Beyond Truth-Telling: Preference Estimation with Centralized School Choice and College Admissions*

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September 2017

Abstract

We propose novel approaches and tests for estimating student preferences with data from centralized matching mechanisms, like the Gale-Shapley Deferred Acceptance, when students are strictly ranked by, e.g., test scores. Without requiring truth-telling to be the unique equilibrium, we show that the matching is (asymptotically) stable, or justified-envy-free, implying that every student is matched with her favorite school/college among those she is qualified for ex post. Having illustrated the approaches in simulations, we apply them to school choice data from Paris and demonstrate evidence supporting stability but not truth-telling. We discuss when each approach is more appropriate in real-life settings.

Centralized mechanisms are common in the placement of students to public schools and colleges. Over the past decade, the Gale-Shapley Deferred Acceptance (DA) has become the leading mechanism to match students with schools at all education levels and is now used in many education systems around the world, including Amsterdam, Boston, Hungary, New York, Paris, and Taiwan.

One of the reasons for the growing popularity of DA is its strategy-proofness (Abdulkadiroğlu and Sönmez, 2003). When applying for admission, students are asked to submit rank-order lists (ROLs) of schools, and it is in their best interest to rank schools truthfully. The mechanism therefore releases students and their parents from strategic considerations. As a consequence, it provides education authorities *"with more credible data about school choices, or parent 'demand' for particular schools,"* as argued by the former Boston Public Schools superintendent Thomas Payzant when recommending DA in 2005. Indeed, such rank-ordered data contain rich information on student preferences over schools, and are increasingly used in the empirical literature.

Due to the strategy-proofness of DA, one may be tempted to assume that the submitted ROLs of schools reveal students' true preferences. However, a clarification of the setting is necessary, as its institutional specifics determine the plausibility of this truth-telling assumption. One is the "lottery" setting, in which an education authority prioritizes students into a limited number of groups for admissions and breaks ties with a post-application lottery (Pathak, 2011). Another is the "strict-priority" setting in which students are strictly ranked by some priority index, e.g., a standardized test score, which is known to students when they submit their ROL. Distinguishing between the two settings highlights the boundary of strategy-proofness. Consider a student who likes a highly selective school but has a low test score. In the strict-priority setting, this student may "skip the impossible" and choose not to apply to this school, as she rationally expects a zero admission probability. Such behavior implies that not all students have strong incentives to rank all schools truthfully in their ROLs. On the contrary, the same student may choose to apply to the highly selective school in the lottery setting, because of the positive admission probability.

In contrast to the lottery setting, the strict-priority setting remains largely unexplored in the empirical literature on school choice and college admissions.[1] Based on theoretical

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[1] There are a few exceptions, Ajayi (2017), Akyol and Krishna (2017), and Burgess, Greaves, Vignoles and Wilson (2014), which we discuss in the literature review.

Table 1: Centralized School Choice and College Admissions based on the Deferred Acceptance Mechanism with Strict Priority Indices: Examples

Country/city	Assignment mechanism	Choice restrictions	Sources
Panel A. Secondary Education			
Boston (selective schools)[a]	Student-proposing DA	Unrestricted	[1]
Chicago (selective schools)[a]	DA (Serial dictatorship)[c]	Up to 6 choices	[2]
NYC (selective schools)[a]	DA (Serial dictatorship)[c]	Unrestricted	[1]
Finland	School-proposing DA	Up to 5 choices	[3]
Ghana	DA (Serial dictatorship)[c]	Up to 6 choices	[4]
Paris	School-proposing DA	Up to 8 choices	[5]
Romania	DA (Serial dictatorship)[c]	Unrestricted	[6]
Singapore	DA (Serial dictatorship)[c]	Up to 6 choices	[7]
Turkey	DA (Serial dictatorship)[c]	Up to 12 choices	[8] [9]
Panel B. Higher Education			
Australia (Victoria)	College-proposing DA	Up to 12 choices	[10]
Chile	College-proposing DA	Up to 8 choices	[11]
Hungary	Student-proposing DA	Unrestricted[b]	[12]
Ireland	College-proposing DA	Up to 10 choices	[13]
Norway	College-proposing DA	Up to 15 choices	[14]
Spain	Student-proposing DA	Region-specific	[15] [16]
Taiwan	College-proposing DA	Up to 100 choices	[17]
Tunisia	College-proposing DA	Up to 10 choices	[18] [19]
Turkey	College-proposing DA	Up to 24 choices	[20] [21]

Notes: [a] For exam schools in Boston, selective enrollment high schools in Chicago, and specialized high schools in NYC, strict priority indices are used in the admission. In contrast, admissions to other schools often do not use strict priority indices. [b] In Hungary, students may apply for any number of programs but they are charged a fee (of approximately 10 euros) for every program after the third application. [c] In all of the countries/cities listed in this table, students' priorities are based on various combinations of grades, entrance/exit exams, and other criteria (aptitude tests, interviews, etc.). When priority indices are not school-specific, i.e., schools/universities rank students in the same way, DA, whether student-proposing or school/college-proposing, is equivalent to serial dictatorship, under which students, in the order of their priority indices, are allowed to choose among the remaining schools or universities.

Sources: [1] Abdulkadiroğlu, Angrist and Pathak (2014); [2] Pathak and Sönmez (2013); [3] Salonen (2014); [4] Ajayi (2017); [5] Hiller and Tercieux (2014); [6] Pop-Eleches and Urquiola (2013); [7] Teo, Sethuraman and Tan (2001); [8] Balinski and Sönmez (1999); [9] Akyol and Krishna (2017); [10] VTAC (2016a; 2016b); [11] Hastings, Neilson and Zimmerman (2013); [12] Biró (2011); [13] Chen (2012); [14] Kirkebøen, Leuven and Mogstad (2016); [15] Mora and Romero-Medina (2001); [16] Calsamiglia, Haeringer and Klijn (2010); [17] UAC (2014); [18] Luflade and Zaiem (2017); [19] Tunisian Ministry of Higher Education and Scientific Research (2017); [20] Balinski and Sönmez (1999); [21] Saygin (2013).

investigations of student incentive and behavior, our paper aims to provide empirical approaches to estimating student preferences with data from school choice and college admissions in the strict-priority setting. These approaches can potentially be applied in many real-life cases. Specifically, Table 1 provides a list of examples, which include school choice in Finland, Paris, and Turkey (Panel A) as well as college admissions in Chile, Norway, and Taiwan (Panel B).

The first contribution of this paper is to clarify the implications of the truth-telling assumption, which requires that students always report true preferences. Given the flour-

ishing empirical literature on the lottery setting (Pathak and Shi, 2014; Abdulkadiroğlu, Agarwal and Pathak, Forthcoming), it would seem natural to extend the truth-telling-based approaches used in that literature to the strict-priority setting. Unfortunately, strategy-proofness only implies that truth-telling is a *weakly* dominant strategy, which leads to the issue of multiple equilibria because some students may achieve the same payoff by opting for non-truth-telling strategies—as illustrated by the "skipping the impossible" example above. Making truth-telling even less likely, many applications of DA restrict the length of submittable ROLs, which destroys strategy-proofness (Haeringer and Klijn, 2009; Calsamiglia et al., 2010).

These arguments are formalized in a theoretical model. Deviating from the theoretical literature, we introduce an application cost that students have to pay when submitting ROLs, and the model therefore has the common real-life applications of DA as special cases. Conditional on both preferences and priorities being private information, we show that for truth-telling to be the unique equilibrium, two conditions are needed: no application cost and large uncertainty in admission outcomes. Neither is easily satisfied in our setting. Even without constraints on the length of submittable ROLs, students may find it costly to rank a long list of schools. As students know their own priority indices, uncertainty in admission outcomes can also be limited.

Going beyond truth-telling, our paper's second contribution is to propose a set of novel empirical approaches that are theoretically founded. We consider a weaker assumption implied by truth-telling: stability, or justified-envy-freeness, of the matching outcome (Abdulkadiroğlu and Sönmez, 2003), which means that every student is matched with her favorite school out of all the feasible ones. A school is feasible for a student if its ex post cutoff is lower than the student's priority index. These cutoffs are well-defined and often observable to the researcher: given an outcome, each school's cutoff is the lowest priority index of the students accepted there. Conditional on the cutoffs, stability therefore defines a discrete choice model with personalized choice sets.

We show that stability is a plausible assumption, as there exists an equilibrium outcome of the game that is asymptotically stable under certain conditions. When school capacities and the number of students increase proportionally while the number of schools is fixed, the fraction of students not matched with their favorite feasible school tends to zero. Although stability, as an ex post optimality condition, is not guaranteed in such

an incomplete-information game if the market size is arbitrary, we provide numerical evidence suggesting that typical real-life markets are sufficiently large for this assumption to be almost exactly satisfied.

Based on the theoretical results, we propose a menu of approaches for preference estimation. We start by formalizing the truth-telling assumption under which one can apply rank-ordered models on submitted ROLs to estimate student preferences. In practice, students rarely rank all available schools, and, therefore, the truth-telling assumption often implicitly imposes the exogeneity of the length of a submitted ROL.[2]

Stability leads to a discrete choice model with personalized choice sets, so the identification and estimation in the discrete-choice literature can be applied (e.g., Matzkin, 1993). Moreover, an important advantage of this approach is that it enables estimation with data on matching outcomes only, without requiring information on students' submitted ROLs.

We also provide a solution if neither truth-telling nor stability is satisfied: as long as students do not play dominated strategies, the submitted ROLs reveal true partial preference orders of schools (Haeringer and Klijn, 2009),[3] which allows us to derive probability bounds for one school being preferred to another. The corresponding moment inequalities can be used for inference by applying the related methods (for a survey, see Tamer, 2010). When stability is satisfied and identifies student preferences, these inequalities provide over-identifying information that can improve estimation efficiency (Moon and Schorfheide, 2009).

To guide the choice between these identifying assumptions, we consider several statistical tests. Truth-telling, leading to more restrictions than stability, can be tested against stability using a Hausman-type test (Hausman, 1978) or a test of over-identifying restrictions (Hansen, 1982). Moreover, stability can be tested against undominated strategies: if the outcome is unstable, the stability restrictions are incompatible with the moment inequalities implied by undominated strategies, allowing us to use tests such as Bugni, Canay and Shi (2015).

Our third contribution is to evaluate the performance of the different approaches based on simulated and real-life data. Having illustrated the main theoretical results with Monte Carlo simulations, we apply the empirical approaches to school choice data

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[2]As a result, we emphasize the difference between *strict* and *weak* truth-telling. The former assumes that all students rank all schools truthfully, while the latter requires students to rank their most-preferred schools truthfully and allows them to omit the least-preferred schools.

[3]A ROL is a true partial preference order if the listed schools are ranked according to true preferences.

from Paris. The data contain 1,590 middle school students applying for admission to 11 academic-track high schools in the Southern District of Paris through a version of the DA mechanism. Schools rank students by their academic grades but give priority to those from low-income families. Our findings are more consistent with stability than truth-telling. Reduced-form evidence on students' ranking behavior suggests that some students may have omitted the most selective schools from their lists because of low admission probabilities. Our proposed statistical tests reject truth-telling in favor of stability but fail to reject stability against undominated strategies. The truth-telling-based estimator tends to underestimate students' valuation of popular schools, and is outperformed by the stability-based estimator when it comes to predicting matching outcomes and student preferences.

To highlight the differences between the proposed approaches and their underlying behavioral assumptions, we summarize our theoretical results and describe the nesting structure of the assumptions in Section 5. We also emphasize the key features of school choice and college admissions in practice that can help us to choose the most appropriate empirical approach to preference estimation.

Other Related Literature. Since the seminal paper by Abdulkadiroğlu and Sönmez (2003), the theoretical investigation of student behavior and matching properties under the DA mechanism has been extensive. Specifically, large markets are commonly used in theoretical studies to explore the properties of mechanisms (see the survey by Kojima, 2015). Closely related to our study is Azevedo and Leshno (2016), who show the asymptotics of the cutoffs of stable matchings. Our paper extends their results to outcomes of Bayesian Nash equilibrium, whereas they implicitly assume that students are always truth-telling. Our results on asymptotic stability in equilibrium are in line with Romero-Medina (1998) and Haeringer and Klijn (2009), who prove the stability of Nash equilibrium outcomes under (variants of) the DA mechanism.

There is a burgeoning literature on preference estimation using centralized school choice data. One strand of this literature uses data from settings in which researchers argue that truth-telling behavior by students is plausible. For example, Hastings, Kane and Staiger (2008) use data from Charlotte-Mecklenburg public school district, and Abdulkadiroğlu et al. (Forthcoming) study school choice data from New York City, which

is a "lottery" setting.[4] Both papers estimate student preferences under the assumption that students truthfully report their preferences. In the same spirit, assuming students report their true preferences in surveys, Budish and Cantillon (2012) and De Haan, Gautier, Oosterbeek and Van der Klaauw (2015) use reported student ordinal preferences to conduct analysis without estimating preferences.

The second strand of the empirical literature explicitly considers strategic behavior of students when estimating student preferences, especially when the mechanism is known not to be strategy-proof, e.g., the (Boston) immediate-acceptance mechanism (Agarwal and Somaini, 2014; Calsamiglia, Fu and Güell, 2014; He, 2015; Hwang, 2014; Kapor, Neilson and Zimmerman, 2016). Due to the lack of strategy-proofness, observed ROLs are sometimes considered as solutions to the maximization of students' expected utility. Taking estimated admission probabilities as students' beliefs, one could adopt the same approach to our setting, i.e., a discrete choice problem defined on the set of possible ROLs. However, we do not apply this expected-utility-maximization approach for several reasons. First, degenerate admission probabilities can occur in various cases, leading to multiple solutions to a student's expected-utility maximization problem (He, 2015). This creates issues for identification of student preferences. In the strict-priority setting, admission probabilities are more likely to be degenerate. Second, the choice probability of a given ROL has to be evaluated against a large number of possible ROLs, which can be computationally cumbersome.[5] Third, the cost of ranking/applying to additional schools, especially the part related to cognitive load, is often not observed, and one would need to impose additional assumptions to take the cost into account in the maximization of expected utility.

As to the strict-priority setting, there are only a handful of empirical studies. Among them, the majority use ad-hoc solutions to the potential problem of students' non-truth-telling behavior. In analyzing school choice in the U.K., where proximity to schools is used as a tie-breaker in determining admission to oversubscribed primary schools, Burgess et al. (2014) restrict each student's set of schools to those that are in close proximity to the student's residence. In the context of admissions to secondary schools in Ghana, where priority is determined by exam scores, Ajayi (2017) only considers a subset of schools

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[4]The authors perform robustness checks on the truth-telling assumption, e.g., only considering students' top three submitted choices.

[5]For example, there are $S!/(S-K)!$ lists ranking $1 \leqslant K \leqslant S$ schools.

with similar selectivity. Akyol and Krishna (2017) is an exception in this literature. Observing the matching outcome and the cutoffs of high school admissions in Turkey, the authors estimate preferences based on the assumption that every student is assigned to her favorite feasible school, which amounts to assuming stability of the matching outcome. We formalize and clarify this stability assumption, along with other extensions. Although stability is a rather common identifying assumption in the two-sided matching literature (see the surveys by Fox, 2009; Chiappori and Salanié, 2016), it is new in empirical studies of school choice and college admissions.

Lastly, estimation of student preferences with college admissions data remains under-explored. It is a challenging problem partly because college admissions are sometimes decentralized. Avery, Glickman, Hoxby and Metrick (2013) overcome this difficulty by focusing on students' choice among admission offers. Alternatively, Long (2004) assumes that a student can enroll in any college and attends her most-preferred one, while Black, Cortes and Lincove (2015) investigate college application behavior of students. With data from a decentralized system where applications are relayed by a platform, Drewes and Michael (2006) assume that students rank programs truthfully when submitting the non-binding ROLs. The centralized college admissions include many applications of the DA mechanism (see Table 1).[6] The specifics of the mechanism have led to numerous studies on the causal effects of education (e.g., Hastings, Neilson and Zimmerman, 2015; Kirkebøen, Leuven and Mogstad, 2016), but not yet on preference estimation. The only exception is Kirkebøen (2012) who applies a rank-ordered logit model under the truth-telling assumption and sometimes excludes from a student's choice set every college program at which the student does not meet the formal requirements or is below its previous-year cutoff.

Organization of the Paper. The paper proceeds as follows. Section 1 presents the model that provides our theoretical foundation for preference estimation. Section 2 discusses the corresponding empirical approaches and tests, which are illustrated in Monte Carlo simulations in Section 3. School choice in Paris, and our results on estimation and testing with the Parisian data, are shown in Section 4. Section 5 discusses practical considerations for applying our approaches to real-world data and outlines some extensions. Section 6 concludes.

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[6]Some centralized college admissions do not use the DA mechanism. For example, studying the mechanism allocating students to majors at colleges in Brazil, Carvalho, Magnac and Xiong (2014) model the joint decision of exam taking and college choice.

1   The Model

To study student behavior, we extend the model in Azevedo and Leshno (2016) by introducing cardinal preferences. An economy, as a school choice/college admissions problem, consists of a finite set of schools or colleges, $\mathcal{S} \equiv \{1, \ldots, S\}$, and a set of students. Student i has a type $\theta_i = (u_i, e_i) \in \Theta \equiv [0,1]^S \times [0,1]^S$, where $u_i = (u_{i,1}, \cdots, u_{i,S}) \in [0,1]^S$ is a vector of von Neumann-Morgenstern (vNM) utilities of being assigned to schools, and $e_i = (e_{i,1}, \cdots, e_{i,S}) \in [0,1]^S$ is a vector of priority indices at schools, a student with a higher index having a higher priority at a school. To simplify notation, we assume that all schools and students are acceptable.[7] Students are matched with schools through a centralized mechanism.

The continuum economy with a unit mass of students is denoted by $E = \{G, q, C\}$, where G is an atomless probability measure over Θ representing the distribution of student population over types; $q = (q_1, \cdots, q_S)$ are masses of seats available at each school, where $0 < q_s < 1$ for all s; lastly, C represents an application cost to be specified shortly. G being atomless implies a measure-zero set of students with indifference in either student utilities or priority indices.

A random finite economy of size I is denoted by $F^{(I)} = \{G^{(I)}, q^{(I)}, C\}$. $F^{(I)}$ is constructed by independently drawing from the distribution (G) I students, indexed by $i \in \{1, \cdots, I\}$, and adjusting the numbers of seats to integers. More specifically, $G^{(I)}$ is the (random) empirical distribution of student types for a sample of I students;[8] $q^{(I)} = [q \cdot I]/I$ is the supply of seats per student, where $[x]$ is the vector of integers nearest to x (with a rounding down in case of a tie). We use $\hat{F}^{(I)} = \{\hat{G}^{(I)}, q^{(I)}, C\}$ to denote a realization of $F^{(I)}$.

In the following, we start with $F^{(I)}$ to specify the matching process and to analyze student behavior, because empirical studies deal with finite economies; the extension to the continuum economy E is deferred to Section 1.4.

In a realization of the random economy, $\hat{F}^{(I)}$, schools first announce their capacities, and every student then submits a rank-order list (ROL) of $1 \leqslant K_i \leqslant S$ schools, denoted

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[7]Assuming acceptability of all schools justifies the normalization of $u \in [0,1]^S$. Although we could extend the preference domain to allow for negative values, this would create the possibility that students avoid being assigned to schools with negative vNM utilities when maximizing expected utility.

[8]For a realized economy with realized student types $(\theta_1, \cdots, \theta_I)$, the realized empirical distribution $\hat{G}^{(I)}$ is defined as $\hat{G}^{(I)}(\theta) = \frac{1}{I} \sum_{i=1}^{I} \mathbb{1}(\theta_i \leqslant \theta) \ \forall \theta \in \mathbb{R}^{2S}$, where $\mathbb{1}(\cdot)$ is an indicator function.

by $L_i = \left(l_i^1, \ldots, l_i^{K_i}\right)$, where $l_i^k \in \mathcal{S}$ is i's kth choice. L_i also represents the set of schools being ranked in L_i. We define $>_{L_i}$ such that $s >_{L_i} s'$ if and only if school s is ranked above school s' in L_i. The set of all possible ROLs is $\mathcal{L}$, which includes all ROLs ranking at least one school. Student i's true ordinal preference induced by her vNM utilities u_i is denoted by $r(u_i) = \left(r_i^1, \ldots, r_i^S\right) \in \mathcal{L}$.

When submitting an ROL, a student incurs a cost $C\left(|L|\right)$, which depends on the number of schools being ranked in L, $|L|$. Furthermore, $C\left(|L|\right) \in [0, +\infty]$ for all L and is weakly increasing in $|L|$. To simplify students' participation decisions, we set $C\left(1\right) = 0$.

Such a cost function flexibly captures many common applications of school choice mechanisms. If $C\left(|L|\right) = 0$ for all L, we are in the traditional setting without costs (e.g., Abdulkadiroğlu and Sönmez, 2003); if $C\left(|L|\right) = \infty$ for $|L|$ greater than a constant $\overline{K}$, it corresponds to the constrained school choice where students cannot rank more than $\overline{K}$ schools (e.g., Haeringer and Klijn, 2009); when $C\left(|L|\right) = c(|L| - \underline{K})$, students have to pay a constant marginal cost c for each choice beyond the first $\underline{K}$ choices, as in Hungarian college admissions (Biró, 2011); the monotonic cost function can simply reflect that it is cognitively burdensome to rank too many schools.

The student-school match is then solved by a mechanism that takes into account students' ROLs and schools' rankings over students. Our main analysis focuses on the student-proposing Gale-Shapley Deferred Acceptance (DA), leaving the discussion of school-proposing Deferred Acceptance to Section 5.2. DA, as a computerized algorithm, works as follows:

Round 1. Every student applies to her first choice. Each school rejects the lowest-ranked students in excess of its capacity and temporarily holds the other students.

Generally, in:

Round k. Every student who is rejected in Round $(k-1)$ applies to the next choice on her list. Each school, pooling together new applicants and those it holds from Round $(k-1)$, rejects the lowest-ranked students in excess of its capacity. Those who are not rejected are temporarily held by the schools.

The process terminates after any Round k when no rejections are issued. Each school is then matched with the students it is currently holding.

1.1 Information Structure and Decision-Making

In any realization of the finite economy, $\hat{F}^{(I)}$, consistent with how the economy is constructed, every student's preferences and priority indices are private information, and are i.i.d. draws from G, which is common knowledge (but $\hat{G}^{(I)}$, the realization of $G^{(I)}$, remains unknown).[9]

Let us start by analyzing the game from student i's point of view. Conditional on others' priority indices and submitted ROLs (L_{-i}, e_{-i}), as well as i's submitted list L_i and priority index e_i, i's admission outcome is deterministic because of the algorithm. Specifically, the outcome at school s is:

$$a_s\left(L_i, e_i; L_{-i}, e_{-i}\right)$$
$$\equiv \begin{cases} \mathbb{1}\left(i \text{ is rejected by } l_i^1, \dots, l_i^k \text{ and accepted by } l_i^{k+1} = s \mid L_i, e_i; L_{-i}, e_{-i}\right) & \text{if } s \in L_i \\ 0 & \text{if } s \notin L_i \end{cases}$$

where $\mathbb{1}(\cdot \mid L_i, e_i; L_{-i}, e_{-i})$ is an indicator function. Moreover, due to the centralized mechanism, a student can receive at most one offer, so $\sum_{s=1}^{S} a_s\left(L_i, e_i; L_{-i}, e_{-i}\right) = 0$ or 1.

Of course, L_{-i} and e_{-i} are unknown to i at the time of submitting her ROL, so i takes into account the distribution when choosing an action.

A pure strategy is $\sigma : \Theta \to \mathcal{L}$. Given σ, the (ex ante) admission probabilities are $\int a_s\left(\sigma(\theta_i), e_i; \sigma_{-i}(\theta_{-i}), e_{-i}\right) dG(\theta_{-i})$ for all i and s, where $\sigma_{-i}(\theta_{-i}) \equiv \{\sigma(\theta_j)\}_{j \neq i}$. We consider a (type-)symmetric equilibrium σ^* in pure strategies such that σ^* solves the following maximization problem for every student type:[10]

$$\sigma^*(\theta_i) \in \underset{\sigma(\theta_i) \in \mathcal{L}}{\arg\max} \left\{\sum_{s \in \mathcal{S}} u_{i,s} \int a_s\left(\sigma(\theta_i), e_i; \sigma_{-i}^*(\theta_{-i}), e_{-i}\right) dG(\theta_{-i}) - C\left(|\sigma(\theta_i)|\right)\right\}. \tag{1}$$

The existence of pure-strategy Bayesian Nash equilibrium can be established by applying Theorem 4 (Purification Theorem) in Milgrom and Weber (1985), although there might be multiple equilibria. For ease of exposition, the following analysis focuses on pure-strategy equilibrium. While economy $F^{(I)}$ is random, it should be emphasized that a strategy σ is "deterministic" in the sense that it only depends on (G, I, C) but not the

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[9]In the previous literature, some papers assume complete information, for example Ergin and Sönmez (2006), Haeringer and Klijn (2009), Kojima (2008), and Pathak and Sönmez (2008); student preferences and priorities are common knowledge. Incomplete information, similar to ours, is also common, for example, Abdulkadiroğlu, Che and Yasuda (2011), Miralles (2008), and He (2015).

[10]It is innocuous to focus on symmetric equilibrium, because it does not restrict the strategy of any student given that they all have different priority indices (almost surely).

realization of $F^{(I)}$.

We define a realized matching $\hat{\mu}$ as a mapping from Θ to $\mathcal{S} \cup \{\varnothing\}$ such that: (i) $\hat{\mu}(\theta_i) = s$ if student i is matched with s; (ii) $\hat{\mu}(\theta_i) = \varnothing$ if student i is unmatched; and (iii) $\hat{\mu}^{-1}(s)$ is the set of students matched with s, while $|\hat{\mu}^{-1}(s)|$ is the number of students matched with s and does not exceed s's capacity.

$\hat{F}^{(I)}$ and σ together lead to a ROL profile as inputs into the DA mechanism and result in a matching, $\mu_{(\hat{F}^{(I)},\sigma)}$, which is uniquely determined by the DA mechanism. Note that $\mu_{(F^{(I)},\sigma)}$ is a random matching because $F^{(I)}$ is a random economy.

Moreover, the (random) cutoff of school s in random matching $\mu_{(F^{(I)},\sigma)}$ is defined as:

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P_s\left(\mu_{(F^{(I)},\sigma)}\right) = \begin{cases} \min\left\{e_{i,s} \mid \mu_{(F^{(I)},\sigma)}(\theta_i) = s\right\} & \text{if } |\mu^{-1}_{(F^{(I)},\sigma)}(s)| = q_s^{(I)} \\ 0 & \text{if } |\mu^{-1}_{(F^{(I)},\sigma)}(s)| < q_s^{(I)} \end{cases}
$$

That is, $P_s\left(\mu_{(F^{(I)},\sigma)}\right)$ is zero if s does not meet its capacity; otherwise, it is the lowest priority index among all accepted students. The vector of cutoffs is denoted by $P\left(\mu_{(F^{(I)},\sigma)}\right)$, and its realization in $\hat{F}^{(I)}$ is $P\left(\mu_{(\hat{F}^{(I)},\sigma)}\right)$.

1.2 Truth-Telling Behavior in Equilibrium

To assess the plausibility of the truth-telling assumption in empirical studies, we begin by investigating students' truth-telling behavior in equilibrium.

A clarification of the concepts is in order. Student i is **weakly truth-telling** (WTT, hereafter) if $\sigma(\theta_i) = (r_i^1, r_i^2, \ldots, r_i^K)$ for $K \leqslant S$. That is, i ranks her K most-preferred schools according to her true preference order but may not rank all available schools. If a WTT strategy always truthfully ranks all S schools and thus $\sigma(\theta_i) = r(u_i)$, i is **strictly truth-telling** (STT, hereafter).[11]

We emphasize the difference between weak and strict truth-telling because the theoretical result of strategy-proofness concerns the latter. However, WTT is often considered in empirical studies because in practice, students rarely rank all available schools, as we shall revisit in Section 2.2.

It is well known that DA is strategy-proof when there is no application cost (Dubins

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[11]Related to the distinction between STT and WTT, the literature on lab experiments on school choice sometimes also defines truth-telling as being different from STT. For example, Chen and Sönmez (2006) call a student truth-telling under the DA mechanism if she ranks her most-preferred schools up to her district school, at which she has guaranteed admission.

and Freedman, 1981; Roth, 1982). That is, when $C(|L|) = 0$ for all $L \in \mathcal{L}$, STT is a weakly-dominant strategy for all students.

