Triggering the Technological Revolution in Carbon Capture and Sequestration Costs. [1]

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Abstract

The nature of optimal environmental policies able to induce sufficient technical progress in pollution abatement technologies has raised a vivid debate between economics over the last decade. Some emphasize the importance of learning-by-doing on these technologies, an argument in favor of early action. Other insisted upon the time needed for R&D to identify the best abatement options, an incentive to delay action in the future. Either triggering technical progress from learning effects of research, all analysis conclude to ambiguous effects of environmental policies on the speed of technical change. One strong limitation of previous approaches is that they do not endogenize the best ways to improve the efficiency of abatement technologies, either through learning on existing techniques or through research to discover new ones. We consider an economy that can trigger some cost breakdown in CCS costs thanks to both learning and R&D. We first reconsider the results of the literature about the extreme cases of a pure learning induced technical revolution and a pure R&D induced cost breakdown in the context of an atmospheric carbon ceiling framework. We show how this setting helps to clarify the existing results from the literature and remove some of their ambiguities. In particular we perform a sensitivity analysis of the optimal policies with respect to relevant parameters, providing strong intuitions about the various effects affecting their dynamics. We next examine the case of a combined learning and R&D policy. We show that the economy may initially perform only research efforts or rely only upon learning to trigger the cost breakdown. A combined policy may only follow pure R&D or learning policies. Combining learning and R&D requires to increase both research efforts and the use of the abatement technology, but the growth rate of pollution abatement must be higher than the growth rate of the research efforts. Contrarily to what is commonly observed in models with constant average and marginal costs of abatement, the use of cleaning technologies may begin before the atmospheric constraint begins to bind. In such situation, the time constraints upon technological development outweighs the environmental constraints and result in early introduction of abatement technologies. But the contrary may also be optimal and we provide a complete discussion of the relevance of these various scenarios.

Keywords: Carbon capture and storage; Energy substitution; Learning-by-doing; Research and development; Carbon stabilization cap.

JEL classifications: Q32, Q42, Q54, Q55, Q58.

Contents

1 Introduction

Technology plays a prominent role in all reflections about finding ways out of the global warming problem. This issue has been forcefully raised by Scott Barret in several occasions (Barret, 2006). For him, instead of seeking for an almost impossible international agreement on carbon emissions mitigation, governments should better cooperate over a common target of triggering a technological revolution in clean energy generation. The help of technical progress is particularly expected in three domains: the enhancement of the productive efficiency of fossil fuels, the development of non carbon based energy production techniques and the improvement of the efficiency of carbon pollution abatement technologies, the future of carbon capture and storage (CCS) technologies appearing as an important issue in this respect. If the economic literature fully agrees to this general statement, it largely diverges in assessing both the policy implications of technical progress opportunities in carbon emissions mitigation and the effects of environmental policies upon technical change in the energy production and consumption sectors. These issues have provoked a vivid debate among economist during the last decade.

Two main reasons may explain this difficulty to reach an agreement about the nature of the relationships between environmental policy and technological development. First, the topic of technical change, or more precisely of endogenous technical change, has emerged only recently in the economic literature. A lot remains to be done on this issue, especially to build a consistent view of the various advances coming both from the microeconomic approaches developed in industrial organization economics and the macroeconomic approaches of the endogenous growth literature. Second, global warming and technical progress are two dynamical processes with their own drivers and constraints, and reaching a reasonable understanding of the time links between these two processes is a modeling challenge, both on the theoretical and on the empirical side.

A first motive of dissent relies in the desirable speed of action to introduce pollution abatement technologies. One main set of arguments in favor of delaying abatement roots in discounting arguments, the abatement options being today typically costly and thus be favorably delayed in the future (Wigley *et al.*, 1996). A second set of arguments advances that in their present sate, existing abatement technologies are too costly, and time should be given to research to develop new and more affordable technical options.

This line of thought has been heavily criticized by Van der Zwaan *et al.*, (2002), and Kverndokk and Rosendhal (2007), among others, because it does not take into account the potential of experience and learning-by-doing in pollution abatement technologies. Taking benefit of such learning opportunities rather requires early action than delay. The argument is reinforced by a capital accumulation motive, the replacement of old and costly vintages by new and cheaper one is a costly process requiring a significant time. This time to build issue appears to be particularly relevant for CCS technologies, their development being submitted to costly capacity expansion constraints. The early action these is also endorsed by industrial organization views. By announcing sufficiently early a credible path of action, in terms of an announced increasing time schedule of a carbon tax for example, the industry will react to this incentive scheme by investing today in abatement technologies, the uncertainty about what the regulator plans to do in the future being removed.

However, learning-by-doing is not the only way to induce technical advances. Another main option is R&D. R&D has two main advantages with respect to learning. First, it does not require to actually use the technology. With sufficient time and effort, it is possible to achieve in the lab potential cost cuts without bearing the high initial cost of using non mature technologies. Second, R&D can span much more potential technical options than actual use, which requires specific technical choices before beginning the exploitation of a given technique, the risk being to be trapped into inferior options or inappropriate initial choices.

It appears immediately that in an R&D induced technical change world, early development of infant abatement technologies may be counterproductive. In policy terms, this means that subsidizing non mature abatement technologies in the hope that learning can reduce their costs in the future may be suboptimal. It would be better to give more time to research to assert the economic potential of different technological options. This issue has been examined carefully by Goulder and Matthai (2000). Comparing a learning induced technical change model with a R&D induced technical change model, they conclude that in a R&D world, delaying actual abatement is optimal while the interest to advance or delay a policy action promoting pollution abatement is usually ambiguous in a learning world.

Induced technical change in carbon emission mitigation technologies is only one aspect of a more general problem involving also alternative clean energy production, like solar energy. These alternative energy sources may also benefit from both learning and R&D cost cutting progress. Such technical advances possibilities should modify the timing of optimal transitions between energy sources, as shown by Chakravorty, Leach and Moreaux in a recent paper (Chakravorty *et al.*, 2012) in the case of learning-by-doing. The same type of conclusions emerges from the study of Henriet for a R&D induced technical breakthrough in a clean energy source production cost (Henriet, 2012).

As remarked by Gerlagh *et al.*(2009), the Goulder-Matthai analysis does not exhaust the concern expressed by Jaffe, Newell and Stavins (2001) concerning the need of a better understanding of the impact of environmental policies upon the nature of induced technological change and the feedback effect of technical change upon the environment itself and thus upon environmental policies.

Contributing to this understanding is the main objective of the present work. Many confusions arise in the previous literature because of the usual incremental way of modeling technical progress. This is especially true when comparing learning-by-doing and research induced technical change. In an incremental model, technical change is a sequence of small improvements progressively reducing the cost of the abatement technology. But incremental actual cost cuts achieved through learning and potential cost cuts achieved by research activity are not really comparable. This is one of the main reason for the ambiguous effects of an environmental policy in a learning-by-doing model shown by Goulder and Matthai.

To escape this difficulty, we adopt a drastic view of technical change, more in line of the Barret initial proposal. Thanks to a combination of R&D activity and learning-by-doing, it is possible to increase over time some know-how index. Once the index has reached a given target, it induces an abrupt revolution in abatement technologies, taking the form of a cost drop from a high level to a low level. To simplify, we assume only one revolution of this kind, meaning that future learning or research activity will become worthless after the revolution.

We make a parallel simplification concerning the dynamics of the environment. Most papers model the environmental dynamics as a progressive accumulation of carbon into the atmosphere, the size of the carbon stock generating welfare damages at each point of time. These damages are increasing with the size of the pollution stock. We depart from this approach by using an alternative route pioneered by Chakravorty *et al.* (2006, 2008) . We assume that the atmospheric carbon stock does not harm directly welfare, but be crossed over some critical threshold level in carbon concentration, earth climate conditions would become catastrophic. This echoes the current policy proposals of targeting a temperature rise of no more than 2^0 C, that is actually trying to stabilize the carbon concentration to a constant level by the end of the century. Hence the environmental policy takes the form of a given mandate over the maximum level of the atmospheric carbon stock.

To simplify farther, we assume in the present paper that fossil fuels are not exhaustible. Introducing depletion constraints over fossil fuels will result in complex Hotelling effects blurring both the rate of carbon accumulation and the timing of technological development.

One drawback of the Goulder-Matthai analysis is that they focus on the polar cases of learning and R&D induced technical change, but these polar cases are extreme situations where the economy would be constrained to use only one device to trigger technological advances. We encompass this limitation by examining a model where both activities contribute to technical progress. This will allow a much better understanding of the delay problem raised in the earlier literature. In particular we show how can be endogenously determined time periods during which the economy should perform only R&D to enhance the technical efficiency of pollution abatement and time periods during which a combination of learning processes and research activity is optimal.

The model is laid down in section 2. In order to drive interesting comparisons with previous results of the literature, we study in section 3 the case of a pure learning-by-doing induced technological revolution and in section 4 the case of a pure R&D induced technological break. We improve on earlier studies by performing rather systematically a sensitivity analysis of the main variables. Usually, one finds such sensitivity analysis in simulation models, but their results are typically hard, if not impossible, to interpret. Our simple setting allows us to derive sensitivity results in the analytical domain, providing strong intuitions on our findings. In particular, we shall exhibit the similarities and the differences between the cases of a learning induced or a R&D induced technical change. In section 5, we examine the general case of a combined learning and R&D process. We derive the implications of such a process for an optimal environmental policy. We also describe the optimal technological development policy which may be of a combined type, a pure learning or a pure R&D type depending upon the model fundamentals. The last section 6 concludes.

2 The model

The economy has access to two primary energy sources. The first one is a polluting resource (let say coal). We assume an infinite supply of this resource, meaning that it will never be exhausted, that is we treat coal as a kind of a renewable polluting resource, or equivalently assume that coal is abundant. Let $x(t)$ be the rate of coal extraction. The second energy source is a clean renewable resource (let say solar) and we denote by $y(t)$ the used flow of solar energy.

Assuming for the sake of simplicity a one to one transformation process of primary energy sources units into energy services units, the production of solar energy services bears a cost $c_y y(t)$. The processing of coal into the generation of energy services may take two forms. Coal may be processed without consideration for the environmental consequences of burning this fossil fuel to produce energy. We call dirty coal processing this energy generation process and $x_d(t)$ denotes the corresponding coal energy services generation rate. The cost of dirty coal processing is $c_x x_d(t)$. It results into a pollution flow $\zeta x_d(t)$ assumed proportional to the dirty coal energy generation. Under our one to one transformation process assumption, x_d is also that fraction of coal extraction involved into dirty processing and ζ is the polluting content of coal.

The pollution flow accumulates into the environment and $Z(t)$ is the pollution stock size at time t. There exists a self-cleaning capacity of the environment, assumed to simplify proportional to the pollution stock size[1] , so that the motion of $Z(t)$ over time is given by:

$$\dot{Z}(t) = \zeta x_d(t) - \alpha Z(t) \quad .$$

The initial pollution stock is $Z(0) = Z^0$.

Coal may be also processed through a clean energy generation process,

[1]See Toman and Withagen (2000) for more general self-regenerating functions generating non-convex programs, the solutions of which necessitating global rather than local comparisons.

thanks for example to CCS effort, resulting in no carbon emissions into the atmosphere. $x_c(t)$ is the rate of clean coal services generation and $(c_x + c_s(t))x_c$ is the cost of clean coal energy services. Under our transformation assumption, $x_c(t)$ is also that fraction of coal extraction involved in clean coal energy production, so that: $x(t) = x_d(t) + x_c(t)$.

Energy services differ by their primary sources (coal or solar) and by their type of coal processing (either dirty or clean coal energy generation) but they are perfect substitutes for the final users once the due costs have been taken into account. Let $q(t) \equiv x_d(t) + x_c(t) + y(t)$ be the instantaneous consumption rate of energy services. This consumption generates a gross surplus, $u(q)$, assumed increasing and concave and satisfying the first Inada condition: $\lim_{q \downarrow 0} u'(q) = +\infty$.

As in Chakravorty *et al.* (2006), we assume that pollution does not harm directly welfare but be crossed over some critical threshold $\bar{Z}$, earth climate conditions would become catastrophic. Thus the society decides to maintain the carbon concentration below this critical level. To give content to the problem we have to assume that $Z^0 \leq \bar{Z}$.

Operating clean coal energy production equipments benefits both from learning-by-doing and dedicated research efforts. The cost reduction that may be achieved through these two processes may be defined in different ways. Here we adopt a drastic view of technical progress. The combination of the accumulation of experience with R&D efforts results into a technological revolution in the clean coal energy generation process. To describe this combined process, we adopt the simplest formulation able to retain the main aspects of the problem. Both learning-by-doing and R&D contribute to the accumulation over time of some stock of know-how. Let $A(t)$ be the level of this stock at time t. $A(t)$ grows over time at a rate depending upon the production scale of clean coal energy, $x_c(t)$, and upon the R&D effort rate, $r(t)$, through the following relation:

$$\dot{A}(t) = a(x_c(t), r(t)) .$$

$a(x_c, r)$ is twice continuously differentiable and both $a_c \equiv \partial a / \partial x_c > 0$, $a_r \equiv \partial a / \partial r > 0$. Know-how may be increased through only learning or R&D, that is: $a(0, r) > 0$ if $r > 0$ together with $a(x_c, 0) > 0$ if $x_c > 0$, while $a(0, 0) = 0$. Assume that $A(0) = 0$, that is normalize to zero the initial know-how index. Once some sufficient level of know-how, $\bar{A}$, has been attained, the technological revolution occurs, resulting into a sudden drop down of the cost of clean coal energy generation, from a high level $\bar{c}_s$, to a low level $\underline{c}_s$.

Thus the additional clean coal energy cost is a function of $A(t)$, $c_s(A(t))$, such that:

$$c_s(A(t)) \quad = \quad \begin{cases} \bar{c}_s & \text{if} \quad A(t) < \bar{A} \\ \\ \underline{c}_s & \text{if} \quad A(t) \geq \bar{A} \end{cases} \quad .$$

We assume to simplify only one technological revolution of this kind, meaning that future learning will be worthless after the revolution and that further R&D efforts will not allow for future cost breaks.

R&D activity has a cost $C_r(r)$, a cost function we assume twice continuously differentiable over $r \in (0, \infty)$, increasing and convex in r while $C_r(0) = 0$. So the marginal cost function $c_r(r) \equiv dC_r(r)/dr$ defined over $(0, \infty)$ verifies: $c_r(r) \geq 0$, $c_r'(r) > 0$ and in addition $\lim_{r \downarrow 0} c_r(r) = c_r^0 \geq 0$, the right end limit of the marginal R&D cost at zero is not necessarily zero.

The society has to determine a primary resources policy use, a split between dirty and clean coal energy generation, together with a R&D policy maximizing a discounted sum of instantaneous net surpluses, $\rho > 0$ being the constant level of the social discount rate, while taking into account the atmospheric carbon concentration constraint, $Z \leq \bar{Z}$.

This problem may be given different formulations depending upon the model fundamentals. If the cost of the clean solar energy is lower than the cost of dirty coal generation, then coal is never used and the pollution problem disappears. So we assume that $c_x < c_y$. It may be the case that clean coal energy generation is so costly even after the revolution that the society will prefer to produce only dirty coal energy services. In such a case, there will be no learning about the clean coal technology and R&D efforts will be worthless and thus no cost breakthrough can occur. This scenario where $x(t) = x_d(t)$ has been already studied by Chakravorty *et al.*(2006). It may also be the case that the pollution ceiling is never attained, a scenario where the more costly clean coal option would never be engaged.

In order to drive an interesting discussion, we assume first that the ceiling constraint binds eventually along the optimal path and, second, that the clean cost option is not too costly to be used at least over some time interval, maybe only after the technological revolution. We shall be more precise about the relevant assumptions for that to be the case in the sequel. If clean coal energy generation is profitable it will be used permanently after its introduction inside the energy mix. Either as a pure consequence of learning-by-doing, in case of no R&D efforts, or as a result of the combination of learning and R&D, the level of know-how will permanently rise, triggering the revolution at some time, $\bar{t}_A$. Then the optimal program may be designed as a sequential optimal control problem composed of two phases: a first phase $[0, \bar{t}_A)$ before the break and a second phase $[\bar{t}_A, \infty)$ after the break.

An optimal policy is hence a solution of the following program OP:

$$\begin{aligned}
\max_{x_c, x_d, y, r, \bar{t}_A} \quad & \int_0^{\bar{t}_A} \left[u(q(t)) - c_x x(t) - \bar{c}_s x_c(t) - c_y y(t) - C_r(r(t)) \right] e^{-\rho t} dt + e^{-\rho \bar{t}_A} \bar{V} \\
s.t. \quad & \dot{Z}(t) = \zeta x_d(t) - \alpha Z(t) \quad Z(0) = Z^0 \ \text{ given} \\
& \dot{A}(t) = a(x_c(t), r(t)) \quad A(0) = 0 \\
& x_c(t) \geq 0 \ , \ x_d(t) \geq 0 \ , \ y(t) \geq 0 \ , \ r(t) \geq 0 \\
& x_c(t) + x_d(t) \leq x(t) \\
& Z(t) \leq \bar{Z} \\
& A(\bar{t}_A) \geq \bar{A} \ .
\end{aligned}$$

$\bar{V}$ is the continuation value obtained by solving the following continuation problem after the technological revolution:

$$\begin{aligned}
\max_{x_c, x_d, y} \quad & \int_{\bar{t}_A}^{\infty} \left[u(q(t)) - c_x x(t) - \underline{c}_s x_c(t) - c_y y(t) \right] e^{-\rho (t - \bar{t}_A)} dt \\
s.t. \quad & \dot{Z}(t) = \zeta x_d(t) - \alpha Z(t) \quad Z(\bar{t}_A) = Z^A \ \text{ given} \\
& x_c(t) \geq 0 \ , \ x_d(t) \geq 0 \ , \ y(t) \geq 0 \\
& x_c(t) + x_d(t) \leq x(t) \\
& Z(t) \leq \bar{Z} \ .
\end{aligned}$$

Before examining the policies solving the program OP, it is useful to consider as benchmarks two polar cases, the case of a pure learning-by doing know-how generation and the case of a pure R&D generation of know-how. We devote the next two sections to these polar cases before turning to the general case.

3 Technological revolution induced by learning

Assume no R&D opportunities. The economy has to rely only upon learning-by-doing, that is on experience accumulation, to trigger the technological revolution. The simplest way to define experience, and thus here the know-how index, is to identify it with the cumulated number of clean coal energy services units generated since the beginning of clean coal energy production. Let $\underline{t}_c$ be the beginning time of clean coal energy production, then:

$$A(t) \equiv \int_{\underline{t}_c}^{t} x_c(\tau)d\tau \ .$$

Denote by $\lambda_A(t)$ and $\lambda_Z(t)$ respectively the costate variables associated to the state variables $A(t)$ and $Z(t)$. Denote also by ν_{xc}, ν_{xd}, ν_y, the Lagrange multipliers associated to the positivity constraints over x_c, x_d, y, respectively, and by ν_Z, the Lagrange multiplier associated to the constraint $Z(t) \leq \bar{Z}$; The optimality conditions over the time interval $[0, \bar{t}_A)$ are:

$$u'(q) = c_x + \bar{c}_s - \lambda_A - \nu_{xc} \quad (3.1)$$

$$u'(q) = c_x + \zeta\lambda_Z - \nu_{xd} \quad (3.2)$$

$$u'(q) = c_y - \nu_y \quad (3.3)$$

$$\dot{\lambda}_Z = (\rho + \alpha)\lambda_Z - \nu_Z \quad (3.4)$$

$$\dot{\lambda}_A = \rho\lambda_A \ . \quad (3.5)$$

Let us first sketch as a benchmark the optimal policy absent any learning abilities, the extra cost of producing clean coal energy with respect to dirty coal energy being c_s . A relevant scenario involves hitting the ceiling at some finite time $\underline{t}_Z$. We have to consider two possibilities. Either $c_y > c_x + c_s$, the high solar cost case, either $c_y < c_x + c_s$, the low solar cost case.

In the high solar cost case, solar energy is never introduced inside the energy mix. When at the ceiling, dirty coal energy generation is constrained by the natural regeneration capacity, so that the production of dirty coal energy is given by $\bar{x}_d \equiv \alpha\bar{Z}/\zeta$. Let $\bar{p} \equiv u'(\bar{x}_d)$. If $\bar{p} < c_x + c_s < c_y$, then the economy prefers to rely only upon dirty coal energy generation and never uses either clean coal energy generation or solar energy generation. The optimal path is a two phases path. During the first phase $[0, \underline{t}_Z)$, the economy produces only dirty coal energy, pollution accumulates and $x_d(t)$ is the solution of $u'(x) = c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)t}$, where $\lambda_{Z0} = \lambda_Z(0)$. This phase ends when the pollution stock reaches the ceiling $\bar{Z}$. Then begins a phase of infinite duration, $[\underline{t}_Z, \infty)$ of dirty coal generation at the level $\bar{x}_d$.

If $c_x + c_s < \bar{p}$, clean coal energy is never introduced before the beginning of the ceiling phase. Identifying (3.2) and (3.1), we get during any time interval below the ceiling where dirty and clean coal generation would be simultaneously operated: $\zeta\lambda_Z(t) = c_s$, which is incompatible with $\lambda_Z(t)$ growing at the rate $\rho + \alpha$. But since solar energy is even more costly than clean coal energy, solar energy is also never used before the ceiling. Thus, the economy produces only dirty energy until the ceiling constraint begins to be binding. During this first phase $[0, \underline{t}_Z)$, the implicit energy price $p(t) \equiv u'(q(t)$ is defined through (3.2) and (3.4) by: $p(t) = c_x + \lambda_{Z0}e^{(\rho+\alpha)t}$. Thus, the implicit energy price increases over time while dirty coal energy generation is progressively reduced. When at the ceiling, the economy starts to produce clean coal energy. The energy implicit price $p(t)$ is now constant and equal to the marginal cost of clean coal energy production $c_x + c_s$. Since solar energy is more expensive than clean coal energy, it is also never used during this time phase. The optimal policy when at the ceiling combines the production of dirty coal energy at the constant rate $\bar{x}_d$ with the production of clean coal energy at the constant rate $\bar{x}_c$, solution of $u'(\bar{x}_d + x_c) = c_x + c_s$. Note that the energy price continuity at $\underline{t}_Z$ requires that the dirty coal energy production path jumps down at $\underline{t}_Z$ from the level $\bar{x}_d + \bar{x}_c$ to the level $\bar{x}_d$ while the clean coal energy production rate jumps up from 0 to $\bar{x}_c$.

Last, in the low solar cost case, we have to distinguish the possibilities $c_y < \bar{p}$ and $\bar{p} < c_y$. If $c_y < \bar{p}$, solar energy is introduced when the ceiling constraint becomes to be binding and clean coal energy generation is never put in operation. Thus after $\underline{t}_Z$, energy production combines dirty coal energy generation at the rate $\bar{x}_d$ and solar energy generation at the rate $\bar{y}$, solution of $u'(\bar{x}_d + y) = c_y$. If $\bar{p} < c_y$, the economy prefers to rely only upon dirty coal energy generation and never uses clean energy in any form: clean coal energy or solar energy.

We turn now to a sensitivity analysis of the optimal policy with respect to some relevant parameters. Consider the optimal scenario in the high solar cost case with $c_x + c_s < \bar{p}$. To completely characterize the optimal policy, one has to identify two variables, λ_{Z0}, the initial level of the pollution opportunity cost, and $\underline{t}_Z$, the time at which the ceiling is attained. Let $x_d(t, \lambda_{Z0})$ be implicitly defined as the solution of $u'(x) = c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)t}$. $(\lambda_{Z0}, \underline{t}_Z)$ are solutions of the following system of conditions:

$$\begin{aligned}\zeta\lambda_{Z0}e^{(\rho+\alpha)t} &= c_s \\ \bar{Z}e^{\alpha\underline{t}_Z} &= Z^0 + \zeta\int_0^{\underline{t}_Z} x_d(t,\lambda_{Z0})e^{\alpha t}dt \ .\end{aligned}$$

We concentrate upon the parameters c_s, Z^0 and $\bar{Z}$.

