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A stochastic model for speculative dynamics

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Abstract. The aim of this paper is to provide a simple model for describing speculative dynamics and to focus on the study of some associated quantities of interest. Starting from a description of individual speculative behaviors, we build a second order non-reversible Markov process, which after simple transformations can be viewed as a *turning* two-dimensional Gaussian process. Then, our main problem is to obtain some bounds for the *persistence rate* relative to the return time to a given level. We prove with both spectral and probabilistic methods that this rate is almost proportional to the turning frequency ω of the model and provide some explicit bounds. The persistence rate being strongly linked to the quasi-stationary distribution of the problem, we also prove its existence. The main results are established with a careful manipulation of a strongly degenerate hypoelliptic second order Markov operator, which significantly complicates the spectral analysis.

1. Introduction

One commonly talks of financial speculation when, due to an anticipating phenomenon, an asset price strongly exceeds its *asset fundamental value*. The investors, "predicting" an increase of the price, choose to buy some assets with the objective to resell it at an even higher price in the future. Financial bubbles can be seen

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as some typical consequences of such anticipative behavior. There exists a lot of famous historical examples: the Dutch Tulip Mania (1634-1637), the Mississippi speculative frenzy (1718-1720), the Roaring '20s (1920-1929). We refer to Garber (1990) for a general background. Through the recent history, one can observe that this phenomena is still actual: the dot-com boom that occurred in March 2000 after having led astronomical heights and lost more than 75% of its value, the housing bubble encountered in the United States (2000-2010) or in Europe (Spain, Ireland, France ...) (see *e.g.* Shiller, 2000 or Friggit, 2015). There exists a large literature on the modelling of speculative bubbles. For instance, the interested reader can find a series of economics references in Scheinkman and Xiong (2003) and a more abstract approach in a recent work of Protter (2013).

Here, our starting point of view already departs from this last reference: our main objective is to give a dynamical interpretation of the speculative behavior of the investors. According to Shiller Shiller (2000), one of the main features of this behavior is the following:

"If asset prices start to rise strongly, the success of some investors attracts public attention that fuels the spread of the enthusiasm for the market: (often, less sophisticated) investors enter the market and bid up prices. (...) If prices begin to sag, pessimism can take hold, causing some investors to exit the market. Downward price motion begets expectations of further downward motion, and so on, until the bottom is eventually reached."

Hence, we propose a mathematical modelling of the following tendency of thought of the investors: "if the prices have been increasing (respectively decreasing) for some times, they will keep on increasing (resp. decreasing) in the next future". Of course, economic reality will prevent that this trend leads to explosion or to implosion and here our objective is to build a class of dynamics accounting for this phenomenon and integrating some noise.

Our model is obtained as the limit of the mean dynamics of a large number of agents (see next paragraph for more details). This limiting model is a one-dimensional diffusion process, which after simple transformations can be viewed as a two-dimensional hypoelliptic Gaussian diffusion. Furthermore, under some assumptions on the speculative parameter, this process is "turning", *i.e.* the transformation matrix of its deterministic part has complex eigenvalues. This turning property certainly expresses a *periodic structure* for the one-dimensional dynamics. As pointed by Evans (1991), such a phenomenon is a characteristic of markets which are submitted to speculation.

Among the numerous questions generated by such a model, we choose to mainly focus on the effects of this turning property on the return time τ of the process to the equilibrium (or asset fundamental price). Our main contribution is then to prove that the tail of the distribution τ is strongly different to that of a non-speculative market. More precisely, this tail is bounded by some exponential tails whose rate is essentially proportional to the frequency ω of the model and independent of the variance of the noise component. Furthermore, the bounds are quantitative and even if the results are mainly stated according to the particular setting of speculative dynamics, the proofs apply for any turning two-dimensional Gaussian diffusion.

Finally, we connect this problem with the quasi-stationary distributions (QSD) associated to the process and to the return time. In this setting, the rate introduced in the previous paragraph is usually called *persistence* (or *extinction*) rate. We then prove the existence of the QSD, which leads to a correct definition of the persistence rate. Before going further, let us remark that the model we will describe is not flexible enough to take into account several economic realities: we omit some considerations on inflation, credit crunch Farhi and Tirole (2012), regulating effect of any federal bank Bernanke and Gertler (2001). Moreover, it is empirically observed that the bubbles bursts are fasten than their formations and our model misses the "loss aversion". We leave all these potential improvements to a future work.

1.1. *Modelling of speculation.* **The model.** Let us denote by $X \coloneqq (X_t)_{t\geq 0}$ the dynamics of the price of an asset. Without loss of generality, we assume that the price is relative: it is the difference of the "true" price of a commodity with another one, which is the equilibrium price (this will lead us to a simple definition of the return time in the sequel). We assume that three mechanisms are at work for the evolution of X:

- Economic reality plays the role of a restoring force, and draws X back toward zero. As a first approximation, it is natural to assume that this force is linear. The associated rate will be denoted $a > 0$.
- Speculation is reinforcing a tendency observed for some times in the past. We make the hypothesis that the weight of past influences is decreasing exponentially fast in time, with rate $b > 0$.
- Uncertainty is modeled by a Brownian motion of volatility $c > 0$, which is a traditional assumption for randomness coming from a lot of small unpredictable and independent perturbations.

Putting together these three leverages, the evolution of X is described by the S.D.E.

$$\forall\, t \geq 0, \qquad dX_t \quad = \quad -aX_t dt + \left(b\int_0^t \exp(b(s-t))\, dX_s\right) dt + c dB_t. \tag{1.1}$$

Before going further, let us explain more precisely its construction: for the sake of simplicity, let us assume in the sequel of this section that initially, $X_0 = 0$.

Because of the presence of the Brownian motion $(B_t)_{t\geq 0}$ in the r.h.s., the trajectories of X are not differentiable with respect to the time parameter $t \geq 0$. But for the purpose of an heuristic interpretation, let us pretend they are, so we can consider $X'_t \coloneqq \frac{dX_t}{dt}$. Assume furthermore that the "origin" of time was chosen so that before it, X was zero, namely, in the above economic interpretation, the two commodities had their prices tied up at their relative equilibrium point before time 0. This enables us to define $X'_t = X_t = 0$ for any $t \leq 0$. The middle term of the r.h.s. of (1.1) can then be rewritten as

$$\begin{aligned} b\int_0^t \exp(b(s-t))\, dX_s &= \int_0^{+\infty} X'_{t-s}\, b\exp(-bs) ds \\ &= \mathbf{E}[X'_{t-\sigma}] \end{aligned} \tag{1.2}$$

where σ is distributed as an exponential variable of parameter b and where $\mathbf{E}$ stands for the expectation with respect to σ (i.e. not with respect to the randomness underlying X, the corresponding expectation will be denoted $\mathbb{E}$). Thus X has a drift taking into account its past tendencies X', but very old ones are almost forgotten, due to the exponential weight.

Microscopic interpretation. Equation (1.1) can be seen as the limit dynamics of the means of (relative) prices predicted by a large number $N \in \mathbb{N}$ of speculative agents. Assume that each agent $n \in [\![N]\!] \coloneqq \{1, ..., N\}$ has his own idea of the evolution of the prices, designated by $X(n) \coloneqq (X_t(n))_{t\geq 0}$. The mean process $\bar{X} \coloneqq (\bar{X}_t)_{t\geq 0}$ is defined by

$$\forall\ t \geq 0, \qquad \bar{X}_t \ \coloneqq \ \frac{1}{N} \sum_{n\in[\![N]\!]} X_t(n).$$

For simplicity, we assume as above that all these processes were also defined for negative times:

$$\forall\ t \leq 0, \forall\ n \in [\![N]\!], \qquad X_t(n) \ = \ \bar{X}_t \ = \ 0.$$

At any time $t \geq 0$, we also assume that each agent $n \in [\![N]\!]$ has access to the whole past history $(\bar{X}_s)_{s\leq t}$ of the mean prices (say, which is published by a particular institute or website). But to handle this wealth of information, agent n has chosen, once for all, a time window length $\Upsilon(n) > 0$ and computes the ratio $(\bar{X}_t - \bar{X}_{t-\Upsilon(n)})/\Upsilon(n)$ to estimate the present tendency of the prices. Then he interferes that this tendency contributes to the infinitesimal evolution of his estimate of prices $dX_t(n)$ via the term $(\bar{X}_t - \bar{X}_{t-\Upsilon(n)})/\Upsilon(n)\, dt$, speculating that what has increased (respectively decreased) will keep on increasing (resp. decreasing). Nevertheless, he also undergoes the strength of the economic reality with rate $a > 0$, which adds a term $-aX_t(n)dt$ to his previsions. Furthermore, we assume that these evaluations can be pertubed by some random events which are modeled through the infinitesimal increment $c\sqrt{N}dB_t(n)$, where $B(n) \coloneqq (B_t(n))_{t\geq 0}$ is a standard Brownian motion. The factor $\sqrt{N}$ accounts for the fact that the consequences of random events are amplified by a large population. Alternately, it could be argued that $\sqrt{N}dB_t(n)$ decomposes into $\sum_{m\in[\![N]\!]} dB_t(n, m)$, where $(B_t(n, m))_{t\geq 0}$, for $n, m \in [\![N]\!]$, are independent Brownian motions standing respectively for the random perturbations induced by m on n (including a self-influence). The previous description leads to the following individual dynamics:

$$\forall\ t \geq 0, \qquad dX_t(n) \ = \ -aX_t(n)dt + \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)}dt + c\sqrt{N}dB_t(n).$$

It follows that

$$\forall\ t \geq 0, \quad d\bar{X}_t \ = \ -a\bar{X}_t dt + \left(\frac{1}{N}\sum_{n\in[\![N]\!]} \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)}\right) dt + c\frac{1}{\sqrt{N}} \sum_{n\in[\![N]\!]} dB_t(n)$$

Let us assume that all the $\Upsilon(n)$, for $n \in [\![N]\!]$, and all the $B(m)$, for $m \in [\![N]\!]$ are independent. A first consequence is that the process $\bar{B} = (\bar{B}_t)_{t\geq 0}$ defined by

$$\forall\ t \geq 0, \qquad \bar{B}_t \ \coloneqq \ \frac{1}{\sqrt{N}} \sum_{n\in[\![N]\!]} B_t(n)$$

is a standard Brownian motion. Next, under the hypothesis that all the $\Upsilon(n)$, $n \in [\![N]\!]$ have the same law as a random variable Υ and that $\int_0^1 s^{-1}\mathbb{P}_\Upsilon(ds) < +\infty$, we get by the law of large numbers (which can be applied under the previous assumptions), that almost surely,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\in[\![N]\!]} \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)} \quad = \quad \mathbf{E}\left[\frac{\bar{X}_t - \bar{X}_{t-\Upsilon}}{\Upsilon}\right]$$

where $\mathbf{E}$ stands for the expectation with respect to Υ only. Thus letting N go to infinity, $\bar{X}$ ends up satisfying the same evolution equation as X, if the law of Υ is such that

$$\forall\, t \geq 0, \qquad \mathbf{E}\left[\frac{X_t - X_{t-\Upsilon}}{\Upsilon}\right] \quad = \quad b\int_0^t \exp(b(s-t))\, dX_s \tag{1.3}$$

almost surely with respect the trajectory $(X_s)_{s\in\mathbb{R}}$.
Contrary to the first guess which could be made, Υ should not be distributed according to an exponential law of parameter b:

Lemma 1. Assume that $\int_0^{+\infty} s^{-1}\mathbb{P}_\Upsilon(ds) < +\infty$. Then, for any continuous semi-martingale $X = (X_t)_{t\in\mathbb{R}}$ with $X_t = 0$ for $t \leq 0$, (1.3) is satisfied if and only if Υ is distributed as a gamma law $\Gamma_{2,b}$ of shape 2 and scale b, namely if

$$\forall\, t \geq 0, \qquad \mathbf{P}[\Upsilon \in dt] \;=\; \Gamma_{2,b}(dt) \;\coloneqq\; b^2 t \exp(-bt)\, dt$$

The proof of this lemma is achieved in Appendix A.

Remark 2 The law $\Gamma_{2,b}$ has the same rate b of exponential decrease of the queues at infinity as the exponential distribution $\Gamma_{1,b}$ of σ in (1.2). The most notable difference between these two distributions is their behavior near zero: it is much less probable to sample a small values under $\Gamma_{2,b}$ than under $\Gamma_{1,b}$. Furthermore, $\Gamma_{2,b}$ is a little more concentrated around its mean $2/b$ than $\Gamma_{1,b}$ around its mean $1/b$, their respective relative standard deviations being $1/2$ and 1. These features are compatible with the previous modelling: the chance is small that an agent looks shortly in the past to get an idea of the present tendency of X and the dispersion of the lengths of the windows used by the agents may not be very important. These behaviors would be amplified, if instead of $\Gamma_{2,b}$, we had chosen a gamma distribution $\Gamma_{k,b}$ of shape k and scale b, with $k \in \mathbb{N}\setminus\{1,2\}$, for the law of Υ. The limit evolution in this situation is dictated by the stochastic differential equation in $X^{[k]}$ given by

$$\forall\, t \geq 0, \qquad dX_t^{[k]} \quad = \quad -aX_t^{[k]}dt + \left(b(k-2)!\int_0^t g_{b,k}(t-s)\, dX_s^{[k]}\right) dt + c dB_t$$

(starting again from $X_0^{[k]} = 0$), where $g_{b,k}$ is the function defined by

$$g_{b,k} \;:\; \mathbb{R}_+ \ni s \quad\mapsto\quad \exp(-bs) \sum_{l\in[\![0,k-2]\!]} \frac{(bs)^l}{l!}$$

For $k = 2$, we recover (1.1) and $X^{[2]} = X$. It can be shown that $X^{[k]}$ is a Markov process of order k: it can be represented as the first component of a k-dimensional Markov process. While this observation provides opportunities of better modellings, the investigation of $X^{[k]}$ for $k > 2$ (as well as the extension to non-integer values of k) is deferred to a future paper.

□

Here we will concentrate on the properties of the 2-order Markov process $X^{[2]} = X$ (see below for the construction of the associated two-dimensional Markov process).

In order to get a first idea of its behavior, one begins by representing several simulations of X in Figure 1.1.

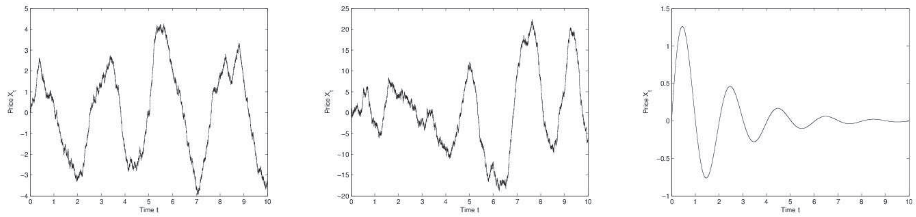

FIGURE 1.1. Several paths for various parameters (left: $a = 1, b = 5, c = 1$, middle: $a = 1, b = 10, c = 5$, right: $a = 1, b = 10, c = 0$).

A periodic structure appears, as that observed in practice in the forming of speculative behaviors. In particular, even in the presence of noise, the process X has a tendency to return periodically to its equilibrium position. In some sense, the variety of the paths is less rich than that experienced by traditional Ornstein-Uhlenbeck (O.U. for short) processes, suggesting a concentration of the trajectory laws around some periodic patterns. Figure 1.2 shows the density of the return time of the process $(X_t)_{t\geq 0}$ to its equilibrium price 0. These results have been obtained using a large number of Monte-Carlo simulations. One may remark in Figure 1.2 that the tail of the return time to equilibrium state is much smaller for our speculative process than the one of the O-U. process with the same invariant measure on the X coordinate and with the same amount of injected randomness (namely through a standard Brownian motion). The purpose of this paper is to quantify these behaviors.

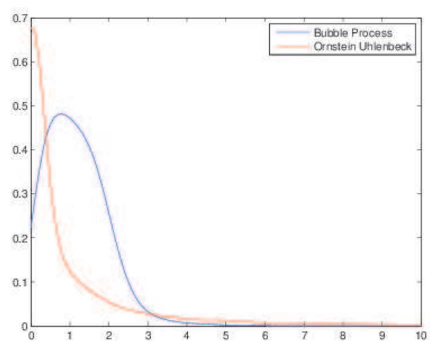

FIGURE 1.2. Numerical comparison between the densities of the return time of the O.U. process and of the speculative dynamics.

1.2. *Results.* Our main result is Theorem 4. Before stating it, let us give a series of properties of the process X.

General properties. We briefly propose in this paragraph a study of the mixing property of the process X and compute the convergence rate to its "invariant" distribution. As already mentioned, the process X whose evolution is driven by (1.1) is not Markovian. Nevertheless, it is not so far away from being Markovian: consider the process $Y \coloneqq (Y_t)_{t\geq 0}$ defined by

$$\forall\, t \geq 0, \qquad Y_t \;\coloneqq\; b\int_0^t \exp(b(s-t))\, dX_s - bX_t.$$

The process $Z \coloneqq (Z_t)_{t\geq 0} \coloneqq ((X_t, Y_t)^*)_{t\geq 0}$ (where * stands for the transpose operation) is then Markovian and its evolution is dictated by the simple 2-dimensional stochastic differential equation

$$\forall\, t \geq 0, \qquad dZ_t \;=\; AZ_t\, dt + C\, dB_t \tag{1.4}$$

starting from $Z_0 = 0$ and where

$$A \;\coloneqq\; \begin{pmatrix} b-a & 1 \\ -b^2 & -b \end{pmatrix} \quad \text{and} \quad C \;\coloneqq\; \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{1.5}$$

The linearity of (1.4) and the fact that the initial condition is deterministic imply that at any time $t \geq 0$ the distribution of Z_t is Gaussian. As it will be checked in next section, this distribution converges for large time $t \geq 0$ toward μ, a centered normal distribution and whose variance matrix Σ is positive definite.

