyer $b = b_1, b_2$ and each $\ell = 1, 2$, we have $v_b^\ell(t) = \frac{4}{\ell} t_b + \sum_{i \neq b} t_i$.

Assume $p_k = 1.9$. For the seller's side, we have $|S_k^*(\bar{t})| = 2$. We study how the trades and prices change as $t_{b_1} \in [0, 1]$ varies. Given t_{b_1},

$$
\begin{aligned}
v_{b_1}^\ell(\underline{t}) &= \frac{4}{\ell} t_{b_1} + 0.3, \\
v_{b_1}^\ell(t) &= \frac{4}{\ell} t_{b_1} + 0.5, \\
v_{b_2}^\ell(\underline{t}) &= \frac{1.2}{\ell} + t_{b_1}, \\
v_{b_2}^\ell(t) &= \frac{1.2}{\ell} + t_{b_1} + 0.2.
\end{aligned}
$$

If we gradually increase t_{b_1} from 0, then at $t_{b_1} = 0.4$, we have $v_{b_1}^1(\underline{t}) = 1.9$. Because $v_{b_2}^\ell(\underline{t}) < 1.9$ for $\ell = 1, 2$, we have $\tilde{t}_{b_1}^1(\underline{t}_{-b_1}) = 0.4$. If we increase t_{b_1} further, at $t_{b_1} = 0.7$, we have $v_{b_2}^1(\underline{t}) = 1.9$. Thus, for buyer b_1 to buy the second unit, t_{b_1} needs to be so high that $v_{b_1}^2(\underline{t}) \geq v_{b_2}^1(\underline{t})$, or equivalently, $t_{b_1} \geq 0.9$. Because $v_{b_2}^2(\underline{t}) < 1.9$ at $t_{b_1} = 0.9$, we have $\tilde{t}_{b_1}^2(\underline{t}_{-b_1}) = 0.9$.

Therefore, the price of each unit for buyer b_1 is the following: $v_{b_1}^1(0.4, t_{-b_1}) = 2.1$ for the first unit, and $v_{b_1}^2(0.9, t_{-b_1}) = 2.3$ for the second unit. In this example, one can verify that this mechanism satisfies ex post incentive compatibility, individual rationality, feasibility, non-wastefulness, and no budget deficit. In the next section, we show that these desirable properties hold generally under groupwise-price mechanisms. $\square$

3. Results with arbitrary number of buyers and sellers

In this section, we show that our groupwise-price mechanism has desirable properties introduced in Section 2.1. In particular, this mechanism is ex post incentive compatible.

[17] Although this example violates our assumptions that all valuations must lie in the unit interval and that $v_i^\ell \neq v_i^{\ell'}$ for each i and $\ell \neq \ell'$, this is just for notational simplicity. We can modify the example to satisfy these assumptions without changing the conclusion.

Theorem 1. *The groupwise-price mechanism satisfies*

(1) *ex post incentive compatibility,*
(2) *individual rationality,*
(3) *feasibility,*
(4) *non-wastefulness, and*
(5) *no budget deficit.*[18]

Recall that all of the above properties are satisfied in Example 1. Theorem 1 shows that these properties hold generally under groupwise-price mechanisms.

Proof. **Ex post incentive compatibility and individual rationality.** We consider only the case of buyers, but a similar argument applies to the case of sellers as well. Suppose that $t_b \in [\tilde{t}_b^{\ell}(\underline{t}_{-b}), \tilde{t}_b^{\ell+1}(\underline{t}_{-b})]$, with the convention that $\tilde{t}_b^{0}(\underline{t}_{-b}) = 0$ and $\tilde{t}_b^{m+1}(\underline{t}_{-b}) = 1$.

Then, for each $\ell' \leq \ell$,

$$v_b^{\ell'}(t_b, t_{-b}) - v_b^{\ell'}(\tilde{t}_b^{\ell'}(\underline{t}_{-b}), t_{-b}) \geq 0,$$

and for each $\ell' > \ell$,

$$v_b^{\ell'}(t_b, t_{-b}) - v_b^{\ell'}(\tilde{t}_b^{\ell'}(\underline{t}_{-b}), t_{-b}) \leq 0.$$

If the buyer b reports truthfully, then her utility is

$$\sum_{\ell'=1}^{\ell} \Big(v_b^{\ell'}(t_b, t_{-b}) - v_b^{\ell'}(\tilde{t}_b^{\ell'}(\underline{t}_{-b}), t_{-b}) \Big),$$

which is nonnegative, showing individual rationality. Moreover, the inequalities above imply that misreporting b's type does not increase her utility, demonstrating ex post incentive compatibility.

Feasibility and Non-wastefulness. Suppose that $|B_k^*(\underline{t})| \geq |S_k^*(\overline{t})|$. In this case, the number of units sold is $|S_k^{**}(\overline{t})| = |S_k^*(\overline{t})|$, and the number of units bought is $|B_k^{**}(\underline{t})| = |S_k^*(\overline{t})|$. Similarly, if $|B_k^*(\underline{t})| \leq |S_k^*(\overline{t})|$, then the number of units sold is $|S_k^{**}(\overline{t})| = |B_k^*(\underline{t})|$, while the number of units bought is $|B_k^{**}(\underline{t})| = |B_k^*(\underline{t})|$. Thus, we have shown both feasibility and non-wastefulness.

[18]The mechanism can be easily modified to satisfy budget balance as well if we specify any agent (possibly randomly) as a residual claimant of the budget surplus at the beginning of the mechanism. See Section 4.1 for detail.

No budget deficit. By construction, for any b, ℓ, t, if $(b,\ell) \in B_k^{**}(\underline{t})$ so that b trades her ℓ-th unit, then the price she pays for that unit is

$$v_b^\ell(\tilde{t}_b^\ell(\underline{t}_{-b}), t_{-b}) \geq v_b^\ell(\tilde{t}_b^\ell(\underline{t}_{-b}), \underline{t}_{-b}) \geq p_k.$$

Similarly, for any s, ℓ, t, if $(s,\ell) \in S_k^{**}(\overline{t})$ so that s trades ℓ-th unit, then the price she receives for that unit is

$$v_s^\ell(\tilde{t}_s^\ell(\overline{t}_{-s}), t_{-s}) \leq v_s^\ell(\tilde{t}_s^\ell(\overline{t}_{-s}), \overline{t}_{-s}) \leq p_k.$$

Therefore, given the feasibility and non-wastefulness established above, the total monetary transfer from the buyers is no smaller than the total monetary transfer to the sellers, implying that the groupwise-price mechanism runs no budget deficit. This completes the proof. □

4. Approximate efficiency

Next, we show that the double-auction mechanism constructed in the previous section approximates an efficient allocation as the number of market participants goes to infinity.

In this section we consider a sequence of markets, where each market is indexed by a positive integer N which we refer to as the market size. The number of sellers n_S and the number of buyers n_B depend (deterministically) on N and grow at the same asymptotic speed as N: Formally, there exist constants $\underline{\gamma}, \overline{\gamma} \in (0,\infty)$ such that for each N, $\underline{\gamma}N < n_S, n_B < \overline{\gamma}N$ (here we are suppressing dependence of n_S and n_B on N for notational simplicity only). The case in which $n_S = n_B = N$ is a special case, but note that the condition is more general and allows for the number of sellers and buyers to be different from each other even asymptotically.

Buyers have the same valuation function to one another and similarly for sellers. For each $i \in B$, $v_i^\ell(t_i, (t_j)_{j\in B\setminus\{i\}}, (t_j)_{j\in S}) = v_i^\ell(t_i, (t'_j)_{j\in B\setminus\{i\}}, (t'_j)_{j\in S})$ if $(t'_j)_{j\in B\setminus\{i\}}$ is a permutation of $(t_j)_{j\in B\setminus\{i\}}$ and $(t_j)_{j\in S}$ is a permutation of $(t'_j)_{j\in S}$. Thus, we are assuming that buyers are ex ante homogeneous, although their valuations can be distinct to one another ex post because of different type realizations. We impose a similar symmetry condition for each seller as well.

Next, we introduce two assumptions that regulate how agent valuations are affected by type profiles in large markets.

Assumption 1. *There exists a constant $\alpha \geq 0$ such that, for every sufficiently large market size N, any pair of agents i and $j \neq i$, any index $\ell \in \{1,\dots,m\}$, and any*

t_j, t'_j, t_{-j}, we have

$$|v_i^\ell(t_{-j}, t_j) - v_i^\ell(t_{-j}, t'_j)| \le \frac{\alpha}{n_B}, \text{ if } j \in B, \tag{4.1}$$
$$|v_i^\ell(t_{-j}, t_j) - v_i^\ell(t_{-j}, t'_j)| \le \frac{\alpha}{n_S}, \text{ if } j \in S.$$

This assumption implies that the influence of any one agent's type on another agent's utility becomes small in large markets.[19]

Assumption 2. *There exist β and β' with $\beta' \ge \beta > 0$ such that, for every sufficiently large market size N, any pair of agents i and $j \neq i$ on the same side of the market (i.e., both i and j are buyers, or both are sellers), any index $\ell \in \{1, \dots, m\}$, and any t,*

$$\beta|t_i - t_j| \le |v_i^\ell(t) - v_j^\ell(t)| \le \beta'|t_i - t_j|. \tag{4.2}$$

The part $\beta|t_i - t_j| \le |v_i^\ell(t) - v_j^\ell(t)|$ in (4.2) requires that a difference in types has a first-order effect on the values. The part $|v_i^\ell(t) - v_j^\ell(t)| \le \beta'|t_i - t_j|$ in (4.2) requires that two persons with similar types have similar values. Throughout this section, we maintain Assumptions 1 and 2.

Remark 1. If the valuation functions are differentiable, the following conditions (i) and (ii) together imply (4.1) and (4.2): (i) for any N, and for any pair of agents i and $j \neq i$, any index $\ell \in \{1, \dots, m\}$, and all t,

$$\frac{\partial v_i^\ell(t)}{\partial t_j} \le \frac{\alpha}{n_S}, \text{ if } j \in S, \tag{4.3}$$
$$\frac{\partial v_i^\ell(t)}{\partial t_j} \le \frac{\alpha}{n_B}, \text{ if } j \in B,$$

and (ii) there exist δ and δ' with $\delta' \ge \delta > 0$ such that, for any N, any agent i, any index $\ell \in \{1, \dots, m\}$, and all t,

$$\delta \le \frac{\partial v_i^\ell(t)}{\partial t_i} \le \delta'. \tag{4.4}$$

In this differentiable case, the part $\delta \le \frac{\partial v_i^\ell(t)}{\partial t_i}$ in (4.4) can be interpreted as requiring that an agent's own type influences her own value in a non-negligible manner everywhere, and the part $\frac{\partial v_i^\ell(t)}{\partial t_i} \le \delta'$ can be interpreted as excluding some pathological cases by assuming

[19]Of course, it does not mean that the interdependence vanishes away in large markets. Agent i's value can vary with t_{-i} in a non-negligible manner even in a large market, even though the effect of each single t_j, $j \neq i$, is vanishing.

that there is a bound on the change in an agent's utility for a small change in her own type.[20] □

Remark 2. While excluding some cases, conditions (4.1) and (4.2) are satisfied by most models in the literature. For example, let $n_B = n_S = N \geq 2$, and for each buyer's utility function (and similarly for each seller's), we may assume that there is a differentiable function $u^\ell : [0,1] \to \mathbb{R}$ for each $\ell = 1, \ldots, m$ such that $v_b^\ell(t) = u^\ell(t_b + \gamma \frac{\sum_{b' \neq b} t_{b'}}{N-1})$ for some constant $\gamma \in (0,1)$, and that, for some $\delta' \geq \delta > 0$, $\frac{du^\ell(x)}{dx} \in (\delta, \delta')$ for all x. Then, (4.4) is satisfied. Moreover, for any $b' \neq b$,

$$
\begin{aligned}
\frac{\partial v_b^\ell(t)}{\partial t_{b'}} &= \frac{du^\ell(t_b + \gamma \frac{\sum_{b'' \neq b} t_{b''}}{N-1})}{dx} \times \frac{\gamma}{N-1} \\
&\leq \frac{\delta' \gamma}{N-1} \\
&= \frac{\delta' \gamma}{N} \times \frac{N}{N-1} \\
&\leq \frac{2\delta' \gamma}{N},
\end{aligned}
$$

thus condition (4.3) is satisfied with respect to $\alpha = 2\delta'\gamma$. Recall, then, that (4.3) and (4.4) imply (4.1) and (4.2).

Another example is an environment with "unobservable fundamentals". Let $n_B = n_S = N$, and let $\theta \in \Theta$ be an unobservable variable that affects every agent's valuation, and assume that each agent i's value for the ℓ-th unit of the trade is a function of only t_i and θ, which we denote by $w_i^\ell(t_i, \theta)$. We also assume that each type t_i is identically and independently distributed conditional on θ, where the conditional distribution of t_i given θ is assumed to be common knowledge. Then, $v_i^\ell(t)$ can be defined as the conditional expectation of $w_i^\ell(t_i, \theta)$ given t, i.e., $v_i^\ell(t) = E(w_i^\ell(t_i, \theta)|t)$. To be specific, let $\Theta = [0,1]$, assume that θ is distributed uniformly over $[0,1]$, and $w_i^\ell(t_i, \theta) = \frac{1}{\ell}(t_i + \theta)$. Given θ, t_i is independently distributed with a density $g(t_i|\theta)$ such that $g(t_i|\theta) = 2 - 2\theta$ for $t_i \in [0, \frac{1}{2}]$,

[20] It is straightforward that (4.3) implies (4.1). For (4.2), observe that

$$
\begin{aligned}
v_i^\ell(t) - v_j^\ell(t) &= v_i^\ell(t_1, \ldots, t_i, \ldots, t_j, \ldots, t_{n_B+n_S}) - v_j^\ell(t_1, \ldots, t_i, \ldots, t_j, \ldots, t_{n_B+n_S}) \\
&= v_i^\ell(t_1, \ldots, t_i, \ldots, t_j, \ldots, t_{n_B+n_S}) - v_i^\ell(t_1, \ldots, t_j, \ldots, t_j, \ldots, t_{n_B+n_S}) \\
&\quad + v_i^\ell(t_1, \ldots, t_j, \ldots, t_j, \ldots, t_{n_B+n_S}) - v_i^\ell(t_1, \ldots, t_j, \ldots, t_i, \ldots, t_{n_B+n_S}),
\end{aligned}
$$

so $v_i^\ell(t) - v_j^\ell(t) \in \left[\delta(t_i - t_j) - \frac{\alpha}{n_B}(t_i - t_j), \delta'(t_i - t_j)\right]$ for $t_i \geq t_j$ and $v_i^\ell(t) - v_j^\ell(t) \in \left[\delta'(t_i - t_j), \delta(t_i - t_j) - \frac{\alpha}{n_B}(t_i - t_j)\right]$ for $t_i < t_j$ if $i, j \in B$ (if $i, j \in S$, then analogous bounds replacing n_B with n_S hold). Thus, (4.2) is satisfied for any sufficiently large N by taking $\beta = \frac{\delta}{2}$ and $\beta' = \delta'$.

and $g(t_i|\theta) = 2\theta$ for $t_i \in (\frac{1}{2}, 1]$. Then, letting $N_1 = |\{j|t_j > \frac{1}{2}\}|$, we have[21]

$$\begin{aligned} v_i^\ell(t) &= \frac{1}{\ell}(t_i + E(\theta|t)) \\ &= \frac{1}{\ell}\left(t_i + \frac{N_1+1}{N+2}\right). \end{aligned}$$

Therefore, for each $j \neq i$, any t, and any $\ell = 1, \ldots, m$, we have $|v_i^\ell(t_{-j}, t_j) - v_i^\ell(t_{-j}, t_j')| \leq \frac{1}{N}$, and $v_i^\ell(t) - v_j^\ell(t) = \frac{t_i - t_j}{\ell}$, and thus, both of the conditions (4.1) and (4.2) are satisfied. □

We assume that agent types are conditionally independent given a state variable. More formally, there is a state variable σ that is drawn randomly from a finite distribution. For each realization of σ, there is a pair of type distributions with cdfs F_σ and G_σ with everywhere positive and continuous pdf's, and buyer and seller types are independently distributed from F_σ and G_σ, respectively, conditional on σ. Note that the case with i.i.d. type distributions is a special case of this model in which the distribution of the state variable is degenerate.

