

“Almost” subsidy-free spatial pricing in a multi-dimensional setting*

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Abstract

Consider a population of citizens uniformly spread over the entire plane, that faces a problem of locating public facilities to be used by its members. The cost of every facility is financed by its users, who also face an idiosyncratic private access cost to the facility. We assume that the facilities' cost is independent of location and access costs are linear with respect to the Euclidean distance. We show that an external intervention that covers a tiny portion of the facility cost is sufficient to guarantee *secession-proofness* or *no cross-subsidization*, where no group of individuals is charged more than its stand alone cost incurred if it had acted on its own. Moreover, we demonstrate that in this case the Rawlsian access pricing is the only secession-proof allocation.

Keywords: Secession-proofness, optimal jurisdictions, Rawlsian allocation, hexagonal partition, cross-subsidization.

JEL Classification Numbers: D70, H20, H73.

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1 Introduction

Consider a society that faces a problem of locating public facilities (or public projects as in Mas-Colell, 1980) to serve its members. The facilities, say libraries, are to be located on the plane and could be visited by citizens at some private “transportation” cost related to distance between their residence and the facility they are assigned to.¹

Assuming that setting up and operating a facility entails a fixed set-up and operational cost, the following problems arise:

- how many facilities should be built;
- where to locate the facilities;
- how to assign citizens to the facilities;
- how to allocate the facilities costs (in the form of access fees) to citizens-users.

In this paper we examine the case where

- the demand for use of services is uniformly distributed over the plane, independently of the access fee;
- the cost of setting up a facility is independent of location;
- transportation cost is proportional to Euclidean distance.

We assume that for any number and location of facilities, assignment of users to facilities and access fees, all citizens-users enjoy a “free entry/exit” option: *any group can build a new facility for their own benefit at the standard fixed cost, and locate it at will.* A threat of free entry and exit leads us imperatively to impose the “secession-proofness” or “core” property: at equilibrium, no group of users should be able to benefit by defecting from the proposed arrangement to set up and operate its own facility.² Note that we do not restrict the set of potentially seceding groups and allow for users assigned to different facilities to form a secession prone group. The secession-proofness also can be

¹Note that the framework we consider here belongs to the type of “horizontal differentiation” models where individuals display distinct preferences over geographical locations of public facilities. This is in contrast to the “vertical differentiation” framework, where individuals exhibit identical preferences over quantity or quality attributes of public projects, see, e.g., Westhoff (1977), Guesnerie and Oddou (1981, 1988), Wooders (1978, 1980), Guesnerie (1995), Weber and Zamir (1985), Greenberg and Weber (1986), Jéhiel and Scotchmer (1997, 2001), Konishi, Le Breton and Weber (1998).

²Since a geographical area served by a public facility can be identified as a political jurisdiction, the secession-proofness could then be viewed as a requirement of political stability.

considered as a “no cross-subsidization” condition where no group of users is required to contribute more than its stand-alone cost. In other words, the equilibrium cost allocation should ensure the voluntary participation of any group of citizens.

The secession-proofness immediately yields the total cost minimization requirement: *the society should minimize the total burden of setting up and operating of all facilities and of the aggregate access fees of all citizens.* This requirement, in turn, leads to two simple but important observations: since the facilities costs are location-independent, every facility should be placed at the location that minimizes the aggregate transportation costs of the group of citizens assigned to the facility; and every citizen should be assigned to the facility closest to her residence.

The characterization of efficient, or cost-minimizing, partitions in this geometric setting is a well-documented problem in mathematics. An efficient partition consists of identical regular hexagons,³ whose size is calculated as a function of the ratio of fixed costs per facility to access or transportation costs per unit of distance. However, the area over which total costs per user are minimized is not a hexagon but a *disk!* Since the plane cannot be partitioned into disks, it helps to explain the first result of this paper that demonstrates that the set of secession-proof allocations is empty. This simply means that it is impossible to allocate facilities’ cost over hexagons in a manner that rules out a threat of secession by all disk-shaped jurisdictions.

The non-existence of secession-proof allocations implies that the stability can be ensured only at some cost and we consider the situation where an external source is willing to finance a fraction δ of the total cost incurred by jurisdictions if they follow the prescribed agreement. Suppose that the total costs (set-up plus operation plus access fees) for the jurisdiction-to-be at the proposed equilibrium are subject to the discount factor $1 - \delta$, whereas forming a new jurisdiction to set up and operate an independent facility requires a full non-subsidized cost. Then the allocation will be δ -secession-proof if the savings reaped by the seceding jurisdiction fall short of the subsidy obtained by members of that jurisdiction at the proposed access fee allocation.

We then determine a minimal subsidy δ^* that rectifies stability failure and demonstrate that the

³See discussion below.

set of δ -secession-proof allocations is non-empty if and only if $\delta \geq \delta^*$. It turns out that the cut-off subsidy value δ^* is determined by the ratio of per capita costs to users in an optimal disk and in an optimal hexagon. A tiny value of the threshold (less than 0.2%) lends an additional credence to the δ -secession-proofness as a stability concept.

The second result of the paper is the characterization of the δ^* -secession-proof allocations. We apply the so-called Rawls principle that requires the minimization of the total cost incurred by the least privileged citizen-user and produces the complete equalization of total cost for all citizens-users. A transparent characterization of that principle requires to subject access to a fee that declines linearly (with the unit slope) in the residence-to-facility distance and to adjust access fees so that operators of the facilities break even. We show that Rawls principle defines uniquely the δ^* -secession-proof allocation. (For higher values of δ , the set of δ -secession-proof allocations contains other allocations as well.) The Rawlsian policies are often advocated on the basis of *justice* considerations, whereas our result offers a *stability* argument in support of the Rawls principle.