Since the Markov process Z is Feller, μ is an invariant probability measure for Z. It is in fact the only one, because the generator L associated to Equation (1.4) and given by

$$L \;\coloneqq\; ((b-a)x + y)\partial_x - (b^2x + by)\partial_y + \frac{c}{2}\partial_x^2 \tag{1.6}$$

is hypoelliptic (also implying that Σ is positive definite).
The study of the ergodicity of Z begins with the spectral resolution of A. Three situations occur:
• If $a > 4b$, A admits two real eigenvalues, $\lambda_\pm \coloneqq (-a \pm \sqrt{a^2 - 4ab})/2$.
• If $a = 4b$, A is similar to the 2×2 Jordan matrix associated to the eigenvalue $-a/2$.
• If $a < 4b$, A admits two conjugate complex eigenvalues, $\lambda_\pm \coloneqq (-a \pm i\sqrt{4ab - a^2})/2$.
But in all cases, let $l < 0$ be the largest real part of the eigenvalues, namely

$$l \;\coloneqq\; \frac{-a + \sqrt{(a^2 - 4ab)_+}}{2} \tag{1.7}$$

This quantity is the exponential rate of convergence of μ_t, the law of Z_t, toward μ, in the $\mathbb{L}^2$ sense: for $t > 0$, measure the discrepancy between μ_t and μ through

$$J(\mu_t, \mu) \;\coloneqq\; \sqrt{\int \left(\frac{d\mu_t}{d\mu} - 1\right)^2 d\mu} \tag{1.8}$$

Since X was our primary object of interest, let us also denote by ν and ν_t the first marginal distributions of μ and μ_t respectively.

Proposition 3. We have

$$\lim_{t\to+\infty} \frac{1}{t}\ln(J(\mu_t, \mu)) \;=\; 2l \;=\; \lim_{t\to+\infty} \frac{1}{t}\ln(J(\nu_t, \nu))$$

The proof of this proposition is given in Subsection 2.1. The convergences obtained can be extended to other discrepancy measures or to more general initial distributions μ_0 of Z_0 (at least under the assumption that $J(\mu_0, \mu) < +\infty$). Thus if we look at $X_{r/|l|}$ for large $r > 0$, it has almost forgotten its starting point and its law is close to the Gaussian distribution ν, up to an error $\exp(-(1+\circ(1))r)$.

Now, remark that the periodicity phenomenon of each trajectory cannot be easily deduced from the mixing property of Proposition 3. Furthermore, the periodicity features only appear for $a < 4b$, as it can be guessed from the existence of non-real eigenvalues, which suggests $2\pi/\omega$ as period, where

$$\omega \quad := \quad \sqrt{ab - \frac{a^2}{4}}. \tag{1.9}$$

In the regime where $b \gg a$, we have $\omega \gg 2|l|$: a lot of periods has to alternate before stationarity is approached. This phenomenon is often encountered in the study of ergodic Markov processes which are far from being reversible, *e.g.* a diffusion on a circle with a strong constant drift (for instance turning clockwise).

Bound for the return to equilibrium $\{X = 0\}$. In view to applications in economics for instance (see the introduction above for more details), it thus seems of larger interest to focus on the (first) return time τ to zero of X, defined by:

$$\tau \quad := \quad \inf\{t \geq 0 \,:\, X_t = 0\} \tag{1.10}$$

Of course it is no longer relevant to assume that $Z_0 = 0$ and thus, we suppose that $(X_0, Y_0) = (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}$. In practice, τ appears through a temporal shift: we are at time $s > 0$ which is such that $X_s > 0$ and we are wondering when in the future X will return to its equilibrium position 0. Up to the knowledge of (X_s, Y_s), the time left before this return has the same law as τ if (x_0, y_0) is initialized with the value (X_s, Y_s). The next result shows that up to universal factors, the exponential rate of concentration of τ is given by $1/\omega$, confirming that when $b \gg a$, the return to zero happens much before the process reaches equilibrium.

Theorem 4. For any $0 < a < 4b$, $c > 0$, $x_0 > 0$ and $y_0 \in \mathbb{R}$, we have

$$\mathbb{P}_{(x_0,y_0)}[\tau > t] \;\leq\; 2\exp\left(-\frac{\ln(2)}{\pi}\omega t\right).$$

Furthermore, if $(1 + \frac{1}{\sqrt{2}})a \leq b$, there exists a quantity $\epsilon(x_0, y_0) > 0$ (which in addition to x_0 and y_0, depends on the parameters a, b, c) such that

$$\forall\, t \geq 0, \qquad \mathbb{P}_{(x_0,y_0)}[\tau > t] \quad \geq \quad \epsilon(x_0, y_0)\exp\left(-4\omega t\right).$$

More generally, to any initial distribution m_0 on $D := \{(x, y) \in \mathbb{R}^2 \,:\, x > 0\}$, we can associate a quantity $\epsilon(m_0)$ such that

$$\forall\, t \geq 0, \qquad \mathbb{P}_{m_0}[\tau > t] \quad \geq \quad \epsilon(m_0)\exp\left(-4\omega t\right).$$

Let us emphasize that the lower bound is much more difficult to obtain than the upper bound, while in reversible situations it is often the opposite which is experienced. The main problem relies on the degeneracy of the dynamics on the Y coordinate, which leads to a significant amount of difficulty to find a suitable control function for the spectral study of second order Markov operator. The finding of such control function is related to the understanding of the use of the drift vector field in comparison with the partial Laplacian on the x coordinate (see Section 3).

Remark 5 In Section 5, it will be shown that a QSD ν^D and a corresponding rate $\lambda_0(D) > 0$ can be associated to D: the support of ν^D is the closure of D and under

$\mathbb{P}_{\nu^D}$, τ is distributed as an exponential random variable of parameter $\lambda_0(D)$:

$$\forall\, t \geq 0, \qquad \mathbb{P}_{\nu^D}[\tau \geq t] \;=\; \exp(-\lambda_0(D)t) \tag{1.11}$$

In the sequel, the quantity $\lambda_0(D)$ will be called the persistence rate of D. It can be seen as the smallest eigenvalue (in modulus) of the underlying Markov generator with a Dirichlet condition on the boundary of the domain D, when it is interpreted as acting on $\mathbb{L}^2(\mu^D)$, where μ^D is the restriction of μ on D. The above theorem then provides lower and upper bounds on $\lambda_0(D)$, essentially proportional to ω: at least for $0 < (1 + \frac{1}{\sqrt{2}})a \leq b$,

$$\frac{\ln(2)}{\pi}\omega \;\leq\; \lambda_0(D) \;\leq\; 4\omega \tag{1.12}$$

According to Figure 1.2, starting from other initial distributions on D, the law of τ will no longer be exponential, nevertheless we believe that for any $(x_0, y_0) \in D$, the following limit takes place

$$\lim_{t\to+\infty} \frac{1}{t}\ln(\mathbb{P}_{x_0,y_0}[\tau > t]) \;=\; -\lambda_0(D)$$

The difficulty in obtaining this convergence stems from the non-reversibility of the process under consideration. In the literature, it is the reversible and elliptic situations which are the most thoroughly investigated. For a general reference on quasi-stationarity, see e.g. the book Collet et al. (2013), as well as the bibliography therein.

□

The previous result provides a good picture for large values of τ, but is there a precursor sign that τ will be much shorter than expected? Indeed we cannot miss it, because in this situation of a precocious return to zero, the system has a strong tendency to first explode. To give a rigorous meaning of this statement, we need to introduce the bridges associated to Z. For $z, z' \in \mathbb{R}^2$ and $T > 0$, denote by $\mathbb{P}^{(T)}_{z,z'}$ the law of the process Z evolving according to (1.4), conditioned by the event $\{Z_0 = z,\; Z_T = z'\}$. Note that there is no difficulty to condition by this negligible set, because the process Z starting from z is Gaussian and the law of Z_T is non-degenerate.

For fixed $z, z' \in \mathbb{R}^2$ and $T > 0$ small, we are interested in the behavior of the scaled process $\xi^{(T)} \coloneqq (\xi^{(T)}_t)_{t\in[0,1]}$, the process defined by

$$\forall\, t \in [0,1], \qquad \xi^{(T)}_t \;\coloneqq\; TZ_{Tt}$$

Let us define the trajectory $\varphi_{z,z'} : [0,1] \to \mathbb{R}^2$ by

$$\forall\, t \in [0,1], \qquad \varphi_{z,z'}(t) \;\coloneqq\; \begin{pmatrix} \frac{6\omega}{b^2}t(1-t)(y-y') \\ 0 \end{pmatrix} \tag{1.13}$$

where $z = (x, y)$ and $z' = (x', y')$.

Theorem 6. For fixed $z, z' \in \mathbb{R}^2$, as T goes to 0_+, $\xi^{(T)}$ converges in probability (under $\mathbb{P}^{(T)}_{z,z'}$) toward the deterministic trajectory $\varphi_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1], \mathbb{R}^2)$.

In particular, if $z, z' \in \mathbb{R}^2$ are such that $\Re(z) > 0$, $\Re(z') \leq 0$ and $\Im(z) \neq \Im(z')$, the bridge $(Z_t)_{t\in[0,T]}$ relying z to z' for small $T > 0$ explodes as $1/T$. From the definition of $\varphi_{z,z'}$ given in (1.13), we can see that the explosion is in the x-direction, toward $+\infty$ or $-\infty$, depending on the sign of $\Im(z) - \Im(z')$, as it is illustrated by the pictures of Figure 1.3.

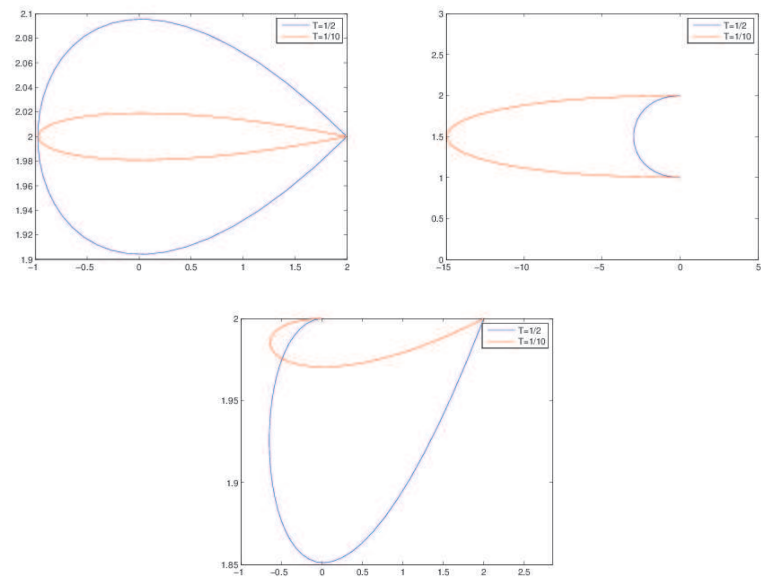

FIGURE 1.3. Expected trajectories of the hypo-elliptic bridge from z to z' within time $1/2$ and within small time $1/10$. Left: Non explosion when $z' = z$. Middle: explosion when $\Re(z) = \Re(z')$ and $\Im(z) \neq \Im(z')$. Right: Non explosion when $\Im(z) = \Im(z')$ and $\Re(z) \neq \Re(z')$.

Remark 7 Note that the sharp behavior of the bridge when $T \to 0$ leads in Section 4 to a probabilistic proof of a lower-bound for $\mathbb{P}(\tau > t)$ (see Proposition 32). The interest of this alternative proof is that the approach is maybe more intuitive. However, we have not been able to provide some explicit constants following this method.

□

The sequel of the paper is constructed on the following plan. In Section 2, we state some preliminary results for Z, especially its Gaussian features that enable to obtain Proposition 3. We will also see how to parametrize the process Z under a simpler form. The exit time is investigated in Section 3, where Theorem 4 is obtained. Section 4 is devoted to the study of bridges and to the proof of Theorem 6 whereas in Section 5, we prove the existence of the persitence rate through the QSD approach.

2. Preliminaries and simplifications

We remind some basic results about the O.U. diffusion Z described by Equations (1.4-1.5).

2.1. *Gaussian computations.* Our main goal here is to prove Proposition 3.

We begin by checking that the process Z is Gaussian. Indeed, considering the process $\widetilde{Z}$ defined by

$$\forall\, t \geq 0, \qquad \widetilde{Z}_t \;\; \coloneqq \;\; \exp(-At)Z_t$$

we get that

$$\forall\, t \geq 0, \qquad d\widetilde{Z}_t \;=\; \exp(-At)(-AZ_t\,dt + dZ_t) \;=\; \exp(-At)C\,dB_t$$

It follows that

$$\begin{aligned} \forall\, t \geq 0, \qquad Z_t \;&=\; \exp(At)Z_0 + \int_0^t \exp(A(t-s))C\,dB_s \\ &=\; \int_0^t \exp(A(t-s))C\,dB_s \end{aligned} \tag{2.1}$$

since we assumed that $Z_0 = 0$. It appears on this expression that for any $t \geq 0$, the law of Z_t is a Gaussian distribution of mean 0 and variance matrix Σ_t given by

$$\Sigma_t \;\;\coloneqq\;\; \int_0^t \exp(A(t-s))CC^* \exp(A^*(t-s))\,ds = \int_0^t \exp(As)CC^*\exp(A^*s)\,ds$$

For $a, b > 0$, the eigenvalues of A have negative real parts, so that the above r.h.s. converges as t goes to infinity toward a symmetric positive definite matrix Σ. As announced in the introduction, the Gaussian distribution μ of mean 0 and variance Σ is then an invariant measure for the evolution (1.4). It is a consequence of the fact that the underlying semi-group is Feller (i.e. it preserves the space of bounded continuous functions), as it can be seen from (2.1), where Z_t depends continuously on Z_0, for any fixed $t \geq 0$. Note furthermore that the above computations show that for any initial law of Z_0, the law of Z_t converges toward μ for large t, because $\exp(At)Z_0$ converges almost surely toward 0. It follows that μ is the unique invariant measure associated to (1.4). To obtain more explicit expressions for the above variances, we need the spectral decomposition of A. The characteristic polynomial of A being $X^2 + aX + ab$, we immediately obtain the results presented in the beginning of Subsection 1.2 about the eigenvalues of A. Let us treat in detail the case $a < 4b$, which is the most interesting for us: there are two conjugate eigenvalues, $\lambda_\pm = l \pm \omega i$, where $l = -a/2$ (see (1.7)) and ω is defined in (1.9).

Lemma 8. If $a < 4b$, there exist two angles $\alpha \in (\pi/2, 3\pi/2)$ and $\beta \in [0, 2\pi)$ such that

$$\forall t \geq 0 \qquad \Sigma_t \;\;=\;\; R_0 - \exp(-at)R_t$$

where

$$R_t \coloneqq \frac{c^2}{4ab-a^2}\begin{pmatrix} \frac{b^2}{a}\left(2+\sqrt{\frac{a}{b}}\cos(2\beta-\alpha-2\omega t)\right) & \frac{b}{a}\left(2\cos(\beta)+\cos(\beta-\alpha-2\omega t)\right) \\ \frac{b}{a}\left(2\cos(\beta)+\cos(\beta-\alpha-2\omega t)\right) & \frac{b^4}{a}\left(2+\sqrt{\frac{a}{b}}\cos(-\alpha-2\omega t)\right) \end{pmatrix}$$

Passing to the limit as $t \to +\infty$, we get $\Sigma = R_0$ and we deduce more precisely that

$$\Sigma \quad = \quad \frac{c^2}{2a^2}\begin{pmatrix} a+b & -b^2 \\ -b^2 & b^3 \end{pmatrix}.$$

Proof: The first line of A shows that an eigenvector associated to $\lambda_\pm$ is $(1, \lambda_\pm + a - b)^*$. So writing

$$\triangle \; \coloneqq \; \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \quad \text{and} \quad M \; \coloneqq \; \begin{pmatrix} 1 & 1 \\ \lambda_- + a - b & \lambda_+ + a - b \end{pmatrix}$$

we have that $A = M\triangle M^{-1}$, where

$$M^{-1} \;=\; \frac{1}{\lambda_+ - \lambda_-}\begin{pmatrix} \lambda_+ + a - b & -1 \\ -\lambda_- + b - a & 1 \end{pmatrix}$$

In view of (2.2), we need to compute for any $s \geq 0$,

$$\exp(As)CC^* \exp(A^*s) \quad = \quad M\exp(s\triangle)M^{-1}CC^*(M^*)^{-1}\exp(s\triangle^*)M^*$$

where * is now the conjugate transpose operation. A direct computation leads to

$$\frac{\lambda_+ - \lambda_-}{c} M\exp(s\triangle)M^{-1}C \;=\; \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \;\coloneqq\; \begin{pmatrix} z\exp(s\lambda_-) - \bar{z}\exp(s\lambda_+) \\ |z|^2\left(\exp(s\lambda_-) - \exp(s\lambda_+)\right) \end{pmatrix}$$

where $z \coloneqq \lambda_+ + a - b = a/2 - b + i\omega$ and $\overline{z} \coloneqq a - b + \lambda_- = a/2 - b - i\omega$. So we get that

$$\begin{aligned}
\exp(As)CC^*\exp(A^*s) \quad &= \quad \frac{c^2}{|\lambda_+ - \lambda_-|^2}\begin{pmatrix} |\sigma_1|^2 & \sigma_1\overline{\sigma_2} \\ \sigma_2\overline{\sigma_1} & |\sigma_2|^2 \end{pmatrix} \\
&= \quad \frac{c^2}{4\omega^2}\begin{pmatrix} 2|z|^2 e^{2ls} - 2\Re(z^2e^{2\lambda_- s}) & 2\Re(z)e^{2ls} - 2\Re(ze^{2\lambda_- s}) \\ 2\Re(z)e^{2ls} - 2\Re(ze^{2\lambda_- s}) & 2|z|^4\left(e^{2ls} - \Re(e^{2\lambda_- s})\right) \end{pmatrix}
\end{aligned}$$

Integrating this expression with respect to s, we obtain, first for any $t \geq 0$,

$$\Sigma_t \quad = \quad \frac{c^2}{4\omega^2}\begin{pmatrix} |z|^2\frac{e^{2lt}-1}{l} - \Re(z^2\frac{e^{2\lambda_- t}-1}{\lambda_-}) & \Re(z)\frac{e^{2lt}-1}{l} - \Re(z\frac{e^{2\lambda_- t}-1}{\lambda_-}) \\ \Re(z)\frac{e^{2lt}-1}{l} - \Re(z\frac{e^{2\lambda_- t}-1}{\lambda_-}) & |z|^4\left(\frac{e^{2lt}-1}{l} - \Re(\frac{e^{2\lambda_- t}-1}{\lambda_-})\right) \end{pmatrix}$$

and next, recalling that $\Re(\lambda_-) = \Re(\lambda_+) = l < 0$,

$$\Sigma \coloneqq \lim_{t\to+\infty}\Sigma_t \quad = \quad \frac{c^2}{4\omega^2}\begin{pmatrix} -|z|^2\frac{1}{l} + \Re(z^2\frac{1}{\lambda_-}) & -\Re(z)\frac{1}{l} + \Re(z\frac{1}{\lambda_-}) \\ -\Re(z)\frac{1}{l} + \Re(z\frac{1}{\lambda_-}) & -|z|^4\left(\frac{1}{l} - \Re(\frac{1}{\lambda_-})\right) \end{pmatrix}$$

Thus it appears that $\forall\ t \geq 0, \Sigma_t = \Sigma - e^{-at}R_t$ where the last term is the matrix defined by

$$R_t \quad \coloneqq \quad \frac{c^2}{4\omega^2}\begin{pmatrix} -|z|^2\frac{1}{l} + \Re(z^2\frac{e^{-2\omega it}}{\lambda_-}) & -\Re(z)\frac{1}{l} + \Re(z\frac{e^{-2\omega it}}{\lambda_-}) \\ -\Re(z)\frac{1}{l} + \Re(z\frac{e^{-2\omega it}}{\lambda_-}) & -|z|^4\left(\frac{1}{l} - \Re(\frac{e^{-2\omega it}}{\lambda_-})\right) \end{pmatrix}$$

Note that $\Sigma = R_0$. To recover the matrices given in the statement of the lemma, we remark that $|\lambda_-|^2 = ab$ and $|z|^2 = b^2$, so there exist angles $\alpha, \beta \in [0, 2\pi)$ such that

$$\lambda_- \;=\; \sqrt{ab}\exp(i\alpha) \quad \text{and} \quad z \;=\; b\exp(i\beta)$$

Since $\Re(\lambda_-) < 0$, we have $\alpha \in (\pi/2, 3\pi/2)$ and the first announced results follow at once. Concerning the more explicit computation of Σ, just take into account that

$\cos(\alpha) = -\frac{\sqrt{a}}{2\sqrt{b}}$, $\sin(\alpha) = -\frac{\omega}{\sqrt{ab}}$ and $\cos(\beta) = \frac{a-2b}{2b}$, $\sin(\beta) = \frac{\omega}{b}$. and expand the matrix

$$R_0 = \frac{c^2 b}{(4b-a)a^2}\begin{pmatrix} b\left(2+\sqrt{\frac{a}{b}}\cos(2\beta-\alpha)\right) & (2\cos(\beta)+\cos(\beta-\alpha)) \\ (2\cos(\beta)+\cos(\beta-\alpha)) & b^3\left(2+\sqrt{\frac{a}{b}}\cos(\alpha)\right) \end{pmatrix}$$

■

In particular, $X_t \xrightarrow[t\to+\infty]{\mathcal{L}} \mathcal{N}(0, c^2(b+a)/(2a^2))$. We precise the functional J defined in (1.8).