Let $I_Z^Z \equiv -\int_0^{\underline{t}_Z}(1/u''(q(t))e^{(\rho+2\alpha)t}dt$ and $\Delta_0 \equiv \zeta\left[\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z + x_c(\underline{t}_Z)e^{\alpha\underline{t}_Z}\right]$. Then it is easily verified that:

$$\begin{aligned}\frac{d\lambda_{Z0}}{dc_s} &= \frac{x_c(\underline{t}_Z)e^{-\rho\underline{t}_Z}}{\Delta_0} > 0 \quad ; \quad \frac{d\underline{t}_Z}{dc_s} = \frac{\zeta I_Z^Z e^{-(\rho+\alpha)\underline{t}_Z}}{\Delta_0} > 0 \\ \frac{d\lambda_{Z0}}{dZ^0} &= \frac{(\rho+\alpha)\lambda_{Z0}}{\Delta_0} > 0 \quad ; \quad \frac{d\underline{t}_Z}{dZ^0} = -\frac{1}{\Delta_0} < 0 \\ \frac{d\lambda_{Z0}}{d\bar{Z}} &= -\frac{(\rho+\alpha)\lambda_{Z0}e^{\alpha t_Z}}{\Delta_0} < 0 \quad ; \quad \frac{d\underline{t}_Z}{d\bar{Z}} = \frac{e^{\alpha\underline{t}_Z}}{\Delta_0} > 0 \ .\end{aligned}$$

As expected, a higher clean coal energy cost translates into a larger opportunity cost of pollution. This is an immediate consequence of the fact that a higher clean coal cost means a lower clean coal energy production and thus a lower energy consumption rate when at the ceiling. Since the energy price level at the ceiling (equal to the clean coal marginal cost) is increased while the rise of the opportunity cost of pollution makes increase the energy price also before the ceiling, the overall effect over the time length before the ceiling could be ambiguous. However, the analysis shows that it must increase, the direct effect over the energy price at the ceiling being larger than the indirect effect over the energy price before the ceiling. The effects of either a larger initial pollution stock or a stricter ceiling constraint are straightforward. Both result in an increased opportunity cost of pollution and a faster attainment of the ceiling.

Next, we examine the changes introduced by learning abilities to this benchmark scenario. Learning abilities do not modify our original result that clean coal energy is never introduced before the ceiling phase. They do not change either our conclusion that solar energy is never introduced inside the energy mix if $c_x + \bar{c}_s < c_y$ and eliminates the clean coal energy option in the reverse case. Let us thus assume that solar energy is more costly than clean coal energy and that $c_x + \bar{c}_s < \bar{p}$.

The optimal path is composed of three phases. During a first phase $[0, \underline{t}_Z)$, the economy produces only dirty coal energy at a declining rate, the energy price growing at the rate $(\rho + \alpha)$. The pollution threshold $\bar{Z}$ is attained at $\underline{t}_Z$, the end of this phase. Then begins a second phase $[\underline{t}_Z, \bar{t}_A)$ during which the environmental constraint binds, the economy combines dirty coal energy generation constrained by the constant rate $\bar{x}_d = \alpha\bar{Z}/\zeta$ and clean coal energy generation. (3.5) defines $\lambda_A(t) = \lambda_{A0}e^{\rho t}$, $\lambda_{A0} = \lambda_A(0)$, $t \leq \bar{t}_A$. Then, $u'(\bar{x}_d + x_c(t)) = c_x + \bar{c}_s - \lambda_{A0}e^{\rho t}$ defines implicitly $x_c(t, \lambda_{A0})$ during the time interval $[\underline{t}_Z, \bar{t}_A)$ and $\dot{x}_c(t) = -\rho\lambda_{A0}e^{\rho t}/u''(\bar{x}_d + x_c) > 0$. Clean coal energy generation increases before the cost break while the implicit energy price decreases. The use of the pollution abatement technology results in experience accumulation up to the level $\bar{A}$, attained at time $\bar{t}_A$, at which the technological revolution occurs and the pollution abatement cost falls from the level $\bar{c}_s$ down to $\underline{c}_s$. Last, the economy enters an infinite duration phase $[\bar{t}_A, \infty)$ combining dirty and clean coal energy generation, the energy price being constant and equal to the post-revolution clean coal energy marginal cost $c_x + \underline{c}_s$.

After the cost breakdown, the economy produces clean coal energy at the constant rate $\underline{x}_c$ solution of: $u'(\bar{x}_d + x_c) = \underline{c}_s$. Thus, $\bar{V}$ the continuation value after the cost break in current terms at $\bar{t}_A$ is given by:

$$\bar{V} = \frac{1}{\rho}\left[u(\bar{x}_d + \underline{x}_c) - c_x(\bar{x}_d + \underline{x}_c) - \underline{c}_s\underline{x}_c\right] .$$

$\bar{t}_A$ must verify the following transversality condition:

$$\mathcal{H}(\bar{t}_A) = -\frac{\partial}{\partial \bar{t}_A}\bar{V}e^{-\rho\bar{t}_A} .$$

Since the economy is blockaded at the ceiling during the first phase of clean coal energy generation $[\underline{t}_Z, \bar{t}_A)$, $\dot{Z}(\bar{t}_A) = 0$. Denote by $\lim_{t\uparrow\bar{t}_A} x_c(t) = x_c^-$, the above condition is equivalent to:

$$\begin{aligned} u(\bar{x}_d + x_c^-) - c_x(\bar{x}_d + x_c^-) - \bar{c}_s x_c^- + \lambda_A(\bar{t}_A)x_c^- &= u(\bar{x}_d + \underline{x}_c) - c_x(\bar{x}_d + \underline{x}_c) \\ &\quad -\underline{c}_s\underline{x}_c . \end{aligned}$$

Simplifying the $c_x\bar{x}_d$ term on both sides and taking into account (3.1): $u'(\bar{x}_d + x_c^-) = c_x + \bar{c}_s - \lambda_A(\bar{t}_A)$ while $u'(\bar{x}_d + \underline{x}_c) = c_x + \underline{c}_s$, we get:

$$u(\bar{x}_d + x_c^-) - u'(\bar{x}_d + x_c^-)x_c^- = u(\bar{x}_d + \underline{x}_c) - u'(\bar{x}_d + \underline{x}_c)\underline{x}_c .$$

Let $\Gamma(x) \equiv u(\bar{x}_d + x) - u'(\bar{x}_d + x)x$. Then $d\Gamma(x)/dx = -u''(\bar{x}_d + x)x > 0$ shows that $\Gamma(x)$ is a monotonously increasing function of x, hence is bijective, showing that $x_c^- = \underline{x}_c$. The clean coal energy generation rate is a continuous time

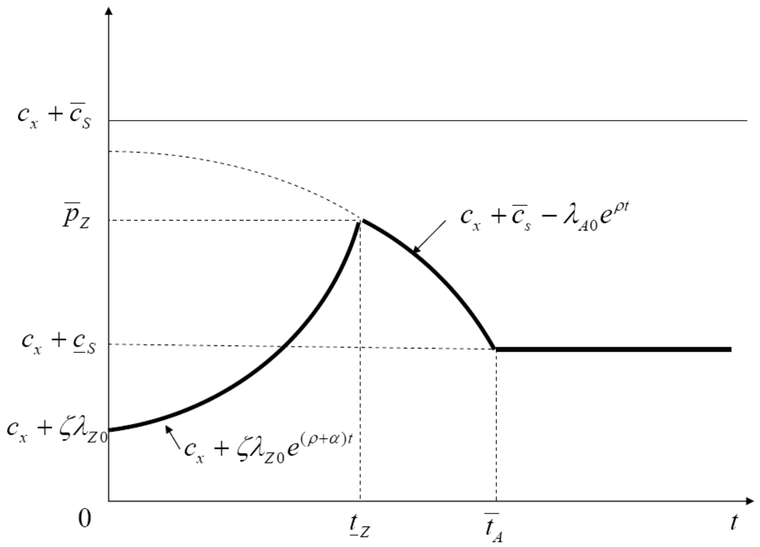

Figure 1: **Price Dynamics in the Pure Learning Case**

function at $\bar{t}_A$. The Figure 1 illustrates the dynamics of the corresponding energy price path.

During the first phase $[0, \underline{t}_Z)$ before the ceiling constraint begins to be binding, only dirty coal energy is produced and the energy price rises at the rate $\rho + \alpha$, just as in the no learning abilities benchmark. But because of the learning process, the energy price at the beginning of the ceiling phase, $p(\underline{t}_Z) \equiv \bar{p}_Z$ is now lower than $c_x + \bar{c}_s$. The energy price next decreases before the cost breakthrough until the break occurs and the price stabilizes forever at the level $c_x + \underline{c}_s$. The energy production rates experience the same kind of jumps described in the benchmark scenario without learning abilities. The use of clean coal energy jumps from 0 up to $x_c(\underline{t}_Z) \equiv x_c^Z$, solution of $u'(\bar{x}_d + x_c) = \bar{p}_Z$. The production of dirty coal energy makes a parallel jump down, from the level $\bar{x}_d + x_c^Z$ to the level $\bar{x}_d$.

In terms of policy tools implementation, an optimal account of learning abilities requires two instruments. The first one is a carbon tax (or a carbon price in a cap and trade system) upon dirty coal energy generation. The tax must be rising at the rate $\rho + \alpha$ before the ceiling begins to be binding and clean energy generation is introduced inside the energy mix. Then the carbon tax should decline over time before stabilizing at the level $\underline{c}_s$ after the cost breakdown occurs. The use of clean energy must be subsidized at the consumption stage during the first phase of clean coal energy generation $[\underline{t}_Z, \bar{t}_A)$. The clean energy production sector supplies clean energy at its marginal cost $c_x + \bar{c}_s$ during this phase, resulting into a clean energy production price equal to this marginal cost. The subsidy is given by $\lambda_{A0}e^{\rho t}$ and allows for an energy price reduction at the consumption stage. Since the carbon tax is equal to $(\bar{c}_s - \lambda_{A0}e^{\rho t})/\zeta$, the dirty coal consumption is maintained to its mandated level $\bar{x}_d$. The subsidy increases over time, allowing for a permanent increase of clean coal energy consumption until the cost breakdown occurs. After $\bar{t}_A$, the subsidy is removed and the production and consumption prices of clean energy are identical and equal to $c_x + \underline{c}_s$. The maximum level of the subsidy, attained at time $\bar{t}_A$ is the cost gap $\bar{c}_s - \underline{c}_s$. The current value level of the subsidy is thus $(\bar{c}_s - \underline{c}_s)e^{-\rho(\bar{t}_A - t)}$ at time t, $\underline{t}_Z \leq t \leq \bar{t}_A$.

We have shown previously that without learning abilities, the economy prefers to rely only upon dirty coal generation if $\bar{p} < c_x + c_s$. The same happens with learning abilities if $\bar{p} < c_x + \underline{c}_s$. In this situation, the technological breakthrough is unable to induce a sufficiently low level of the pollution abatement marginal cost to justify beginning clean coal energy generation. In the intermediate case: $c_x + \underline{c}_s < \bar{p} < c_x + \bar{c}_s$, a new possible scenario emerges. In this scenario, clean coal energy use is delayed after the attainment of the ceiling until some time $\underline{t}_c$. Then clean coal energy generation expands until the cost breakdown occurs. Remark that there should be no quantity discontinuity in this scenario. The use of dirty coal energy is maintained to the level $\bar{x}_d$ while the use of clean coal energy is initially nill at $\underline{t}_c$. However, Appendix A.1 shows that the economy cannot improve over a policy based upon the sole use of dirty coal energy by adopting such a combined policy.

Let us retain the case $c_x + \bar{c}_s < \bar{p}$. To characterize the optimal policy with learning abilities, we have to identify four variables, the initial values of λ_Z and λ_A together with the optimal time to attain the ceiling $\underline{t}_Z$ and the optimal time to trigger the technological revolution, $\bar{t}_A$. Let $x_d(t, \lambda_{Z0})$ be implicitly defined by $u'(x) = c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)t}$ over the time interval $[0, \underline{t}_Z)$ and $x_c(t, \lambda_{A0})$ be implicitly defined by $u'(\bar{x}_d + x_c) = c_x + \bar{c}_s - \lambda_{A0}e^{\rho t}$ over the time interval $[\underline{t}_Z, \bar{t}_A)$. $(\lambda_{Z0}, \lambda_{A0}, \underline{t}_Z, \bar{t}_A)$ are solutions of the following system of four conditions:

- The ceiling attainment condition, $Z(\underline{t}_Z) = \bar{Z}$:

$$\bar{Z}e^{\alpha \underline{t}_Z} = Z^0 + \zeta \int_0^{\underline{t}_Z} x_d(t, \lambda_{Z0}) e^{\alpha t} dt$$

- The critical experience level attainment condition at the revolution time, $A(\bar{t}_A) = \bar{A}$:

$$\bar{A} = \int_{\underline{t}_Z}^{\bar{t}_A} x_c(t, \lambda_{A0}) dt$$

- The price continuity requirement at $\underline{t}_Z$:

$$\zeta \lambda_{Z0} e^{(\rho+\alpha)\underline{t}_Z} = \bar{c}_s - \lambda_{A0} e^{\rho \underline{t}_Z}$$

- The price continuity requirement at the cost break time:

$$\lambda_{A0} e^{\rho \bar{t}_A} = \bar{c}_s - \underline{c}_s$$

Denote by:

$$I_Z^Z \equiv -\int_0^{\underline{t}_Z} \frac{e^{(\rho+2\alpha)t}}{u''(q(t))} dt > 0 \ ; \ I_A \equiv -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{e^{\rho t}}{u''(q(t))} dt > 0$$
$$J_A^c \equiv -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{dt}{u''(q(t))} > 0 \ ; \ x_c^Z \equiv x_c(\underline{t}_Z) \ ; \ x_c^A \equiv x_c(\bar{t}_A)$$
$$T_A \equiv \bar{t}_A - \underline{t}_Z \ ; \ \pi_Z \equiv \zeta(\rho+\alpha)\lambda_{Z0} e^{\alpha \underline{t}_Z} + \rho \lambda_{A0}$$
$$\Delta_0 \equiv \zeta \left[\zeta(\rho+\alpha)\lambda_{Z0} I_Z^Z + x_c^Z e^{\alpha \underline{t}_Z} \right] .$$

We refer to Appendix A.2 for calculation details. We show the following effects of a higher initial pollution stock over the optimal path features:

$$\frac{d\lambda_{Z0}}{dZ^0} = \frac{(\rho+\alpha)\lambda_{Z0}}{\Delta_0} > 0 \quad ; \quad \frac{d\underline{t}_Z}{dZ^0} = \frac{d\bar{t}_A}{dZ^0} = -\frac{1}{\Delta_0} < 0$$
$$\frac{d\lambda_{A0}}{dZ^0} = \frac{\rho \lambda_{A0}}{\Delta_0} > 0$$

The impact of a higher Z^0 is qualitatively the same as in the case without learning abilities: the pollution opportunity cost is higher and the attainment of the ceiling is accelerated. We remark that even if the levels of the variables are of course different, the qualitative expressions of the partial derivatives are the same with and without learning abilities. This is a consequence of the fact that a higher initial pollution stock mainly affects the optimal path before the ceiling phase. Computing the effect of a higher Z^0 over the price level at the beginning of the ceiling phase, $\bar{p}_Z$, confirms this result:

$$\begin{aligned} \frac{d\bar{p}_Z}{dZ^0} &= \left[\frac{d\lambda_{Z0}}{dZ^0} + (\rho+\alpha)\lambda_{Z0}\frac{d\underline{t}_Z}{dZ^0}\right] e^{(\rho+\alpha)\underline{t}_Z} \\ &= [(\rho+\alpha)\lambda_{Z0} - (\rho+\alpha)\lambda_{Z0}]\,\frac{e^{(\rho+\alpha)\underline{t}_Z}}{\Delta_0} = 0 \end{aligned}$$

A higher initial pollution stock has no effect over the energy price at the beginning of the ceiling phase, and thus no effect over the production rate of clean coal energy at $\underline{t}_Z$. Furthermore, $d\underline{t}_Z/dZ^0 = d\bar{t}_A/dZ^0 < 0$ shows that the first phase at the ceiling keeps the same length, thus the production plan of clean coal energy is just translated sooner in time by a higher initial pollution level. This property induces an upper shift of the initial level of the learning rent λ_A completely neutral in terms of current value subsidy levels throughout the phase $[\underline{t}_Z, \bar{t}_A)$.

One could be tempted to think that a stricter ceiling constraint would have the same qualitative effects than a higher initial pollution stock. But the induced change over dirty coal energy production modifies the comparative advantage of dirty coal versus clean coal energy generation and thus the value of experience acquisition. More precisely:

$$\begin{aligned} \frac{d\lambda_{Z0}}{d\bar{Z}} &= -\frac{(\rho+\alpha)\lambda_{Z0}e^{\alpha\underline{t}_Z}}{\Delta_0} + \frac{\alpha\rho\lambda_{A0}T_A}{\zeta\Delta_0}\ ? \\ \frac{d\underline{t}_Z}{d\bar{Z}} &= \frac{e^{\alpha\underline{t}_Z}}{\Delta_0} + \frac{\alpha\rho\lambda_{A0}T_A I_Z^Z}{x_c^Z\Delta_0 e^{\alpha\underline{t}_Z}} > 0 \\ \frac{d\lambda_{A0}}{d\bar{Z}} &= -\frac{\rho\lambda_{A0}e^{\alpha\underline{t}_Z}}{\Delta_0} - \frac{\alpha\rho\lambda_{A0}T_A(\pi_Z I_Z^Z + x_c^Z e^{2\alpha\underline{t}_Z})}{x_c^Z\Delta_0 e^{\alpha\underline{t}_Z}} < 0 \\ \frac{d\bar{t}_A}{d\bar{Z}} &= \frac{e^{\alpha\underline{t}_Z}}{\Delta_0} + \frac{\alpha T_A(\pi_Z I_Z^Z + x_c^Z e^{2\alpha\underline{t}_Z})}{x_c^Z\Delta_0 e^{\alpha\underline{t}_Z}} > 0 \end{aligned}$$

A stricter ceiling constraint ($d\bar{Z} < 0$) has an ambiguous effect over the initial pollution opportunity cost. This translates into an ambiguous consequence over the energy implicit price path before the beginning of the ceiling phase. However the analysis shows a faster attainment of the ceiling, a higher learning rent together with a sooner technological revolution time. The effect of a stricter environmental standard combines two components shown in the above expressions. The first component is the effect of a change of $\bar{Z}$ for a given clean energy production path. As expected, this component works in the same direction as the effect of higher initial Z^0. It increases the pollution opportunity cost, fastens the ceiling attainment, increases the value of learning and fastens the revolution. The second component expresses the induced effect of a stricter ceiling upon $x_c(t)$, the clean coal energy production rate, during the pre-revolution phase at the ceiling, $[\underline{t}_Z, \bar{t}_A)$. This effect is depending upon T_A, the length of this time phase. This effect has a negative impact over the pollution opportunity cost, it reduces $\underline{t}_Z$, makes increase the value of learning and fastens the revolution. Thus these two effects work in the same direction for $\underline{t}_Z$, λ_{A0} and $\bar{t}_A$ but in an opposite direction for λ_{Z0}.

This does not mean that the effect of a ceiling modification over the energy price at the beginning of the ceiling phase, $\bar{p}_Z$, and upon T_A, the time length of the first phase of clean coal production, is indeterminate. More precisely:

$$\begin{aligned}
\frac{d\bar{p}_Z}{d\bar{Z}} &= \frac{\alpha\rho\lambda_{A0}T_A}{\zeta x_c^Z} e^{\rho \underline{t}_Z} > 0 \\
\frac{dT_A}{d\bar{Z}} &= \frac{\alpha T_A}{\zeta x_c^Z} > 0 \ .
\end{aligned}$$

Thus a stricter ceiling constraint results into a lower energy price at the beginning of the ceiling phase together with a shorter time period before the revolution once clean coal production begins. This implies that the production of clean coal energy is increased by a stricter ceiling constraint and hence that the energy implicit price is lower during the time phase $[\underline{t}_Z, \bar{t}_A)$. To this lower price level correspond both a higher subsidy level to clean coal energy and a lower carbon price. Note that a lower $\bar{p}_Z$ and a lower $\underline{t}_Z$ are compatible with either a higher or a lower level of $\zeta\lambda_{Z0}$, the initial level of the carbon price. Note also that learning abilities reverses the usual result that a stricter environmental constraint should translate into a higher optimal carbon tax. The analysis shows to the contrary that, in between the beginning of the ceiling period and the technological revolution, the carbon tax level is lowered by a stricter ceiling constraint.

We have shown also that a stricter ceiling means an increased use of clean coal energy and thus a faster technological revolution. This is reminiscent of the Porter, Van der Linde hypothesis (1995). Following Michael Porter, imposing 'tight' environmental regulation (that is 'more' than Pigouvian) should spur more R&D efforts from the energy industry. In the present context, improving the efficiency of pollution abatement requires an increased use of the clean coal energy generation technology. A stricter environmental standard, though perfectly 'Pigouvian', achieves this outcome quite naturally, by reducing the comparative advantage of dirty energy with respect to clean energy. However, this does not mean that before the beginning of clean coal energy use, a stricter environmental standard should translate into a higher carbon tax.

Turn to the consequences of a higher experience threshold triggering the technological revolution in clean coal energy generation. After computations, we get:

$$\frac{d\lambda_{Z0}}{d\bar{A}} = \frac{\rho\lambda_{A0}}{\Delta_0} > 0 \;\; ; \;\; \frac{d\lambda_{A0}}{d\bar{A}} = -\frac{\rho\lambda_{A0}}{x_c^Z}\left[1 + \frac{\zeta\rho\lambda_{A0}I_Z^Z}{x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}}\right] < 0$$
$$\frac{d\underline{t}_Z}{d\bar{A}} = \frac{\zeta\rho\lambda_{A0}I_Z^Z}{x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} > 0 \;\; ; \;\; \frac{d\bar{t}_A}{d\bar{A}} = \frac{1}{x_c^Z}\left[1 + \frac{\zeta\rho\lambda_{A0}I_Z^Z}{\Delta_0 e^{\alpha \underline{t}_Z}}\right] > 0$$
$$\frac{d\bar{p}_Z}{d\bar{A}} = \frac{\rho\lambda_{A0}e^{\rho \underline{t}_Z}}{x_c^Z} > 0 \;\; ; \;\; \frac{dT_A}{d\bar{A}} = \frac{1}{x_c^Z} > 0 \; .$$

As expected, a higher experience requirement to trigger the cost break results into a lower level of the learning rent and a longer time before the cost break. The opportunity cost of pollution is increased together with the energy price at the beginning of the ceiling phase but because the learning rent is decreased in a higher proportion than λ_Z is increased, the attainment of the ceiling is delayed.

The sensitivity analysis of a small increase of the clean energy cost before the break is more intricate since it affects simultaneously the price convergence condition towards the ceiling, the relative profitability of clean coal energy before the break and the value of learning in getting an increased cost cut. Denote by:

$$I_c \equiv \int_{\underline{t}_Z}^{\bar{t}_A} x_c(t)e^{-\rho t}dt > 0 \; .$$

Then we get first:

$$\frac{d\lambda_{Z0}}{d\bar{c}_s} = \frac{\rho I_c}{\Delta_0} > 0 \; ;$$
$$\frac{d\underline{t}_Z}{d\bar{c}_s} = \frac{\zeta\rho I_Z^Z I_c}{x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} > 0 \; ;$$
$$\frac{d\bar{p}_Z}{d\bar{c}_s} = \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{d\lambda_{Z0}}{d\bar{c}_s} + (\rho+\alpha)\lambda_{Z0}\frac{d\underline{t}_Z}{d\bar{c}_s}\right] > 0 \; .$$

Hence a higher clean energy cost induces a higher pollution opportunity cost together with a delayed attainment of the ceiling. The energy price at the beginning of the ceiling is also increased. These results are expected, note that the direct effect of the cost increase dominates the indirect effect over the energy price before the ceiling phase, resulting into a slower move towards the ceiling.