The last result is a multi-dimensional counterpart of Drèze, Le Breton and Weber (2007). The latter paper shows that in the unidimensional setting the Rawlsian distribution is the unique secession-proof allocation. In fact, when the population is uniformly spread over the entire real line, both efficient and stable partitions consist of optimal-size intervals. Thus, there is no gap between efficiency and optimality and the value of δ^* is equal to zero. In an earlier paper Le Breton, Weber and Drèze (2005) examine the case of the population uniformly distributed over a bounded interval. They show that large intervals shrink the gap between efficiency and stability and every secession-proof allocation turns to be “almost” Rawlsian.

In general, the related literature deals almost exclusively with the uni-dimensional case. Cremer, de Kerchove and Thisse (1985), Alesina and Spolaore (1997) examine the existence of secession-proof allocations in the case where the population is uniformly spread over a *bounded* interval and the unique cost sharing rule available for each jurisdiction is the *equal-share* scheme, when all citizens in the same jurisdiction are subject to an identical access fee. Casella (2001) studies a model where individuals are uniformly distributed over the circle. Le Breton and Weber (2003) and Haimanko,

Le Breton and Weber (2004) address the existence and characterization of secession-proof access fee allocations in the case of general distributions. Bogomolnaia et al. (2005, 2007) examine secession-proofness under various notions of stability.

The paper is organized as follows. The next section contains the model and introduces the needed definitions. The main results, whose proofs are relegated to the Appendix, are stated in Section 3. Section 4 concludes.

2 The Model

We consider a society with a continuum of individuals that has to determine a partition into multiple groups (jurisdictions). Each jurisdiction has to be assigned a public facility accessible to its members and an allocation of access fees to share the facility cost among them. The facilities will be located in a multi-dimensional space:

Assumption A.1 — Multidimensionality: The space of facilities' locations is the two-dimensional Euclidean space $X = \mathfrak{R}^2$.

Citizens have idiosyncratic preferences (or transportation costs) over possible facilities they could be assigned to. We assume that for every individual the transportation cost is represented by the Euclidean distance from her residence to the facility in her jurisdiction:

Assumption A.2 — Euclidean transportation costs: For every individual located at

$l = (l_1, l_2) \in X$, her accession cost to every $t = (t_1, t_2) \in X$ is given by

$$\|t - l\| = \sqrt{|t_1 - l_1|^2 + |t_2 - l_2|^2}. \quad (1)$$

This formalization allows us to identify an individual with her location and to characterize the society by the distribution of individuals' locations. We assume that the citizens are uniformly distributed over the entire space X :

Assumption A.3 — Uniform distribution: The citizens' distribution is given by the two-dimensional Lebesgue measure⁴ λ over \mathfrak{R}^2 .

⁴See Halmos (1950), p. 153.

The area of a measurable⁵ set S will be denoted by $\lambda(S)$, i.e., $\lambda(S) = \int_S dt$. In what follows, the null-measured sets with $\lambda(S) = 0$ will be disregarded, so that the qualification “up to a null-set” should be added to almost all our results.

In our set-up, every jurisdiction is a measurable bounded subset of X with positive measure. The collection of such sets will be denoted by $\mathcal{M}(X)$. We assume that the cost of each facility is independent of its location and consists of a fixed cost, independent of the size of a jurisdiction, and a variable operational cost proportional to the jurisdiction size:

Assumption A.4 — Facility cost: For a facility assigned to a jurisdiction, the cost is given by

$$f(S) = g + \alpha\lambda(S), \quad (2)$$

where $g > 0, \alpha \geq 0$ are two constants.

We now formally introduce the notion of a partition of a measurable subset $S \subset X$:

Definition 2.1: A partition P of a (possibly infinite-measured) set S is a jurisdiction structure that consists of sets from $\mathcal{M}(X)$ which are “almost” pairwise disjoint: $\lambda(T \cap T') = 0$ for all $T \neq T'$ in P , and whose union covers the entire set S : $\bigcup_{T \in P} T = S$. The set of partitions of S is denoted by $\mathcal{P}(S)$. Obviously, if the measure of S is infinite, then every $P \in \mathcal{P}(S)$ consists of an infinite number of jurisdictions.

Now let us turn to the determination of facility choices. For each $S \in \mathcal{M}(X)$ and a location $l \in X$ we denote by $D(S, l)$ the value of total transportation cost in S (with respect to location l):

$$D(S, l) = \int_S \|t - l\| dt. \quad (3)$$

In what follows, the efficiency requires that the facility location in each jurisdiction S would minimize the total transportation cost of its members. That is, if a jurisdiction S is assigned to the facility located at point m , then m is a solution of the following problem:⁶

$$D(S, m) = \min_{l \in X} D(S, l). \quad (4)$$

⁵A subset of X is measurable if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

⁶In operations research, the value of this program is called $MAT(S)$ (Minimal Aggregate Transportation cost of the set S).

Note that since the integral in (3) is continuous in l , and for $l \rightarrow \infty$ the value of the program in (3) tends to infinity, the problem in (4) has a solution, called a *central location* of S . We use the following lemma:

Lemma 2.2: For every jurisdiction $S \in \mathcal{M}(X)$, there is a unique central location, denoted by $m(S)$.⁷

Lemma 2.2 resolves the issue of an efficient facility location choice for every jurisdiction and we denote by $D(S)$ the aggregate transportation cost of members of S :

$$D(S) = D(S, m(S)). \quad (5)$$

Every measurable set $S \subset X$ can be partitioned into several jurisdictions. We define the stand alone average total cost⁸ in S as the minimum over all partitions P of S :

$$K(S) = \inf_P \frac{\sum_{T \in P} [D(T) + f(T)]}{\lambda(S)} = \alpha + \inf_P \frac{\sum_{T \in P} [D(T) + g]}{\lambda(S)}. \quad (6)$$

We have

Definition 2.3: A partition P is S -efficient if it is a solution to (6). An X -efficient partition will be simply called an efficient partition.