Lemma 9. Let μ and $\widetilde{\mu}$ be two Gaussian distributions in $\mathbb{R}^d$, $d \geq 1$, of mean 0 and respective variance matrices Σ and $\widetilde{\Sigma}$, assumed to be positive definite. If $\widetilde{\Sigma}^{-1} - \Sigma^{-1}/2$ is positive definite

$$J(\widetilde{\mu}, \mu) = \sqrt{\frac{1}{\sqrt{\det(\mathrm{Id} - S^2)}} - 1}$$

where $S \coloneqq \Sigma^{-1}\widetilde{\Sigma} - \mathrm{Id}$ and $J(\widetilde{\mu}, \mu) = +\infty$ otherwise.

Proof: From the above assumptions, we have

$$\forall\, x \in \mathbb{R}^d, \qquad \frac{d\widetilde{\mu}}{d\mu}(x) = \sqrt{\frac{\det(\Sigma)}{\det(\widetilde{\Sigma})}}\exp\left(-x^*\frac{(\widetilde{\Sigma}^{-1} - \Sigma^{-1})}{2}x\right).$$

Thus the function $\frac{d\widetilde{\mu}}{d\mu}$ belongs to $\mathbb{L}^2(\mu)$ (property itself equivalent to the finiteness of $J(\widetilde{\mu}, \mu)$), if and only if the symmetric matrix $\widetilde{\Sigma}^{-1} - \Sigma^{-1} + \Sigma^{-1}/2$ is positive definite. In this case, we have

$$\begin{aligned}
\int \left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 d\mu &= \frac{\det(\Sigma)}{\det(\widetilde{\Sigma})}\sqrt{\frac{\det((2\widetilde{\Sigma}^{-1} - \Sigma^{-1})^{-1})}{\det(\Sigma)}} \\
&= \frac{\sqrt{\det(\Sigma)}}{\det(\widetilde{\Sigma})}\sqrt{\det(\widetilde{\Sigma})\det((2\mathrm{Id} - \Sigma^{-1}\widetilde{\Sigma})^{-1})} \\
&= \sqrt{\frac{\det(\Sigma)}{\det(\widetilde{\Sigma})}}\sqrt{\det((\mathrm{Id} - S)^{-1})} = \frac{1}{\sqrt{\det(\mathrm{Id}+S)\det(\mathrm{Id}-S)}}
\end{aligned}$$

where we used that $\widetilde{\Sigma} = \Sigma(\mathrm{Id} + S)$. It remains to note that

$$J^2(\widetilde{\mu}, \mu) = \int \left[\left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 - 2\frac{d\widetilde{\mu}}{d\mu} + 1\right] d\mu = \int \left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 d\mu - 1$$

■

Still in the case $4b > a$, we can now proceed to the

Proof of Proposition 3 (Section 1)

In the view to Lemma 8, we want to apply Lemma 9 with $\Sigma = R_0$ and $\widetilde{\Sigma} = R_0 - \exp(-at)R_t$, for $t \geq 0$. This amounts to take $S \coloneqq -\exp(-at)R_0^{-1}R_t$, matrix converging to zero exponentially fast as t goes to $+\infty$. It follows that for t large enough, $\widetilde{\Sigma}^{-1} - \Sigma^{-1}/2$ is positive definite and we can apply Lemma 9. Taking into account that the matrices R_t are bounded uniformly over $t \in \mathbb{R}_+$ and

$$\begin{aligned} \frac{1}{\sqrt{\det(\mathrm{Id} - S^2)}} &= \frac{1}{\sqrt{1 - \mathrm{tr}(S^2) + \mathcal{O}(\|S^2\|_{\mathrm{HS}}^2)}} \\ &= 1 + \frac{1}{2}\mathrm{tr}(R_0^{-1}R_tR_0^{-1}R_t)\exp(-2at) + \mathcal{O}(\exp(-4at)) \end{aligned}$$

(where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm, i.e. the square root of the sum of the squares of the entries of the matrix). We will be able to conclude to

$$\lim_{t\to+\infty} \frac{1}{t}\ln(J(\mu_t,\mu)) = -a \tag{2.2}$$

if we can show that

$$\liminf_{t\to+\infty} \mathrm{tr}(R_0^{-1}R_tR_0^{-1}R_t) > 0 \tag{2.3}$$

(since it is clear that $\limsup_{t\to+\infty} \mathrm{tr}(R_0^{-1}R_tR_0^{-1}R_t) < +\infty$). Taking advantage of the fact that R_0 is a symmetric and positive definite matrix, we consider for $t \geq 0$, $\widehat{R}_t \coloneqq R_0^{-1/2}R_tR_0^{-1/2}$, which is also symmetric. Since $\mathrm{tr}(R_0^{-1}R_tR_0^{-1}R_t) = \mathrm{tr}(\widehat{R}_t^2) = \left\|\widehat{R}_t\right\|_{\mathrm{HS}}^2$, this quantity is nonnegative and can only vanish if $\widehat{R}_t$, or equivalently R_t, is the null matrix. This never happens, because the first entry of R_t, namely $(cb/a)^2(2+\sqrt{a/b}\cos(2\beta-\alpha-2\omega t))/(4b-a)$, is positive. The continuity and the periodicity of the mapping $\mathbb{R}_+ \ni t \mapsto R_t$ enables to check the validity of (2.3) and next of (2.2).

The same result for the first marginal ν_t (the law of X_t) is obtained in the same way. From Lemma 8, for any $t \geq 0$, ν_t is the real Gaussian law of mean 0 and variance $r_0 - \exp(-at)r_t$, where

$$\forall\, t \geq 0, \qquad r_t \coloneqq \frac{(cb)^2}{4a^2b - a^3}\left(2 + \sqrt{\frac{a}{b}}\cos(2\beta - \alpha - 2\omega t)\right)$$

with the angles $\alpha \in (\pi/2, 3\pi/2)$ and $\beta \in [0, 2\pi)$ described in Lemma 8. In particular $\nu \coloneqq \lim_{t\to+\infty} \nu_t$ is the real Gaussian law $\mathcal{N}(0, r_0)$. Lemma 9 with $d = 1$ implies (2.2).

■

The remaining situations $a = 4b$ and $a > 4b$ can be treated in the same way. In view of the previous arguments, it is sufficient to check that it is possible to write

$$\forall\, t \geq 0, \qquad \Sigma_t = \Sigma - \exp(-2lt)R_t$$

where l is defined in (1.7) and where the family $(R_t)_{t\geq 0}$ is such that

$$\lim_{t\to+\infty} \frac{1}{t}\ln(\|R_t\|) = 0$$

for any chosen norm $\|\cdot\|$ on the space of 2×2 real matrices, due to their mutual equivalence. Obtaining the family $(R_t)_{t\geq 0}$ also relies on the spectral decomposition of A, with R_t converging for large times t if $a > 4b$ and exploding like t^2 if $a = 4b$.

2.2. *Simplifications with the view to Theorem 4*. Let us begin this subsection by emphasizing two important properties of Theorem 4:
• The result does not depend on the variance coefficient c.
• The exponent is proportional to $\omega = \sqrt{ab - a^2/4}$ which denotes the mean angular speed of the deterministic system $\dot{z} = Az$.
These properties can be understood through some linear and scaling transformations of the process $(Z_t)_{t\geq 0}$. More precisely, these transformations will be used in the sequel to reduce the problem to the study of a process with mean constant angular speed and a normalized diffusion component.

We choose to first give the idea in a general case and then, apply it to our model.

Let $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_\pm = -\rho \pm i\omega$ where $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$. Let us consider the two-dimensional Gaussian differential system given by

$$d\zeta_t = A\zeta_t dt + \Sigma dB_t \tag{2.4}$$

where $\Sigma \in \mathbb{M}_2(\mathbb{R})$ and $(B_t)_{t\geq 0}$ is a standard two dimensional Brownian motion. For such a process, the precise transformation is given in Proposition 11. This proposition is based on the next lemma.

Lemma 10. Let $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_\pm = -\rho \pm i\omega$ where $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$. There exists $P \in GL_2(\mathbb{R})$ such that

$$A = P(-\rho I_2 + \omega J_2)P^{-1}$$

where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.5}$$

Furthermore, for every $v \in \mathbb{R}^2 \backslash \{0\}$, $P := P_v$ given by $P_v = (v, \frac{A+\rho I_2}{\omega} v)$ is an admissible choice.

Proof: Set $B = \frac{A+\rho I_2}{\omega}$. The eigenvalues of B are $\pm i$ so that $B^2 = -I_2$. For any $v \in \mathbb{R}^2$, set $P_v = (v, Bv)$. The matrix P_v is clearly invertible and using that $B^2 v = -v$, one obtains that $B = P_v J_2 P_v^{-1}$.

■

□

Proposition 11. Let $(\zeta_t)_{t\geq 0}$ be a solution to (2.4) where $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_\pm = -\rho \pm i\omega$ (with $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$). For any $\alpha \in \mathbb{R}^*$ and $v \in \mathbb{R}^2 \backslash \{0\}$, set $\hat{\zeta}_t = \sqrt{\omega}\alpha P_v^{-1} \zeta_{\frac{t}{\omega}}$. The process $(\hat{\zeta}_t)_{t\geq 0}$ is a solution to

$$d\hat{\zeta}_t = -\frac{\rho}{\omega}\hat{\zeta}_t + J_2\hat{\zeta}_t + \alpha P_v^{-1}\Sigma dW_t \tag{2.6}$$

where (W_t) is a standard two-dimensional Brownian motion.

Proof: First, set $\tilde{\zeta}_t^v = P_v^{-1}\zeta_t$. Owing to the preceding lemma, $(\tilde{\zeta}_t^v)_{t\geq 0}$ is a solution to

$$d\tilde{\zeta}_t^v = -\rho\tilde{\zeta}_t^v + \omega J_2 \tilde{\zeta}_t^v + P_v^{-1}\Sigma dB_t.$$

For any $\alpha \in \mathbb{R}^*$, set $\hat{\zeta}_t = \sqrt{\omega}\alpha\tilde{\zeta}^v_{\frac{t}{\omega}}$. Setting $W_t = \sqrt{\omega}B_{\frac{t}{\omega}}$ (which is a Brownian motion), one checks that

$$d\hat{\zeta}_t = -\frac{\rho}{\omega}\hat{\zeta}_t + J_2\hat{\zeta}_t + \alpha P_v^{-1}\Sigma dW_t.$$

■

□

We now apply this proposition to our problem.

Corollary 12. Let $(Z_t)_{t\geq 0}$ be a solution to (1.4) and assume that $a < 4b$. Set $\omega = \sqrt{ab - \frac{a^2}{4}}$. Let $v \in \mathbb{R}^2 \setminus \{0\}$ and set $P_v = (v, Bv)$ with $B = \frac{1}{\omega}(A + \frac{a}{2}I_2)$.

i) Then for any $\alpha \in \mathbb{R}\setminus\{0\}$, the process $(\hat{Z}_t)_{t\geq 0}$ defined by $\hat{Z}_t = \sqrt{\omega}\alpha P_v^{-1} Z_{\frac{t}{\omega}}$ is a solution to

$$d\hat{Z}_t = -\frac{a}{2\omega}\hat{Z}_t + J_2\hat{Z}_t + \alpha c P_v^{-1}\Sigma dW_t \quad \text{with} \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.7}$$

where W is a standard two-dimensional Brownian motion.

ii) In particular, if $v = (\frac{1}{b^2}(\frac{a}{2} - b), 1)^*$ and $\alpha = \frac{\sqrt{2\omega}}{cb^2}$, then $(\hat{Z}_t) := (U_t, V_t)$ is a solution to

$$\begin{cases} dU_t = -\frac{a}{2\omega}U_t - V_t dt \\ dV_t = -\frac{a}{2\omega}V_t + U_t + \sqrt{2}dW_t. \end{cases} \tag{2.8}$$

where W is now a standard one-dimensional Brownian motion.

Remark 13 In the second part of the corollary, one remarks that one chooses v in order that the transformed process has only a (normalized) diffusive component on the second coordinate.

Furthermore, if Z has $(x, y) \in \mathbb{R}^2$ for initial deterministic condition, then $\hat{Z}$ starts from the point $\sqrt{\omega}c^{-1}b^{-2}(\omega y, b^2 x + (b - a/2)y)$. The images of $(1, 0)^*$ and $(0, 1)^*$ by P_v^{-1} are particularly important for our purposes, since they enable to see that the half-plane $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ for Z is transformed into the half-plane $\{(u, v) \in \mathbb{R}^2 : v > \frac{2b-a}{2\omega}u\}$ for $\hat{Z}$. Note that in the setting $a << b$, the latter half-plane is quite similar to the former one, since $\omega \sim \sqrt{ab} << b$.

□

Proof: For the first part, recall that $(Z_t) = (X_t, Y_t)$ satisfies $dZ_t = AZ_t + c\Sigma dB_t$ with $A = \begin{pmatrix} b - a & 1 \\ -b^2 & -b \end{pmatrix}$ and Σ is defined in (2.7). When $a < 4b$, the eigenvalues of A are given by

$$\lambda_\pm = -\frac{a}{2} \pm i\omega.$$

For any $v \in \mathbb{R}^2$, set $P_v := (v, Bv)$ with $B = \frac{1}{\omega}(A + \frac{a}{2}I_2)$. Applying the previous proposition, we deduce that for any $\alpha \in \mathbb{R}^*$, $(\hat{Z}_t)_{t\geq 0} := (\alpha\sqrt{\omega}P_v^{-1}Z_{\frac{t}{\omega}})_{t\geq 0}$ is a solution to

$$d\hat{Z}_t = -\frac{a}{2\omega}\hat{Z}_t + J_2\hat{Z}_t + \alpha c P_v^{-1}\Sigma d\hat{W}_t$$

where $(\hat{W}_t)$ is a two-dimensional Brownian motion. For the second part, we choose v and α so that

$$\alpha c P_v^{-1}\Sigma = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \tag{2.9}$$

If $v = (u_1, u_2)^*$, then

$$P_v = \begin{pmatrix} u_1 & \frac{1}{\omega}\left((b-\frac{a}{2})u_1 + u_2\right) \\ \\ u_2 & -\frac{1}{\omega}\left(b^2 u_1 + (b-\frac{a}{2})u_2\right) \end{pmatrix} \tag{2.10}$$

The fact that $(P_v^{-1}\Sigma)_{1,1} = 0$ implies $b^2u_1 + (b-\frac{a}{2})u_2 = 0$. Setting $v = (\frac{1}{b^2}(\frac{a}{2}-b), 1)^*$, we have

$$P_v = \begin{pmatrix} \frac{1}{b^2}\left(\frac{a}{2}-b\right) & \frac{\omega}{b^2} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P_v^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{b^2}{\omega} & \frac{1}{\omega}\left(\frac{a}{2}-b\right) \end{pmatrix}.$$

Condition (2.9) is then satisfied when $\alpha = \frac{\sqrt{2}\omega}{cb^2}$.

■

□

3. Dirichlet eigenvalues estimates

This section is devoted to the proof of Theorem 4. We aim to obtain successively upper and lower bounds for $\mathbb{P}_{(x_0,y_0)}(\tau > t)$ where $\tau := \inf\{t \geq 0, X_t \leq 0\}$. In fact, some of the results will be stated for exit times of more general domains. For a given (open) domain $\mathcal{S}$ of $\mathbb{R}^2$, we denote

$$\tau_{\mathcal{S}} := \inf\{t \geq 0, (X_t, Y_t) \in \mathcal{S}^c\}.$$

3.1. *Upper-bound for the exit time of an angular sector $\mathcal{S}$.*

3.1.1. *The case $\mathcal{S} = \{(x,y), x > 0\}$.* First, we focus on the stopping time τ that corresponds to the exit time of $D = \{(x,y), x > 0\}$.