The effect of a higher initial clean coal energy cost over the learning rent is indeterminate but it is possible to show that:

$$\begin{aligned}\frac{d\bar{t}_A}{d\bar{c}_s} &= \frac{\zeta}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}}\left[(\pi_Z I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z})(J_A^c - I_A e^{-\rho \bar{t}_A}) + x_c^Z I_Z^Z (e^{-\rho \underline{t}_Z} - e^{-\rho \bar{t}_A})\right] > 0 \\ \frac{dT_A}{d\bar{c}_s} &= \frac{J_A^c - I_A e^{-\rho \bar{t}_A}}{x_c^Z} > 0\end{aligned}$$

Hence a higher initial clean coal energy cost means a longer period before the cost breakdown. Since the energy price is shifted upward at the beginning of the ceiling phase, the exploitation of clean coal energy is reduced, implying a slower learning process and thus a delayed, although more significant, cost cut.

Last, considering the consequences of a lower clean energy cost after the break, we find that:

$$\begin{aligned}\frac{d\lambda_{Z0}}{d\underline{c}_s} > 0 \quad &; \quad \frac{d\lambda_{A0}}{d\underline{c}_s} < 0 \\ \frac{d\underline{t}_Z}{d\underline{c}_s} > 0 \quad &; \quad \frac{d\bar{t}_A}{d\underline{c}_s} > 0 \\ \frac{d\bar{p}_Z}{d\underline{c}_s} > 0 \quad &; \quad \frac{dT_A}{d\underline{c}_s} > 0\end{aligned}$$

These effects fit the intuition. A higher clean energy cost level after the break, that is a lower cost cut thanks to learning, results into a higher pollution opportunity cost together with a delayed attainment of the ceiling. On the other hand, the learning rent is reduced and the time length before the cost break to occur is enlarged.

The following propositions summarize our findings:

Proposition 1 *(i) With only learning abilities, clean coal energy generation is never introduced before the atmospheric ceiling constraint begins* *to be binding. If solar energy production is cheaper than clean coal energy, it eliminates this option and is itself eliminated in the reverse case. The optimal energy policy is a three phases path composed of a first phase of only dirty energy generation at a declining rate until the carbon ceiling level is attained. Then clean coal energy generation begins at an increasing rate while the production of dirty energy is constrained by the ceiling. This second phase ends at the technological revolution time. After the revolution, the economy stays permanently at the ceiling and produces a constant rate of clean coal energy at the post-revolution low marginal cost.*

(ii) *The possibility of a learning induced technological revolution in CCS lowers the carbon opportunity cost before the attainment of the ceiling, this cost being rising exponentially at the rate $(\rho + \alpha)$. During the pre-revolution phase at the ceiling, the energy implicit price decreases over time. There is no price cut at the technological breakthrough. After the break, the energy price remains constant and equal to the clean coal energy marginal production cost.*

(iii) *The implementation of the optimal policy requires to combine a carbon tax (or a carbon price in a cap and trade system) and a subsidy at the consumption stage to clean energy use. The carbon tax increases before the atmospheric ceiling constraint begins to be binding , decreases during the second phase and stays constant after the cost revolution. The subsidy is introduced whence the ceiling is attained, it increases exponentially at the rate ρ before the revolution and is suppressed after the cost breakthrough. Its current value level at time t is given by $(\bar{c}_s - \underline{c}_s)e^{-\rho(\bar{t}_A - t)}$, the cost gap in present value from t.*

Concerning the sensitivity analysis with respect to some relevant parameters, we show that:

Proposition 2 *(i) The pollution opportunity cost, or the optimal carbon tax, is increased by a higher initial pollution stock, a higher know-how requirement to trigger the technological revolution or a higher CCS cost before the revolution. A stricter environment standard, that is a lower $\bar{Z}$, has an ambiguous effect over the carbon price before the ceiling constraint begins to be binding and lowers this price whence the ceiling is binding.*

(ii) *The learning rent, or equivalently the subsidy needed to induce the optimal level of clean coal energy generation, is reduced by a higher initial* *pollution stock or a higher required know-how level to trigger the technological revolution. It is increased by a stricter environmental standard while a higher pre-revolution CCS cost has ambiguous effects over the learning rent.*

(iii) *The ceiling constraint binds earlier if the initial pollution stock is higher, the ceiling constraint more stringent, the know-how requirement to trigger the cost cut in CCS operation less stringent or the abatement cost before the revolution less expensive.*

(iv) *The length of the learning phase between the beginning of the ceiling phase and the revolution time is independent from the initial pollution stock. It decreases with a stricter ceiling constraint, a less stringent know-how target and a lower pre-revolution CCS abatement cost.*

4 The R&D induced technical revolution in abatement

Consider the reverse case of no learning abilities. The technical revolution only results from sufficient efforts in R&D. Take the simplest form for the consequences of such efforts over the accumulation of know-how, that is assume that $\dot{A} = r$. Then the optimality conditions before the revolution become:

$$u'(q) = c_x + \bar{c}_s - \nu_{xc} \quad (4.1)$$
$$u'(q) = c_x + \zeta\lambda_Z - \nu_{xd} \quad (4.2)$$
$$u'(q) = c_y - \nu_y \quad (4.3)$$
$$\lambda_A = c_r(r) - \nu_r \quad (4.4)$$
$$\dot{\lambda}_Z = (\rho + \alpha)\lambda_Z - \nu_Z \quad (4.5)$$
$$\dot{\lambda}_A = \rho\lambda_A\ . \quad (4.6)$$

Assume that $c_y > \bar{p} > c_x + \bar{c}_s$, hence solar energy never enters the energy mix. As in the preceding case, clean coal energy generation should not be introduced before the beginning of the ceiling phase. Since R&D only affects the time of the revolution, whatever be the cost level, the pre-revolution high cost level or the post-revolution low cost level, the constancy of unit costs is incompatible with the rise of the pollution opportunity cost before the ceiling.

This in turn implies that the revolution should not occur strictly before the beginning of the ceiling phase. Assume to the contrary that $\bar{t}_A < \underline{t}_Z$. Then clean coal energy production begins at $\underline{t}_Z$ with the best technology of cost $\underline{c}_s$. Since there are no learning abilities, we are in the benchmark case exposed at the beginning of section 3. After reaching the ceiling, the energy price is constant and given by $c_x + \underline{c}_s$. Before $\underline{t}_Z$, the energy price is given by: $p(t) = c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)t}$ until $\bar{p}_Z = c_x + \underline{c}_s$ is attained at time $\underline{t}_Z$. Consider a small decrease of the research effort $dr < 0$ at each time during the time interval $[0, \bar{t}_A)$. The revolution time $\bar{t}_A$ would be slightly delayed but for a sufficiently low dr, the revolution occurs before $\underline{t}_Z$. Thus nothing would be changed to the resource use policy before and after $\underline{t}_Z$, the only consequence being a reduction in the R&D cost. Thus such a reduction would be beneficial, resulting in an optimal time of the revolution happening at least whence the ceiling has been attained, that is $\underline{t}_Z \leq \bar{t}_A$ in any optimal scenario.

Next let us assume that the know-how requirement is sufficiently stringent to have $\underline{t}_Z < \bar{t}_A$. We shall be more precise later over the conditions for the revolution to occur only strictly after the beginning of the ceiling phase. After the revolution, the economy remains blockaded at the ceiling, the energy price is given by $c_x + \underline{c}_s$, the production of dirty coal energy by $\bar{x}_d$ and the production of clean coal energy by $\underline{x}_c$, thus, $\bar{V}$, the continuation value after the revolution is the same as before and given by:

$$\bar{V} = \frac{1}{\rho}\left[u(\bar{x}_d + \underline{x}_c) - c_x(\bar{x}_d + \underline{x}_c) - \underline{c}_s\underline{x}_c\right] .$$

Denote $h^- \equiv \lim_{t\uparrow\bar{t}_A} h(t)$ for any time function $h(t)$. The transversality condition at $\bar{t}_A$ is now expressed as:

$$u(\bar{x}_d + x_c^-) - c_x(\bar{x}_d + x_c^-) - \bar{c}_s x_c^- - C_r(r^-) + \lambda_A^- r^- = u(\bar{x}_d + \underline{x}_c) - c_x(\bar{x}_d + \underline{x}_c) - \underline{c}_s\underline{x}_c .$$

Taking (4.1) and (4.4) into account, this is equivalent to:

$$u(q^-) - u'(q^-)x_c^- - C_r(r^-) + c_r(r^-)r^- = u(\bar{x}_d + \underline{x}_c) - u'(\bar{x}_d + \underline{x}_c)\underline{x}_c$$

Denote by: $\Gamma_r(r) \equiv c_r(r)r - C_r(r)$. Since $C_r(0) = 0$, $\Gamma_r(0) = 0$ and $\Gamma_r' = c_r'(r)r > 0$ under our cost convexity assumption. Thus $\Gamma_r(r) > 0$ if $r > 0$.

Denote by $\Gamma(x_c) \equiv u(\bar{x}_d + x_c) - u'(\bar{x}_d + x_c)x_c$, an increasing function of x_c, as shown before. Then the transversality condition is equivalent to:

$$\Gamma(\underline{x}_c) - \Gamma(x_c^-) \quad = \quad \Gamma_r(r^-) > 0$$

At the revolution time, $\bar{t}_A$, the energy price jumps down from the level $c_x + \bar{c}_s$ to the level $c_x + \underline{c}_s$. This corresponds to an upward jump of clean coal energy generation from the level x_c^-, solution of $u'(\bar{x}_d + x_c) = c_x + \bar{c}_s$, up to the level $\underline{x}_c$, itself solution of: $u'(\bar{x}_d + x_c) = c_x + \underline{c}_s$. The transversality condition shows that to this quantity jump corresponds a unique level of r^-, the research effort just before the revolution. Since $\lambda_A^-(\bar{t}_A) = c_r(r^-)$, the terminal level of the R&D knowledge rent is thus also determined. Let $\bar{\lambda}_A$ be this level, then: $\lambda_{A0}e^{\rho\bar{t}_A} = \bar{\lambda}_A$, taking (4.6) into account.

This shows a first difference between the R&D strategy to trigger the technological revolution and the learning strategy. Inducing the right level of experience acquisition, and thus the right level of clean coal energy production, required a specific subsidy in the preceding case. Such a subsidy device is no more needed to induce the optimal level of R&D investments, the carbon tax being a sufficient tool to implement the optimal scenario. The optimal time profile of the subsidy also resulted into a continuous energy price trajectory despite the cost revolution. This is no more the case under a R&D induced technological revolution and one obtains the usual conclusion that energy services are permanently priced at their marginal cost, the cost break resulting into a price breakdown at the revolution time.

A second difference appears in the computation of the R&D rent. While the learning rent simply identified with the cost gap in the preceding section, the R&D rent at the revolution time is defined through the transversality condition by a complex relation depending upon not only the shape of the R&D cost function but also upon the energy gross surplus function.

Third, the relative independency between the dynamics of energy use and the dynamics of know-how induced by the R&D policy widens the space of possible energy scenarios. Assume that the solar energy cost, c_y, is such that: $\underline{c}_s < c_y < \bar{c}_s$. In the learning induced revolution framework, solar energy would eliminate the use of clean coal energy and thus the possibility of a revolution. In the R&D induced revolution framework, the corresponding optimal scenario is the following. During a first phase $[0, \underline{t}_Z)$, the economy only relies upon the use of dirty coal energy until the ceiling is attained. Then solar energy is introduced in combination with dirty coal energy generation up to a level $\bar{y}$ solution of: $u'(\bar{x}_d + y) = c_y$. Thus, before the technological revolution, that is during the time interval $[\underline{t}_Z, \bar{t}_A)$, the economy is constrained by the ceiling but never produces clean coal energy, this one being more costly than solar energy. After the revolution, solar energy is eliminated from the energy mix and the economy combines the production of dirty energy at the rate $\bar{x}_d$ and of clean coal energy at the rate $\underline{x}_c$. Thus the energy transition scenario is composed of a first phase using only coal, a second phase using both coal and solar energy and a third phase using only coal but with a positive amount of clean coal energy.

Next, let us turn to the description of the optimal R&D policy. If $c_r(0) = c_r^0 > 0$, R&D investment may be delayed. Taking (4.4) into account, $\lambda_A(t) \geq c_r^0$ appears as a necessary condition for strictly positive R&D efforts, $r(t) > 0$. Since $\lambda_{A0} = \lambda_A(0)$ is determined by the whole model structure, this may or not be the case at time $t = 0$. If $\lambda_{A0} > c_r^0$, R&D effort is set immediately at a positive level. Since $c_r'(r) > 0$, the optimal R&D effort is implicitly defined by $\lambda_A(t) = c_r(r(t))$ as an increasing function of λ_A and thus an increasing function of time since $\lambda_A(t)$ is growing exponentially. Let $r(t) = r_A(\lambda_A(t))$ be that function. Then $r(t)$ grows permanently over time. If $\lambda_{A0} < c_r^0$, R&D investments are delayed until $\underline{t}_A$ solution of $\lambda_A(t) = c_r^0$. At $\underline{t}_A$, R&D activity shows a smooth start from a zero level and then increases permanently over time as in the preceding case. One may observe that this feature goes in the opposite direction of many endogenous growth models where R&D efforts should be set initially at a high level and then be decreased. This is because these models usually assume the existence of increasing returns to scale in the knowledge generation process, returns to scale resulting from an inheritance effect of previously accumulated knowledge. Such effects are absent in the present model and we obtain the usual conclusion that because of discounting, R&D costs should be delayed in time. The result is an increasing R&D effort path, maybe from a zero initial level after some time period without research activity.

Let us first consider the optimal scenario in a situation where $\lambda_{A0} > c_r^0$. If $c_r^0 = 0$, this is the only optimal solution. It is identified by computing the vector of variables $(\lambda_{Z0}, \lambda_{A0}, \underline{t}_Z, \bar{t}_A)$, a vector solution of the following set of conditions:

- The ceiling attainment condition, $Z(\underline{t}_Z) = \bar{Z}$:

$$\bar{Z}e^{\alpha \underline{t}_Z} = Z^0 + \zeta \int_0^{\underline{t}_Z} x_d(t) e^{\alpha t} dt \ .$$

- The know-how requirement condition, $A(\bar{t}_A) = \bar{A}$:

$$\bar{A} = \int_0^{\bar{t}_A} r_A(\lambda_{A0} e^{\rho t}) dt \ .$$

- The price continuity requirement at $\underline{t}_Z$:

$$\zeta \lambda_{Z0} e^{(\rho+\alpha)\underline{t}_Z} = \bar{c}_s$$

- The R&D rent condition at $\bar{t}_A$:

$$\Gamma(\underline{x}_c) - \Gamma(x_c^-) = \Gamma_r(r_A(\lambda_{A0} e^{\rho \bar{t}_A})) \ .$$

It is worth contrasting the pure learning and pure R&D induced technological breakthrough models through a parallel comparative dynamics exercise to the one performed in Section 3. The computations details are presented in Appendix A.3. The main conclusions are the following.

Denote as before $\Delta_0 = \zeta \left[\zeta(\rho+\alpha)\lambda_{Z0} + x_c^Z e^{\alpha \underline{t}_Z}\right]$ and by: $\bar{r} = \lim_{t \uparrow \bar{t}_A} r(t)$ and $r_0 \equiv r(0)$.

The effects of a larger initial pollution stock Z^0 or a stricter ceiling constraint are the following:

$$\frac{d\lambda_{Z0}}{dZ^0} = -\frac{d\lambda_{Z0}}{d\bar{Z}} e^{-\alpha \underline{t}_Z} = \frac{(\rho+\alpha)\lambda_{Z0}}{\Delta_0} > 0 \ ; \ \frac{d\underline{t}_Z}{dZ^0} = -\frac{d\underline{t}_Z}{d\bar{Z}} e^{-\alpha \underline{t}_Z} = -\frac{1}{\Delta_0}$$

$$\frac{d\lambda_{A0}}{dZ^0} = \frac{d\bar{t}_A}{dZ^0} = 0$$

$$\frac{d\lambda_{A0}}{d\bar{Z}} = -\frac{\bar{r}\rho\lambda_{A0}}{\bar{r} - r_0} \frac{d\bar{t}_A}{d\bar{Z}} = -\frac{\alpha(\bar{c}_s - \underline{c}_s) e^{-\rho \bar{t}_A}}{\zeta r_0} < 0$$

The decoupling of the know-how dynamics from the economic arbitrages driving the energy policy removes the indeterminacy problem identified in the learning model. It appears clearly that a larger initial pollution stock or a stricter ceiling constraint have the same qualitative effects over the energy implicit price trajectory. Both make rise the pollution opportunity cost, and thus the energy price before the ceiling, and fasten the attainment of the ceiling.

The differences between a stricter ceiling an a higher initial pollution stock appear when considering the R&D optimal policy. There is no effect of the initial pollution stock over the R&D policy. Since the economy is permanently constrained by the ceiling after $\underline{t}_Z$, a stricter ceiling results into a higher R&D effort and thus in earlier technological revolution. We observed a similar accelerating effect in the learning model. Imposing a stricter environmental standard makes rise the R&D effort to trigger the cost revolution.

The consequences of a higher know-how requirement, $\bar{A}$, to trigger the technical revolution offer another illustration of the relative independency between the R&D policy and the energy policy. After computations, we get:

$$\frac{d\lambda_{Z0}}{d\bar{A}} = \frac{d\underline{t}_Z}{d\bar{A}} = 0$$
$$\frac{d\lambda_{A0}}{d\bar{A}} = -\rho\lambda_{A0}\frac{d\bar{t}_A}{d\bar{A}} = -\frac{\rho\lambda_{A0}}{r_0} < 0$$

A higher knowledge target induces a slow down of the research efforts and thus a delayed technical revolution. A larger know-how requirement has no effect over the energy consumption policy, the ceiling being attained at the same time and the pollution opportunity cost being unaffected by a higher $\bar{A}$.

The independency feature disappears when considering the additional cost of clean coal energy production before the revolution, since this cost both affects the convergence condition of the energy price towards its ceiling level and the size of the cost breakdown. The calculus shows that:

$$\frac{d\lambda_{Z0}}{d\bar{c}_s} = \frac{x_c^z}{\Delta_0}e^{-\rho\underline{t}_Z} > 0 \quad ; \quad \frac{d\underline{t}_Z}{d\bar{c}_s} = \frac{\zeta I_Z^Z}{\Delta_0}e^{-(\rho+\alpha)\underline{t}_Z} > 0$$
$$\frac{d\lambda_{A0}}{d\bar{c}_s} = \frac{x_c^-}{r_0}e^{-\rho\bar{t}_A} > 0 \quad ; \quad \frac{d\bar{t}_A}{d\bar{c}_s} = -\frac{x_c^-(\bar{r}-r_0)}{r_0\bar{r}\rho\lambda_{A0}}e^{-\rho\bar{t}_A} < 0$$

As in the pure learning model, a higher initial clean coal energy cost means a higher pollution opportunity cost together with a delayed arrival at the ceiling. As before, the direct effect over the energy price after $\underline{t}_Z$ resulting from a higher $\bar{c}_s$ dominates the indirect effect over the energy price of a higher pollution opportunity cost resulting in a longer time before the beginning of the ceiling phase. Contrarily to the learning model where the time length between the beginning of the learning process and the revolution time was enlarged by a higher $\bar{c}_s$, R&D is accelerated by the perspective of a larger cost breakthrough and the revolution comes earlier.

Since the clean coal cost level affects only the post-revolution phase, one should expect that a higher $\underline{c}_s$ has no effect upon the energy use before the revolution. Furthermore, the perspective of a smaller cost breakthrough should discourage research and delay the revolution time. The calculus confirms these straightforward intuitions.

Up to now we considered only scenarios where $\lambda_{A0} > c_r^0$, but we need to make precise the domain of validity of such policies. Let us consider an R&D policy starting at time 0 from $r(0) = 0$, that is $\lambda_{A0} = c_r^0$. Then the cost breakthrough occurs at a time $\bar{T}_A$ solution of:

$$\bar{A} = \int_0^T r_A(c_r^0 e^{\rho t})dt$$

$\bar{T}_A$ is the maximum time delay to get the breakthrough since the economy starts from the lowest possible level of research efforts. Let $(\lambda_{Z0}^0, \underline{t}_Z^0)$ be defined as the solutions of:

$$\begin{aligned} \bar{Z}e^{\alpha \underline{t}_Z} &= Z^0 + \zeta \int_0^{\underline{t}_Z} x_d(t)e^{\alpha t}dt \\ \zeta \lambda_{Z0} e^{(\rho+\alpha)\underline{t}_Z} &= \underline{c}_s \; . \end{aligned}$$

$(\lambda_{Z0}^0, \underline{t}_Z^0)$ are the optimal initial levels of the pollution opportunity cost and time delay before the ceiling in a situation where the cost breakthrough would occur just when the ceiling constraint begins to be binding. If $\underline{t}_Z^0 > \bar{T}_A$, the active R&D phase has to be delayed until $\underline{t}_A = \underline{t}_Z^0 - \bar{T}_A$ as noticed before. In this scenario, clean energy production is introduced at $\underline{t}_Z^0$ with the best technology, the cost breakthrough occurring at $\underline{t}_Z^0$.

In the contrary case, triggering the technological revolution when attaining the ceiling requires to set λ_{A0} above c_r^0 and thus $r(0) > 0$. To $\underline{t}_Z^0$ corresponds a unique value of λ_{A0}, we denote by λ_{A0}^0 solution of:

$$\bar{A} = \int_0^{\underline{t}_Z^0} r_A(\lambda_{A0}e^{\rho t})dt \; .$$

Let $\bar{\lambda}_A^0 \equiv \lambda_{A0}^0 e^{\rho \underline{t}_Z^0}$.

We have to consider the transversality condition while taking explicitly into account the constraint: $\underline{t}_Z \leq \bar{t}_A$, which requires to modify this condition as such. Denote by μ_Z, the Lagrange multiplier associated to the constraint $\underline{t}_Z \leq \bar{t}_A$. Then optimality requires that:

$$\mathcal{H}(\bar{t}_A) + \mu_Z = -\frac{\partial}{\partial \bar{t}_A}\bar{V}e^{-\rho \bar{t}_A}$$

with $\mu_Z \geq 0$ and $\mu_Z(\bar{t}_A - \underline{t}_Z) = 0$. This is equivalent to:

$$\bar{\Gamma} \;\equiv\; \Gamma(\underline{x}_c) - \Gamma(x_c^-) \;=\; \Gamma_r(r^-) + \mu_Z \;\geq\; \Gamma_r(r^-)$$

Note that $\bar{\Gamma}$ is given by the cost parameters and the energy demand shape, and is independent from either the R&D policy or the ceiling attainment condition. Let $\bar{r}$ be the solution of $\bar{\Gamma} = \Gamma_r(r)$ and $\bar{\lambda}_A = c_r(\bar{r}) \equiv \bar{\lambda}_A(\bar{\Gamma})$. Then, $\Gamma_r(r)$ being an increasing function of r, $\bar{r}$ is an increasing function of $\bar{\Gamma}$ and hence $\bar{\lambda}_A(\bar{\Gamma})$ is an increasing function of $\bar{\Gamma}$. This implies that if $\bar{\lambda}_A(\bar{\Gamma}) < \bar{\lambda}_A^0$, the constraint $\underline{t}_Z \leq \bar{t}_A$ does not bind, while it is binding in the reverse case.