From now on, we will focus our analysis on efficient partitions, which is a well documented problem in mathematics. The qualitative result (re)discovered by many authors states that there is a unique “shape” of efficient partitions which consists of identical regular hexagons:⁹

Result 2.4: Partition P is efficient if and only if it is comprised of identical regular hexagons, whose stand-alone cost is minimal among all regular hexagons.

The size of hexagons in efficient partitions, derived in the Appendix, obviously depends upon the value of the fixed component of facility cost: the smaller the cost, the smaller are jurisdictions in an efficient partition.

⁷This result is essentially multi-dimensional. In the unidimensional setting, the set of central locations of a bounded set T coincides with the set of its medians, which may contain a continuum of points.

⁸Since the total cost for an infinite-measured set is infinite, in this case we will take a limit of the sequence of sets that uniformly approach S .

⁹See Fejes Toth (1953), Haimovich and Magnanti (1988), Morgan and Bolton (2002), as well as Christaller (1933), Lösch (1954), Bollobas and Stern (1972), and Stern (1972) in the economic geography context.

Let us turn to the examination of accession fees. In every potential jurisdiction $S \in \mathcal{M}(X)$, a sharing rule y is a measurable function on S that specifies accession fees of members of S , if this jurisdiction forms. We impose the budget balancedness condition:

Assumption A.5 — Budget balancedness: The accession fees of members of S cover the costs of the facility:

$$\int_S y(t) dt = f(S). \quad (7)$$

It would be useful to consider the notion of *consistent sharing rule*. Since the whole plane is partitioned into identical (hexagonal) jurisdictions, it makes sense to demand that the individuals in identical locations within different jurisdictions bear the same costs. We impose a weak form of consistency that requires that any two individuals in any two different jurisdictions, whose location is identical with respect to their corresponding central points, are assigned the same accession fees.¹⁰

Assumption A.6 — Consistent sharing rule: For every efficient partition P^* , every two different (hexagonal) jurisdictions $H, H' \in P^*$ and every two individuals $t \in H, t' \in H'$ satisfying $t - m(S) = t' - m(S')$, we have $y(t) = y(t')$.

A sharing rule y associated with partition P^* determines the cost allocation for every $t \in X$

$$c(t) = y(t) + \|t - m(H^t)\|, \quad (8)$$

where $m(H^t)$ is the central location of the hexagon H^t in P^* that contains t .

We will now fix one of the (fully equivalent to each other) efficient partitions, say P^* . The cost sharing rule chosen by the society satisfies a requirement of *voluntary participation* that no group of individuals contributes more than the cost incurred if it had acted on its own. Thus, the formation of jurisdictions and the allocation of accession fees within each of them rules out secession threats by groups of individuals. It is important to point out that we consider secession threats by all measurable sets of individuals, including re-combination of members of different jurisdictions. Formally,

¹⁰While this assumption is not essential for the main result, it substantially simplifies the calculus of the proof.

Definition 2.5: Let a cost allocation c be given. A set $S \in \mathcal{M}(X)$ is *prone to secession* (given c) if

$$c(S) = \frac{1}{\lambda(S)} \int_S c(t) dt > K(S). \quad (9)$$

A cost allocation c is *secession-proof* if no set $S \in \mathcal{M}(X)$ is prone to secession (given c).

The set of secession-proof cost allocations on X will be denoted by \mathcal{A} .

The next definition introduces the allocations that satisfy the *Rawls principle* by minimizing the total cost of the most disadvantaged individual in each jurisdiction. It implies the *cost equalization* across the entire society:

Definition 2.6: A cost allocation r is called *Rawlsian* if the value $r(t)$ is constant within each $H \in P^*$, and, hence, on X . That is, for every $t, t' \in X$ we have $r(t) = r(t')$.

3 Results

We now state the main results of the paper. First, we demonstrate that under our assumptions, a secession-proof allocation fails to exist.

Proposition 3.1: Suppose that assumptions A.1-A.6 hold. Then the set of secession-proof allocations \mathcal{A} is empty.

In absence of secession-proof allocations, we search for a solution “closest” to secession-proofness. For instance, we may assume that there is a fixed per capita secession cost for any subgroup $S \subset X$; alternatively, one can consider government intervention to subsidize a certain fraction of the total cost of every citizen to prevent the formation of groups prone to secession. Both approaches are essentially equivalent and yield the following definition of *δ -secession-proofness*:

Definition 3.2: Let $\delta > 0$ be given. A cost allocation c is *δ -secession-proof* if for all $S \in \mathcal{M}(X)$ the following inequality holds:

$$(1 - \delta)c(S) \leq K(S). \quad (10)$$

The set of δ -secession-proof allocations for X will be denoted by $\mathcal{A}(\delta)$.

In other words, if individuals follow the prescribed agreement, then the δ -part of their total cost is covered “from outside”. If, however, a jurisdiction wants to secede, then its members will have to bear all costs on their own.