Proposition 14. Let $(Z_t)_{t\geq 0}$ be a solution to (1.4) with $a < 4b$. Then, for every $(x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$\mathbb{P}_{(x_0,y_0)}(\tau > t) \leq 2\exp\left(-\frac{\log 2}{\pi}\omega t\right).$$

Proof: Set $z(t) = (x(t), y(t)) = (\mathbb{E}[X_t], \mathbb{E}[Y_t])$. The function $(z(t))_{t\geq 0}$ being a solution to $\dot{z} = Az$, we have

$$\forall\, t \geq 0, \quad \ddot{x}(t) + a\dot{x} + abx(t) = 0.$$

Since $a < 4b$, the roots of the characteristic equation associated with the previous equation are: $\lambda_\pm = -\frac{a}{2} \pm \omega$ where $\omega = \sqrt{ab - a^2/4}$. Hence, there exists $C > 0$ and $\varphi_0 \in (-\pi, \pi]$ such that

$$x(t) = C\cos(\omega t + \varphi_0), \quad t \geq 0.$$

Reminding that $x_0 > 0$, we deduce that $\varphi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, at time $T_\omega = \frac{\pi}{\omega}$,

$$\forall x_0 > 0, \quad \omega T_\omega + \phi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2}) \implies x(T_\omega) < 0.$$

But X_{T_ω} is Gaussian and has a symmetric distribution around its mean. Thus, we deduce that

$$\forall x_0 > 0,\ y_0 \in \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(X_{T_\omega} < 0) \geq \frac{1}{2}$$

which in turn implies that

$$\forall x_0 > 0,\ y_0 \in \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau \geq T_\omega) \leq \frac{1}{2}. \tag{3.1}$$

Thus, we have an upper-bound at time T_ω which does not depend on the initial value (x_0, y_0). As a consequence, we can use a Markov argument: the Markov property and (3.1) lead to

$$\begin{aligned} \mathbb{P}(\tau > kT_\omega | \tau > (k-1)T_\omega) &= \frac{\mathbb{E}[\mathbb{P}_{(X_{(k-1)T_\omega}, Y_{(k-1)T_\omega})}(\tau > T_\omega) 1_{\tau > (k-1)T_\omega}]}{\mathbb{P}(\tau > (k-1)T_\omega)} \\ &\leq \sup_{x_0 > 0, y_0 \in \mathbb{R}} \mathbb{P}_{(x_0,y_0)}(\tau > T_\omega). \end{aligned}$$

An iteration of this property yields

$$\forall n \in \mathbb{N},\ \forall (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau > nT) \leq \left(\frac{1}{2}\right)^n.$$

It follows that

$$\forall t \geq 0,\ \forall (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau > t) \leq \left(\frac{1}{2}\right)^{\lfloor \frac{t}{T_\omega} \rfloor} \leq 2\exp\left(-\frac{\log 2}{T_\omega} t\right).$$

This concludes the proof. □

3.1.2. *Extension to general angular sectors.* We now consider an angular sector $\mathcal{S}_{\alpha_1,\alpha_2}$ defined as

$$\mathcal{S}_{\alpha_1,\alpha_2} = \{(x, y) \in \mathbb{R}^2,\ x > 0\,,\ \alpha_1 x < y < \alpha_2 x\} \tag{3.2}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. The set $\mathcal{S}_{\alpha_1,\alpha_2}$ can also be written $\mathcal{S}_{\alpha_1,\alpha_2} = \{(r\cos\theta, r\sin\theta), r > 0, \theta_1 < \theta < \theta_2\}$ with $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$. Note that for the sake of simplicity, we only consider angular sectors which are included in $\{(x, y), x > 0\}$. The results below can be extended to any angular sectors for which the angular size is lower than π. For such domains, we first give a result when the model has a constant (mean) angular speed even if such a result does not apply to the solutions of (1.4) for sake of completeness. This is the purpose of Lemma 15 below.

Concerning now our initial motivation, we also derive an extension of Proposition 14 for any general angular sector, and this result is stated in Proposition 16.

Lemma 15. Let $(Z_t)_{t\geq 0}$ be a solution of

$$dZ_t = -\rho Z_t + \omega J_2 Z_t + \Sigma dW_t$$

where $\rho \in \mathbb{R}$, $\omega \in \mathbb{R}_+^*$, $\Sigma \in \mathbb{M}_2(\mathbb{R})$ and W is a two-dimensional Brownian motion. Let $\mathcal{S}_{\alpha_1,\alpha_2}$ be defined by (3.2) where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. Then, for any $(x_0, y_0) \in \mathcal{S}_{\alpha_1,\alpha_2}$,

$$\mathbb{P}(\tau_{\mathcal{S}_{\alpha_1,\alpha_2}} \geq t) \leq 2\exp\left(-\frac{\ln(2)}{\theta_2 - \theta_1}\omega t\right)$$

with $\theta_1 = \mathrm{Arctan}(\alpha_1)$ and $\theta_2 = \mathrm{Arctan}(\alpha_2)$.

Proof: Let $z(t) = (\mathbb{E}[X_t], \mathbb{E}[Y_t])$ and define $(u(t))_{t\geq 0} := (e^{\rho t} z(t))_{t\geq 0}$, u is a solution of $\dot{u} = \omega J_2 u$. We deduce that

$$e^{\rho t} z(t) = (A\cos(\omega t + \varphi), A\omega \sin(\omega t + \varphi))$$

where $A \geq 0$ and $\varphi \in [-\pi, \pi)$. This implies that the angular rate of $(z(t))_{t\geq 0}$ is constant and is equal to ω. Thus, it follows that for every starting point $(x, y) \in \mathcal{S}_{\alpha_1,\alpha_2}$,

$$z(T_\omega) \in \mathcal{S}^c_{\alpha_1,\alpha_2} \quad \text{with} \quad T_\omega = \omega(\theta_2 - \theta_1).$$

One can then find a line passing through 0 and dividing $\mathbb{R}^2$ into two half-planes D^+ and D^- such that $\mathcal{S}_{\alpha_1,\alpha_2}$ is included in D^- and $z(T_\omega) \in D^+$. Owing to the symmetry of a one-dimensional centered Gaussian distribution, we have

$$\mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in D^-) = \mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in D^+) = \frac{1}{2}.$$

One finally deduces that for every $(x, y) \in \mathcal{S}_{\alpha_1,\alpha_2}$,

$$\mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in \mathcal{S}^c_{\alpha_1,\alpha_2}) \geq \frac{1}{2}.$$

and thus that

$$\forall\, (x, y) \in \mathcal{S}_{\alpha_1,\alpha_2}, \quad \mathbb{P}_{(x,y)}(\tau_{\mathcal{S}_{\alpha_1,\alpha_2}} > T_\omega) \leq \frac{1}{2}.$$

The end of the proof is then identical to that of Proposition 14.

□

We now consider our initial speculative process $(Z_t)_{t\geq 0}$ which is solution of Equation (1.4). We have the following result.

Proposition 16. Let $(Z_t)_{t\geq 0}$ be a solution to (1.4). Let $\mathcal{S}_{\alpha_1,\alpha_2}$ be defined by (3.2) where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. Then, for any $(x_0, y_0) \in \mathcal{S}_{\alpha_1,\alpha_2}$,

$$\mathbb{P}(\tau_{\mathcal{S}_{\alpha_1,\alpha_2}} \geq t) \leq 2\exp\left(-\frac{\ln 2}{\tilde{\theta}_1 - \tilde{\theta}_2}\omega t\right)$$

with $\tilde{\theta}_i = \mathrm{Arctan}\left(\frac{a/2 - b - \alpha_i}{\omega}\right)$, $i = 1, 2$.

Remark 17 Taking $\alpha_1 = -\infty$ and $\alpha_2 = +\infty$, we retrieve Proposition 14 since $\mathcal{S}_{-\infty,+\infty}$ then corresponds to the half-plane $\{x > 0\}$. Note that contrary to Lemma 15, the exponential rate is not directly proportional to ω. More precisely, due to the non constant angular speed, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ depend on ω. For the particular domain of Proposition 14 this dependence does not appear since, even if the the angular rate is not constant, the time to do a *U-turn* is still proportional to ω.

□

Proof: By Corollary 12, for any v of $\mathbb{R}^2 \setminus \{0\}$, $(\tilde{Z}_t)_{t\geq 0} = (P_v^{-1} Z_{\frac{t}{\omega}})_{t\geq 0}$ (where $P_v = (v, Bv)$) satisfies

$$d\tilde{Z}_t = (-\rho I + J_2)\tilde{Z}_t + \tilde{\Sigma} dW_t$$

where $\tilde{\Sigma}$ is a constant real matrix and where $\rho = a/(2\omega)$. In the new basis $\tilde{\mathcal{B}} = (v, Bv)$,

$$\mathcal{S}_{\alpha_1,\alpha_2} = \{\tilde{z} = (\tilde{x}, \tilde{y})_{\tilde{\mathcal{B}}} \in \mathbb{R}^2, \alpha_1 (P_v \tilde{z})_1 < (P_v \tilde{z})_2 < \alpha_2 (P_v \tilde{z})_1\}.$$

Setting $v = (1, \frac{a}{2} - b)$, we deduce from Equation (2.10) that

$$P_v = \begin{pmatrix} 1 & 0 \\ (\frac{a}{2} - b) & -\omega \end{pmatrix}$$

In such a case

$$P_v \tilde{z} = \begin{pmatrix} \tilde{x} \\ \\ (\frac{a}{2} - b)\tilde{x} - \omega \tilde{y} \end{pmatrix}$$

so that

$$\mathcal{S}_{\alpha_1,\alpha_2} = \{\tilde{z} = (\tilde{x}, \tilde{y})_{\tilde{\mathcal{B}}} \in \mathbb{R}^2, \left(\frac{a}{2} - b - \alpha_2\right) \tilde{x} < \omega \tilde{y} < \left(\frac{a}{2} - b - \alpha_1\right) \tilde{x}\}.$$

Thus, we deduce from Lemma 15 that

$$\mathbb{P}_{\tilde{x},\tilde{y}}(\tilde{\tau}_{\mathcal{S}_{\alpha_1,\alpha_2}} \geq t) \leq 2 \exp\left(-\frac{\ln 2}{\tilde{\theta}_1 - \tilde{\theta}_2} t\right)$$

where for a given domain A, $\tilde{\tau}_A := \inf\{t \geq 0, \tilde{Z}_t \in A^c\}$ and $\tilde{\theta}_i = \mathrm{Arctan}\left(\frac{a/2 - b - \alpha_i}{\omega}\right)$, $i = 1, 2$. □

3.2. *Lower-bound.*

3.2.1. *General tool.* In this second part, our aim is to obtain the lower-bound part of Theorem 4, in particular we want to derive a upper-bound on:

$$\bar{\lambda} \quad := \quad \limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}(\tau \geq t)).$$

Our results are based on the following result valid for general Markov processes (see *e.g* Collet et al. 2013).

Proposition 18. Let $(X_t)_{t \geq 0}$ be a $\mathbb{R}^d$-valued Markov process with infinitesimal generator L and initial distribution m_0. Let $\mathcal{S}$ be an (open) domain of $\mathbb{R}^d$ and assume that $m_0(\mathcal{S}) = 1$. Let $\tau := \inf\{t > 0, X_t \in \mathcal{S}^c\}$. Then, if there exists a bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\begin{cases} f_{/\partial\mathcal{S}} = 0 \quad \text{and} \quad f_{/\mathcal{S}} > 0 \\ \forall x \in \mathcal{S}, \quad Lf(x) \geq -\lambda f(x) \end{cases} \tag{3.3}$$

then, $\mathbb{E}_{m_0}[e^{\lambda\tau}] = +\infty$. As a consequence,

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau \geq t)) \quad \leq \quad \lambda.$$

The end of Section 3.2 is devoted to the construction of a function f satisfying (3.3). In fact, for this part, the degeneracy of the process described by Equation (1.4) *implies a significant amount of difficulties*. In the next subsection, we briefly treat the (easier) elliptic case and some of the ideas developed in this framework will then be extended to the initial hypoelliptic setting.

3.2.2. *The elliptic case.* From Corollary 12, we can reduce the problem to the study of a process (U_t, V_t) solution to (2.8). In this part we focus on its elliptic counterpart and consider $(\xi_t)_{t\geq 0}$ solution of

$$d\xi_t = (-\rho\xi_t + J_2\xi_t)dt + \sqrt{2}dW_t \tag{3.4}$$

where $\rho \in \mathbb{R}$ and W is a standard two-dimensional Brownian motion. We switch to polar coordinates for (3.3) on $D = \mathcal{S}_0 = \{(x,y), x > 0\}$. Proposition 37 shows that L_ρ is given by

$$\forall f \in \mathcal{C}^2(\mathbb{R}_+^* \times \mathbb{R}) \qquad L_\rho(f) = -\rho r\partial_r(f) + \partial_\theta(f) + \partial_r^2(f) + \frac{1}{r}\partial_r(f) + \frac{1}{r^2}\partial_\theta^2(f). \tag{3.5}$$

If we formally omit the derivatives ∂_r by fixing $r > 0$, (3.3) is reduced to finding G_r and λ_r such that $r^{-2}G_r''(\theta) + G_r'(\theta) = -\lambda_r G_r(\theta), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ with $G_r(\pi/2) = G_r(-\pi/2) = 0$. The solutions are $G_r(\theta) = \alpha_1 e^{\rho_1\theta} + \alpha_2 e^{\rho_2\theta}$ where (ρ_1, ρ_2) are the complex roots of the characteristic equation $X^2/r^2 + X + \lambda_r = 0$. We check that $\Re(\rho_1) = \Re(\rho_2) = -r^2/2$ and the boundary conditions imply the choice of λ_r such that $\Im(\rho_1) = -\Im(\rho_2) = 1$. This is possible *iff* $\lambda_r = \frac{1}{r^2} + \frac{r^2}{4}$ and the solutions of this spectral problem are proportional to $G_r(\theta) \coloneqq e^{-\frac{r^2}{2}\theta}\cos\theta$. This construction cannot be extended to the initial problem (3.3) with $L = L_\rho$ but this suggests to consider

$$g(r,\theta) = re^{\beta(\theta)r^2}\cos(\theta), \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \ r \geq 0, \quad \beta \in \mathcal{C}^2([-\pi/2, \pi/2], \mathbb{R}) \tag{3.6}$$

Proposition 19. For any $g \in \mathcal{C}^2\left(\mathbb{R}_+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \mathbb{R}\right)$ given by (3.6), one has

$$\forall (r,\theta) \in \mathbb{R}_+^* \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad L_\rho g(r,\theta) = \left[\psi_1(\theta)r^2 + \psi_2(\theta)\right] g(r,\theta)$$

where

$$\begin{aligned}\psi_1(\theta) &= -2\rho\beta(\theta) + \beta'(\theta) + \left(4\beta^2(\theta) + (\beta'(\theta))^2\right)\\ \psi_2(\theta) &= -\rho + 8\beta(\theta) - (1 + 2\beta'(\theta))\tan\theta + \beta''(\theta).\end{aligned}$$

The problem is reduced to find $\beta \leq 0$ (due to the boundedness condition in Proposition 18) such that $\frac{L_\rho g}{g}$ is lower-bounded on $\mathbb{R}_+^* \times]-\frac{\pi}{2}, \frac{\pi}{2}[$. We need to satisfy the following constraint $\psi_1 \geq 0$ on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$. The inequality $L_\rho g \geq -\lambda_\rho g$ is then obtained by with $\lambda_\rho := \inf_{\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[} \psi_2(\theta) > -\infty$. Note that this implies in particular that

$$\limsup_{\theta \to \frac{\pi}{2}} 1 + 2\beta'(\theta) \leq 0 \quad \text{and} \quad \liminf_{\theta \to -\frac{\pi}{2}} 1 + 2\beta'(\theta) \geq 0.$$

A solution of the problem is given in the next proposition.

Proposition 20. (i) Let $\rho \geq 0$ and let g be given by (3.6) with

$$\beta(\theta) = \begin{cases} \frac{1}{4}(1 - \sqrt{3}) & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ \frac{1}{4}(\sin(2\theta) - \sqrt{3}) & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}]. \end{cases} \tag{3.7}$$

Then, for every $r > 0$ and $\theta \in [-\pi/2, \pi/2]$ such that $\theta \neq \pi/4$,

$$L_\rho g(r,\theta) \geq -\lambda_\rho g(r,\theta) \quad \text{with } \lambda_\rho = 2\sqrt{3} + \rho.$$

(ii) Let $\rho \geq 0$ and consider $(\xi_t)_{t\geq 0}$ solution to (3.4) and $\tau := \inf\{t \geq 0, (\xi_t)_1 < 0\}$.

$$\forall (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R} \qquad \limsup_{t \to +\infty} -\frac{1}{t}\log(\mathbb{P}_{(x_0,y_0)}(\tau \geq t)) \leq \lambda_\rho$$

Proof: (i) First, assume that $\rho = 0$,since β is a piecewise $\mathcal{C}^2$-function on $[-\pi/2, \pi/2]$, we can use this function since ItÃ´'s formula is still available in this case. Furthermore, we check that

$$\psi_1(\theta) = \begin{cases} 1 - \frac{\sqrt{3}}{2} & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ 1 + \cos(\frac{\pi}{3} + 2\theta) & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases} \tag{3.8}$$

so that ψ_1 is non-negative on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. As well, easy computations yield:

$$\psi_2(\theta) = \begin{cases} 2 - 2\sqrt{3} - \tan\theta & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ -2\sqrt{3} & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases} \tag{3.9}$$

It follows that ψ_2 is lower-bounded by $-2\sqrt{3}$ and the result follows when $\rho = 0$. The extension to the case $\rho > 0$ is obvious using that $-\rho\beta$ is a non-negative function.

(ii) This statement follows from Proposition 18, since g is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$. □

Remark 21 Using the scaling and linear transformations previously described, this result can be transferred to general elliptic two-dimensional O.U. evolutions whose drift is given via a matrix admitting complex conjugate eigenvalues (trajectories that have a tendency to turn around $(0, 0)$).

The function g is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$ and belong to the domain of L_ρ so that Proposition 18 still holds. If we now switch to cartesian coordinates, the counterpart of g has the following form:

$$f(x, y) = xe^{-\frac{\sqrt{3}-1}{4}(x^2+y^2)} e^{-\frac{(x-y)^2}{4} 1_{\{y \geq x\}}}.$$

Figure 3.4 represents the partition of the state space $\mathbb{R}^2$ (seen as $\mathbb{R}_+ \times [0, 2\pi)$ in the second picture) for the construction of the function β (and g) as well as the function $g(r, \theta)$ for several values of r. We should understand the function g as follows: $g(r, \theta)$ is be large when the dynamical system is suspected to take long time to exit the set $\mathcal{S}_0$ starting from (r, θ). Conversely, it should be small when the vector field of the dynamical system push the trajectories out of $\mathcal{S}_0$. As pointed out by Figure 3.4, we do not need to consider sub-domain of $\mathcal{S}_0$: the action of the Brownian motion is elliptic and we can always build some trajectories starting from any point of $\mathcal{S}_0$ and staying an arbitrarily long time in $\mathcal{S}_0$. Note that when r is small, the starting point is near the origin, whatever the value of θ is and hence, the function $g(r, \theta)$ is small (see the right side of Figure 3.4).