Strategy-proofness, or weak dominance of STT, leaves open the possibility of multiple equilibria when every student can rank all schools at no cost. In fact, even when all others play STT, there might exist multiple best responses for a given student. There is no hope of finding conditions to make STT a strictly-dominant strategy either, because it would require STT to be strictly better than all other possible strategies against all possible action profiles of other students.

If we assume that the equilibrium where everyone plays STT is always selected, we implicitly impose a selection rule. It is therefore useful to clarify the conditions under which STT is the unique equilibrium. The following example highlights two sources of equilibrium multiplicity in a complete-information environment.

Example 1 (Multiple Equilibria under DA without Application Cost). *Let us consider a finite economy with three students (i_1, i_2, i_3), three one-seat schools (s_1, s_2, s_3), and no application cost. As common knowledge, school priority ranking and student ordinal preferences are as follows:*

School priority ranking (high to low)	Student ordinal preferences (more to less preferred)
s_1: i_1, i_2, i_3	i_1: s_1, s_2, s_3
s_2: i_1, i_2, i_3	i_2: s_1, s_2, s_3
s_3: i_1, i_2, i_3	i_3: s_2, s_1, s_3

*The game has many equilibria in addition to STT, coming from two sources: "**irrelevance at the bottom**" and "**skipping the impossible**." Both arise when some admission probabilities are zero.*

In this complete-information setting, for i_1, the lower part of her submitted ROL is "payoff-irrelevant" as long as s_1 is top-ranked, because i_1 has zero probability of getting into any school other than s_1. In fact, any ROL of the form (s_1, s', s'') for any $s', s'' \in \{s_2, s_3\} \cup \{\varnothing\}$, is weakly dominant for i_1. This happens when a student is certain to be accepted by her earlier choices. This same applies to i_2 as well: it is payoff-irrelevant whether or not i_2 ranks s_3 after s_2, given s_2 is ranked.

For i_2 and i_3, "skipping the impossible" comes into play. Both students can omit s_1 from their submitted ROLs without harming their payoffs, because the seat at s_1 will

be taken by i_1 in any equilibrium. Making things worse, for both i_2 and i_3, how they rank s_1 does not matter for their equilibrium payoffs. Additionally, for i_3, s_2 is also an "impossible" school, which means that her equilibrium strategy can be anything as long as s_3 is included in her submitted ROL.

One may conjecture that STT might survive as the unique equilibrium when information is incomplete. Indeed, specifying the incompleteness of information, the following proposition provides a sufficient condition.

Proposition 1. *STT is the unique Bayesian Nash equilibrium under DA if (i) there is no application cost: $C(|L|) = 0$, $\forall L \in \mathcal{L}$; and (ii) the joint distribution of preferences and priorities G has full support.*

All proofs can be found in Appendix A. The first condition is violated if students cannot rank as many schools as they wish, or if they suffer a cognitive burden when ranking too many schools. It should also be emphasized that the cost need not be large, because the marginal benefit of ranking an additional school can be close to zero. When a student considers her admission probability at her kth choice, she may face a very large probability of being accepted by at least one of her earlier choices. This is in the same spirit as the "irrelevance at the bottom" in Example 1.

The full-support condition makes all admission probabilities non-zero by introducing uncertainties, and therefore any deviation from STT is costly. This is more plausible when the priority index is determined by an ex post lottery and when the information on others' preferences over schools is less precise.

Proposition 1 specifies when students have incentives to rank *all* schools truthfully, but this result does not extend to WTT. As the equilibrium condition, Equation (1) implies that a student may omit her most-preferred school if the admission probability is close to zero, thus violating WTT. This is precisely the "skipping the impossible" behavior discussed in Example 1.

We may take one step back and focus on whether students have incentives to rank the *included* schools truthfully. We call L_i, $|L_i| \leq S$, a **partial preference order** of schools if L_i respects the true preference ordering among those ranked in L_i. That is, s is ranked before s' in L_i, only if $u_{i,s} > u_{i,s'}$; when s is not ranked in L_i, there is no information on how s is ranked relative to any other school according to i's true preferences.

Proposition 2. *Under DA with application cost, it is a weakly-dominated strategy to submit an ROL that is not a partial preference order.*

Proposition 2 can be considered as a corollary of Proposition 4.2 in Haeringer and Klijn (2009), and thus we omit its proof. The key is to note that such an ROL is dominated by the ROL that ranks the same schools according to their true preference order.

1.3 Matching Outcome: Stability

The above results speak to the plausibility of the truth-telling assumptions WTT and STT in empirical studies. In particular, WTT is not theoretically supported as a weakly-dominant strategy even in DA with no application cost; whenever there is any form of application cost, STT is no longer a dominant strategy.

Taking a different perspective, we note that all equilibria have the same matching outcome in Example 1. This motivates us to investigate the properties of equilibrium matching outcomes of DA. Intuitively, the degree of multiplicity in equilibrium outcomes must be smaller than that in equilibrium strategies. In the two-sided matching literature, stability is the leading concept for equilibrium outcome and the main identifying assumption (Chiappori and Salanié, 2016). We investigate whether it can also be the unique equilibrium outcome in school choice and college admissions.

Unfortunately, we shall demonstrate that having stability as the unique equilibrium outcome requires similar conditions to those for STT being the unique equilibrium. In fact, whenever there is application cost, stability is not guaranteed in equilibrium. This is because Bayesian Nash equilibrium implies ex ante optimality of student strategy, while stability requires ex post optimality.

Because we study the ex post properties of a matching, let us consider $\hat{\mu}$, a realization of the random matching. (i, s) form a **blocking pair** if (i) i prefers s over her matched school $\hat{\mu}(\theta_i)$ while s has an empty seat $(|\hat{\mu}^{-1}(s)| < q_s^{(I)})$, or if (ii) i prefers s over $\hat{\mu}(\theta_i)$ while s has no empty seats $(|\hat{\mu}^{-1}(s)| = q_s^{(I)})$ but i's priority index is higher than its cutoff, $e_{i,s} > \min_{\{j:\hat{\mu}(\theta_j)=s\}}(e_{j,s})$. $\hat{\mu}$ is **stable** if there is no blocking pair.

Stability is a concept borrowed from two-sided matching and is also known as **elimination of justified envy** in school choice (Abdulkadiroğlu and Sönmez, 2003). In our setting, stability can be conveniently linked to school cutoffs. Given a realized matching $\hat{\mu}$, school s is ex post **feasible** for i if $e_{i,s} \geqslant P_s(\hat{\mu})$, and we denote the set of feasible

schools for i by $\mathcal{S}(e_i, P(\hat{\mu}))$. We then have the following lemma, combining Lemmata 1 and 2 in Balinski and Sönmez (1999), whose straightforward proof is omitted here.

Lemma 1. *A realized matching $\hat{\mu}$ is stable if and only if $\hat{\mu}(\theta_i) = \arg\max_{s \in \mathcal{S}(e_i, P(\hat{\mu}))} u_{i,s}$ for all $i = 1, \cdots, I$; i.e., every student is matched with her favorite feasible school.*

Given that the cutoffs of a matching are observed ex post by the researcher, we can define every student's set of feasible schools; Lemma 1 therefore implies a discrete choice model with personalized choice sets. We further formalize this idea in Section 2.3.

We may also be interested in stability being an outcome of equilibrium in dominant strategies, which would free us from specifying the information structure and from imposing additional equilibrium conditions. The following lemma provides the necessary and sufficient conditions, which are similar to those for STT to be the unique equilibrium.

Lemma 2. *Under DA, stable matching is an outcome of Bayesian Nash equilibrium in dominant strategy if and only if $C(|L|) = 0$ for all L. It is the unique equilibrium outcome if additionally G has full support.*

The "if and only if" statement of the lemma is implied by strategy-proofness of DA without application cost, while the uniqueness statement is a result of Proposition 1.

It is well known that DA always produces a stable matching when students are STT (Gale and Shapley, 1962), but not when they are only WTT. The following results, clarifying the relationship between WTT and stability, have implications for our empirical approaches in the next section.

Proposition 3. *Suppose everyone is WTT under DA. Given a realized matching:*

(i) every assigned student is matched with her favorite feasible school; and

(ii) if everyone who has at least one feasible school is matched, the matching is stable.

1.4 Asymptotic Stability in Bayesian Nash Equilibrium

So far, we have shown that neither truth-telling (STT and WTT) nor stability can emerge in the game as equilibrium outcome without some potentially restrictive assumptions. Following the literature on large markets, we study whether stability of the equilibrium outcome can be asymptotically satisfied.

We now revisit the continuum economy, E, and additionally introduce a sequence of random finite economies $\{F^{(I)}\}_{I \in \mathbb{N}}$ that are constructed from E as before.

We extend our analysis to the continuum economy. The definitions of matching, DA, and stability can be naturally extended to continuum economies as in Abdulkadiroğlu, Che and Yasuda (2015) and Azevedo and Leshno (2016), which is discussed in Appendix A.2.1. These definitions are similar to their counterparts in finite economies. For example, a matching in E when everyone adopts σ is $\mu_{(E,\sigma)} : \Theta \to \mathcal{S} \cup \{\varnothing\}$, which satisfies (i) $\mu_{(E,\sigma)}(\theta_i) = s$ when type θ_i is matched with s and (ii) $G(\mu_{(E,\sigma)}^{-1}(s)) \leqslant q_s$.

It is known that, generically, there exists a unique stable matching in the continuum economy (Azevedo and Leshno, 2016);[12] we impose the conditions for the uniqueness and denote this stable matching in E as μ^∞ and the corresponding cutoffs as P^∞. Although μ^∞ is unique, there may exist some Nash equilibrium that leads to an unstable matching. We present such a continuum-economy example in Appendix A.2.5, which extends the discrete version in Haeringer and Klijn (2009). Citing the results in Haeringer and Klijn (2009), Appendix A.2.5 further shows that every Nash equilibrium outcome in the continuum economy is stable if and only if student priority indices at all schools satisfy the so-called Ergin acyclicity condition. Ergin (2002) calls priority indices acyclical if they never give rise to situations where a student can block a potential settlement between any other two students without affecting her own match. This condition is satisfied when all schools rank every student by a single priority index. In the following, we assume that all Nash equilibria of the continuum economy E result in the stable matching, which in practice can be checked using Proposition A3.

Because we are interested in equilibrium outcomes, we augment the sequence of economies with equilibrium strategies, $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$, where $\sigma^{(I)}$ is a pure-strategy Bayesian

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[12]A sufficient condition for the uniqueness of stable outcome in E is that G has full support. Even when G does not have full support, the uniqueness can be achieved when $\sum_{s=1}^{S} q_s < 1$. Let σ^{STT} be the STT strategy. That is, $\sigma^{STT}(\theta_i) = r(u_i)$ for all θ_i. We define the demand for each school in (E, σ^{STT}) as a function of the cutoffs:

$$D_s(P \mid E, \sigma^{STT}) = \int \mathbb{1}(u_{i,s} = \max_{s' \in \mathcal{S}(e_i, P)} u_{i,s'}) dG(\theta_i),$$

where $\mathbb{1}(\cdot)$ is an indicator function. Let $D(P \mid E, \sigma^{STT}) = [D_s(P \mid E, \sigma)]_{s \in \mathcal{S}}$. E admits a unique stable matching if the image under $D(P \mid E, \sigma^{STT})$ of the closure of the set

$$\{P \in (0,1)^S : D(P \mid E, \sigma^{STT}) \text{ is not continuously differentiable at } P\}$$

has Lebesgue measure zero.

Nash equilibrium in $F^{(I)}$ and satisfies the following assumption.

Assumption 1. *There exists σ^∞ such that $\lim_{I \to \infty} G\left(\{\theta_i \in \Theta \mid \sigma^{(I)}(\theta_i) = \sigma^\infty(\theta_i)\}\right) = 1$.*

Although $F^{(I)}$ is a random economy, $\sigma^{(I)}$ is fixed given the size of the economy. In other words, $\sigma^{(I)}$ remains as an equilibrium strategy in any realization of $F^{(I)}$. Assumption 1 regulates how the equilibria evolve with market size, which is necessary as there are multiple equilibria. By this assumption, in the sequence $\{\sigma^{(I)}\}_{I \in \mathbb{N}}$, fewer and fewer student types need to adjust their optimal actions when the market size grows.

We provide some justifications for this assumption in Appendix A.2.4. Lemma A2 shows that a strategy that does not lead to μ^∞ in the continuum economy cannot survive as an equilibrium when the market size grows. This immediately implies that in sufficiently large economies, every student includes in her ROL the school prescribed by μ^∞ (Lemma A3). Moreover, students do not pay a cost to rank more schools in large economies (Lemma A4). These results imply strong restrictions on the sequence of Bayesian Nash equilibria in the direction of satisfying Assumption 1. Lastly, as Lemma A5 shows, Assumption 1 is exactly satisfied when it is costly to rank more than one school ($C(2) > 0$).

1.4.1 Asymptotic Stability: Definition and Results

Let the random matching $\mu_{\left(F^{(I)}, \sigma^{(I)}\right)}$ be $\mu^{(I)}$, and the associated random cutoffs $P\left(\mu^{(I)}\right)$ be $P^{(I)}$. The following definition formalizes the concept of asymptotic stability.[13]

Definition 1. *A sequence of random matchings, $\{\mu^{(I)}\}_{I \in \mathbb{N}}$, associated with the sequence of random economies and equilibrium strategies, $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$, is asymptotically stable if,*

$$\lim_{I \to \infty} G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \notin \arg\max_{s \in \mathcal{S}(e_i, P^{(I)})} u_{i,s}\}) = 0, \text{ almost surely;}$$

or, equivalently, the fraction of students who are not matched with their favorite feasible school in a random finite economy converges to zero almost surely.

We are now ready to introduce our result.

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[13] We define the probability space, $(\Omega, \mathcal{F}, \mathbb{P})$. Specifically, $\Omega = \prod_{I \in \mathbb{N}} \Theta^I$, and an element in Ω is denoted by $\omega = (\omega_1, \omega_2, \cdots)$, where ω_I is a possible realization of student types in the random economy $F^{(I)}$. $\mathcal{F}$ is a Borel σ-algebra of Ω, and $\mathbb{P}$ is a probability measure from $\mathcal{F}$ to $[0, 1]$.

Proposition 4. *In the sequence of random economies and equilibrium strategies, $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$, if Assumption 1 is satisfied, then*

(i) the random cutoffs converge to those of the stable matching in the continuum economy: $\lim_{I \to \infty} P^{(I)} = P^{\infty}$, almost surely;

(ii) the sequence of random matchings, $\{\mu^{(I)}\}_{I \in \mathbb{N}}$, is asymptotically stable.

1.4.2 Probability of Being in a Blocking Pair for a Given Student

Based on the above results, we can discuss the probability of a given student being in a blocking pair. The following proposition shows how economy size, the cost of submitting a list, and other factors play a role.

Proposition 5. *Suppose student i exists in all economies in the sequence $\{F^{(I)}\}_{I \in \mathbb{N}}$ which is associated with a sequence of Bayesian Nash equilibria in pure strategies $\{\sigma^{(I)}\}_{I \in \mathbb{N}}$.*

(i) Let $\sigma^{(I)}(\theta_i) = L^{(I)}$; then $L^{(I)}$ is a partial order of i's ordinal preferences. If ex post i forms a blocking pair with s, s must not be included in $L^{(I)}$, $s \in \mathcal{S} \backslash L^{(I)}$.

The probability that i is in a blocking pair with any school in the random matching $\mu^{(I)}$, denoted by $B_i^{(I)} = \Pr(\exists s \in \mathcal{S}, u_{i,s} > u_{i,\mu^{(I)}(\theta_i)}, e_{i,s} \geqslant P_s^{(I)})$, satisfies:

(ii) $B_i^{(I)}$ is bounded above: $B_i^{(I)} \leqslant |\mathcal{S} \backslash L^{(I)}| \frac{C(|L^{(I)}|+1) - C(|L^{(I)}|)}{\max_{s \in \mathcal{S} \backslash L^{(I)}} u_{i,s}}$;

(iii) If $\{\sigma^{(I)}\}_{I \in \mathbb{N}}$ satisfies Assumption 1, $B_i^{(I)}$ converges to zero almost surely.

Because in equilibrium student i reports a partial order of her true preferences, she can only form a blocking pair with a school that she did not rank (part i). Therefore, whenever it is less costly to rank more schools, the probability that i is in a blocking pair decreases with the marginal application cost (part ii). Together, Proposition 5 shows that stability is more plausible when the cost of ranking more schools is lower and/or the market is large. Moreover, in the case of constrained/truncated DA where there is a limit on the length of ROLs, the higher the number of schools that can be ranked, the more likely stability is to be satisfied.

2 Empirical Approaches

Building on the theoretical results from the previous section, we formalize the estimation of student preferences under different sets of assumptions and propose a series of tests to

select the appropriate approach. To be more concrete, we consider a logit-type random utility model, although our approaches can be extended to other specifications.

Throughout this section, we consider a random finite economy $F^{(I)}$ in which I students compete for admission into S distinct schools. Each school s has a positive capacity $I \times q_s$, and students are assigned through some version of the student-proposing DA. In addition to submitted ROLs and matching outcomes, the researcher also observes students' priority indices, student characteristics, and school attributes. Given these observables, we discuss the "conditional choice probability" of a student submitting a given ROL or being matched with a given school from the researcher's perspective.

2.1 Model Setting

As is traditional and more convenient in empirical analysis, we now let the student utility functions take any value on the real line.[14] With some abuse, we still use the same notation for utility functions. That is, student i's utility from attending schools s is defined as:

$$u_{i,s} = V_{i,s} + \epsilon_{i,s} = V(Z_{i,s}, \beta) + \epsilon_{i,s}, \tag{2}$$

where $V(\cdot, \cdot)$ is a known function, taking as arguments $Z_{i,s}$, a vector of observable student-school characteristics, and β, a vector of unknown constants to be estimated; $\epsilon_{i,s}$ is the unobservable student heterogeneity.

We further define $Z_i \equiv \{Z_{i,s}\}_{s=1}^S$, and $\epsilon_i \equiv \{\epsilon_{i,s}\}_{s=1}^S$. It is assumed that $\epsilon_i \perp Z_i$ and that $\epsilon_{i,s}$ is i.i.d. over i and s with the type-I extreme value (Gumbel) distribution. Such a formulation rules out outside options, although this assumption can be relaxed. Furthermore, this specification implicitly imposes the following assumption:

Assumption (Maintained).

Let μ be the random matching outcome. Then, $u_{i,s} \perp \mu$; in other words, $u_{i,s}$, student i's preference of being (hypothetically) matched with s, does not depend on the matching.

This assumption implies that student preferences are not affected by other students'

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[14]In the theoretical discussion, we restricted the vNM utility functions to be in $[0, 1]$. One can use, for example, the inverse of standard normal distribution to transform all utility functions to be on the real line. It should be emphasized that we cannot apply the expected utility theory with the transformed utility functions, and we do not.

school assignments (no peer effects). Moreover, statistics that are associated with the matching outcome, such as cutoffs, do not enter the utility function either.

2.2 Truth-Telling

Recall that empirical studies with application data from college admissions and school choice with strict priority ranking are still rare, while the empirical literature on school choice with coarse priority ranking is fast-growing. In this literature, some proposed approaches are based on the truth-telling assumption (Hastings et al., 2008; Abdulkadiroğlu et al., Forthcoming). Given that similar mechanisms are commonly used in both settings, we discuss how these approaches can be extended to our setting and clarify the assumptions embedded within.

We start with WTT instead of STT because students in school choice and college admissions usually do not rank the same number of choices in reality (Abdulkadiroğlu et al., Forthcoming; He, 2015; Artemov, Che and He, 2017). Under the assumption of truth-telling in the absence of outside option, this can only be consistent with students' WTT behavior but not STT, because STT requires everyone to rank all schools. We discuss STT with outside options in Appendix A.3.

For notational convenience, we make it explicit that student i's type θ_i is described by (u_i, e_i). Let $\sigma^W : \mathbb{R}^S \times [0,1]^S \to \mathcal{L}$ be a WTT pure strategy.[15] More precisely, the WTT assumption amounts to the following two assumptions:

Assumption (Weak Truth-Telling).
WTT1. Suppose $\sigma^W(u_i, e_i) = L = (l^1, \ldots, l^K)$. $\sigma^W(u_i, e_i)$ ranks i's top K preferred schools according to her true preferences: $u_{i,l^1} > \cdots > u_{i,l^K}$, and $u_{i,l^K} > u_{i,s'}$ for all s' not ranked in L;
WTT2. The number of schools ranked by a student is exogenous: $u_i \perp |\sigma^W(u_i, e_i)|, \forall i$.

We are interested in the choice probability of L conditional on observables, where the uncertainty from the researcher's perspective is due to the utility shocks (ϵ_i). Note that:

$$\Pr\left(\sigma^W(u_i, e_i) = L \mid Z_i; \beta\right)$$
$$= \Pr\left(\sigma^W(u_i, e_i) = L) \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K\right) \times \Pr\left(|\sigma^W(u_i, e_i)| = K \mid Z_i; \beta\right).$$

---

[15]Because the preference space is transformed from $[0,1]^S$ to $\mathbb{R}^S$, a strategy is now defined on the transformed strategy space. Moreover, it will be clear that σ^W does not depend on priority indices, e_i.

The probabilities are calculated by integrating out the unobservables (ϵ_i) in u_i. Assumption WTT2 implies that $\Pr\left(|\sigma^W(u_i, e_i)| = K \mid Z_i; \beta\right)$ does not depend on preferences and is therefore equal to $\Pr\left(|\sigma^W(u_i, e_i)| = K\right)$. This justifies the decision to focus on the following conditional probability:

$$
\begin{aligned}
&\Pr\left(\sigma^W(u_i, e_i) = L \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K\right) \\
&= \Pr\left(u_{i,l^1} > \cdots > u_{i,l^K} > u_{i,s'} \ \forall\, s' \in \mathcal{S}\backslash L \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K\right) \\
&= \prod_{s \in L}\left(\frac{\exp(V_{i,s})}{\sum_{s' \nmid_L s}\exp(V_{i,s'})}\right),
\end{aligned}
$$

where $s' \nmid_L s$ indicates that s' is not ranked before s in L, which includes s itself and the schools not ranked in L. This rank-ordered (or "exploded") logit model can be seen as a series of conditional logit models: one for the top-ranked school (l^1) being the most preferred; another for the second-ranked school (l^2) being preferred to all schools except the one ranked first, and so on.

The model can be estimated by maximum likelihood estimation (MLE) with the log-likelihood function:

$$
\ln L_{TT}\left(\beta \mid \mathbf{Z}, |\sigma^W|\right) = \sum_{i=1}^{I}\sum_{s \in \sigma^W(u_i, e_i)} V_{i,s} - \sum_{i=1}^{I}\sum_{s \in \sigma^W(u_i, e_i)} \ln\left(\sum_{s' \nmid_{\sigma^W(u_i, e_i)} s} \exp(V_{i,s'})\right),
$$

where $|\sigma^W|$ is the vector of lengths of all submitted ROLs. The estimator is denoted by $\hat{\beta}_{TT}$. Alternatively, the generalized method of moment (GMM) can be employed, for which the moment conditions based on the choice probabilities can be derived as in Section 2.5. The identification of such a model is discussed in Beggs, Cardell and Hausman (1981).

2.3 Stability

We now assume that the matching is stable and explore how to identify and estimate student preferences. The discussion abstracts away from the DA mechanism and ignores how the stable matching is obtained. We first formulate the matching as the outcome of a discrete choice model and then clarify the conditions that are needed for identification and estimation.

Consider that we observe the matching outcome μ and the associated cutoffs $P(\mu)$,

both of which are random variables determined by the unobserved utility shocks (ϵ). We define $\infty \cdot 0 = 0$ and formulate a modified utility function:

$$\overline{u}_{i,s} \equiv u_{i,s} - \infty \times \mathbb{1}(e_{i,s} < P_s(\mu)) = V(Z_{i,s}, \beta) - \infty \times \mathbb{1}(e_{i,s} < P_s(\mu)) + \epsilon_{i,s}, \qquad (3)$$

where the term $-\infty \times \mathbb{1}_{\{e_{i,s} < P_s(\mu)\}}$ is zero for feasible schools but equal to $-\infty$ for infeasible ones, thus making them always less desirable.

The following lemma shows the equivalence between stable matching and the discrete choice model based on the modified utility functions.

Lemma 3. *A realized matching $\hat{\mu}$ is stable if and only if $\hat{\mu}(\theta_i) = \hat{\mu}(u_i, e_i) = \arg\max_{s \in \mathcal{S}} \overline{u}_{i,s}$.*

This lemma is implied by Lemma 1. The formal proof is omitted but a sketch is as follows. Because of the "penalty" on infeasible schools, the discrete choice problem, $\max_{s \in \mathcal{S}} \overline{u}_{i,s}$, is equivalent to the utility maximization problem among the feasible schools with the true utility functions, $\max_{s \in \mathcal{S}(e_i, P(\hat{\mu}))} u_{i,s}$.

The above result motivates us to link the identification and estimation of student preferences under the stability assumption to those in discrete choice models. We can consider that the set of schools are offered to every student at personalized "prices" that are either zero or infinite, depending on the student's priority.[16] A stable matching is equivalent to the equilibrium outcome in which every student makes the optimal choice given the market-clearing "prices."

This formulation requires the following assumptions.

Assumption (Stability).

ST1. For all i, $e_i \perp \epsilon_i$. That is, students do not, or cannot, affect their priority index at any school based on their preferences.

ST2. $\mathbb{1}(e_{i,s} < P_s(\mu)) \perp \epsilon_i$ for all i and s, or equivalently $\mathcal{S}(e_i, P(\mu)) \perp \epsilon_i$: No student can influence her own set of feasible schools.

Assumption ST1 is analogous to the exogeneity of observables in discrete choice models. When priority indices (e_i) are determined by test scores, the assumption implies that no student intentionally under-performs or over-performs in exams.

Assumption ST2 deserves some discussion. Essentially, it assumes that students are "price takers," or more precisely, that they take school feasibility as given and they cannot

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[16]Representing matching, in particular stable matching, with student demand and personalized "prices" is also discussed in He, Li and Yan (2015b) and He, Miralles, Pycia and Yan (2015a).

change $\mathbb{1}(e_{i,s} < P_s(\mu))$. One may be concerned that, in a finite market, an individual student can affect some cutoffs by applying to a school or not, and may change the feasibility of some schools. Another concern is that given student preferences, there can be multiple stable matchings in a finite market. If a single student can influence the selection among the stable matchings, Assumption ST2 is also violated.

These concerns diminish as the economy grows large, because the potential influence on cutoffs by individual students decreases and there tends to be a unique stable matching. Moreover, even in small markets, Assumption ST2 can be satisfied. Importantly, the assumption requires $\mathbb{1}(e_{i,s} < P_s(\mu))\perp\epsilon_i$, or equivalently $\mathcal{S}(e_i, P(\mu))\perp\epsilon_i$, instead of $P(\mu)\perp\epsilon_i$; there are cases in which a student can influence cutoffs but not her set of feasible schools. The following lemma gives an example of such a case.

Lemma 4. *When every school ranks students in the same way, or $e_{i,s} = \bar{e}_i$ for all s and i, Assumption ST2 is always satisfied.*

We skip the formal proof but the outline is as follows. In this case, DA is equivalent to serial dictatorship in which students choose the remaining schools one by one in the order determined by their priority indices. There will be only one stable matching for each realization of student types. Moreover, the set of feasible schools for student i is determined by the students with higher priority indices. Because preferences are independent across students by assumption, we have $\mathbb{1}(e_{i,s} < P_s(\mu))\perp\epsilon_i$, or $\mathcal{S}(e_i, P(\mu))\perp\epsilon_i$.

It should be noted that $\epsilon_i \not\perp P(\mu)$ even in this case. For example, when i chooses s among the feasible schools, the cutoff of s will possibly increase; similarly, i may decrease s's cutoff by choosing a different school. However, we always have $\mathbb{1}(e_{i,s} < P_s(\mu))\perp\epsilon_i$, because s will remain feasible to i either way.[17]

Building on Lemma 3 and Assumptions ST1 and ST2, we can write the probability that student i is matched with s, or, in other words, chooses s conditional on $\mathcal{S}(e_i, P(\mu))$,

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[17]We provide an example in which $\mathbb{1}(e_{i,s} < P_s(\mu))\not\perp\epsilon_i$ for some i. Let $I = 3$ and $S = 3$, each with one seat. Students have the same preferences, $(u_{i,1}, u_{i,2}, u_{i,3}) = (0.9, 0.6, 0.3)$ for $i \in \{1, 2, 3\}$; the priority index vectors $(e_{i,1}, e_{i,2}, e_{i,3})$ are $(0.8, 0.5, 0.8)$ for $i = 1$, $(0.5, 0.8, 0.3)$ for $i = 2$, and $(0.3, 0.3, 0.5)$ for $i = 3$.