This definition relaxes the constraints which determine secession-proof allocations and, obviously, if δ is large enough, the set $\mathcal{A}(\delta)$ is nonempty. Moreover, it is easy to see that if $\mathcal{A}(\delta)$ is nonempty for some δ , it is also the case for all $\delta' > \delta$. Hence, there is a threshold value δ^* defined by

$$\delta^* = \inf\{\delta > 0 \mid \mathcal{A}(\delta) \neq \emptyset\}. \quad (11)$$

The value δ^* therefore can represent the *cost of stability*, which is the minimal per-capita subsidy required to sustain secession-proofness. We show below that the value of δ^* is very small, in fact, less than 0.2% of the total cost. We have:

Proposition 3.3: Under Assumptions A.1-A.6, the set $\mathcal{A}(\delta^*)$ is nonempty, and, moreover, the Rawlsian allocation is the only δ^* -secession-proof allocation.

Before proceeding with the discussion of this result, it would be useful to point out that in order to determine an optimal jurisdictional shape, one has to solve the following minimization problem:

$$\min_{S \in \mathcal{M}(X)} K(S). \quad (12)$$

It is not surprising that, due to transportation costs, a solution for (12) is a disk. To provide an informal argument, consider a set H and the disk S on Figure 1. By replacing a small disk B in H by the same size disk B' in S , one strictly decreases the average (per capita) transportation cost of the jurisdiction H as the total transportation cost of the set $H \setminus B \cup B'$ is lower than that of the set S .

Insert Figure 1

Thus, no hexagon could be optimal in terms of per capita cost of its members. Denote by $K(B)$ the value of the problem in (12) and consider a hexagon H in an efficient partition. Obviously, $K(B) < K(H)$, and we show that the cost of stability, δ^* , is given by

$$\delta^* = 1 - \frac{K(B)}{K(H)} > 0. \quad (13)$$

Thus, the cost differential between an efficient hexagon and an optimal disk necessitates government intervention and subsidization of efficient partitions. In fact, the value of the cost differential, and, therefore, of δ^* , is very small:

Corollary 3.4: $\delta^* \approx 0.0019$.

Insert Figure 2

Indeed, the difference in average cost between the disk and the hexagon in Figure 2 is generated by a small variation of transportation costs incurred by a tiny mass of individuals.

It is important to point out the absence of the cost differential between the optimal jurisdictional shape and elements of an efficient partition in the uni-dimensional setting. In this case efficient and optimal jurisdictions are intervals of the same size and Drèze, Le Breton and Weber (2007) have shown that no subsidy is needed to obtain the stability of efficient partitions, yielding $\delta^* = 0$ in this case.

We would also like to provide an intuition for the result in Proposition 3.3, and, by doing so, to highlight the nature of its proof presented in the Appendix. To explain why the Rawlsian allocation is a unique δ^* -secession-proof allocation, consider an arbitrary δ^* -secession-proof allocation c . By applying Fubini's theorem, we show that under c , the average cost incurred by every optimal-sized disk is the same and is given by $K(H)$. Otherwise, there would exist an optimal disk-shaped set B whose average cost $c(B)$ exceeds $K(H)$ and $(1 - \delta^*)c(B) > K(B)$, a violation of (10) and a contradiction to secession-proofness. Now, suppose that there is a group of individuals T , each contributing below the value $K(H)$. Consider an optimal-sized disk S that contains T (or its subset) at its boundary (see Figure 3).

Insert Figure 3

Then the members of S , who do not belong to T , should on average contribute more than $K(H)$. On the other hand, we show that the average cost of individuals in a set "close" to an optimal-sized disk S , are almost the same as in the disk itself, where the average cost is given by $K(B)$. More precisely, we demonstrate that the average cost differential between optimal disk S and a slightly

smaller disk S' with the same center is of the second order of the difference between the radiuses of S and S' . This allows us to show that the set $S \setminus T$ will be prone to secession. Thus, c must satisfy the Rawls principle.

4 Conclusions

In this paper we study the stability of jurisdiction formation in the model with a population of citizens spread over the plane that has to determine the location of multiple public facilities. The cost of every facility is independent of location and is financed by its users, who face an idiosyncratic private access cost to the facility. It is shown that, under linear transportation costs and uniform population distribution, unlike in the one-dimensional case (Drèze, Le Breton and Weber, 2007), stability is unattainable without outside subsidies. The main reason is that the optimal shape of a jurisdiction is a disk with the facility at its center. However, it is impossible to accommodate the entire population in disks. Instead, the efficient partition is into regular identical hexagons but then there would be some users who would incur a high cost and form an efficient disk. The incentive to secede can be overcome if agents are subsidized to forego this opportunity and we show that such a subsidy is very small, indeed. Moreover, we demonstrate that at the minimal subsidy level that guarantees stability, the only secession-proof allocation is Rawlsian, where agents with high transportation costs are fully compensated through a reduced contribution towards the facility cost.

The results of the paper could be generalized in several directions. First, instead of linear transportation costs we can use a general strictly increasing continuous function of the distance to facility. We can also use a general distance metric rather than the Euclidean one, which would not affect the results of Proposition 3.3. However, the proofs become much more involved and tedious, without adding much to the essence of our results. Moreover, it would be very difficult to determine the precise value of the minimal subsidy δ^* in the general case.

Another possible line of a future research is the examination of the case where the population is distributed over a large bounded convex set. The results by Haimovitch and Magnanti (1988) and Morgan and Bolton (2002) indicate that a hexagonal partition is, indeed, approximately efficient if the

support of the distribution becomes sufficiently large. The question concerning the link between the Rawlsian allocation and stability remains however open. The same applies to a possible consideration of the cases with a general population distribution and a location space which spans over more than two dimensions. Some preliminary results have been obtained in these cases and will be further examined in our future research.