□

3.2.3. *The hypoelliptic case.* We now come back to the study of the lower-bound of Theorem 4. This result is proved in Proposition 27 stated below. We know from Corollary 12 that up to linear changes of variables in time and space, the initial dynamic may be reduced to the simplified stochastic evolution described by Equation (2.8). Again let us write down the corresponding infinitesimal generator $\mathcal{L}_\rho$ in polar coordinates (see Proposition 37 given in the appendix):

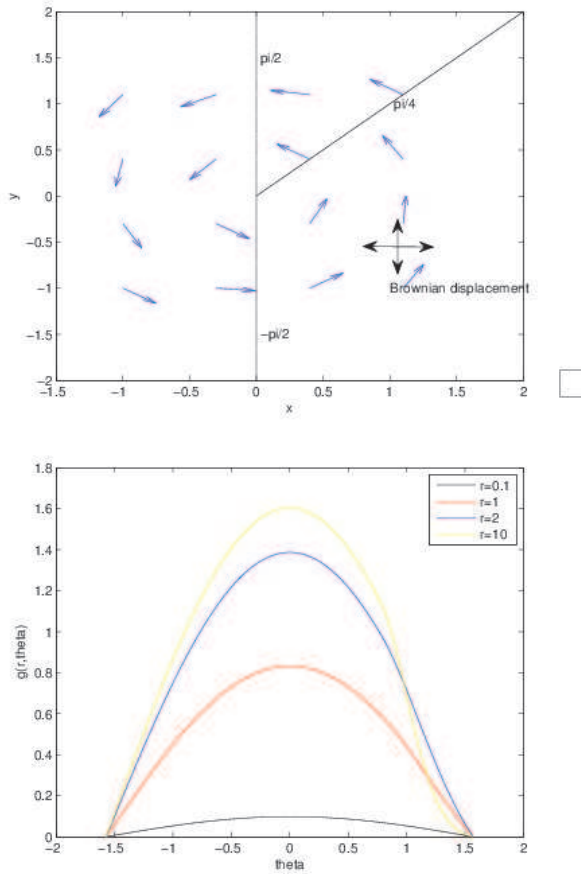

FIGURE 3.4. Left: the domain to avoid is $\theta \in [\pi/2, 3\pi/2]$. The elliptic situation is illustrated by the full rank black double arrow: the Brownian motion always move in all directions. In blue: rotation + homothety vector field. Right: function $\theta \mapsto g(r,\theta)$ for several values of r.

$$\mathcal{L}_\rho = -\rho r \partial_r + \partial_\theta + \frac{\sin^2\theta}{2}\partial^2_{rr} - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta + \frac{\sin\theta\cos\theta}{r}\partial^2_{r\theta} + \frac{\cos^2\theta}{2r}\partial_r + \frac{\cos^2\theta}{2r^2}\partial^2_\theta \tag{3.10}$$

with $\rho = -\frac{a}{2\omega}$. As mentioned before, we would like to use a strategy similar to the one considered in the elliptic case. However, the hypoelliptic problem is more involved. Roughly speaking, the degeneracy of the diffusive component implies that in the neighborhood of $\pi/2$, the paths of the solutions to (2.8) can not be strongly slowed down by the action of the Brownian motion (see Remark 21 and the study on Brownian bridges below). In other words, we are not able (and it seems indeed impossible) to build a function β such that the function ψ_2 defined in the previous subsection is lower-bounded. Thus, the idea is to reduce the domain to a smaller angular sector $\mathcal{S}$ included in $\{(x,y), x>0\}$ where the diffusive action of the Brownian motion is more likely to keep the process in $\mathcal{S}$.

Consequently, we consider a more general class of functions g (which must be calibrated in the sequel) and define

$$g(r,\theta) = r^n \gamma(\theta) e^{\beta(\theta) r^2} \tag{3.11}$$

where n is a positive integer and γ and β are some sufficiently smooth functions. Now β should be bounded above by a negative constant for g to have a chance to be bounded. The new function γ will be chosen in order that g is positive in the interior of the angular sector and vanishes on the boundary of $\mathcal{S}$. We first describe the effect of $\mathcal{L}_\rho$ on such a function g (the computations are deferred to the appendix).

Proposition 22. For any $g \in \mathcal{C}^2\left(\mathbb{R}_+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \mathbb{R}\right)$ given by (3.11), one has

$$\forall (r,\theta) \in \mathbb{R}_+^* \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad L_\rho g(r,\theta) = \left[\varphi_1(\theta) r^2 + \varphi_2(\theta) + \frac{\varphi_3(\theta)}{r^2}\right] g(r,\theta)$$

where

$$\begin{aligned}
\varphi_1(\theta) &= -2\rho\beta(\theta) + \beta'(\theta) + \left(2\sin\theta\beta(\theta) + \cos\theta\beta'(\theta)\right)^2, \\
\varphi_2(\theta) &= -n\rho + \beta(\theta)\left((4n+2)\sin^2\theta + 2\cos^2\theta\right) \\
&\quad + (1 + 2\beta'(\theta)\cos^2\theta + 4\beta(\theta)\sin\theta\cos\theta)\frac{\gamma'}{\gamma}(\theta) + \beta''(\theta)\cos^2(\theta) \\
&\quad + 2(n+1)\cos\theta\sin\theta\beta'(\theta), \\
\varphi_3(\theta) &= (n^2 - n)\sin^2\theta + \cos^2\theta(n + \frac{\gamma''(\theta)}{\gamma(\theta)}) + 2(n-1)\sin\theta\cos\theta\frac{\gamma'(\theta)}{\gamma(\theta)}.
\end{aligned}$$

We now need to find an (open) angular sector $\mathcal{S} = \{(r\cos\theta, r\sin\theta), \theta_1 < \theta < \theta_2\}$, a positive integer n, some functions γ and β such that

(1) $\gamma(\theta) > 0$ on (θ_1, θ_2), $\gamma(\theta_1) = \gamma(\theta_2) = 0$, $\beta(\theta) \le 0$ on $[\theta_1, \theta_2]$,
(2) φ_1 and φ_3 are non-negative on $\mathcal{S}$,
(3) φ_2 is lower-bounded.
(4) β is bounded above by a negative constant.

This is the purpose of the next proposition.

Proposition 23. Let $\rho \ge 0$.
(i) Let g be defined by (3.11) with $n = 2$,

$$\gamma(\theta) = \begin{cases} -\sin(2\theta) & \text{if } \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}] \\ \cos^2(\pi/4 + \theta) & \text{if } \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \end{cases} \tag{3.12}$$

and $\beta(\theta) = -\frac{1}{2}$. Then, for every $r > 0$ and $\theta \in]-\frac{\pi}{2}, \frac{\pi}{4}[$ with $\theta \neq -\frac{\pi}{4}$,

$$\mathcal{L}_\rho g(r,\theta) \quad \ge \quad -(3 + 2\rho) g(r,\theta).$$

(ii) As a consequence, for any open half-plane H such that $\mathcal{S} := \{(r\cos\theta, r\sin\theta), r > 0, \theta \in]-\frac{\pi}{2}, \frac{\pi}{4}[\} \subset H$, for any probability measure m_0 on $\mathbb{R}^2$ such that $m_0(H) = 1$, we have

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau_H \ge t)) \quad \le \quad 3 + 2\rho.$$

Remark 24 The vector field corresponding to the drift part of the stochastic evolution under study, as well as the most favorable positions (which are expected to be the points where g is large) for the starting point in order to keep the process in $\mathcal{S}$

for large times are illustrated in Figure 3.5. As pointed out above, the angular sector $[\pi/4, \pi/2]$ is now avoided to keep the process in the half-plane $x > 0$. Moreover, the right side of Figure 3.5 shows that excessive values of r (too large or too small ones) are also prohibited: small values are unfavourable since it corresponds to starting positions very close to the origin (and naturally close to the axis $x = 0$). Large values of r are also disadvantageous owing to the large norm of the drift vector field against which the Brownian motion has to fight to keep the process in $\mathcal{S}$.

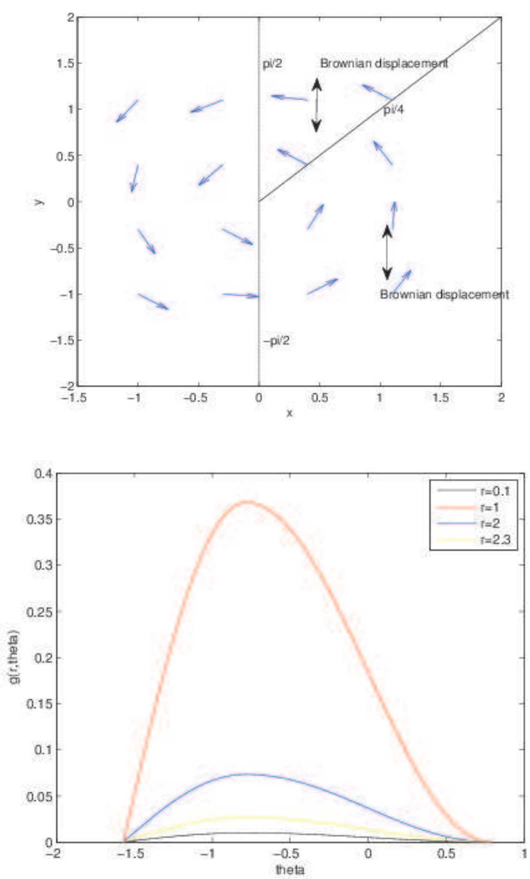

FIGURE 3.5. Left: the domain to avoid is $\theta \in [\pi/4, 3\pi/2]$. The hypo-elliptic situation is illustrated by the rank 1 double arrow: the Brownian motion can only move in vertical directions. In blue: rotation + homothety vector field. Right: function $\theta \mapsto g(r, \theta)$ for several values of r.

□

Proof: With the proposed choices of n and γ, one checks that

$$\varphi_3(\theta) = \begin{cases} 0 & \text{if } \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}) \\ \frac{2}{1-\sin(2\theta)} & \text{if } \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \end{cases}$$

so that φ_3 is non-negative. Since β is constant and ρ is non-negative, the fact that φ_1 is non-negative is obvious. Thus, it remains to focus on φ_2. In fact, easy computations show that $\varphi_2(\theta) = -(3+2\rho)$ on $[-\frac{\pi}{4}, \frac{\pi}{4})$ whereas

$$\forall \theta \in (-\frac{\pi}{2}, -\frac{\pi}{4}], \quad \varphi_2(\theta) = -(3+2\rho) + \frac{2}{\tan(2\theta)}.$$

The conclusion of the first assertion follows.

(ii) By Proposition 18 and what precedes, for any probability measure $m_{\mathcal{S}}$ on $\mathbb{R}^2$ such that $m_{\mathcal{S}}(\mathcal{S}) = 1$,

$$\limsup_{t\to+\infty} -\frac{1}{t}\log\left(\mathbb{P}_{m_{\mathcal{S}}}(\tau_{\mathcal{S}} \geq t)\right) \leq 3+2\rho. \tag{3.13}$$

Now, consider the general case. Let m_0 be a probability such that $m_0(H) = 1$. Then, for every $t > 0$, for every $a.s.$ finite stopping time T,

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{P}_{m_0}(\tau_H \geq T+t) \geq \mathbb{P}_{m_0}(\tau_H > T, Z_s \in \mathcal{S}\ \forall s \in [T, T+t]).$$

Thus,

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{E}_{m_0}\left[1_{\{\tau_H > T, Z_T \in \mathcal{S}\}} \mathbb{P}(Z_{s+T} \in \mathcal{S}, \forall s \in [0,t] | \mathcal{F}_T)\right].$$

and it follows from the Markov property that

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{E}_{m_0}[1_{\{\tau_H > T, Z_T \in \mathcal{S}} \mathbb{P}_{Z_T}(\tau_{\mathcal{S}} \geq t)].$$

If we assume for a moment that T is such that

$$\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S}) > 0, \tag{3.14}$$

then,

$$-\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau_H \geq t)) \leq -\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})) - \frac{1}{t}\log\left(\mathbb{P}_{m_{\mathcal{S}}}(\tau_{\mathcal{S}} \geq t)\right),$$

where $m_{\mathcal{S}}$ is the probability measure defined for every bounded measurable function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$m_{\mathcal{S}}(h) = \frac{1}{\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})} \mathbb{E}_{m_0}[h(Z_T) 1_{\{\tau_H > T, Z_T \in \mathcal{S}\}}].$$

By (3.13) and the (strict) positivity of $\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})$, we obtain that

$$\limsup_{t\to+\infty} -\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau_H \geq t)) \leq 3+2\rho.$$

Thus, it remains to prove (3.14). It is certainly enough to show that for every $(x_0, y_0) \in H$, there exists a deterministic positive $T(x,y)$ such that

$$\mathbb{P}_{(x_0,y_0)}(\tau_H > T(x_0,y_0), Z_{T(x_0,y_0)} \in \mathcal{S}) > 0.$$

The idea is to build some "good" controlled trajectories: let $\varphi \in L^{2,\text{loc}}(\mathbb{R}_+, \mathbb{R})$ and denote by $(z_\varphi(t))_{t\geq 0}$ the solution of the controlled system

$$\begin{cases} \dot{x}(t) = -\rho x(t) - y(t) \\ \dot{y}(t) = -\rho y(t) + x(t) + \varphi(t) \end{cases}$$

starting from $z_0 = (x_0, y_0) \in H$. The classical Support Theorem (see Stroock and Varadhan 1972) can be applied since the coefficients of the diffusion are Lipschitz continuous. This implies that (3.14) is true as soon as there exists such a φ for which the solution $(z_\varphi(t))_{t\in[0,T(x_0,y_0)]}$ belongs to H and such that $z_\varphi(T(x_0,y_0))$ belongs to $\mathcal{S}$. Such a controlled trajectory can be built through the following lemma.

□

Lemma 25. Let $\kappa \in (0, +\infty]$ and set $H_\kappa = \{(x,y), y < \kappa x\}$ and $H_\infty = D$ $(= \{(x,y), x > 0\})$.

(i) Let $(x_0, y_0) \in H_\kappa$ with $y_0 \geq 0$. Then, for every $v \in (-\infty, y_0]$, there exists a controlled trajectory $(x_\varphi(t), y_\varphi(t))_{t\geq 0}$ starting from (x_0, y_0) and a positive T_v such that $\{z_\varphi(t) : t \geq 0\} \subset H_\kappa \cap H_\infty$, $x_\varphi(T_v) > 0$ and $y_\varphi(T_v) = v$.

(ii) Let $(x_0, y_0) \in H_\kappa$ with $y_0 \leq 0$ and consider $(x(t), y(t))_{t\geq 0}$ the solution to the free dynamical system (*i.e.* the controlled trajectory with $\varphi \equiv 0$) starting from (x_0, y_0). Then, there exists $T > 0$ such that $(x(t), y(t))_{t\in[0,T]} \subset H_\kappa$ and such that $(x(T), y(T)) = (a_T, 0)$ with $a_T > 0$. Furthermore, writing $(x_0, y_0) = (r_0 \cos(-\theta_0), r_0 \sin(-\theta_0))$ (with $r_0 > 0$ and $\theta_0 \in (\pi - \text{Arctan}(\kappa), 0]$), this property holds with $T = \theta_0$ and $a_T = r_0 e^{-\rho\theta_0}$.

Remark 26 Note that this lemma will be also used in the proof of Proposition 32 (see Step 3). This is the reason why its statements are a little sharper than what we need for the proof of the previous proposition.

□

Proof: (i) Without loss of generality, we only prove the result when $\kappa < +\infty$. The idea is to build φ such that the derivative of the second component is large enough. More precisely, for every $M > 0$,

$$\begin{cases} \dot{x}_M(t) = -\rho x_M(t) - y_M(t) \\ \dot{y}_M(t) = -M \end{cases}$$

is certainly an equation of a controlled trajectory (by setting $\varphi(t) = -M + \rho y_M(t) + x_M(t)$). Furthermore, denoting by $z_0 = (x_0, y_0)$ its starting point, we have

$$y_M(t) = -Mt + y_0 \quad \text{and} \quad x_M(t) = \left(x_0 + \frac{M}{\rho^2} + \frac{y_0}{\rho}\right) e^{-\rho t} + \frac{M}{\rho} t - \frac{M}{\rho^2} - \frac{y_0}{\rho}.$$

First, let us choose M large enough in order that for all $t \geq 0$, $x_M(t) > 0$ and $(x_M(t), y_M(t)) \in H_\kappa$, *i.e.* such that $x_M(t) > 0$ and $\kappa x_M(t) - y_M(t) \geq 0$ for all $t \geq 0$. A simple study of the derivative of $t \to x_M(t)$ yields

$$\forall t \geq 0, \quad x_M(t) \geq x_M(t_M^*) \quad \text{with} \quad t_M^* = \frac{1}{\rho} \log\left(1 + \frac{\rho}{M}(y_0 + \rho x_0)\right) \quad \text{and}$$

$$x_M(t_M^*) = \frac{M}{\rho^2} \log\left(1 + \frac{\rho}{M}(y_0 + \rho x_0)\right) - \frac{y_0}{\rho} \xrightarrow{M\to+\infty} x_0.$$

Thus, for every $\varepsilon > 0$, there exists M_ε large enough such that $x(t^*_{M_\varepsilon}) \geq \kappa x_0 - \varepsilon$. Using that for any $M > 0$ and $t \geq 0$, $y_M(t) \leq y_0$ and setting $\varepsilon = \frac{\kappa x_0 - y_0}{2}$, we obtain that

$$\forall t \geq 0, \quad x_{M_\varepsilon}(t) > 0 \quad \text{and} \quad \kappa x_{M_\varepsilon}(t) - y_{M_\varepsilon}(t) > 0.$$

Since y_{M_ε} is a continuous function such that $y_{M_\varepsilon}(t) \to -\infty$ as $t \to +\infty$, it follows that for every $v \in (-\infty, y_0]$, there exists $T_v > 0$ such that $y_{M_\varepsilon}(T_v) = v$.
(ii) The result is obvious since the solution to the free dynamical system satisfies

$$(x(t), y(t)) = r_0 e^{-\rho t}(\cos(t - \theta_0), \sin(t - \theta_0)), \quad t \geq 0.$$

□

We are now able to prove the lower-bound of Theorem 4.