In this setting, we specialize our groupwise-price mechanism by providing a particular procedure to set the parameters, namely, the number of submarkets K and the reference prices $(p_1, \ldots, p_K)$, as follows. We call the resulting mechanism the **canonical groupwise-price mechanism**.

- Set K to be an integer depending on N such that $K \to \infty$ and $\frac{K^5}{N} \to 0$ as $N \to \infty$ and, for notational simplicity, such that n_B and n_S are multiples of K.[22] The agents are divided into K submarkets, each with $a_B = \frac{n_B}{K}$ buyers and $a_S = \frac{n_S}{K}$ sellers.
- Given reported t, let $\hat{t}_k = ((t_i)_{i \notin (S_k \cup B_k)}, (1)_{i \in (S_k \cup B_k)})$, and let $\hat{v}_B^{(q)}$ be the q-th highest value among $\{v_b^\ell(\hat{t}_k)\}_{b \in B, \ell}$, and $\hat{v}_S^{(q)}$ be the q-th lowest value among $\{v_s^\ell(\hat{t}_k)\}_{s \in S, \ell}$.

[21]Note that

$$E(\theta|t) = \frac{\int_0^1 \theta \left(\prod_{j|t_j \leq \frac{1}{2}}(1-\theta)\right)\left(\prod_{j|t_j > \frac{1}{2}} \theta\right) d\theta}{\int_0^1 \left(\prod_{j|t_j \leq \frac{1}{2}}(1-\theta)\right)\left(\prod_{j|t_j > \frac{1}{2}} \theta\right) d\theta} = \frac{\int_0^1 (1-\theta)^{N-N_1}\theta^{N_1+1} d\theta}{\int_0^1 (1-\theta)^{N-N_1}\theta^{N_1} d\theta},$$

where, for the denominator,

$$\begin{aligned} \int_0^1 (1-\theta)^{N-N_1}\theta^{N_1} d\theta &= \frac{1}{N_1+1}(1-\theta)^{N-N_1}\theta^{N_1+1}\Big|_0^1 + \int_0^1 \frac{N-N_1}{N_1+1}(1-\theta)^{N-N_1-1}\theta^{N_1+1} \\ &= \ldots = \frac{(N-N_1)!(N_1)!}{(N+1)!}, \end{aligned}$$

and similarly, for the numerator, $\int_0^1 (1-\theta)^{N-N_1}\theta^{N_1+1} d\theta = \frac{(N-N_1)!(N_1+1)!}{(N+2)!}$. Therefore, $E(\theta|t) = \frac{N_1+1}{N+2}$.

[22]For example, K may be an integer of the order N^c with $c \in (0, \frac{1}{5})$ or of the order $\log(N)$.

The reference price in submarket k, p_k, is given by

$$p_k = \min\left\{\hat{v}_B^{(q)}, \hat{v}_S^{(q+1)}\right\},$$

where q is an integer such that $\hat{v}_S^{(q)} \leq \hat{v}_B^{(q)}$ and $\hat{v}_S^{(q+1)} > \hat{v}_B^{(q+1)}$.[23] Note that we include all buyers and sellers in computing p_k.

We define asymptotic efficiency in terms of trade outcomes. We say that a mechanism is **asymptotically efficient** if the ex ante non-monetary payoff of each agent in that mechanism approaches that in a competitive mechanism, i.e., as N goes to infinity, for any agent i, $E[|\sum_{\ell=1}^{\zeta_i(t)} v_i^\ell(t) - \sum_{\ell=1}^{\zeta_i^*(t)} v_i^\ell(t)|] \to 0$, where ζ denotes the object allocation rule of our mechanism, and ζ^* denotes that of a competitive mechanism. In this sense, the trade outcome in the mechanism becomes "arbitrarily close" to the first-best level in large economies.[24]

Theorem 2. *The canonical groupwise-price mechanism is asymptotically efficient.*

Proof. See Appendix A. □

The formal proof of this result is involved, so we offer some intuition here while deferring the proof to the Appendix. To get the first intuition, recall that our groupwise-price mechanism sets a reference price p_k for each submarket k. This suggests that most mutually beneficial trades can be realized if the reference prices approximate the market clearing price in large economies.

However, whether this intuition goes through is far from obvious. More specifically, there are at least two challenges. First, the reference price for a submarket must be *independent of reported types of agents in that submarket* in order to keep ex post incentive compatibility of the mechanism. This implies that the relevant information from agents in a submarket should be ignored when setting that submarket's reference price. This poses a problem, because the reference price does not converge to the market-clearing price even in a large market if private information from too many agents is ignored. Second, even if the reference prices are close to the market-clearing prices, additional efficiency loss can occur because agents in a submarket can trade only with those in the same submarket. This can prevent some beneficial trades from happening between agents in different submarkets.

[23]If $\hat{v}_S^{(q)} \leq \hat{v}_B^{(q)}$ for all q, then we set $p_k = \hat{v}_B^{(m n_S)}$. If $\hat{v}_S^{(q)} > \hat{v}_B^{(q)}$ for all q, then we set $p_k = \hat{v}_S^{(1)}$.

[24]Recall that our mechanism may generate budget surplus while a competitive mechanism balances the budget. Hence the difference in the ex ante "total" (i.e., the sum of monetary and non-monetary) payoffs may not vanish. Section 4.1 studies this issue and offers possible solutions to obtain stronger convergence results.

Our proof shows that inefficiencies from both of these sources can be appropriately bounded. Regarding the first challenge, our approach is to divide the market into a sufficiently large number of submarkets (i.e., $K \to \infty$ as $N \to \infty$). Doing so makes the effect of ignoring types of one submarket for calculating reference prices negligible in large markets, by which we show that the reference prices approximate a market-clearing price in large economies (with high probability). Regarding the second challenge, our approach is to keep the number of submarkets *sufficiently small* relative to N (i.e., $\frac{K}{N} \to 0$ or equivalently $a_B = \frac{n_B}{K}, a_S = \frac{n_S}{K} \to \infty$ as $N \to \infty$), so that the number of beneficial trades prevented from happening across different submarkets is sufficiently small. Clearly, there is a potential conflict between these two approaches. Our formal proof shows that there is an appropriate growth rate of the number of submarkets such that these conflicting forces can be balanced in such a way that both challenges are addressed. Furthermore, given such an appropriate choice of the growth rate, a lower bound of the convergence rate is obtained as a polynomial function of the size of the economy, N (see Remark 3 in the Appendix). The existence of such a growth rate is not obvious, and we refer interested readers to the proof in Appendix A.

4.1. **Asymptotic Budget Balance.** In the preceding section, we have established that the trading pattern of the objects converges to an efficient one under the canonical groupwise-price mechanism. However, this does not imply that the expected payoff of each agent converges to the efficient level in the competitive mechanism, because a groupwise-price mechanism can run budget surplus. Unless the budget surplus is included in the welfare, this implies that the welfare level including transfer in groupwise-price mechanisms can be lower than that in a competitive equilibrium.

To present a formal analysis on this issue, we begin by defining asymptotic budget balance. A mechanism $\varphi = (\zeta, \tau)$ is **asymptotically budget-balanced** if

$$\lim_{N\to\infty} E\left[\frac{\sum_{b\in B}\tau_b(t) - \sum_{s\in S}\tau_s(t)}{n_B + n_S}\right] = 0.$$

Note that n_B and n_S depend on N and $\underline{\gamma}N < n_B, n_S < \overline{\gamma}N$ for some constants $\underline{\gamma}, \overline{\gamma} \in (0, \infty)$, so asymptotic budget balance is equivalent to

$$\lim_{N\to\infty} E\left[\frac{\sum_{b\in B}\tau_b(t) - \sum_{s\in S}\tau_s(t)}{N}\right] = 0.$$

This condition ensures that the per-capita budget imbalance converges to zero in expectation as the market size approaches infinity.

The following example shows that even the canonical groupwise-price mechanism can violate asymptotic budget balance.

Example 2. Suppose $m = 2$, $n_B = n_S = N$, the values for each seller s is given by $v_s^1(t) = t_s$ and $v_s^2(t) = t_s + 2$, and the values for each buyer b are given by $v_b^\ell(t) = t_b + 1$ for both $\ell = 1, 2$ (thus, agents in this example have private values).[25] Note that, for each seller, the value of her first unit of the object is in $[0, 1]$, and the value for the second unit is in $[2, 3]$. Thus, for the sellers, there is a "gap" between possible values of the first and second units of the object.

Consider the canonical groupwise-price mechanism. By the definition of the reference price, with a large N, it is very likely that p_k is close to 1.5 for each k. The probability that $|B_k^*(\underline{t})| < a_B = \frac{N}{K}$ is bounded away from zero even if N goes to infinity. On the other hand, the probability that $|S_k^*(\bar{t})| = a_S = \frac{N}{K}$ approaches one. Thus, the probability that the sellers in submarket k are on the long side of the market, i.e., $|S_k^*(\bar{t})| > |B_k^*(\underline{t})|$, is bounded away from zero. In such a case, each seller who trades earns at most 1, while each buyer who trades pays p_k. Since at least a fraction of agents bounded away from zero trade in expectation, and p_k is higher than 1.5 with probability bounded away from zero, this implies that the expected budget surplus per capita does not converge to zero as $N \to \infty$. That is, the canonical groupwise-price mechanism is not asymptotically budget-balanced. □

This example shows that the groupwise-price mechanism does not necessarily achieve asymptotic budget balance. However, as we present below, there are at least two approaches that enable us to have the mechanism achieve asymptotic budget balance, thereby enabling each agent to asymptotically enjoy the same level of ex ante expected utility as in a competitive equilibrium.

One solution is to randomly choose one agent independently from agents' reports and give all the budget surplus to that agent while prohibiting her from trading. In other words, we can achieve asymptotic efficiency by augmenting the canonical groupwise-price mechanism by exogenously appointing one revenue absorber. By construction, this modified mechanism is budget-balanced. Moreover, it is easy to verify that the above modification does not invalidate any of our preceding results so all other desirable properties of the original mechanism continue to hold.

Nevertheless, it is conceivable that a social planner does not want to use a revenue absorber.[26] This concern motivates the second solution. Let us begin by imposing an

[25]Although this example violates our assumptions that all valuations must lie in the unit interval and that $v_i^\ell \neq v_i^{\ell'}$ for each i and $\ell \neq \ell'$, this is just for notational simplicity. We can modify the example to satisfy these assumptions without changing the conclusion.

[26]For instance, agents' payoff functions may fail to be quasi-linear (which we assume throughout the paper) because of income effect if the monetary transfer is large. Under income effect, agents' payoffs

additional assumption. We say that agent valuations **allow no gaps** if $v_s^\ell(1, t_{-s}) \geq v_s^{\ell+1}(0, t_{-s})$ and $v_b^{\ell+1}(1, t_{-b}) \geq v_b^\ell(0, t_{-b})$ for each $b \in B$, $s \in S$, $\ell \in \{1, \ldots, m-1\}$, and $t \in [0,1]^{B \cup S}$. This assumption may be interpreted as imposing certain "smoothness" of supply and demand functions. Note that the sellers' valuations in Example 2 violate this requirement. Note also that this condition is automatically satisfied if agents have single-unit demand and supply, i.e., $m = 1$, as assumed in most existing studies.

Theorem 3. *Suppose that agent valuations allow no gaps. Then, the canonical groupwise-price mechanism is asymptotically budget-balanced.*

Proof. See Appendix B. □

An immediate corollary of Theorems 2 and 3 is a stronger form of asymptotic efficiency. More specifically, the ex-ante expected utility of each buyer and seller converges to the level achieved with a competitive mechanism under truthtelling as the market size approaches infinity.[27]

5. Conclusion

This paper investigated whether desirable properties can be achieved in double auction environments with value interdependence. We showed that there exists a mechanism that satisfies ex post incentive compatibility, individual rationality, feasibility, non-wastefulness, no budget deficit, and asymptotic efficiency. To our knowledge, our mechanism is the first double auction mechanism with these properties in the interdependent values setting.

We do not necessarily regard our mechanism as an immediately applicable solution, but rather as a step toward understanding what desirable properties can be achieved in practice. In fact, there are still several important gaps between our current knowledge and practical use. First, the social planner is assumed to know the functional form of the agents' payoff functions. Second, the trading prices can vary across agents under our mechanism. Both features are shared by most mechanisms in the literature,[28] but they

may exhibit risk aversion, and hence randomly awarding one agent with a large amount of money may be inefficient.

[27]Strictly speaking, the statement of Theorem 3 merely states that the *aggregate* budget surplus per capita vanishes, and is silent about the distribution of transfer across different agents. However, the proof of the theorem reveals that each agent's transfer converges to its competitive level. This fact and Theorem 2 imply this corollary.

[28]Even in the one-sided auction under private values, VCG payments can vary across agents with multi-unit demand. In the interdependent values setting, the mechanism by Ausubel (1999) shares this

may pose challenges in some applications. We hope that our analysis stimulates future studies aimed at practical applications.

In addition, there are a number of possible directions of future research. One possibility is to examine the speed of convergence (i.e., how quickly efficiency is approximated) in more detail, or the possibility of a stronger form of asymptotic efficiency (e.g., whether efficiency in the "absolute term" is possible, rather than in the "per-capita term" as in this paper). These issues appear to be technically challenging exercises.[29] We leave them as topics for future research.