5 Appendix

Proof of Lemma 2.2: Let $S \in \mathcal{M}(X)$ be given and assume that S has two different central points, m and m' . Let L be the straight line connecting m and m' . Denote $S' = S \setminus L$ and $\bar{m} = \frac{m + m'}{2}$. Obviously m and m' are central points of S' as well and $D(S) = D(S')$. Then for every $t \in S'$ we have

$$\frac{1}{2} (\|t - m\| + \|t - m'\|) > \|t - \bar{m}\| \quad (14)$$

and, since $\lambda(S) = \lambda(S') > 0$, this implies that

$$\int_{S'} \|t - \bar{m}\| dt < \frac{1}{2} \left(\int_{S'} \|t - m\| dt + \int_{S'} \|t - m'\| dt \right). \quad (15)$$

However, by (3) and (4), the right-hand side of (15) is equal to $D(S) = D(S')$, a contradiction to m and m' being central points of S' . \square

Before proceeding with the proof of Propositions 3.1 and 3.3, we need a notation to state some preliminary results. Without loss of generality, we shall assume, that the variable component of facility costs α in Assumption A.4 is equal to zero.

Lemma A.1: The solution of (12) is a disk of radius l^* , where the value of l^* is given by

$$l^* = \left(\frac{3g}{\pi} \right)^{\frac{1}{3}}. \quad (16)$$

Moreover, the value of the solution of (12), $K(B)$, the per capita cost in an optimal disk is equal to l^* .

Proof: Take any set S that solves (12) and consider the (uniquely defined) disk B with the radius l and the center at $m(S)$ so that B and S have the same measure. Then, since S solves (12), we must have

$$0 \geq D(S) - D(B) = \int_{S \setminus B} \|p - m(S)\| dp - \int_{B \setminus S} \|p - m(S)\| dp. \quad (17)$$

However, for any $p \in S \setminus B$ we have $\|p - m(S)\| \geq l$ and for any $p \in B \setminus S$ we have $\|p - m(S)\| \leq l$. Note that both sets $S \setminus B$ and $B \setminus S$ have the same measure denoted by μ . We have

$$\int_{S \setminus B} \|p - m\| dp \geq l\mu \geq \int_{B \setminus S} \|p - m\| dp. \quad (18)$$

Combined with (17) this implies that $\lambda(\{p \in S \setminus B : \|p - m\| > l\}) = 0$, and similarly $\lambda(\{p \in B \setminus S : \|p - m\| < l\}) = 0$. Since obviously $\lambda(\{p : \|p - m\| = l\}) = 0$, we conclude that $\mu = 0$ and, up to a set of measure zero, S coincides with the disk B .

It is left to show that the optimal radius l is equal to l^* and to derive $K(B)$. Notice that for every disk B^l with the radius l , the total transportation cost $D(B^l) = \frac{2\pi l^3}{3}$. Since the area of B^l is πl^2 , the average cost within B^l is $K(B^l) = \frac{g}{\pi l^2} + \frac{2l}{3}$. It is straightforward to verify that the last expression attains its minimum at l^* determined by (12), yielding the minimal average cost $K(B) = l^*$. \square

We utilize the lemma that evaluates the average cost of jurisdictions which are “close” to optimal disks:

Lemma A.2: Let $\gamma > 0$ is sufficiently small and the set S be located between two disks with the same center, $B_a^{l^*-\gamma}$ and $B_a^{l^*}$, i.e. $B_a^{l^*-\gamma} \subset S \subset B_a^{l^*}$. Then $K(S)$, the aggregate average cost over S , differs from the aggregate average cost over optimal disk $K(B)$ only in the second order term:

$$K(S) < l^* + \frac{4}{l^*} \gamma^2. \quad (19)$$

Proof: Let $\tilde{S} = S \cap (B_a^{l^*} \setminus B_a^{l^*-\gamma})$. In our derivations below we take into account that the total transportation cost within S (weakly) increases if we replace the $m(S)$ by a , and that the distance between any point in \tilde{S} to a is bounded from above by l^* . Denote $z = \frac{3}{\pi} \lambda(\tilde{S})$. By utilizing (??) we

have:

$$\begin{aligned}
K(S) &= \frac{g + D(S)}{\lambda(S)} \leq \frac{g + \int \|a - t\| dt}{\lambda(S)} \leq \frac{g + D(B_a^{l^* - \gamma}) + l^* \lambda(\tilde{S})}{\lambda(B_a^{l^* - \gamma}) + \lambda(\tilde{S})} = \\
&= \frac{(l^*)^3 + 2(l^* - \gamma)^3 + zl^*}{3(l^* - \gamma)^2 + z} < \frac{3(l^*)^3 - 6(l^*)^2\gamma + 6l^*\gamma^2 + zl^*}{3(l^* - \gamma)^2 + z} = \\
&= \frac{3l^*(l^* - \gamma)^2 + zl^*}{3(l^* - \gamma)^2 + z} + \frac{3l^*\gamma^2}{3(l^* - \gamma)^2 + z} < l^* + \frac{3l^*\gamma^2}{3(l^*/2)^2} = l^* + \frac{4}{l^*}\gamma^2,
\end{aligned} \tag{20}$$

as for γ small enough $l^* - \gamma > \frac{1}{2}l^*$. \square

Lemma A.3: Let H be an efficient hexagon. The per capita cost over H is given by

$$K(H) = \frac{\sqrt{3}}{2} \left(\frac{2}{3} + \ln \sqrt{3} \right)^{\frac{2}{3}} g^{\frac{1}{3}}. \tag{21}$$