Proposition 27. Let $(Z_t)_{t\geq 0}$ be a solution of (1.4) with $(1+\frac{1}{\sqrt{2}})a \leq b$ and let $\tau = \inf\{t > 0, X_t = 0\}$. Then, for every probability measure m_0 on $\mathbb{R}^2$ such that $m_0(\{(x,y), x > 0\}) = 1$,

$$\limsup_{t\to+\infty} -\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau \geq t)) \leq \left(3 + \frac{a}{\omega}\right)\omega.$$

Remark 28 Since $\frac{a}{\omega} = (\frac{b}{a} - \frac{1}{4})^{-\frac{1}{2}}$,

$$\sup_{(a,b), 0<(1+\frac{1}{\sqrt{2}})a\leq b} \left(3 + \frac{a}{\omega}\right) = 3 + (\frac{3}{4} + \frac{1}{\sqrt{2}})^{-\frac{1}{2}} \leq 4.$$

This corresponds to the bound given in Theorem 4. However, the reader can remark that the above result yields some sharper bounds. In particular, when a tends to 0, $3 + a/\omega$ tends to 3.

□

Proof: Let $z_0 = (x_0, y_0) \in \mathbb{R}^2$ such that $x_0 > 0$. Owing to the symmetry of the Brownian motion, one can check that

$$\mathbb{P}_{z_0}(\tau \geq t) = \mathbb{P}_{-z_0}(\tau_{D_-} \geq t)$$

where $z_0 = (x_0, y_0)^*$, $D_- = \{(x,y), x < 0\}$ and $\tau_{D_-} = \inf\{t \geq 0, Z_t \in D_-^c\}$. Second, set $v = (\frac{1}{b^2}(\frac{a}{2} - b), 1)^*$ and $P_v = (v, Bv)$ with $B = \frac{1}{\omega}(A + \frac{a}{2}I_2)$. By Corollary 12, there exists $\alpha > 0$ such that $(\tilde{Z}_t)_{t\geq 0} := (\sqrt{\omega}\alpha P_v^{-1} Z_{\frac{t}{\omega}})_{t\geq 0}$ is a solution of (2.8). Denote respectively by (x,y) and by $(\tilde{x},\tilde{y})$, the coordinates in the canonical basis and in the basis $\tilde{\mathcal{B}} = (v, Bv)$. Computing $P_v(\tilde{x},\tilde{y})^*$, one checks that in the new basis, the set D_- corresponds to the half-plane H_κ defined by

$$H_\kappa = \{(\tilde{x},\tilde{y}), \tilde{y} < \kappa\tilde{x}\} \quad \text{with} \quad \kappa = \frac{1}{\omega}\left(b - \frac{a}{2}\right).$$

Furthermore, from the very definition of $(\tilde{Z}_t)_{t\geq 0}$, we have

$$\tau_{D_-}^{-z_0} = \tilde{\tau}_{H_\kappa}^{-\tilde{z}_0} \quad \text{with } \tilde{\tau}_{H_\kappa} = \inf\{t \geq 0, \tilde{Z}_t \in H_\kappa^c\} \text{ and } \tilde{z}_0 = \sqrt{\omega}\alpha P_v^{-1} z_0.$$

In particular, $\mathbb{P}_{-z_0}(\tau_{D_-} \geq t) = \mathbb{P}_{-\tilde{z}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)$ so that for any probability m_0 on $\mathbb{R}^2$ such that $m_0(\{(x,y), x > 0\}) = 1$,

$$\mathbb{P}_{m_0}(\tau_{D_-} \geq t) = \mathbb{P}_{\tilde{m}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)$$

where $\tilde{m}_0 := m_0 \circ (z \mapsto -\sqrt{\omega}\alpha P_v^{-1} z)$ satisfies $\tilde{m}_0(H_\kappa) = 1$. Now, when $(1+\frac{1}{\sqrt{2}})a \leq b$, one checks that $\kappa \geq 1$ so that H_κ contains the set $\mathcal{S} = \{((\tilde{x},\tilde{y}), \tilde{x} > 0, \tilde{y} < \tilde{x}\}$ of Proposition 23 (written in polar coordinates). Applying the second item of this proposition with $\rho = a/(2\omega)$, we finally obtain

$$\limsup_{t\to+\infty} -\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau \geq t)) = \omega \limsup_{t\to+\infty} -\frac{1}{\omega t}\log(\mathbb{P}_{\tilde{m}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)) \leq \omega\left(3 + \frac{a}{\omega}\right).$$

□

4. Bridges at small times and persistence rate

After considerations on bridges associated to (1.1), we briefly sketch how to recover a lower-bound for $\mathbb{P}_{(x_0,y_0)}(\tau \geq t)$ (where $\tau := \inf\{t \geq 0, Z_t \in \{(x,y), x < 0\}\}$).

4.1. *Explosion of bridges at small times.* We will prove Theorem 6 and discuss some related results. From the Gaussian feature of the problem, we could have worked directly with the process Z whose evolution is given by (1.4). But the computations presented in Subsection 2.1 suggest that it is easier to consider the simplifications of Subsection 2.2. We thus consider the two-dimensional O.U. process $(Z_t)_{t\geq 0} \coloneqq (X_t, Y_t)_{t\geq 0}$

$$\left\{ \begin{array}{rcl} dX_t &=& (-\rho X_t - Y_t)\, dt \\ dY_t &=& (-\rho Y_t + X_t)\, dt + \sqrt{2} dW_t \end{array} \right. \tag{4.1}$$

where $\rho \in \mathbb{R}$ and $(W_t)_{t\geq 0}$ is a standard real Brownian motion. Let us assume that the initial condition of Z is a deterministic point $z_0 = (x_0, y_0)^* \in \mathbb{R}^2$, we then have

Lemma 29. Z_t is distributed as a Gaussian law of mean $m_t(z_0)$ and variance Σ_t, with

$$\begin{array}{rcl} m_t(z_0) &\coloneqq& \exp(-\rho t) \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) \\ x_0 \sin(t) + y_0 \cos(t) \end{pmatrix} \\ \Sigma_t(1,1) &\coloneqq& \dfrac{1-e^{-2\rho t}}{2\rho} - \dfrac{e^{-2\rho t}}{2(1+\rho^2)}(\sin(2t) - \rho\cos(2t)) - \dfrac{\rho}{2(1+\rho^2)} \\ \Sigma_t(1,2) = \Sigma_t(2,1) &\coloneqq& \dfrac{e^{-2\rho t}}{2(1+\rho^2)}(\cos(2t) + \rho\sin(2t)) - \dfrac{1}{2(1+\rho^2)} \\ \Sigma_t(2,2) &\coloneqq& \dfrac{1-e^{-2\rho t}}{2\rho} + \dfrac{e^{-2\rho t}}{2(1+\rho^2)}(\sin(2t) - \rho\cos(2t)) + \dfrac{\rho}{2(1+\rho^2)} \end{array}$$

Proof: Let us denote

$$A \coloneqq \begin{pmatrix} -\rho & -1 \\ 1 & -\rho \end{pmatrix} \quad \text{and} \quad C \coloneqq \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

From the beginning of Subsection 2.1, we get that for any $t \geq 0$, on the one hand

$$m_t(z_0) = \exp(At)z_0 = \exp(-\rho t) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

and on the other hand, the validity of (2.2). We compute that for any $s \geq 0$,

$$\begin{array}{rcl} \exp(As)CC^* \exp(A^* s) &=& 2\exp(-2\rho s) \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ && \times \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \\ &=& \exp(-2\rho s) \begin{pmatrix} 1-\cos(2s) & -\sin(2s) \\ -\sin(2s) & 1+\cos(2s) \end{pmatrix} \end{array}$$

The announced expressions for the entries of Σ_t follow from immediate integrations. For instance for $\Sigma_t(1,1)$, we have

$$\begin{aligned}\Sigma_t(1,1) &= \int_0^t \exp(-2\rho s)(1-\cos(2s))\,ds \\ &= \frac{1-\exp(-2\rho t)}{2\rho} - \Re\left(\int_0^t \exp(2(i-\rho)s)\,ds\right) \\ &= \frac{1-\exp(-2\rho t)}{2\rho} - \Re\left(\frac{\exp(2(i-\rho)t)-1}{2(i-\rho)}\right) \\ &= \frac{1-\exp(-2\rho t)}{2\rho} + \frac{1}{2(1+\rho^2)}\Re\left((\rho+i)(\exp(2(i-\rho)t)-1)\right) \\ &= \frac{1-\exp(-2\rho t)}{2\rho} + \frac{1}{2(1+\rho^2)}(\exp(-2\rho t)(\rho\cos(2t)-\sin(2t))-\rho)\end{aligned}$$

■

Let us denote by $p_t(z_0,z)\,dz$ the law of Z_t knowing that $Z_0=z_0$, the above lemma yields

$$\forall\ t>0,\ \forall\ z_0,z\in\mathbb{R}^2,\qquad p_t(z_0,z) = \frac{1}{2\pi\det(\Sigma_t)}\exp(-(z-m_t(z_0))^*(2\Sigma_t)^{-1}(z-m_t(z_0)))$$

The Bayes formula shows that the law of Z_t conditioned by $Z_T=z_T$ and $Z_0=z_0$ is a non-degenerate Gaussian whose density is proportional to $z\mapsto p_t(z_0,z)p_{T-t}(z,z_T)$. Let $\eta_t^{(T)}(z_0,z_T)$ (resp. $\sigma_t^{(T)}$) be its mean (resp. its covariance), next definition permits to derive some technicalities

$$\forall\ u\in[0,1],\qquad \varphi_{z_0,z_T}(u) \coloneqq \begin{pmatrix} 0 \\ 6u(1-u)(x_0-x_T)\end{pmatrix}$$

Proposition 30. For all $z_0,z_T\in\mathbb{R}^2$ and $u\in(0,1)$, we have

$$\lim_{T\to 0_+} T\eta_{uT}^{(T)}(z_0,z_T) = \varphi_{z_0,z_T}(u) \qquad\text{and}\qquad \lim_{T\to 0_+}\sigma_{uT}^{(T)}(z_0,z_T)=0$$

Proof: For $u\in(0,1)$, we denote $v\coloneqq 1-u$, $\eta_u\coloneqq\eta_{uT}^{(T)}(z_0,z_T)$, $\sigma_u\coloneqq\sigma_{uT}^{(T)}(z_0,z_T)$. Lemma 29 leads to

$$\begin{aligned}(z-\eta_u)^*\sigma_u^{-1}(z-\eta_u) = &(z-m_{uT}(z_0))^*\Sigma_{uT}^{-1}(z-m_{uT}(z_0)) \\ &+ (z_T-m_{vT}(z))^*\Sigma_{vT}^{-1}(z_T-m_{vT}(z)) + C(z_0,z_T),\end{aligned}$$

where $C(z_0,z_T)$ is a normalizing term which is independent of z. It follows that

$$\eta_u = \sigma_u(e^{-\rho uT}S_uB_uz_0 + e^{-\rho vT}B_v^*S_vz_T) \tag{4.2}$$

$$\sigma_u = \left(S_u + e^{-2\rho vT}B_v^*S_vB_v\right)^{-1} \tag{4.3}$$

where for any $w\geq 0$,

$$\begin{aligned}B_w &\coloneqq \begin{pmatrix}\cos(wT) & -\sin(wT)\\ \sin(wT) & \cos(wT)\end{pmatrix} \\ S_w &\coloneqq \Sigma_{wT}^{-1} = \frac{1}{D_w}\begin{pmatrix}\Sigma_{wT}(2,2) & -\Sigma_{wT}(1,2)\\ -\Sigma_{wT}(1,2) & \Sigma_{wT}(1,1)\end{pmatrix} \\ D_w &\coloneqq \det(\Sigma_{wT}) = \Sigma_{wT}(1,1)\Sigma_{wT}(2,2) - (\Sigma_{wT}(1,2))^2\end{aligned}$$

These expressions depend on $T > 0$ and the proof will be obtained by expanding them for small $T > 0$. Indeed, simple computations show that for $w \in (0,1)$, as $T \to 0_+$,

$$\begin{pmatrix} \Sigma_{wT}(1,1) & \Sigma_{wT}(1,2) \\ \Sigma_{wT}(1,2) & \Sigma_{wT}(2,2) \end{pmatrix} = \begin{pmatrix} \frac{2(wT)^3}{3} + \mathcal{O}((wT)^4) & -(wT)^2 + \mathcal{O}((wT)^3) \\ -(wT)^2 + \mathcal{O}((wT)^3) & 2wT + \mathcal{O}((wT)^2) \end{pmatrix}$$

where $\mathcal{O}((wT)^p)$, for $p \in \mathbb{R}$, stands for a quantity bounded by $A(wT)^p$, uniformly over $\rho \in [-1,1]$ and for wT small enough. It follows that

$$\begin{aligned} D_w &= \frac{(wT)^4}{3} + \mathcal{O}((wT)^4) \\ S_w &= \begin{pmatrix} \frac{6}{(wT)^3} + \mathcal{O}((wT)^{-2}) & \frac{3}{(wT)^2} + \mathcal{O}((wT)^{-1}) \\ \frac{3}{(wT)^2} + \mathcal{O}((wT)^{-1}) & \frac{2}{wT} + \mathcal{O}(1) \end{pmatrix} \end{aligned}$$

Using furthermore that for $v \in (0,1)$, we have $e^{-2\rho vT} = 1 + \mathcal{O}(vT)$ and that

$$B_v = \begin{pmatrix} 1 + \mathcal{O}((vT)^2) & -vT + \mathcal{O}((vT)^3) \\ vT + \mathcal{O}((vT)^3) & 1 + \mathcal{O}((vT)^2) \end{pmatrix}$$

we deduce that

$$e^{-2\rho vT} B_v^* S_v B_v = \begin{pmatrix} \frac{6}{(vT)^3} + \mathcal{O}((vT)^{-2}) & -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) \\ -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) & \frac{2}{vT} + \mathcal{O}(1) \end{pmatrix}$$

If $d(u,v) = 12(u^3+v^3)(u+v) - 9(u^2-v^2)^2 + \mathcal{O}(T^{-1})$, we obtain from (4.3) $\forall u \in (0,1)$

$$\sigma_u = \frac{1}{d(u,v)} \begin{pmatrix} 2(u+v)(uv)^3T^3 + \mathcal{O}(T^4) & 3(u^2-v^2)(uv)^2T^2 + \mathcal{O}(T^3) \\ 3(u^2-v^2)(uv)^2T^2 + \mathcal{O}(T^3) & 6(u^3+v^3)uvT + \mathcal{O}(T^2) \end{pmatrix}$$

Since $v = 1-u$, $d(u,v) = 12 + \mathcal{O}(T^{-1})$ and

$$\sigma_u = \begin{pmatrix} \frac{2}{3}(uv)^3T^3 + \mathcal{O}(T^4) & (u-v)(uv)^2T^2 + \mathcal{O}(T^3) \\ (u-v)(uv)^2T^2 + \mathcal{O}(T^3) & 2(u^3+v^3)uvT + \mathcal{O}(T^2) \end{pmatrix} \tag{4.4}$$

We obtain the second convergence announced. To deduce the first one, we begin by checking that

$$\begin{aligned} \forall u \in (0,1) \qquad e^{-\rho uT} S_u B_u &= \begin{pmatrix} \frac{6}{(uT)^3} + \mathcal{O}((uT)^{-2}) & -\frac{3}{(uT)^2} + \mathcal{O}((uT)^{-1}) \\ \frac{3}{(uT)^2} + \mathcal{O}((uT)^{-1}) & -\frac{1}{uT} + \mathcal{O}(1) \end{pmatrix} \\ e^{-\rho vT} B_v^* S_v &= \begin{pmatrix} \frac{6}{(vT)^3} + \mathcal{O}((vT)^{-2}) & \frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) \\ -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) & -\frac{1}{vT} + \mathcal{O}(1) \end{pmatrix} \end{aligned}$$

In conjunction with (4.4), we get

$$\begin{aligned} \sigma_u e^{-\rho uT} S_u B_u &= \begin{pmatrix} 1 - 3u^2 + 2u^3 + \mathcal{O}(T) & -u(1-u)^2T + \mathcal{O}(T^2) \\ \frac{6u(1-u)}{T} + \mathcal{O}(1) & 1 - 4u + 3u^2 + \mathcal{O}(T) \end{pmatrix} \\ \sigma_u e^{-\rho vT} B_v^* S_v &= \begin{pmatrix} 3u^2 - 2u^3 + \mathcal{O}(T) & u^2(1-u)T + \mathcal{O}(T^2) \\ -\frac{6u(1-u)}{T} + \mathcal{O}(1) & -2u + 3u^2 + \mathcal{O}(T) \end{pmatrix} \end{aligned}$$

In these expression, the $(2,1)$-entries explode as $T \to 0_+$, it explains the renormalization by T considered in the above proposition for $\eta_{uT}^{(T)}(z_0, z_T)$ and resulting convergence.

■

Remark 31 When $x_0 = x_T$ (if z_0 and z_T are on the same vertical line), it is simpler for the underlying vertical Brownian motion to link z_0 to z_T: the second component of φ_{z_0,z_T} is equal to 0. The second component of $(\eta_{uT}^{(T)})$ is then convergent when $T \to 0$ and we can obtain

$$(\sigma_u e^{-\rho uT} S_u B_u)_{2,1} = \frac{6u(1-u)}{T} - 2\rho u(1-u)(2-u) + \mathcal{O}(T) \tag{4.5}$$

and

$$(\sigma_u e^{-\rho vT} B_v^* S_v)_{2,1} = -\frac{6u(1-u)}{T} - 2\rho u(1-u^2) + \mathcal{O}(T). \tag{4.6}$$

Combined with the previous results, if $x_0 = x_T$ no renormalization is needed and we get

$$\lim_{T\to 0_+} \eta_{uT}^{(T)}(z_0, z_T) = \begin{pmatrix} x_0 \\ (1-4u+3u^2)y_0 - (2u-3u^2)y_T - 6\rho u(1-u)x_0 \end{pmatrix}$$

Even in the case when $z_0 = z_T$, the asymptotic bridge does not stay still (except if $y_0 = 0$), since

$$\lim_{T\to 0_+} \eta_{uT}^{(T)}(z_0, z_0) = \begin{pmatrix} x_0 \\ (1-6u+6u^2)y_0 - 6\rho u(1-u)x_0 \end{pmatrix}.$$

□

End of the proof of Theorem 6. Similarly to the notational conventions endorsed in the introduction, for $T > 0$ and $z, z' \in \mathbb{R}^2$, let $\mathbb{P}_{z,z'}^{(T)}$ be the law of the process Z evolving according to (4.1), conditioned by the event $\{Z_0 = z,\ Z_T = z'\}$ and consider the process $\xi^{(T)} \coloneqq (\xi_u^{(T)})_{u\in[0,1]}$ defined by

$$\forall\ u \in [0,1], \qquad \xi_u^{(T)} \coloneqq TZ_{Tu}$$

Under $\mathbb{P}_{z,z'}^{(T)}$ this process is Gaussian and Proposition 30 enables to see that for fixed $z, z' \in \mathbb{R}^2$, as T goes to 0_+, $\xi^{(T)}$ converges in probability (under $\mathbb{P}_{z,z'}^{(T)}$) toward the deterministic trajectory $\varphi_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1],\mathbb{R}^2)$). Indeed, $\lim_{T\to 0_+} T^2\sigma_{uT}^{(T)}(z_0, z_T) = 0$ would even have been sufficient for this behavior. Using the linear space-time transformation described in Subsection 2.2, this result can be retranscripted under the form of Theorem 6.