Another direction is to consider more general environments, such as those with multidimensional signals,[30] multiple types of objects, complementarity in agents' valuations, dispensing with the assumption that each agent is predetermined to be a buyer or a seller (that is, allowing agents to buy or sell depending on signals and prices), and so on. Generalizations in these directions are not straightforward, and we leave them for future research. However, we believe that some intuitions obtained in our study may be useful in designing desirable mechanisms in these more general environments.

feature, and it presumes the knowledge of the social planner about the payoff functions. Note, however, important advance such as Perry and Reny (2002) who implement a desired outcome as an equilibrium of a game whose form does not depend on the functional form of the payoff functions.

[29]Measuring efficiency in the per-capita term is standard in the literature, especially with interdependence. For example, see Reny and Perry (2006) for double auction, Vives (2002) for Cournot oligopoly, and Lee and Yariv (2014) for matching.

[30]In the one-sided, object allocation setting, a recent contribution by Hashimoto (2013) obtains positive results even with multi-dimensional signals. Although there does not appear to be an obvious way to adapt his idea to our double-auction setting, studying multidimensional signals in double auction would be an important future research.

Appendix A. Proof of Theorem 2

We prove the theorem in three steps. In the first step, we prove the result under the assumption that each agent's type is drawn i.i.d. according to the uniform distribution. In the second step, we build on this result to establish the desired result for the case with more general type distributions while retaining the i.i.d. assumption. In the last step, we use this result to obtain the desired result for the general case of conditionally independent types.

A.1. **Proof for the uniform distribution case.** In this subsection, we prove the result under the assumption that each agent's type is drawn i.i.d. according to the uniform distribution.

Lemma 1 shows that, by a law of large numbers, the sup-norm distance between the empirical cdf of types and the true cdf of types in each submarket k is small in an event that occurs with a high probability. Focusing on that event, Lemmas 2 to 5 evaluate how many efficiency-enhancing trades are left unrealized. Building on these lemmas, we complete the proof by bounding the overall expected efficiency loss.

We first look at each submarket k. Let $\Lambda = K$, and consider $\lambda \in \{1, \ldots, \Lambda\}$.

Let

$$\begin{aligned}
\underline{x}_k^\lambda &= \left\{ s \in S_k | t_s \leq \frac{\lambda}{\Lambda} - \frac{1}{K} \right\}, \\
\overline{x}_k^\lambda &= \left\{ s \in S_k | t_s \geq \frac{\lambda}{\Lambda} + \frac{1}{K} \right\}, \\
\underline{y}_k^\lambda &= \left\{ b \in B_k | t_b \leq \frac{\lambda}{\Lambda} - \frac{1}{K} \right\}, \\
\overline{y}_k^\lambda &= \left\{ b \in B_k | t_b \geq \frac{\lambda}{\Lambda} + \frac{1}{K} \right\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{X}_k^\lambda &= \frac{|\underline{x}_k^\lambda|}{a_S}, \\
\overline{X}_k^\lambda &= \frac{|\overline{x}_k^\lambda|}{a_S}, \\
\underline{Y}_k^\lambda &= \frac{|\underline{y}_k^\lambda|}{a_B}, \\
\overline{Y}_k^\lambda &= \frac{|\overline{y}_k^\lambda|}{a_B},
\end{aligned}$$

where $a_S = \frac{n_S}{K}, a_B = \frac{n_B}{K}$. Each of $\underline{X}_k^\lambda, \overline{X}_k^\lambda, \underline{Y}_k^\lambda$, and $\overline{Y}_k^\lambda$ is a binomially distributed variable, with means and variances as follows.

$$\begin{aligned}
E(\underline{X}_k^\lambda) = E(\underline{Y}_k^\lambda) &= \frac{\lambda}{\Lambda} - \frac{1}{K}, \\
E(\overline{X}_k^\lambda) = E(\overline{Y}_k^\lambda) &= 1 - \frac{\lambda}{\Lambda} - \frac{1}{K}, \\
V(\underline{X}_k^\lambda), V(\overline{X}_k^\lambda) &\leq \frac{1}{4a_S}, \\
V(\underline{Y}_k^\lambda), V(\overline{Y}_k^\lambda) &\leq \frac{1}{4a_B}.
\end{aligned}$$

Let E_k be the event that all of the following four inequalities hold,

$$\left|\underline{X}_k^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| < \frac{1}{2K},$$
$$\left|\overline{X}_k^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| < \frac{1}{2K},$$
$$\left|\underline{Y}_k^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| < \frac{1}{2K},$$
$$\left|\overline{Y}_k^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| < \frac{1}{2K},$$

for every $\lambda \in \{1, \dots, \Lambda\}$. Note that, given event E_k, $\underline{X}_k^{\lambda+1} - \underline{X}_k^\lambda > 0$ by the assumption $\Lambda = K$ (and similarly for $\overline{X}_k$, $\underline{Y}_k$, $\overline{Y}_k$).

By Chebyshev's inequality, we obtain the following.

Lemma 1. $\Pr(E_k) > 1 - \frac{2\Lambda K^2}{a_S} - \frac{2\Lambda K^2}{a_B}$.

Proof. By Chebyshev's inequality, for each λ,

$$\Pr\left(\left|\underline{X}_k^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| > \eta\right) < \frac{1}{4a_S\eta^2},$$
$$\Pr\left(\left|\overline{X}_k^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| > \eta\right) < \frac{1}{4a_S\eta^2},$$
$$\Pr\left(\left|\underline{Y}_k^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| > \eta\right) < \frac{1}{4a_B\eta^2},$$
$$\Pr\left(\left|\overline{Y}_k^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K}\right)\right| > \eta\right) < \frac{1}{4a_B\eta^2},$$

where $\eta = \frac{1}{2K}$. Because the probability of the union of events is weakly smaller than the sum of the probabilities of those events by Boole's inequality, these inequalities imply that $\Pr(E_k) > 1 - \frac{\Lambda}{2a_S\eta^2} - \frac{\Lambda}{2a_B\eta^2}$. Substituting in $\eta = \frac{1}{2K}$, we obtain the desired conclusion. □

Now, consider the event F that $v_i^\ell(t) \neq v_j^{\ell'}(t)$ for every ℓ, ℓ', i, j such that $\ell \neq \ell'$ or $i \neq j$. Because types are drawn i.i.d. from the uniform distribution on $[0,1]$, F is a probability one event. From this and the fact that there are K submarkets, we have $\Pr(E) > 1 - \frac{2\Lambda K^3}{a_S} - \frac{2\Lambda K^3}{a_B} = 1 - \frac{2\Lambda K^4}{n_S} - \frac{2\Lambda K^4}{n_B}$, where $E := (\bigcap_k E_k) \cap F$. Thus, if $\frac{\Lambda K^4}{n_S}$ and $\frac{\Lambda K^4}{n_B}$ converge to zero as N goes to infinity, this probability converges to one.

We observe that the overall type distributions satisfy similar properties. Let

$$\begin{aligned}
\underline{x}^\lambda &= \left\{ s \in S | t_s \leq \frac{\lambda}{\Lambda} - \frac{1}{K} \right\}, \\
\overline{x}^\lambda &= \left\{ s \in S | t_s \geq \frac{\lambda}{\Lambda} + \frac{1}{K} \right\}, \\
\underline{y}^\lambda &= \left\{ b \in B | t_b \leq \frac{\lambda}{\Lambda} - \frac{1}{K} \right\}, \\
\overline{y}^\lambda &= \left\{ b \in B | t_b \geq \frac{\lambda}{\Lambda} + \frac{1}{K} \right\},
\end{aligned}$$

and

$$\begin{aligned}
\underline{X}^\lambda &= \frac{|\underline{x}^\lambda|}{n_S}, \\
\overline{X}^\lambda &= \frac{|\overline{x}^\lambda|}{n_S}, \\
\underline{Y}^\lambda &= \frac{|\underline{y}^\lambda|}{n_B}, \\
\overline{Y}^\lambda &= \frac{|\overline{y}^\lambda|}{n_B}.
\end{aligned}$$

Lemma 2. *Given that E has occurred, we have*

$$\begin{aligned}
\left| \underline{X}^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K} \right) \right| &< \frac{1}{2K}, \\
\left| \overline{X}^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K} \right) \right| &< \frac{1}{2K}, \\
\left| \underline{Y}^\lambda - \left(\frac{\lambda}{\Lambda} - \frac{1}{K} \right) \right| &< \frac{1}{2K}, \\
\left| \overline{Y}^\lambda - \left(1 - \frac{\lambda}{\Lambda} - \frac{1}{K} \right) \right| &< \frac{1}{2K},
\end{aligned}$$

for every $\lambda \in \{1, \ldots, \Lambda\}$.

Proof. Given that E_k has occurred,

$$\underline{X}_k^\lambda \in \left(\frac{\lambda}{\Lambda} - \frac{3}{2K}, \frac{\lambda}{\Lambda} - \frac{1}{2K} \right).$$

Thus, given that $E = \bigcap_k E_k$ has occurred,

$$\underline{X}^\lambda = \frac{1}{n_S}\sum_{k=1}^{K}(a_S \underline{X}_k^\lambda) \in \left(\frac{\lambda}{\Lambda} - \frac{3}{2K}, \frac{\lambda}{\Lambda} - \frac{1}{2K}\right).$$

We obtain the desired conclusions about $\overline{X}^\lambda, \underline{Y}^\lambda$, and $\overline{Y}^\lambda$ by symmetric arguments. $\square$

Given reported t, let $v_B^{(q)}$ be the q-th highest value among $\{v_b^\ell(t)\}_{b\in B,\ell}$, and $v_S^{(q)}$ be the q-th lowest value among $\{v_s^\ell(t)\}_{s\in S,\ell}$. Let p^{MC} be a market-clearing price, defined as

$$p^{MC} = \min\left\{v_B^{(q)}, v_S^{(q+1)}\right\},$$

where q is an integer such that $v_S^{(q)} \leq v_B^{(q)}$ and $v_S^{(q+1)} > v_B^{(q+1)}$.[31]

Let s^ℓ (b^ℓ) denote the seller (the buyer) who has the highest (lowest) type among those trading at least ℓ units under p^{MC}.[32]

By symmetry among sellers, each type of the seller with $t < t_{s^\ell}$ sells at least ℓ units under p^{MC}, and similarly for the buyers (we set $t_{s^\ell} = 0$ ($t_{b^\ell} = 1$) if no seller type (buyer type) trades at least ℓ units).[33] Let λ_B^ℓ and λ_S^ℓ be integers such that $t_{b^\ell} \in \left[\frac{\lambda_B^\ell}{\Lambda}, \frac{\lambda_B^\ell+1}{\Lambda}\right)$ and $t_{s^\ell} \in \left[\frac{\lambda_S^\ell-1}{\Lambda}, \frac{\lambda_S^\ell}{\Lambda}\right)$.

In the remainder of the proof, we show that the level of trades approaches an efficient level as $N \to \infty$, implying that per-capita inefficiency caused by failed trades converges to zero. By ex ante symmetry across buyers and across sellers, this suffices for the proof of the theorem. We begin with the following lemma, which shows that the reference price p_k for each submarket k is close to the market clearing price p^{MC}.

Lemma 3. *Given that E has occurred, $p_k \in \left[p^{MC}, p^{MC} + (\frac{2}{K} + \frac{4}{\Lambda})\beta' + \frac{2\alpha}{K}\right]$ for each k.*

Proof. Let D_0 and S_0 be the demand and supply under the true type profile at price p^{MC}, and let $D'(p)$ and $S'(p)$ be the demand and supply under the modified type profile (i.e., $\hat{t}_k = ((t_i)_{i\notin(S_k\cup B_k)}, (1)_{i\in(S_k\cup B_k)})$) at price p, respectively. Formally, define

$$\begin{aligned}
D_0 &= \#\{(b,\ell) \in B \times \{1,\dots,m\} | v_b^\ell(t) \geq p^{MC}\},\\
S_0 &= \#\{(s,\ell) \in S \times \{1,\dots,m\} | v_s^\ell(t) < p^{MC}\},\\
D'(p) &= \#\{(b,\ell) \in B \times \{1,\dots,m\} | v_b^\ell(\hat{t}_k) \geq p\},\\
S'(p) &= \#\{(s,\ell) \in S \times \{1,\dots,m\} | v_s^\ell(\hat{t}_k) < p\}.
\end{aligned}$$

[31] If $v_S^{(q)} \leq v_B^{(q)}$ for all q, then we set $p^{MC} = v_B^{(mn_B)}$. If $v_S^{(q)} > v_B^{(q)}$ for all q, then we set $p^{MC} = v_S^{(1)}$.

[32] Hence, for example, if there is no type of the seller who trades exactly two units, we have $t_{s^2} = t_{s^3}$.

[33] Recall that no two buyers or sellers have the same type with each other under event E.

Because the demand under the modified type profile is weakly larger than the one under the true type profile, we have $D'(p^{MC}) \geq D_0$ at price p^{MC}. On the other hand, the supply under the modified type profile is weakly smaller, and hence $S'(p^{MC}) \leq S_0$. Hence, we have $D'(p^{MC}) \geq D_0 = S_0 \geq S'(p^{MC})$, which implies that p_k is no smaller than p^{MC}.

In the rest of this proof, we shall show

$$p_k \leq \bar{p}_k := p^{MC} + \left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{2\alpha}{K}. \tag{A.1}$$

In the following, we first show that $S'(\tilde{p}_k) \geq S_0 = D_0 \geq D'(\tilde{p}_k)$, where

$$\begin{aligned}\tilde{p}_k &= p^{MC} + \left(\frac{2}{K} + \frac{3}{\Lambda}\right)\beta' + \frac{2\alpha}{K} \\ &= \bar{p}_k - \frac{\beta'}{\Lambda}.\end{aligned}$$

Then, we show that this implies the desired inequality (A.1).

We first consider sellers. For each ℓ, define $S^\ell(\tilde{p}_k) = |\{s \in S | v_s^\ell(\hat{t}_k) < \tilde{p}_k\}|$ and $S_0^\ell = |\{s \in S | v_s^\ell(t) < p^{MC}\}|$. We shall show that $S^\ell(\tilde{p}_k) \geq S_0^\ell$. To show this, consider the following cases.

(1) Suppose $S_0^\ell = 0$. Then trivially $S^\ell(\tilde{p}_k) \geq S_0^\ell$.
(2) Suppose $S^\ell(\tilde{p}_k) = n_S$. Then trivially $S^\ell(\tilde{p}_k) \geq S_0^\ell$.
(3) Suppose $S_0^\ell > 0$ and $S^\ell(\tilde{p}_k) < n_S$. We first show the following claim.