Proof: Consider a regular hexagon H_l , where l denotes the distance between the center $m(H_l)$ and a midpoint of its side. The total transportation cost in H_l is

$$\begin{aligned}
D(H_l) &= 12 \int_0^l \int_0^{\frac{x}{\sqrt{3}}} \sqrt{x^2 + y^2} dx dy = 6 \int_0^l \left[y \sqrt{x^2 + y^2} + x^2 \ln \left(y + \sqrt{x^2 + y^2} \right) \right]_0^{\frac{x}{\sqrt{3}}} dx \\
&= 6 \int_0^l \left[\frac{x}{\sqrt{3}} \sqrt{x^2 + \frac{x^2}{3}} + x^2 \ln \left(\frac{x}{\sqrt{3}} + \sqrt{x^2 + \frac{x^2}{3}} \right) - x^2 \ln x \right] dx \\
&= 6 \int_0^l x^2 \left[\frac{2}{3} + \ln \sqrt{3} \right] dx = 2l^3 \left[\frac{2}{3} + \ln \sqrt{3} \right].
\end{aligned} \tag{22}$$

Since the area of H_l is $2\sqrt{3}l^2$, the average cost per citizen in jurisdiction H_l is given by

$$K(H_l) = \frac{g}{2\sqrt{3}l^2} + \frac{l}{\sqrt{3}} \left[\frac{2}{3} + \ln \sqrt{3} \right], \tag{23}$$

which attains its minimum at the efficient hexagon H , i.e., when

$$l = \tilde{l} = \left(\frac{2}{3} + \ln \sqrt{3} \right)^{-\frac{1}{3}} g^{\frac{1}{3}}. \tag{24}$$

It is easy to verify that then the per capita average cost $K(H) = K(H_{\tilde{l}})$ is given by (21) which at the same time represents the average cost of the whole plane X under an efficient partition. \square

Take efficient partition P^* of X . For every positive integer N , consider a subset G_N of P^* that consists of N^2 adjacent hexagons.

Insert Figure 4

Let the sequence $\{G_N\}_{N=1,\dots,\infty}$ be nested, i.e., each G_N is imbedded into G_{N+2} “symmetrically”, such that the set $G_{N+2} \setminus G_N$ is a “hexagonal ring” comprised of $4N + 4$ regular hexagons.

Insert Figure 5

We have the following result:

Lemma A.4: For every $a \in G_N$, the disk B_a is contained in G_{N+2} .

Proof: Denote by \bar{l} the side of a hexagon in partition P^* . Since the minimal width of the hexagonal ring F_N is equal to \bar{l} , it suffice to demonstrate that $\bar{l} > l^*$. Note that $\bar{l} = \frac{2}{\sqrt{3}}\tilde{l}$, where \tilde{l} is the distance between the center of the efficient hexagon and the middle point of one of its sides, which has been derived in (24). Thus,

$$\bar{l} = \frac{2}{\sqrt{3}} \left(\frac{2}{3} + \ln \sqrt{3} \right)^{-\frac{1}{3}} g^{\frac{1}{3}}, \quad (25)$$

which, by (16), exceeds the value l^* . \square

Let the efficient partition P^* be endowed with the sharing rule y , that generates cost allocation c , and H is a (hexagonal) jurisdiction in P^* . Denote by λ^H the Lebesgue measure of H and by λ^B the Lebesgue measure of an optimal disk.

For every $a \in X$ denote by the value $\varphi(a)$ the aggregated cost incurred by the members of the disk B_a :

$$\varphi(a) = c(B_a) = \int_{B_a} c(t)dt \quad (26)$$

Define $\bar{\varphi}$ as the aggregated cost incurred by the allocation c on all disks of optimal size whose centers belong to the hexagon H :

$$\bar{\varphi} := \int_H \varphi(a)da. \quad (27)$$

Note that, due to the consistency assumption A.6, the value $\bar{\varphi}$ is invariant to a choice of a hexagon in P^* . We need the following result:

Lemma A.5:

$$\bar{\varphi} = I, \quad \text{where } I := \lambda^B \int_H c(t) dt. \quad (28)$$

Proof: Define the function $\Psi(a, t)$ on $G_N \times G_{N+2} \subset \mathfrak{R}^4$ by

$$\Psi(a, t) = \begin{cases} c(t), & \text{if } t \in B_a; \\ 0, & \text{otherwise.} \end{cases} \quad (29)$$

We will integrate the function $\Psi(a, t)$ over the set $G_N \times G_{N+2}$. According to Fubini's theorem (Halmos (1950), p.148), two different orders of integration yield the same result. First, we integrate with respect to t and then to a . By (26) and (27), and using Lemma A.4 we have

$$\int_{G_N} \left[\int_{G_{N+2}} \Psi(a, t) dt \right] da = \int_{G_N} \left[\int_{B_a} c(t) dt \right] da = \int_{G_N} \varphi(a) da = N^2 \int_H \varphi(a) da = N^2 \bar{\varphi}. \quad (30)$$

Before integrating in the reverse order, note that the following duality property

$$\{a | t \in B_a\} \equiv B_t \quad (31)$$

holds for every $t \in X$. This is a simple consequence of the symmetry of the distance $\|t - p\|$ as a function of two arguments, and the circle B_t being the set of points p for which $\|p - t\| = \|t - p\| \leq l^*$.