Consider the case of a binding constraint, that is $\bar{\lambda}_A(\bar{\Gamma}) \geq \bar{\lambda}_A^0$. Let $r_0(t) \equiv r_A(\lambda_{A0}^0 e^{\rho t})$, the optimal R&D policy may be of two types. If $\underline{t}_Z^0 < \bar{T}_a$, it is defined by $r_0(t)$ over the time interval $[0, \underline{t}_Z^0)$. The research effort is initially strictly positive ($r_0(0) > 0$) and the cost break occurs at $\underline{t}_Z^0$. If $\underline{t}_Z^0 > \bar{T}_a$, then the active research phase is delayed until some time $\underline{t}_A = \underline{t}_Z^0 - \bar{T}_A$.

Turn now to the case of a non binding constraint: $\bar{\lambda}_A(\bar{\Gamma}) < \bar{\lambda}_A^0$. Then $\bar{\lambda}_A(\bar{\Gamma}) < \bar{\lambda}_A^0$ is equivalent to $r(\bar{t}_A) < r(\underline{t}_Z^0)$, implying that $\bar{t}_A > \underline{t}_Z^0$ to satisfy the knowledge accumulation constraint. Hence, $\lambda_{A0} < \lambda_{A0}^0$. The economy follows a less active R&D policy. To lower levels of $\bar{\Gamma}$ correspond lower levels of $\bar{\lambda}_A(\bar{\Gamma})$ and thus lower paths of R&D efforts. If $\bar{\Gamma}$ is such that $\bar{\lambda}_A(\bar{\Gamma}) < c_r^0$, R&D efforts become unprofitable and there is no cost breakthrough. We conclude that the optimal policy is one of the four possible types described in the following Proposition:

Proposition 3

1. *If $\bar{\lambda}_A(\bar{\Gamma}) < c_r^0$, there is no R&D activity and trivially the cost break never occurs, the society prefers to use clean coal energy when at the ceiling at the high cost level.*

2. *If $c_r^0 < \bar{\lambda}_A(\bar{\Gamma}) < \bar{\lambda}_A^0$, the active R&D policy starts immediately at time 0. R&D efforts increase over time and the cost breakthrough occurs strictly after the beginning of the ceiling phase, resulting in a time phase $[\underline{t}_Z, \bar{t}_A)$ where the economy uses the clean coal energy technology at its highest cost $\bar{c}_s$.*

3. *If $\bar{\lambda}_A^0 < \bar{\lambda}_A$ and $t_Z^0 < \bar{T}_A$, then the economy starts to perform R&D efforts right from $t = 0$, the optimal R&D effort is given by $r_0(t)$ resulting in a cost breakthrough occurring just at the time $\underline{t}_Z^0$ when the ceiling constraint begins to bind.*

4. *If* $\bar{\lambda}_A^0 < \bar{\lambda}_A$ *and* $t_Z^0 > \bar{T}_A$*, then the active R&D phase is delayed until some time* $\underline{t}_A$ *such that* $\underline{t}_A = \underline{t}_Z^0 - \bar{T}_A$*, also triggering the technological revolution just at the arrival at the ceiling.*

Concerning the sensitivity of the optimal path to some relevant parameters in the case $c_r^0 < \bar{\lambda}_A(\bar{\Gamma}) < \bar{\lambda}_A^0$, we have shown that:

Proposition 4 *(i) The pollution opportunity cost before the ceiling constraint begins to bind, or equivalently the optimal carbon tax, is increased by a larger initial pollution stock, a stricter ceiling constraint or a higher CCS cost before the technological revolution. It is unaffected by the know-how target to trigger the revolution.*

(ii) The ceiling constraint binds earlier with a higher initial pollution stock, a stricter ceiling constraint, a higher CCS cost before the revolution and is unaffected by the revolution know-how target.

(iii) The R&D rent, or equivalently the intensity of R&D efforts, is unaffected by the initial pollution stock. It is increased by a stricter ceiling constraint, a less stringent know-how requirement or a higher CCS cost before the revolution.

(iv) The technological breakthrough is delayed by a less stringent ceiling constraint, a more stringent know-how requirement to trigger the cost break or a lower initial CCS cost. The revolution time is independent from the initial pollution stock.

It is interesting to contrast the sensitivity analysis of the learning induced and the R&D induced technological revolution. The following Table 1 summarizes our main findings, the pure R&D case qualitative effects appearing between parenthesis in the table.

The table shows that the two technological breakthrough triggering devices, learning-by-doing or R&D, behave more or less the same in qualitative terms. A part from the independency property of the R&D way to trigger

	$d\lambda_{Z0}$	$dt_{\underline{Z}}$	$d\lambda_{A0}$	$d\bar{t}_A$
dZ^0	$+\ (+)$	$-\ (-)$	$+\ (0)$	$-\ (0)$
$d\bar{Z}$	$?\ (-)$	$+\ (+)$	$-\ (-)$	$+\ (+)$
$d\bar{A}$	$+\ (0)$	$+\ (0)$	$-\ (-)$	$+\ (+)$
$d\bar{c}_s$	$+\ (+)$	$+\ (+)$	$?\ (-)$	$+\ (-)$

Table 1: Comparing the learning and R&D sensitivity analysis

the revolution with respect to the energy policy we already noticed, we remark that one important difference lies in the effect of the initial CCS cost. A higher initial cost delays the revolution in a learning model while it accelerates it in a R&D model. This is a fairly straightforward consequence of the fact that a learning-by-doing process is dependent upon the profitability conditions over the use of clean coal energy, a higher CCS cost reducing the use of clean energy and thus delaying the revolution, while a higher initial CCS cost widens the cost gap that may be achieved thanks to R&D, thus creating an incentive to trigger the revolution sooner in time.

5 Combining learning and R&D to trigger the technological revolution

To characterize the optimal policy, we put more structure upon the know-how accumulation process. Assume that:

Assumption 1 *1. $a(x_c, r)$ is a concave function of (x_c, r), that is:*

$$a_{cc} \equiv \partial^2 a/\partial x_c^2 < 0 \quad ; \quad a_{rr} \equiv \partial^2 a/\partial r^2 < 0$$
$$a_{cc}a_{rr} - (a_{cr})^2 > 0 \quad \textit{where} \quad a_{cr} \equiv \partial^2 a/\partial x_c \partial r$$

2. x_c and r are weak complements: $a_{cr} \geq 0$.

3. $a(x_c, r)$ exhibits non increasing returns to scale, that is:

$$a(c_x, r) \geq a_c(x_c, r)x_c + a_r(x_c, r)r$$

The current value Lagrangian of the first phase problem OP defined in section 2 (dropping the time index for the ease of reading) is:

$$\begin{aligned}\mathcal{L} &= u(x_c + x_d + y) - c_x(x_c + x_d) - \bar{c}_s x_c - c_y y - C_r(r) - \lambda_Z(\zeta x_d - \alpha Z) \\ &\quad + \lambda_A a(x_c, r) + \nu_{xc} x_c + \nu_{xd} x_d + \nu_y y + \nu_r r + \nu_Z(\bar{Z} - Z)\ .\end{aligned}$$

The optimal policy must be a solution of the following set of conditions:

$$u'(q) = c_x + \bar{c}_s - \lambda_A a_c(x_c, r) - \nu_{xc} \quad (5.1)$$

$$u'(q) = c_x + \zeta\lambda_Z - \nu_{xd} \quad (5.2)$$

$$u'(q) = c_y - \nu_y \quad (5.3)$$

$$\lambda_A a_r(x_c, r) = c_r(r) - \nu_r \quad (5.4)$$

$$\dot{\lambda}_Z = (\rho + \alpha)\lambda_Z - \nu_Z \quad (5.5)$$

$$\dot{\lambda}_A = \rho\lambda_A\ . \quad (5.6)$$

To these conditions must be added the usual complementary slackness conditions and a transversality condition at $\bar{t}_A$ that we discuss later.

The main difference with the preceding sections is that it is now possible to begin the production of clean coal energy before the ceiling constraint begins to bind. This is a consequence of the non linear link between knowledge accumulation and the intensity of learning or R&D efforts triggering the revolution together with the complementarity effects between learning and R&D. Let us concentrate upon the high solar cost case: $c_y > c_x + \bar{c}_s$, so that coal is the only exploited primary energy source. First, we prove the following important result.

Proposition 5 *Along an optimal energy and know-how accumulation policy, the cost breakthrough happens either strictly after the ceiling has been attained or either at the time when the ceiling begins to bind, that is* $\underline{t}_Z \leq \bar{t}_A$ *in all optimal scenarios.*

Proof: Assume to the contrary that $\bar{t}_A < \underline{t}_Z$. Over a time interval $[t_0, \bar{t}_A)$ we may be in three possible situations: either the cost break is triggered only by research, either it is triggered only through learning, or either it is triggered by a combination of research and learning. In the first case, slowing down slightly the research effort is beneficial, as noticed in section 3. In the second and third cases, λ_A will be zero after the cost break and since the production of clean coal energy is positive in these two cases, $\zeta\lambda_Z = \underline{c}_s$ is incompatible with λ_Z growing exponentially during the time interval $[\bar{t}_A, \underline{t}_Z)$. Hence clean coal exploitation should be interrupted after the cost break, meaning that delaying the revolution by reducing the use of clean coal energy before the revolution is beneficial.

5.1 Know-how accumulation scenarios

An optimal policy of knowledge accumulation is a sequence of time phases composed of the three following types of transitory tails:

(i) *A Combined tail* during which both research and learning are used to accumulate know-how. Let $\mathcal{T}^C$ be such a time phase, then $x_c(t) > 0$ and $r(t) > 0$, $t \in \mathcal{T}^C$.

(ii) *A Pure R&D tail* during which there is no exploitation of clean coal energy and the economy performs only research activity. Let $\mathcal{T}^R$ be such a time phase, then $x_c(t) = 0$ and $r(t) > 0$, $t \in \mathcal{T}^R$.

(iii) *A Pure learning tail* during which there is no research activity and know-how accumulates only because of learning. Let $\mathcal{T}^L$ be such a time phase, $x_c(t) > 0$ and $r(t) = 0$, $t \in \mathcal{T}^R$.

Such transitory phases can happen indifferently before or during the ceiling phase. The main complexity now is that there is no more a simple link between the energy production path, and thus the timing of the pre-ceiling and ceiling phases, and the structure of the combined learning and R&D know-how accumulation path.

We thus have to describe the main features of the possible transitory tails in the two cases of a pre-ceiling phase and a ceiling phase.

Combined tails

Assume first that $\mathcal{T}^C \subset [0, \underline{t}_Z)$. During this time phase, $(x_c(t), x_d(t), r(t))$ are defined as functions of $(\lambda_Z(t), \lambda_A(t))$ by (5.1), (5.2) and (5.4) with $\nu_{xd} = \nu_{xc} = \nu_r = 0$. Let $x_c(\lambda_Z, \lambda_A)$, $x_d(\lambda_Z, \lambda_A)$, $r(\lambda_Z, \lambda_A)$ be the corresponding implicit functions. Let $\delta \equiv a_{cc}a_{rr} - (a_{cr})^2$. $\delta > 0$ through the concavity assumption over $a(x_c, r)$ and let $\Delta_1 \equiv -u''(q(t))\lambda_A(t)\left[\lambda_A(t)\delta - a_{cc}c_r'\right] > 0$. Differentiating the relevant optimality conditions and dropping the arguments of the functions for the ease of reading gets:

$$\frac{\partial x_d}{\partial \lambda_Z} = -\frac{\zeta}{\Delta_1}\left[u''(\lambda_A a_{rr} - c_r') + \lambda_A^2\delta - \lambda_A a_{cc}c_r'\right] < 0 \; ; \tag{5.7}$$

$$\frac{\partial x_d}{\partial \lambda_A} = -\frac{u''}{\Delta_1}\left[a_c(\lambda_A a_{rr} - c_r') - \lambda_A a_r a_{cr}\right] < 0 \; ; \tag{5.8}$$

$$\frac{\partial x_c}{\partial \lambda_Z} = \frac{\zeta u''}{\Delta_1}(\lambda_A a_{rr} - c_r') > 0 \; ; \tag{5.9}$$

$$\frac{\partial x_c}{\partial \lambda_A} = -\frac{\partial x_d}{\partial \lambda_A} > 0 \; ; \tag{5.10}$$

$$\frac{\partial r}{\partial \lambda_Z} = -\frac{\zeta u''}{\Delta_1}\lambda_A a_{cr} > 0 \; ; \tag{5.11}$$

$$\frac{\partial r}{\partial \lambda_A} = -\frac{u''\lambda_A}{\Delta_1}\left[a_c a_{cr} - a_r a_{cc}\right] > 0 \; . \tag{5.12}$$

Since $\lambda_Z(t) = \lambda_{Z0}e^{(\rho+\alpha)t}$ and $\lambda_A(t) = \lambda_{A0}e^{\rho t}$ during the time phase $[0, \underline{t}_Z)$, we conclude that both $\lambda_Z(t)$ and $\lambda_A(t)$ are time increasing and thus:

$$\frac{dx_d}{dt} < 0 \;\; ; \;\; \frac{dx_c}{dt} > 0 \;\; ; \;\; \frac{dr}{dt} > 0 \; .$$

Before the ceiling phase, the use of dirty coal energy should decline while the use of clean coal energy should expand, together with an ever increasing level of R&D efforts. Note however that λ_Z being increasing, $q(t)$ has to decrease. The increased use of clean energy does not compensate for the declining rate of use of dirty energy, the aggregate use of energy being strictly decreasing with time before the ceiling is attained.

Next, assume that $\mathcal{T}^C \subset [\underline{t}_Z, \bar{t}_A)$. Now $x_d(t) = \bar{x}_d$ and $(x_c(t), r(t))$ are implicitly defined by (5.1) and (5.4) as functions of λ_A only. Let $x_c(\lambda_A)$, $r(\lambda_A)$ be these functions. Denote:

$$\Delta_2 \equiv u''(q)(\lambda_A a_{rr} - c_r') - \lambda_A a_{cc}c_r' + \lambda_A^2\delta > 0 \; .$$

Differentiating (5.1) and (5.4), we get:

$$\frac{dx_c(\lambda)}{d\lambda_A} = \frac{1}{\Delta_2}\left[\lambda_A a_{cr}a_r - a_c(\lambda_A a_{rr} - c_r')\right] > 0 \; ; \tag{5.13}$$

$$\frac{dr(\lambda_A)}{d\lambda_A} = \frac{1}{\Delta_2}\left[\lambda_A a_c a_{cr} - (u''(q) + \lambda_A a_{cc})a_r\right] > 0 \; . \tag{5.14}$$

This shows that $x_c(\lambda_A)$ and $r(\lambda_A)$ are increasing functions of λ_A, and thus of time, also during the ceiling phase. This implies in turn that $a(\lambda_A) \equiv a(x_c(\lambda_A), r(\lambda_A))$ is also an increasing function of λ_A. Since λ_A increases over time, $a(\lambda_A)$ increases over time, the accumulation of know-how through the combined effect of learning and R&D accelerates over time. Last, we conclude that $x_c(t)$ being increasing through time, the energy implicit price should permanently decrease during such a $\mathcal{T}^C$ time phase.

Let us denote by $\sigma \equiv a_c/a_r$ the marginal rate of substitution (MRS) between learning and R&D. We are going to show that during a combined tail, σ is declining over time. Consider first the case $\mathcal{T}^C \subset [0, \underline{t}_Z)$. The MRS is defined implicitly as a function of (λ_Z, λ_A) during $\mathcal{T}_c$. Let $\sigma(\lambda_Z, \lambda_A)$ be this implicit function:

$$\sigma(\lambda_Z, \lambda_A) \;=\; \frac{a_c(x_c(\lambda_Z, \lambda_A), r(\lambda_Z, \lambda_A))}{a_r(x_c(\lambda_Z, \lambda_A), r(\lambda_Z, \lambda_A))}.$$

Making use of (5.9)-(5.12), we get the following expressions of the partial derivatives of the function $\sigma(\lambda_Z, \lambda_A)$ with respect to (λ_Z, λ_A):

$$\frac{\partial \sigma(\lambda_Z, \lambda_A)}{\partial \lambda_Z} \;=\; \frac{\zeta u''}{a_r \Delta_1}\left[\lambda_A \delta + c_r'(\sigma a_{rc} - a_{cc})\right] \;< 0 \qquad (5.15)$$

$$\frac{\partial \sigma(\lambda_Z, \lambda_A)}{\partial \lambda_A} \;=\; \frac{\sigma c_r' u''}{\Delta_1}\left[\sigma a_{rc} - a_{cc}\right] \;< 0 \;. \qquad (5.16)$$

Hence we conclude from $\dot{\lambda}_Z(t) > 0$ and $\dot{\lambda}_A(t) > 0$, $t \in [0, \underline{t}_Z)$, that:

$$\dot{\sigma}(t) \;=\; \frac{\partial \sigma}{\partial \lambda_Z}\dot{\lambda}_Z(t) + \frac{\partial \sigma}{\partial \lambda_A}\dot{\lambda}_A(t) \;< 0 \;.$$

In the case $\mathcal{T}^C \subset [\underline{t}_Z, \bar{t}_A)$, Appendix A.4 shows that $\dot{\sigma} < 0$. This property of combined tails does not translate to the other types of phases, a point we check below. The dynamics of $\sigma(t)$ has important implications over the dynamics of $x_c(t)$ and $r(t)$ during a combined phase.

Proposition 6 *During any time phase accumulating know-how through both learning and R&D, either before the ceiling phase or either during the ceiling phase, the learning effort increases at a higher rate than the R&D effort.*

Proof: Since $\dot{\sigma}(t) < 0$, $t \in \mathcal{T}^C$:

$$\dot{\sigma}(t) < 0 \iff \frac{\dot{a}_c}{a_c} < \frac{\dot{a}_r}{a_r} \;.$$

This implies that in the (x_c, r) plane, the optimal trajectory cuts lower an lower isoclines. Under our assumptions concerning the $a(x_c, r)$ function, the isoclines in the plane (x_c, r) are increasing functions of x_c, describing lower and lower levels of σ when moving in the east direction. Thus, the combined path cuts lower an lower rays r/x_c, the path bending more and more in the direction of experience with respect to R&D. In other words, while both r and x_c increase over time, $\dot{r}/r < \dot{x}_c/x_c$ during a combined learning and R&D phase. This phenomenon applies indifferently during the pre-ceiling phase or the ceiling phase.

The Figure 2 illustrates the corresponding dynamics in the (x_c, r) plane.

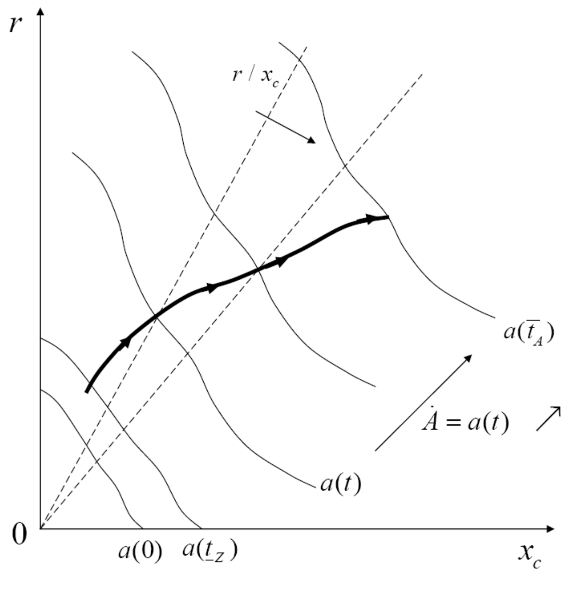

Figure 2: **Learning and R&D Dynamics in the (x_c, r) Plane**

The difference between the respective dynamics of x_c and r may be explained by the fact that triggering the revolution through R&D relies upon costly efforts that will be recovered only when the revolution occurs. To the contrary, even if being more costly to use than dirty energy, clean coal energy generation generates a positive surplus when in use. This is the case before the ceiling constraint begins to be binding, the use of clean coal energy helping to alleviate the environmental burden upon the use of dirty energy. This is also the case during the ceiling phase, clean coal energy use allowing for an energy consumption increase above the constrained level $\bar{x}_d$. The consequence is that the economy should rely more and more over time upon learning with respect to research activity to trigger the technological revolution. The interesting aspect of this result is its degree of generality. It does not involve any kind of specific experience capital in learning or particular assumptions upon the relative productiveness of learning and R&D in boosting the acquisition of know-how.

Pure R&D tails

Let us now consider a time phase $\mathcal{T}^R$ involving only R&D activity without clean energy production and hence no learning on the CCS technology. Then (5.4) defines $r(t)$ as an implicit function of λ_A and:

$$\frac{dr(\lambda_A)}{d\lambda_A} = \frac{a_r}{c_r' - \lambda_A a_{rr}} > 0 \ .$$

$\lambda_A(t)$ being increasing over time, $r(t)$ grows also over time. Since $a_{cr} > 0$, this implies that $\dot{a}_c(0, r) = a_{cr}\dot{r} > 0$ and hence that $\lambda_A a_c(0, r)$ is an increasing function of time. Assume first that $\mathcal{T}^R \subset [0, \underline{t}_Z)$. Then over the time interval $\mathcal{T}^R$:

$$\nu_{xc}(t) = \bar{c}_s - [\lambda_A(t)a_c(0, r(t)) + \zeta\lambda_Z(t)] \ .$$

It appears that $\nu_{xc}(t)$ is a strictly decreasing time function during a time interval $\mathcal{T}^R$. This implies that a corner path where $x_c = 0$ and $r > 0$ cannot follow an interior path where both $x_c > 0$ and $r > 0$. Furthermore we remark that $r(t)$ being increasing over time, $a_c(0, r)$ increases over time, since $a_{cr} > 0$, while $a_r(0, r)$ decreases over time, since $a_{rr} < 0$. Hence $\sigma = a_c/a_r$ increases over time during the time interval $\mathcal{T}^R$. Last, note that the border condition:

$$\zeta\lambda_Z = \bar{c}_s - \lambda_A a_c(0, r(\lambda_A))$$

defines an implicit relation between λ_Z and λ_A such that:

$$\left.\frac{d\lambda_A}{d\lambda_Z}\right|_{x_c=0} = -\frac{\zeta(c_r' - \lambda_A a_{rr})}{a_r a_{cr}\lambda_A + a_c(c_r' - \lambda_A a_{rr})} < 0 \ .$$

Assume now that $\mathcal{T}_r \subset [\underline{t}_Z, \bar{t}_A)$. Since $x_c = 0$, $u'(\bar{x}_d) = \bar{p}$ then $\nu_{xc} = \bar{c}_s - \lambda_A(t)a_c(0, r(\lambda_A)) - \bar{p}$, also a decreasing time function. Note that, as in the case $\mathcal{T}_r \subset [0, \underline{t}_Z)$, $\sigma(t)$ increases over time. The border condition is now:

$$\bar{p} = \bar{c}_s - \lambda_A a_c(0, r(\lambda_A)) \ .$$

This condition defines a unique level of λ_A, we denote by $\bar{\lambda}_A^R$. $\lambda_A(t) > \bar{\lambda}_A^R$ is incompatible with a policy performing only research activity without clean coal energy production.