Claim 1. *Suppose $S_0^\ell > 0$ and $S^\ell(\tilde{p}_k) < n_S$. Then $\lambda_S^\ell(p^{MC}) < \Lambda - 1 - \frac{2\Lambda}{K}$.*

Proof. Suppose for contradiction that $\lambda_S^\ell(p^{MC}) \geq \Lambda - 1 - \frac{2\Lambda}{K}$. Take $s' \in S$ as the seller whose type is the highest among the sellers s with $v_s^\ell(t) < p^{MC}$. Then $t_{s'} \geq \frac{\lambda_S^\ell(p^{MC})-1}{\Lambda}$; Note that such a seller s' exists because of the assumption $S_0^\ell > 0$. Consider the following cases.

(a) Suppose $s' \notin S_k$. In this case, for any $\tilde{s} \in S_k$, we have

$$\begin{aligned}v_{\tilde{s}}^\ell(\hat{t}_k) &\leq v_{s'}^\ell(\hat{t}_k) + \beta'(1 - t_{s'}) \\ &\leq v_{s'}^\ell(t) + \beta'(1 - t_{s'}) + \frac{2\alpha}{K} \\ &< p^{MC} + \beta'\left(1 - \frac{\lambda_S^\ell(p^{MC}) - 1}{\Lambda}\right) + \frac{2\alpha}{K} \\ &\leq p^{MC} + \beta'\left(1 - \frac{\Lambda - 1 - \frac{2\Lambda}{K} - 1}{\Lambda}\right) + \frac{2\alpha}{K} \\ &= p^{MC} + \beta'\left(\frac{2}{\Lambda} + \frac{2}{K}\right) + \frac{2\alpha}{K} \\ &\leq \tilde{p}_k.\end{aligned}$$

Because the modified type for $\tilde{s} \in S_k$ at $\hat{t}_k$ is the highest possible type by the definition of $\hat{t}_k$, the above inequality implies $S^\ell(\tilde{p}_k) = n_S$, a contradiction.

(b) Suppose $s' \in S_k$. In this case, we have[34]

$$\begin{aligned}
v^\ell_{s'}(\hat{t}_k) &\leq v^\ell_{s'}(t) + \beta'(1 - t_{s'}) + \frac{2\alpha}{K} \\
&< p^{MC} + \beta'\left(1 - \frac{\lambda^\ell_S(p^{MC}) - 1}{\Lambda}\right) + \frac{2\alpha}{K} \\
&\leq p^{MC} + \beta'\left(1 - \frac{\Lambda - 1 - \frac{2\Lambda}{K} - 1}{\Lambda}\right) + \frac{2\alpha}{K} \\
&= p^{MC} + \beta'\left(\frac{2}{\Lambda} + \frac{2}{K}\right) + \frac{2\alpha}{K} \\
&\leq \tilde{p}_k.
\end{aligned}$$

Thus, we obtain $S^\ell(\tilde{p}_k) = n_S$, a contradiction.

$\square$

To prove the desired conclusion $S^\ell(\tilde{p}_k) \geq S^\ell_0$ for this case, let $s \in S$ be the seller whose type is the lowest among those with $v^\ell_s(\hat{t}_k) \geq \tilde{p}_k$; Note that such a seller s exists because $S^\ell(\tilde{p}_k) < n_S$. Consider the following cases.

(a) Suppose $s \notin S_k$. Let $\hat{\lambda}^\ell_S(\tilde{p}_k)$ be an integer such that the interval $\left[\frac{\hat{\lambda}^\ell_S(\tilde{p}_k)-1}{\Lambda}, \frac{\hat{\lambda}^\ell_S(\tilde{p}_k)}{\Lambda}\right)$ contains the type of the seller whose valuation at $\hat{t}_k$ is the highest among those whose value at $\hat{t}_k$ is lower than $\tilde{p}_k$. By event E, we have $t_s \in \left[\frac{\hat{\lambda}^\ell_S(\tilde{p}_k)-1}{\Lambda}, \frac{\hat{\lambda}^\ell_S(\tilde{p}_k)+1}{\Lambda}\right)$.

[34] The first inequality of the display inequalities below is obtained as follows: letting $s \notin S_k$ be an arbitrary seller outside S_k,

$$\begin{aligned}
v^\ell_{s'}(\hat{t}_k) &= v^\ell_{s'}(1, (1)_{\hat{s} \in (S_k \cup B_k) \setminus \{s'\}}, (t_{\hat{s}})_{\hat{s} \notin (S_k \cup B_k)}) \\
&\leq v^\ell_{s'}(1, t_{s'}, t_{-s,s'}) + \left(\frac{a_S - 1}{n_S} + \frac{a_B}{n_B}\right)\alpha \\
&\leq v^\ell_{s}(1, t_{s'}, t_{-s,s'}) + \left(\frac{a_S - 1}{n_S} + \frac{a_B}{n_B}\right)\alpha + \beta'(1 - t_{s'}) \\
&\leq v^\ell_{s}(t_s, t_{s'}, t_{-s,s'}) + \frac{2\alpha}{K} + \beta'(1 - t_{s'}) \\
&= v^\ell_{s'}(t_{s'}, t_s, t_{-s,s'}) + \frac{2\alpha}{K} + \beta'(1 - t_{s'}) \\
&= v^\ell_{s'}(t) + \frac{2\alpha}{K} + \beta'(1 - t_{s'}).
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{1}{n_S}(S^\ell(\tilde{p}_k) - S_0^\ell) &> \frac{1}{n_S}\left(\left(\frac{\hat{\lambda}_S^\ell(\tilde{p}_k) - 1}{\Lambda} - \frac{1}{2K}\right)n_S - a_S - \left(\frac{\lambda_S^\ell(p^{MC})}{\Lambda} + \frac{1}{2K}\right)n_S\right) \\
&\geq \frac{\hat{\lambda}_S^\ell(\tilde{p}_k) - \lambda_S^\ell(p^{MC}) - 1}{\Lambda} - \frac{2}{K} \\
&\geq \frac{v_s^\ell(\hat{t}_k) - v_{s'}^\ell(t) - \frac{2\alpha}{K}}{\beta'} - \frac{3}{\Lambda} - \frac{2}{K} \\
&> \frac{\tilde{p}_k - p^{MC} - \frac{2\alpha}{K}}{\beta'} - \frac{3}{\Lambda} - \frac{2}{K} \\
&= 0.
\end{aligned}$$

(b) Suppose $s \in S_k$. In this case, $S^\ell(\tilde{p}_k) \geq n_S - a_S = n_S\left(1 - \frac{1}{K}\right)$ and $S_0^\ell < \left(\frac{\lambda_S^\ell(p^{MC})}{\Lambda} + \frac{1}{2K}\right)n_S$, and thus, $S^\ell(\tilde{p}_k) > S_0^\ell$ if

$$1 - \frac{1}{K} > \frac{\lambda_S^\ell(p^{MC})}{\Lambda} + \frac{1}{2K},$$

or $1 - \frac{\lambda_S^\ell(p^{MC})}{\Lambda} > \frac{3}{2K}$. This inequality is satisfied because $\lambda_S^\ell(p^{MC}) < \Lambda - 1 - \frac{2\Lambda}{K}$ by Claim 1, and hence $1 - \frac{\lambda_S^\ell(p^{MC})}{\Lambda} > \frac{1}{\Lambda} + \frac{2}{K} > \frac{3}{2K}$.

Hence we have shown that $S^\ell(\tilde{p}_k) \geq S_0^\ell$ for each ℓ. Therefore $S'(\tilde{p}_k) = \sum_{\ell=1}^m S^\ell(\tilde{p}_k) \geq \sum_{\ell=1}^m S_0^\ell = S_0$. By an analogous argument, we obtain $D'(\tilde{p}_k) \leq D_0$. Therefore $S'(\tilde{p}_k) \geq S_0 = D_0 \geq D'(\tilde{p}_k)$. To complete the proof, consider the following cases.

(1) Suppose $D^0 > 0$. Then, because $S'(\bar{p}_k) \geq S'(\tilde{p}_k) \geq D^0$, it follows that $S'(\bar{p}_k) > 0$. If $D'(\tilde{p}_k) = 0$, then because $D'(\tilde{p}_k) \geq D'(\bar{p}_k)$, it follows that $D'(\bar{p}_k) = 0$, and hence $S'(\bar{p}_k) > D'(\bar{p}_k)$, as desired. If $D'(\tilde{p}_k) > 0$, then since $\bar{p}_k = \tilde{p}_k + \frac{\beta'}{\Lambda}$, under event E, $D'(\tilde{p}_k) > D'(\bar{p}_k)$. Therefore we have $S'(\bar{p}_k) \geq S'(\tilde{p}_k) \geq D'(\tilde{p}_k) > D'(\bar{p}_k)$, as desired.

(2) Suppose $D^0 = 0$. Then, by definition of p^{MC}, it follows that $p^{MC} = v_S^{(1)}$. Because $\hat{v}_S^{(1)} \leq v_S^{(1)} + \frac{2\alpha}{K}$, we obtain that $\bar{p}_k > \hat{v}_S^{(1)}$, which implies $S'(\bar{p}_k) > 0$. Because $D^0 \geq D'(\tilde{p}_k) \geq D'(\bar{p}_k)$ from an earlier argument, it follows that $D'(\bar{p}_k) = 0$. Therefore $S'(\bar{p}_k) > 0 = D'(\bar{p}_k)$, as desired.

□

Lemma 4. *Assume E holds. In the canonical groupwise-price mechanism, at least*

$$\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} - \max\{n_B, n_S\}m\left[\frac{\left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} + \frac{2}{\Lambda} + \frac{1}{2K}\right]$$

units of the object are traded.

Proof. For each $b \in B_k$ and $\ell \in \{1, \ldots, m\}$, let $\tilde{v}_b^\ell := v_b^\ell(\underline{t}) = v_b^\ell((0)_{i \in S_k}, t_{-S_k})$. If (b, ℓ) satisfies $\tilde{v}_b^\ell \geq p_k$, then $(b, \ell) \in B_k^*(\underline{t})$. Because

$$\tilde{v}_b^\ell \geq v_b^\ell(t) - a_S \cdot \frac{\alpha}{n_S} = v_b^\ell(t) - \frac{\alpha}{K},$$

if

$$v_b^\ell(t) - \frac{\alpha}{K} \geq \bar{p}_k, \tag{A.2}$$

then $\tilde{v}_b^\ell \geq p_k$, and hence $(b, \ell) \in B_k^*(\underline{t})$.

Let λ^ℓ be the smallest nonnegative integer that satisfies

$$\lambda^\ell \geq \Lambda \left[\frac{\left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} + \frac{\lambda_B^\ell + 1}{\Lambda} - \frac{1}{K} \right]. \tag{A.3}$$

Then, by rearranging terms,

$$\frac{\lambda^\ell}{\Lambda} + \frac{1}{K} - \frac{\lambda_B^\ell + 1}{\Lambda} \geq \frac{\left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta}.$$

Consider an arbitrary buyer $b \in \overline{y}_k^{\lambda^\ell}$. Using the above inequality, and noting that $t_b \geq t_{b^\ell}$ because $t_b \geq \frac{\lambda^\ell}{\Lambda} + \frac{1}{K} \geq \frac{\lambda_B^\ell + 1}{\Lambda} \geq t_{b^\ell}$, we obtain that

$$\begin{aligned}
v_b^\ell(t) - \frac{\alpha}{K} &\geq v_{b^\ell}^\ell(t) + \beta(t_b - t_b^\ell) - \frac{\alpha}{K} \\
&\geq p^{MC} + \beta\left(\frac{\lambda^\ell}{\Lambda} + \frac{1}{K} - \frac{\lambda_B^\ell + 1}{\Lambda}\right) - \frac{\alpha}{K} \\
&\geq p^{MC} + \left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K} - \frac{\alpha}{K} \\
&= p^{MC} + \left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{2\alpha}{K} \\
&= \bar{p}_k,
\end{aligned}$$

thus $(b, \ell) \in B_k^*(\underline{t})$. So we obtain

$$|B_k^*(\underline{t})| \geq a_B \sum_\ell \overline{Y}_k^{\lambda^\ell} \geq \sum_\ell a_B \left[1 - \frac{\lambda^\ell}{\Lambda} - \frac{3}{2K}\right]. \tag{A.4}$$

Because λ^ℓ is defined as the smallest integer satisfying (A.3), we have

$$\lambda^\ell \leq \Lambda \left[\frac{(\frac{2}{K} + \frac{4}{\Lambda})\beta' + \frac{3\alpha}{K}}{\beta} + \frac{\lambda_B^\ell + 1}{\Lambda} - \frac{1}{K} \right] + 1.$$

Substituting this inequality to inequality (A.4), we obtain

$$\begin{aligned}|B_k^*(\underline{t})| &\geq \sum_\ell a_B\left[1-\left[\frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta'+\frac{3\alpha}{K}}{\beta}+\frac{\lambda_B^\ell+1}{\Lambda}-\frac{1}{K}\right]-\frac{1}{\Lambda}-\frac{3}{2K}\right]\\ &= \sum_\ell a_B\left[1-\frac{\lambda_B^\ell}{\Lambda}-\frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta'+\frac{3\alpha}{K}}{\beta}-\frac{2}{\Lambda}-\frac{1}{2K}\right].\end{aligned}$$

For each $s \in S_k$ and $\ell \in \{1,\dots,m\}$, let $\tilde{v}_s^\ell := v_s^\ell(\bar{t}) = v_s^\ell(t_s,(1)_{i\in B_k},t_{-B_k})$. If (s,ℓ) satisfies $\tilde{v}_s^\ell < p_k$, then $(s,\ell)\in S_k^*(\bar{t})$. Because

$$\tilde{v}_s^\ell \leq v_s^\ell(t)+a_B\cdot\frac{\alpha}{n_B} = v_s^\ell(t)+\frac{\alpha}{K},$$

if

$$v_s^\ell(t)+\frac{\alpha}{K} < p^{MC}, \tag{A.5}$$

then $\tilde{v}_s^\ell < p_k$, and hence $(s,\ell)\in S_k^*(\bar{t})$.