Suppose students are strictly truth-telling. That is, everyone ranks every school truthfully, and therefore the matching outcome is stable. The cutoffs are $P = (0.8, 0.8, 0.5)$, which leads to $\mathcal{S}(e_1, P) = \{1, 3\}$. However, if $(u_{i,1}, u_{i,2}, u_{i,3}) = (0.6, 0.9, 0.3)$ for $i = 1$, then $P' = (0.5, 0.5, 0.5)$ and $\mathcal{S}(e_1, P') = \{1, 2, 3\}$. Therefore, for $i = 1$, $\mathbb{1}(e_{1,s} < P_s(\mu))\not\perp\epsilon_1$.

In this example, the endogeneity of the set of feasible schools occurs for students who are close to the observed cutoffs. We conjecture that the fraction of such students can be small in large markets.

as follows:

$$\Pr \left(s = \arg\max_{s \in \mathcal{S}} \overline{u}_{i,s} \mid Z_i, e_i, \mathcal{S}(e_i, P(\mu)); \beta \right)$$

$$= \Pr \left(s = \mu(u_i, e_i) = \arg\max_{s \in \mathcal{S}(e_i, P(\mu))} u_{i,s} \mid Z_i, e_i, \mathcal{S}(e_i, P(\mu)); \beta \right),$$

in which s is a given school and $\arg\max_{s \in \mathcal{S}(e_i, P(\mu))} u_{i,s}$ or $\mu(u_i, e_i)$ are random variables depending on the unobserved ϵ_i. Given the parametric assumptions on utility functions, the corresponding (conditional) log-likelihood function is:

$$\ln L_{ST} \left(\beta \mid \mathbf{Z}, \mathbf{e}, \mathcal{S}(e_i, P(\mu)) \right) = \sum_{i=1}^{I} \sum_{s=1}^{S} V_{i,s} \times \mathbb{1}(\mu(u_i, e_i) = s) - \sum_{i=1}^{I} \ln \left(\sum_{s' \in \mathcal{S}(e_i, P(\mu))} \exp(V_{i,s'}) \right).$$

(4)

The MLE estimator under the stability assumption is denoted by $\hat{\beta}_{ST}$. Similarly, a GMM estimator can also be applied, as detailed in Section 2.5.

Identification. The above discussion transforms the matching game into a discrete choice model.[18] Therefore, the (nonparametric) identification arguments for discrete choice models still apply (Matzkin, 1993).[19]

An important feature in the stability-based estimation is that students face *personalized* choice sets. In principle, as long as the choice sets are determined exogenously, which is true in our model, the identification goes through.

Another concern is that a student's priority index may enter her utility functions directly, when, for example, priority indices are determined by test score or student ability. In this case, the stability assumption does not reveal information about low-scoring students' preferences over popular schools, because such schools are often infeasible to them. This may lead to a failure of identifying how test scores determine student preferences.

This problem can be mitigated if we have another measure of student ability, which is

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[18]A simplification is that we ignore the restrictions implied by the cutoffs $P(\mu)$, which may lead to efficiency loss in estimation. That is, even when the sets of feasible schools are exogenous to every single student's preferences, $P(\mu)$ is endogenously determined by the model's parameters. However, the additional information in these restrictions may be negligible, given we use the information on the whole matching already. An earlier version of the paper relaxes this assumption and uses the restrictions implied by the cutoffs. Our estimation results from simulated data and school choice data from Paris show that using the cutoff restrictions makes a negligible difference in the estimation results.

[19]Moreover, as we shall see in Section 2.4, WTT implies additional restrictions beyond those from stability. Therefore, student preferences are also (non-parametrically) identified under the WTT assumption.

the case in our empirical exercise. As an exclusion restriction, we assume that conditional on student ability, student priority indices do not determine preferences and only affect school feasibility. If, additionally, student priority indices have full support (i.e., can take any possible value) at each given level of student ability, we will observe low-ability students having all schools feasible. This will restore nonparametric identification in discrete choice models as in Matzkin (1993).

Relative to WTT or STT, the stability assumption uses unambiguously less information from the data. Both WTT and STT utilize all information implied by the submitted ROLs, while stability only implies restrictions on matching outcome. Naturally, one may be concerned that taking the approach based on stability leads to a substantial loss of identification power; in particular, we may lose identification of substitution patterns when we allow for more flexible random utility models (Berry, Levinsohn and Pakes, 2004; Abdulkadiroğlu et al., Forthcoming). Indeed, as we shall see in our Monte Carlo simulations and analysis of the data on school choice in Paris, there is a clear bias-variance tradeoff: stability tolerates non-truth-telling behavior at the cost of yielding less precise estimates.

Identification and Estimation with Asymptotic Stability. When taking the above results to real-life data, one may be concerned that the observed matching is not exactly stable. Indeed, our theoretical results only prove *asymptotic* stability. This raises the question of whether MLE and GMM are still consistent, and the answer relies on the convergence rates of the estimators and the matching outcome. We bound the speed of convergence to stability in Appendix A.2.3. For any constant $0 < \eta < 1$, the probability that the fraction of students who can form a blocking pair in $F^{(I)}$ is more than η converges to zero exponentially (Proposition A2). This implies that the MLE and GMM estimators are consistent, as they have a convergence rate of root-I when stability is exactly satisfied (Newey and McFadden, 1994).

2.4 Testing Truth-Telling against Stability

Having two distinct estimators, $\hat{\beta}_{TT}$ and $\hat{\beta}_{ST}$, for the parameters of the random utility model makes it possible to test the truth-telling assumption against stability. We now maintain the assumption of identification given stability; we shall see shortly that WTT

thus provides over-identifying restrictions and can be tested. In practice, one should check that the conditions for identification (for example, those in Matzkin, 1993) are satisfied before conducting the test.

As summarized in Proposition 3, if every student is WTT and is assigned to a school, the matching outcome is stable. Stability, however, does not imply that students are WTT and is therefore a less restrictive assumption.

To see the additional restrictions from WTT, let us consider student i who submits a K-choice list L and is matched with a school s. Therefore, s must be ranked in L. WTT implies the following conditions on the choice probability:

$$\Pr\left(\sigma^W(u_i, e_i) = L \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K\right)$$

$$= \Pr\left(u_{i,l^1} > \cdots > u_{i,l^K} > u_{i,s'}, \forall s' \in \mathcal{S}\backslash L \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K; s = \underset{s \in \mathcal{S}(e_i, P(\mu))}{\arg\max}\ u_{i,s}\right)$$

$$\times \Pr\left(s = \mu(u_i, e_i) = \underset{s \in \mathcal{S}(e_i, P(\mu))}{\arg\max}\ u_{i,s} \mid Z_i; \beta; \mathcal{S}(e_i, P(\mu))\right) \tag{5}$$

This equality uses the fact that the event, $\left(u_{i,l^1} > \cdots > u_{i,l^{K_i}} > u_{i,s'}\forall s' \in \mathcal{S}\backslash L\right)$, implies $\left(s = \arg\max_{s \in \mathcal{S}(e_i, P(\mu))} u_{i,s}\right)$ but not the reverse.[20] This is because i's feasible schools are either ranked below s in L or are omitted from L; in either case, WTT requires that s is preferred to any other feasible school. Therefore, the first conditional probability on the right-hand side of the equality cannot always be one. As the restrictions implied by stability are just $\Pr\left(s = \mu(u_i, e_i) = \arg\max_{s \in \mathcal{S}(e_i, P(\mu))} u_{i,s} \mid Z_i; \beta; \mathcal{S}(e_i, P(\mu))\right)$, the additional restrictions from WTT are summarized in the first term. When the model is identified under stability, Equation (5) summarizes the over-identifying restrictions.

Hausman Test. Our estimator $\hat{\beta}_{TT}$ uses all the restrictions implied by WTT. Therefore, under the null hypothesis that students are WTT, both estimators $\hat{\beta}_{TT}$ and $\hat{\beta}_{ST}$ are consistent but only $\hat{\beta}_{TT}$ is asymptotically efficient. Under the alternative that the matching outcome is stable but students are not WTT, only $\hat{\beta}_{ST}$ is consistent.

In this setting, the general specification test developed by Hausman (1978) can be

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[20]We also make use of the exogeneity of the set of feasible schools (Assumption ST2) and the exogeneity of the length of submitted ROL (Assumption WTT2). Therefore,

$$\Pr\left(s = \mu(u_i, e_i) = \underset{s \in \mathcal{S}(e_i, P(\mu))}{\arg\max}\ u_{i,s} \mid Z_i; \beta; \mathcal{S}(e_i, P(\mu))\right) = \Pr\left(s = \underset{s \in \mathcal{S}(e_i, P(\mu))}{\arg\max}\ u_{i,s} \mid Z_i; \beta; |\sigma^W(u_i, e_i)| = K\right).$$

applied by computing the following test statistic:

$$T_H = (\hat{\beta}_{ST} - \hat{\beta}_{TT})'(\hat{\mathbf{V}}_{ST} - \hat{\mathbf{V}}_{TT})^{-1}(\hat{\beta}_{ST} - \hat{\beta}_{TT}),$$

where $(\hat{\mathbf{V}}_{ST} - \hat{\mathbf{V}}_{TT})^{-1}$ is the inverse of the difference between the asymptotic covariance matrices of $\hat{\beta}_{ST}$ and $\hat{\beta}_{TT}$. Under the null hypothesis, $T_H \sim \chi^2(d_\beta)$, where d_β is the dimension of β. If the model is correctly specified and the matching is stable, the rejection of the null hypothesis implies that (weak) truth-telling is violated in the data.

Testing Over-identifying Restrictions. The key assumption in the above Hausman test is that we have a consistent and efficient estimator, $\hat{\beta}_{TT}$. When relying on MLE or GMM, this may require parametric assumptions to achieve efficiency. An alternative is to construct a test for over-identifying restrictions (Hansen, 1982), which is made feasible because of the nesting structure of WTT and stability due to Proposition 3. Instead of requiring $\hat{\beta}_{TT}$ to be (completely) efficient, the test for over-identifying restrictions only requires that $\hat{\beta}_{TT}$ utilizes more restrictions than $\hat{\beta}_{ST}$. Based on Equation (5), we can separate out the additional restrictions and test whether they are satisfied based on the test proposed by Hansen (1982).

2.5 Undominated Strategies and Stability

2.5.1 Undominated Strategies without Assuming Stability

The estimation method described in Section 2.3 is only valid when the matching outcome is stable. However, as we have shown theoretically, stability can fail. Without stability, one may consider the undominated-strategy assumption. Under the rationality assumption that students play undominated strategies, observed ROLs are students' true partial preference orders in the context of the student-proposing DA. That is, every submitted ROL, L_i, respects student i's true preference ordering among the schools ranked in L_i.

These partial orders provide information that can be used to identify student preferences, but only partially, because the econometric structure is now incomplete (Tamer, 2003). In other words, for a student with type (u_i, e_i), the assumption of undominated strategies does not predict a unique ROL for the student. As we shall see, undominated strategies lead to a set of inequality restrictions that can be satisfied by a set of β's, instead of a unique vector of β. Therefore, we lose point identification.

Moment inequalities. Students' submitted ROLs can be used to form *conditional moment inequalities*. Without loss of generality, consider two schools s_1 and s_2. Since not everyone ranks both schools, the probability of i, who adopts the strategy $\sigma(u_i, e_i)$, ranking s_1 before s_2, i.e., $s_1 \succ_{\sigma(u_i, e_i)} s_2$, is:

$$\Pr\left(s_1 \succ_{\sigma(u_i, e_i)} s_2 \mid Z_i; \beta\right) = \Pr\left(u_{i,s_1} > u_{i,s_2} \text{ and } s_1, s_2 \in \sigma(u_i, e_i) \mid Z_i; \beta\right)$$
$$\leqslant \Pr\left(u_{i,s_1} > u_{i,s_2} \mid Z_i; \beta\right) \tag{6}$$

The first equality is because of undominated strategy, and the inequality defines a lower bound for the conditional probability of $u_{i,s_1} > u_{i,s_2}$. Similarly, one can derive an upper bound:

$$\Pr\left(u_{i,s_1} > u_{i,s_2} \mid Z_i; \beta\right) \leqslant 1 - \Pr\left(s_2 \succ_{\sigma(u_i, e_i)} s_1 \mid Z_i; \beta\right). \tag{7}$$

Inequalities (6) and (7) yield the following conditional moment inequalities:

$$\Pr(u_{i,s_1} > u_{i,s_2} \mid Z_i; \beta) - E\left[\mathbb{1}(s_1 \succ_{\sigma(u_i, e_i)} s_2) \mid Z_i; \beta\right] \geqslant 0;$$
$$1 - E\left[\mathbb{1}(s_2 \succ_{\sigma(u_i, e_i)} s_1) \mid Z_i; \beta\right] - \Pr\left(u_{i,s_1} > u_{i,s_2} \mid Z_i; \beta\right) \geqslant 0.$$

Similar inequalities can be computed for any pair of schools, and the above inequalities can be generalized to any n schools in $\mathcal{S}$, where $2 \leqslant n \leqslant S$. In the simulations and empirical analysis, we focus on inequalities derived with two schools. The bounds become uninformative if $n \geqslant 3$, because not many schools are simultaneously ranked by the majority of students. We further interact Z_i with the above conditional inequalities and thus obtain M_1 unconditional moment inequalities, $(m_1, \ldots, m_{M_1})$.[21]

Estimation with Moment Inequalities. To obtain consistent point estimates with moment inequalities, one can follow the approach of Andrews and Shi (2013), which is valid for both point and partial identifications. The objective function is a test statistic, $T_{MI}(\beta)$, of the Cramer-von Mises type with the modified method of moments (or sum function). This test statistic is constructed as follows from the previously defined unconditional moment inequalities:

$$T_{MI}(\beta) = \sum_{j=1}^{M_1} \left[\frac{\bar{m}_j(\beta)}{\hat{\sigma}_j(\beta)}\right]_-^2 \tag{8}$$

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[21]Such variables in Z_i are known as instruments in the method of moments literature.

where $\bar{m}_j(\beta)$ and $\hat{\sigma}_j(\beta)$ are the sample mean and standard deviation of the jth moment, $m_j(\beta)$, respectively; and the operator $[\ ]_-$ is such that $[a]_- = \min\{0, a\}$. We denote the point estimate $\hat{\beta}_{MI}$, which minimizes $T_{MI}(\beta)$, and, to construct the *marginal confidence intervals*, one can use the method in Bugni, Canay and Shi (2017). For a given coordinate β_k of β, the authors provide a test for the null hypothesis $H_0 : \beta_k = \beta_0$, for any given $\beta_0 \in \mathbb{R}$. The confidence interval for the true value of β_k is the convex hull of all β_0's for which H_0 is not rejected.

While it may seem attractive to assume that students play undominated strategies, it should be noted that only relying on moment inequalities often leads to uninformative bounds on parameters of interest, given the current state of the art of econometric techniques. This motivates us to consider combining the inequalities with restrictions implied by stability, when there are reasons to believe that stability can be satisfied.

2.5.2 Stability and Undominated Strategies

An important advantage of the stability assumption is that it only requires data on the matching outcomes. However, as submitted ROLs are often observed, one might prefer to use the information contained in such data as well. Under the assumption that stability provides point identification of student preferences, these partial orders provide over-identifying information that can be used in combination with stability to estimate student preferences.

The potential benefits from this approach can be illustrated through a simple example. Consider a constrained/truncated DA where students are only allowed to rank up to three schools out of four. With personalized sets of feasible schools under the stability assumption, the preferences over two schools, say s_1 and s_2, are estimated mainly from the sub-sample of students who are assigned to either of these schools while having priority indices above the cutoffs of both. Yet it is possible that all students include s_1 and s_2 in their ROLs, even if these schools are not ex post feasible for some students. In such a situation, all students could be used to estimate the preference ranking of s_1 and s_2, rather than just a sub-sample. As shown below, this argument can be extended to the case where two or more schools are observed being ranked by a subset of students.

Moment equalities. To combine the above over-identifying information in ROLs with that from stability, we reformulate the likelihood function described in Equation (4) into *moment equalities*. The "choice" probability of the matched school can be rewritten as a moment condition by equating theoretical and empirical probabilities:

$$\sum_{i=1}^{I} \Pr\left(s = \arg\max_{s' \in \mathcal{S}(e_i, P)}(u_{i,s'}) \mid Z_i, P(\mu); \beta\right) - E\left(\sum_{i=1}^{I} \mathbb{1}(\mu(u_i, e_i) = s)\right) = 0, \forall s \in \mathcal{S},$$

where $\mathbb{1}(\mu(u_i, e_i) = s)$ is an indicator function taking the value of one if and only if $\mu(u_i, e_i) = s$. We again interact the variables in Z with the above conditions, leading to M_2 moment equalities, $(m_{M_1+1}, \ldots, m_{M_1+M_2})$.

Estimation with Moment (In)equalities. To obtain consistent point estimates with both equality and inequality moments (henceforth, moment (in)equalities), we can augment the test statistic defined in Equation (8) to incorporate the additional M_2 unconditional moment equalities:

$$T_{MEI}(\beta) = \sum_{j=1}^{M_1} \left[\frac{\bar{m}_j(\beta)}{\hat{\sigma}_j(\beta)}\right]_{-}^{2} + \sum_{j=M_1+1}^{M_1+M_2} \left[\frac{\bar{m}_j(\beta)}{\hat{\sigma}_j(\beta)}\right]^{2}. \tag{9}$$

We denote the point estimate $\hat{\beta}_{MEI}$, which minimizes $T_{MEI}(\beta)$, and we can take the same approach as in Section 2.5.1 to construct marginal confidence intervals for β.

2.6 Testing Stability against Undominated Strategies

Given identification of student preferences under stability, the moment inequalities add over-identifying information to the moment equalities implied by stability, which constitutes a test of stability, provided that students do not play dominated strategies. More precisely, if both assumptions are satisfied, the moment (in)equality model in Section 2.5 should yield a point estimate that fits the data relatively well; otherwise, there should not exist a point β that satisfies all moment (in)equalities. Formally, we follow the specification test in Bugni et al. (2015).

It should be noted that, for the above test, we maintain the undominated-strategies assumption, which might raise concerns, because students could make mistakes as documented in several high-stake real-life contexts; moreover, untrue partial preference ordering is not dominated under school-proposing DA. We revisit these issues in Section 5.2.

The discussion in Section 2.5.1 provides another test of the undominated-strategies assumption, which also relies on the non-emptyness of the identified set under the null hypothesis (Bugni et al., 2015). That is, if there is no value of β satisfying the moment inequalities, the undominated-strategies assumption is not satisfied. It should be noted that the available methods of moment (in)equalities tend to result in conservative confidence sets of parameters, which implies that this test may lack power.

3   Results from Monte Carlo Simulations

To illustrate the proposed estimation approaches and tests, we carry out Monte Carlo (MC) simulations, the details of which are consigned to Appendix C.

Bayesian Nash equilibrium of the school choice game is simulated in two distinct settings where I students compete for admission to 6 schools with per capita capacities $\{q_s\}_{s=1}^6 = \{0.1, 0.1, 0.05, 0.1, 0.3, 0.3\}$.[22] The first is the **constrained/truncated DA** where students are allowed to rank up to K schools ($K < 6$). The second setting, which we refer to as **DA with cost**, allows students to rank as many schools as they wish but imposes a constant marginal cost c per additional school in the list after the first choice.

Student preferences over schools are generated using a parsimonious version of the random utility model in Section 2.1:

$$u_{i,s} = \alpha_s - d_{i,s} + 3(a_i \cdot \bar{a}_s) + \epsilon_{i,s}, \tag{10}$$

where α_s is school s's fixed effects; $d_{i,s}$ is the walking distance from student i's residence to school s; a_i is student i' ability; $\bar{a}_s$ is school s's quality; and $\epsilon_{i,s}$ is an error term that is drawn from the type-I extreme value distribution. Student priority indices are constructed such that (i) student i's priority index at each school is correlated with her ability a_i with a correlation coefficient of 0.7; (ii) i's priority indices at any two schools s and s' are also correlated with correlation coefficient 0.7.

Several lessons can be drawn from these simulations. The first key result is that in both settings, the distribution of school cutoffs is close to jointly normal and degenerates as the number of seats and the number of students I increase proportionally while holding

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[22]The details on solving equilibrium are described in Appendix C.2. In general, there are multiple equilibria. The results presented in the paper are from the equilibrium that is found by our computational algorithm.
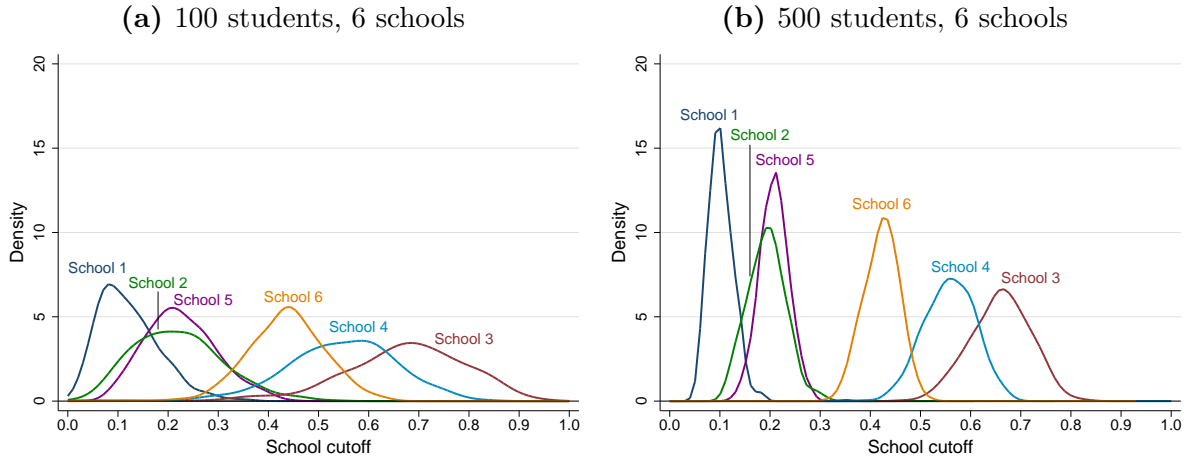
(a) 100 students, 6 schools   **(b)** 500 students, 6 schools

Figure 1: Monte Carlo Simulations: Impact of Economy Size on the Equilibrium Distribution of Cutoffs (Constrained/Truncated DA)

Notes: This figure shows the marginal distribution of school cutoffs in equilibrium under the constrained/truncated DA (ranking 4 out of 6 schools) when varying the number of students, I, who compete for admission to 6 schools with a total enrollment capacity of $I \times 0.95$ seats. Using 500 simulated samples, the line fits are from a Gaussian kernel with optimal bandwidth using MATLAB's ksdensity command. See Appendix C for details on the Monte Carlo simulations.

constant the number of schools (see Figure 1); the matching outcome is "almost" stable (i.e., almost every student is assigned to her favorite feasible school) even in moderately-sized economies. By contrast, WTT is often violated among the majority of the students, even when they can rank 5 out of 6 schools (constrained DA) or when the cost of including an extra school is negligibly small (DA with cost).[23] When the cost of ranking more schools becomes larger, Bayesian Nash equilibrium of the game can result in all students submitting fewer than 6 schools even when they are allowed to rank all of them. Based on these results, observing that only a few students make full use of their ranking opportunities may not be viewed as a compelling argument in favor of truth-telling when the application cost is a legitimate concern.

The second important insight from the MC simulations is that the stability assumption leads to estimates much closer to the true values than WTT. Table 2 reports the results from estimating student preferences under each of the following assumptions: (i) weak truth-telling (columns 2–4); (ii) stability (columns 5–7); and (iii) stability and undominated strategies (columns 8–10). The results in Panel A are for the constrained/truncated DA where students are allowed to rank up to 4 schools. Those in Panel B are for the

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[23]Consistent with the theoretical predictions from Section 1.4.2, our simulations show that the fraction of students who are matched with their favorite feasible school decreases with the cost of ranking more schools. However, violations of stability are rare unless students face very large application costs (see Figure C4 in the Appendix).

Table 2: Monte Carlo Results (500 Students, 6 Schools, 500 Samples)

		Identifying assumptions								
		Weak Truth-telling			Stability			Stability and undominated strategies		
	True value	Mean	SD	CP	Mean	SD	CP	Mean	SD	CP
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Panel A. Constrained/Truncated DA (ranking up to 4 out of 6 schools)										
PARAMETERS										
School 2	0.50	−0.13	0.06	0.00	0.51	0.29	0.94	0.50	0.28	1.00
School 3	1.00	−2.08	0.14	0.00	1.05	0.58	0.96	1.02	0.57	1.00
School 4	1.50	−1.29	0.12	0.00	1.54	0.52	0.96	1.52	0.50	1.00
School 5	2.00	0.56	0.07	0.00	2.02	0.31	0.96	2.01	0.29	1.00
School 6	2.50	0.23	0.12	0.00	2.53	0.45	0.96	2.51	0.43	1.00
Own ability × school quality	3.00	9.40	0.64	0.00	2.97	2.29	0.96	3.05	2.26	1.00
Distance	−1.00	−0.71	0.08	0.08	−1.01	0.20	0.95	−1.01	0.20	1.00

MODEL SELECTION TESTS

Truth-Telling (H_0) vs. Stability (H_1): H_0 rejected in 100% of samples (at 0.05 significance level).

Stability (H_0) vs. Undominated strategies (H_1): H_0 rejected in 0% of samples (at 0.05 significance level).

	True value	Mean	SD	CP	Mean	SD	CP	Mean	SD	CP
Panel B. DA with application cost (constant marginal cost $c = 10^{-6}$)										
PARAMETERS										
School 2	0.50	0.41	0.09	0.88	0.51	0.29	0.94	0.49	0.28	1.00
School 3	1.00	0.57	0.16	0.23	1.05	0.58	0.96	1.00	0.53	1.00
School 4	1.50	1.17	0.15	0.37	1.54	0.52	0.96	1.49	0.48	1.00
School 5	2.00	1.74	0.11	0.32	2.02	0.30	0.96	1.99	0.29	1.00
School 6	2.50	2.24	0.14	0.50	2.54	0.45	0.96	2.48	0.41	1.00
Own ability × school quality	3.00	2.19	0.72	0.77	2.96	2.29	0.96	3.16	2.29	1.00
Distance	−1.00	−0.93	0.09	0.88	−1.01	0.20	0.95	−1.00	0.30	1.00

MODEL SELECTION TESTS

Truth-Telling (H_0) vs. Stability (H_1): H_0 rejected in 37% of samples (at 0.05 significance level).

Stability (H_0) vs. Undominated strategies (H_1): H_0 rejected in 0% of samples (at 0.05 significance level).

Notes: This table reports Monte Carlo results from estimating students' preferences under different set of identifying assumptions: (i) weak truth-telling; (ii) stability; (iii) stability and undominated strategies. 500 Monte Carlo samples of school choice data are simulated under two data generating processes for an economy in which 500 students compete for admission into 6 schools: a constrained/truncated DA where students are allowed to rank up to 4 schools out of 6 (Panel A); an unconstrained DA where students can rank as many schools as they wish, but incur a constant marginal cost $c = 10^{-6}$ for including an extra school in their ROL beyond the first choice (Panel B). Under assumptions (i) and (ii), student preferences are estimated using maximum likelihood estimation. Under assumption (iii), they are estimated using Andrews and Shi (2013)'s method of moment (in)equalities. Column 1 reports the true values of the parameters. The mean and standard deviation (SD) of point estimates across the Monte Carlo samples are reported in columns 2, 5 and 8, and in columns 3, 6 and 9, respectively. Columns 4, 7 and 10 report the coverage probabilities (CP) for the 95 percent confidence intervals. The confidence intervals in models (i) and (ii) are the Wald-type confidence intervals obtained from the inverse of the Hessian matrix. The marginal confidence intervals in model (iii) are computed using the method proposed by Bugni et al. (2017). Truth-telling is tested against stability by constructing a Hausman-type test statistic from the estimates of both approaches. Stability is tested against undominated strategies by checking if the identified set of the moment (in)equality model is empty, using the test proposed by Bugni et al. (2015). See Appendix C for details on the Monte Carlo simulations.

unconstrained DA where the marginal application cost is set equal to 10^{-6}.