Take a point $t \in G_{N-2}$. By Lemma A.4, $B_t \subset G_N$, and

$$\int_{G_N} \Psi(a, t) da = \int_{B_t} c(t) da = c(t) \int_{B_t} da = \lambda^B c(t). \quad (32)$$

We have:

$$\int_{G_{N+2}} \left[\int_{G_N} \Phi(a, t) da \right] dt = \int_{G_{N-2}} \left[\int_{G_N} \Phi(a, t) da \right] dt + L_N, \quad (33)$$

where

$$L_N := \int_{G_{N+2} \setminus G_{N-2}} \left[\int_{G_N} \Phi(a, t) da \right]. \quad (34)$$

By using (32), the first term in (33) can be presented as:

$$\int_{G_{N-2}} \left[\int_{G_N} \Phi(a, t) da \right] dt = \int_{G_{N-2}} \lambda^B c(t) dt = (N-2)^2 I. \quad (35)$$

Fubini's theorem allows us to rewrite (33) as

$$N^2 \bar{\varphi} = (N-2)^2 I + L_N = N^2 I + L_N - 4(N-1)I. \quad (36)$$

Let us estimate the absolute value of the last two terms. Since $\int_{G_N} \Phi(a, t) da = \int_{G_N \cap B_t} c(t) da \leq \int_{B_t} c(t) da = \lambda^B c(t)$ for any $t \in G_{N+2}$, and hence, for any $t \in G_{N+2} \setminus G_{N-2}$, it follows that

$$|L_N - 4(N-1)I| \leq |L_N| + 4(N-1)I \leq 4(N-1)I + \int_{G_{N+2} \setminus G_{N-2}} \lambda^B c(t) dt = (12N-4)I < 12NI. \quad (37)$$

Thus,

$$|N^2 \bar{\varphi} - N^2 I| \leq 12NI, \quad \text{or} \quad |\bar{\varphi} - I| \leq \frac{12I}{N}. \quad (38)$$

Since N can be made arbitrarily large, we immediately obtain the desired equality $\bar{\varphi} = I$. \square

Since Proposition 3.1 is a corollary of Proposition 3.3, we proceed with the

Proof of Proposition 3.3: Let us show first that the value of δ^* in (11) is given by (13), the right side of which, by Lemmas A.1 and A.3, is equal to

$$1 - \delta^* = \frac{2}{\pi^{\frac{1}{3}} 3^{\frac{1}{6}} \left(\frac{2}{3} + \ln \sqrt{3}\right)^{\frac{2}{3}}} \approx 0.0019. \quad (39)$$

We will demonstrate that the set of δ -secession-proof allocations is empty if and only if $\delta < \delta^*$ defined in (13).

Consider an arbitrary positive number δ and consider a δ -secession-proof allocation c . The budget balancedness assumption A.5 implies that the value of I , determined by (28), is equal to $\lambda^B \lambda^H K(H)$, and by Lemma A.4. so is the value of $\bar{\varphi}$. Hence, there exists $a \in H$ such that $\varphi(a) \geq \lambda^B K(H)$. On the other hand, the stand alone aggregate cost in B_a is $\lambda^B K(B)$. Since c is δ -secession-proof, Definition 3.2 implies that $(1 - \delta)\lambda^B K(H) \leq \lambda^B K(B)$, or $\delta \geq 1 - \frac{K(B)}{K(H)}$.

Let us show that the Rawlsian allocation r is δ -secession-proof whenever $\delta \geq \delta^*$. Indeed, $\delta \geq \delta^*$. Since the total individual cost under r , $r(t)$, is equal to $K(H)$ for every $t \in X$, then for an optimal disk B we have $(1 - \delta)K(H) \leq K(B)$. Moreover, for any $S \in \mathcal{M}(X)$ we have $K(S) \geq K(B)$ and therefore $(1 - \delta)K(H) \leq K(B) \leq K(S)$, i.e, r is δ -secession-proof.

To complete the proof of the proposition, it remains to demonstrate that the Rawlsian allocation (which assigns every individual in X the access fee of $K(H)$) is the only δ^* -secession-proof. For this end, consider an arbitrary δ^* -secession-proof allocation $c(\cdot)$ and estimate the number of individuals whose access fee is “substantially” below the level $K(H)$.

Take a positive number $\varepsilon > 0$. Consider first an arbitrary ring $B_a \setminus B_a^{l^*-\gamma}$ and evaluate the measure of individuals t whose cost $c(t)$ satisfies $c(t) < K(H) - \varepsilon$. Denote this set by U , and consider the set $S = B_a \setminus U$, for which, by Lemma A.2, we have $K(S) < l^* + \frac{4}{l^*}\gamma^2$. On the other hand,

$$c(S) = c(B^{l^*}) - c(U) \geq \lambda^B K(H) - \lambda(U)K(H) + \lambda(U)\varepsilon = \lambda(S)K(H) + \lambda(U)\varepsilon. \quad (40)$$

The δ^* -secession-proofness of $c(\cdot)$ implies that the average per capita cost in group S , adjusted by $1 - \delta^*$, does not exceed its stand-alone value, $K(S)$:

$$(1 - \delta^*)\frac{c(S)}{\lambda(S)} = (1 - \delta^*)(K(H) + \frac{\lambda(U)}{\lambda(S)}\varepsilon) \leq K(S) < l^* + \frac{4}{l^*}\gamma^2. \quad (41)$$

Since $K(B) = l^* = (1 - \delta^*)K(H)$, we have:

$$\lambda(U) \leq \frac{4\lambda(S)}{l^*(1 - \delta^*)\varepsilon}\gamma^2 < \frac{4\pi(l^*)^2}{l^*(1 - \delta^*)\varepsilon}\gamma^2 = W\gamma^2, \quad (42)$$

where W is a constant independent of γ .