Pure learning tails

Next, consider the case of a time phase $\mathcal{T}^L$. Note that this requires $c_r^0 > 0$. Assume first that $\mathcal{T}^L \subset [0, \underline{t}_Z)$. Then (5.1), (5.2) define implicitly x_c and x_d as functions of (λ_Z, λ_A). Let $x_c(\lambda_Z, \lambda_A)$ and $x_d(\lambda_Z, \lambda_A)$ be the corresponding implicit functions. Denote by $\Delta_3 \equiv \lambda_A u''(q) a_{cc}(x_c, 0) > 0$. Then it is easily checked that:

$$\frac{\partial x_c}{\partial \lambda_Z} = -\frac{\zeta u''(q)}{\Delta_3} > 0 \quad ; \quad \frac{\partial x_c}{\partial \lambda_A} = -\frac{a_c u''(q)}{\Delta_x} > 0 \ ;$$
$$\frac{\partial x_d}{\partial \lambda_Z} = \frac{\zeta(u''(q) + \lambda_A a_{cc})}{\Delta_3} < 0 \quad ; \quad \frac{\partial x_d}{\partial \lambda_A} = \frac{u''(q) a_c}{\Delta_x} < 0 \ .$$

Since $\lambda_Z(t)$ and $\lambda_A(t)$ are increasing time functions before $\underline{t}_Z$, this shows that $x_d(t)$ declines over time while $x_c(t)$ increases over time. Hence $a_c(x_c, 0)$ decreases over time, since $a_{cc} < 0$ and $a_r(x_c, 0)$ increases over time, since $a_{cr} > 0$. We conclude that $\sigma(t) = a_c(t)/a_r(t)$ decreases during a $\mathcal{T}^L$ type time interval. Furthermore $\nu_r(t)$ is given by:

$$\nu_r(t) \quad = \quad c_r^0 - \lambda_A(t) a_r(x_c(\lambda_Z(t), \lambda_A(t)), 0) \ .$$

$a_r(x_c, 0)$ being increasing over time, $\nu_r(t)$ decreases, implying that a time phase where $x_c > 0$ and $r = 0$ cannot follow a time phase where both $x_c > 0$ and $r > 0$. Next consider the border condition:

$$\lambda_A a_r(x_c(\lambda_Z, \lambda_A), 0) \quad = \quad c_r^0 \ .$$

This condition defines an implicit relationship between λ_Z and λ_A such that:

$$\left.\frac{d\lambda_A}{d\lambda_Z}\right|_{r=0} \quad = \quad -\frac{\zeta a_{rc}}{a_c a_{rc} - a_r a_{cc}} \ < 0 \ .$$

Next, assume that $\mathcal{T}^L \subset [\underline{t}_Z, \bar{t}_A)$. Then $x_d = \bar{x}_d$ and (5.1) defines an implicit function $x_c(\lambda_A)$ such that:

$$\frac{dx_c}{d\lambda_A} \quad = \quad -\frac{a_c}{u'' + \lambda_A a_{cc}} > 0 \ .$$

We thus conclude that λ_A being an increasing time function, $x_c(t)$ should also grow over time. Hence $\lambda_A(t) a_r(x_c(\lambda_A(t)), 0)$ increases over time, implying that $\nu_r(t)$ should decrease over time. Once again, we have verified that a time phase during which $x_c(t) > 0$ and $r(t) = 0$ cannot follow a time phase where both $x_c > 0$ and $r > 0$. The border condition:

$$\lambda_A a_r(x_c(\lambda_A), 0) \;=\; c_r^0 \;,$$

now defines a unique value of λ_A we denote by $\bar{\lambda}_A^L$. If $\lambda_A > \bar{\lambda}_A^L$ a pure learning policy is no more optimal.

Taking stock, we can now describe the optimal know-how acquisition policies. Denote by PR (Pure R&D) a time phase of the $\mathcal{T}^R$ type, where the know-how index increases thanks to only research efforts, by PL (Pure Learning) a time phase of the $\mathcal{T}^L$ type and by C (Combined) a time phase combining the use of research efforts and the clean coal energy technology. We have shown that x_c, x_d and r are defined as continuous functions of either both λ_Z and λ_A during the pre-ceiling phase, and that x_c and r are defined as continuous functions of λ_A during the ceiling phase if $\underline{t}_Z < \bar{t}_A$. Since $\lambda_Z(t)$ and $\lambda_A(t)$ are continuous time functions over the time interval $[0, \bar{t}_A)$, that is to the exception of the revolution time $\bar{t}_A$, we conclude that x_c, x_d and r must be continuous time functions. This implies in turn that ν_{xc} and ν_r are also continuous time functions. A transition from a PR phase to a LR phase requires an upward jump down of ν_{xc} from zero to some strictly positive level since ν_{xc} will have to decrease strictly during the LR phase. This cannot be optimal. The same argument applies to a transition from a LR phase to a PR phase, such a transition requiring an upward jump of ν_r at the transition time. It applies also to transitions from a C phase to either a PR phase or a LR phase, the first transition requiring an upward jump of ν_r and the second one an upward jump of ν_{xc}. Hence we conclude that a PR phase or a PL phase can only precede a combined C phase. Of course, it remains possible that the optimal path begins with an inactive phase, during which the economy makes no efforts at all to trigger the technological revolution.

Let us first consider the case $\underline{t}_Z < \bar{t}_A$, that is the revolution occurs only strictly after the beginning of the ceiling phase. Then the optimal know-how active acquisition policy is one of the following scenarios:

(i) A PR phase followed until $\bar{t}_A$, the revolution time;

(ii) A PL phase followed until $\bar{t}_A$;

(iii) A PR phase followed by a C phase until $\bar{t}_A$;

(iv) A PL phase followed by a C phase until $\bar{t}_A$;

(v) A C phase followed until $\bar{t}_A$.

The relevance of the preceding scenarios depends upon the models fundamentals, in particular the knowledge generation function. Let us briefly sketch the main features of these different scenarios.

Scenario 1 : Pure R&D policies

In this scenario, the research efforts are constantly increasing until the cost breakthrough. The energy implicit price is given by $c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)t}$ before $\underline{t}_Z$, growing exponentially while the use of dirty energy declines. Then the price stabilizes at $\bar{p}$ until the cost break occurs and clean energy is introduced inside the energy mix in combination with dirty energy.

Scenario 2 : Pure learning policies

Such a scenario requires a sufficiently high level of $c_r^0 > 0$ to prevent R&D to be profitable. The use of clean energy starts at time 0 and increases permanently until the technological revolution occurs. Before the ceiling phase, the aggregate energy production declines, the expansion in the use of clean energy does not compensate for the sharper decline of the use of dirty energy. After $\underline{t}_Z$, the continuous expansion of the use of clean energy makes decrease the energy price until the revolution occurs and the price stabilizes at the level $c_x + \underline{c}_s$.

Scenario 3 : Pure research then combined policies

These policies may be of two kinds, depending upon the ceiling beginning to bind during the pure R&D phase or during the combined phase. In the first case, the energy price increases up to $\bar{p}$, a level attained at the beginning of the ceiling phase. Then begins a first phase at the ceiling $[\underline{t}_Z, \bar{t}_r)$ during which only dirty coal energy is exploited. At $\bar{t}_r$, the economy begins to produce clean coal energy and the energy price decreases until $\bar{t}_A$, when the cost break occurs and the energy price stabilizes forever at the level $c_x + \underline{c}_s$. The research efforts increase over time until the technological revolution time. Thus the MRS between learning intensity and research first increases until $\bar{t}_r$ before decreasing during the time phase $[\bar{t}_r, \bar{t}_A)$.

In the second case, the clean coal energy option is introduced before the ceiling constraint begins to be binding, thus during a time phase where the energy price grows at the rate $(\rho + \alpha)$. During a first phase $[0, \bar{t}_r)$, only dirty energy is produced at a declining rate and the economy performs increasing R&D efforts. At $\bar{t}_r$, clean coal energy is introduced from a null level. Then clean coal energy production expands while the use of dirty energy decreases, the aggregate trend being a decreasing energy use until the ceiling is attained. During this second time phase $[\bar{t}_r, \underline{t}_Z)$, the research effort continues to increase while the MRS between learning and R&D decreases, implying a higher rate of growth of clean coal production than the research efforts growth rate. Then begins a first phase at the ceiling $[\underline{t}_Z, \bar{t}_A)$ until the cost revolution. Know-how accumulation accelerates also during this phase, learning expansion remaining higher than research efforts increases.

Scenario 4 : Pure learning then combined policies

As in the preceding scenario, the ceiling constraint may bind before or after the beginning of the combined phase of knowledge accumulation. In the first case, the learning process relies on a continuous expansion of the use of clean coal energy until $\underline{t}_Z$. The energy price increases exponentially during the time phase $[0, \underline{t}_Z)$, implying a declining energy consumption, the use of dirty energy being decreasing at a higher rate than the growth rate of use of clean energy. The MRS between learning and R&D declines during this time interval. Then begins a second phase $[\underline{t}_Z, \bar{t}_c)$, during which the economy does not perform R&D efforts, clean coal energy use continues to increase, the use of dirty energy is constrained at the level $\bar{x}_d$ and the energy price decreases. After this phase, the economy enter a combined regime of know-how accumulation based upon the use of clean coal energy and research activities. Such a scenario supposes that $c_r(0) = c_r^0$ be strictly positive and sufficiently high to prevent research activities before the ceiling has been attained. During the combined phase $[\bar{t}_c, \bar{t}_A)$ the energy price continues to decrease, the know-how accumulation accelerates, the MRS between learning and R&D decreases, the rate of growth of x_c being larger than the rate of growth of $r(t)$. At the end of this phase, the cost revolution occurs and the energy use stabilizes to its optimal post-revolution level.

In the second case, the optimal path is composed of a first phase below the ceiling $[0, \bar{t}_c)$ without research activity but with a combined use of clean energy at an increasing rate and dirty energy at a declining rate. The aggregate energy use decreases while the energy price increases. Then begins a second phase below the ceiling $[\bar{t}_c, \underline{t}_Z)$ where the economy accumulates know-how both from learning and R&D activity. The energy price continues to increase and the MRS between learning and R&D being decreasing, $r(t)$ increases at a lower rate than $x_c(t)$. This phase is followed by a phase at the ceiling until the cost breakdown occurs, $[\underline{t}_Z, \bar{t}_A)$. The energy price now decreases during this phase, clean coal energy use and R&D activity continue to grow until $\bar{t}_A$ and the MRS being also decreasing, clean coal use grows at a higher rate than the research effort level.

Scenario 5 : Combined Policies

In this scenario, the economy accumulates know-how by using clean coal energy and an active R&D policy right from the beginning of time. During a first phase $[0, \underline{t}_Z)$, the energy price increases, the energy supply decreases but the use of dirty energy declines while the use of clean energy increases, together with the intensity of R&D efforts. Then the ceiling is reached and the economy enters a second phase of combined learning and R&D know-how accumulation, $[\underline{t}_Z, \bar{t}_A)$ until the technological revolution occurs. The energy price now decreases. Clean coal energy use continues to grow together with intensity of research activity. The MRS between learning and R&D being decreasing, the growth rate of $x_c(t)$ is higher than the growth rate of $r(t)$.

Figure 3 illustrates the shape of the energy price path if $\underline{t}_Z < \bar{t}_A$ for all scenarios expected a pure R&D policy followed until the cost break. In this last case, the energy price grows up to $\bar{p}$, attained at $\underline{t}_Z$. Then it stays constant at this level until the cost break occurs. At $\bar{t}_A$, the energy price jumps down from the level $\bar{p}$ to the level $c_x + \underline{c}_s$, last it stays permanently at this level, clean coal energy generation beginning after $\bar{t}_A$. The proof that the energy price should jump down at $\bar{t}_A$ is presented in the next subsection.

Figure 3 shows that the energy implicit price path combines in a straightforward way the features of the energy use dynamics exposed in the preceding sections. Before the ceiling constraint begins to bind, the energy price rises exponentially at the rate $(\rho + \alpha)$, and thus the aggregate supply of energy decreases. This means that even if clean coal energy is used during the preceiling phase, its expansion over time does not compensate for the reduction of the use of dirty coal energy. During the first phase at the ceiling preceding the revolution, the use of dirty energy is constrained at the constant level $\bar{x}_d$ while the use of clean coal energy continues to expand. The result is a decreasing energy price until the revolution occurs. At the revolution time, as in the pure R&D case, the price jumps down while the use of clean energy

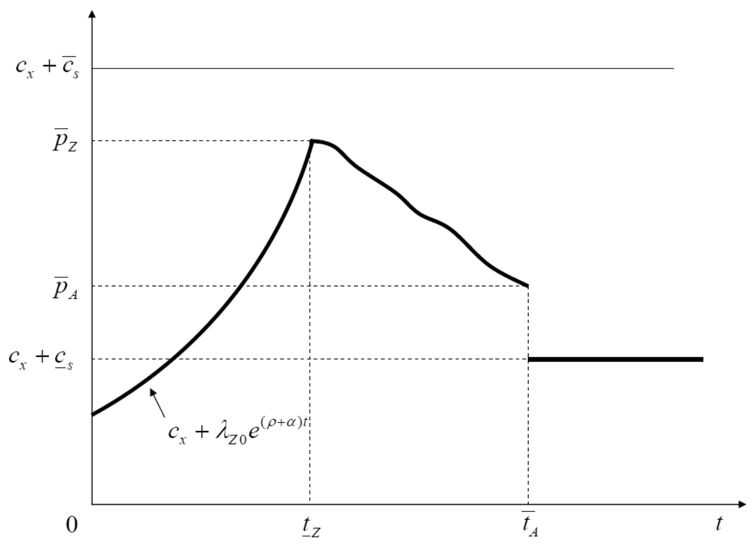

Figure 3: **Price Dynamics with Combined Learning and R&D Know-how Accumulation**

jumps up. But because of the learning effect before the revolution, the price jump is no more equal to the cost gap $\bar{c}_s - \underline{c}_s$, as was the case in the pure R&D model. The jump is reduced thanks to learning.

As in the pure learning model, the implementation of the optimal policy requires to combine a carbon tax, its rate being given by $\zeta\lambda_Z(t)$, together with a subsidy to clean energy consumption, its rate being given by $\lambda_A(t)a_c(t)$. During the pre-ceiling phase, the tax rate must increase exponentially. The subsidy level must decrease over time if clean energy generation is put in operation. During the ceiling phase preceding the revolution, $[\underline{t}_Z, \bar{t}_A)$, the carbon tax must decrease while the subsidy must increase to sustain an increased use of clean energy during this time phase.

In the case of a technological revolution triggered just when the economy reaches the ceiling, that is $\underline{t}_Z = \bar{t}_A$, the already mentioned possible sequences of phase may be optimal. The only difference of course is that the sequence happens during the pre-ceiling phase.

To make progress in the determination of the optimal scenario, we need to describe the terminal conditions at $\bar{t}_A$. This discussion will allow to set conditions for the revolution to be triggered just when attaining the ceiling and for the reverse case $\underline{t}_Z < \bar{t}_A$.

5.2 Terminal condition at the revolution time $\bar{t}_A$

Thanks to Proposition 5, we know that two cases have to be considered, the case of a revolution happening during a ceiling phase, $\underline{t}_Z = \bar{t}_A$, and the case of a revolution happening just at the time at which the ceiling is attained, $\underline{t}_Z = \bar{t}_A$.

Let us compute the transversality condition at $\bar{t}_A$, the revolution time. The continuation value after the break, $\bar{V}$, keeps the same expression as before, being independent from the device triggering the breakthrough. Denote by $h^- \equiv \lim_{t\uparrow\bar{t}_A} h(t)$ and by $h^+ \equiv \lim_{t\downarrow\bar{t}_A} h(t)$ for any time function $h(t)$. Denote also by μ_Z the multiplier associated to the constraint $\bar{t}_A \geq \underline{t}_Z$. Then the condition reads:

$$u(q^-) - c_x q^- - \bar{c}_s x_c^- - C_r(r^-) + \lambda_A^- a(x_c^-, r^-) + \mu_Z \;\; = \;\; u(q^+) - c_x q^+ - \underline{c}_s x_c^+ \; .$$

Simplifying on both sides the $c_x \bar{x}_d$ term while taking into account (5.1) and $u'(q^+) = c_x + \underline{c}_s$, we obtain:

$$\Gamma(x_c^-) - C_r(r^-) + \lambda_A^- a(x_c^-, r^-) - \lambda_A^- a_c(x_c^-, r^-) x_c^- + \mu_Z \;\; = \;\; \Gamma(x_c^+) \; .$$

Since x_c^+ is independent from the devices used to trigger the revolution, the r.h.s is independent from what happens before $\bar{t}_A$. Denote it by $\bar{\Gamma} \equiv \Gamma(x_c^+)$. Then adding and subtracting $c_r(r^-)r^-$, remembering the expression of Γ_r and taking (5.4) into account:

$$\begin{aligned}
\bar{\Gamma} - \Gamma(x_c^-) - \mu_Z \;\; &= \;\; c_r(r^-)r^- - C_r(r^-) \\
&\quad + \lambda_A \left[a(x_c^-, r^-) - a_c(x_c^-, r^-) x_c^-\right] - c_r(r^-) r^- \\
&= \;\; \Gamma_r(r^-) + \lambda_A^- \left[a(x_c^-, r^-) - a_c(x_c^-, r^-) x_c^- - a_r(x_c^-, r^-) r^-\right] \; .
\end{aligned}$$

Assume $\underline{t}_Z < \bar{t}_A$, then $\mu_Z = 0$. The *r.h.s.* is positive under our assumptions, showing that $x_c^+ > x_c^-$ and thus that the energy price should jump down at the revolution time. Figure 3 illustrates this feature of the energy price path. Furthermore, the transversality condition when $\underline{t}_Z < \bar{t}_A$ appears as an equation linking together $(x_c^-, r^-, \lambda_A^-)$ of the form:

$$\begin{aligned}\Phi(x_c^-, r^-, \lambda_A^-) &\equiv u(\bar{x}_d + x_c^-) - c_x(\bar{x}_d + x_c^-) - \bar{c}_x x_c^- - C_r(r^-) + \lambda_A^- a(x_c^-, r^-) \\ &= \rho \bar{V} \ .\end{aligned}$$

Differentiating we obtain:

$$d\Phi = \left[u'(q^-) - c_x - \bar{c}_s - \lambda_A a_c\right] dx_c + \left[\lambda_A a_r - c_r\right] dr - d\lambda_A^- a(x_c^-, r^-) \ .$$

Taking (5.1) and (5.4) into account, $d\Phi = d\lambda_A^- a(x_c^-, r^-) > 0$ if $x_c^- > 0$ and $r^- > 0$. The same applies if either $x_c^- > 0, r^- = 0$ or either $x_c^- = 0, r^- > 0$. Hence $\Phi = \rho\bar{V}$ defines a unique value of $\bar{\lambda}_A$ at $\bar{t}_A$, a value we denote by $\bar{\lambda}_A$. Furthermore, since $d\Phi/d\lambda_A > 0$, $\bar{\lambda}_A$ is an increasing function of $\bar{V}$, the continuation value. Let $\bar{\lambda}_A(\bar{V})$ be the corresponding implicit function.

Next, consider the case $\underline{t}_Z = \bar{t}_A$. As noticed before, the technological revolution may be triggered through any of the possible sequence of phases already described. It results that the revolution may occur from five possible paths before $\underline{t}_Z$. To these five possible paths correspond five possible terminal levels of λ_A, a vector $(\underline{\lambda}_A^R, \underline{\lambda}_A^L, \underline{\lambda}_A^C, \underline{\lambda}_A^{RC}, \underline{\lambda}_A^{LC})$. The conditions allowing to determine this vector are presented in Appendix A.5.

Under the constraint $\underline{t}_Z = \bar{t}_A$, only one of these scenarios is an optimum. To identify the optimal path, remark that the Hamilton-Bellman-Jacobi equation defines the value function from time 0 as $W = \rho H^*(0)$, where $H^*(0)$ is the optimized hamiltonian function at time 0. Differentiating and remembering that $\partial H^*/\partial x_d = \partial H^*/\partial x_c = \partial H^*/\partial r = 0$ we obtain:

$$\frac{dW}{\rho} = -d\lambda_{Z0}\dot{Z}(0) + d\lambda_{A0}\dot{A}(0) \ .$$

This shows that the value function is a decreasing function of λ_Z and an increasing function of λ_A. Since $\lambda_Z(t)$ and $\lambda_A(t)$ are defined as exponentially increasing time functions at two different rates $(\rho + \alpha)$ and ρ, the $\{\lambda_Z(t), \lambda_A(t)$ trajectories never cross themselves during a pre-ceiling phase. From the fact that $\lambda_Z(\underline{t}_Z) = (\underline{c}_s - c_x)/\zeta$ in all scenarios, we conclude that the optimal scenario is the scenario giving the higher value of λ_A at $\underline{t}_Z$. Let $\underline{\lambda}_A \equiv \max((\underline{\lambda}_A^R, \underline{\lambda}_A^L, \underline{\lambda}_A^C, \underline{\lambda}_A^{RC}, \underline{\lambda}_A^{LC})$.

Since $\bar{\lambda}_A(\bar{V})$ is an increasing function of $\bar{V}$, it appears that the constraint $\underline{t}_Z \leq \bar{t}_A$ does not bind if $\bar{\lambda}_A \leq \underline{\lambda}_A$ while it is binding in the reverse case $\underline{\lambda}_A < \bar{\lambda}_A$.

We are now in position of examining the relevance of the previously sketched scenarios. We proceed by considering the dual space $(\lambda_{Z0}, \lambda_{A0})$. In the next section we describe the optimal policy in the case $\bar{\lambda}_A < \underline{\lambda}_A$, that is we consider the optimal scenario in a situation where $\underline{t}_Z < \bar{t}_A$, the technological revolution occurs only after the beginning of the ceiling phase. Then we study the case of a technological revolution occurring just when the ceiling is attained.

5.3 Optimal policies triggering the revolution during the ceiling phase

To the terminal value $\bar{\lambda}_A$ corresponds a unique value of λ_Z, defined implicitly by: $\zeta\lambda_Z = \bar{c}_s - a_c(x_c(\bar{\lambda}_A), r(\bar{\lambda}_A))$. Let $\underline{\lambda}_Z$ be this value. Remark that because of the price jump at $\bar{t}_A$, λ_Z will also jump down from $\underline{\lambda}_Z$ to the level $(\underline{c}_s - c_x)/\zeta$ after the break. In a situation where the cost breakthrough occurs only after the ceiling has been attained, we have shown previously that $\bar{\lambda}_A < \underline{\lambda}_A$.

To identify the domain of validity of the different scenarios in the (λ_Z, λ_A) plane, we now take into account the definitions of the $x_c = 0$ and $r = 0$ borders. The border $x_c = 0$ defines an implicit relation between λ_Z and λ_A we denote by $\hat{\lambda}_A^x(\lambda_Z)$. Furthermore:

$$\frac{d\hat{\lambda}_A^x}{d\lambda_Z} = -\frac{\zeta(c_r' - \lambda_A a_{rr})}{a_r a_{cr}\lambda_A + a_c(c_r' - \lambda_A a_{rr})} < 0$$

On the other hand, the border $r = 0$ defines another implicit relation between λ_Z and λ_A, a relation we denote by $\hat{\lambda}_A^r(\lambda_Z)$ and:

$$\frac{d\hat{\lambda}_A^r}{d\lambda_Z} = -\frac{\zeta a_{cr}}{a_r a_{cc} - a_c a_{cr}} < 0$$

The curves $\hat{\lambda}_A^x(\lambda_Z)$ and $\hat{\lambda}_A^r(\lambda_Z)$ cross themselves at $(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)$ solution of:

$$\begin{aligned} \zeta\lambda_Z &= \bar{c}_s - \lambda_A a_c(0,0) \\ \lambda_A a_r(0,0) &= c_r^0 \end{aligned}$$

Note that is $c_x + \bar{c}_s < \bar{p}$, $\hat{\lambda}_Z^0 < \bar{c}_s/\zeta$ implies that $\hat{\lambda}_Z^0 < (\bar{p} - c_x)/\zeta \equiv \bar{\lambda}_Z$. Differentiating around the point $(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)$, it is easily verified that:

$$\left|\frac{d\hat{\lambda}_A^x}{d\lambda_Z}\right|_{(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)} > \left|\frac{d\hat{\lambda}_A^r}{d\lambda_Z}\right|_{(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)}.$$

Three possibilities have to be considered:

(i) Either $\hat{\lambda}_Z^0 < (\underline{c}_s - c_x)/\zeta$ and the point $(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)$ is located to the left of the vertical $\lambda_Z = (\underline{c}_s - c_x)/\zeta$.

(ii) Either $(\underline{c}_s - c_x)/\zeta < \lambda_Z < (\bar{p} - c_x)/\zeta$ and the point $(\hat{\lambda}_Z^0, \lambda_A^0)$ is located in between the vertical borders $\lambda_Z = (\underline{c}_s - c_x)/\zeta$ and the vertical border $(\bar{p} - c_x)/\zeta$.