In both settings, the WTT-based estimator ($\hat{\beta}_{TT}$) is severely biased (see column 2). Particularly in Panel A, we note that low-ability students' valuation of the most popular schools tends to be underestimated (e.g., School 6), because such schools are more likely to be omitted from low-priority students' ROLs due to their low admission probabilities. This bias is also present among small schools (e.g., Schools 3 and 4), which are often left out of ROLs because their cutoffs tend to be higher than those of equally desirable but

larger schools.

By contrast, the average of the stability-based estimates ($\hat{\beta}_{ST}$) is reasonably close to the true parameter values. Its standard deviations, however, are larger than those obtained under WTT. This efficiency loss is a direct consequence of restricting the choice sets to feasible schools and of ignoring the information content of ROLs.[24] Under the assumption that the matching outcome is stable, the Hausman test presented in Section 2.4 strongly rejects truth-telling in our simulations.

The estimator from the moment (in)equalities approach ($\hat{\beta}_{MEI}$), which incorporates the over-identifying information contained in students' ROLs, is also consistent (column 8). Compared with using stability alone, the inclusion of moment inequalities is informative to the extent that these inequalities define sufficiently tight bounds for the probability of a preference ordering over some pairs of schools. This is more likely when the constraint on the length of ROLs is mild and/or when the cost of ranking an extra school is low, since these situations increase the chances of observing subsets of schools being ranked by a large fraction of students. A limitation of this approach, however, is that the currently available methods for conducting inference based on moment (in)equality models are typically conservative. As a result, the 95 percent marginal confidence intervals based on moment (in)equalities cover the true values too often (coverage probability, or CP, is close to one). Besides, the intervals tend to be wider than those obtained using moment equalities alone, although the point estimates are closer to the true parameter values.

4   School Choice in Paris

Since 2008, the Paris Education Authority (*Rectorat de Paris*) assigns students to public high schools based on a version of the school-proposing DA called AFFELNET (Hiller and Tercieux, 2014). At the district level, student priority indices are not school-specific (as detailed below) and the mechanism is equivalent to a serial dictatorship.

Towards the end of the Spring term, final-year middle school students who are admitted to the upper secondary academic track (*Seconde Générale et Technologique*)[25]

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[24]We further quantify the efficiency loss in Appendix C.5 in simulations where students are strictly truth-telling. In this case, the WTT-based estimator is consistent and efficient, while the stability-based estimator is still consistent but inefficient.

[25]In the French educational system, students are tracked at the end of the final year of *collège* (equiv-

are requested to submit a ROL of up to 8 public high schools to the Paris Education Authority. Students' priority indices are determined as follows:

(i) Students' academic performance during the last year of middle school is graded on a scale of 400 to 600 points.

(ii) Paris is divided into four districts. Students receive a "district" bonus of 600 points for each school in their list that is located in their home district, and no bonus for the others. Therefore, students applying to a school in their home district have full priority over out-of-district applicants to the same school.

(iii) Low-income students are awarded an additional bonus of 300 points. As a result, these students are given full priority over all other students from the same district.[26]

The DA algorithm is run at the end of the academic year to determine school assignment for the following academic year. Unassigned students can participate in a supplementary round of admissions by submitting a new ROL of schools among those with remaining seats, the assignment mechanism being the same as for the main round.

Note that the mechanism would be strategy-proof if there were no constraints on the length of ROLs. Nonetheless, under the current mechanism, it is still a dominated strategy to submit a ROL that is not a partial order of true preferences (Proposition 2).

4.1 Data

For our empirical analysis, we use data from Paris' Southern District (*Sud*) and study the choices of 1,590 within-district middle school students who applied for admission to the district's 11 public high schools for the academic year starting in 2013. Owing to the 600-point "district" bonus, this district is essentially an independent market.[27]

Along with information on socio-demographic characteristics and home addresses, our data contain all the relevant variables to replicate the matching outcome, including the schools' capacities, the students' ROLs of schools, and their priority indices (converted into percentiles between 0 and 1). Individual examination results for the *Diplôme national*

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alent to middle school), at the age of 15, into vocational or academic upper secondary education.

[26]The low-income status is conditional on a student applying for and being granted the means-tested low-income financial aid in the last year of middle school. A family with two children would be eligible for this financial aid in 2013 if its taxable income was below 17,155 euros. The aid ranges from 135 to 665 euros per year.

[27]Out-of-district applicants could affect the availability of school seats in the supplementary round, but this is of little concern since, in the district, only 22 students out of 1,590 were unassigned at the end of the main round (for the comparison of assigned and unassigned students, see Appendix Table E4).

du brevet (DNB)—a national exam that all students take at the end of middle school—are used to construct different measures of academic ability (math, French, and composite score), which are normalized as percentiles between 0 and 1.[28] Table 3 reports students' characteristics, choices, and admission outcomes. Almost half of the students are of high socioeconomic status (SES), while 15 percent receive the low-income bonus. 99 percent are assigned to a within-district school in the main admission round, but only half obtain their first choice. Compared to their assigned schools, applicants' first-choice schools tend to have students of higher ability and more privileged background.

More detailed summary statistics for the 11 academic-track high schools in the district are presented in Table 4. Columns 1–4 show that there is a high degree of stratification among schools, both in terms of the average ability of students enrolled in 2012 and of their social background (measured by the fraction of high SES students). Columns 5–8 report a number of outcomes from the 2013 round of assignment. The district's total capacity (1,692 seats) is unevenly distributed across schools: the smallest school has 62 seats while the largest has 251. School cutoffs in 2013 are strongly correlated with the different measures of school quality, albeit not perfectly. The last column shows the fraction of ROLs in which each school appears. The least popular three schools are ranked by less than 24 percent of students, and two of them remain under-subscribed (Schools 1 and 3) and thus have cutoffs equal to zero. Consistent with our Monte Carlo results, we note that smaller schools are omitted by more students, even if they are of high quality. Likewise, a sizeable fraction of students (20 percent) do not rank the best-performing school (School 11) in their lists.

More detailed statistics on enrollment outcomes (see Appendix Table E4) indicate a high level of compliance with the matching outcome, as 96 percent of assigned students attend the school they were matched with. Very few students (around 1 percent) attend a public high school different from their assignment school, and less than 3 percent opt out from the public school system to enroll in a private school.

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[28]See Appendix B for a detailed description of the data sources. A map of the district is provided in Appendix Figure E5.

Table 3: High School Applicants in the Southern District of Paris: Summary Statistics

Variable	Mean	SD	Min	Max	N
Panel A. Student characteristics					
Age	15.0	0.4	13	17	1,590
Female	0.51	0.50	0	1	1,590
French score	0.56	0.25	0.00	1.00	1,590
Math score	0.54	0.24	0.01	1.00	1,590
Composite score	0.55	0.21	0.02	0.99	1,590
High SES	0.48	0.50	0	1	1,590
With low-income bonus	0.15	0.36	0	1	1,590
Panel B. Choices and outcomes					
Number of choices within district	6.6	1.3	1	8	1,590
Assigned to a within-district school	0.99	0.12	0	1	1,590
Assigned to first choice school	0.56	0.50	0	1	1,590
Panel C. Attributes of first choice school					
Distance (km)	1.52	0.93	0.01	6.94	1,590
Mean student French score	0.62	0.11	0.32	0.75	1,590
Mean student math score	0.61	0.13	0.27	0.78	1,590
Mean student composite score	0.61	0.12	0.31	0.77	1,590
Fraction high SES in school	0.53	0.15	0.15	0.71	1,590
Panel D. Attributes of assigned school					
Distance (km)	1.55	0.89	0.06	6.94	1,568
Mean student French score	0.56	0.12	0.32	0.75	1,568
Mean student math score	0.54	0.14	0.27	0.78	1,568
Mean student composite score	0.55	0.13	0.31	0.77	1,568
Fraction high SES in school	0.48	0.15	0.15	0.71	1,568

Notes: This table provides summary statistics on the choices of middle school students from the Southern District of Paris who applied for admission to the district's 11 public high schools for the academic year starting in 2013, based on administrative data from the Paris Education Authority (*Rectorat de Paris*). All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in [0, 1]. The composite score is the average of the scores in French and math. The correlation coefficient between French and math scores is 0.50. School attributes, except distance, are measured by the average characteristics of students enrolled in each school in the previous year (2012).

Table 4: High Schools in the Southern District of Paris: Summary Statistics

School	School attributes (2012)				Assignment outcomes (2013)			
	Mean French score (1)	Mean math score (2)	Mean composite score (3)	Fraction high SES students (4)	Capacity (5)	Count (6)	Admission cutoffs (7)	Fraction ROLs ranking it (8)
School 1	0.32	0.31	0.31	0.15	72	19	0.000	0.22
School 2	0.36	0.27	0.32	0.17	62	62	0.015	0.23
School 3	0.37	0.34	0.35	0.16	67	36	0.000	0.14
School 4	0.44	0.35	0.39	0.46	140	140	0.001	0.59
School 5	0.47	0.44	0.46	0.47	240	240	0.042	0.83
School 6	0.47	0.46	0.46	0.32	171	171	0.069	0.71
School 7	0.58	0.54	0.56	0.56	251	251	0.373	0.91
School 8	0.58	0.66	0.62	0.30	91	91	0.239	0.39
School 9	0.65	0.62	0.63	0.66	148	148	0.563	0.83
School 10	0.68	0.66	0.67	0.49	237	237	0.505	0.92
School 11	0.75	0.78	0.77	0.71	173	173	0.705	0.80

Notes: This tables provides summary statistics on the attributes of high schools in the Southern District of Paris and on the outcomes of the 2013 assignment round, based on administrative data from the Paris Education Authority (*Rectorat de Paris*). School attributes in 2012 are measured by the average characteristics of the schools' enrolled students in 2012–2013. All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in [0, 1]. The composite score is the average of the scores in French and math. The correlation coefficient between school-average math and French scores is 0.97.

4.2 Evaluating Identifying Assumptions: Reduced-Form Evidence

We analyze applicants' ranking behavior by investigating whether students are less likely to rank schools at which they might expect low admission probabilities. As illustrated in the "skipping the impossible" Example 1 (Section 1.2), this type of behavior would be inconsistent with the (weakly) truth-telling assumption.
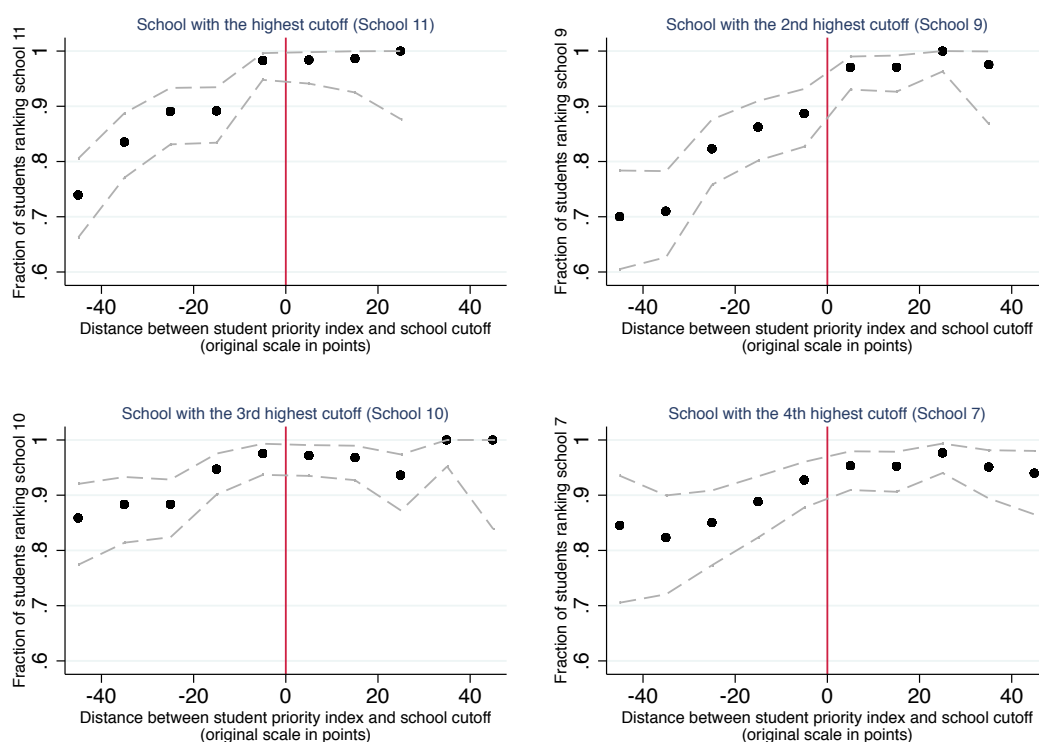

Figure 2: Fraction of Students Ranking Each of the Four Most Selective Schools in the Southern District of Paris, by Distance to School Cutoff

Notes: The results are calculated with administrative data from the Paris Education Authority (*Rectorat de Paris*) for students who applied to the 11 high schools of Paris's Southern District for the academic year starting in 2013. The figure shows the ranking behavior of students as a function of the distance (using the original scale in points) between each school's cutoff and students' priority index. The sample is restricted to students with a priority index within -50 and $+50$ points of the cutoffs and, for each school, students are grouped into distance-to-cutoff bins of 10-point width. Bins with less than 10 observations are excluded. Each point represents the fraction of students in a given bin who rank the school in their list. The dotted lines show the 95 percent confidence interval. Low-income students are not included in the samples because the low-income bonus of 300 points places them well above the cutoffs.

Figure 2 focuses on the district's four most selective schools (as measured by their cutoffs). For each school separately, we plot the fraction of students who rank it in their ROL as a function of their distance to the school cutoff, which is computed as the difference (using the original scale in points) between the student's priority index and

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the cutoff.[29] The observed pattern indicates that almost all students with a priority index above a school's cutoff include that school in their list, whereas the fraction of students ranking the school decreases rapidly when the priority index falls below the cutoff. Irrespective of strategic considerations, one might expect high priority students to have a stronger preference for the most selective schools—since priorities are positively correlated with academic performance—and hence to rank them more often. However, the kink around the cutoffs is consistent with some students omitting the most selective schools from their list because they expect low admission probabilities. Such a pattern is difficult to be reconciled with the truth-telling assumption.
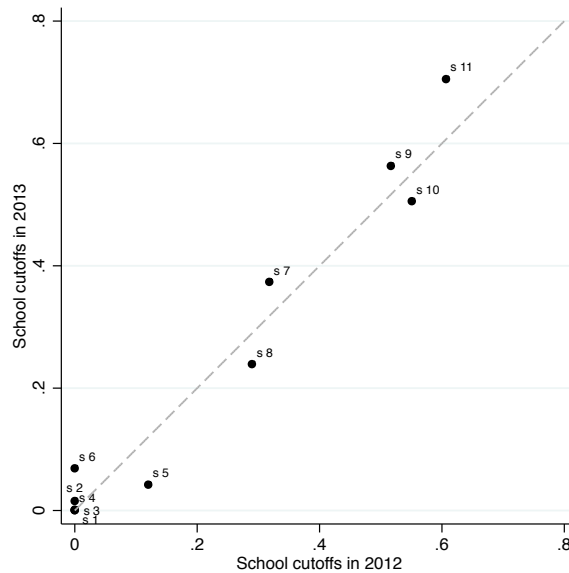

Figure 3: School Cutoffs in 2012 and 2013

Notes: The results are calculated with administrative data from the Paris Education Authority (*Rectorat de Paris*). The 11 schools of the Southern district are represented by dots, with the cutoff in 2013 on the Y-axis and the cutoff in 2012 on the X-axis. The dashed line denotes the 45-degree line.

The reduced-form evidence in Figure 2 suggests that students' ranking behavior could be influenced by their expected admission probabilities. At the time of application, students know their academic grades and low-income status but not their priority ranking nor the ex post cutoffs.[30] They can, however, gather information on past year cutoffs, which may help them assess their admission probabilities. While we do not have direct

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[29]We restrict the sample to students whose score falls between -50 and $+50$ points of each school's cutoff (using the original scale). Due to the low-income bonus of 300 points, low-income students' priority indices are always well above the cutoffs. They are therefore not considered in the analysis.

[30]This uncertainty in both priority ranking and cutoffs may explain why some students find it optimal to rank multiple schools, given that the cost of ranking up to 8 choices is arguably negligible.

information on students' beliefs, Figure 3 shows that the current year (2013) cutoffs are similar those observed in the previous year (2012). This lends support to the assumption that students have some ability to predict their admission probabilities at the different schools.[31] Although not a necessary condition for the matching outcome to be stable, this feature makes the stability assumption more likely to be satisfied in the Parisian setting.

4.3 Estimation and Test Results

We assume that student i's utility from attending school s can be represented by the following random utility model:

$$u_{i,s} = \alpha_s - d_{i,s} + Z'_{i,s}\gamma + \lambda\epsilon_{i,s}, \ s = 1, \ldots, 11; \tag{11}$$

where α_s is the school fixed effect, $d_{i,s}$ is the distance to school s from i's place of residence, and $Z_{i,s}$ is a vector of student-school-specific observables. As observed heterogeneity, $Z_{i,s}$ includes two variables that capture potential non-linearities in the disutility of distance and control for potential behavioral biases towards certain schools: "closest school" is a dummy variable equal to one if s is the closest to student i among all 11 schools; "high school co-located with middle school" is another dummy that equals one if high school s and the student's middle school are co-located at the same address.[32] To account for students' heterogeneous valuation of school quality, interactions between student scores and school scores are introduced separately for French and math, as well as an interaction between own SES and the fraction of high SES students in the school. These school attributes are measured among the entering class of 2012, whereas our focus is on students applying for admission in 2013. We normalize the variables in $Z_{i,s}$ so that each school's fixed effect can be interpreted as the mean valuation, relative to School 1, of a non-high-SES student who has median scores in both French and math, whose middle school is not co-located with that high school, and for whom the high school is not the closest to her residence.

The idiosyncratic error term $\epsilon_{i,s}$ is assumed to be an i.i.d. type-I extreme value, and the variance of unobserved heterogeneity is λ^2 multiplied by the variance of $\epsilon_{i,s}$. The effect of distance is normalized to -1, and, therefore, the fixed effects and γ are all

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[31]The comparison could not be performed for earlier years due to changes in the computation of the priority index and small changes in the set of available schools.

[32]There are five such high schools in the district.

measured in terms of willingness to travel. As a usual position normalization, $\alpha_1 = 0$. We do not consider outside options because of students' almost perfect compliance with the matching outcome from the centralized mechanism.

Using the same procedures as in the Monte Carlo simulations (described in Appendix C), we obtain the results summarized in Table 5, where each column reports estimates under a given set of identifying assumptions: (i) weak truth-telling (column 1); (ii) stability (column 2); and (iii) stability with undominated strategies (column 3).[33]

The results show clearly that the WTT-based estimates (column 1) are rather different from the others. More specifically, a downward bias is apparent when a popular school is not ranked by many students, especially for Schools 8 and 11. School 8, which is omitted by 61 percent of students, is assumed under the WTT assumption to be less desirable than all the schools included in the ROL, which leads to a lower estimated fixed effect than under the stability assumption. Similarly, there is a noticeable difference in the quality estimate of School 11, which is one of the most popular schools. This under-estimation is mitigated when the model is estimated under the other two sets of assumptions.

Provided that the model is correctly specified, the Hausman test rejects truth-telling in favor of stability (p-value < 0.01); the test based on moment (in)equalities does not reject the null hypothesis that stability is consistent with undominated strategies at a 5 percent significance level.

The results show that "closest school" has no significant effect, but students significantly prefer co-located schools. Compared with low-score students, those with high French (math) scores have a stronger preference for schools with higher French (math) scores. Moreover, high SES students prefer schools that have admitted a larger fraction of high SES students in the previous year (2012).

It is worth noting that the WTT-based estimates of the covariates (Panel B) are not

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[33]For the estimates reported in column 3, we rely on the method of moment (in)equalities where inequalities are constructed as described in Section 2.5. Determined by our selection of $Z_{i,s}$, we interact French score, math score, and distances to Schools 1 and 2 with the conditional moments. Although one could use more variables, e.g., SES status and distance to other schools, they provide little additional variation. In principle, the assumption of undominated strategies alone implies partial identification (see Section 2.5.1). Because stability is not rejected by our test, we do not present results based on this approach (available upon request). It should be noted, however, that the marginal confidence intervals that we obtain for each parameter using only moment inequalities turn out to be very wide in our empirical setting, and hence are relatively uninformative. The possible reasons are that the derived bounds for the probability of a preference ordering over the different pairs of schools (55 moment inequalities) are fairly wide, and that the available methods to conduct inference based on moment inequalities are typically conservative.

Table 5: Estimation Results under Different Sets of Identifying Assumptions

	Identifying assumptions		
	Weak Truth-telling	Stability of the matching outcome	Stability and undominated strategies
	(1)	(2)	(3)
Panel A. School fixed effects			
School 2	−0.71	1.46	1.21
	[−1.17, −0.24]	[0.64, 2.28]	[0.14, 2.29]
School 3	−2.12	1.03	0.84
	[−2.66, −1.58]	[0.19, 1.86]	[−0.56, 2.01]
School 4	3.31	2.91	2.90
	[2.75, 3.86]	[2.07, 3.76]	[2.36, 3.39]
School 5	5.13	4.16	4.16
	[4.41, 5.84]	[3.22, 5.10]	[3.71, 4.49]
School 6	4.87	4.24	4.30
	[4.21, 5.54]	[3.29, 5.18]	[3.73, 4.82]
School 7	7.32	6.81	6.24
	[6.47, 8.18]	[5.65, 7.98]	[5.76, 7.28]
School 8	1.59	4.46	4.27
	[1.10, 2.08]	[3.46, 5.47]	[2.98, 5.26]
School 9	6.84	7.77	6.57
	[6.07, 7.61]	[6.55, 8.99]	[5.84, 7.26]
School 10	7.84	7.25	6.44
	[6.94, 8.75]	[6.01, 8.49]	[5.87, 7.05]
School 11	5.35	7.28	5.61
	[4.62, 6.08]	[6.06, 8.51]	[4.98, 7.33]
Panel B. Covariates			
Closest school	−0.37	−0.19	−0.15
	[−0.63, −0.11]	[−0.47, 0.10]	[−0.75, 0.57]
High school co-located with middle school	2.54	1.76	1.54
	[2.02, 3.07]	[1.19, 2.32]	[0.17, 3.12]
Student French score [×10] × school French score [×10]	0.20	0.18	0.23
	[0.16, 0.23]	[0.13, 0.24]	[0.10, 0.35]
Student math score [×10] × school math score [×10]	0.30	0.27	0.30
	[0.26, 0.34]	[0.21, 0.32]	[0.18, 0.40]
High SES × fraction high SES in school	6.79	4.92	8.12
	[5.62, 7.97]	[3.31, 6.54]	[4.18, 12.55]
Scaling parameter	3.09	1.33	1.50
	[2.79, 3.38]	[1.16, 1.50]	[1.20, 1.64]
Number of students	1,590	1,568	1,590

Notes: This table reports the estimates of the model described by Equation (11) for the Southern District of Paris, with the coefficient on distance being normalized to −1. The point estimates in columns 1 and 2 are based on maximum likelihood whereas those in column 3 are based on moment equalities and inequalities, with 95 percent confidence intervals indicated in parentheses.

Model selection tests: A Hausman test, testing (1) against (2) reject the null hypothesis that the truth-telling assumption is satisfied in favor of stability (p-value < 0.01); a test based on moment equalities and inequalities does not reject the null hypothesis that stability is consistent with undominated strategies at the 95 percent level.

markedly different from the stability-based estimates. However, one cannot conclude that the truth-telling assumption produces reasonable results, as the estimates of fixed effects have shown. Moreover, we emphasize in the next section the relatively poor fit of the WTT-based estimates to the data.

4.4   Goodness of Fit

On three sets of outcomes (cutoffs, assignment, and revealed preferences), we compare the observed values to those predicted using the estimates from Table 5. These tests show that the stability-based estimates fit the data well, as opposed to those based on WTT, whose predictions are further away from the observed outcomes (see Appendix D for computational details).
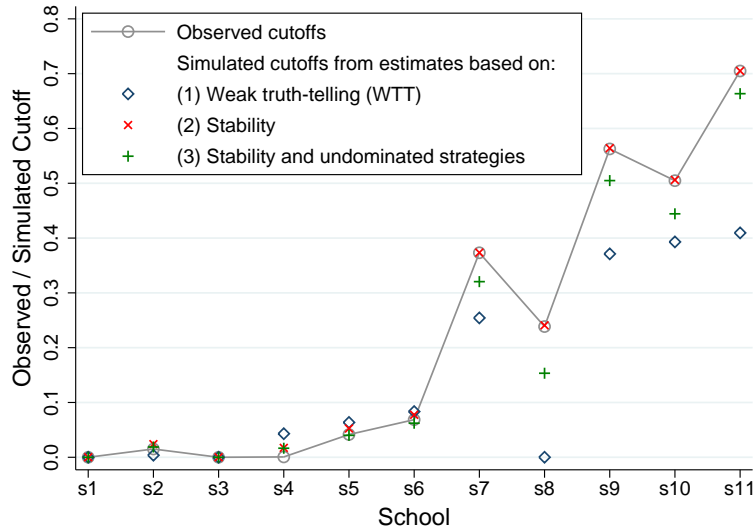

Figure 4: Goodness of Fit: Observed vs. Simulated Cutoffs

Notes: This figure compares the cutoffs observed for the 11 high schools of the Southern District of Paris in 2013, to those simulated under different sets of identifying assumptions as in Table 5. The reported values for the simulated cutoffs are averaged over 300 simulated samples. See Appendix D for details.

In particular, the results show clearly that estimates based on stability (with or without undominated strategies) predict cutoffs close to the observed ones (see Figure 4 and Appendix Table D3). By contrast, the WTT-based estimates substantially under-predict the cutoffs of the most popular schools.

Panel A of Table 6 compares students' predicted assignment to the observed one, using 300 simulated samples. The stability-based estimates estimates have between 33 and 38 percent successful prediction rates, whereas the WTT-based estimates accurately

Table 6: Goodness-of-Fit Measures Based on Different Sets of Identifying Assumptions

	Estimates from		
	Weak Truth-telling	Stability of the matching outcome	Stability and undominated strategies
	(1)	(2)	(3)
Panel A. Simulated vs. observed assignment (300 simulated samples)			
Mean predicted fraction of students assigned to observed assignment	0.220 (0.011)	0.383 (0.010)	0.326 (0.012)
Panel B. Predicted vs. observed partial preference order			
Predicting observed ordering of top two choices	0.553	0.615	0.595
Predicting observed ordering of all submitted choices	0.012	0.025	0.022

Notes: This table reports two sets of goodness-of-fit measures comparing the observed outcomes to those predicted under the different sets of identifying assumptions as in Table 5, for the high school assignment of students in the Southern District of Paris. Panel A compares students' observed assignment with their predicted assignment in 300 simulated samples. In all simulations, we vary only the utility shocks, which are kept common across columns 1–3 (see Appendix D for details). Predicted and observed assignments are compared by computing the average predicted fraction of students who are assigned to their observed assignment school, with standard deviations across the simulation samples reported in parentheses; in other words, this is the average fraction of times each student is assigned to her observed assignment in the 300 simulated samples. Panel B uses two measures to compare students' observed partial preference order (revealed by their submitted ROL) with the prediction, among students who rank at least two schools: (i) mean predicted probability of the observed ordering of students' top two choices, which is averaged across students; and (ii) mean predicted probability of the observed ordering of students' full list of choices. Due to the logit specification, those probabilities can be calculated without simulation.

predict only 22 percent of assignments.

Panel B calculates the predicted probability that students have their observed partial preference order (revealed by their submitted ROL), which can be calculated without simulation due to the logit specification. The observed ordering of students' top two choices has a mean predicted probability of between 60 and 62 percent when using the stability-based estimates, which is higher than the 55 percent achieved by the WTT-based estimates. We next consider the observed ordering of students' full list of choices. Again, the stability-based estimates outperform the WTT-based estimates, with an average predicted probability between 2.2 and 2.5 percent for the former vs. 1.2 percent for the latter. The predictive performance of the stability-based estimates along the two measures reported in Panel B is noteworthy given that the prediction is partly out of sample.[34]

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[34]In the data, 54 percent of students ranked at least one infeasible school among their top two choices (34 percent ranked exactly one infeasible school, while 20 percent ranked exactly two). The average fraction of infeasible schools among all submitted choices is 30 percent.

5   Summary and Discussion

In this section, we summarize our results. In centralized school choice and colleges admissions where students are strictly ranked by some priority index, we clarify when each approach is more appropriate for empirical analysis. We also discuss whether the results can be extended to the school-proposing DA, to the case with non-equilibrium behavior, and to settings beyond school choice and college admissions.