Following Remark 31, if $z = (x, y)$ and $z' = (x', y')$ are such that $x = x'$, then the process $\widetilde{\xi}^{(T)} \coloneqq (\widetilde{\xi}_u^{(T)})_{u\in[0,1]}$, defined by

$$\forall\ u \in [0,1], \qquad \widetilde{\xi}_u^{(T)} \coloneqq Z_{Tu}$$

converges in probability (under $\mathbb{P}_{z,z'}^{(T)}$) toward the deterministic trajectory $\widetilde{\varphi}_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1],\mathbb{R}^2)$), where

$$\forall\ u \in [0,1], \qquad \widetilde{\varphi}_{z,z'}(u) \coloneqq \begin{pmatrix} x \\ (1-4u+3u^2)y - (2u-3u^2)y' - 6\rho u(1-u)x \end{pmatrix}$$

Using the linear space-time transformation described in Subsection 2.2, this result can also rewritten in the original setting of the Introduction.

■

4.2. *Probabilistic proof of a persistence rate upper-bound.* Theoretical derivations on the bridges associated to (4.1) permit to retrieve a lower-bound of $\mathbb{P}(\tau \geq t)$, for τ defined in (1.10). We sketch the main arguments (a complete proof can be found in Gadat et al. 2013).

Proposition 32. (i) Let $(Z_t)_{t\geq 0}$ be a solution of (4.1) and for $\kappa \geq 1$, let $H_\kappa = \{(x,y), y < \kappa x\}$. Then, for any positive ρ_0, there exists a constant $\tilde{\lambda} > 0$ such that for any $\rho \in [0, \rho_0]$ satisfying $\kappa \geq 3\rho$ and any $z_0 \in H_\kappa$, one can find a constant $C := C(z_0, \rho_0, \kappa)$ such that

$$\mathbb{P}_{z_0}(\tau_{H_\kappa} \geq t) \geq C \exp(-\tilde{\lambda} t), \quad t > 0.$$

(ii) If (Z_t) is a solution of (1.4) and $\omega = \sqrt{ab - a^2/4}$ with $0 < 2a \leq b$, then

$$\exists \tilde{\lambda} > 0 \quad \forall z_0 \in D = \{(x,y), x > 0\} \qquad \mathbb{P}_{z_0}(\tau \geq t) \geq C_{z_0,a,b,c} \exp(-\tilde{\lambda}\omega t), \quad t > 0,$$

Proof: (i) The proof is divided into three steps. Firstly, we build a subset $\mathcal{S}$ of H_κ for which any bridge associated to (4.1), starting and ending in $\mathcal{S}$ (at a time T which will be chosen small) stays in H_κ with a high probability. Then, a Markov-type argument used in Proposition 14 to obtain the announced result when starting point in $\mathcal{S}$. Finally, we get the result to any initial point in H_κ.

Step 1. *Lower-bound for* $\inf_{z,z'\in\mathcal{S}} \mathbb{P}^{(T)}_{z,z'}(\tau_{H_\kappa} > T)$ *for a particular* $T > 0$. The first key ingredient relies on the asymptotic expansion for small time T of the mean of the bridge $(\eta^1_{uT}(z_0, z_T))_{0\leq u\leq 1}$ given in Equations (4.5) and (4.6). We conclude that $(\eta^1_{uT}(z_0, z_T))_{0\leq u\leq 1}$ is kept at a distance greater than 1 from ∂H_κ if (z_0, z_T) are chosen in the half plane $[1; +\infty) \times \mathbb{R}$. The second main argument originates from Theorem V.5.3 of Adler (1990): a universal constant C exists such that

$$\forall T \geq 0,\ \forall h \geq 1, \quad \mathbb{P}^{(T)}_{z_0,z_T}\left(\sup_{u\in[0,T]} |Z_{uT} - \eta^{(T)}_{uT}(z_0, z_T)| > h\right) \leq Ch \exp\left(-\frac{h^2}{2\bar{\sigma}_T}\right)$$

Applying the previous inequality with $h = \sqrt{\sigma_T}$, we deduce that there exists $T_1 \in (0, T_0]$ such that for every $T \in (0, T_1]$, for every $z_0, z_T \in \mathbb{R}^2$ and every $\rho \in [0, \rho_0]$,

$$\mathbb{P}^{(T)}_{z_0,z_T}\left(\sup_{u\in[0,T]} |Z_{uT} - \eta^{(T)}_{uT}(z_0, z_T)| \leq \sqrt{\sigma_T}\right) \geq \frac{1}{2}.$$

Again, the bound on $\sigma^{(T)}_{uT}(z_0, z_T)$ obtained in Proposition 30 (uniform in z_0 and z_T) permits to conclude that if $\mathcal{S}$ is chosen as $\mathcal{S} = [1, 1 + h_1] \times [-h_2, h_2]$ with h_1 and h_2 small enough, then for every $\rho \in [0, \rho_0]$ and $\kappa \geq 1$ verifying $\kappa \geq 3\rho$, we have

$$\inf_{z,z'\in\mathcal{S}} \mathbb{P}^{(T)}_{z,z'}(\tau_{H_\kappa} > T) \quad \geq \quad \frac{1}{2}. \tag{4.7}$$

Step 2. *Lower-bound for* $\mathbb{P}_{z_0}(\tau_{H_\kappa} > t)$ *when* $z_0 \in \mathcal{S}$. From (4.7), forr every $\ell \geq 1$, we have

$$\begin{aligned}\mathbb{P}_{z_0}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S}) &\geq \mathbb{P}_{z_0}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S} | \tau_{H_\kappa} > (\ell-1)T, Z_{(\ell-1)T} \in \mathcal{S}) \\ &\quad \times \mathbb{P}_{z_0}(\tau_{H_\kappa} > (\ell-1)T, Z_{(\ell-1)T} \in \mathcal{S}).\end{aligned}$$

The Markov property, the compactness of $\mathcal{S}$ and the smoothness of the conditionned law of $Z_{\ell t}$ lead to $\mathbb{P}_z(\tau_{H_\kappa} > t) \geq \mathbb{P}_z(\tau_{H_\kappa} > k_T T, Z_{k_T T} \in \mathcal{S})$, where $k_T = \lfloor t/T \rfloor + 1$. An induction ensures that for every $z \in \mathcal{S}$ $\mathbb{P}_z(\tau_{H_\kappa} > t) \geq \varsigma^{k_T} \geq C \exp(-\tilde{\lambda} t)$, where

$\tilde{\lambda} = -\log(\varsigma)/T$ and $\varsigma := \frac{1}{2}\inf_{z\in\mathcal{S}} \mathbb{P}_z(Z_T \in \mathcal{S})$.

Step 3. *Lower-bound for $\mathbb{P}_z(\tau_{H_\kappa} > t)$ when $z \in H_\kappa$.* To extend the lower-bound to any $z_0 = (x_0, y_0) \in H_\kappa$, it is enough to build a controlled trajectory $(z_\varphi(t))_{t\geq 0}$ such that $z_\varphi(0) = z_0$, $z_\varphi(t_0)$ belongs to $\mathcal{S}$ and such that $z_\varphi(t) \in H_\kappa$ for every $t \in [0, t_0]$. This can be done using either Proposition 23 with a careful inspection of the differential system

$$\begin{cases} \dot{x}(t) = -\rho x(t) - y(t) \\ \dot{y}(t) = -\rho y(t) + x(t) + \varphi(t) \end{cases}$$

(ii) By Corollary 12, there exists α such that $(\hat{Z}_t)_{t\geq 0} := \left(\sqrt{\omega}\alpha P_v^{-1} Z_{\frac{t}{\omega}}\right)_{t\geq 0}$ is a solution of (2.8) and we conclude using (i) and a similar proof of Proposition 27.

■

□

5. On the persistence rate

Our goal here is to prove the existence of the quasi-stationary distribution and its persistence rate, as alluded to in Remark 5. Recall that $D := \{(x, y) \in \mathbb{R}^2 : x > 0\}$ and let ∂D be its boundary. We are interested in L_D, the realization on D of the differential operator L given by (1.6) with Dirichlet boundary condition on ∂D. From a probabilist point of view, it is constructed in the following way. For any $z \in \mathbb{R}^2$, let $(Z_t^z)_{t\geq 0}$ be a diffusion process whose evolution is dictated by L and whose initial condition is $Z_0^z = z$. Starting from z, $(Z_t^z)_{t\geq 0}$ can be obtained by solving the stochastic differential equation (1.4) with coefficients given by (1.5). Let τ be the stopping time defined by (1.10), namely

$$\tau \quad := \quad \inf\{t \geq 0 : Z_t^z \in \partial D\}$$

For any $t \geq 0$, any $z \in D$ and any measurable and bounded function f defined on D, consider

$$P_t^D[f](z) \quad := \quad \mathbb{E}[f(Z_t^z)\mathbb{1}_{t<\tau}] \tag{5.1}$$

Recall that μ is the invariant Gaussian probability measure of L and denote by μ^D its restriction to D. Then P_t^D can be extended into a contraction operator on $\mathbb{L}^2(\mu^D)$. Indeed, let P_t be the full operator associated to L: any $z \in D$ and any measurable and bounded function f defined on $\mathbb{R}^2$, we have

$$P_t[f](z) \quad := \quad \mathbb{E}[f(Z_t^z)] \tag{5.2}$$

Since μ is invariant for P_t, for any measurable and bounded function f defined on D (which can also be seen as a function on $\mathbb{R}^2$ by assuming that it vanishes outside D), we get by Cauchy-Schwarz inequality,

$$\mu^D[(P_t^D[f])^2] \leq \mu^D[P_t^D[f^2]] \leq \mu^D[P_t[f^2]] \leq \mu[P_t[f^2]] = \mu[f^2] = \mu^D[f^2]$$

This bound enables to extend P_t^D as a contraction on $\mathbb{L}^2(\mu^D)$. The Markov property implies that $(P_t^D)_{t\geq 0}$ is a semi-group, which is easily seen to be continuous in $\mathbb{L}^2(\mu^D)$. The operator L^D is then defined as the generator of this semi-group (in the Hille-Yoshida sense): its domain $\mathcal{D}(L_T^D)$ is the dense subspace of $\mathbb{L}^2(\mu^D)$

consisting of functions f such that $(P_t^D[f] - f)/t$ converges in $\mathbb{L}^2(\mu^D)$ as t goes to 0_+ and the limit is $L^D[f]$ by definition.

The spectrum of $-L^D$ admits a smallest element (in modulus) $\lambda_0(D)$. It is a positive real number and the main objective of this appendix is to justify the assertions made in Remark 5. We begin by being more precise about the existence of $\lambda_0(D)$:

Proposition 33. There exists a number $\lambda_0(D) > 0$ and two functions $\varphi, \varphi^* \in \mathcal{D}(L^D) \cap \bigcap_{r \geq 1} \mathbb{L}^r(\mu^D) \setminus \{0\}$, which are positive on D, such that

$$\begin{aligned} L^D[\varphi] &= -\lambda_0(D)\varphi \\ L^{D*}[\varphi^*] &= -\lambda_0(D)\varphi^* \end{aligned}$$

where L^{D*} is operator adjoint of L^D in $\mathbb{L}^2(\mu^D)$.

Essentially, this result is a consequence of the Krein-Rutman theorem (which is an infinite version of the Perron-Frobenius theorem, see for instance the paper of Du 2006) and the fact that the eigenfunctions belong to $\mathbb{L}^p(\mu^D)$ instead of $\mathbb{L}^2(\mu^D)$ comes from the hyperboundedness of the underlying Dirichlet semi-group.

The rigorous proof relies on a simple technical lemma about the kernels of the operators P_t^D for $t > 0$. To check their existence, we first come back to P_t for a given $t > 0$: from the computations of Section 2, this operator is indeed given by a kernel

$$\forall\, z \in \mathbb{R}^2,\, \forall\, f \in \mathbb{L}^2(\mu), \qquad P_t[f](z) \quad = \quad \int p_t(z, z') f(z')\, \mu(dz')$$

where

$$\forall\, z,\, z' \in \mathbb{R}^2, \quad p_t(z, z') \quad := \quad \sqrt{\frac{\det(\Sigma)}{\det(\Sigma_t)}} \exp\left(-(z' - z_t)^* \Sigma_t (z' - z_t) + (z')^* \Sigma z'\right) \tag{5.3}$$

with

$$z_t \quad := \quad \exp(At)z$$

It follows easily from (5.1) and (5.2) that the same is true for P_t^D: there exists a function $D^2 \ni (z, z') \mapsto p_t^D(z, z') \geq 0$ such that

$$\forall\, z \in D,\, \forall\, f \in \mathbb{L}^2(\mu^D), \qquad P_t^D[f](z) \quad = \quad \int p_t^D(z, z') f(z')\, \mu(dz')$$

and satisfying

$$\forall\, z, z' \in D, \qquad p_t^D(z, z') \quad \leq \quad p_t(z, z') \tag{5.4}$$

More refined arguments based on the hypoellipticity of L^D enable to see that the mapping p_t^D is continuous and positive on D^2. We can now state a simple but crucial observation:

Lemma 34. For any $r > 1$, there exists a time $T_r > 0$ such that

$$\forall\, t \geq T_r, \qquad \int (p_t^D(z, z'))^r\, \mu^D(dz)\, \mu^D(dz') \quad < \quad +\infty$$

Proof: From (5.4), it is sufficient to prove that

$$\int (p_t(z, z'))^r\, \mu(dz)\, \mu(dz') \quad < \quad +\infty$$

and this can be obtained without difficulty from (5.3) and from the explicit computations of $\exp(tA)$ of Σ_t and of Σ presented in Section 2.

■

We can now come to the

Proof of Proposition 33

We begin by applying Lemma 34 with $r = 2$ to find some $T_2 > 0$ such that for $t \geq T_2$ we have

$$\int (p_t^D(z, z'))^2 \, \mu^D(dz) \, \mu^D(dz') \quad < \quad +\infty$$

which implies that P_t^D is of Hilbert-Schmidt class and thus a compact operator. Note furthermore that the spectral radius of P_t^D is positive for all $t \geq 0$. Indeed, this feature can be deduced from the second bound of Theorem 4, which implies that for all $z \in D$, $P_t^D[\mathbb{1}_D](z) = \mathbb{P}_z[\tau > t] > 0$. Thus we are in position to apply Krein-Rutman theorem (see Theorems 1.1 and 1.2 of Du 2006, where the abstract Banach X space should be $\mathbb{L}^2(\mu^D)$ and the cone K should consist of the nonnegative elements of $\mathbb{L}^2(\mu^D)$): if $\theta_t > 0$ is the spectrum radius of P_t^D, then there exists a positive function $\varphi_t \in \mathbb{L}^2(\mu^D) \setminus \{0\}$ such that $P_t[\varphi_t] = \theta_t \varphi_t$. This property characterizes θ_t and φ_t (up to a constant factor): if θ is a positive real and if $\varphi \in \mathbb{L}^2(\mu^D)$ is a positive function such that $P_t[\varphi] = \theta\varphi$ then is $\theta = \theta_t$ and φ is proportional to φ_t. This suggests to consider the renormalization $\mu^D[\varphi_t^2] = 1$, so that φ_t is uniquely determined (being positive). From the previous property, we deduce that for all $t \geq T_2$ and all $n \in \mathbb{N}$, $\varphi_{nt} = \varphi_t$ and $\theta_{nt} = \theta_t^n$. Indeed, it is sufficient to note that

$$P_{nt}^D[\varphi_t] = (P_t^D)^n[\varphi_t] = \theta_t^n \varphi_t.$$

We deduce that for any $r \in \mathbb{Q} \cap [1, +\infty)$, $\varphi_{T_2 r} = \varphi_{T_2}$ and $\theta_{T_2 r} = \theta_{T_2}^r$: write $r = p/q$ with $p, q \in \mathbb{N}$ and note that $\varphi_{T_2} = \varphi_{pT_2} = \varphi_{qrT_2} = \varphi_{rT_2}$ and similarly $\theta_{T_2}^p = \theta_{rT_2}^q = \theta_{pT_2}$. Let us define $\varphi \coloneqq P_{T_2}^D \varphi_{T_2} = \theta_{T_2} \varphi_{T_2}$. Since $T_2 > 0$ and $P_{T_2}^D(\mathbb{L}^2(\mu^D))$ is included in the domain of L^D, we have $\varphi \in \mathcal{D}(L^D)$. Furthermore from the general Hille-Yoshida theory we have in $\mathbb{L}^2(\mu^D)$,

$$\lim_{t \to 0_+} \frac{P_{T_2+t}^D[\varphi_{T_2}] - P_{T_2}^D[\varphi_{T_2}]}{t} \quad = \quad L^D[P_{T_2}[\varphi_{T_2}]].$$

Thus considering t of the form qT_2 with $q \in \mathbb{Q}_+$ going to zero, we deduce that

$$L^D[\varphi] = \lim_{q \in \mathbb{Q},\, q \to 0_+} \frac{\theta_{T_2}^{q+1} - \theta_{T_2}}{T_2 q} \varphi_{T_2} = \theta_{T_2} \frac{\ln(\theta_{T_2})}{T_2} \varphi_{T_2} = \frac{\ln(\theta_{T_2})}{T_2} \varphi.$$

It remains to set $\lambda_0(D) = -\ln(\theta_{T_2})/T_2$. Since θ_{T_2} is the spectral norm of the contraction operator P_{T_2}, it appears that $\lambda_0(D) \geq 0$. The first bound of Theorem 4 enables to check that $\lambda_0(D) > 0$: from Cauchy-Schwarz inequality, we get that for all $f \in \mathbb{L}^2(\mu^D)$ and all $z \in D$,

$$\begin{aligned} (P_{T_2}^D[f])^2(z) \quad &\leq \quad P_{T_2}^D[f^2](z) P_{T_2}^D[\mathbb{1}_D](z) \\ &\leq \quad P_{T_2}^D[f^2](z) \sup_{z' \in D} P_{T_2}^D[\mathbb{1}_D](z') \end{aligned}$$

it follows that

$$\begin{aligned}\mu^D[(P^D_{T_2}[f])^2] &\le \sup_{z\in D} P^D_{T_2}[\mathbb{1}_D](z)\mu^D[P^D_{T_2}[f^2]]\\ &\le \sup_{z\in D} P^D_{T_2}[\mathbb{1}_D](z)\mu^D[f^2]\end{aligned}$$

So the norm operator of $P^D_{T_2}$ satisfies

$$\theta_{T_2} = \left\|P^D_{T_2}\right\|_{\mathbb{L}^2(\mu^D)\to\mathbb{L}^2(\mu^D)} \le \sup_{z\in D} P^D_{T_2}[\mathbb{1}_D](z) = \sup_{z\in D} \mathbb{P}_z[\tau > T_2] \tag{5.5}$$

which itself is strictly less than 1 for T_2 large enough. Up to the choice of such a T_2 in the above arguments, we conclude that $\lambda_0(D) > 0$.