(iii) Either $\hat{\lambda}_Z^0 > (\bar{p} - c_x)/\zeta$ and the point $(\hat{\lambda}_Z^0, \hat{\lambda}_A^0)$ is located to the right of the border $(\bar{p} - c_x)/\zeta$.

Note that the implicit energy price is at most equal to $\bar{p}$ in any optimal scenario. This means that $\lambda_Z(t) < (\bar{p} - c_x)/\zeta \equiv \bar{\lambda}_Z$. The vertical $\lambda_Z = \bar{\lambda}_Z$ defines the upper border of possible value of λ_Z in all optimal scenario. The Figure 4 illustrates the three possible cases.

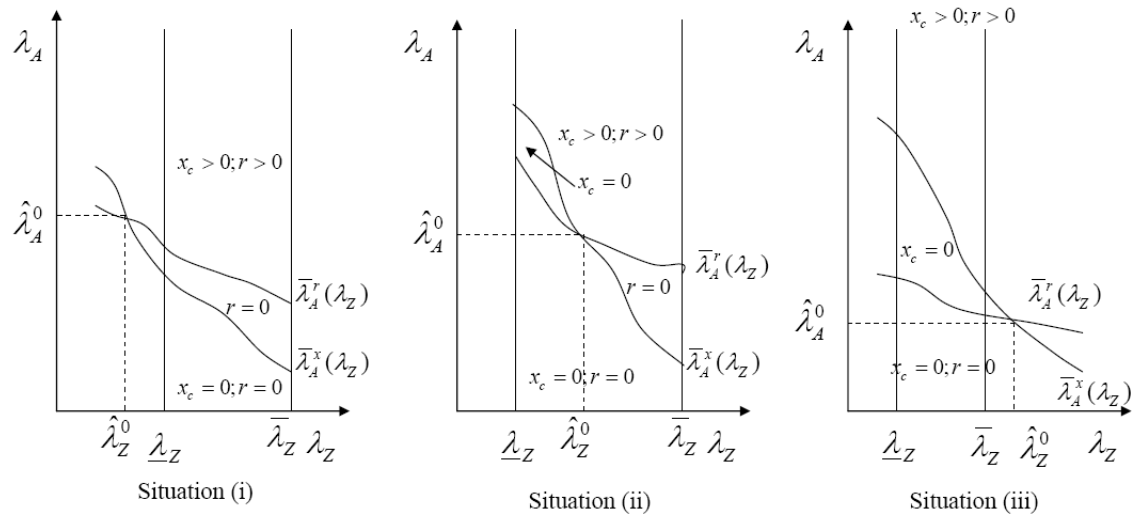

Figure 4: **Activity Constraints in the (λ_Z, λ_A) Plane**

Optimal combined policies

Let us first consider the combined policies. The condition $\bar{x}_d = x_d(\lambda_Z, \lambda_A)$ defines an implicit relation between λ_Z and λ_A we denote by $\bar{\lambda}^c_A(\lambda_Z)$. Taking (5.7), (5.8) into account, it is immediately verified that: $d\bar{\lambda}^c_A(\lambda_Z)/d\lambda_Z < 0$. In the case $\underline{t}_Z < \bar{t}_A$, $\lambda_Z \in [\underline{\lambda}_Z, \bar{\lambda}_Z]$ defines the relevant domain of possible values of λ_Z. On one hand, $\bar{\lambda}^c_A(\underline{\lambda}_Z) = \bar{\lambda}_A$ in a combined scenario and on the other hand we denote by $\bar{\lambda}^c_A \equiv \bar{\lambda}^c_A(\bar{\lambda}_Z)$.

To a combined policy corresponds a $\{\lambda_Z(t), \lambda_A(t)\}$ trajectory initiating below the curve $\bar{\lambda}^c_A(\lambda_Z)$ at some point $(\lambda_{Z0}, \lambda_{A0})$. Before $\underline{t}_Z$, the trajectory is defined by $(\lambda_{Z0}e^{(\rho+\alpha)t}, \ \lambda_{A0}e^{\rho t})$. hence $\{\lambda_Z(t), \lambda_A(t)\}$ moves in the north east direction in the dual plane. At $\underline{t}_Z$, the trajectory hits the border $\bar{\lambda}^c_A(\lambda_Z)$. Then it follows this curve until $\bar{t}_A$ is reached, that is at the point $\lambda_Z(\bar{t}_A) = \underline{\lambda}_Z$, $\lambda_A(\bar{t}_A) = \bar{\lambda}_A$. The following Figure 5 illustrates this construction.

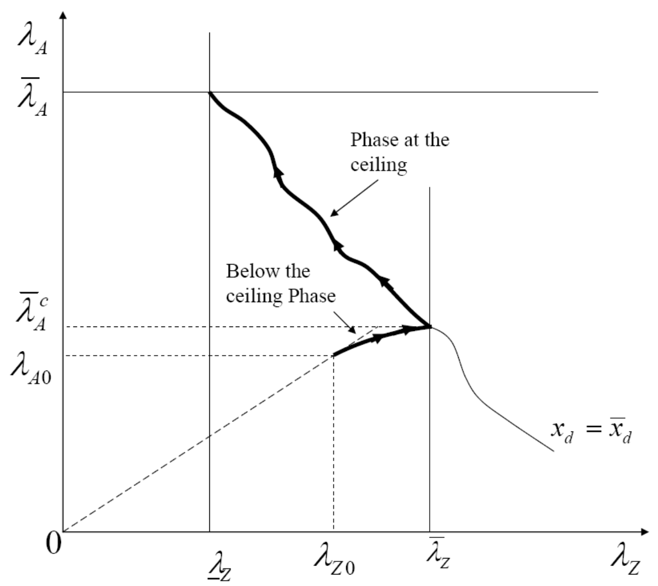

Figure 5: **Combined Policies**

The combined scenario is optimal from any point $(\lambda_{Z0}, \lambda_{A0})$ located above the border $\max(\hat{\lambda}^x_A(\lambda_Z), \hat{\lambda}^r_A(\lambda_Z))$, that is the region where both $x_c > 0$ and $r > 0$.

Initial R&D optimal policies

Turn to the scenarios involving at least initially pure R&D policies. We call these policies *Initial R&D Policies*. Within a pure R&D tail, the ceiling constraint is defined by: $\zeta\lambda_Z = \bar{p} - c_x$, that is the vertical line $\lambda_Z = \bar{\lambda}_Z$. $\bar{\lambda}^c_A(\lambda_Z)$, the curve defining the ceiling constraint in a combined policy cuts the vertical $\lambda_Z = \bar{\lambda}_Z$ at a point where $x_c = 0$. Thus the curves $\bar{\lambda}^c_A(\lambda_Z)$ and $\hat{\lambda}^x_A(\lambda_Z)$ intersect themselves at $\lambda_Z = \bar{\lambda}_Z$, that is along the vertical $\lambda_Z = \bar{\lambda}_Z$ at the point $(\bar{\lambda}^c_A, \bar{\lambda}_Z)$. Furthermore, computing the derivatives $d\bar{\lambda}^c_A/d\lambda_Z$ and $d\hat{\lambda}^x_A/d\lambda_Z$ around the point $(\bar{\lambda}_Z, \bar{\lambda}^c_A)$ we observe that the ceiling border for a combined path is located above the $x_c = 0$ locus.

Next consider the situation (i) depicted in Figure 4. The curve $\bar{\lambda}^r_A(\lambda_Z)$ is located above the curve $\bar{\lambda}^c_A(\lambda_Z)$ inside the whole domain $\underline{\lambda}_Z < \lambda_Z < \bar{\lambda}_Z$. Hence the curve $\bar{\lambda}^r_A(\lambda_Z)$ intersects the vertical $\lambda_Z = \bar{\lambda}_Z$ above $(\bar{\lambda}_Z, \bar{\lambda}^c_A)$. It results that it is impossible to follow an initial R&D policy in the relevant domain. In the situation (ii), an initial R&D policy is the only optimal scenario from any point $(\lambda_{Z0}, \lambda_{A0})$ located above the curve $\bar{\lambda}^x_A$ and to the left of the vertical $\lambda_Z = \hat{\lambda}^0_Z$. In the situation (iii), the border $x_c = 0$, that is the curve $\bar{\lambda}^x_A(\lambda_Z)$, is located above the $r = 0$ border, that is the curve $\bar{\lambda}^r_A(\lambda_Z)$ in the relevant domain $[\underline{\lambda}_Z, \bar{\lambda}_Z]$. Hence an initial R&D policy is the only optimal scenario in this situation.

Let us begin by describing the optimal scenario in the situation (iii). Consider the $\{\lambda_Z, \lambda_A\}$ trajectory where $\lambda_Z(t) = \lambda_{Z0}e^{(\rho+\alpha)t}$, $\lambda_A(t) = \lambda_{A0}e^{\rho t}$ going through the point $(\bar{\lambda}_Z, \bar{\lambda}^c_A)$. Denote by (S_r) this separating curve.

To complete the discussion we need to take into account the terminal condition. If $\bar{\lambda}_A < \bar{\lambda}^c_A$, the optimal scenario is a type 1 scenario. The optimal $\{\lambda_Z(t), \lambda_A(t)\}$ trajectory is located below the separatrix (S_r). During the pre-ceiling phase, the trajectory moves in the north east direction until the ceiling is attained. At $\underline{t}_Z$, the trajectory hits the vertical $\lambda_Z = \bar{\lambda}_Z$. Then it moves upward along this vertical until $\bar{\lambda}_A$ is reached at $\bar{t}_A$.

If $\bar{\lambda}_A > \bar{\lambda}^c_A$, the optimal policy is a type 3 scenario. If $(\lambda_{Z0}, \lambda_{A0})$ is located below the separating curve (S_r), then the economy hits the ceiling constraint while performing only R&D activity, that is $\underline{t}_Z < \bar{t}_r$. Then the trajectory follows the vertical $\bar{\lambda}_Z$ up to $\bar{\lambda}^c_A$. Next the optimal trajectory follows the curve corresponding to the ceiling constraint in a combined path, performing both research activity and clean coal energy production. If $(\lambda_{Z0}, \lambda_{A0})$ is located above the separating curve (S_r), then the economy moves from a pure R&D regime to a combined regime before attaining the ceiling, that is $\bar{t}_r < \underline{t}_Z$. After crossing the $x_c = 0$ border, the optimal trajectory enters the combined regime zone until the ceiling border is reached. This border is then followed up to $(\underline{t}_Z, \bar{\lambda}_A)$. The Figure 6 illustrates this construction.

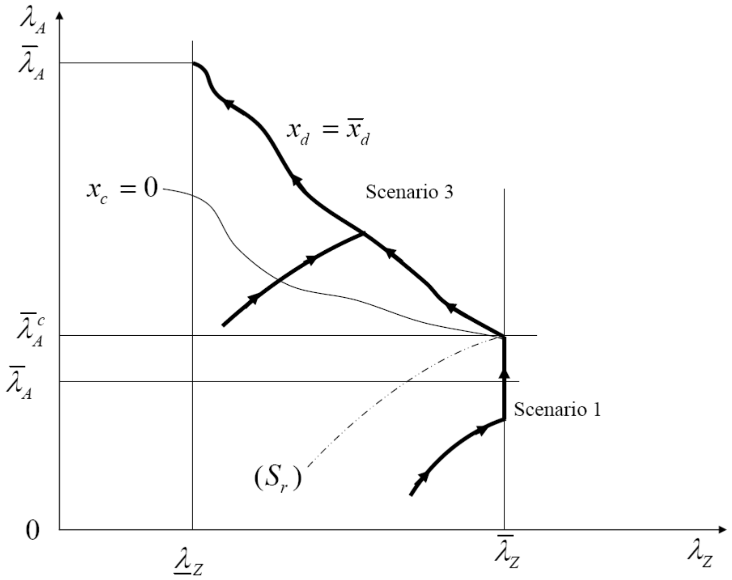

Figure 6: **Initial R&D Policies**

In the situation (ii) the initial research optimal policies correspond to a type 3 scenario. The economy starts performing only R&D before introducing the exploitation of clean coal before the ceiling is attained. Then the economy performs both R&D and clean coal energy production until the technological revolution. In the situation (i), as noticed before, an initial R&D policy is not optimal.

Initial learning policies

The scenarios involving pure learning tails follow the same principle of construction. We call these policies *Initial Learning Policies.* The ceiling constraint along a pure learning tail defines an implicit relation between λ_Z and λ_A, we denote by $\bar{\lambda}_A^L(\lambda_Z)$ and such that:

$$\frac{d\bar{\lambda}_A^L}{d\lambda_Z} = -\frac{\zeta(u'' + \lambda_A a_{cc})}{u'' a_c} < 0$$

It is easily checked that the curve $\bar{\lambda}_A^L(\lambda_Z)$ and the curve $\bar{\lambda}_A^c(\lambda_Z)$, corresponding respectively to the ceiling constraint in a pure learning regime and to the ceiling constraint in a combined R&D and learning regime, cross themselves along the locus $r = 0$, that is the curve $\hat{\lambda}_A^r(\lambda_Z)$. Let $(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)$ be the intersection point of these three curves. It may also be verified that:

$$\left|\frac{d\hat{\lambda}_A^r}{d\lambda_Z}\right|_{(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)} < \left|\frac{d\bar{\lambda}_A^c}{d\lambda_Z}\right|_{(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)} < \left|\frac{d\bar{\lambda}_A^L}{d\lambda_Z}\right|_{(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)}$$

In the situation (i), an initial learning policy is the only optimal policy. In the situation (ii), it is the optimal policy for $\lambda_{Z0} > \hat{\lambda}_Z^0$ and $\lambda_{A0} > \hat{\lambda}_A^r(\lambda_{Z0})$. In the situation (iii), it cannot be an optimal policy.

Let us begin by the situation (i). We have to consider the implications of the terminal condition. If $\bar{\lambda}_A < \bar{\lambda}_A^L$, then the optimal policy is a type 2 scenario. In the reverse case, it is a type 4 scenario. One can define a separating curve (S_c) corresponding to the $\{\lambda_Z, \lambda_A\}$ trajectory going through $(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)$, the intersection point between the three curves. Initiated below the separating curve (S_c), the optimal path hits the ceiling border for a pure learning tail $\bar{\lambda}_A^L$ whence the ceiling is attained. Then it follows this curve until the point $(\bar{\lambda}_Z^L, \bar{\lambda}_A^L)$ is attained, a point at which a combined phase at the ceiling begins. This combined phase identifies in the (λ_Z, λ_A) plane to a motion of the $\{\lambda_Z(t), \lambda_A(t)\}$ trajectory along the curve $\bar{\lambda}_A^c(\lambda_Z)$ until the cost breakthrough. Hence when starting from below the separatrix (S_c), the economy begins to perform research activity only strictly after the ceiling has been attained, that is $\underline{t}_Z < \bar{t}_c$ in this scenario.

Initiated above (S_c), the optimal path crosses the $r = 0$ border, that is the curve $\hat{\lambda}_A^r(\lambda_Z)$, before the ceiling constraint begins to be binding. Then it enters a region of combined learning and R&D accumulation of know-how policies until it hits the ceiling curve for such combined policies. Hence $\bar{t}_c < \underline{t}_Z$ in this scenario. Last the optimal $\{\lambda_Z(t), \lambda_A(t)\}$ trajectory follows the curve $\bar{\lambda}_A^c(\lambda_Z)$ until $(\underline{\lambda}_Z, \bar{\lambda}_A)$ is attained at $\bar{t}_A$. The Figure 7 illustrates the construction.

In the situation (ii) only the type 4 scenario may be valid in a case where the combined process of know-how accumulation through both learning and R&D begins before the ceiling constraint is binding, that is $\bar{t}_c < \underline{t}_Z$. Last, as noticed before, an initial learning policy is never optimal in the situation (iii).

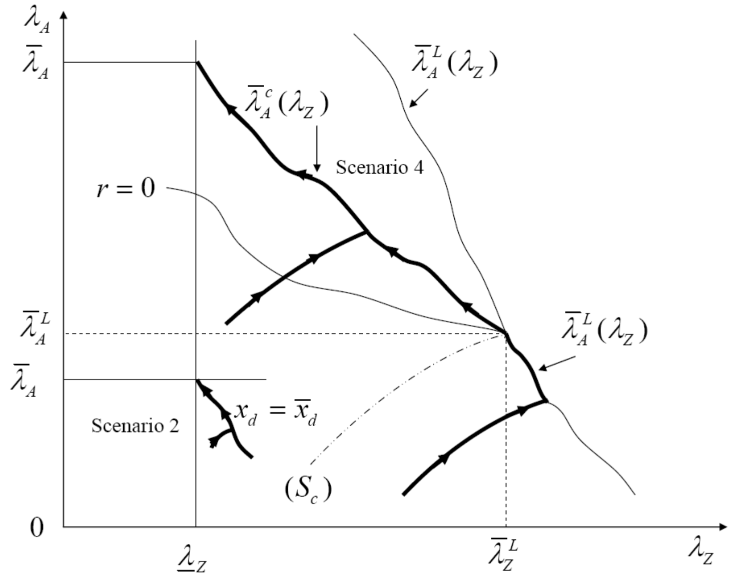

Figure 7: **Initial Learning Policies**

5.4 Optimal policies triggering the revolution when the ceiling is attained

In the situation (i), a pure research policy is not optimal. The optimal policy is the scenario of the possible type 2-5 giving the highest level of λ_A when the vertical $\lambda_z = (\underline{c}_s - c_x)/\zeta$ is attained. In the situations (ii) and (iii), a pure learning policy is excluded, the optimal scenario among the types 1, 3,4, 5 is the one giving the highest level of λ_A at $\underline{t}_Z$. Note that these active policies may be preceded by a time phase without any effort to trigger the technological revolution. The same applies to scenarios where $\underline{t}_Z < \bar{t}_A$.

5.5 Identification of the optimal policies

The previous description shows how to build the optimal policies in the dual space from some given $(\lambda_{Z0}, \lambda_{A0})$. The initial level of the costate variables are themselves endogenously determined by the models fundamentals and the initial conditions. We thus modify slightly the original approach by assuming that the initial know-how is some $A^0 \geq 0$ and we proceed to the identification of the optimal policies in the (Z^0, A^0) plane. Four main variables have to be determined, the initial levels of the costate variables, λ_{Z0} and λ_{A0}, together with the ceiling attainment time $\underline{t}_Z$ and the revolution time $\bar{t}_A$. Appendix A.6 shows that in the case $\underline{t}_Z < \bar{t}_A$, these variables are uniquely determined by a system of four conditions: the ceiling attainment condition $Z(\underline{t}_Z) = \bar{Z}$, the know-how target attainment condition $A(\bar{t}_A) = \bar{A} - A^0$, the continuity condition at $\underline{t}_Z$, $\zeta\lambda_Z(\underline{t}_Z) = \bar{c}_s - a_c(\underline{t}_Z)\lambda_A(\underline{t}_Z)$ and the terminal condition $\lambda_A(\bar{t}_A) = \bar{\lambda}_A(\bar{V})$.

It results that for some initial pair (Z^0, A^0) it is possible to determine an optimal policy starting from $(\lambda_{Z0}, \lambda_{A0}$ in the dual space. Next we build the equivalent of the border conditions in the primal space (Z^0, A^0). The $x_c = 0$ border is equivalent to $x_c(\lambda_Z(Z^0, A^0), \lambda_A(Z^0, A^0)) = 0$. Denote this condition as $x_c(A^0, Z^0) = 0$. It is easily checked that:

$$\frac{\partial \lambda_{Z0}}{\partial Z^0} > 0 \quad ; \quad \frac{\partial \lambda_{Z0}}{\partial A^0} < 0$$
$$\frac{\partial \lambda_{A0}}{\partial Z^0} > 0 \quad ; \quad \frac{\partial \lambda_{A0}}{\partial A^0} > 0$$

It is next possible to show that the condition $x_c(Z^0, A^0) = 0$ defines an implicit decreasing relation between Z^0 and A^0, a relation we denote by $A^L(Z)$. Above this curve, the optimal scenario is a combined policy while below it, the learning process is delayed in time.

The constraint $\underline{t}_Z \leq \bar{t}_A$ defines another frontier in the (A, Z) plane. This frontier intersects the point $(\bar{A}, \bar{Z})$ and separates initial (A^0, Z^0) pairs whose corresponding optimal scenarios trigger the technological revolution just when the ceiling is attained or strictly after the ceiling phase. Let $Z^0(A)$ be the corresponding curve, describing an increasing relation between A and Z.

The border $r = 0$ defines another decreasing relation between A and Z implicitly defined as: $\lambda_A(A, Z)a_r(x_c(A, Z)) = c_r^0$. Denote by $A^R(Z)$ the corresponding curve. Initial pairs (A^0, Z^0) located below this curve result in initial learning policies without R&D. The curves $A^L(Z)$ and $A^R(Z)$ cross themselves at $(\hat{A}^0, \hat{Z}^0)$. The construction is first illustrated at the Figure 8 for scenarios where $\underline{t}_Z < \bar{t}_A$.

The Figure illustrate two possible scenarios. The trajectory pictured at the left of the graph, the trajectory (I) is an initial R&D policy. Starting

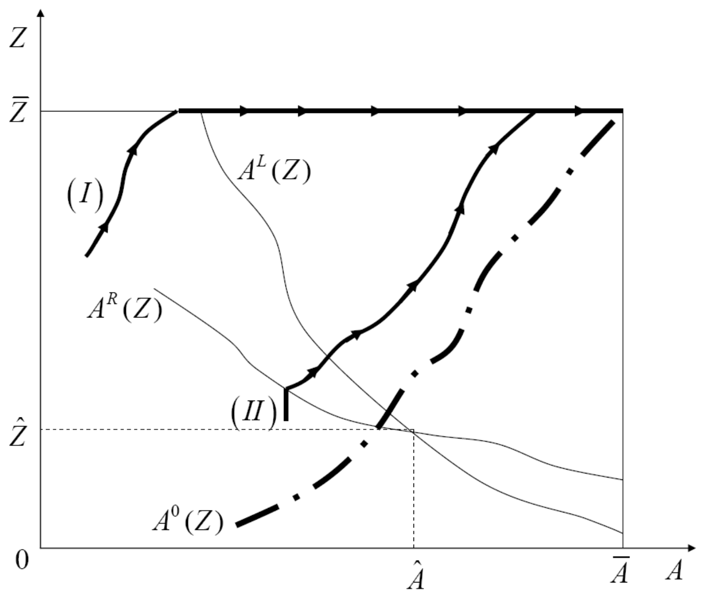

Figure 8: **Optimal Policies in the** (A, Z) **Plane**

from a given (A^0, Z^0) the trajectory hits the $\bar{Z}$ ceiling constraint at a level $A < A^L(\bar{Z})$. Hence the economy encounters the ceiling constraint while performing only R&D efforts. Then the system moves along the horizontal $\bar{Z}$. The $x_c = 0$ border is attained and then begins a phase at the ceiling combining both learning and R&D to trigger the revolution. The process ends when the point $\bar{A}, \bar{Z})$ is attained.

The second trajectory, the trajectory (II), starts with an initial phase without research or learning efforts. Thus it moves as a vertical, A being constant and Z being increasing. Then it enters a phase during which the economy performs R&D activity without using clean coal energy and thus without learning. The third phase is a combined phase below the ceiling. Then the ceiling $\bar{Z}$ is attained and the economy continues to accumulate know-how in a combined way until the technological revolution occurs.

6 Conclusion

Since Goulder, Matthai (2000), it is frequently advanced in the economic literature that carbon pricing policies should not be too stringent in order for R&D to have sufficient time to select and develop better abatement technological options. The present analysis invites to reconsider seriously such statements. Instead of an incremental process of technical improvement, we have modeled technical change as a drastic process able to trigger a technological revolution in the form of an abrupt cost break in pollution abatement, provided that a sufficiently high level of know-how has been previously accumulated. This framework allows for a much clearer view of the effects of an environmental policy upon the trend of efforts to trigger a technological improvement in abatement technologies. The Goulder-Matthai analysis proceeded by contrasting situations where technical progress resulted from learning-by-doing in pollution abatement techniques from situations where technical advances could be obtained only through dedicated R&D efforts. We have first followed this approach by studying the polar cases of a pure learning induced technological revolution and a pure R&D induced breakthrough.