Now consider the rectangular Q with sides of $2l^*$ and l^* centered at the origin. For any small positive number γ , denote by $R[i, \gamma]$ the ring $B_{p_i} \setminus B_{p_i}^{l^*-\gamma}$ centered at the point $p_i = (i\gamma, 0)$, where i is any (positive or negative) integer. For large enough positive integer N we have the following inclusion:

$$Q \subset \bigcup_{i=-N}^N R\left[i, \frac{l^*}{N}\right]. \quad (43)$$

Indeed, it is easy to see that $\forall x \in Q$ there exist at least one i such that $x \in B_{p_i}$, and at least one j such that $x \notin B_{p_j}$. Hence, there exist such i and $j = i \pm 1$ that the two statements

$$x \in B_{p_i}; \quad x \notin B_{p_j} \quad (44)$$

hold simultaneously. As obviously $B_{p_i}^{l^*-\gamma} \subset B_{p_j}$ for $j = i \pm 1$, we have that $x \in B_{p_i} \setminus B_{p_i}^{l^*-\gamma} = R[i, \gamma]$, with $\gamma = \frac{l^*}{N}$.

For $i = -N, \dots, -1, 0, 1, \dots, N$, denote by U and U_i the sets of individuals in Q and $R\left[i, \frac{l^*}{N}\right]$, respectively, who incur the cost less than $K(H) - \varepsilon$ under the allocation c . By utilizing (42), we have $\lambda(U_i) \leq W\frac{(l^*)^2}{N^2}$. Thus, since $U \subset \bigcup_{i=-N}^N U_i$, we have

$$\lambda(U) \leq (2N + 1)W\frac{(l^*)^2}{N^2} < \frac{3}{N}W(l^*)^2. \quad (45)$$

Since N can be chosen arbitrarily large, (45) implies that $\lambda(U) = 0$. Note that this argument actually implies that for any rectangular with the sides of $2l^*$ and l^* , the Lebesgue measure of the set of individuals who incur the cost less than $l^* - \varepsilon$ under the allocation c has the zero measure.

Finally, consider an arbitrary hexagon H in the efficient partition P^* . It is contained in the union of several rectangles with the sides of $2l^*$ and l^* . Hence, the measure of the set of individuals in H whose cost is less than $K(H) - \varepsilon$ is zero. But the set of individuals in H who contribute less than $K(H)$ is the union of the sets in H whose members contribute less than $K(H) - 1/n$ for $n = 1, 2, \dots$, and as the countable union of null-sets, this set has zero measure. Hence, the budget balancedness implies that set of those incurring the cost higher than $K(H)$ has zero measure as well. Finally, Assumption A.6 guarantees that every $t \in X$ contributes $K(H)$, implying that the only δ^* -secession-proof allocation is Rawlsian. \square

Proof of Corollary 3.4: Follows immediately from Lemmas A.1 and A.3. Indeed, by (16) and (21), we have

$$\delta^* = 1 - \frac{K(B)}{K(H)} = 1 - \frac{\left(\frac{3}{\pi}\right)^{\frac{1}{3}}}{\frac{\sqrt{3}}{2} \left(\frac{2}{3} + \ln \sqrt{3}\right)^{\frac{2}{3}}} \approx 0.0019. \quad (46)$$

\square

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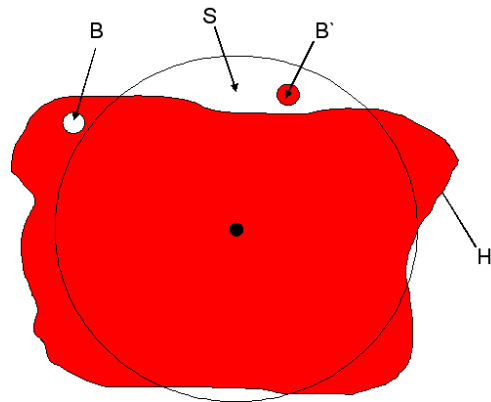


Figure 1.

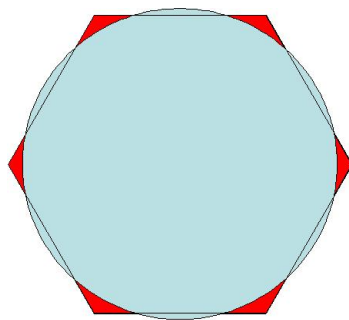


Figure 2.

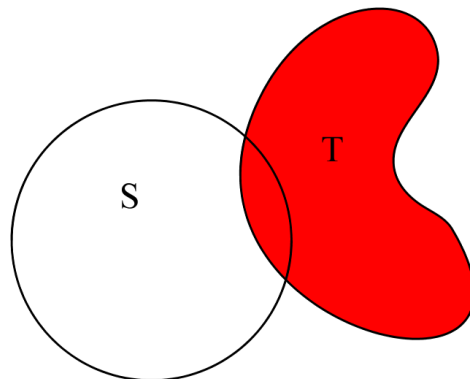


Figure 3.

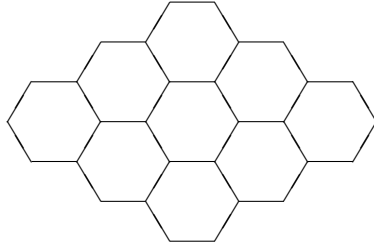


Figure 4: Hexagonal rectangular G_N for $N = 3$.

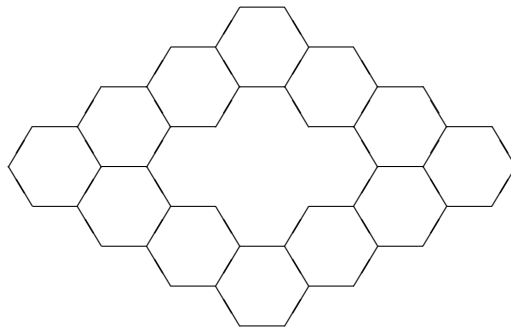


Figure 5: Hexagonal ring $G_{N+2} \setminus G_N$ for $N = 2$.