Let us now check that $\varphi \in \bigcap_{r\ge 1}\mathbb{L}^r(\mu^D)$, since a priori we only know that $\varphi \in \mathbb{L}^2(\mu^D) = \bigcap_{r\in[1,2]}\mathbb{L}^r(\mu^D)$. This is due to the hyperboundedness of $(P^D_t)_{t\ge 0}$. Let $r > 2$ be given and a corresponding $T_r > 0$ such that the conclusion of Lemma 34 is satisfied. Let $f \in \mathbb{L}^2(\mu^D)$ be given. Cauchy-Schwarz and Hölder inequalities imply that for all $z \in D$ and all $t \ge T_r$,

$$\begin{aligned}(P^D_t[f](z))^r &= \left(\int f(z')p^D_t(z,z')\,\mu^D(dz')\right)^r\\ &\le \left(\int f^2(z')\,\mu^D(dz')\right)^{\frac{r}{2}}\left(\int (p^D_t(z,z'))^2\,\mu^D(dz')\right)^{\frac{r}{2}}\\ &\le \left(\int f^2(z')\,\mu^D(dz')\right)^{\frac{r}{2}}\left(\int (p^D_t(z,z'))^r\,\mu^D(dz')\right)\end{aligned}$$

Integrating this bound with respect to $\mu^D(dz)$, it follows that

$$\left(\int (P^D_t[f])^r\,d\mu^D\right)^{\frac{1}{r}} \le \left(\int (p^D_t(z,z'))^r\,\mu^D(dz)\mu^D(dz')\right)^{\frac{1}{r}}\left(\int f^2(z')\,\mu^D dz'\right)^{\frac{1}{2}}$$

namely P^D_t send continuously $\mathbb{L}^2(\mu^D)$ into $\mathbb{L}^r(\mu^D)$. If furthermore t is of the form T_2q with $q \in \mathbb{Q}\cap[1,+\infty)$, we get from $\varphi_{T_2q} = P^D_{T_2q}[\varphi_{T_2q}]/\theta_{T_2q}$ that $\varphi = \varphi_{T_2q}$ belongs to $\mathbb{L}^r(\mu^D)$.

The same arguments are also valid for the adjoint semigroup $(P^{D*}_t)_{t\ge 0}$. Its elements for $t > 0$ admit the kernels p^{D*}_t where

$$\forall\, t > 0, \forall\, z, z' \in D, \qquad p^{D*}_t(z,z') := \frac{\mu^D(z')p^D_t(z',z)}{\mu^D(z)} = \frac{\mu(z')p^D_t(z',z)}{\mu(z)}$$

We end up with the same quantity $\lambda_0(D)$, since for any $t > 0$ the operators P^D_t and $P^{D^*}_t$ have the same spectral radius.

■

Let ν^D be the probability measure on D which admits $\varphi^*/\mu^D[\varphi^*]$ as density with respect to μ^D. The validity of (1.11) is ensured by the following proposition (see *e.g.* Méléard and Villemonais 2012 for a proof).

Proposition 35. The probability measure ν^D is a quasi-stationary distribution for L^D and under $\mathbb{P}_{\nu^D}$, τ is distributed as an exponential law of parameter $\lambda_0(D)$.

The bounds (1.12) are now easy to deduce. Indeed recalling the definition of $\lambda_0(D)$ in terms of θ_{T_2} given in the proof of Proposition 33 (and the fact that T_2

can be chosen arbitrary large), we get from the first bound of Theorem 4 that $\lambda_0(D) \geq \ln(2)\omega/\pi$.

The second bound of Theorem 4 applied with $m_0 = \nu^D$ gives that $\lambda_0(D) \leq 4$.

Remark 36 Is ν^D the unique quasi-stationary probability measure associated to L^D? A priori one has to be careful since this is wrong for the usual one-dimensional Ornstein-Uhlenbeck process with respect to a half-line. Nevertheless we believe there is uniqueness in our situation, because it is easy for the underlying process to get out of D uniformly over the starting point (as shown by the first bound of Theorem 4) and this should be a sufficient condition (in the spirit of Section 7.7 of the book Collet et al. (2013) of Collet, Martínez and San Martín, which unfortunately only treat the case of one-dimensional diffusions). At least from the uniqueness statement included in Krein-Rutman theorem (cf. again Theorem 1.2 of Du 2006), we deduce that ν^D is the unique quasi-stationary measure admitting a density with respect to μ^D which is in $\mathbb{L}^2(\mu^D)$. By hyperboundedness of $(P_t^D)_{t\geq 0}$, the latter condition can be relaxed by only requiring that the density belongs to $\bigcap_{p>1} \mathbb{L}^p(\mu^D)$.

□

6. Conclusion

In this paper a model of speculative evolution was proposed. The dynamics has to be at least of second order, to have a chance to display a weak periodic behavior typical of this kind of phenomena. This second order is induced by the way the process under consideration weights its past evolution to infer its future behavior (increase/decrease in the close past favoring an immediate tendency to follow the same trend). Dynamics of all orders (including non-integer ones) could be obtained in the same fashion, by modifying the weights. At the “microscopic level”, the latter are related to the distribution of the backward time windows used by a multitude of agents in order to speculate on the future evolution.

From the mathematical point of view, our main interest was in the return time to the equilibrium “price” and we have shown that it is more concentrated than the relaxation to the equilibrium distribution of the prices. This feature explains the almost periodic aspect of the typical trajectories. We have obtained some lower and upper bounds of this concentration rate in Theorem 4. Even if there is still a gap between our lower and upper bounds, numerous simulations (not shown here) *via* Fleming-Viot’s algorithm described in Del Moral and Miclo (2003) leads to the conjecture that $\lambda_0(D) = \frac{\log(2)}{\pi}\omega$.

Several questions are left open by our work. Among them, one can thought of a finer modelling that would take into account the time-inhomogeneity of a, b and c. Such an extension remains Gaussian but need more effort from a stochastic analysis point of view. Some difficult statistical questions can also be raised: how can we estimate (a, b, c) when only the price coordinate is observed, at a regular frequency? Is there any good strategy to exploit the periodic structure of the trajectories?

As a conclusion, let us just consider the example of the home price index relative to disposable income per household in France from 1965 to 2015 shown in Figure 6.6. If $t_0 \coloneqq 1965$, $t_1 \coloneqq 1999$ and $t_2 \coloneqq 2000$. Figure 6.6 suggests that between t_0 and

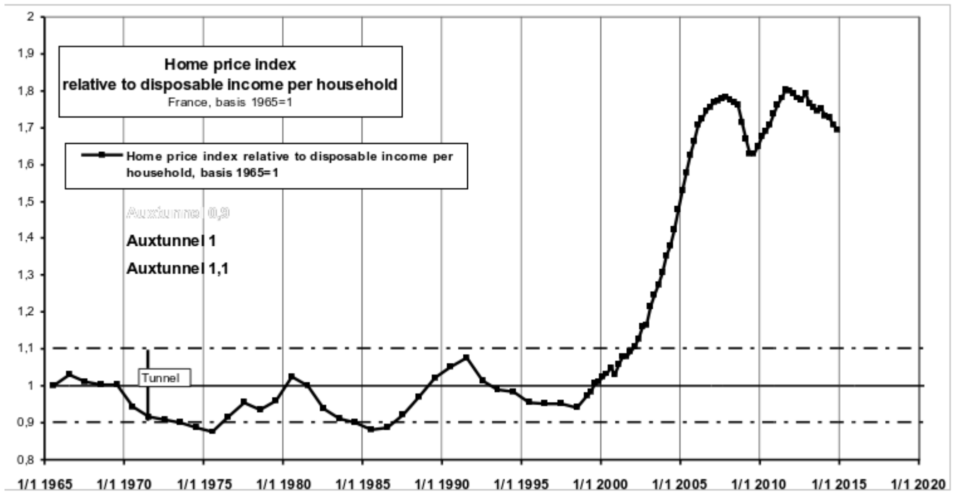

FIGURE 6.6. Friggit's curve Friggit (2015) of the index of asset price relatively to the disposable income in France (1965-2015).

t_1 there were 3 periods with a first set of coefficients (a_1, b_1, c_1) and that between t_1 and t_2 there was one quarter of a period under the coefficients (a_2, b_2, c_2). It follows that $1/\omega_1$ and $1/\omega_2$ should respectively be proportional to $(t_1 - t_0)/3$ and $4(t_2 - t_1)$. Using the lower bound obtained in Theorem 4, we may postulate that the probability that the index price X hits the equilibrium level 1 (see Figure 6.6) before year 2018 (which corresponds to an average annual loss of around 13%) is at least 50%. We can thus wonder if the famous *kiss landing* generally announced by estate agents may not more probably end in a crash ...

Appendix A. Proof of Lemma 1

Proof: First, let us prove that it is a sufficient condition. By continuity of X, it is sufficient to check the almost sure equality of (1.3) for any fixed $t \geq 0$. Then denote $\widetilde{X}_s = X_t - X_{t-s}$, for $s \geq 0$, so that

$$\mathbf{E}\left[\frac{X_t - X_{t-\Upsilon}}{\Upsilon}\right] = b^2 \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} s \exp(-bs)\, ds = b^2 \int_0^{+\infty} \widetilde{X}_s \exp(-bs)\, ds$$

The fact that X is a semi-martingale enables to integrate by parts and we find

$$\begin{aligned} b^2 \int_0^{+\infty} \widetilde{X}_s \exp(-bs)\, ds &= -b\Big[\widetilde{X}_s \exp(-bs)\Big]_0^{+\infty} + b \int_0^{+\infty} \exp(-bs)\, d\widetilde{X}_s \\ &= b \int_{-\infty}^{t} \exp(-b(t-s))\, dX_s = b \int_0^{t} \exp(-b(t-s))\, dX_s \end{aligned}$$

Now, we focus on the converse implication. Denoting by G the distribution of Υ, (1.3) reads

$$\forall\, t \geq 0, \quad \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} G(ds) = \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} \Gamma_{2,b}(ds) \quad a.s. \qquad \text{(A.1)}$$

Now, let $t > 0$. Since the above equality holds almost surely, it follows from Girsanov Theorem (see *e.g.* Revuz and Yor 1999, Chapter 8), that we can replace

$(X_s)_{s\in[0,t]}$ by c times a Brownian motion (and next by linearity take $c=1$). Then, the main argument is the support Theorem (see *e.g.* Stroock and Varadhan 1972), which yields in particular that for every positive t and ε, for every $\mathcal{C}^1$-function $\varphi:(-\infty,t]\to\mathbb{R}$ such that $\varphi(u)=0$ on $\mathbb{R}_-$,

$$\mathbb{P}(\sup_{s\in[0,t]}|X_s-\varphi(s)|\le\varepsilon)>0. \tag{A.2}$$

Let φ be such a function. By (A.1) and (A.2), we obtain that for every positive ε,

$$\left|\int_0^{+\infty}\frac{\varphi(t)-\varphi(t-s)}{s}(G(ds)-\Gamma_{2,b}(ds))\right| \quad \le 2\varepsilon\int_0^{+\infty}\tfrac{1}{s}(G(ds)+\Gamma_{2,b})(ds)$$

and it follows that for every $\mathcal{C}^1$-function with $\varphi(u)=0$ on $\mathbb{R}_-$,

$$\int_0^{+\infty}\frac{\varphi(t)-\varphi(t-s)}{s}G(ds)=\int_0^{+\infty}\frac{\varphi(t)-\varphi(t-s)}{s}\Gamma_{2,b}(ds), \tag{A.3}$$

the result being available for all positive t. Denoting $r=\varphi(t)$ and $h(s)=\varphi(t)-\varphi(t-s)$ for all $s\in[0,t]$, we get that for all $r\in\mathbb{R}$ and all $\mathcal{C}^1$ function $h:[0,t]\to\mathbb{R}$ with $h(0)=0$,

$$r\int_t^{+\infty}\frac{1}{s}(G-\Gamma_{2,b})(ds)+\int_0^t h(s)\frac{G-\Gamma_{2,b}}{s}\,ds \quad = \quad 0$$

namely

$$\int_t^{+\infty}\frac{1}{s}(G-\Gamma_{2,b})(ds) \quad = \quad 0$$

and G and $\Gamma_{2,b}$ coincide on $(0,t]$. This being true for all $t>0$, we get that $G=\Gamma_{2,b}$ on $(0,+\infty)$. Because they are both probability measures, they cannot differ only on $\{0\}$, so $G=\Gamma_{2,b}$. This proof can be extended to any continuous semi-martingale whose martingale part is non-degenerate. □

Appendix B. Computations in polar coordinates

For the sake of completeness, we give below a series of elementary but tedious computations which are omitted in Section 3. We start with the proof of (3.5) and (3.10)

Proposition 37. In the usual polar coordinates (r,θ), the infinitesimal generator L_ρ of the elliptic diffusion whose evolution is described by (3.4) is given by

$$\begin{aligned}\forall g\in\mathcal{C}^2(\mathbb{R}_+^*\times\mathbb{R}),\quad L_\rho g(r,\theta) \quad &= \quad -\rho r\partial_r g(r,\theta)+\partial_\theta g(r,\theta)+\partial_r^2 g(r,\theta)\\ &\quad +\frac{1}{r}\partial_r g(r,\theta)+\frac{\partial_\theta^2}{r^2}g(r,\theta).\end{aligned}$$

In a similar way, the action infinitesimal generator $\mathcal{L}_\rho$ of the hypo-elliptic diffusion described by (2.8) is given by

$$\mathcal{L}_\rho=-\rho r\partial_r+\partial_\theta+\frac{\sin^2\theta}{2}\partial_{rr}^2-\frac{\sin\theta\cos\theta}{r^2}\partial_\theta+\frac{\sin\theta\cos\theta}{r}\partial_{r\theta}^2+\frac{\cos^2\theta}{2r}\partial_r+\frac{\cos^2\theta}{2r^2}\partial_\theta^2.$$

Proof: Write $g(r,\theta)=f(r\cos\theta,r\sin\theta)$. Using that

$$\partial_x f=\cos\theta\partial_r g-\frac{\sin\theta}{r}\partial_\theta g,\quad \partial_y f=\sin\theta\partial_r g+\frac{\cos\theta}{r}\partial_\theta g,$$

one checks that, $-(\rho x + y)\partial_x f + (x - \rho y)\partial_y f = -\rho r \partial_r g + \partial_\theta g$. Expressions (3.5) and (3.10) follow from

$$\partial_x^2 f = \cos^2\theta \partial_r^2 g + 2\frac{\sin\theta\cos\theta}{r^2}\partial_\theta g - 2\frac{\sin\theta\cos\theta}{r}\partial_{r\theta}^2 g + \frac{\sin^2\theta}{r}\partial_r g + \frac{\sin^2\theta}{r^2}\partial_\theta^2 g,$$

$$\partial_y^2 f = \sin^2\theta \partial_r^2 g - 2\frac{\sin\theta\cos\theta}{r^2}\partial_\theta g + 2\frac{\sin\theta\cos\theta}{r}\partial_{r\theta}^2 g + \frac{\cos^2\theta}{r}\partial_r g + \frac{\cos^2\theta}{r^2}\partial_\theta^2 g.$$

□

Proofs of Proposition 19 and Proposition 22 In Subsections 3.2.2 and 3.2.3, we need to compute $\frac{L_\rho g}{g}$ and $\frac{\mathcal{L}_\rho g}{g}$ where g has the form: $g(r,\theta) = r^n\gamma(\theta)e^{\beta(\theta)r^2}$. Note that in Subsection 3.2.2, $n = 1$ and $\gamma(\theta) = \cos\theta$. Then, $\frac{L_\rho g}{g}$ and $\frac{\mathcal{L}_\rho g}{g}$ are expressed in terms of some functions denoted by ψ_1, ψ_2, φ_i, $i = 1, 2, 3$. The computation of these functions follows from those of the derivatives of g given below:

$$\begin{aligned}
&\frac{\partial_r g}{g}(r,\theta) = \left(\frac{n}{r} + 2\beta(\theta)r\right),\\
&\frac{\partial_r^2 g}{g}(r,\theta) = \frac{n^2-n}{r^2} + 4r^2\beta^2(\theta) + (4n+2)\beta(\theta),\\
&\frac{\partial_\theta g}{g}(r,\theta) = \beta'(\theta)r^2 + \frac{\gamma'(\theta)}{\gamma(\theta)},\\
&\frac{\partial_{r\theta}^2 g}{rg}(r,\theta) = 2\beta(\theta)\beta'(\theta)r^2 + \left((2+n)\beta'(\theta) + 2\frac{\gamma'(\theta)}{\gamma(\theta)}\beta(\theta)\right) + n\frac{\gamma'(\theta)}{r^2\gamma(\theta)}\\
&\frac{1}{r^2}\frac{\partial_\theta^2 g}{g}(r,\theta) = \beta'(\theta)^2 r^2 + \left(\beta''(\theta) + 2\beta'(\theta)\frac{\gamma'(\theta)}{\gamma(\theta)}\right) + \frac{1}{r^2}\frac{\gamma''(\theta)}{\gamma(\theta)}.
\end{aligned}$$

We can now use carefully the expressions of the elliptic (resp. hypo-elliptic) generator L_ρ (resp. $\mathcal{L}_\rho$) given by (3.5) (resp. (3.10)).

■

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