When the revolution is triggered by learning-by-doing, the optimal policy implementation requires to combine a price upon carbon emissions and a subsidy to clean energy generation. This subsidy must grow over time during the learning period preceding the technological revolution, inducing a permanent rise of the use of clean energy. The optimal carbon tax should increase before the beginning of the learning process and decrease afterwards. We show also that a stricter environmental standard, here modeled in terms of a critical atmospheric carbon concentration not to be crossed over, has the effect of increasing the use of abatement technologies before the technical breakthrough, resulting in an earlier revolution. However, this does not mean that the optimal carbon tax corresponding to a stricter atmospheric concentration mandate needs to be increased. A stricter mandate has an ambiguous effect over the carbon tax level before the beginning of the clean energy generation phase and reduces this level during the learning phase.

In a R&D induced technological revolution framework, the optimal policy implementation no more requires specific subsidies, an optimal carbon price being a sufficient tool to induce the optimal level of R&D efforts. Under the reasonable assumption of increasing and convex costs of research, discounting favors delaying the R&D efforts, resulting in an increasing time pattern for such efforts until the technological breakthrough occurs. In all cases, it is not optimal to trigger the revolution strictly before the atmospheric concentration constraint begins to be binding. The optimal carbon tax should rise until the carbon concentration ceiling has been attained and then should be maintained at a constant level before jumping down at the technological revolution time. As for the learning induced technical revolution, a stricter environmental standard spurs more R&D efforts from the carbon abatement industry and reduces the time delay before the cost breakthrough. This is reminiscent of the Porter hypothesis.

However the initial cost level of the abatement technology has ambiguous effects upon the intensity of R&D efforts. This was not the case in the learning induced technical break. A higher initial pollution abatement cost reduces the use of clean energy before the revolution, slowing down the learning process and delaying the breakthrough. In a R&D induced technical break context, a higher initial pollution abatement cost widens the cost gap that can be achieved thanks to R&D, an incentive to increase R&D efforts. On the other hand, it also increases the cost of using clean energy before the break, an incentive to reduce the research efforts, in order to diminish the total costs of the energy policy.

A main drawback of the Goulder-Matthai analysis is that the pure R&D and the pure learning-by-doing technical change models are not really comparable. These extreme cases describe situations where the economy is constrained to rely upon only one of these devices to achieve a technological improvement of the pollution abatement technologies. A correct account of the effects of an environmental policy upon induced technical progress in abatement technologies requires a framework where both R&D and learning can contribute to technology advances. We thus turn to the study of such a combined process. We show that the R&D efforts should permanently increase before the technical revolution. This is a straightforward consequence of discounting and our increasing marginal cost of research assumption. The use of clean energy should also increase over time, meaning an accelerating learning process.

Even under the assumption of constant average and marginal costs of producing clean energy, it may now be the case that clean coal production starts before the atmospheric carbon concentration constraint begins to bind. But this has no qualitative consequences over the optimal time profile of the carbon price. The carbon price must increase before the atmospheric ceiling constraint is attained and decrease afterwards until the technological revolution occurs. As in the pure learning case, the carbon price tool has to be completed by a subsidy to the consumption of clean energy. The subsidy level rises all along the pre-revolution phase of clean energy production.

Concerning the priority that may be given to research with respect to learning in generating technical advances, we show that the growth rate of use of clean energy should be higher than the growth rate of research efforts. In a drastic technical progress framework, research activity bears only costs before the revolution, the prize in terms of cost cut being ripped only at the end of the research process. This is not the case for learning, since the use of the abatement technology, even at the high pre-revolution cost level, allows to generate some positive surplus. The consequence is that the economy gives more an more weight to learning with respect to R&D in achieving the technological breakthrough, independently of the respective marginal contributions of learning and research to the accumulation of know-how.

Technical progress paths combining research and learning are not the only optimal ones. We show also that an optimal policy may involve an initial period of only R&D activity before launching the use of the pollution abatement technology or an initial period based only upon learning, research being too costly to be justified until the time before the revolution be sufficiently short.

This work may be extended in several directions. The first one is to take explicitly into account the scarcity of fossil fuels. We assume an infinite supply of such resources, an assumption frequently made in the relevant literature. However fuel scarcity should result in Hotelling effects, affecting both the timing of the environmental policy and the timing of the technological investment policy before the breakthrough. The drastic technical progress framework is useful to obtain clear cut results concerning the relationships between an environmental policy and a technical development policy. It appears interesting to compare its conclusions with the results derived from incremental technical progress models. This should allow to shed some light on the various puzzles which have been identified in this literature. We focus primarily upon technical progress in pollution abatement technologies, but the analysis could be extended to technological competition between abatement techniques and clean energy generation process, like solar energy production for example.

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APPENDIX

A.1 Appendix 1: Proof that pure dirty and combined dirty and clean energy scenarios may be welfare equivalent

In the scenario under consideration, $p(\underline{t}_Z) = \bar{p}$. By a standard markovian argument nothing is changed in the scenarios comparison by assuming that $Z(0) = \bar{Z}$ and $\underline{t}_Z = 0$. Then the present value at time 0 of a policy using only dirty coal energy generation over $[0, \infty)$ is given by:

$$V_d = \frac{1}{\rho} \left[u(\bar{x}_d) - c_x \bar{x}_d \right]$$

The present value at 0 of a policy using both dirty and clean coal energy from some time $\underline{t}_c \geq 0$ is given by:

$$\begin{aligned} V_c &= \left[u(\bar{x}_d) - c_x \bar{x}_d \right] \left(\frac{1 - e^{-\rho \underline{t}_c}}{\rho} \right) \\ &+ \int_{\underline{t}_c}^{\bar{t}_A} \left[u(\bar{x}_d + x_c(t)) - c_x(\bar{x}_d + x_c(t)) - \bar{c}_s x_c(t) \right] e^{-\rho t} dt + \bar{V} e^{-\rho \bar{t}_A} \ . \end{aligned}$$

Let:

$$\int_{\underline{t}_c}^{\bar{t}_A} \left[u(\bar{x}_d + x_c(t)) - c_x(\bar{x}_d + x_c(t)) - \bar{c}_s x_c(t) \right] e^{-\rho t} dt \equiv \int_{\underline{t}_c}^{\bar{t}_A} \Phi(t) e^{-\rho t} dt \equiv I \ .$$

Integrating by parts:

$$I = - \left. \frac{\Phi(t) e^{-\rho t}}{\rho} \right|_{\underline{t}_c}^{\bar{t}_A} + \frac{1}{\rho} \int_{\underline{t}_c}^{\bar{t}_A} \dot{\Phi}(t) e^{-\rho t} dt \ .$$

Taking (3.1) into account, it is easily checked that:

$$\dot{\Phi}(t) = \left[u'(q(t)) - c_x - \bar{c}_s \right] \dot{x}_c(t) = -\lambda_{A0} \dot{x}_c(t) e^{\rho t} \ .$$

Remembering that $x_c(\underline{t}_c) = 0$ while $x_c(\bar{t}_A) = \underline{x}_c$ and making use of the previously computed expression of $\dot{\Phi}(t)$:

$$I = \left[u(\bar{x}_d) - c_x \bar{x}_d \right] \frac{e^{-\rho \underline{t}_c}}{\rho} - \left[u(\bar{x}_d + \underline{x}_c) - c_x(\bar{x}_d + \underline{x}_c) - \bar{c}_s \underline{x}_c \right] \frac{e^{-\rho \bar{t}_A}}{\rho} - \frac{\lambda_{A0} \underline{x}_c}{\rho} \ .$$

Remembering the expression of $\bar{V}$, V_c simplifies to:

$$V_c \;=\; \frac{1}{\rho}\left[u(\bar{x}_d) - c_x\bar{x}_d\right] + \frac{1}{\rho}\left[(\bar{c}_s - \underline{c}_s)e^{-\rho\bar{t}_A} - \lambda_{A0}\right]\underline{x}_c \;.$$

Since $\lambda_{A0}e^{\rho\bar{t}_A} = \bar{c}_s - \underline{c}_s$ through the transversality condition at $\bar{t}_A$, we conclude that:

$$V_c \;=\; \frac{1}{\rho}\left[u(\bar{x}_d) - c_x\bar{x}_d\right] \;=\; V_d$$

The society is indifferent between sticking to the sole use of dirty coal energy once the ceiling constraint begins to be binding or follow some combined policy of dirty and clean coal energy generation started at any moment after the beginning of the ceiling phase.

A.2 Appendix 2: Comparative dynamics in the pure learning case

Denote by:

$$\begin{aligned}
I_Z^Z \;&\equiv\; -\int_0^{\underline{t}_Z} \frac{e^{(\rho+2\alpha)t}}{u''(q(t))}dt \;>\; 0 \\
I_A \;&\equiv\; -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{e^{\rho t}}{u''(q(t))}dt \;>\; 0 \\
J_A^c \;&\equiv\; -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{dt}{u''(q(t))} \;>\; 0
\end{aligned}$$

$$x_c^Z \equiv x_c(\underline{t}_Z) \quad ; \quad x_c^A \equiv x_c(\bar{t}_A)$$

$$T_A \equiv \bar{t}_A - \underline{t}_Z \quad ; \quad \pi_Z \equiv \zeta(\rho+\alpha)\lambda_{Z0}e^{\alpha\underline{t}_Z} + \rho\lambda_{A0}$$

Then after linearizing the set of conditions defining $(\lambda_{Z0}, \lambda_{A0}, \underline{t}_Z, \bar{t}_A)$, we get the following system in matrix form:

$$\begin{bmatrix} -\zeta^2 I_Z^Z & 0 & \zeta x_c^Z e^{\alpha \underline{t}_Z} & 0 \\ 0 & I_A & -x_c^Z & x_c^A \\ \zeta e^{\alpha \underline{t}_Z} & 1 & \pi_Z & 0 \\ 0 & 1 & 0 & \rho\lambda_{A0} \end{bmatrix} \begin{bmatrix} d\lambda_{Z0} \\ d\lambda_{A0} \\ d\underline{t}_Z \\ d\bar{t}_A \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} dZ^0 + \begin{bmatrix} e^{\alpha \underline{t}_Z} \\ \alpha T_A/\zeta \\ 0 \\ 0 \end{bmatrix} d\bar{Z}$$
$$+ \begin{bmatrix} 0 \\ J_A^c \\ e^{-\rho \underline{t}_Z} \\ e^{-\rho \bar{t}_A} \end{bmatrix} d\bar{c}_s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -e^{-\rho \bar{t}_A} \end{bmatrix} d\underline{c}_s + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} d\bar{A} \ .$$

The determinant of the system, we denote by Δ is:

$$\Delta \;=\; x_c^A \zeta^2 \left[I_Z^Z \pi_Z + e^{2\alpha \underline{t}_Z} x_c^Z \right] - \rho\lambda_{A0}\zeta^2 \left[I_Z^Z (I_A \pi_Z + x_c^Z) + I_A x_c^Z e^{2\alpha \underline{t}_Z} \right] \ . \tag{A.2.1}$$

Note that $\dot{q} = \dot{x}_c = -\rho\lambda_{A0} e^{\rho t}/u''(q)$. Thus:

$$\begin{aligned} I_A &= -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{e^{\rho t}}{u''(q)} dt = -\frac{1}{\rho\lambda_{A0}} \int_{\underline{t}_Z}^{\bar{t}_A} \frac{\rho\lambda_{A0} e^{\rho t}}{u''(q)} dt \\ &= \frac{1}{\rho\lambda_{A0}} \int_{\underline{t}_Z}^{\bar{t}_A} \dot{x}_c(t) dt = \frac{x_c^A - x_c^Z}{\rho\lambda_{A0}} \ . \end{aligned} \tag{A.2.2}$$

Substituting for I_A its expression (A.2.2) into (A.2.1) we obtain:

$$\begin{aligned} \Delta/\zeta^2 &= (x_c^A - \rho\lambda_{A0} I_A) \left[I_Z^Z \pi_Z + e^{2\alpha \underline{t}_Z} x_c^Z \right] - \rho\lambda_{A0} I_Z^Z x_c^Z \\ &= x_c^Z \left[I_Z^Z (\zeta(\rho+\alpha)\lambda_{Z0} + \rho\lambda_{A0}) + x_c^Z e^{2\alpha \underline{t}_Z} - \rho\lambda_{A0} I_Z^Z \right] \\ &= x_c^Z \left[\zeta(\rho+\alpha)\lambda_{Z0} I_Z^Z e^{\alpha \underline{t}_Z} + x_c^Z e^{2\alpha \underline{t}_Z} \right] \ > 0 \ . \end{aligned}$$

Denote by $\Delta_0 = \zeta \left[\zeta(\rho+\alpha)\lambda_{Z0} I_Z^Z + x_c^Z e^{\alpha \underline{t}_Z} \right]$, so that $\Delta = \zeta x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}$. Next, applying Cramer rule, we get first:

$$
\begin{aligned}
\frac{d\lambda_{Z0}}{dZ^0} &= -\frac{1}{\Delta}\left[-x_c^A\pi_Z + \rho\lambda_{A0}I_A\pi_Z + \rho\lambda_{A0}x_c^Z\right] \\
&= -\frac{1}{\Delta}\left[-x_c^A\pi_Z + (x_c^A - x_c^Z)\pi_Z + \rho\lambda_{A0}x_c^Z\right] \\
&= -\frac{x_c^Z}{\Delta}\left[-\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z e^{\alpha \underline{t}_Z} - \rho\lambda_{A0} + \rho\lambda_{A0}\right] \\
&= \frac{\zeta(\rho+\alpha)x_c^Z\lambda_{A0}e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} \\
&= \frac{(\rho+\alpha)\lambda_{Z0}}{\Delta_0} > 0 \ .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\underline{t}_Z}{dZ^0} &= \frac{\zeta e^{\alpha \underline{t}_Z}}{\Delta}\left[\rho\lambda_{A0}I_A - x_c^A\right] \\
&= -\frac{\zeta x_c^Z e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} = -\frac{1}{\Delta_0} < 0 \ .
\end{aligned}
$$

$$
\frac{d\lambda_{A0}}{dZ^0} = \frac{\zeta x_c^Z\rho\lambda_{A0}e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} = \frac{\rho\lambda_{A0}}{\Delta_0} > 0 \ .
$$

$$
\frac{d\bar{t}_A}{dZ^0} = -\frac{\zeta x_c^Z e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} = -\frac{1}{\Delta_0} \ .
$$

The computation shows that $d\underline{t}_Z/dZ^0 = d\bar{t}_Z/dZ^0$ and thus that $dT_A/dZ^0 = 0$. Furthermore:

$$
\begin{aligned}
\frac{d\bar{p}_Z}{dZ^0} &= -e^{\rho \underline{t}_Z}\left[\frac{d\lambda_{A0}}{dZ^0} + \rho\lambda_{A0}\frac{d\underline{t}_Z}{dZ^0}\right] \\
&= -\frac{e^{\rho \underline{t}_Z}}{\Delta_0}\left[\rho\lambda_{A0} - \rho\lambda_{A0}\right] = 0 \ .
\end{aligned}
$$

Turning to the effects of a higher $\bar{Z}$, we find:

$$
\begin{aligned}
\frac{d\lambda_{Z0}}{d\bar{Z}} &= \frac{1}{\Delta}\left\{e^{\alpha \underline{t}_Z}\left[\rho\lambda_{A0}x_c^Z + \pi_Z(\rho\lambda_{A0}I_A - x_c^A)\right] - \frac{\alpha T_A}{\zeta}\left[-\zeta\rho\lambda_{A0}x_c^Z e^{\alpha \underline{t}_Z}\right]\right\} \\
&= \frac{x_c^Z}{\Delta}\left\{e^{\alpha \underline{t}_Z}\left[\rho\lambda_{A0} - \pi_Z\right] + \alpha T_A\rho\lambda_{A0}e^{\alpha \underline{t}_Z}\right\} \\
&= -\frac{\zeta(\rho+\alpha)\lambda_{Z0}e^{2\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} + \alpha T_A\frac{\rho\lambda_{A0}x_c^Z e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} \\
&= -\frac{(\rho+\alpha)\lambda_{Z0}e^{\alpha \underline{t}_Z}}{\Delta_0} + \frac{\alpha T_A}{\zeta}\frac{\rho\lambda_{A0}}{\Delta_0}(?) \ .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\underline{t}_Z}{d\bar{Z}} &= \frac{1}{\Delta}\left\{-\zeta e^{2\alpha \underline{t}_Z}\left(\rho\lambda_{A0}I_A - x_c^A\right) - \frac{\alpha T_A}{\zeta}\rho\lambda_{A0}(-\zeta^2 I_Z^Z)\right\} \\
&= \frac{\zeta x_c^Z e^{2\alpha \underline{t}_Z}}{\zeta x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} + \frac{\zeta^2 \alpha\rho\lambda_{A0}T_A I_Z^Z}{\zeta^2 x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} \\
&= \frac{e^{\alpha \underline{t}_Z}}{\Delta_0} + \frac{\alpha\rho\lambda_{A0}T_A I_Z^Z}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} \;>\; 0\ .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\lambda_{A0}}{d\bar{Z}} &= \frac{1}{\Delta}\left\{\zeta e^{2\alpha \underline{t}_Z}(-\rho\lambda_{A0}x_c^Z) + \frac{\alpha T_A}{\zeta}\left[-\zeta^2 I_Z^Z \pi_Z - \zeta^2 x_c^Z e^{2\alpha \underline{t}_Z}\right]\right\} \\
&= -\frac{\zeta x_c^Z \rho\lambda_{A0} e^{2\alpha \underline{t}_Z}}{\zeta x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} - \frac{\zeta\alpha\rho\lambda_{A0}T_A\left[\pi_Z I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z}\right]}{\zeta x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} \\
&= -\frac{\rho\lambda_{A0}e^{\alpha \underline{t}_Z}}{\Delta_0} - \frac{\alpha\rho\lambda_{A0}T_A\left[\pi_Z I_Z^Z + x_c^Z e^{\alpha \underline{t}_Z}\right]}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}} \;<\; 0\ .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\bar{t}_A}{d\bar{Z}} &= \frac{1}{\Delta}\left\{\zeta x_c^Z e^{2\alpha \underline{t}_Z} + \alpha\zeta T_A\left[\pi_Z I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z}\right]\right\} \\
&= \frac{e^{\alpha \underline{t}_Z}}{\Delta_0} + \alpha T_A \frac{\pi_Z I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z}}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\bar{p}_Z}{d\bar{Z}} &= \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{d\lambda_{Z0}}{d\bar{Z}} + (\rho+\alpha)\lambda_{Z0}\frac{d\underline{t}_Z}{d\bar{Z}}\right] \\
&= \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{\alpha\rho\lambda_{A0}T_A}{\zeta\Delta_0} + (\rho+\alpha)\lambda_{Z0}\frac{\alpha\rho\lambda_{A0}T_A I_Z^Z}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}}\right] \\
&= \frac{\alpha\rho\lambda_{A0}T_A e^{(\rho+\alpha)\underline{t}_Z}}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}}\left[x_c^Z e^{\alpha \underline{t}_Z} + \zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z\right] \\
&= \frac{\alpha\rho\lambda_{A0}T_A}{\zeta x_c^Z}e^{\rho \underline{t}_Z} \;> 0\ .
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\bar{T}_A}{d\bar{Z}} &= \frac{d\bar{t}_A}{d\bar{Z}} - \frac{d\underline{t}_Z}{d\bar{Z}} \\
&= \frac{\alpha T_A}{x_c^Z \Delta_0 e^{\alpha \underline{t}_Z}}\left[(\pi_Z - \rho\lambda_{A0})I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z}\right] \\
&= \frac{\alpha T_A}{x_c^Z \Delta_0}\left[\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z + x_c^Z e^{\alpha \underline{t}_Z}\right] \\
&= \frac{\alpha T_A}{\zeta x_c^Z} \;> 0\ .
\end{aligned}
$$

Next turning to the effects of a higher initial clean coal energy cost, we obtain:

$$\begin{aligned}
\frac{d\lambda_{Z0}}{d\bar{c}_s} &= \frac{\zeta x_c^Z e^{\alpha \underline{t}_Z}}{\Delta}\left[x_c^A(e^{-\rho\underline{t}_Z} - e^{-\rho\bar{t}_A}) + \rho\lambda_{A0}(J_A^c - I_A e^{-\rho\underline{t}_Z})\right] \\
&= \frac{1}{\Delta_0}\left[(x_c^A - \rho\lambda_{A0} I_A)e^{-\rho\underline{t}_Z} - x_c^A e^{-\rho\bar{t}_A} + \rho\lambda_{A0} J_A^c\right] \\
&= \frac{1}{\Delta_0}\left[x_c^Z e^{-\rho\underline{t}_Z} - x_c^A e^{-\rho\bar{t}_A} + \rho\lambda_{A0} J_A^c\right] \qquad \text{(A.2.3)}
\end{aligned}$$

This expression is of indeterminate sign. However taking into account the expression of $\dot{x}_c$:

$$\rho\lambda_{A0} J_A^c = -\int_{\underline{t}_Z}^{\bar{t}_A} \frac{\rho\lambda_{A0} e^{\rho t}}{u''(q(t))} e^{-\rho t} dt = \int_{\underline{t}_Z}^{\bar{t}_A} \dot{x}_c(t) e^{-\rho t} dt$$

Integrating by parts:

$$\begin{aligned}
\rho\lambda_{A0} J_A^c &= x_c(t)e^{-\rho t}\Big|_{\underline{t}_Z}^{\bar{t}_A} + \rho\int_{\underline{t}_Z}^{\bar{t}_A} x_c(t)e^{-\rho t} dt \\
&= x_c^A e^{-\rho\bar{t}_A} - x_c^Z e^{-\rho\underline{t}_Z} + \rho\int_{\underline{t}_Z}^{\bar{t}_A} x_c(t)e^{-\rho t} dt\ . \qquad \text{(A.2.4)}
\end{aligned}$$

Inserting the expression (A.2.4) of $\rho\lambda_{A0} J_A^c$ into (A.2.3), we obtain:

$$\frac{d\lambda_{Z0}}{d\bar{c}_s} = \frac{\rho}{\Delta_0}\int_{\underline{t}_Z}^{\bar{t}_A} x_c(t)e^{-\rho t} dt \equiv \frac{\rho I_c}{\Delta_0} > 0$$

The effect of a higher $\bar{c}_s$ over λ_{A0} is indeterminate. Next turning upon the impact over $\underline{t}_Z$ and $\bar{t}_A$, we obtain:

$$\begin{aligned}
\frac{d\underline{t}_Z}{d\bar{c}_s} &= -\frac{\zeta^2 I_Z^Z}{\Delta}\left\{x_c^A(e^{-\rho\bar{t}_A} - e^{-\rho\underline{t}_Z}) + \rho\lambda_{A0}(I_A e^{-\rho\underline{t}_Z} - J_A^c)\right\} \\
&= -\frac{\zeta^2 I_Z^Z}{\Delta}\left\{x_c^A e^{-\rho\bar{t}_A} + (\rho\lambda_{A0} I_A - x_c^A)e^{-\rho\underline{t}_Z} - \rho\lambda_{A0} J_A^c\right\} \\
&= -\frac{\zeta^2 I_Z^Z}{\Delta}\left\{x_c^A e^{-\rho\bar{t}_A} - x_c^Z e^{-\rho\underline{t}_Z} - \left[x_c^A e^{-\rho\bar{t}_A} - x_c^Z e^{-\rho\underline{t}_Z} + \rho I_c\right]\right\} \\
&= \frac{\zeta^2 \rho I_Z^Z I_c}{\Delta} > 0
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{d\bar{t}_A}{d\bar{c}_s} &= \frac{1}{\Delta}\Big\{-\zeta^2 I_Z^Z\Big[x_c^Z(e^{-\rho\bar{t}_A}-e^{-\rho\underline{t}_Z})+\pi_Z(I_A e^{-\rho\bar{t}_A}-J_A^c)\Big] \\
&\quad -\zeta^2 x_c^Z e^{2\alpha\underline{t}_Z}\Big[I_A e^{-\rho\bar{t}_A}-J_A^c\Big]\Big\} \\
&= \frac{1}{\Delta}\Big\{-\zeta^2 I_Z^Z x_c^Z(e^{-\rho\bar{t}_A}-e^{-\rho\underline{t}_Z})-(I_A e^{-\rho\bar{t}_A}-J_A^c)\zeta^2\big[\pi_Z I_Z^Z+x_c^Z e^{2\alpha\underline{t}_Z}\big]\Big\} \\
&= \frac{\zeta^2}{\Delta}\Big\{I_Z^Z x_c^Z(e^{-\rho\underline{t}_Z}-e^{-\rho\bar{t}_A})+(J_A^c-I_A e^{-\rho\bar{t}_A})\big[\pi_Z I_Z^Z+x_c^Z e^{2\alpha\underline{t}_Z}\big]\Big\}\ .
\end{aligned}$$

Since:

$$J_A^c - I_A e^{-\rho\bar{t}_A} = -\int_{\underline{t}_Z}^{\bar{t}_A}\frac{1}{u''(q(t))}\Big[1-e^{-\rho(\bar{t}_A-t)}\Big]\,dt > 0\ ,$$

while $\underline{t}_Z < \bar{t}_A$ implies that $e^{-\rho\underline{t}_Z}-e^{-\rho\bar{t}_A}>0$, we conclude that $d\bar{t}_A/d\bar{c}_s>0$.