5.1   Choosing among the Approaches: A Summary

In the estimation of preferences based on real-life data from centralized school choice and college admissions, there are a number of practical considerations that should be taken into account. Recall that we focus on the strict-priority setting in which students are ranked based on strict priority indices that are ex ante privately known. Building on the results from our theoretical and empirical analyses, this section emphasizes some of the key market features that deserve careful examination when one decides which approach to use in a given context.

The Nesting Structure of Identifying Assumptions. Our theoretical framework makes clear that estimating preferences from school choice and college admissions data involves choosing among alternative identifying assumptions. In the strict-priority setting, the candidate assumptions follow a nesting structure, as depicted in Figure 5.

Truth-telling is the first natural candidate identifying assumption because of DA's strategy-proofness. However, as discussed in Section 2.2, strict truth-telling (students truthfully rank all schools) is the unique equilibrium of the game under DA, only if students are able to rank all schools at no cost and face sufficient uncertainty (Proposition 1 in Section 1.2). In real-life data, students seldom rank all schools, which calls for a weaker version of the truth-telling assumption. As clarified in the theoretical analysis, weak truth-telling (students truthfully rank their most preferred schools and omit some least preferred ones) does not follow directly from the DA mechanism being strategy-proof, as it requires additional restrictive assumptions such as the length of ROLs being independent of preferences.

Stability is an even weaker assumption on students' behavior, while still allowing for the identification of preferences. It states that every student is assigned to her favorite ex
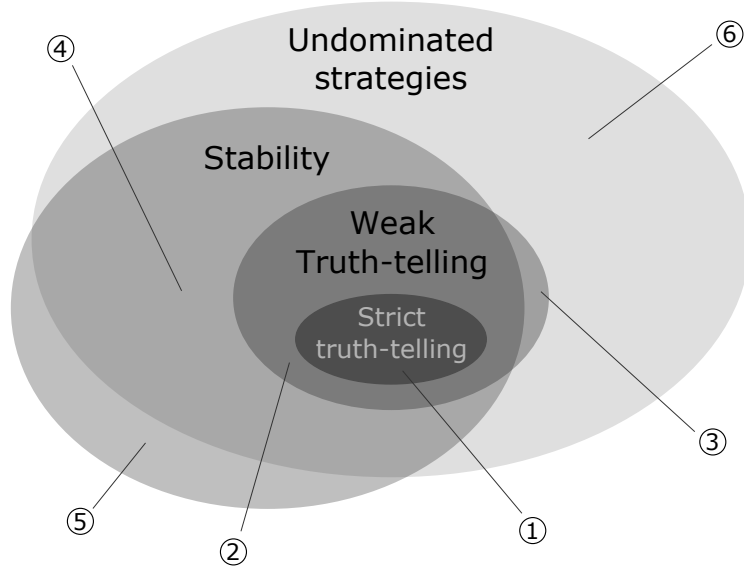
Figure 5: Nesting Structure of Identifying Assumptions

Notes: This figure shows the nesting structure of the identifying assumptions that can be used to analyze data generated by DA and its variants in the strict-priority setting. The numbered areas correspond to different combinations of identifying assumptions: (1) strict truth-telling; (2) weak truth-telling and stability; (3) weak truth-telling without stability; (4) stability and undominated strategies; (5) stability without undominated strategies; (6) undominated strategies without stability.

post feasible school, and is always satisfied when students are strictly truth-telling. Although it is not guaranteed in all Bayesian Nash equilibria, even when students are weakly truth-telling, it is asymptotically satisfied when the economy grows large (Proposition 4).

The third candidate identifying assumption for preference estimation is that students do not play dominated strategies (Proposition 2), so that submitted ROLs reveal students' partial preference orders of schools. Weak truth-telling is a special case of this more general assumption, whereas stability may hold even if some students play dominated strategies.

The Choice of Empirical Approaches and Tests. When choosing among the candidate identifying assumptions, consideration should be given to the particular features of the problem under study, as well as the available data. For each assumption, Table 7 summarizes the features making it more plausible, the required data, and the appropriate estimation methods and tests.

Truth-telling is more likely to be satisfied when students can rank as many schools as they wish at no cost, and are subject to large uncertainty about each school's exact ranking of students. Conditional on students' submitted ROLs being observed, preferences

Table 7: Summary of Empirical Approaches and Tests

Identifying assumption	What makes the assumption more plausible?	Required data	Estimation method	Testing
Weak Truth-Telling: Schools in a submitted ROL are ranked in true preference order and omitted ones are less preferred.	(1) No cost of ranking more schools, e.g., no restriction on the length of submitted ROL and choice set not being too large. (2) At the time of application, each student knows her own priority index but not others', and the distribution of priority indices has a large variance.	Submitted ROLs	MLE/GMM	H_0: Both weak truth-telling and stability are satisfied; H_1: Only stability is satisfied.
Stability: Every student is assigned to her favorite feasible school.	Stability is satisfied if truth-telling holds and (almost) everyone is assigned. Otherwise, it is more likely to be true when (1) market is large (many students, big schools); (2) students are less constrained when applying to more schools; (3) students face limited uncertainty about how schools rank them at the time of application; (4) students know more about others' preferences; or (5) cutoffs are easy to predict.	Matching outcome, school capacities, priority indices	MLE/GMM	This can be tested, e.g., using the Hausman (1978) or Hansen (1982) tests, under the condition that (almost) everyone is assigned.
Undominated strategies: Submitted ROLs are true partial preference orders.	(1) No "safety school" so that "irrelevancy at the bottom" of one's ROL is less likely. (2) No "impossible school" so that students do not rank impossible school arbitrarily.	Submitted ROLs	Moment inequalities (partial identification)	H_0: Both stability and undominated strategies are satisfied H_1: Only undominated strategies is satisfied
Stability and Undominated strategies: Every student is assigned to her favorite feasible school, and submitted ROLs are true partial preference orders.	See the conditions laid out separately for stability and undominated strategies	Submitted ROLs, matching outcome, school capacities, priority indices	Moment equalities + moment inequalities	This can be tested, e.g., with the approach in Bugni et al. (2015), assuming that students never play dominated strategies.

Notes: This table describes the empirical approaches and tests that can be used to analyze data generated by DA and its variants in the strict-priority setting.

can be estimated using either MLE or GMM. The choice between weak truth-telling and strict truth-telling depends on whether students rank all schools (Section 2.2) and on the importance of outside options (Appendix A.3).

When students face some cost of ranking more schools (e.g., if the length of ROLs is restricted), stability can be a more plausible assumption than truth-telling. This assumption is more likely to hold when the market is larger (i.e., many students and many seats per school), when students are less constrained in applying to multiple schools (e.g., longer ROLs), when they are less uncertain about each school's ranking of all students at the time of application, the more they know about others' preferences, or when it is easier for them to predict school cutoffs (Proposition 5). Our Monte Carlo simulations additionally provide numerical evidence suggesting that stability is a plausible assumption even when students face non-negligible application costs (Appendix C.3).

Estimating preferences under the stability assumption requires knowledge of the matching outcome, the school capacities and the students' priority indices, but has the advantage of not requiring data on submitted ROLs.

Weak truth-telling does not always imply stability, but it does if (almost) all students are assigned to a school (Proposition 3). In this case, weak truth-telling can be tested against stability using the Hausman (1978) and Hansen (1982) tests.

If it is believed that neither truth-telling nor stability assumptions are likely to be satisfied in the context being studied, preferences can be still be partially identified under the assumption that students do not play dominated strategies. This assumption will be more plausible if no school is either "safe" or "impossible" for students, making it less likely that students rank some schools in an arbitrary manner. Submitted ROLs can then be used to form conditional moment inequalities that allow for the partial identification of preferences.

When the conditions for both stability and undominated-strategies assumptions are jointly satisfied, the moment inequalities from the latter assumption provide over-identifying information that can be used in combination with the stability assumption to estimate preferences based on all of the available data (ROLs, matching outcome, school capacities, and priority indices). Additionally, the stability assumption can be tested against the undominated-strategies assumption using the specification test in Bugni et al. (2015).

5.2 Discussion and Extension

The School-Proposing DA. Our main results can be extended to the school-proposing DA, which is also commonly used in practice (see Table 1). Under this mechanism, schools "propose" to students following the order of student priority indices. Proposition 2 no longer holds; that is, students might have incentives not to report a true partial preference order (Haeringer and Klijn, 2009). Nonetheless, the asymptotic stability result (Proposition 4) still holds, as its proof does not rely on Proposition 2. Indeed, it is known that the matching outcome can be stable in Nash equilibrium under the school-proposing DA (Haeringer and Klijn, 2009).

To summarize, if the matching market under the school-proposing DA has features making the matching outcome stable (see Table 7), we can formulate identification and estimation of student preferences based on stability. However, the truth-telling assumption no longer has theoretical support, as the school-proposing DA is not strategy-proof for students (Roth, 1982). Additionally, the approach relying on undominated strategies does not apply, since there are no dominated strategies under this mechanism (Haeringer

and Klijn, 2009).

Non-Equilibrium Strategies. We have thus far focused on the case in which everyone plays an equilibrium strategy with a common prior, an assumption that one may want to relax. More realistically, some students could have different information and make mistakes when strategizing.[35]

Indeed, a growing number of studies find that strategic mistakes are not uncommon even in strategy-proof environments. Laboratory experiments have shown that a significant fraction of subjects do not report their preferences truthfully in strategy-proof mechanisms (Chen and Sönmez, 2006). More relevantly, mistakes occur in high-stake real-world contexts, e.g., the admissions to Israeli graduate programs in psychology (Hassidim, Romm and Shorrer, 2016), the medical resident match in the U.S. (Rees-Jones, 2016), and the Australian college admissions (Artemov et al., 2017). Without estimating preferences, these studies show that a non-negligible fraction of participants make unambiguous mistakes in their ROLs.

However, the vast majority of these mistakes are not payoff relevant. In other words, although some students play dominated strategies, the matching outcome is still close to stable—which corresponds to area ⑤ in Figure 5. Based on these observations, the results in Artemov et al. (2017) imply that, as identifying restrictions, assuming stability can be more robust and more plausible than the assumption of undominated strategies.

Beyond School Choice and College Admissions. Although the analysis has focused on school choice and college admissions, our results can apply to certain assignment/matching procedures based on DA. Let us call agents on the two sides "applicants" and "recruiters," respectively. The key requirement is that at the time of applying, applicants have sufficiently precise information on how recruiters rank them and that researchers have information on how recruiters exactly rank applicants.[36] Examples include teacher assignment to public schools in France (Terrier, 2014; Combe, Tercieux and Terrier, 2016) and the Scottish Foundation Allocation Scheme matching medical school

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[35]Under some non-DA mechanisms, Calsamiglia et al. (2014) and He (2015) consider the possibility that students make mistakes when submitting ROLs.

[36]When researchers have no information on how either side ranks the other, we are in the classic setting of two-sided matching, where additional assumptions are often needed for identification and estimation (Chiappori and Salanié, 2016).

graduates to training programs (Irving, 2011), which are both centralized. The estimation approaches discussed in Section 2 could be implemented in these settings.

6   Conclusion

We present novel approaches to estimating student preferences with school choice or college admissions data generated by the popular Deferred Acceptance mechanism when applicants are ranked strictly by some ex-ante known priority index. We provide theoretical and empirical evidence showing that, in this commonly observed setting, it is rather restrictive to assume that students truthfully rank schools when applying for admission. Instead, stability (or justified-envy-freeness) of the matching outcome provides rich identifying information, while being a weaker assumption on students' behavior. Assuming that students do not play dominated strategies, we also discuss methods with moment inequalities, which can be useful whether stability is satisfied or not. A series of tests are proposed to guide the selection of the appropriate approach.

The estimation and testing methods are illustrated with Monte Carlo simulations. When applied to school choice data from Paris, our results are more consistent with stability than with truth-telling. Reduced-form evidence on ranking behavior suggests that some students omit the most selective schools from their list because of low admission probabilities. Our proposed tests reject truth-telling but not stability and, compared with our preferred estimates based on stability (with or without moment inequalities from undominated strategies), assuming truth-telling leads to an under-estimation of preferences for popular or small schools. Moreover, the stability-based estimators outperform the truth-telling-based estimator in predicting matching outcomes and student preferences.

Our approaches are applicable to many school choice and college admissions systems around the world, as well as to other matching schemes such as teacher assignment in France and medical matching in Scotland.

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(For Online Publication)

Appendix to

Beyond Truth-Telling: Preference Estimation with Centralized School Choice and College Admissions

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September 2017

List of Appendices

Appendix A Proofs and Additional Results

Section A.1 collects the proofs and additional results for a finite economy, while those related to asymptotics and the continuum economy are presented in Section A.2.

A.1 Finite Economy: Proofs from Sections 1.2 and 1.3

Proof of Proposition 1.

Without application cost, STT is a dominant-strategy Bayesian Nash equilibrium (Dubins and Freedman, 1981; Roth, 1982), so we only need to prove its uniqueness.

Suppose that a non-STT strategy, $\overline{\sigma}$, is another equilibrium. Without loss of generality, let us assume $\overline{\sigma}$ is in pure strategy.

Since STT is a weakly dominant strategy, it implies that, for any i and any $\theta_{-i} \in \Theta^{I-1}$,

$$\sum_{s=1}^{S} u_{i,s} a_s \left(r(u_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right) \geqslant \sum_{s=1}^{S} u_{i,s} a_s \left(\overline{\sigma}(\theta_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right),$$

in which both terms are non-negative given the assumptions on G. Moreover, $\overline{\sigma}$ being an equilibrium means that, for any i:

$$\sum_{s=1}^{S} u_{i,s} \int a_s \left(r(u_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right) dG(\theta_{-i}) \leqslant \sum_{s=1}^{S} u_{i,s} \int a_s \left(\overline{\sigma}(\theta_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right) dG(\theta_{-i}).$$

We therefore must have that, for any i and any $\theta_{-i} \in \Theta^{I-1}$ except a measure-zero set of θ_{-i},

$$\sum_{s=1}^{S} u_{i,s} a_s \left(r(u_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right) = \sum_{s=1}^{S} u_{i,s} a_s \left(\overline{\sigma}(\theta_i), e_i; \overline{\sigma}(\theta_{-i}), e_{-i} \right). \tag{A.1}$$

Through the following claims, we then show that $\overline{\sigma}$ must be STT, i.e., $\overline{\sigma}(\theta_i) = r(u_i)$.

Claim 1: $\overline{\sigma}(\theta_i)$ and $r(u_i)$ have the same top choice.

Proof of Claim 1: Given the full support of G, there is a positive probability that i's priority indices at all schools are the highest among all students. In this event, i is accepted by r_i^1 (her most preferred school) when submitting $r(u_i)$ and accepted by the top choice in $\overline{\sigma}(\theta_i)$ when submitting $\overline{\sigma}(\theta_i)$. As preferences are strict, $\overline{\sigma}(\theta_i)$ must have r_i^1 as the top choice to have Equation (A.1) satisfied.

Claim 2: $\overline{\sigma}(\theta_i)$ and $r(u_i)$ have the same top two choices.

Proof of Claim 2: From Claim 1, we know that $\overline{\sigma}(\theta_i)$ and $r(u_i)$ agree on their top choices. Given the full support of G, there is a positive probability that i's type and

others' types satisfy the following conditions: (i) i's priority index is the lowest among all students at school r_i^1; (ii) i's priority index is the highest among all students at all other schools; and (iii) all other students have r_i^1 as their most preferred school. In this event, by Claim 1, all students rank r_i^1 as top choice. Therefore, i is rejected by r_i^1, but she is definitely accepted by her second choice. Because STT means she is accepted by r_i^2, Equation (A.1) implies that $\overline{\sigma}(\theta_i)$ must also rank r_i^2 as the second choice. This proves the claim.

We can continue proving a series of similar claims that $\overline{\sigma}(\theta_i)$ and $r(u_i)$ must agree on top S choices. In other words, $\overline{\sigma}(\theta_i) = r(u_i)$. This proves that there is no non-STT equilibrium, and, therefore, STT is the unique Bayesian Nash equilibrium. ∎

Proof of Lemma 2.

The sufficiency of the first statement is implied by the strategy-proofness of DA and by DA producing a stable matching when everyone is STT. That is, STT is a dominant strategy if $C(|L|) = 0$ for all L, which always leads to stability.

To prove its necessity, it suffices to show that there is no dominant strategy when $C(|L|) > 0$ for some $L \in \mathcal{L}$.

If $C(|L|) = +\infty$ for some L, we are in the case of the constrained/truncated DA, and it is well known that there is no dominant strategy (see, e.g., Haeringer and Klijn, 2009).

Now suppose that $0 < C(|L|) < +\infty$ for some $L \in \mathcal{L}$. If a strategy ranks fewer than S schools with a positive probability, we know that it cannot be a dominant strategy for the same reason as in the contrained/truncated DA. If a strategy does always rank all schools, then it is weakly dominated by STT. We therefore need to show that STT is not a dominant strategy for all student types, for which we can construct an example where it is profitable for a student to drop some schools from her ROL to save application costs for some profiles of ROLs submitted by other students.

Therefore, there is no dominant strategy when $C(|L|) > 0$ for some $L \in \mathcal{L}$, and hence stability cannot be an equilibrium outcome in dominant strategy.

The second statement is implied by Proposition 1 and that DA produces a stable matching when everyone is STT. ∎

Proof of Proposition 3.

(i) Suppose that given a realized matching $\hat{\mu}$, there is a student-school pair (i, s) such

that $\hat{\mu}(\theta_i) \neq \varnothing$, $u_{i,s} > u_{i,\hat{\mu}(\theta_i)}$, and $e_{i,s} \geqslant P_s(\hat{\mu})$. That is, i is not matched with her favorite feasible school.

Since i is weakly truth-telling, she must have ranked all schools that are more preferred to $\hat{\mu}(\theta_i)$, including s. The DA algorithm implies that i must have been rejected by s at some round given that she is accepted by a lower-ranked school $\hat{\mu}(\theta_i)$. As i is rejected by s in some round, the cutoff of s must be higher than $e_{i,s}$. This contradiction rules out the existence of such matchings.

(ii) Given the result in part (i), when every student who has at least one feasible school is matched, everyone must be assigned to her favorite feasible school. Moreover, unmatched students have no feasible school. Therefore, the matching is stable. ∎

A.2 Asymptotics: Proofs and Additional Results

We now present the proofs of results in the main text as well as some additional results on the asymptotics and the continuum economy.

A.2.1 Matching and the DA Mechanism in the Continuum Economy

We follow Abdulkadiroğlu et al. (2015) and Azevedo and Leshno (2016) to extend the definitions of matching and DA to the continuum economy, E.

Similar to that in finite economies, a matching in E is a function $\mu : \Theta \to \mathcal{S} \cup \{\varnothing\}$, such that (i) $\mu(\theta_i) = s$ if student i is matched with s; (ii) $\mu(\theta_i) = \varnothing$ if student i is unmatched; (iii) $\mu^{-1}(s)$ is measurable and is the set of students matched with s, while $G(\mu^{-1}(s)) \leqslant q_s$; and (iv) (open on the right) for any $s \in \mathcal{S}$, the set $\{\theta_i \in \Theta : u_{i,\mu(\theta)} \leqslant u_{i,s}\}$ is open.

The last condition is imposed because in the continuum model it is always possible to add a measure-zero set of students to a school without exceeding its capacity. This would generate multiplicities of stable matchings that differ only in sets of measure zero. Condition (iv) rules out such multiplicities. The intuition is that the condition implies that a stable matching always allows an extra measure zero set of students into a school when this can be done without compromising stability.

The DA algorithm works almost the same as in a finite economy. Abdulkadiroğlu et al. (2015) formally define the algorithm, and prove that it converges. A sketch of the mechanism is as follows. At the first step, each student applies to her most preferred

school. Every school tentatively admits up to its capacity from its applicants according to its priority order, and rejects the rest if there are any. In general, each student who was rejected in the previous step applies to her next preferred school. Each school considers the set of students it has tentatively admitted and the new applicants. It tentatively admits up to its capacity from these students in the order of its priority, and rejects the rest. The process converges when the set of students that are rejected has zero measure. Although this process might not complete in finite time, it converges in the limit (Abdulkadiroğlu et al., 2015).

A.2.2 Proofs of Propositions 4 and 5

We start with some auxiliary results that are useful to prove the propositions.

Similar to Azevedo and Leshno (2016), we define the convergence of $\{F^{(I)}\}_{I \in \mathbb{N}}$ to E if $q^{(I)}$ converges to q and if $G^{(I)}$ converges to G in the weak-* topology.[A.1] We similarly define the convergence of $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ to (E, σ^∞), additionally requiring the empirical distributions of ROLs prescribed by $\sigma^{(I)}$ in finite economies to converge to those in E prescribed by σ^∞.

Lemma A1. *For a sequence of random economies and equilibrium strategies $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ satisfying Assumption 1, $P^{(I)}$, the random cutoff associated with $(F^{(I)}, \sigma^{(I)})$, converges to $P(\mu_{(E,\sigma^\infty)})$ almost surely.*

Proof of Lemma A1.

First, we note that the sequence of random economies $\{F^{(I)}\}_{I \in \mathbb{N}}$ converges to E almost surely. By construction, $q^{(I)}$ converges to q. Moreover, by the Glivenko-Cantelli Theorem, the empirical distribution functions $G^{(I)}$ converge to G in the weak-* topology almost surely. Therefore, we have that $\{F^{(I)}\}_{I \in \mathbb{N}}$ converges to E almost surely.

Second, we show that $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ converges to (E, σ^∞) almost surely. As $\sigma^{(I)}$ and σ^∞ map student types to ROLs of schools, $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ is a sequence of random economies that are defined with ROLs. A student's "type" is now characterized by $(L_i, e_i) \in \mathcal{L} \times [0, 1]^S$. Let M^∞ be the probability measure on the modified student types in (E, σ^∞). That is, for any $\Lambda \subset \mathcal{L} \times [0, 1]^S$, $M^\infty(\Lambda) = G(\{\theta_i \in \Theta \mid (\sigma^\infty(\theta_i), e_i) \in \Lambda\})$.

Similarly, $M^{(I)}$ is the empirical distribution function of the modified types in the random economy $\{F^{(I)}, \sigma^{(I)}\}$. We want to show that $M^{(I)}$ converges to M^∞ in the weak-* topology almost surely.

Let $X : \mathcal{L} \times [0,1]^S \to [\underline{x}, \overline{x}] \subset \mathbb{R}$ be a bounded continuous function. We also define $M^{(I)}_{\sigma^\infty}$ the random probability measure on $\mathcal{L} \times [0,1]^S$ when students play σ^∞ in random economy $F^{(I)}$. Because the strategy is fixed at σ^∞ for all I, by the same arguments as above (i.e., the convergence of $q^{(I)}$ to q and the Glivenko-Cantelli Theorem), $M^{(I)}_{\sigma^\infty}$ converges to M^∞ almost surely.

Let $\Theta^{(I)} = \{\theta_i \in \Theta \mid \sigma^{(I)}(\theta_i) \neq \sigma^\infty(\theta_i)\}$. We have the following results:

$$
\left| \int X dM^{(I)} - \int X dM^\infty \right|
$$
$$
\leqslant \left| \int X dM^{(I)} - \int X dM^{(I)}_{\sigma^\infty} \right| + \left| \int X dM^{(I)}_{\sigma^\infty} - \int X dM^\infty \right|
$$
$$
= \left| \int X(\sigma^{(I)}(\theta_i), e_i) dG^{(I)} - \int X(\sigma^\infty(\theta_i), e_i) dG^{(I)} \right| + \left| \int X dM^{(I)}_{\sigma^\infty} - \int X dM^\infty \right|
$$
$$
= \left| \int_{\theta_i \in \Theta^{(I)}} \left[X(\sigma^{(I)}(\theta_i), e_i) - X(\sigma^{(\infty)}(\theta_i), e_i) \right] dG^{(I)} \right| + \left| \int X dM^{(I)}_{\sigma^\infty} - \int X dM^\infty \right|
$$
$$
\leqslant (\overline{x} - \underline{x}) G^{(I)}(\Theta^{(I)}) + \left| \int X dM^{(I)}_{\sigma^\infty} - \int X dM^\infty \right|,
$$

where the first inequality is due to the triangle inequality; the equalities are because of the definitions of $M^{(I)}$ and $M^{(I)}_{\sigma^\infty}$ and because $X(\sigma^{(I)}(\theta_i), e_i) = X(\sigma^\infty(\theta_i), e_i)$ whenever $\theta_i \notin \Theta^{(I)}$; the last inequality comes from the boundedness of X.

Because $\lim_{I \to \infty} G(\Theta^{(I)}) = 0$ by Assumption 1 and $G^{(I)}$ converges to G almost surely, $\lim_{I \to \infty} G^{(I)}(\Theta^{(I)}) = 0$ almost surely. Moreover, $M^{(I)}_{\sigma^\infty}$ converges to M^∞ almost surely, and thus the above inequalities implies $\int X dM^{(I)}$ converges to $\int X dM^\infty$ almost surely. By the Portmanteau theorem, $M^{(I)}$ converge to M^∞ in the weak-* topology almost surely.

This proves $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ converges to (E, σ^∞) almost surely. By Proposition 3 of Azevedo and Leshno (2016), $P^{(I)}$ converges to $P(\mu_{(E,\sigma^\infty)})$ almost surely. $\blacksquare$

Proposition A1. *In a sequence of random economies and equilibrium strategies $\{F^{(I)}, \sigma^{(I)}\}_{I \in \mathbb{N}}$ satisfying Assumption 1, $\mu(E, \sigma^\infty) = \mu^\infty$ and thus $\sigma^\infty(\theta_i)$ ranks $\mu^\infty(\theta_i)$ for all $\theta_i \in \Theta$ except a measure-zero set of student types.*

Proof of Proposition A1.

Suppose that the first statement in the proposition is not true, $G(\{\theta_i \in \Theta \mid \mu_{(E,\sigma^\infty)}(\theta_i) \neq \mu^\infty(\theta_i)\}) > 0$ and therefore $P(E, \sigma^\infty) \neq P^\infty$. Because there is a unique stable matching

in E, which is the unique equilibrium outcome, by assumption, $\mu_{(E,\sigma^\infty)}$ is not stable and thus is not an equilibrium outcome.

Recall that $P(E,\sigma^\infty)$, P^∞, $\mu_{(E,\sigma^\infty)}$, and μ^∞ are fixed constants, although their counterparts in finite economies are random variables. Moreover, $\sigma^{(I)}$ and σ^∞ are not random variables either.

For some $\eta, \xi > 0$, we define:

$$
\Theta_{(\eta,\xi)} = \left\{ \theta_i \in \Theta \left|
\begin{array}{c}
e_{i,\mu^\infty(\theta_i)} - P^\infty_{\mu^\infty(\theta_i)} > \eta, \\[4pt]
e_{i,\mu_{(E,\sigma^\infty)}(\theta_i)} - P_{\mu_{(E,\sigma^\infty)}(\theta_i)}(\mu_{(E,\sigma^\infty)}) > \eta, \\[4pt]
e_{i,s} - P_s(\mu_{(E,\sigma^\infty)}) < -\eta, \text{ for all } s \text{ ranked above } \mu_{(E,\sigma^\infty)}(\theta_i) \text{ by } \sigma^\infty(\theta_i); \\[4pt]
u_{i,\mu^\infty(\theta_i)} - u_{i,\mu_{(E,\sigma^\infty)}(\theta_i)} > \xi.
\end{array}
\right. \right\},
$$

$\Theta_{(\eta,\xi)}$ must have a positive measure for some $\eta, \xi > 0$ and is a subset of students who can form a blocking pair in $\mu_{(E,\sigma^\infty)}$. Clearly, $\sigma^\infty(\theta_i)$ ranks $\mu_{(E,\sigma^\infty)}(\theta_i)$ but not $\mu^\infty(\theta_i)$ for all $\theta_i \in \Theta_{(\eta,\xi)}$. We further define:

$$
\Theta^{(I)}_{(\eta,\xi)} = \Theta_{(\eta,\xi)} \cap \{\theta_i \in \Theta \mid \sigma^{(I)}(\theta_i) \text{ ranks } \mu_{(E,\sigma^\infty)}(\theta_i) \text{ but not } \mu^\infty(\theta_i)\}.
$$

By Assumption 1, $\sigma^{(I)}$ converges to σ^∞, and thus $\Theta^{(I)}_{(\eta,\xi)}$ converges to $\Theta_{(\eta,\xi)}$ and has a positive measure when I is sufficiently large.