Let λ^ℓ be defined as the largest integer such that

$$\lambda^\ell \leq \lambda_S^\ell - 1 - \frac{\alpha\Lambda}{\beta K}+\frac{\Lambda}{K}. \tag{A.6}$$

Then, by rearranging terms,

$$\frac{\lambda^\ell}{\Lambda}-\frac{1}{K}-\frac{\lambda_S^\ell}{\Lambda} \leq -\frac{1}{\Lambda}-\frac{\alpha}{\beta K}.$$

Consider an arbitrary seller $s \in \underline{x}_k^{\lambda^\ell}$. Using the above equality, and noting $t_s < t_{s^\ell}$ (because $t_s < \frac{\lambda^\ell}{\Lambda}-\frac{1}{K} \leq \frac{\lambda_S^\ell-1}{\Lambda}-\frac{\alpha}{\beta K} < \frac{\lambda_S^\ell-1}{\Lambda} \leq t_{s^\ell}$), we obtain

$$\begin{aligned}v_s^\ell(t)+\frac{\alpha}{K} &\leq v_{s^\ell}^\ell(t)-\beta(t_{s^\ell}-t_s)+\frac{\alpha}{K}\\ &\leq v_{s^\ell}^\ell(t)-\beta\left(\frac{\lambda_S^\ell-1}{\Lambda}-\left(\frac{\lambda^\ell}{\Lambda}-\frac{1}{K}\right)\right)+\frac{\alpha}{K}\\ &\leq v_{s^\ell}^\ell(t)-\frac{\alpha}{K}+\frac{\alpha}{K}\\ &< p^{MC},\end{aligned}$$

thus showing that relation (A.5) holds, and hence $(s,\ell)\in S_k^*(\bar{t})$. So we obtain

$$|S_k^*(\bar{t})| \geq a_S\sum_\ell \underline{X}_k^{\lambda^\ell} \geq \sum_\ell a_S\left[\frac{\lambda^\ell}{\Lambda}-\frac{3}{2K}\right]. \tag{A.7}$$

Because λ^ℓ is defined as the largest integer satisfying (A.6), we have

$$\lambda^\ell \geq \lambda_S^\ell - 1 - \frac{\alpha\Lambda}{\beta K}+\frac{\Lambda}{K}-1, \tag{A.8}$$

so

$$\frac{\lambda^\ell}{\Lambda} \geq \frac{\lambda_S^\ell}{\Lambda} - \frac{\alpha}{\beta K} - \frac{2}{\Lambda} + \frac{1}{K}, \tag{A.9}$$

Substituting this inequality into inequality (A.7), we obtain

$$|S_k^*(\bar{t})| \quad \geq \quad \sum_\ell a_S \left[\frac{\lambda_S^\ell}{\Lambda} - \frac{\alpha}{\beta K} - \frac{2}{\Lambda} - \frac{1}{2K}\right].$$

Therefore, the number of trades in submarket k is at least

$$\min\left\{a_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda} - \frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} - \frac{2}{\Lambda} - \frac{1}{2K}\right], a_S \sum_\ell \left[\frac{\lambda_S^\ell}{\Lambda} - \frac{\alpha}{\beta K} - \frac{2}{\Lambda} - \frac{1}{2K}\right]\right\}.$$

Summing across all submarkets (and noting that this lower bound does not depend on k), there are at least

$$\begin{aligned} &\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda} - \frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} - \frac{2}{\Lambda} - \frac{1}{2K}\right], n_S \sum_\ell \left[\frac{\lambda_S^\ell}{\Lambda} - \frac{\alpha}{\beta K} - \frac{2}{\Lambda} - \frac{1}{2K}\right]\right\} \\ \geq \quad &\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} + \max\{n_B, n_S\} m \left[-\frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} - \frac{2}{\Lambda} - \frac{1}{2K}\right], \end{aligned}$$

trades under our double auction mechanism.[35] □

Lemma 5. *Assume E holds. In the efficient allocation, under p^{MC}, at most*

$$\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} + \max\{n_B, n_S\} \frac{3m}{2K}$$

units are traded.

Proof. First, consider the buyers. By definition, if buyer b satisfies

$$t_b < \frac{\lambda_B^\ell}{\Lambda} = \frac{(\lambda_B^\ell + \frac{\Lambda}{K})}{\Lambda} - \frac{1}{K},$$

or equivalently, if $b \in \underline{y}^{\lambda_B^\ell + \frac{\Lambda}{K}}$, then b does not trade her ℓ-th unit in the efficient trade. The number of such buyers b is $n_B \underline{Y}^{\lambda_B^\ell + \frac{\Lambda}{K}}$, which we know is no smaller than $n_B \left(\frac{\lambda_B^\ell}{\Lambda} - \frac{3}{2K}\right)$. Therefore the number of the buyers who buy their ℓ-th units of the object is bounded from above by $n_B \left(1 - \frac{\lambda_B^\ell}{\Lambda} + \frac{3}{2K}\right)$.

Next, consider the sellers. By definition, if seller s satisfies

$$t_s > \frac{\lambda_S^\ell}{\Lambda} = \frac{(\lambda_S^\ell - \frac{\Lambda}{K})}{\Lambda} + \frac{1}{K},$$

[35] Note that $\frac{\left(\frac{2}{K}+\frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} \geq \frac{3\alpha}{\beta K} \geq \frac{\alpha}{\beta K}$.

or equivalently, if $s \in \overline{x}^{\lambda_S^\ell - \frac{\Lambda}{K}}$, then s does not trade her ℓ-th unit in the efficient trade. The number of such sellers s is $n_S \overline{X}^{\lambda_S^\ell - \frac{\Lambda}{K}}$, which we know is no smaller than $n_S\left(1 - \frac{\lambda_S^\ell}{\Lambda} - \frac{3}{2K}\right)$. Therefore the number of the sellers who buy their ℓ-th units of the object is bounded from above by $n_S\left(\frac{\lambda_S^\ell}{\Lambda} + \frac{3}{2K}\right)$.

Therefore the number of trades is at most

$$\begin{aligned} & \min\left\{n_B \sum_\ell \left(1 - \frac{\lambda_B^\ell}{\Lambda} + \frac{3}{2K}\right), n_S \sum_\ell \left(\frac{\lambda_S^\ell}{\Lambda} + \frac{3}{2K}\right)\right\} \\ \leq \quad & \min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} + \max\{n_B, n_S\}\frac{3m}{2K}, \end{aligned}$$

which completes the proof.

□

Now we shall complete the proof of the Theorem. By Lemmata 4 and 5, the "per-capita" welfare loss for buyers is bounded from above by

$$\begin{aligned} & \frac{1}{n_B}\Bigg[\Pr(E)\Bigg\{\left[\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} + \max\{n_B, n_S\}\frac{3m}{2K}\right] \\ & - \left[\min\left\{n_B \sum_\ell \left[1 - \frac{\lambda_B^\ell}{\Lambda}\right], n_S \sum_\ell \frac{\lambda_S^\ell}{\Lambda}\right\} + \max\{n_B, n_S\} m \left[-\frac{\left(\frac{2}{K} + \frac{4}{\Lambda}\right)\beta' + \frac{3\alpha}{K}}{\beta} - \frac{2}{\Lambda} - \frac{1}{2K}\right]\right]\Bigg\} \\ & + (1 - \Pr(E)) m n_B\Bigg] \\ \leq \; & \frac{1}{n_B}\left(m \max\{n_B, n_S\}\left[\frac{3}{2K} + \frac{(\frac{2}{K} + \frac{4}{\Lambda})\beta' + \frac{3\alpha}{K}}{\beta} + \frac{2}{\Lambda} + \frac{1}{2K}\right] + \left[\frac{2\Lambda K^4}{n_S} + \frac{2\Lambda K^4}{n_B}\right] \times m n_B\right) \\ \leq \; & \frac{\overline{\gamma} m}{\underline{\gamma}}\left[\frac{3}{2K} + \frac{(\frac{2}{K} + \frac{4}{\Lambda})\beta' + \frac{3\alpha}{K}}{\beta} + \frac{2}{\Lambda} + \frac{1}{2K}\right] + \left[\frac{2m\Lambda K^4}{n_S} + \frac{2m\Lambda K^4}{n_B}\right], \end{aligned} \tag{A.10}$$

where the first inequality follows because $\Pr(E) \leq 1$ (since $\Pr(E)$ is a probability) and $(1 - \Pr(E))$ is bounded from above by replacing $\Pr(E)$ with its lower bound, $1 - \frac{2\Lambda K^4}{n_S} - \frac{2\Lambda K^4}{n_B}$, while the second inequality comes from simplifying terms. Because $\Lambda = K$, $K \to \infty$, $\frac{K^5}{N} \to 0$ as $N \to \infty$, and $\underline{\gamma} N < n_B, n_S < \overline{\gamma} N$ for all N by assumption, the right-most expression of inequality (A.10) approaches zero as $N \to \infty$. This implies that the per-capita inefficiency for buyers from failed trades also approaches zero as $N \to \infty$. A symmetric argument shows that the per-capita inefficiency for sellers from failed trades approaches zero as $N \to \infty$, completing the proof under the assumption that each agent's type is drawn i.i.d. according to the uniform distribution.

A.2. **Proof for the general independent type case.** In this subsection, suppose that all types are independently distributed, and each buyer's type follows a distribution with cdf F with an everywhere positive and continuous pdf f while each seller's type follows a distribution with cdf G with an everywhere positive and continuous pdf g. We shall show that the conclusion of the theorem holds under these assumptions.

To show the result for this case, first note that F and G admit inverse functions F^{-1} and G^{-1}, and $\tau_b = F(t_b)$ and $\tau_s = G(t_s)$ follow the uniform distribution over $[0,1]$. For each $\tau = (\tau_j)_{j \in B \cup S}$, define $\tilde{v}_i(\tau) = v_i((F^{-1}(\tau_b))_{b \in B}, (G^{-1}(\tau_s))_{s \in S})$ as the new valuation function of agent i. Given that $v_i(\cdot)$ satisfies conditions (4.1) and (4.2), the valuation function $\tilde{v}_i(\cdot)$ satisfies the same conditions too, as shown in the next paragraph. Therefore, the conclusion of the theorem holds by the analysis in Subsection A.1.

It remains to show that $\tilde{v}_i(\cdot)$ satisfies conditions (4.1) and (4.2). To see this, let $\overline{f}, \underline{f} \in (0,\infty)$ be such that $f(\tau_b) \in [\underline{f}, \overline{f}]$ for all $\tau_b \in [0,1]$ and $\overline{g}, \underline{g} \in (0,\infty)$ be such that $g(\tau_s) \in [\underline{g}, \overline{g}]$ for all $\tau_s \in [0,1]$ (note that f and g are strictly positive and continuous on $[0,1]$, so such $\underline{f}, \overline{f}, \underline{g}$, and $\overline{g}$ exist). Then, for each i and $j \neq i$,

$$|\tilde{v}_i^\ell(\tau_{-j}, \tau_j) - \tilde{v}_i^\ell(\tau_{-j}, \tau_j')| \leq \frac{\underline{f}^{-1}\alpha}{n_B}, \text{if } j \in B,$$
$$|\tilde{v}_i^\ell(\tau_{-j}, \tau_j) - \tilde{v}_i^\ell(\tau_{-j}, \tau_j')| \leq \frac{\underline{g}^{-1}\alpha}{n_S}, \text{if } j \in S,$$

thus condition (4.1) is satisfied, and

$$|\tilde{v}_i^\ell(\tau) - \tilde{v}_j^\ell(\tau)| \in [\beta \overline{f}^{-1}|\tau_i - \tau_j|, \beta' \underline{f}^{-1}|\tau_i - \tau_j|] \text{ if } i, j \in B,$$
$$|\tilde{v}_i^\ell(\tau) - \tilde{v}_j^\ell(\tau)| \in [\beta \overline{g}^{-1}|\tau_i - \tau_j|, \beta' \underline{g}^{-1}|\tau_i - \tau_j|] \text{ if } i, j \in S,$$

thus condition (4.2) is satisfied.

A.3. **Proof for the general conditional independence case.** Finally, we will complete the proof for the general conditionally independent type as assumed in the main text of the paper. To do so, recall that there is a state variable σ that is drawn randomly from a certain finite distribution. For each realization of σ, there is a pair of type distributions, one for the buyers and another for the sellers, and types are independently distributed conditional on σ. It is clear that the expected efficiency loss in this model is simply a weighted average of expected inefficiencies conditional on σ. Because the proof in Subsection A.2 shows that expected inefficiency goes to zero for any fixed σ, we conclude that the expected inefficiency that is averaged over the state variables also converges to zero. This completes the proof of the theorem.

Remark 3. While the rate of convergence to efficiency is not our main focus, our proof sheds light on this issue. For that purpose, first recall that $\Lambda = K$. This and the last inequality imply that the expected per-capita inefficiency in the canonical groupwise-price mechanism is $O\left(\frac{1}{K} + \frac{K^5}{N}\right)$. By taking $K = N^{\frac{1}{6}}$, this is $O\left(N^{-\frac{1}{6}}\right)$. In particular, a canonical groupwise-price mechanism can diminish the per-capita inefficiency at a polynomial rate in N. Whether a mechanism with a better convergence rate exists is an open question.

Appendix B. Proof of Theorem 3

In this section, we prove Theorem 3 under the assumption that each agent's type is drawn i.i.d. according to the uniform distribution. Extension to the general case of conditionally independent types is omitted because the argument is analogous to the one for Theorem 2, which is presented in Subsections A.2 and A.3.

In the following, we fix one realization of type profile t under event E. Let $\left[\frac{\lambda_S^\ell(p)-1}{\Lambda}, \frac{\lambda_S^\ell(p)}{\Lambda}\right)$ be the interval that contains the highest type of the sellers in S whose ℓ-th unit value given t is smaller than or equal to p. Similarly, let $\left[\frac{\lambda_B^\ell(p)}{\Lambda}, \frac{\lambda_B^\ell(p)+1}{\Lambda}\right)$ be the interval that contains the lowest type of the buyers in B whose ℓ-th unit value given t is higher than or equal to p.

Also, we use the following notation.

$$A_S^\lambda = \frac{1}{n_S}\left|\left\{s \in S | t_s \leq \frac{\lambda}{\Lambda}\right\}\right|,$$
$$A_B^\lambda = \frac{1}{n_B}\left|\left\{b \in B | t_b \leq \frac{\lambda}{\Lambda}\right\}\right|.$$

Under event E, $A_S^\lambda, A_B^\lambda \in \left(\frac{\lambda}{\Lambda} - \frac{1}{2K}, \frac{\lambda}{\Lambda} + \frac{1}{2K}\right)$.

At a fixed submarket k, we introduce analogous notation. Let $\left[\frac{\lambda_{S_k}^\ell(p)-1}{\Lambda}, \frac{\lambda_{S_k}^\ell(p)}{\Lambda}\right)$ be the interval that contains the highest type of the sellers in S_k whose ℓ-th unit value given $\bar{t}$ is smaller than or equal to p. Similarly, let $\left[\frac{\lambda_{B_k}^\ell(p)}{\Lambda}, \frac{\lambda_B^\ell(p)+1}{\Lambda}\right)$ be the interval that contains the lowest type of the buyers in B_k whose ℓ-th unit value given $\underline{t}$ is higher than or equal to p. Also, let

$$A_{S_k}^\lambda = \frac{1}{a_S}\left|\left\{s \in S_k | t_s \leq \frac{\lambda}{\Lambda}\right\}\right|,$$
$$A_{B_k}^\lambda = \frac{1}{a_B}\left|\left\{b \in B_k | t_b \leq \frac{\lambda}{\Lambda}\right\}\right|.$$

Under event E, $A_{S_k}^\lambda, A_{B_k}^\lambda \in \left(\frac{\lambda}{\Lambda} - \frac{1}{2K}, \frac{\lambda}{\Lambda} + \frac{1}{2K}\right)$. Let $S_k(p)$ and $D_k(p)$ denote the supply and demand functions in submarket k, i.e., $S_k(p) = |\{(s,\ell)|v_s^\ell(\bar{t}) < p\}|$, and $D_k(p) = |\{(b,\ell)|v_b^\ell(\underline{t}) \geq p\}|$.