We show below that $\{\sigma^{(I)}\}_{I \in \mathbb{N}}$ is not a sequence of equilibrium strategies. Consider a unilateral deviation for $\theta_i \in \Theta^{(I)}_{(\eta,\xi)}$ from $\sigma^{(I)}(\theta_i)$ to L_i such that the only difference between the two actions is that $\mu_{(E,\sigma^\infty)}(\theta_i)$, ranked in $\sigma^{(I)}(\theta_i)$, is replaced by $\mu^\infty(\theta_i)$ in L_i while L_i is kept as a partial order of i's true preferences.

By Lemma A1, for $0 < \phi < \xi/(1+\xi)$ there exists $n \in \mathbb{N}$ such that, in all $F^{(I)}$ with $I > n$, i is matched with $\mu_{(E,\sigma^\infty)}(\theta_i)$ with probability at least $(1-\phi)$ if submitting $\sigma^{(I)}(\theta_i)$ but would have been matched with $\mu^\infty(\theta_i)$ if instead L_i had been submitted.

Let $\mathrm{EU}(\sigma^{(I)}(\theta_i))$ be the expected utility when submitting $\sigma^{(I)}(\theta_i)$. Then $\mathrm{EU}(\sigma^{(I)}(\theta_i)) \leqslant (1-\phi)u_{i,\mu_{(E,\sigma^\infty)}(\theta_i)} + \phi$ because $\max_s\{u_{i,s}\} \leqslant 1$ by assumption, and $\mathrm{EU}(L_i) \geqslant (1-\phi)u_{i,\mu^\infty(\theta_i)}$.

The difference between the two actions is:

$$\mathrm{EU}(L_i) - \mathrm{EU}(\sigma^{(I)}(\theta_i))$$

$$\geqslant (1-\phi)u_{i,\mu^\infty(\theta_i)} - (1-\phi)u_{i,\mu_{(E,\sigma^\infty)}(\theta_i)} - \phi$$

$$\geqslant (1-\phi)\xi - \phi$$

$$> 0,$$

which proves that $\{\sigma^{(I)}\}_{I\in\mathbb{N}}$ is not a sequence of equilibrium strategies. This contradiction further shows that $G(\{\theta_i \in \Theta \mid \mu_{(E,\sigma^\infty)}(\theta_i) \neq \mu^\infty(\theta_i)\}) = 0$ and that $\sigma^\infty(\theta_i)$ ranks $\mu^\infty(\theta_i)$ for all $\theta_i \in \Theta$ except a measure-zero set of student types. ∎

We are now ready to prove Proposition 4.

Proof of Proposition 4.

Part (i) is implied by Lemma A1 and Proposition A1. Because $F^{(I)}$ converges to E almost surely and $\sigma^{(I)}$ converges to σ^∞, $P^{(I)}$ converges to $P(\mu_{(E,\sigma^\infty)})$ almost surely. Moreover, $\mu_{(E,\sigma^\infty)} = \mu^\infty$ except a measure-zero set of students implies that $P(\mu_{(E,\sigma^\infty)}) = P^\infty$. Therefore, $\lim_{I\to\infty} P^{(I)} = P^\infty$ almost surely.

To show part (ii), we first define $\Theta^{(I)} = \{\theta_i \in \Theta \mid \sigma^{(I)}(\theta_i) \neq \sigma^\infty(\theta_i)\}$. By Assumption 1, $G^{(I)}(\Theta^{(I)})$ converges to zero almost surely. We have the following inequalities:

$$G^{(I)}\left(\left\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^{(I)})}{\arg\max} u_{i,s}\right\}\right)$$

$$\leqslant \left| G^{(I)}\left(\left\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^{(I)})}{\arg\max} u_{i,s}\right\}\right) - G^{(I)}\left(\left\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^\infty)}{\arg\max} u_{i,s}\right\}\right) \right|$$

$$+ G^{(I)}\left(\left\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^\infty)}{\arg\max} u_{i,s}\right\}\right)$$

$$\leqslant G^{(I)}\left(\{\theta_i \in \Theta \mid \mathcal{S}(e_i, P^\infty) \neq \mathcal{S}(e_i, P^{(I)})\}\right) + G^{(I)}\left(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)\}\right),$$

where the first inequality is due to the triangle inequality; the second inequality is because $\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \arg\max_{s\in\mathcal{S}(e_i, P^{(I)})} u_{i,s}\}$ and $\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \arg\max_{s\in\mathcal{S}(e_i, P^\infty)} u_{i,s}\}$ can possibly differ only when $\mathcal{S}(e_i, P^\infty) \neq \mathcal{S}(e_i, P^{(I)})$ and because:

$$\left\{\theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^\infty)}{\arg\max} u_{i,s}\right\} = \{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)\}.$$

Furthermore,

$$G^{(I)} \left(\{ \theta_i \in \Theta \mid \mathcal{S}(e_i, P^\infty) \neq \mathcal{S}(e_i, P^{(I)}) \} \right)$$
$$= G^{(I)} \left(\{ \theta_i \in \Theta \mid \min(P_s^\infty, P_s^{(I)}) \leqslant e_{i,s} < \max(P_s^\infty, P_s^{(I)}), \exists s \in \mathcal{S} \} \right).$$

The right hand side converges to zero almost surely, because almost surely $G^{(I)}$ converges to G, which is atomless, and $\lim_{n \to \infty} P^{(I)} = P^\infty$ almost surely. Therefore,

$$\lim_{I \to \infty} G^{(I)} \left(\{ \theta_i \in \Theta \mid \mathcal{S}(e_i, P^\infty) \neq \mathcal{S}(e_i, P^{(I)}) \} \right) = 0 \text{ almost surely.} \qquad (A.2)$$

Moreover,

$$G^{(I)} \left(\{ \theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i) \} \right)$$
$$\leqslant G^{(I)}(\Theta^{(I)}) + G^{(I)} \left(\{ \theta_i \in \Theta \backslash \Theta^{(I)} \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i) \} \right)$$
$$\leqslant G^{(I)}(\Theta^{(I)}) + G^{(I)} \left(\left\{ \theta_i \in \Theta \backslash \Theta^{(I)} \mid u_{i,\mu^{(I)}(\theta_i)} > u_{i,\mu^\infty(\theta_i)} \ \& \ e_{i,\mu^{(I)}(\theta_i)} \geqslant P^{(I)}_{\mu^{(I)}(\theta_i)} \right\} \right)$$
$$+ G^{(I)} \left(\left\{ \theta_i \in \Theta \backslash \Theta^{(I)} \mid u_{i,\mu^{(I)}(\theta_i)} < u_{i,\mu^\infty(\theta_i)}, e_{i,\mu^\infty(\theta_i)} < P^{(I)}_{\mu^\infty(\theta_i)}, \ \& \ e_{i,\mu^{(I)}(\theta_i)} \geqslant P^{(I)}_{\mu^{(I)}(\theta_i)} \right\} \right)$$

In the first inequality, we decompose the student type space into two, $\Theta^{(I)}$ and $\Theta \backslash \Theta^{(I)}$. In the former, students do not adopt σ^∞, while those in the latter set do and thus rank the school prescribed by μ^∞. The second inequality consider the events when $\mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)$ can possibly happen.

The almost-sure convergence of $G^{(I)}$ to G and that of $P^{(I)}$ to P^∞ implies that:

$$\lim_{I \to \infty} G^{(I)} \left(\left\{ \theta_i \in \Theta \backslash \Theta^{(I)} \mid u_{i,\mu^{(I)}(\theta_i)} > u_{i,\mu^\infty(\theta_i)} \ \& \ e_{i,\mu^{(I)}(\theta_i)} \geqslant P^{(I)}_{\mu^{(I)}(\theta_i)} \right\} \right) = 0 \text{ almost surely,}$$

because for any s such that $u_{i,s} > u_{i,\mu^\infty(\theta_i)}$, we must have $e_{i,s} < P_s^\infty$.

Similarly, almost surely,

$$\lim_{I \to \infty} G^{(I)} \left(\left\{ \theta_i \in \Theta \backslash \Theta^{(I)} \mid u_{i,\mu^{(I)}(\theta_i)} < u_{i,\mu^\infty(\theta_i)}, e_{i,\mu^\infty(\theta_i)} < P^{(I)}_{\mu^\infty(\theta_i)}, \ \& \ e_{i,\mu^{(I)}(\theta_i)} \geqslant P^{(I)}_{\mu^{(I)}(\theta_i)} \right\} \right) = 0.$$

Therefore, $G^{(I)} \left(\{ \theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i) \} \right) = 0$ almost surely. Together with (A.2), it implies that $G^{(I)} \left(\{ \theta_i \in \Theta : \mu^{(I)}(\theta_i) \notin \arg\max_{s \in \mathcal{S}(e_i, P^{(I)})} u_{i,s} \} \right)$ converges to zero almost surely. In other words, $\{ \mu^{(I)} \}_{I \in \mathbb{N}}$ is asymptotically stable. ∎

Proof of Proposition 5.

The first statement in part (i) is implied by Proposition 2. Suppose that i is in a blocking pair with some school s. It means that the ex post cutoff of s is lower than i's

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priority index at s. Therefore, if $s \in L_i^{(I)}$, the stability of DA (with respect to ROLs) implies that i must be accepted by s or by schools ranked above and thus preferred to s. Therefore, i and s cannot form a blocking pair if $s \in L_i^{(I)}$, which proves the second statement in part (i).

Part (iii) is implied by Proposition 4 (part (i)).

To show part (ii), we let $\mathcal{S}_0^{(I)} = \mathcal{S} \backslash L_i^{(I)}$ and therefore:

$$
\begin{aligned}
B_i^{(I)} &= \Pr(\exists s \in \mathcal{S}_0, u_{i,s} > u_{i,\mu^{(I)}(\theta_i)} \text{ and } e_{i,s} \geqslant P_s^{(I)}) \\
&\leqslant \sum_{s \in \mathcal{S}_0^{(I)}} \Pr\left(e_{i,t} < P_t^{(I)}, \forall t \in L_i^{(I)}, s.t., u_{i,t} > u_{i,s}; e_{i,s} \geqslant P_s^{(I)} \right).
\end{aligned}
$$

Let $B_{i,s}^{(I)}$ be $\Pr\left(e_{i,t} < P_t^{(I)}, \forall t \in L_i^{(I)}, s.t., u_{i,t} > u_{i,s}; e_{i,s} \geqslant P_s^{(I)} \right)$ for $s \in \mathcal{S}_0^{(I)}$. Since $s \in \mathcal{S}_0$ and $L_i^{(I)}$ is ex ante optimal for i in $F^{(I)}, \sigma^{(I)}$, it implies:

$$
\begin{aligned}
&\sum_{s \in \mathcal{S}} u_{i,s} \int a_s \left(L_i^{(I)}, e_i; \sigma^{(I)}(\theta_{-i}), e_{-i} \right) dG(\theta_{-i}) - C\left(\left| L_i^{(I)} \right| \right) \\
&\geqslant \sum_{s \in \mathcal{S}} u_{i,s} \int a_s \left(L, e_i; \sigma^{(I)}(\theta_{-i}), e_{-i} \right) dG(\theta_{-i}) - C\left(\left| L_i^{(I)} \right| + 1 \right)
\end{aligned}
$$

where L ranks all schools in $L_i^{(I)}$ and s while respecting their true preference rankings, i.e., adding s to the true partial preference order $L_i^{(I)}$ while keeping the new list a true partial preference order.

For notational convenience, we relabel the schools such that school k is the kth choice in L and that s is k^*th school in L. Those not ranked in L are labeled as $|L_i^{(I)}| + 2, \cdots, S$. It then follows that:

$$
\begin{aligned}
&C\left(\left| L_i^{(I)} \right| + 1 \right) - C\left(\left| L_i^{(I)} \right| \right) \\
&\geqslant \sum_{t=1}^{k^*-1} 0 + B_{i,s}^{(I)} u_{i,s} \\
&\quad + \sum_{t=k^*+1}^{|L_i^{(I)}|+1} u_{i,t} \cdot \left(\begin{array}{c} \Pr\left(e_{i,t} \geqslant P_t^{(I)}; e_{i,\tau} < P_\tau^{(I)}, \tau = 1, \ldots, t-1 \right) - \\ \Pr\left(e_{i,t} \geqslant P_t^{(I)}; e_{i,\tau} < P_\tau^{(I)}, \tau = 1, \ldots, k^*-1, k^*+1, \cdots, t-1 \right) \end{array} \right), \\
&= \sum_{t=1}^{k^*-1} 0 + B_{i,s}^{(I)} u_{i,s} \\
&\quad - \sum_{t=k^*+1}^{|L_i^{(I)}|+1} u_{i,t} \cdot \Pr\left(e_{i,t} \geqslant P_t^{(I)}; e_{i,s} \geqslant P_s^{(I)}; e_{i,\tau} < P_\tau^{(I)}, \tau = 1, \ldots, k^*-1, k^*+1, \cdots, t-1 \right),
\end{aligned}
$$

where the zeros in the first term on the right come from the upper invariance of DA. That is, the admission probability at any school ranked above s is the same when i submits either $L_i^{(I)}$ or L.

Also note that $u_{i,s} > u_{i,t}$ for all $t \geq k^* + 1$ and that:

$$\sum_{t=k^*+1}^{|L_i^{(I)}|+1} \Pr\left(e_{i,t} \geq P_t^{(I)}; e_{i,s} \geq P_s^{(I)}; e_{i,\tau} < P_\tau^{(I)}, \tau = 1, \ldots, k^* - 1, k^* + 1, \cdots, t - 1\right) \leq B_{i,s}^{(I)}.$$

Besides, $u_{i,k^*+1} \geq u_{i,t}$ for all $t = k^* + 2, \cdots, |L_i| + 1$. Therefore, for all $s \in \mathcal{S}_0^{(I)}$,

$$C\left(\left|L_i^{(I)}\right| + 1\right) - C\left(\left|L_i^{(I)}\right|\right) \geq B_{i,s}^{(I)} u_{i,s} - B_{i,s}^{(I)} u_{i,k^*+1}$$

This leads to:

$$B_{i,s}^{(I)} \leq \frac{C\left(\left|L_i^{(I)}\right| + 1\right) - C\left(\left|L_i^{(I)}\right|\right)}{u_{i,s} - u_{i,k^*+1}} \leq \frac{C\left(\left|L_i^{(I)}\right| + 1\right) - C\left(\left|L_i^{(I)}\right|\right)}{u_{i,s}}.$$

Finally, $B_i^{(I)} \leq \sum_{s \in \mathcal{S}_0^{(I)}} B_{i,s}^{(I)} \leq |\mathcal{S} \backslash L_i| \frac{C(|L_i|+1)-C(|L_i|)}{\max_{s \in \mathcal{S} \backslash L_i^{(I)}} u_{i,s}}.$ ∎

A.2.3 Asymptotic Distribution of Cutoffs and Convergence Rates

For the next result, we define the demand for each school in (E, σ) as a function of the cutoffs:

$$D_s(P \mid E, \sigma) = \int \mathbb{1}\left(u_{i,s} = \max_{s' \in \mathcal{S}(e_i, P) \bigcap \sigma(\theta_i)} u_{i,s'}\right) dG(\theta_i),$$

where $\sigma(\theta_i)$ also denotes the set of schools ranked by i; $\mathbb{1}()$ is an indicator function. Let $D(P \mid E, \sigma) = [D_s(P \mid E, \sigma)]_{s \in \mathcal{S}}$.

Assumption 2.

(i) There exists $n \in \mathbb{N}$ such that $\sigma^{(I)} = \sigma^\infty$ for all $I > n$;

(ii) $D(\cdot \mid E, \sigma^\infty)$ is C^1 and $\partial D(P^\infty \mid E, \sigma^\infty)$ is invertible;

(iii) $\sum_{s=1}^{S} q_s < 1$.

Part (i) says that σ^∞ maintains as an equilibrium strategy in any economy of a size that is above a threshold. This is supported partially by the discussion in Section A.2.4. In particular, when $C(2) > 0$, part (i) is satisfied. $D(\cdot \mid E, \sigma^\infty)$ being C^1 (in part ii) holds when G admits a continuous density. In this case, the fraction of students whose demand is affected by changes in P is continuous. Part (iii) guarantees that every school has a positive cutoff in the stable matching of E.

Because of Assumption 2, our setting with cardinal preferences can be transformed into one defined by students' ROLs that is identical to that in Azevedo and Leshno (2016). Therefore, some of their results are also satisfied in our setting.

Proposition A2. *In a sequence of matchings, $\{\mu^{(I)}\}_{I\in\mathbb{N}}$, of the sequence of random economies and equilibrium strategies, $\{F^{(I)}, \sigma^{(I)}\}_{I\in\mathbb{N}}$, satisfying Assumption 2, we have the following results:*

(i) The distribution of cutoffs in $(F^{(I)}, \sigma^{(I)})$ satisfies:

$$\sqrt{I}\left(P^{(I)} - P^\infty\right) \xrightarrow{d} N(0, V(\sigma^\infty))$$

where $V(\sigma^\infty) = \partial D(P^\infty \mid E, \sigma^\infty)^{-1} \Sigma \left(\partial D(P^\infty \mid E, \sigma^\infty)^{-1}\right)'$, and

$$\Sigma = \begin{pmatrix} q_1(1-q_1) & -q_1q_2 & \cdots & -q_1q_S \\ -q_2q_1 & q_2(1-q_2) & \cdots & \vdots \\ \vdots & \vdots & \ddots & -q_{S-1}q_S \\ -q_Sq_1 & \cdots & -q_Sq_{S-1} & q_S(1-q_S) \end{pmatrix}.$$

(ii) For any $\eta > 0$ and $I > n$, there exist constants γ_1 and γ_2 such that the probability that the matching $\mu^{(I)}$ has cutoffs $||P^{(I)} - P^\infty|| > \eta$ is bounded by $\gamma_1 e^{-\gamma_2 I}$:

$$\Pr\left(||P^{(I)} - P^\infty|| > \eta\right) < \gamma_1 e^{-\gamma_2 I}.$$

(iii) Moreover, suppose that G admits a continuous density. For any $\eta > 0$ and $I > n$, there exist constants γ_1' and γ_2' such that, in matching $\mu^{(I)}$, the probability of the fraction of students who can form a blocking pair being greater than η is bounded by $\gamma_1' e^{-\gamma_2' I}$:

$$\Pr\left(G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \notin \underset{s\in\mathcal{S}(e_i, P^{(I)})}{\arg\max} u_{i,s}\}) > \eta\right) < \gamma_1' e^{-\gamma_2' I}.$$

Parts (i) and (ii) are from Azevedo and Leshno (2016) (Proposition G1 and part 2 of Proposition 3), although our part (iii) is new and extends their part 3 of Proposition 3. Proposition A2 describes convergence rates and thus has implications for empirical approaches based on stability (see Section 2.3).

Proof of Proposition A2 (part iii).

To show part (iii), we use similar techniques as in the proof for Proposition 3 (part 3)

in Azevedo and Leshno (2016). We first derive the following results.

$$G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \notin \underset{s \in \mathcal{S}(e_i, P^{(I)})}{\arg\max} u_{i,s}\})$$

$$=G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \notin [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \forall s \in \mathcal{S}; \mu^{(I)}(\theta_i) \notin \underset{s \in \mathcal{S}(e_i, P^\infty)}{\arg\max} u_{i,s}\}$$

$$+ G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}; \mu^{(I)}(\theta_i) \notin \underset{s \in \mathcal{S}(e_i, P^{(I)})}{\arg\max} u_{i,s}\})$$

$$\leqslant G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)\})$$

$$+ G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}\})$$

In the first equality, whenever $e_{i,s} \notin [\min(P_s^\infty, P_s^{(I_m)}), \max(P_s^\infty, P_s^{(I_m)})], \forall s \in \mathcal{S}$, i faces the same set of feasible schools given either $P^{(I)}$ or P^∞, $\mathcal{S}(e_i, P^{(I)}) = \mathcal{S}(e_i, P^\infty)$. Because μ^∞ is stable, $\mu^\infty(\theta_i)$ is i's favorite school in $\mathcal{S}(e_i, P^\infty)$; together with the relaxation of the conditions in the second term, it leads to the inequality.

By Azevedo and Leshno (2016) Proposition 3 (part 3), we can find γ_1' and γ_2' such that:

$$\Pr\left(G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)\}) > \eta/2\right) < \gamma_1' e^{-\gamma_2' I}/2. \tag{A.3}$$

Let $\overline{g}$ be the supremum of the marginal probability density of $e_{i,s}$ across all s. Denote the set of student types with priority indices which have at least one coordinate close to P^∞ by distance $\eta_1/(2S\overline{g})$ (where $\eta_1 = \eta/4$):

$$\Theta_{\eta_1} = \{\theta_i \in \Theta \mid \exists s \in \mathcal{S}, |e_{i,s} - P_s^\infty| \leqslant \eta_1/(2S\overline{g})\}.$$

Then $G(\Theta_{\eta_1}) \leqslant 2S\overline{g} \cdot \eta_1/(2S\overline{g}) = \eta_1$. The fraction of students in $F^{(I)}$ that have types in Θ_{η_1} is then $G^{(I)}(\Theta_{\eta_1})$. Note that $G^{(I)}(\Theta_{\eta_1})$ is a random variable with mean $G(\Theta_{\eta_1})$. By the Vapnik-Chervonenkis Theorem,[A.2]

$$\Pr(G^{(I)}(\Theta_{\eta_1}) > 2\eta_1) < \Pr(|G^{(I)}(\Theta_{\eta_1}) - G(\Theta_{\eta_1})| > \eta_1) < \gamma_1' e^{-\gamma_2' I}/4. \tag{A.4}$$

Moreover, by part (ii), we know that:

$$\Pr\left(||P^{(I)} - P^\infty|| > \eta_1/(2S\overline{g})\right) < \gamma_1' e^{-\gamma_2' I}/4. \tag{A.5}$$

[A.2]See Azevedo and Leshno (2016) and the references therein for more details on the theorem for its application in our context.

We can choose γ_1' and γ_2' appropriately, so that (A.3), (A.4), and (A.5) are all satisfied.

When the event, $||P^{(I)} - P^\infty|| > \eta_1/(2S\bar{g})$, does not happen,

$$\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}\} \subset \Theta_{\eta_1}.$$

When neither $||P^{(I)} - P^\infty|| > \eta_1/(2S\bar{g})$ nor $G^{(I)}(\Theta_{\eta_1}) > 2\eta_1$ happens,

$$G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}\}) \leqslant 2\eta_1 = \eta/2;$$

the probability that neither of these two events happens is at least $1 - \gamma_1' e^{-\gamma_2' I}/4 - \gamma_1' e^{-\gamma_2' I}/4 = 1 - \gamma_1' e^{-\gamma_2' I}/2$. This implies,

$$\Pr(G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}\}) > \eta/2) < \gamma_1' e^{-\gamma_2' I}/2.$$
(A.6)

The events in (A.3) and (A.6) do not happen with probability at least $1 - \gamma_1' e^{-\gamma_2' I}/2 - \gamma_1' e^{-\gamma_2' I}/2 = 1 - \gamma_1' e^{-\gamma_2' I}$; and when they do not happen,

$$G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \notin \arg\max_{s \in \mathcal{S}(e_i, P^{(I)})} u_{i,s}\})$$
$$\leqslant G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \neq \mu^\infty(\theta_i)\})$$
$$\quad + G^{(I)}(\{\theta_i \in \Theta \mid e_{i,s} \in [\min(P_s^\infty, P_s^{(I)}), \max(P_s^\infty, P_s^{(I)})), \exists s \in \mathcal{S}\})$$
$$\leqslant \eta.$$

Therefore,

$$\Pr\left(G^{(I)}(\{\theta_i \in \Theta \mid \mu^{(I)}(\theta_i) \notin \arg\max_{s \in \mathcal{S}(e_i, P^{(I)})} u_{i,s}\}) > \eta \right) < \gamma_1' e^{-\gamma_2' I}.$$

∎

A.2.4 Properties of Equilibrium Strategies in Large Economies

We now discuss the properties of Bayesian Nash equilibria in a sequence of random economies and thus provide some justifications to Assumptions 1 and 2.

We start with Lemma A2 showing that a strategy, which does not result in the stable matching in the continuum economy when being adopted by students in the continuum economy, cannot survive as an equilibrium strategy in sufficiently large economies. This immediately implies that, in finite large economies, every student always includes in her

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ROL the matched school in the continuum-economy stable matching (Lemma A3). More-over, students do not pay a cost to rank more schools in large economies (Lemma A4). Lastly, when it is costly to rank more than one school ($C(2) > 0$), in sufficiently large economies, it is an equilibrium strategy for every students to only rank the matched school prescribed by the continuum-economy stable matching.

Lemma A2. *If a strategy σ results in a matching $\mu_{(E,\sigma)}$ in the continuum economy such that $G(\{\theta_i \in \Theta \mid \mu_{(E,\sigma)}(\theta_i) \neq \mu^\infty(\theta_i)\}) > 0$, then there must exist $n \in \mathbb{N}$ such that σ is not an equilibrium in $F^{(I)}$ for all $I > n$.*

Proof of Lemma A2.

Suppose instead that there is a subsequence of finite random economies $\{F^{(I_n)}\}_{n \in \mathbb{N}}$ such that σ is always an equilibrium. Note that we still have $F^{(I_n)} \to E$ almost surely, and therefore $\{F^{(I_n)}, \sigma\}$ converges to $\{E, \sigma\}$ almost surely.

Given the student-proposing DA, we focus on the student-optimal stable matching (SOSM) in $(F^{(I)}, \sigma)$. By Proposition 3 of Azevedo and Leshno (2016), it must be that $P^{(I_n)} \to P^\sigma$ almost surely, where $P^{(I_n)} = P(\mu_{(F^{(I_n)},\sigma)})$ and $P^\sigma = P(\mu_{(E,\sigma)})$.

Because there is a unique Nash equilibrium outcome in E, which is also the unique stable matching in E by assumption, $G(\{\theta_i \in \Theta_i \mid \mu_{(E,\sigma)}(\theta_i) \neq \mu^\infty(\theta_i)\}) > 0$ in the continuum economy implies that P^σ is not the cutoffs of μ^∞ (the stable matching in E), $P^\sigma \neq P^\infty$.

Because there is a unique stable matching in E by assumption, $\mu_{(E,\sigma^\infty)}$ is not stable and thus is not an equilibrium outcome in E. There exist some $\eta, \xi > 0$, such that:

$$\Theta_{(\eta,\xi)} = \left\{ \theta_i \in \Theta \; \middle| \; \begin{array}{c} e_{i,\mu^\infty(\theta_i)} - P^\infty_{\mu^\infty(\theta_i)} > \eta, \\ e_{i,\mu_{(E,\sigma)}(\theta_i)} - P^\sigma_{\mu_{(E,\sigma)}(\theta_i)} > \eta, \\ e_{i,s} - P^\sigma_s < -\eta, \text{ for all } s \text{ ranked above } \mu_{(E,\sigma)}(\theta_i) \text{ by } \sigma(\theta_i); \\ u_{i,\mu^\infty(\theta_i)} - u_{i,\mu_{(E,\sigma)}(\theta_i)} > \xi. \end{array} \right\}$$

$\Theta_{(\eta,\xi)}$ must have a positive measure for some $\eta, \xi > 0$ and is a subset of students who can form a blocking pair in $\mu_{(E,\sigma)}$. Clearly, $\sigma(\theta_i)$ ranks $\mu_{(E,\sigma)}(\theta_i)$ but not $\mu^\infty(\theta_i)$ for all $\theta_i \in \Theta_{(\eta,\xi)}$.

We show below that σ is not an equilibrium strategy in sufficiently large economies. Consider a unilateral deviation for $\theta_i \in \Theta_{(\eta,\xi)}$ from $\sigma(\theta_i)$ to L_i such that the only difference between the two actions is that $\mu_{(E,\sigma)}(\theta_i)$, ranked in $\sigma(\theta_i)$, is replaced by $\mu^\infty(\theta_i)$ in L_i

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while L_i is kept as a partial order of i's true preferences.

Because $P^{(I_n)} \to P^\sigma$ almost surely, for $0 < \phi < \xi/(1+\xi)$ there exists $n_1 \in \mathbb{N}$ such that, in all $F^{(I_n)}$ with $I_n > n_1$, i is matched with $\mu_{(E,\sigma)}(\theta_i)$ with probability at least $(1-\phi)$ if submitting $\sigma(\theta_i)$ but would have been matched with $\mu^\infty(\theta_i)$ if instead L_i had been submitted.

Let $\mathrm{EU}(\sigma(\theta_i))$ be the expected utility when submitting $\sigma(\theta_i)$. Then $\mathrm{EU}(\sigma(\theta_i)) \leqslant (1-\phi)u_{i,\mu_{(E,\sigma)}(\theta_i)} + \phi$, because $\max_s\{u_{i,s}\} \leqslant 1$ by assumption, and $\mathrm{EU}(L_i) \geqslant (1-\phi)u_{i,\mu^\infty(\theta_i)}$. The difference between the two actions is:

$$
\begin{aligned}
\mathrm{EU}(L_i) &- \mathrm{EU}(\sigma(\theta_i)) \\
&\geqslant (1-\phi)u_{i,\mu^\infty(\theta_i)} - (1-\phi)u_{i,\mu_{(E,\sigma)}(\theta_i)} - \phi \\
&\geqslant (1-\phi)\xi - \phi \\
&> 0,
\end{aligned}
$$

which proves that σ is not an equilibrium strategy in $F^{(I_n)}$ for $I_n > n_1$. This contradiction further implies that there exist $n \in \mathbb{N}$ such that σ is not an equilibrium strategy in all $F^{(I)}$ with $I > n$. ∎

Lemma A3. *If a strategy σ is such that $G(\{\theta_i \in \Theta \mid \sigma(\theta_i)$ does not rank $\mu^\infty(\theta_i)\}) > 0$, then there must exist $n \in \mathbb{N}$ such that σ is not an equilibrium in $F^{(I)}$ for all $I > n$.*

Proof of Lemma A3.

Note that $G(\{\theta_i \in \Theta \mid \sigma(\theta_i)$ does not rank $\mu^\infty(\theta_i)\}) > 0$ implies $G(\{\theta_i \in \Theta \mid \mu_{(E,\sigma)}(\theta_i) \neq \mu^\infty(\theta_i)\}) > 0$, because i cannot be matched with $\mu^\infty(\theta_i)$ if $\sigma(\theta_i)$ does not rank $\mu^\infty(\theta_i)$. Lemma A2 therefore implies the statement in this lemma. ∎

Lemmata A2 and A3 justify that, in large enough finite economies, we only consider equilibrium strategies with which every student ranks her matched school in μ^∞ of the continuum economy. We can further bound the number of choices that a student ranks by the following lemma.

Lemma A4. *Suppose $C(K) = 0$ and $C(K+1) > 0$ for $1 \leqslant K \leqslant (S-1)$. Consider a strategy σ such that $\sigma(\theta_i)$ ranks at least $K+1$ schools for all $\theta_i \in \Theta' \subset \Theta$ and $G(\Theta') > 0$. In the sequence of random economies, $\{F^{(I)}\}_{I \in \mathbb{N}}$, there exists $n \in \mathbb{N}$ such that σ is not an equilibrium strategy in any economy $F^{(I)}$ for $I > n$.*

Proof of Lemma A4.

By Lemma A3, we only need to consider all σ that rank $\mu^\infty(\theta_i)$ for θ_i. Otherwise, the statement is satisfied already.

Let $C(K+1) = \xi$. By Proposition 3 of Azevedo and Leshno (2016), it must be that $P^{(I)} \to P^\sigma$ almost surely in the sequence $\{F^{(I)}, \sigma\}_{I \in \mathbb{N}}$, where $P^{(I)} = P(\mu_{(F^{(I)}, \sigma)})$ and $P^\sigma = P(\mu_{(E,\sigma)})$. For $0 < \phi < 2\xi$, there must exit $n \in \mathbb{N}$ such that i is matched with $\mu^\infty(\theta_i)$ with probability at least $1 - \phi$ in $F^{(I)}$ for all $I > n$.

Let $\mathrm{EU}(\sigma(\theta_i))$ be the expected utility when submitting $\sigma(\theta_i)$. We compare this strategy with any unilateral deviation L_i that keeps ranking $\mu^\infty(\theta_i)$ but drops one of the other ranked schools in $\sigma(\theta_i)$.

Then $\mathrm{EU}(\sigma(\theta_i)) \leqslant (1-\phi)u_{i,\mu^\infty(\theta_i)} + \phi - \xi$, where the right side assumes that i obtains the highest possible utility (equal to one) whenever not being matched with $\mu_\infty(\theta_i)$. Moreover, $\mathrm{EU}(L_i) \geqslant (1-\phi)u_{i,\mu^\infty(\theta_i)} + \xi$. The difference between the two actions is:

$$\mathrm{EU}(L_i) - \mathrm{EU}(\sigma(\theta_i)) \geqslant 2\xi - \phi > 0,$$

which proves that σ is an equilibrium strategy in $F^{(I)}$ for $I > n$. ∎

Moreover, when $C(2) > 0$, we can obtain even sharper results:

Lemma A5. *Suppose $C(2) > 0$ (i.e., it is costly to rank more than one school), and $\sigma(\theta_i) = (\mu^\infty(\theta_i))$ (i.e., only ranking the school prescribed by μ^∞) for all student types. In a sequence of random economies $\{F^{(I)}\}_{I \in \mathbb{N}}$, there exists $n \in \mathbb{N}$ such that σ is a Bayesian Nash equilibrium in $F^{(I)}$ for all $I > n$.*

Proof of Lemma A5. This lemma is implied by Lemma A3 and Lemma A4 (when $K = 1$). ∎

A.2.5 Nash equilibrium and Stable Outcome

Example A1 (An unstable Nash equilibrium outcome in the continuum economy). *Suppose that the continuum of students consists of three types, $\Theta = \{\Theta_1, \Theta_2, \Theta_3\}$, with each type being of the same measure, $1/3$. Let $S = \{s_1, s_2, s_3\}$ be the set of schools and $q = (1/3, 1/3, 1/3)$ the vector of capacities. Student preferences and the priority structure (i.e., student priority indices at each school) are given in the table below. Student priority indices are random draws from a uniform distribution on the interval therein, which*

allows schools to strictly rank students. It is assumed that students are only allowed to rank up to two schools.

	Student preferences			Student priority indices		
	Type 1	Type 2	Type 3	Type 1	Type 2	Type 3
School s_1	**1**	0.5	0	$(1/3, 2/3]$	$[0, 1/3]$	$(2/3, 1]$
School s_2	0	0	**0.5**	$(1/3, 2/3]$	$[0, 1/3]$	$(2/3, 1]$
School s_3	0.5	**1**	1	$(2/3, 1]$	$(1/3, 2/3]$	$[0, 1/3]$

	ROL by type		
	Type 1	Type 2	Type 3
1st choice	s_1	s_1	s_3
2nd choice	s_3	s_2	s_1

Students submit ROLs as shown in the above. One can verify that the outcome is that Type-1 students are matched with s_1, Type-2 with s_2, and Type-3 with s_3, which is indicated by the boxes in the table. This matching is not stable as Type-2 students have justified envy for s_3, i.e., a positive measure of Type-2 students can form a blocking pair with School s_3. However, there is no profitable deviation for any subset of Type-2 students.

The above example is based on the discrete version in Haeringer and Klijn (2009). They further show that, in discrete and finite economies, DA with constraints implements stable matchings in Nash equilibria if and only if the student priority indices at all schools satisfy the so-called Ergin acyclicity condition (Ergin, 2002). We extend this result to the continuum economy and to a more general class of DA mechanisms where the cost function of ranking more than one school, $C(|L|)$, is flexible.

Definition A1. *In a continuum economy, we fix a vector of capacities, $\{q_s\}_{s=1}^{S}$, and a distribution of priority indices, H. An Ergin cycle is constituted of distinct schools (s_1, s_2) and subsets of students $\{\Theta_1, \Theta_2, \Theta_3\}$ (of equal measure $q_0 > 0$), whose elements are denoted by θ_1, θ_2, and θ_3, respectively, and whose "identities" are i_1, i_2, and i_3, such that the following conditions are satisfied:*

(i) Cycle condition: $e_{i_1,s_1} > e_{i_2,s_1} > e_{i_3,s_1}$, and $e_{i_3,s_2} > e_{i_1,s_2}$, for all i_1, i_2, and i_3.

(ii) Scarcity condition: there exist (possibly empty) disjoint sets of agents Θ_{s_1}, $\Theta_{s_2} \in \Theta \setminus \{\Theta_1, \Theta_2, \Theta_3\}$ such that $e_{i,s_1} > e_{i_2,s_1}$ for all $\theta_i \in \Theta_{s_1}$, $|\Theta_{s_1}| = q_{s_1} - q_0$; $e_{i,s_2} > e_{i_1,s_2}$ for

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all $i \in \Theta_{s_1}$, and $|\Theta_{s_2}| = q_{s_2} - q_0$.

A priority index distribution H is Ergin-acyclic if it allows no Ergin cycles.

Note that this acyclicity condition is satisfied if all schools rank students in the same way. Under this acyclicity condition, we can extend Theorem 6.3 in Haeringer and Klijn (2009) to the continuum economy.

Proposition A3. *In the continuum economy E:*

(i) *Every stable outcome is an outcome of some Nash equilibrium.*

(ii) *If $C(2) = 0$, every (pure-strategy) Nash equilibrium outcome is stable if and only if the economy satisfies Ergin-acyclicity (Haeringer-Klijn, 2009).*

(iii) *If $C(2) > 0$, all (pure-strategy) Nash equilibrium outcomes are stable.*

(iv) *If $C(|L|) = 0$ for all L, the unique trembling hand perfect equilibrium (THPE) is when everyone ranks all schools truthfully, and the corresponding outcome is the student-optimal stable matching.*

Proof. Part (i) can be shown by letting every student i submit a one-school list including only $\mu^\infty(\theta_i)$.

To prove parts (ii) and (iii), we can use the proof of Theorem 6.3 in Haeringer and Klijn (2009) and, therefore, that of Theorem 1 in Ergin (2002). They can be directly extended to the continuum economy under more general DA mechanisms, although they focus on discrete economies. We notice the following:

(a) The continuum economy can be "discretized" in a similar manner as in Example A1 and each subset of students can be treated as a single student. When doing so, we do not impose restrictions on the sizes of the subsets, as long as they have a positive measure. This allows us to use the derivations in the aforementioned proofs.

(b) The flexibility in the cost function of ranking more schools does not impose additional restrictions. As we focus on Nash equilibrium, for any strategy with more than one school listed, we can find a one-school list that has the same or higher payoff. And indeed, many steps in the aforementioned proofs involve such a treatment.

To show part (iv), we notice that a perturbed game is a copy of a base school choice game, with the restriction that only totally mixed strategies are allowed to be played.

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It is known that the set of trembling hand perfect equilibria is the set of undominated strategies, which implies that strict truth-telling is the unique THPE. ∎

A.3 Estimation under the Strict Truth-Telling Assumption with Outside Option

The following discussion supplements Section 2.2 in which we present how weak truth-telling (WTT) can be applied to data on school choice and college admissions and what assumptions it entails. However, assuming the length of submitted ROL is exogenous (Assumption WTT2) may seem restrictive. An alternative way to relax this assumption is to introduce an outside option and to make some school unacceptable to some students.

Suppose that i's utility for her outside option is denoted by $u_{i,0} = V_{i,0} + \epsilon_{i,0}$, where $\epsilon_{i,0}$ is a type I extreme value. We then augment the type space of each student with the outside option and let $\sigma^S : \mathbb{R}^{(S+1)} \times [0,1]^S \to \mathcal{L}$ be an STT pure strategy defined on the augmented preference space. More precisely, one version of the STT assumption contains the following two assumptions:

Assumption (Strict Truth-Telling with Outside Option).
STT1. $\sigma^S(u_i, u_{i,0}, e_i)$ ranks all i's acceptable schools according to her true preferences.
STT2. Students do not rank unacceptable schools: $u_{i,0} > u_{i,s'}$ for all s' not ranked by $\sigma^S(u_i, u_{i,0}, e_i)$.

Given these two assumptions, similar to the case with WTT, either MLE or GMM can be applied based on the following choice probabilities:

$$
\Pr\left(\sigma^S(u_i, u_{i,0}, e_i) = L \mid Z_i; \beta\right)
$$
$$
= \Pr\left(u_{i,l^1} > \cdots > u_{i,l^K} > u_{i,0} > u_{i,s'} \; \forall \, s' \in \mathcal{S} \backslash L \mid Z_i; \beta\right)
$$
$$
= \frac{\exp(V_{i,0})}{\exp(V_{i,0}) + \sum_{s' \notin L} \exp(V_{i,s'})} \prod_{s \in L} \left(\frac{\exp(V_{i,s})}{\exp(V_{i,0}) + \sum_{s' \in \mathcal{S}} \exp(V_{i,s'})} \right).
$$

For an example where the STT assumption is imposed, see He and Magnac (2016) in which the authors observe students ranking all available options and have information on the acceptability of each option.

Assumptions STT1 and STT2 can be justified as an equilibrium outcome when there is no application cost. However, there may be an issue of multiple equilibria created by

unacceptable schools. Namely, if a student can always decline to enroll at an unacceptable school, she may not mind including or excluding that school in her ROL and being assigned to it (He, 2015). This is labeled as the issue of multiple best responses in He (2015), because a student is always indifferent between bottom-ranking or omitting unacceptable schools in her ROLs in any equilibrium.

Appendix B Data

B.1 Data Sources

For the empirical analysis, we use three administrative data sets on Parisian students, which are linked using an encrypted version of the French national student identifer (*Identifiant National Élève*).

(i) **Application Data**: The first data set was provided to us by the Paris Education Authority (*Rectorat de Paris*) and contains all the information necessary to replicate the assignment of students to public academic-track high schools in the city of Paris for the 2013-2014 academic year. This includes the schools' capacities, the students' ROLs of schools, and their priority indices at every school. Moreover, it contains information on students' socio-demographic characteristics (age, gender, parents' SES, low-income status, etc.), and their home addresses, allowing us to compute distances to each school in the district.

(ii) **Enrollment Data**: The second data set is a comprehensive register of students enrolled in Paris' middle and high schools during the 2012–2013 and 2013–2014 academic years (*Base Elèves Académique*), which is also from the Paris Education Authority. This data set allows to track students' enrollment status in all Parisian public and private middle and high schools.

(iii) **Exam Data**: The third data set contains all Parisian middle school students' individual examination results for a national diploma, the *Diplôme national du brevet* (DNB), which students take at the end of middle school. We obtained this data set from the statistical office of the French Ministry of Education (*Direction de l'Évaluation, de la Prospective et de la Performance du Ministère de l'Éducation Nationale*).

B.2 Definition of Variables

Priority Indices. Students' priority indices at every school are recorded as the sum of three main components: (i) students receive a "district" bonus of 600 points on each of the schools in their list which are located in their home district; (ii) students' academic performance during the last year of middle school is graded on a scale of 400 to 600

points; (iii) students from low-income families are awarded an additional bonus of 300 points. We convert these priority indices into percentiles between 0 and 1.

Student Scores. Based on the DNB exam data set, we compute several measures of student academic performance, which are normalized as percentiles between 0 and 1 among all Parisian students who took the exam in the same year. Both French and math scores are used, and we also construct the students' composite score, which is the average of the French and math scores. Note that students' DNB scores are different from the academic performance measure used to calculate student priority indices as an input into the DA mechanism. Recall that the latter is based on the grades obtained by students throughout their final year of middle school.

Socio-Economic Status. Students' socio-economic status is based on their parents' occupation. We use the French Ministry of Education's official classification of occupations to define "high SES": if the occupation of the student's legal guardian (usually one of the parents) belongs to the "very high SES" category (company managers, executives, liberal professions, engineers, academic and art professions), the student is coded as high SES, otherwise she is coded as low SES.[A.3]

B.3   Construction of the Main Data Set for Analyses

In our empirical analysis, we use data from the Southern District of Paris (*District Sud*). We focus on public middle school students who are allowed to continue their studies in the academic track of upper secondary education and whose official residence is in the Southern District. We exclude those with disabilities, those who are repeating the first year of high school, and those who were admitted to specific selective tracks offered by certain public high schools in Paris (e.g., music majors, bilingual courses, etc.), as these students are given absolute priority in the assignment over other students. This leads to the exclusion of 350 students, or 18 percent of the total, the majority of whom are grade repeaters. Our data thus include 1,590 students from 57 different public middle schools, with 96 percent of students coming from one of the district's 24 middle schools.

[A.3]There are four official categories: low SES, medium SES, high SES, and very high SES.

Appendix C Monte Carlo Simulations

This appendix provides details on the Monte Carlo simulations that we perform to assess our empirical approaches and model selection tests. Section C.1 specifies the model, Section C.2 describes the data generating processes, Section C.3 reports a number of summary statistics for the simulated data, Section C.4 presents the estimation and testing procedures, and, finally, Section C.5 discusses the main results.

C.1 Model Specification

Economy Size. We consider an economy where $I = 500$ students compete for admission to $S = 6$ schools. The vector of school capacities is specified as follows:

$$I \cdot \{q_s\}_{s=1}^6 = \{50, 50, 25, 50, 150, 150\}.$$

Setting the total capacity of schools (475 seats) to be strictly smaller than the number of students (500) simplifies the analysis by ensuring that each school has a strictly positive cutoff in equilibrium.

Spatial Configuration. The school district is stylized as a disc of radius 1 (Figure C1). The schools (represented by red circles) are evenly located on a circle of radius $1/2$ around the district centroid; the students (represented by blue circles) are uniformly distributed across the district area. The cartesian distance between student i and school s is denoted by $d_{i,s}$.

Student Preferences. To represent student preferences over schools, we adopt a parsimonious version of the random utility model described in Section 2.1. Student i's utility from attending school s is specified as follows:

$$u_{i,s} = 10 + \alpha_s - d_{i,s} + \gamma(a_i \cdot \bar{a}_s) + \epsilon_{i,s}, \; s = 1, \ldots, 6; \tag{A.7}$$

where $10 + \alpha_s$ is school s's fixed effects; $d_{i,s}$ is the walking distance from student i's residence to school s; a_i is student i's ability; $\bar{a}_s$ is school s's quality; and $\epsilon_{i,s}$ is an error term that is drawn from a type-I extreme value distribution. Setting the effect of distance to -1 ensures that other coefficients can be interpreted in terms of willingness to travel.
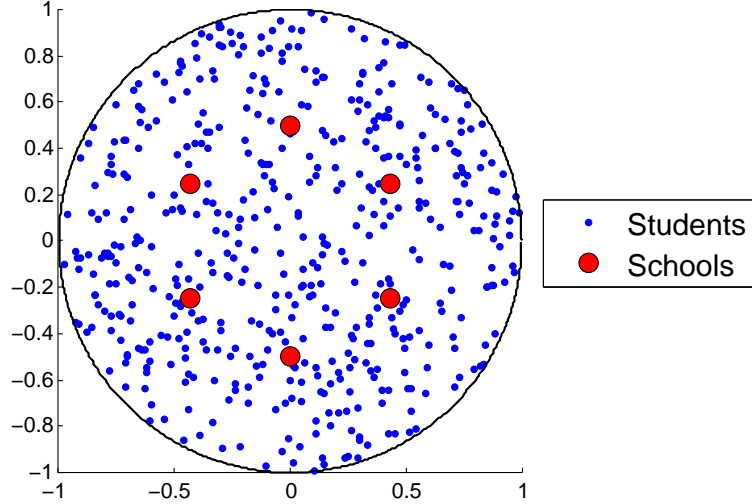
Figure C1: Monte Carlo Simulations: Spatial Distribution of Students and Schools

Notes: This figure shows the spatial configuration of the school district considered in one of the Monte Carlo samples, for the case with 500 students and 6 schools. The school district is represented as a disc of radius 1. The small blue and large red circles show the location of students and of schools, respectively.

The school fixed effects above the common factor, 10, are specified as follows:

$$\{\alpha_s\}_{s=1}^6 = \{0, 0.5, 1.0, 1.5, 2.0, 2.5\}$$

Adding the common value of 10 for every school ensures that all schools are acceptable in the simulated samples.

Students' abilities $\{a_i\}_{i=1}^I$ are randomly drawn from a uniform distribution on the interval $[0, 1]$. School qualities $\{\bar{a}_s\}_{s=1}^S$ are exogenous to students' idiosyncratic preferences $\epsilon_{i,s}$. The procedure followed to ascribe values to the schools' qualities is discussed at the end of this section.

The positive coefficient γ on the interaction term $a_i \cdot \bar{a}_s$ reflects the assumption that high-ability students value school quality more than low-ability students. In the simulations, we set $\gamma = 3$.

Priority Indices. Students are ranked separately by each school based on a school-specific index $e_{i,s}$. The vector of student priority indices at a given school s, $\{e_{i,s}\}_{i=1}^I$ is constructed as correlated random draws with marginal uniform distributions on the interval $[0,1]$, such that: (i) student i's index at each school is correlated with her ability a_i with a correlation coefficient of ρ; (ii) i's indices at any two schools s_1 and s_2 are also correlated with correlation coefficient ρ. When ρ is set equal to 1, a student has

the same priority at all schools. When ρ is set equal to zero, her priority indices at the different schools are uncorrelated. For the simulations presented in this appendix, we choose $\rho = 0.7$. It is assumed that student know their priority indices but not their priority ranking at each school.

School Quality. To ensure that school qualities $\{\bar{a}_s\}_{s=1}^S$ are exogenous to students' idiosyncratic preferences, while being close to those observed in Bayesian Nash equilibrium of the school choice game, we adopt the following procedure: we consider the unconstrained student-proposing DA where students rank all schools truthfully; students' preferences are constructed using random draws of errors and a common prior about the average quality of each school; students rank schools truthfully and are assigned through the DA mechanism; each school's quality is computed as the average ability of students assigned to that school; a fixed-point vector of school qualities, denoted by $\{\bar{a}_s^*\}_{s=1}^S$, is found; the value of each school's quality is set equal to mean value of $\bar{a}_s^*$ across the samples.

The resulting vector of school qualities is:

$$\{\bar{a}_s\}_{s=1}^6 = \{0.28, 0.39, 0.68, 0.65, 0.47, 0.61\}$$

C.2 Data Generating Processes

The simulated data are constructed under two distinct data generating processes (DGPs).

DGP 1: Constrained/Truncated DA. This DGP considers a situation where the student-proposing DA is used to assign students to schools but where the number of schools that students are allowed to rank, K, is strictly smaller than the total number of available schools, S. For expositional simplicity, students are assumed to incur no cost when ranking exactly K schools. Hence:

$$C\left(|L|\right) = \begin{cases} 0 & \text{if } |L| \leqslant K \\ +\infty & \text{if } |L| > K \end{cases}$$

In the simulations, we set $K = 4$ (students are allowed to rank up to 4 schools out of 6).

DGP 2: Unconstrained DA with Cost. This DGP considers the case where students are not formally constrained in the number of schools they can rank but nevertheless incur a constant marginal cost, denoted by $c(>0)$, each time they increase the length of their ROL by one, if this list contains more than one school. Hence:

$$C\left(|L|\right) = c \cdot \left(|L| - 1\right),$$

where the marginal cost c is strictly positive. In the simulations, we set $c = 10^{-6}$.

For each DGP, we adopt a two-stage procedure to solve for a Bayesian Nash equilibrium of the school choice game.

Stage 1: Distribution of Cutoffs Under Unconstrained DA. Students' "initial" beliefs about the distribution of school cutoffs are based on the distribution of cutoffs that arises when students submit unrestricted truthful rankings of schools under the standard DA. Specifically:

(i) For $m = 1, \cdots, M$, we independently generate sample m by drawing students' geographic coordinates, ability $a_i^{(m)}$, school-specific priority indices $e_{i,s}^{(m)}$, and idiosyncratic preferences $\epsilon_{i,s}^{(m)}$ over the S schools for all I students. We then calculate $u_{i,s}^{(m)}$ for all $i = 1, \cdots, I$, $s = 1, \cdots, S$, and $m = 1, \cdots, M$.

(ii) Student i in sample m submits a complete and truthful ranking $r(u_i^{(m)})$ of the schools; i.e., i is strictly truth-telling.

(iii) After collecting $\{r(u_i^{(m)})\}_{i=1}^{I}$, the DA mechanism assigns students to schools taking into account their priority indices in sample m.

(iv) Each matching $\mu^{(m)}$ in sample m determines a vector of school cutoffs $P^{(m)} \equiv \{P_s^{(m)}\}_{s=1}^{S}$.

(v) The cutoffs $\{P^{(m)}\}_{m=1}^{M}$ are used to derive the empirical distribution of school cutoffs under the unconstrained DA, which is denoted by $\hat{F}^0(\cdot \mid \{P^{(m)}\}_{m=1}^{M})$.

In the simulations, we set $M = 500$.

Stage 2: Bayesian Nash Equilibrium. For each DGP, the M Monte Carlo samples generated in Stage 1 are used to solve the Bayesian Nash equilibrium of the school choice game. Specifically:

(i) Each student i in each sample m determines all possible true partial preference orders $\{L_{i,n}^{(m)}\}_{n=1}^{N}$ over the schools, i.e., all potential ROLs of length between 1 and K that respect i's true preference ordering $R_{i,m}$ of schools among those ranked in $L_{i,n}^{(m)}$; for each student, there are $N = \sum_{k=1}^{K} S!/[k!(S-k)!]$ such partial orders. Under the constrained/truncated DA (DGP 1), students consider only true partial preference orderings of length K $(< S)$, i.e., 15 candidate ROLs when they rank exactly 4 schools out of 6;[A.4] under the unconstrained DA with cost (DGP 2), students consider all true partial orders of length up to S, i.e., 63 candidate ROLs when they can rank up to 6 schools.

(ii) For each candidate ROL $L_{i,n}^{(m)}$, student i estimates the (unconditional) probabilities of being admitted to each school by comparing her indices $e_{i,s}$ to the distribution of cutoffs. Initial beliefs on the cutoff distribution are based on $\hat{F}^0(\cdot \mid \{P^{(m)}\}_{m=1}^{M})$, i.e., the empirical distribution of cutoffs under unconstrained DA with strictly truth-telling students.

(iii) Each student selects the ROL $L_i^{(m)*}$ that maximizes her expected utility, where the utilities of each school are weighted by the probabilities of admission according to her beliefs.

(iv) After collecting $\{L_i^{(m)*}\}_{i=1}^{I}$, the DA mechanism is run in sample m.

(v) The matching outcomes across the M samples jointly determine the "posterior" empirical distribution of school cutoffs, $\hat{F}^1(\cdot \mid \cdot)$.

(vi) Students use $\hat{F}^1(\cdot \mid \cdot)$ as their beliefs, and steps (ii) to (v) are repeated until a fixed point is found, which occurs when the posterior distribution of cutoffs ($\hat{F}^t(\cdot \mid \cdot)$) is consistent with students' beliefs $\hat{F}^{t-1}(\cdot \mid \cdot)$. The equilibrium beliefs are denoted by $\hat{F}^*(\cdot \mid \cdot)$.

The simulated school choice data are then constructed based on a new set of M Monte Carlo samples, which are distinct from the samples used to find the equilibrium distribution of cutoffs. In each of these new Monte Carlo samples, submitted ROLs are students' best response to the equilibrium distribution of cutoffs ($\hat{F}^*(\cdot \mid \cdot)$). The school choice data consist of students' priority indices, their submitted ROLs, the student-school matching outcome, and the realized cutoffs in each sample.

[A.4]This is without loss of generality, because in equilibrium the admission probability is non-degenerate and it is, therefore, in students' best interest to rank exactly 4 schools.

C.3 Summary Statistics of Simulated Data

We now present some descriptive analysis on the equilibrium cutoff distributions and the 500 Monte Carlo samples of school choice data that are simulated for each DGP.

Equilibrium Distribution of Cutoffs. The equilibrium distribution of school cutoffs is displayed in Figure C2 separately for each DGP. In line with the theoretical predictions (Proposition A2), the marginal distribution of cutoffs is approximately normal. Because both DGPs involve the same profiles of preferences and produce almost identical matchings, the empirical distribution of cutoffs under the constrained/truncated DA (left panel) is very similar to that observed under the unconstrained DA with cost (right panel).
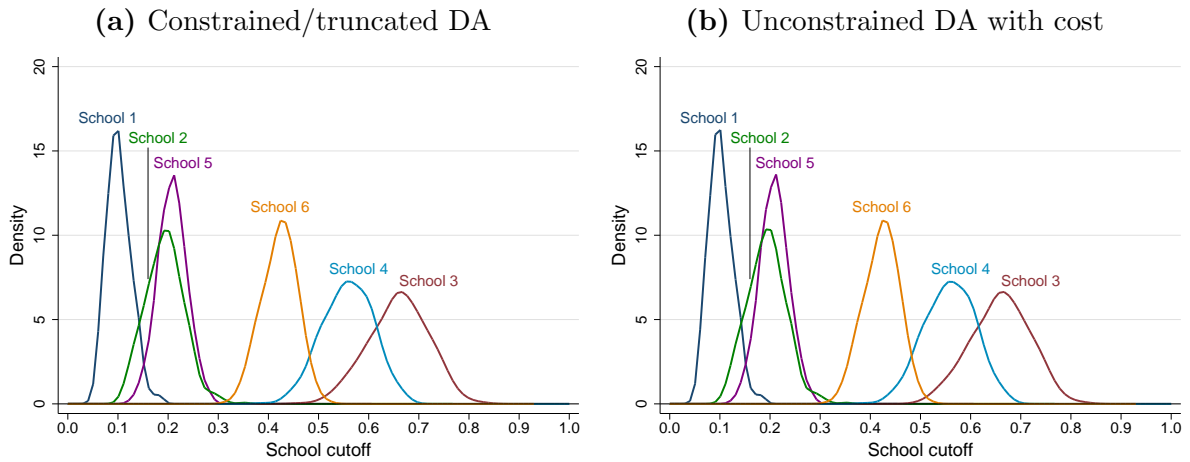

(a) Constrained/truncated DA **(b)** Unconstrained DA with cost

Figure C2: Monte Carlo Simulations: Equilibrium Distribution of School Cutoffs (6 schools, 500 students)

Notes: This figure shows the equilibrium marginal distribution of school cutoffs under the constrained/truncated DA (left panel) and the DA with cost (right panel) in a setting where 500 students compete for admission to 6 schools. With 500 simulated samples, the line fits are from a Gaussian kernel with optimal bandwidth using MATLAB's ksdensity command.