Lemma 6. *Let*

$$\begin{aligned}\overline{q}_k &= p^{MC} + \beta' \left(\frac{3}{\Lambda} + \frac{1}{K}\right) + \frac{\alpha}{K},\\ \underline{q}_k &= p^{MC} - \beta' \left(\frac{3}{\Lambda} + \frac{1}{K}\right) - \frac{\alpha}{K}.\end{aligned}$$

Then, $S_k(\overline{q}_k) > D_k(\overline{q}_k)$ *and* $D_k(\underline{q}_k) > S_k(\underline{q}_k)$.

Proof. We first show $S_k(\overline{q}_k) > D_k(\overline{q}_k)$. To do so, for any ℓ, let ${S_k}^{\ell}(\overline{q}_k)$ be the number of the sellers who supply their ℓ'th unit at $\overline{q}_k$, corresponding to $S_k(\overline{q}_k)$. Similarly, let S_0^{ℓ} be the number of the sellers who supply their ℓ'th unit at p^{MC}, corresponding to S_0. Then, for any given ℓ,

(1) Suppose that $v_s^{\ell}(\overline{t}) \leq \overline{q}_k$ for all s. Then every seller in S_k supplies her ℓ'th unit at price $\bar{q}_k$. So
$$\frac{1}{a_S} S_k^{\ell}(\overline{q}_k) - \frac{1}{n_S} S_0^{\ell} = 1 - \frac{1}{n_S} S_0^{\ell} \geq 0.$$
(2) Suppose that $v_s^{\ell}(t) \geq p^{MC}$ for all s. Then no seller in S supplies her ℓ'th unit at price p^{MC}. So
$$\frac{1}{a_S} S_k^{\ell}(\overline{q}_k) - \frac{1}{n_S} S_0^{\ell} = \frac{1}{a_S} S_k^{\ell}(\overline{q}_k) - 0 \geq 0.$$
(3) Suppose that neither of the above cases holds. Let s be the seller whose type is the lowest among those in submarket k with $v_s^{\ell}(\overline{t}) > \overline{q}_k$, and $s' \in S$ be the seller whose type is the highest among those in the entire market with $v_{s'}^{\ell}(t) < p^{MC}$ (such s and s' exist by the assumption of this case). Then,

$$t_{s'} \geq \frac{\lambda_S^{\ell}(p^{MC}) - 1}{\Lambda}. \tag{B.1}$$

Depending on the realization t, t_s is either in $\left[\frac{\lambda_{S_k}^{\ell}(\overline{q}_k)-1}{\Lambda}, \frac{\lambda_{S_k}^{\ell}(\overline{q}_k)}{\Lambda}\right)$ or $\left[\frac{\lambda_{S_k}^{\ell}(\overline{q}_k)}{\Lambda}, \frac{\lambda_{S_k}^{\ell}(\overline{q}_k)+1}{\Lambda}\right)$. Hence, $t_s \leq \frac{\lambda_{S_k}^{\ell}(\overline{q}_k)+1}{\Lambda}$. Thus,

$$\begin{aligned}\overline{q}_k - p^{MC} &< v_s^{\ell}(\overline{t}) - v_{s'}^{\ell}(t)\\ &\leq v_s^{\ell}(t) - v_{s'}^{\ell}(t) + \frac{\alpha}{K}\\ &\leq \beta' \frac{\lambda_{S_k}^{\ell}(\overline{q}_k) - \lambda_S^{\ell}(p^{MC}) + 2}{\Lambda} + \frac{\alpha}{K},\end{aligned}$$

which implies

$$\beta' \left(\frac{3}{\Lambda} + \frac{1}{K}\right) + \frac{\alpha}{K} \leq \beta' \frac{\lambda_{S_k}^{\ell}(\overline{q}_k) - \lambda_S^{\ell}(p^{MC}) + 2}{\Lambda} + \frac{\alpha}{K},$$

or equivalently

$$\frac{\lambda^{\ell}_{S_k}(\overline{q}_k)-\lambda^{\ell}_{S}(p^{MC})-1}{\Lambda}\geq\frac{1}{K}.$$

This implies

$$\frac{1}{a_S}S^{\ell}_k(\overline{q}_k)-\frac{1}{n_S}S^{\ell}_0\geq\frac{1}{K}>0.$$

By the conclusions of the above cases,

$$\frac{1}{a_S}S_k(\overline{q}_k)-\frac{1}{n_S}S_0=\sum_{\ell}\frac{1}{a_S}S^{\ell}_k(\overline{q}_k)-\frac{1}{n_S}S^{\ell}_0>0,$$

thus we obtain $\frac{1}{a_S}S_k(\overline{q}_k)>\frac{1}{n_S}S_0$. With an analogous argument, we can show $\frac{1}{n_S}D_0>\frac{1}{a_S}D_k(\overline{q}_k)$ as well. These inequalities, together with the relation $S_0=D_0$, imply the desired conclusion, $S_k(\overline{q}_k)>D_k(\overline{q}_k)$.

The proof for $D_k(\underline{q}_k)>S_k(\underline{q}_k)$ is analogous and hence omitted. □

The following lemma is useful for the rest of the proof.

Lemma 7. *Let $p'\geq p$ be prices such that there exist two buyers $b,b'\in B_k$ where $v^1_b(\underline{t})>p'$ and $v^m_{b'}(\underline{t})<p$, and moreover, there exist two sellers $s,s'\in S_k$ where $v^1_s(\overline{t})<p$ and $v^m_{s'}(\overline{t})>p'$. Then,*

$$\begin{aligned}S_k(p')-S_k(p) &\in \left(a_S\left(\frac{p'-p}{\beta'}-\frac{3+2m}{\Lambda}-\frac{1}{K}-\frac{3\alpha m}{\beta' n_S}\right),a_S m\left(\frac{p'-p}{\beta}+\frac{3}{\Lambda}+\frac{1}{K}\right)\right),\\ D_k(p)-D_k(p') &\in \left(a_B\left(\frac{p'-p}{\beta'}-\frac{3+2m}{\Lambda}-\frac{1}{K}-\frac{3\alpha m}{\beta' n_B}\right),a_B m\left(\frac{p'-p}{\beta}+\frac{3}{\Lambda}+\frac{1}{K}\right)\right).\end{aligned}$$

Proof. Let $\underline{s}\in S_k$ be the seller who has the lowest type in S_k, and $\overline{s}\in S_k$ be the seller who has the highest type in S_k.

Proof for a lower bound for $S_k(\cdot)$. We consider the following two cases. First, suppose that there exists $\ell\in\{1,\dots,m\}$ such that $v^{\ell}_{\underline{s}}(\overline{t})<p\leq p'\leq v^{\ell}_{\overline{s}}(\overline{t})$. Let $s\in S_k$ be the seller in submarket k whose type is the highest among those with $v^{\ell}_s(\overline{t})<p$. Then,

$$t_s\geq\frac{\lambda^{\ell}_{S_k}(p)-1}{\Lambda}. \tag{B.2}$$

Similarly, let $s'\in S_k$ be the seller in submarket k whose type is the lowest among those with $v^{\ell}_{s'}(\overline{t})\geq p'$. s' has the next lowest type above the type of the seller, say s'', who is the highest type with value less than p', and $t_{s''}$ is at most $\frac{\lambda^{\ell}_{S_k}(p')}{\Lambda}$ by definition of $\lambda^{\ell}_{S_k}(p')$. Therefore, by event E, there should be at least one seller within each interval of length $\frac{1}{\Lambda}$, which implies

$$t_{s'}\leq\frac{\lambda^{\ell}_{S_k}(p')+1}{\Lambda}. \tag{B.3}$$

Therefore,

$$
\begin{aligned}
S_k(p') - S_k(p) &> a_S \left(\frac{\lambda^{\ell}_{S_k}(p') - 1}{\Lambda} - \frac{1}{2K} - \frac{\lambda^{\ell}_{S_k}(p)}{\Lambda} - \frac{1}{2K} \right) \\
&= a_S \left(\frac{\lambda^{\ell}_{S_k}(p') + 1}{\Lambda} - \frac{1}{2K} - \frac{\lambda^{\ell}_{S_k}(p) - 1}{\Lambda} - \frac{3}{\Lambda} - \frac{1}{2K} \right) \\
&\geq a_S \left(t_{s'} - t_s - \frac{3}{\Lambda} - \frac{1}{K} \right) \\
&\geq a_S \left(\frac{p' - p}{\beta'} - \frac{3}{\Lambda} - \frac{1}{K} \right),
\end{aligned}
$$

where the first inequality follows from the definition of $\lambda^{\ell}_{S_k}(p')$ and event E, the equality follows from calculation, the second inequality follows from inequalities (B.2) and (B.3), and the last inequality follows because $t_{s'} - t_s \geq \frac{v^{\ell}_{s'}(\bar{t}) - v^{\ell}_{s}(\bar{t})}{\beta'} \geq \frac{p'-p}{\beta'}$ by assumption on p, p', s, and s' as well as condition (4.2) in the main text.

Next, suppose that there exists no $\ell \in \{1, \dots, m\}$ such that $v^{\ell}_{\underline{s}}(\bar{t}) < p \leq p' \leq v^{\ell}_{\bar{s}}(\bar{t})$. Let ℓ' and s' be defined by:

$$
\begin{aligned}
\ell' &= \min\{\tilde{\ell} \in \{1, \dots, m\} | v^{\tilde{\ell}}_{\bar{s}}(\bar{t}) \geq p'\}, \\
s' &= \arg\min_{\tilde{s} \in S_k} \{t_{\tilde{s}} | v^{\ell'}_{\tilde{s}}(\bar{t}) \geq p'\}.
\end{aligned}
$$

That is, ℓ' and s' satisfy $v^{\ell'}_{s'}(\bar{t}) \geq p'$ and the pair (ℓ', s') is the smallest of such pairs with respect to the lexicographic order that relies first on the index and then on the agent's type. Similarly, let ℓ and s be

$$
\begin{aligned}
\ell &= \max\{\tilde{\ell} \in \{1, \dots, m\} | v^{\tilde{\ell}}_{\underline{s}}(\bar{t}) < p\}, \\
s &= \arg\max_{\tilde{s} \in S_k} \{t_{\tilde{s}} | v^{\ell}_{\tilde{s}}(\bar{t}) < p\}.
\end{aligned}
$$

That is, (ℓ, s) is the largest index-seller pair satisfying $v^{\ell}_{s}(\bar{t}) < p$ with respect to the lexicographic oder described above. The relation $\ell \geq \ell'$ contradicts the assumption that there exists no $\ell \in \{1, \dots, m\}$ such that $v^{\ell}_{\underline{s}}(\bar{t}) < p \leq p' \leq v^{\ell}_{\bar{s}}(\bar{t})$, so $\ell' > \ell$. Hence,

$$
\begin{aligned}
p' - p &< v^{\ell'}_{s'}(\bar{t}) - v^{\ell}_{s}(\bar{t}) \\
&\leq [v^{\ell'}_{s'}(\bar{t}) - v^{\ell'}_{\underline{s}}(0, \bar{t}_{-\underline{s}})] + [v^{\ell'-1}_{\bar{s}}(1, \bar{t}_{-\bar{s}}) - v^{\ell'-1}_{\underline{s}}(0, \bar{t}_{-\underline{s}})] + \dots + [v^{\ell+1}_{\bar{s}}(1, \bar{t}_{-\bar{s}}) - v^{\ell+1}_{\underline{s}}(0, \bar{t}_{-\underline{s}})] \\
&\quad + [v^{\ell}_{\bar{s}}(1, \bar{t}_{-\bar{s}}) - v^{\ell}_{s}(\bar{t})] + (\ell' - \ell)\frac{\alpha}{n_S} \\
&\leq [v^{\ell'}_{s'}(\bar{t}) - v^{\ell'}_{\underline{s}}(\bar{t})] + [v^{\ell}_{\bar{s}}(\bar{t}) - v^{\ell}_{s}(\bar{t})] + \frac{2\beta'}{\Lambda} + \frac{2\alpha}{n_S} + (\ell' - \ell - 1)[\beta' + 2(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_S})] + (\ell' - \ell)\frac{\alpha}{n_S} \\
&\leq \frac{\lambda^{\ell'}_{S_k}(p') + 1}{\Lambda}\beta' + (1 - \frac{\lambda^{\ell}_{S_k}(p) - 1}{\Lambda})\beta' + \frac{2\beta'}{\Lambda} + \frac{2\alpha}{n_S} + (\ell' - \ell - 1)[\beta' + 2(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_S})] + (\ell' - \ell)\frac{\alpha}{n_S},
\end{aligned}
$$

where the first inequality follows from the definitions of p, p', s', ℓ', s, and ℓ, the second inequality follows because the "no gap" condition implies

$$v_{\overline{s}}^{\tilde{\ell}}(1, \overline{t}_{-\overline{s}}) - v_{\underline{s}}^{\tilde{\ell}+1}(0, \overline{t}_{-\underline{s}}) \geq v_{\overline{s}}^{\tilde{\ell}}(1, \overline{t}_{-\overline{s}}) - v_{\underline{s}}^{\tilde{\ell}+1}(0, \overline{t}_{-\overline{s}}) - \frac{\alpha}{n_S} \geq -\frac{\alpha}{n_S},$$

for each $\tilde{\ell} \in \{\ell, \ell+1, \ldots, \ell'-1\}$, the third inequality follows because, under event E, $t_{\underline{s}} \leq \frac{1}{\Lambda}$ and $t_{\overline{s}} \geq \frac{\Lambda-1}{\Lambda}$, and hence,

$$v_{\underline{s}}^{\ell'}(\overline{t}) - v_{\underline{s}}^{\ell'}(0, \overline{t}_{-\underline{s}}) \leq \frac{\beta'}{\Lambda} + \frac{\alpha}{n_S},$$
$$v_{\overline{s}}^{\ell}(1, \overline{t}_{-\overline{s}}) - v_{\overline{s}}^{\ell}(\overline{t}) \leq \frac{\beta'}{\Lambda} + \frac{\alpha}{n_S},$$

by assumption (4.2) in the main text,[36] and for each $\tilde{\ell} \in \{\ell+1, \ldots, \ell'-1\}$,

$$v_{\overline{s}}^{\tilde{\ell}}(1, \overline{t}_{-\overline{s}}) - v_{\underline{s}}^{\tilde{\ell}}(0, \overline{t}_{-\underline{s}}) \leq \beta' + 2\left[\frac{\beta'}{\Lambda} + \frac{\alpha}{n_S}\right],$$

and the last inequality follows by definitions of $p, p', \ell, s, \ell', s'$, and by assumption (4.2) in the main text. Rearranging terms, we obtain

$$\text{(B.4)} \qquad \begin{aligned} & \frac{\lambda_{S_k}^{\ell'}(p')+1}{\Lambda} + 1 - \frac{\lambda_{S_k}^{\ell}(p)-1}{\Lambda} + (\ell' - \ell - 1) \\ > \; & \frac{p' - p - \frac{2\beta'}{\Lambda} - \frac{2\alpha}{n_S} - 2(\ell'-\ell-1)\left(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_S}\right) - (\ell'-\ell)\frac{\alpha}{n_S}}{\beta'}. \end{aligned}$$

[36] We obtain those inequalities as follows.

$$\begin{aligned} v_{\underline{s}}^{\ell'}(\overline{t}) &= v_{\underline{s}}^{\ell'}(\overline{t}_{\underline{s}}, \overline{t}_s, \overline{t}_{-\underline{s},s}) \\ &\leq v_{\underline{s}}^{\ell'}(\overline{t}_{\underline{s}}, 0, \overline{t}_{-\underline{s},s}) + \frac{\alpha}{n_S} \\ &\leq v_{s}^{\ell'}(\overline{t}_{\underline{s}}, 0, \overline{t}_{-\underline{s},s}) + \frac{\alpha}{n_S} + \frac{\beta'}{\Lambda} \\ &\leq v_{s}^{\ell'}(\overline{t}_s, 0, \overline{t}_{-\underline{s},s}) + \frac{\alpha}{n_S} + \frac{\beta'}{\Lambda} \\ &= v_{\underline{s}}^{\ell'}(0, \overline{t}_s, \overline{t}_{-\underline{s},s}) + \frac{\alpha}{n_S} + \frac{\beta'}{\Lambda} \\ &= v_{\underline{s}}^{\ell'}(0, \overline{t}_{-\underline{s}}) + \frac{\alpha}{n_S} + \frac{\beta'}{\Lambda}, \end{aligned}$$

where, for example, $v_s^{\ell'}(\overline{t}_{\underline{s}}, 0, \overline{t}_{-\underline{s},s})$ represents the value of s for ℓ'-th unit when her own type is 0, the type of $\underline{s}$ is $\overline{t}_{\underline{s}}$, and the types of the others are $\overline{t}_{-\underline{s},s}$, and similarly for the other expressions. Similarly, we obtain

$$v_{\overline{s}}^{\ell}(1, \overline{t}_{-\overline{s}}) - v_{\overline{s}}^{\ell}(\overline{t}) \leq \frac{\beta'}{\Lambda} + \frac{\alpha}{n_S}.$$

Therefore,

$$\begin{aligned}
S_k(p') - S_k(p) &> a_S\left(\frac{\lambda^{\ell'}_{S_k}(p')-1}{\Lambda} - \frac{1}{2K} - 0\right) + a_S\left(1 - \frac{\lambda^{\ell}_{S_k}(p)}{\Lambda} - \frac{1}{2K}\right) + a_S(\ell' - \ell - 1) \\
&\geq a_S\left(\frac{\lambda^{\ell'}_{S_k}(p')+1}{\Lambda} + 1 - \frac{\lambda^{\ell}_{S_k}(p)-1}{\Lambda} - \frac{3}{\Lambda} - \frac{1}{K} + (\ell' - \ell - 1)\right) \\
&\geq a_S\left(\frac{p' - p - \frac{2\beta'}{\Lambda} - \frac{2\alpha}{n_S} - 2(\ell' - \ell - 1)\left(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_S}\right) - (\ell' - \ell)\frac{\alpha}{n_S}}{\beta'} - \frac{3}{\Lambda} - \frac{1}{K}\right) \\
&= a_S\left(\frac{p' - p}{\beta'} - \frac{5}{\Lambda} - \frac{1}{K} - (\ell' - \ell)\frac{\alpha}{\beta' n_S} - 2(\ell' - \ell)\frac{\alpha}{\beta' n_S} - 2\frac{\ell' - \ell - 1}{\Lambda}\right), \\
&\geq a_S\left(\frac{p' - p}{\beta'} - \frac{3 + 2m}{\Lambda} - \frac{1}{K} - \frac{3\alpha m}{\beta' n_S}\right),
\end{aligned}$$

where the term $a_S(\ell' - \ell - 1)$ in the first line corresponds to the supply of the objects from all agents in submarket K for the $(\ell+1)th$, ..., $(\ell'-1)th$ units, the first inequality follows from the definition of $\lambda^{\ell}_{S_k}(p')$ and event E, the second inequality follows from calculation, the third inequality follows from inequality (B.4), the equality follows from calculation, and the fourth inequality follows from the fact $\ell' - \ell \leq m$.

Proof for an upper bound for $S_k(\cdot)$. To obtain an upper bound on the difference in supplies, let $L := \{\ell \in \{1, \ldots, m\} | v^{\ell}_{\underline{s}}(\overline{t}) < p' \text{ and } v^{\ell}_{\bar{s}}(\overline{t}) \geq p\}$. We shall first show

$$\frac{\lambda^{\ell}_{S_k}(p') - \lambda^{\ell}_{S_k}(p) - 2}{\Lambda}\beta \leq p' - p, \tag{B.5}$$

for each $\ell \in L$. To show this, let $s' \in S_k$ be the seller in submarket k whose type is the highest among those with $v^{\ell}_{s'}(\overline{t}) \leq p'$, and $s \in S_k$ be the seller in submarket k whose type is the lowest among those with $v^{\ell}_{s}(\overline{t}) > p$; Note that such s' and s exist since $\ell \in L$. Then, $t_{s'} \geq \frac{\lambda^{\ell}_{S_k}(p')-1}{\Lambda}$. By event E, t_s is either in $\left[\frac{\lambda^{\ell}_{S_k}(p)-1}{\Lambda}, \frac{\lambda^{\ell}_{S_k}(p)}{\Lambda}\right)$ or $\left[\frac{\lambda^{\ell}_{S_k}(p)}{\Lambda}, \frac{\lambda^{\ell}_{S_k}(p)+1}{\Lambda}\right)$. Hence, $t_s \leq \frac{\lambda^{\ell}_{S_k}(p)+1}{\Lambda}$. Thus,

$$p' - p > v^{\ell}_{s'}(\overline{t}) - v^{\ell}_{s}(\overline{t}) \geq \beta\left(\frac{\lambda^{\ell}_{S_k}(p') - \lambda^{\ell}_{S_k}(p) - 2}{\Lambda}\right),$$

as desired.

Consider $\ell \notin L$. Suppose first that $v^{\ell}_{\underline{s}}(\overline{t}) \geq p'$. Then by assumption $p' \geq p$ it follows that $v^{\ell}_{\underline{s}}(\overline{t}) \geq p$, and hence no seller in S_k supplies the ℓ-th unit of the object under either p or p'. Suppose next that $v^{\ell}_{\bar{s}}(\overline{t}) < p$. Then by assumption $p' \geq p$ it follows that $v^{\ell}_{\bar{s}}(\overline{t}) < p'$, and hence every seller in S_k supplies the ℓ-th unit of the object under both p and p'.

We now show $S_k(p') - S_k(p) < a_S m\left(\frac{p'-p}{\beta} + \frac{3}{\Lambda} + \frac{1}{K}\right)$. To show this, applying inequality (B.5) and the above argument we obtain

$$\begin{aligned} S_k(p') - S_k(p) &< a_S \sum_{\ell \in L} \left(\frac{\lambda^{\ell}_{S_k}(p')}{\Lambda} + \frac{1}{2K} - \left(\frac{\lambda^{\ell}_{S_k}(p) - 1}{\Lambda} - \frac{1}{2K} \right) \right) \\ &= a_S \sum_{\ell \in L} \left(\frac{\lambda^{\ell}_{S_k}(p') - \lambda^{\ell}_{S_k}(p) - 2}{\Lambda} + \frac{3}{\Lambda} + \frac{1}{K} \right) \\ &\le a_S m \left(\frac{p'-p}{\beta} + \frac{3}{\Lambda} + \frac{1}{K} \right). \end{aligned}$$

Proof for a lower bound for $D_k(\cdot)$. Let $\underline{b} \in B_k$ be the seller who has the lowest type in B_k, and $\overline{b} \in B_k$ be the seller who has the highest type in B_k. We consider the following two cases.

First, suppose that there exists $\ell \in \{1, \ldots, m\}$ such that $v^{\ell}_{\underline{b}}(\underline{t}) < p \le p' \le v^{\ell}_{\overline{b}}(\underline{t})$. Let $b' \in B_k$ be the seller in submarket k whose type is the lowest among those with $v^{\ell}_{b'}(\underline{t}) \ge p'$. Then we obtain the following inequality:[37]

$$t_{b'} < \frac{\lambda^{\ell}_{B_k}(p') + 1}{\Lambda}. \tag{B.6}$$

Similarly, let $b \in B_k$ be the seller in submarket k whose type is the highest among those with $v^{\ell}_b(\underline{t}) < p$. Then, the following inequality follows:[38]

$$t_b \ge \frac{\lambda^{\ell}_{B_k}(p) - 1}{\Lambda}. \tag{B.7}$$

[37]This inequality follows because $\left[\frac{\lambda^{\ell}_{B_k}(p')}{\Lambda}, \frac{\lambda^{\ell}_{B}(p')+1}{\Lambda}\right)$ is defined as the interval that contains the lowest type of the buyers in B_k whose ℓ-th unit value given $\underline{t}$ is higher than or equal to p'.

[38]To obtain this inequality, recall that t_b is the type just below the lowest type of the buyers in B_k whose ℓ-th unit value given $\underline{t}$ is higher than or equal to p, and the latter type is in the interval $\left[\frac{\lambda^{\ell}_{B_k}(p)}{\Lambda}, \frac{\lambda^{\ell}_{B}(p)+1}{\Lambda}\right)$ by definition. By event E, t_b can be smaller than $\frac{\lambda^{\ell}_{B_k}(p)}{\Lambda}$ at most by $\frac{1}{\Lambda}$, so we obtain inequality (B.7).

Therefore,

$$
\begin{aligned}
D_k(p) - D_k(p') &> a_B \left[\left(1 - \frac{\lambda^{\ell}_{B_k}(p)+1}{\Lambda} - \frac{1}{2K}\right) - \left(1 - \frac{\lambda^{\ell}_{B_k}(p')}{\Lambda} + \frac{1}{2K}\right)\right] \\
&= a_B \left(\frac{\lambda^{\ell}_{B_k}(p')+1}{\Lambda} - \frac{1}{2K} - \frac{\lambda^{\ell}_{B_k}(p)-1}{\Lambda} - \frac{1}{2K} - \frac{3}{\Lambda}\right) \\
&\geq a_B \left(t_{b'} - t_b - \frac{3}{\Lambda} - \frac{1}{K}\right) \\
&\geq a_B \left(\frac{p'-p}{\beta'} - \frac{3}{\Lambda} - \frac{1}{K}\right),
\end{aligned}
$$

where the first inequality follows from the definition of $\lambda^{\ell}_{B_k}(p')$, the equality follows from calculation, the second inequality follows from inequalities (B.6) and (B.7), and the last inequality follows because $t_{b'} - t_b \geq \frac{v^{\ell}_{b'}(\bar{t})-v^{\ell}_{b}(\bar{t})}{\beta'} > \frac{p'-p}{\beta'}$ by assumption on p, p', b, and b' as well as condition (4.2) in the main text.

Next, suppose that there exists no $\ell \in \{1, \dots, m\}$ such that $v^{\ell}_{\underline{b}}(\underline{t}) < p \leq p' \leq v^{\ell}_{\bar{b}}(\underline{t})$. Let ℓ' and b' be defined by:

$$
\begin{aligned}
\ell' &= \min\{\tilde{\ell} \in \{1, \dots, m\} | v^{\tilde{\ell}}_{\bar{b}}(\underline{t}) \geq p'\}, \\
b' &= \arg\min_{\tilde{b}\in B_k}\{t_{\tilde{b}} | v^{\ell'}_{\tilde{b}}(\underline{t}) \geq p'\}.
\end{aligned}
$$

That is, ℓ' and b' satisfy $v^{\ell'}_{b'}(\underline{t}) \geq p'$ and the pair (ℓ', b') is the smallest of such pairs with respect to the lexicographic order that relies first on the index and then on the agent's type. Similarly, let ℓ and b be

$$
\begin{aligned}
\ell &= \max\{\tilde{\ell} \in \{1, \dots, m\} | v^{\tilde{\ell}}_{\underline{b}}(\underline{t}) < p\}, \\
b &= \arg\max_{\tilde{b}\in B_k}\{t_{\tilde{s}} | v^{\ell}_{\tilde{b}}(\underline{t}) < p\}.
\end{aligned}
$$