School cutoffs are not strictly aligned with the school fixed effects, since cutoffs are also influenced by school size. In the simulations, small schools (e.g., Schools 3 and 4) tend to have higher cutoffs than larger schools (e.g., Schools 5 and 6) because, in spite of being less popular, they can be matched only with a small number of students, which pushes their cutoffs upward.[A.5]

Figure C3 reports the marginal distribution of cutoffs in the constrained/truncated DA for various economy sizes. The simulations show that as the number of seats and the

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[A.5]Note that this phenomenon is also observed if one sets $\gamma = 0$, i.e., when students' preferences over schools do not depend on the interaction term $a_i \cdot \bar{a}_s$.

number of students increase proportionally while holding the number of schools constant, the distribution of school cutoffs degenerates and becomes closer to a normal distribution.
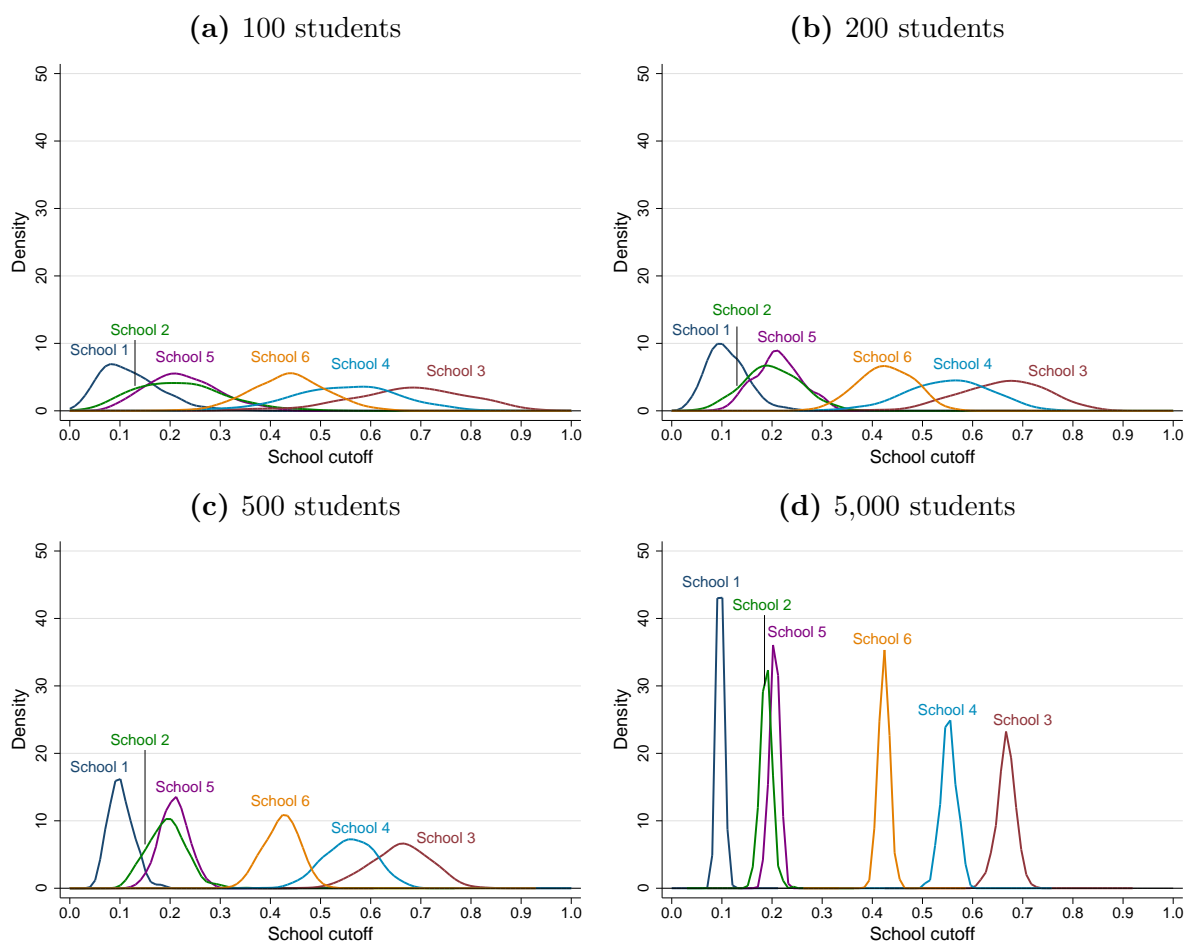

(a) 100 students

(b) 200 students

(c) 500 students

(d) 5,000 students

Figure C3: Monte Carlo Simulations: Impact of Economy Size on the Equilibrium Distribution of Cutoffs (Constrained/Truncated DA)

Notes: This figure shows the equilibrium marginal distribution of school cutoffs under the constrained/truncated DA (ranking 4 out of 6 schools) when varying the number of students, I, who compete for admission into 6 schools with a total enrollment capacity of $I \times 0.95$ seats. Using 500 simulated samples, the line fits are from a Gaussian kernel with optimal bandwidth using MATLAB's ksdensity command.

Summary Statistics. Table C1 shows some descriptive statistics of the simulated data from both DGPs. The reported means are averaged over the 500 Monte Carlo samples.

All students under the constrained/truncated DA submit ROLs of the maximum allowed length (4 schools). Under the unconstrained DA with cost, students are allowed to rank as many schools as they wish but, due to the cost of submitting longer lists, they rank 4.6 schools on average.

A.30

Under both DGPs, all school seats are filled, and, therefore, 95 percent of students are assigned to a school. Weak truth-telling is violated under the constrained/truncated DA, since less than half of submitted ROLs rank truthfully students' most-preferred schools. Although less widespread, violations of WTT are still observed under the unconstrained DA with cost, since about 20 percent of students do not truthfully rank their most-preferred schools. By contrast, almost every student is matched with her favorite feasible school under both DGPs.

Table C1: Monte Carlo Simulations: Summary Statistics

	Data generating process	
	Constrained/truncated DA (1)	Unconstrained DA with cost (2)
Panel A. Outcomes		
Average length of submitted ROL	4.00	4.60
	(0.000)	(0.054)
Assigned to a school	0.950	0.950
	(0.000)	(0.000)
Weakly truth-telling	0.391	0.792
	(0.022)	(0.018)
Assigned to favorite feasible school	1.000	1.000
	(0.000)	(0.000)
Panel B. Parameters		
Number of students	500	500
Number of schools	6	6
Number of simulated samples	500	500
Maximum possible length of ROL	4	6
Marginal application cost (c)	0	10^{-6}

Notes: This table presents summary statistics of simulated data under two DGPs: (i) constrained/truncated DA (column 1): students are only allowed to rank 4 schools out of 6; and (ii) unconstrained DA with cost (column 2): students can rank as many schools as they like, but incur a constant marginal cost of $c = 10^{-6}$ per extra school included in their ROL beyond the first choice. Standard deviations across the 500 simulation samples are in parentheses.

Comparative Statics. To explore how the cost of ranking more schools affects weak truth-telling and ex post stability in equilibrium, we simulated data for DGP 2 (DA with cost) using different values of the cost parameter c, while keeping the other parameters at their baseline values.[A.6]

For each value of the cost parameter, we simulated 500 samples of school choice data and computed the following statistics by averaging across samples: (i) average length of

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[A.6] We performed a similar exercise for DGP 1 (constrained/truncated DA) by varying the number of schools that students are allowed to rank. The results (available upon request) yield conclusions similar to those based on DGP 2 (DA with cost).

submitted ROL; (ii) average fraction of weakly truth-telling students; and (iii) average fraction of students assigned to their favorite feasible school.
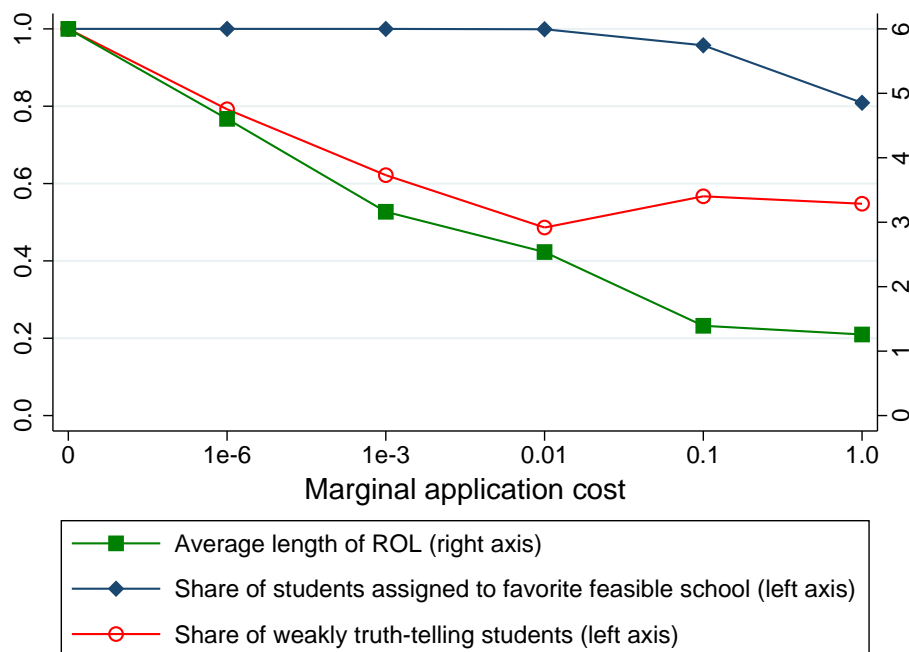

Figure C4: Monte Carlo Simulations: Impact of the Marginal Cost of Applying to Schools on Equilibrium Outcomes (500 Students, 6 Schools)

Notes: This figure presents summary statistics of simulated data under unconstrained DA with cost (DGP 2), in which students can rank as many schools as they like, but incur a constant marginal cost c per extra school included in their ROL beyond the first. The data are simulated using different values of the marginal cost parameter c, while maintaining the other parameters at their baseline values. For each value of the cost parameter, 500 samples of school choice data are simulated. The following statistics are computed by averaging across samples: (i) average length of submitted ROL; (ii) average fraction of weakly truth-telling students; (iii) average fraction of students matched with favorite feasible school.

The results of this comparative statics exercise are displayed in Figure C4. They confirm that, in our simulations, stability is a weaker assumption than WTT whenever students face a cost of ranking more schools: the share of students assigned to their favorite feasible school (blue line) is always larger than the share of WTT students (red line). Consistent with the predictions from Section 1.4.2, the fraction of students who are matched with their favorite feasible school decreases with the marginal cost of ranking more schools (parameter c). In our simulations, violations of this assumption are very rare, except in the extreme case where students face a large marginal application cost c equal to 1 (in which case students rank only 1.3 school on average).

C.4 Estimation and Testing

Identifying Assumptions. With the simulated data at hand, student preferences described by Equation (A.7) are estimated under different sets of identifying assumptions:

(i) *Weak Truth-Telling.* The choice probabilities for individual ROLs can be fully specified and the corresponding rank-ordered logit model is estimated by Maximum Likelihood Estimation (MLE), as discussed in Section 2.2.

(ii) *Stability.* Under the assumption that students are matched with their favorite feasible school given the ex post cutoffs, the model is estimated by MLE based on a conditional logit model where each student's choice set is restricted to the ex post feasible schools and where the matched school is the chosen alternative.[A.7]

(iii) *Stability and Undominated Strategies.* The method of moment (in)equalities in Andrews and Shi (2013) is used to obtain point estimates, where conditional moment inequalities are derived from students' observed orderings of all 15 possible pairs of schools (see Section 2.5). The variables that are used to interact with these conditional moment inequalities and thus to obtain the unconditional ones are student ability (a_i), distance to School 1 ($d_{i,1}$) and distance to School 2 ($d_{i,2}$), which brings the total number of moment inequalities to 120. The approach proposed by Bugni et al. (2017) is used to construct the marginal confidence intervals for the point estimates.

Model Selection Tests. Two tests are implemented:

(i) *Truth-Telling vs. Stability.* This test is carried out by constructing a Hausman-type test statistic from the estimates of the WTT and stability approaches. Under the null hypothesis that students are WTT, both estimators $\hat{\beta}_{TT}$ and $\hat{\beta}_{ST}$ are consistent but only $\hat{\beta}_{TT}$ is asymptotically efficient. Under the alternative that the matching outcome is stable but students are not WTT, only $\hat{\beta}_{ST}$ is consistent.

(ii) *Stability vs. Undominated Strategies.* As shown in Section 2.5, stability implies a set of moment equalities, while undominated strategies lead to another set of moment inequalities. When undominated strategies are assumed to be satisfied, testing

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[A.7]The stability-based estimator can be equivalently obtained using a GMM estimation with moment equalities defined by the first-order conditions of the log-likelihood function.

stability amounts to checking if the identified set of the moment (in)equality model is empty. Specifically, we apply the test proposed by Bugni et al. (2015) (Test RS).

C.5   Results

The results from 500 Monte Carlo samples are reported in the main text (Table 2). They are consistent with the theoretical predictions for both the constrained/truncated DA (Panel A) and the unconstrained DA with cost (Panel B).

Weak Truth-Telling.   The coefficients reported in column 2 of Table 2 show that violation of the WTT assumption leads to severely biased estimates. In both DGPs, students' valuation of popular schools tends to be underestimated. This problem is particularly acute when one considers the smaller schools (e.g., Schools 3 and 4), which often have higher cutoffs than the larger ones (see Figure C2), and are therefore more often left out of students' ROLs due to low admission probabilities. The WTT-based estimator is more biased under the constrained/truncated DA.

Stability.   By contrast, estimation under the assumption that the matching outcome is stable performs well in our simulations. The point estimates (column 5) are reasonably close to the true parameter values, although they are more dispersed than the WTT (column 6 vs. column 3). This efficiency loss is a direct consequence of restricting the choice sets to include only feasible schools and of considering a single choice situation for every matched student.

Test of Truth-Telling vs. Stability.   Under the assumption that the matching outcome is stable, the Hausman test strongly rejects truth-telling in the constrained/truncated DA simulations (last row of Panel A in Table 2) and rejects this assumption in 37 percent of the samples simulated under the unconstrained DA with cost (last row of Panel B), under which truth-telling is violated for only 21 percent of students (Table C1).

Stability and Undominated Strategies.   The results from the moment (in)equalities approach show that in the two specific cases under study, the over-identifying information provided by students' true partial orders of schools has only a small impact on estimates

A.34

(column 8 vs. column 5).[A.8] On average, estimates based on the method of moment (in)equalities are closer to the true values of the parameters (column 9 vs. column 6) but, unfortunately, the marginal confidence intervals obtained using the Bugni et al. (2017) approach tend to be conservative, especially relative to the stability-based estimates from MLE. As a result, the coverage probability (CP) of 95 percent confidence interval is close to one (column 10 vs. column 7).

Stability vs. Undominated Strategies. Stability is tested against undominated strategies by applying the test proposed by Bugni et al. (2015) to check whether the identified set of the moment (in)equality model is empty. This tests fails to reject stability in all samples (last row of Panel A and of Panel B).[A.9]

Efficiency Loss from Stability-Based Estimates. The efficiency loss from estimating the model under stability is further explored by comparing the truth-telling-based and stability-based estimates in a setting where students are strictly truth-telling. To that end, we generate a new set of 500 Monte Carlo samples using the unconstrained DA DGP, after setting the marginal application cost c to zero. In this setting, all students submit truthful ROLs that rank all 6 schools. The estimation results, which are reported in Table C2, show that while both truth-telling-based and stability-based estimates are close to the true parameters values, the latter are much more imprecisely estimated than the former (column 6 vs. column 3): the stability-based estimates have standard deviations 2.5 to 3.8 times larger than the TT-based estimates. Note, however, that the efficiency loss induced by the stability assumption is considerably reduced when combing stability and undominated strategies (column 9 vs. column 3): the standard deviations of estimates based on the moment (in)equality approach are only 1.3 to 1.9 larger than their truth-telling counterparts.

Reassuringly, the Hausman test rejects truth-telling against stability in exactly 5 percent of samples, which is the intended type-I error rate. This test can therefore serve as a useful tool to select the efficient truth-telling-based estimates over the less efficient

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[A.8]Larger improvements are obtained when we relax the constraint on the number of choices than students can submit or when we reduce the marginal cost of ranking an extra school (results available upon request).

[A.9]To evaluate the power of the two tests, especially the test for stability, we constructed some simulation examples in which stability is rejected when 30 percent of students are not assigned to their favorite feasible school. These results are available upon request.

stability-based estimates when both assumptions are satisfied.

Table C2: Monte Carlo Results: Unconstrained DA (500 Students, 6 Schools, 500 Samples)

	True value (1)	Weak Truth-telling			Stability			Stability and undominated strategies		
		Mean (2)	SD (3)	CP (4)	Mean (5)	SD (6)	CP (7)	Mean (8)	SD (9)	CP (10)
PARAMETERS										
School 2	0.50	0.50	0.10	0.94	0.51	0.29	0.94	0.50	0.12	1.00
School 3	1.00	1.01	0.16	0.95	1.05	0.58	0.96	1.00	0.22	1.00
School 4	1.50	1.52	0.15	0.95	1.54	0.52	0.96	1.51	0.21	1.00
School 5	2.00	2.02	0.11	0.96	2.02	0.30	0.96	2.02	0.14	1.00
School 6	2.50	2.52	0.14	0.94	2.54	0.45	0.96	2.53	0.19	1.00
Own ability × school quality	3.00	2.98	0.66	0.95	2.96	2.29	0.96	3.08	0.99	1.00
Distance	−1.00	−1.00	0.08	0.96	−1.01	0.20	0.95	−1.02	0.16	1.00

MODEL SELECTION TESTS

Truth-telling (H_0) vs. Stability (H_1): H_0 rejected in 5.0% of samples (at 0.05 significance level).

Stability (H_0) vs. Undominated strategies (H_1): H_0 rejected in 0% of samples (at 0.05 significance level).

Notes: This table reports Monte Carlo results from estimating students' preferences under different sets of identifying assumptions: (i) weak truth-telling; (ii) stability; (iii) stability and undominated strategies. 500 Monte Carlo samples of school choice data are simulated under the following data generating process for an economy in which 500 students compete for admission to 6 schools: an unconstrained DA where students can rank as many schools as they wish, with no cost for including an extra school in their ROL. Under assumption (iii), the model is estimated using Andrews and Shi (2013)'s method of moment (in)equalities. Column 1 reports the true values of the parameters. The mean and standard deviation (SD) of point estimates across the Monte Carlo samples are reported in columns 2, 5 and 8, and in columns 3, 6 and 9, respectively. Columns 4, 7 and 10 report the coverage probabilities (CP) for the 95 percent confidence intervals. The confidence intervals in models (i) and (ii) are the Wald-type confidence intervals obtained from the inverse of the Hessian matrix. The marginal confidence intervals in model (iii) are computed using the method proposed by Bugni et al. (2017). Truth-telling is tested against stability by constructing a Hausman-type test statistic from the estimates of both approaches. Stability is tested against undominated strategies by checking if the identified set of the moment(in)equality model is empty, using the test proposed by Bugni et al. (2015).

Appendix D Goodness of Fit

This appendix presents the goodness-of-fit statistics that we use to compare the estimates of the model described by Equation (11) in the main text. These estimates are obtained under different sets of identifying assumptions and are reported in Table 5. The goodness-of-fit statistics described in Section D.1 are based on simulation techniques, whereas those described in Section D.2 use closed-form expressions for the choice probabilities (due to the logit specification).

D.1 Simulation-Based Goodness-of-Fit Measures

To compare different estimators' ability to predict school cutoffs and students' assignment, we use several simulation-based goodness-of-fit statistics. We keep fixed the estimated coefficients and $Z_{i,s}$, and draw utility shocks as type-I extreme values. This leads to the simulated utilities for every student in 300 simulation samples. When studying the WTT-based estimates, we let students submit their top 8 schools according to their simulated preferences; the matching outcome is obtained by running DA. For the other sets of estimates, because stability is assumed, we focus on the unique stable matching in each sample, which is calculated using students' priority indices and simulated ordinal preferences.

Predicted Cutoffs. Observed school cutoffs are compared to those simulated using the different estimates. The results, which are averaged over the 300 simulated samples, are reported in Table D3, with standard deviations across the samples in parentheses (see Figure 4 in the main text for a graphical representation).

Predicted Assignment. Students' observed assignment is compared to their simulated assignment by computing the average predicted fraction of students who are assigned to their observed assignment school; in other words, this is the average fraction of times each student is assigned to her observed assignment in the 300 simulated samples, with standard deviations across the simulation samples reported in parentheses. The results are reported in Panel A of Table 6 in the main text.

D.2 Predicted vs. Observed Partial Preference Order

Our final set of goodness of fit measures involves comparing students' observed partial preference order (revealed by their ROL) with the predictions based on different sets of identifying assumptions. We use two distinct measures: (i) the first is the mean predicted probability of the observed ordering of students' top two choices, which is averaged across students; (ii) the second measure is the mean predicted probability of the observed ordering of students' full list of choices. Because of the type-I extreme values, we can exactly calculate these probabilities. The results are reported in Panel B of Table 6 in the main text.

Table D3: Goodness of Fit: Observed vs. Simulated Cutoffs

| | Observed cutoffs | Cutoffs in simulated samples with estimates from | | |
| | | Weak Truth-telling | Stability of the matching outcome | Stability and undominated strategies |
	(1)	(2)	(3)	(4)
School 1	0.000	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
School 2	0.015	0.004 (0.006)	0.024 (0.012)	0.019 (0.013)
School 3	0.000	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
School 4	0.001	0.043 (0.015)	0.017 (0.007)	0.017 (0.008)
School 5	0.042	0.064 (0.016)	0.053 (0.012)	0.040 (0.013)
School 6	0.069	0.083 (0.025)	0.078 (0.022)	0.062 (0.024)
School 7	0.373	0.254 (0.020)	0.373 (0.010)	0.320 (0.012)
School 8	0.239	0.000 (0.001)	0.241 (0.023)	0.153 (0.047)
School 9	0.563	0.371 (0.033)	0.564 (0.017)	0.505 (0.023)
School 10	0.505	0.393 (0.029)	0.506 (0.011)	0.444 (0.014)
School 11	0.705	0.409 (0.040)	0.705 (0.009)	0.663 (0.013)

Notes: This table compares the cutoffs, observed for the 11 high schools of the Southern District of Paris in 2013, to the average cutoffs simulated under various identifying assumptions as in Table 5. The reported values for the simulated cutoffs are averaged over 300 simulated samples, and the standard deviations across the samples are reported in parentheses. In all simulations, we vary only the utility shocks, which are kept common across columns 2–4.
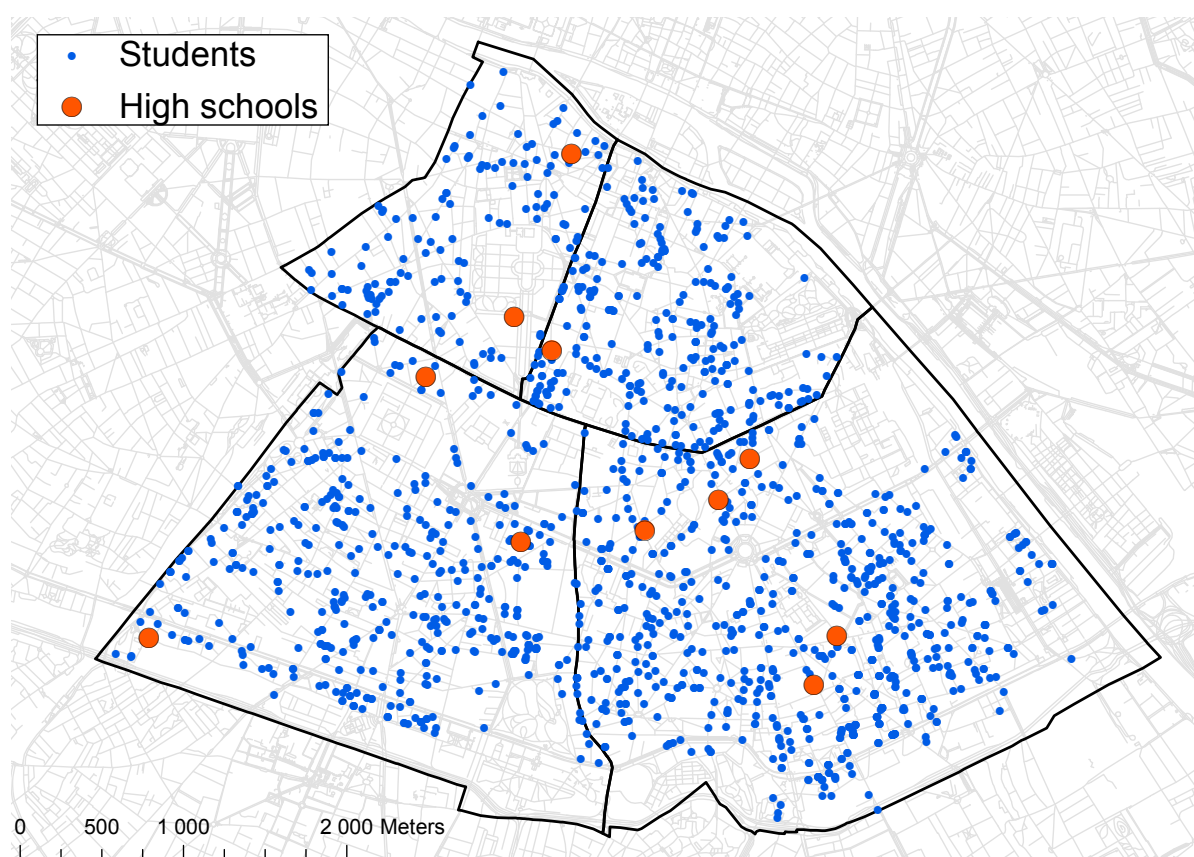
Appendix E   Supplementary Figure and Table

Figure E5: The Southern District of Paris for Public High School Admissions

Notes: The Southern District of Paris covers four of the city's 20 *arrondissements* (administrative divisions): 5th, 6th, 13th and 14th. The large red circles show the location of the district's 11 public high schools (*lycées*). The small blue circles show the home addresses of the 1,590 students in the data.

Table E4: Assigned and Unassigned Students in the Southern District of Paris

	Sample	
	Assigned	Unassigned
Panel A. Student characteristics		
Age	15.0	15.0
Female	0.51	0.45
French score	0.56	0.45
Math score	0.54	0.47
Composite score	0.55	0.46
high SES	0.48	0.73
With low-income bonus	0.16	0.00
Panel B. Enrolment outcomes		
Enrolled in assignment school	0.96	
Enrolled in another public school	0.01	0.65
Enrolled in a private school	0.03	0.35
Number of students	1,568	22

Notes: The summary statistics reported in this table are based on administrative data from the Paris Education Authority (*Rectorat de Paris*), for students who applied to the 11 high schools of Paris's Southern District for the academic year starting in 2013. All scores are from the exams of the *Diplôme national du brevet* (DNB) in middle school and are measured in percentiles and normalized to be in $[0, 1]$. Enrollment shares are computed for students who are still enrolled in the Paris school system at the beginning of the 2013-2014 academic year (97 percent of the initial sample). Students unassigned after the main round have the possibility of participating in a supplementary round, but with choices restricted to schools with remaining seats.