That is, (ℓ, b) is the largest index-seller pair satisfying $v^{\ell}_{b}(\underline{t}) < p$ with respect to the lexicographic oder described above. The relation $\ell \geq \ell'$ contradicts the assumption that there exists no $\ell \in \{1, \ldots, m\}$ such that $v_{\underline{b}}^{\ell}(\underline{t}) < p \leq p' \leq v_{\overline{b}}^{\ell}(\underline{t})$, so $\ell' > \ell$. Hence,

$$\begin{aligned}
p' - p &< v_{b'}^{\ell'}(\underline{t}) - v_{b}^{\ell}(\underline{t}) \\
&\leq [v_{b'}^{\ell'}(\underline{t}) - v_{\underline{b}}^{\ell'}(0, \underline{t}_{-\underline{b}})] + [v_{\overline{b}}^{\ell'-1}(1, \underline{t}_{-\overline{b}}) - v_{\underline{b}}^{\ell'-1}(0, \underline{t}_{-\underline{b}})] + \ldots + [v_{\overline{b}}^{\ell+1}(1, \underline{t}_{-\overline{b}}) - v_{\underline{b}}^{\ell+1}(0, \underline{t}_{-\underline{b}})] \\
&\quad + [v_{\overline{b}}^{\ell}(1, \underline{t}_{-\overline{b}}) - v_{b}^{\ell}(\underline{t})] + (\ell' - \ell)\frac{\alpha}{n_B} \\
&\leq [v_{b'}^{\ell'}(\underline{t}) - v_{\underline{b}}^{\ell'}(\underline{t})] + [v_{\overline{b}}^{\ell}(\underline{t}) - v_{b}^{\ell}(\underline{t})] + \frac{2\beta'}{\Lambda} + \frac{2\alpha}{n_B} + (\ell' - \ell - 1)[\beta' + 2(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_B})] + (\ell' - \ell)\frac{\alpha}{n_B} \\
&\leq \frac{\lambda_{B_k}^{\ell'}(p') + 1}{\Lambda}\beta' + (1 - \frac{\lambda_{B_k}^{\ell}(p) - 1}{\Lambda})\beta' + \frac{2\beta'}{\Lambda} + \frac{2\alpha}{n_B} + (\ell' - \ell - 1)[\beta' + 2(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_B})] + (\ell' - \ell)\frac{\alpha}{n_B},
\end{aligned}$$

where the first inequality follows from the definitions of p, p', b', ℓ', b, and ℓ, the second inequality follows because the "no gap" assumption implies

$$v_{\overline{b}}^{\tilde{\ell}}(1, \underline{t}_{-\overline{b}}) - v_{\underline{b}}^{\tilde{\ell}+1}(0, \underline{t}_{-\underline{b}}) \geq v_{\overline{b}}^{\tilde{\ell}}(1, \underline{t}_{-\overline{b}}) - v_{\underline{b}}^{\tilde{\ell}+1}(0, \underline{t}_{-\overline{b}}) - \frac{\alpha}{n_B} \geq -\frac{\alpha}{n_B},$$

for each $\tilde{\ell} \in \{\ell, \ell + 1, \ldots, \ell' - 1\}$, the third inequality follows because, under event E, $t_{\underline{b}} \leq \frac{1}{\Lambda}$ and $t_{\overline{b}} \geq \frac{\Lambda - 1}{\Lambda}$, and hence,

$$\begin{aligned}
v_{\underline{b}}^{\ell'}(\underline{t}) - v_{\underline{b}}^{\ell'}(0, \underline{t}_{-\underline{b}}) &\leq \frac{\beta'}{\Lambda} + \frac{\alpha}{n_B}, \\
v_{\overline{b}}^{\ell}(1, \underline{t}_{-\overline{b}}) - v_{\overline{b}}^{\ell}(\underline{t}) &\leq \frac{\beta'}{\Lambda} + \frac{\alpha}{n_B},
\end{aligned}$$

by assumption (4.2) in the main text, for each $\tilde{\ell} \in \{\ell + 1, \ldots, \ell' - 1\}$,

$$v_{\overline{b}}^{\tilde{\ell}}(1, \underline{t}_{-\overline{b}}) - v_{\underline{b}}^{\tilde{\ell}}(0, \underline{t}_{-\underline{b}}) \leq \beta' + 2\left[\frac{\beta'}{\Lambda} + \frac{\alpha}{n_B} r\right],$$

and the last inequality follows by definitions of $p, p', \ell, b, \ell', b'$, and by assumption (4.2) in the main text. Rearranging terms, we obtain

$$\begin{aligned}
&\frac{\lambda_{B_k}^{\ell'}(p')+1}{\Lambda} + 1 - \frac{\lambda_{B_k}^{\ell}(p)-1}{\Lambda} + (\ell' - \ell - 1) \\
&> \frac{p' - p - \frac{2\beta'}{\Lambda} - \frac{2\alpha}{n_B} - 2(\ell' - \ell - 1)(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_B}) - (\ell' - \ell)\frac{\alpha}{n_B}}{\beta'}.
\end{aligned} \tag{B.8}$$

Therefore,

$$\begin{aligned}
D_k(p) - D_k(p') &> a_B\left(1 - \frac{\lambda^{\ell}_{B_k}(p)+1}{\Lambda} - \frac{1}{2K}\right) + a_B\left(\frac{\lambda^{\ell'}_{B_k}(p')}{\Lambda} - \frac{1}{2K}\right) + a_B(\ell' - \ell - 1) \\
&\geq a_B\left(\frac{\lambda^{\ell'}_{B_k}(p')+1}{\Lambda} + 1 - \frac{\lambda^{\ell}_{B_k}(p)-1}{\Lambda} - \frac{3}{\Lambda} - \frac{1}{K} + (\ell' - \ell - 1)\right) \\
&\geq a_B\left(\frac{p' - p - \frac{2\beta'}{\Lambda} - \frac{2\alpha}{n_B} - 2(\ell' - \ell - 1)(\frac{\beta'}{\Lambda} + \frac{\alpha}{n_B}) - (\ell' - \ell)\frac{\alpha}{n_B}}{\beta'} - \frac{3}{\Lambda} - \frac{1}{K}\right) \\
&= a_B\left(\frac{p'-p}{\beta'} - \frac{5}{\Lambda} - \frac{1}{K} - (\ell' - \ell)\frac{\alpha}{\beta' n_B} - 2(\ell' - \ell)\frac{\alpha}{\beta' n_B} - 2\frac{\ell' - \ell - 1}{\Lambda}\right), \\
&\geq a_B\left(\frac{p'-p}{\beta'} - \frac{3+2m}{\Lambda} - \frac{1}{K} - \frac{3\alpha m}{\beta' n_B}\right),
\end{aligned}$$

where the term $a_B(\ell' - \ell - 1)$ in the first line corresponds to the supply of the objects from all agents in submarket K for the $(\ell+1)th$, ..., $(\ell'-1)th$ units, the first inequality follows from the definition of $\lambda^{\ell}_{B_k}(p')$, the second inequality follows from calculation, the third inequality follows from inequality (B.8), the equality follows from calculation, and the fourth inequality follows from the fact $\ell' - \ell \leq m$.

Proof for an upper bound for $D_k(\cdot)$. To obtain an upper bound on the difference in demands, let $L := \{\ell \in \{1, \ldots, m\} | v^{\ell}_{\underline{b}}(\underline{t}) < p' \text{ and } v^{\ell}_{\bar{b}}(\underline{t}) \geq p\}$. We shall first show

$$\frac{\lambda^{\ell}_{B_k}(p') - \lambda^{\ell}_{B_k}(p) - 2}{\Lambda}\beta \leq p' - p, \tag{B.9}$$

for each $\ell \in L$. To show this, let $b' \in B_k$ be the seller in submarket k whose type is the highest among those with $v^{\ell}_{b'}(\underline{t}) < p'$, and $b \in B_k$ be the seller in submarket k whose type is the lowest among those with $v^{\ell}_{b}(\underline{t}) \geq p$; Note that such b' and b exist since $\ell \in L$. Then, $t_{b'} \geq \frac{\lambda^{\ell}_{B_k}(p')-1}{\Lambda}$ and $t_b \leq \frac{\lambda^{\ell}_{B_k}(p)+1}{\Lambda}$. Thus,

$$p' - p > v^{\ell}_{b'}(\underline{t}) - v^{\ell}_{b}(\underline{t}) \geq \beta\left(\frac{\lambda^{\ell}_{B_k}(p') - \lambda^{\ell}_{B_k}(p) - 2}{\Lambda}\right),$$

as desired.

Consider $\ell \notin L$. Suppose first that $v^{\ell}_{\underline{b}}(\underline{t}) \geq p'$. Then by assumption $p' \geq p$ it follows that $v^{\ell}_{\underline{b}}(\underline{t}) \geq p$, and hence every buyer in B_k demands the ℓ-th unit of the object under both p and p'. Suppose next that $v^{\ell}_{\bar{b}}(\underline{t}) < p$. Then by assumption $p' \geq p$ it follows that $v^{\ell}_{\bar{b}}(\underline{t}) < p'$, and hence no buyer in B_k demands the ℓ-th unit of the object under either p or p'.

We now show $D_k(p) - D_k(p') < a_B m\left(\frac{p'-p}{\beta} + \frac{3}{\Lambda} + \frac{1}{K}\right)$. To show this, applying inequality (B.9) and the above argument we obtain

$$\begin{aligned}
D_k(p) - D_k(p') &< a_B \sum_{\ell \in L} \left(\left(1 - \frac{\lambda^\ell_{B_k}(p)}{\Lambda} + \frac{1}{2K} \right) - \left(1 - \frac{\lambda^\ell_{B_k}(p') - 1}{\Lambda} - \frac{1}{2K} \right) \right) \\
&= a_B \sum_{\ell \in L} \left(\frac{\lambda^\ell_{B_k}(p') - \lambda^\ell_{B_k}(p) - 2}{\Lambda} + \frac{3}{\Lambda} + \frac{1}{K} \right) \\
&\leq a_B m \left(\frac{p' - p}{\beta} + \frac{3}{\Lambda} + \frac{1}{K} \right).
\end{aligned}$$

□

Lemma 7 has an implication for the shape of the inverse demand and supply functions.

Lemma 8. *Let $x' > x$. Let $D_k^{-1}(x')$ denote an arbitrary price p' such that $D_k(p') = x'$. Similarly, define $D_k^{-1}(x), S_k^{-1}(x'), S_k^{-1}(x)$. Then,*

$$\begin{aligned}
S_k^{-1}(x') - S_k^{-1}(x) &\in \left(\left(\frac{x' - x}{a_S m} - \frac{3}{\Lambda} - \frac{1}{K} \right) \beta, \left(\frac{x' - x}{a_S} + \frac{3 + 2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_S} \right) \beta' \right), \\
D_k^{-1}(x) - D_k^{-1}(x') &\in \left(\left(\frac{x' - x}{a_B m} - \frac{3}{\Lambda} - \frac{1}{K} \right) \beta, \left(\frac{x' - x}{a_B} + \frac{3 + 2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_B} \right) \beta' \right).
\end{aligned}$$

Proof. We first show the bounds for the supply. Let $p' = S_k^{-1}(x')$ and $p = S_k^{-1}(x)$. Observe that $p' \geq p$, $x' = S_k(p')$ and $x = S_k(p)$. By Lemma 7,

$$x' - x \in \left(a_S \left(\frac{p' - p}{\beta'} - \frac{3 + 2m}{\Lambda} - \frac{1}{K} - \frac{3\alpha m}{\beta' n_S} \right), a_S m \left(\frac{p' - p}{\beta} + \frac{3}{\Lambda} + \frac{1}{K} \right) \right),$$

or equivalently,

$$p' - p \in \left(\left(\frac{x' - x}{a_S m} - \frac{3}{\Lambda} - \frac{1}{K} \right) \beta, \left(\frac{x' - x}{a_S} + \frac{3 + 2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_S} \right) \beta' \right).$$

The proof for $D^{-1}(\cdot)$ is symmetric and hence omitted. □

Now we shall prove the theorem. Recall that $S_k(\overline{q}_k) > D_k(\overline{q}_k)$ and $D_k(\underline{q}_k) > S_k(\underline{q}_k)$. These imply $D_k^{-1}(S_k(\underline{q}_k)), S_k^{-1}(D_k(\underline{q}_k)) \geq \underline{q}_k$, and $D_k^{-1}(S_k(\overline{q}_k)), S_k^{-1}(D_k(\overline{q}_k)) \leq \overline{q}_k$. The per-capita budget surplus in the submarket k is at most $\max\{D_k^{-1}(S_k(\underline{q}_k)) - \underline{q}_k, \overline{q}_k -$

$S_k^{-1}(D_k(\overline{q}_k))\}$. We observe that this converges to zero as the market size grows. First,

$$
\begin{aligned}
D_k^{-1}(S_k(\underline{q}_k)) - \underline{q}_k &\leq D_k^{-1}(S_k(\overline{q}_k)) - \underline{q}_k + \left[\frac{S_k(\overline{q}_k) - S_k(\underline{q}_k)}{a_B} + \frac{3+2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_B}\right]\beta' \\
&\leq \overline{q}_k - \underline{q}_k + \left[\frac{a_S m}{a_B}\left(\frac{\overline{q}_k - \underline{q}_k}{\beta} + \frac{3}{\Lambda} + \frac{1}{K}\right) + \frac{3+2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_B}\right]\beta' \\
&\leq (\overline{q}_k - \underline{q}_k)\left(1 + \frac{\beta' a_S m}{\beta a_B}\right) + \left[\frac{3 + 2m + \frac{3a_S m}{a_B}}{\Lambda} + \frac{\frac{a_S m}{a_B} + 1}{K} + \frac{3\alpha m}{\beta' n_B}\right]\beta' \\
&\leq \left(\beta'\left(\frac{2}{K} + \frac{6}{\Lambda}\right) + \frac{2\alpha}{K}\right)\left(1 + \frac{\beta' a_S m}{\beta a_B}\right) + \left[\frac{3 + 2m + \frac{3a_S m}{a_B}}{\Lambda} + \frac{\frac{a_S m}{a_B} + 1}{K} + \frac{3\alpha m}{\beta' n_B}\right]\beta',
\end{aligned}
$$

where the first inequality follows from Lemma 8, the second inequality follows from Lemma 7, and the last inequality follows from the definitions of $\bar{q}_k$ and $\underline{q}_k$. Therefore, it converges to zero as the market size grows.

Similarly,

$$
\begin{aligned}
\overline{q}_k - S_k^{-1}(D_k(\overline{q}_k)) &\leq \overline{q}_k - S_k^{-1}(D_k(\underline{q}_k)) + \left[\frac{D_k(\underline{q}_k) - D_k(\overline{q}_k)}{a_S} + \frac{3+2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_S}\right]\beta' \\
&\leq \overline{q}_k - \underline{q}_k + \left[\frac{a_B m}{a_S}\left(\frac{\overline{q}_k - \underline{q}_k}{\beta} + \frac{3}{\Lambda} + \frac{1}{K}\right) + \frac{3+2m}{\Lambda} + \frac{1}{K} + \frac{3\alpha m}{\beta' n_S}\right]\beta' \\
&\leq (\overline{q}_k - \underline{q}_k)\left(1 + \frac{\beta' a_B m}{\beta a_S}\right) + \left[\frac{3 + 2m + \frac{3a_B m}{a_S}}{\Lambda} + \frac{\frac{a_B m}{a_S} + 1}{K} + \frac{3\alpha m}{\beta' n_S}\right]\beta' \\
&\leq \left(\beta'\left(\frac{2}{K} + \frac{6}{\Lambda}\right) + \frac{2\alpha}{K}\right)\left(1 + \frac{\beta' a_B m}{\beta a_S}\right) + \left[\frac{3 + 2m + \frac{3a_B m}{a_S}}{\Lambda} + \frac{\frac{a_B m}{a_S} + 1}{K} + \frac{3\alpha m}{\beta' n_S}\right]\beta',
\end{aligned}
$$

where the first inequality follows from Lemma 8, the second inequality follows from Lemma 7, and the last inequality follows from the definitions of $\bar{q}_k$ and $\underline{q}_k$. Therefore, it converges to zero as the market size grows. These show the desired conclusion.

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