This gives the following effects over $\bar{p}_Z$ and T_A:

$$\begin{aligned}
\frac{d\bar{p}_Z}{d\bar{c}_s} &= \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{\rho I_c}{\Delta_0}+(\rho+\alpha)\lambda_{Z0}\frac{\rho\zeta^2 I_c I_Z^Z}{\Delta}\right] \\
&= \frac{\zeta^2\rho I_c e^{(\rho+\alpha)\underline{t}_Z}}{\Delta}\left[x_c^Z e^{\alpha\underline{t}_Z}+\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z\right] \\
&= \frac{\zeta\rho I_c e^{(\rho+\alpha)\underline{t}_Z}\Delta_0}{\zeta x_c^Z\Delta_0 e^{\alpha\underline{t}_Z}} \\
&= \frac{\rho I_c}{x_c^Z}e^{\rho\underline{t}_Z} > 0\ .
\end{aligned}$$

$$\frac{dT_A}{d\bar{c}_s} = \frac{\zeta^2}{\Delta}\Big\{I_Z^Z x_c^Z(e^{-\rho\underline{t}_Z}-e^{-\rho\bar{t}_A})+(J_A^c-e^{-\rho\bar{t}_A}I_A)(\pi_Z I_Z^Z+x_c^Z e^{2\alpha\underline{t}_Z})-\rho I_c I_Z^Z\Big\}\ .$$

Since $\rho I_c = \rho\lambda_{A0}J_c^A - x_c^A e^{-\rho\bar{t}_A}+x_c^Z e^{-\rho\underline{t}_Z}$, the expression into brackets is equivalent to:

$$\begin{aligned}
\{\} &= I_Z^Z\Big[x_c^Z(e^{-\rho\underline{t}_Z}-e^{-\rho\bar{t}_A})+\pi_Z(J_A^c-e^{-\rho\bar{t}_A}I_A)-\rho\lambda_{A0}J_A^c+x_c^A e^{-\rho\bar{t}_A}-x_c^Z e^{-\rho\underline{t}_Z}\Big] \\
&\quad +x_c^Z(J_A^c-I_A e^{-\rho\bar{t}_A})e^{2\alpha\underline{t}_Z} \\
&= I_Z^Z\Big[(x_c^A-x_c^Z)e^{-\rho\bar{t}_A}+J_A^c(\pi_Z-\rho\lambda_{A0})-\pi_Z I_A e^{-\rho\bar{t}_A}\Big]+x_c^Z(J_A^c-I_A e^{-\rho\bar{t}_A}) \\
&= I_Z^Z\Big[(x_c^A-x_c^Z)e^{-\rho\bar{t}_A}+\zeta(\rho+\alpha)\lambda_{Z0}e^{\alpha\underline{t}_Z}(J_A^c-I_A e^{-\rho\bar{t}_A})-\rho\lambda_{A0}I_A e^{-\rho\bar{t}_A}\Big]+x_c^Z(J_A^c-I_A e^{-\rho\bar{t}_A})
\end{aligned}$$

Since $\rho\lambda_{A0}I_A = x_c^A - x_c^Z$, we obtain after simplification:

$$\begin{aligned}
\frac{d\bar{T}_A}{d\bar{c}_s} &= \frac{\zeta^2 e^{\alpha \underline{t}_Z}}{\delta}\left[\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z + x_c^Z e^{\alpha\underline{t}_Z}\right](J_A^c - I_A e^{-\rho\bar{t}_A}) \\
&= \frac{\zeta e^{\alpha\underline{t}_Z}\Delta_0(J_A^c - I_A e^{-\rho\bar{t}_A})}{\zeta x_c^Z \Delta_0 e^{\alpha\underline{t}_Z}} \\
&= \frac{J_A^c - I_A e^{-\rho\bar{t}_A} I_A}{x_c^Z} > 0 \ .
\end{aligned}$$

The effects of a higher $\underline{c}_s$ are the following.

$$\frac{d\lambda_{Z0}}{d\underline{c}_s} = \frac{\zeta x_c^Z e^{\alpha\underline{t}_Z} x_c^A e^{-\rho\bar{t}_A}}{\zeta x_c^Z \Delta_0 e^{\alpha\underline{t}_Z}} = \frac{x_c^A}{\Delta_0}e^{-\rho\bar{t}_A} > 0 \ .$$

$$\frac{d\lambda_{A0}}{d\underline{c}_s} = -\frac{\zeta(\pi_Z I_Z^Z + x_c^Z e^{2\alpha\underline{t}_Z})x_c^A e^{-\rho\bar{t}_A}}{x_c^Z \Delta_0 e^{\alpha\underline{t}_Z}} < 0 \ .$$

$$\frac{d\underline{t}_Z}{d\underline{c}_s} = \frac{\zeta^2 I_Z^Z x_c^A e^{-\rho\bar{t}_A}}{\Delta} = \frac{\zeta x_c^A I_Z^Z}{x_c^Z \Delta_0 e^{\alpha\underline{t}_Z}} e^{-\rho\bar{t}_A} > 0 \ .$$

$$\frac{d\bar{t}_A}{d\underline{c}_s} = \frac{\zeta^2 e^{-\rho\bar{t}_A}}{\Delta}\left[I_A(\pi_Z I_Z^Z + x_c^Z e^{2\alpha\underline{t}_Z}) + I_Z^Z x_c^Z\right] > 0 \ .$$

$$\frac{d\bar{p}_Z}{d\underline{c}_s} = \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{d\lambda_{Z0}}{d\underline{c}_s} + (\rho+\alpha)\lambda_{Z0}\frac{d\underline{t}_Z}{d\underline{c}_s}\right] > 0 \ .$$

$$\begin{aligned}
\frac{dT_A}{d\underline{c}_s} &= \frac{\zeta^2 e^{-\rho\bar{t}_A}}{\delta}\left[I_A(\pi_Z I_Z^Z + x_c^Z e^{2\alpha\underline{t}_Z}) + I_Z^Z x_c^Z - x_c^A I_Z^Z\right] \\
&= \frac{\zeta^2 e^{-\rho\bar{t}_A}}{\delta}\left[I_Z^Z(\pi_Z I_A + x_c^Z - x_c^A) + x_c^Z I_A e^{2\alpha\underline{t}_Z}\right] \\
&= \frac{\zeta^2 e^{-\rho\bar{t}_A}}{\delta}\left[I_Z^Z(\zeta(\rho+\alpha)\lambda_{Z0}I_A e^{\alpha\underline{t}_Z} + \rho\lambda_{A0}I_A + x_c^Z - x_c^A) + x_c^Z I_A e^{2\alpha\underline{t}_Z}\right] \\
&= \frac{\zeta^2 I_A e^{-\rho\bar{t}_A} e^{\alpha\underline{t}_Z}}{\delta}\left[\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z + x_c^Z e^{\alpha\underline{t}_Z}\right] \\
&= \frac{\zeta I_A e^{\alpha\underline{t}_Z} e^{-\rho\bar{t}_A}\Delta_0}{x_c^Z \Delta_0 e^{\alpha\underline{t}_Z}} = \frac{I_A}{x_c^Z}e^{-\rho\bar{t}_A} > 0 \ .
\end{aligned}$$

Last, the effects of a higher know-how target $\bar{A}$ are:

$$\frac{d\lambda_{Z0}}{d\bar{A}} = \frac{\zeta x_c^Z e^{\alpha \underline{t}_Z}}{\Delta}\rho\lambda_{A0} = \frac{\rho\lambda_{A0}}{\Delta_0} > 0 \; .$$

$$\frac{d\lambda_{A0}}{d\bar{A}} = -\frac{\zeta\rho\lambda_{A0}}{\Delta}\left[\pi_Z I_Z^Z + x_c^Z e^{2c\alpha \underline{t}_Z}\right] < 0 \; .$$

$$\frac{d\underline{t}_Z}{d\bar{A}} = \frac{\zeta^2\rho\lambda_{A0} I_Z^Z}{\Delta} > 0 \; .$$

$$\frac{d\bar{t}_A}{d\bar{A}} = \frac{\zeta^2}{\Delta}\left[\pi_Z I_Z^Z + x_c^Z e^{2\alpha \underline{t}_Z}\right] > 0 \; .$$

$$\begin{aligned}
\frac{d\bar{p}_Z}{d\bar{A}} &= \zeta e^{(\rho+\alpha)\underline{t}_Z}\left[\frac{d\lambda_{Z0}}{d\bar{A}} + (\rho+\alpha)\lambda_{Z0}\frac{d\underline{t}_Z}{d\bar{A}}\right] \\
&= \frac{\zeta\rho\lambda_{A0}e^{(\rho+\alpha)\underline{t}_Z}}{\Delta_0}\left[1 + \frac{\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z}{x_c^Z e^{\alpha \underline{t}_Z}}\right] \\
&= \frac{\zeta\rho\lambda_{A0}e^{(\rho+\alpha)\underline{t}_Z}}{x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}}\Delta_0 \\
&= \frac{\rho\lambda_{A0}}{x_c^Z}e^{\rho \underline{t}_Z} > 0 \; .
\end{aligned}$$

$$\begin{aligned}
\frac{dT_A}{dbarA} &= \frac{\zeta^2}{\Delta}\left[I_Z^Z(\pi_Z - \rho\lambda_{A0}) + x_c^Z e^{2\alpha \underline{t}_Z}\right] \\
&= \frac{\zeta^2}{\Delta}\left[\zeta(\rho+\alpha)\lambda_{Z0}I_Z^Z e^{\alpha \underline{t}_Z} + x_c^Z e^{2c\alpha \underline{t}_Z}\right] \\
&= \frac{\zeta\delta_0 e^{\alpha \underline{t}_Z}}{\zeta x_c^Z\Delta_0 e^{\alpha \underline{t}_Z}} = \frac{1}{x_c^Z} > 0 \; .
\end{aligned}$$

A.3 Appendix 3: Comparative dynamics in the pure R&D model

The relative independency of the R&D policy with respect to the energy policy results in a pair of two dimensional linearized systems, the first one describing the effects over $(\lambda_{Z0}, \underline{t}_Z)$ while the second one describes the effects over $(\lambda_{A0}, \bar{t}_A)$. Denote by:

$$I_Z^Z \equiv \int_0^{\underline{t}_Z} \frac{e^{(\rho+2\alpha)t}}{u''(q(t))} dt > 0 \quad ; \quad I_A \equiv \int_0^{\bar{t}_A} \frac{e^{\rho t}}{c_r'(r(t))} dt > 0$$
$$\bar{r} = r(\bar{t}_A) \quad ; \quad r(0) = r_0$$

Then the systems are expressed as:

$$\begin{bmatrix} -\zeta^2 I_Z^Z & \zeta x_c^Z e^{\alpha \underline{t}_Z} \\ 1 & (\rho+\alpha)\lambda_{Z0} \end{bmatrix} \begin{bmatrix} d\lambda_{Z0} \\ d\underline{t}_Z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} dZ^0 + \begin{bmatrix} -e^{\alpha \underline{t}_Z} \\ 0 \end{bmatrix} d\bar{Z} + \begin{bmatrix} 0 \\ e^{-(\rho+\alpha)\underline{t}_Z}/\zeta \end{bmatrix} d\bar{c}_s$$

$$\begin{bmatrix} I_A & \bar{r} \\ 1 & \rho\lambda_{A0} \end{bmatrix} \begin{bmatrix} d\lambda_{A0} \\ d\bar{t}_A \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\bar{A} + \begin{bmatrix} 0 \\ -\frac{\alpha(\bar{c}_s - \underline{c}_s)}{\zeta \bar{r}} e^{-\rho \bar{t}_A} \end{bmatrix} d\bar{Z}$$
$$+ \begin{bmatrix} 0 \\ \frac{x_c^-}{\bar{r}} e^{-\rho \bar{t}_A} \end{bmatrix} d\bar{c}_s - \begin{bmatrix} 0 \\ \frac{x_c}{\bar{r}} e^{-\rho \bar{t}_A} \end{bmatrix} d\underline{c}_s$$

The determinant of the first system is $-\zeta(\zeta(\rho+\alpha)\lambda_{Z0} I_Z^Z + x_c^Z e^{\alpha \underline{t}_Z}) \equiv -\Delta_0 < 0$. Applying Cramer rule, we obtain:

$$\frac{d\lambda_{Z0}}{dZ^0} = \frac{(\rho+\alpha)\lambda_{Z0}}{\Delta_0} \quad ; \quad \frac{d\lambda_{Z0}}{d\bar{Z}} = \frac{(\rho+\alpha)\lambda_{Z0} e^{\alpha \underline{t}_Z}}{\Delta_0} \quad ; \quad \frac{d\lambda_{Z0}}{d\bar{c}_s} = \frac{x_c^Z e^{-\rho \underline{t}_Z}}{\Delta_0}$$
$$\frac{d\underline{t}_Z}{dZ^0} = -\frac{1}{\Delta_0} \quad ; \quad \frac{d\underline{t}_Z}{d\bar{Z}} = -\frac{e^{\alpha \underline{t}_Z}}{\Delta_0} \quad ; \quad \frac{d\underline{t}_Z}{d\bar{c}_s} = \frac{\zeta I_Z^Z e^{-(\rho+\alpha)\underline{t}_Z}}{\Delta_0}$$

By construction: $d\lambda_{Z0}/d\bar{A} = d\underline{t}_Z/d\bar{A} = 0$ and $d\lambda_{Z0}/d\underline{c}_s = d\underline{t}_Z/d\underline{c}_s = 0$. The features of the energy policy before the cost breakthrough do not depend of the know-how requirement to trigger the break or of the clean energy additional cost after the break.

Next turn to the second system. The determinant of this system is: $\rho\lambda_{A0} I_A - \bar{r}$. Since $\dot{r}(t) = \rho\lambda_{A0} e^{\rho t}/c_r'(r)$, we obtain:

$$\rho\lambda_{A0} I_A = \rho\lambda_{A0} \int_0^{\bar{t}_A} \frac{e^{\rho t}}{c_r'(r(t))} dt = \int_0^{\bar{t}_A} \dot{r}(t) dt = \bar{r} - r_0$$

Hence: $\rho\lambda_{A0}I_A - \bar{r} = -r_0 < 0$. Remember that the strict negativity of the determinant is a consequence of the assumption $\lambda_{!A0} > c_r(0)$, which implies that $r_0 > 0$. Applying Cramer rule, we then get:

$$\begin{aligned}
\frac{d\lambda_{A0}}{d\bar{A}} &= -\frac{\rho\lambda_{A0}}{r_0} \; ; \; \frac{d\lambda_{A0}}{d\bar{Z}} = -\frac{\alpha(\bar{c}_s - \underline{c}_s)e^{-\rho\bar{t}_A}}{\zeta r_0} \\
\frac{d\lambda_{A0}}{d\bar{c}_s} &= \frac{x_c^- e^{-\rho\bar{t}_A}}{r_0} \; ; \; \frac{d\lambda_{A0}}{d\underline{c}_s} = -\frac{\underline{x}_c e^{-\rho\underline{t}_A}}{r_0} \\
\frac{d\bar{t}_A}{d\bar{A}} &= \frac{1}{r_0} \; ; \; \frac{d\bar{t}_A}{d\bar{Z}} = \frac{\alpha(\bar{c}_s - \underline{c}_s)I_A e^{-\rho\underline{t}_A}}{\zeta r_0 \bar{r}} \\
\frac{d\bar{t}_A}{d\bar{c}_s} &= -\frac{x_c^- I_A e^{-\rho\bar{t}_A}}{r_0\bar{r}} \; ; \; \frac{d\bar{t}_A}{d\underline{c}_s} = \frac{\underline{x}_c I_A e^{-\rho\bar{t}_A}}{r_0\bar{r}}
\end{aligned}$$

A.4 Appendix 4. Proof that $\dot{\sigma} < 0$

We get from (5.4): $\lambda_A = c_r/a_r$. Denote $\sigma = a_c/a_r$, then (5.1) becomes: $u'(q) = c_x + \bar{c}_s - \sigma c_r$ during the time interval $[\underline{t}_Z, \bar{t}_A)$. Time differentiating, we obtain:

$$\dot{\sigma}c_r = -\left[\sigma c_r'\dot{r} + u''(q)\dot{x}_c\right]$$

Taking (5.13) and (5.14) into account, this is equivalent to:

$$\begin{aligned}
\dot{\sigma}c_r &= -\frac{\dot{\lambda}_A}{\Delta_1}\{u''\left[\lambda_A(a_{cr}a_r - a_c a_{rr}) + a_c c_r'\right] \\
&\quad +\sigma c_r'\left[\lambda_A(a_{cr}a_c - a_r a_{cc}) - u'' a_r\right]\} \\
&= -\frac{\dot{\lambda}_A}{\Delta_1}\{\lambda_A\left[u''(a_{cr}a_r - a_c a_{rr}) + \sigma c_r'(a_{cr}a_c - a_r a_{cc})\right] \\
&\quad \left. +u'' a_c c_r' - u''\frac{a_c}{a_r}a_r c_r' \right\} \\
&= -\frac{\rho\lambda_A^2 a_r}{\Delta_1}\left[u''(a_{cr} - \sigma a_{rr}) + \sigma c_r'(a_{cr}\sigma - a_{cc})\right]
\end{aligned}$$

Let: $P(\sigma) \equiv c_r' a_{cr}\sigma^2 - \sigma(u'' a_{rr} + c_r' a_c c) + u'' a_{cr}$, then:

$$\dot{\sigma} = -\frac{\rho\lambda_A^2 a_r}{\Delta_1 c_r}P(\sigma)$$

$c_r' a_{cr} > 0$ and $u'' a_{cr} < 0$ imply that $P(\sigma)$ has two real roots of opposite signs. Denote by $\hat{\sigma}$ the positive root. Then:

$$\dot{\sigma} \gtreqqless 0 \iff P(\sigma) \lesseqqgtr 0 \iff \sigma \lesseqqgtr \hat{\sigma}$$

We conclude that during the time interval $[\underline{t}_Z, \bar{t}_A)$, $\sigma(t)$ must be a monotonous time function, either constantly increasing or either constantly decreasing. Then denoting by $\bar{p}_Z = u'(q(\underline{t}_Z))$ and by $\bar{p}_A = u'(q^-(\bar{t}_A))$, $\sigma_Z \equiv \sigma(\underline{t}_Z)$ and $\sigma_A \equiv \sigma^-(\bar{t}_A)$ may be expressed as:

$$\sigma_Z = \frac{\bar{p}_Z - c_x - \bar{c}_s}{c_r(r(\underline{t}_Z))} \quad \text{and} \quad \sigma_A = \frac{\bar{p}_A - c_x - \bar{c}_s}{c_r(r^-(\bar{t}_A))}$$

And $\bar{p}_A < \bar{p}_Z$ together with $r(\underline{t}_Z) < r^-(\bar{t}_A)$, that is $c_r(r(\underline{t}_Z)) < c_r(r^-(\bar{t}_A))$ imply that $\sigma_A < \sigma_Z$. We thus conclude that $\sigma(t)$ having to be monotonous, $\sigma(t)$ is a decreasing time function over the interval $[\underline{t}_Z, \bar{t}_A)$.

A.5 Appendix 5.

Pure research paths

Let $\underline{t}^R$ be the date of both the arrival at the ceiling and the revolution: $\underline{t}_Z = \bar{t}_A = \underline{t}^R$ in a scenario where the revolution is triggered only through R&D efforts. Assume that initially $r(0) > 0$. Then $(\lambda_{Z0}, \lambda_{A0}, \underline{t}^R)$ are solution of the following system of conditions:

$$\begin{aligned}
\bar{Z}e^{\alpha \underline{t}^R} &= Z^0 + \int_0^{\underline{t}^R} x_d(\lambda_{Z0}e^{(\rho+\alpha)t})e^{\alpha t}dt \\
\bar{A} &= \int_0^{\underline{t}^R} a(0, r(\lambda_{A0}e^{\rho t}))dt \\
\underline{c}_s &= c_x + \zeta\lambda_{Z0}e^{(\rho+\alpha)\underline{t}^R}
\end{aligned}$$

Let $\underline{\lambda}_A^R \equiv \lambda_{A0}e^{\rho \underline{t}^R}$.

Pure learning phases

Let $\underline{t}^L$ be the common date of arrival at the ceiling and the revolution when the policy scenario involves only learning-by-doing. Assume that initially $x_c(0) > 0$, then $(\lambda_{Z0}, \lambda_{A0}, \underline{t}^L)$ are solution of:

$$\begin{aligned}
\bar{Z}e^{\alpha \underline{t}^L} &= Z^0 + \int_0^{\underline{t}^L} x_d(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t})e^{\alpha t}dt \\
\bar{A} &= \int_0^{\underline{t}^L} a(x_c(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t}), 0)dt \\
\underline{c}_s &= c_x + \zeta \lambda_{Z0}e^{(\rho+\alpha)\underline{t}^L}
\end{aligned}$$

Let $\underline{\lambda}_A^L \equiv \lambda_{A0}e^{\rho \underline{t}^L}$.

Combined phases

Let $\underline{t}^C$ be given by $\underline{t}^C = \underline{t}_Z = \bar{t}_A$ in a scenario involving both learning and R&D to trigger the technological breakthrough. In a case where initially $x_c(0) > 0$ and $r(0) > 0$, $(\lambda_{Z0}, \lambda_{A0}, \underline{t}^C$ are solution of:

$$\begin{aligned}
\bar{Z}e^{\alpha \underline{t}^C} &= Z^0 + \int_0^{\underline{t}^C} x_d(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t})e^{\alpha t}dt \\
\bar{A} &= \int_0^{\underline{t}^C} a(x_c(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t}), r(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t}))dt \\
\underline{c}_s &= c_x + \zeta \lambda_{Z0}e^{(\rho+\alpha)\underline{t}^C}
\end{aligned}$$

Let $\underline{\lambda}_A^C \equiv \lambda_{A0}e^{\rho \underline{t}^C}$.

Pure R&D then combined phases

Let $\underline{t}^{RC} = \underline{t}_Z = \bar{t}_A$ in a two phases scenario during which the economy performs only research during a time interval $[0, \bar{t}_r)$ and next both clean coal production and research during a time phase $[\bar{t}_r, \underline{t}^{RC})$. Assume that initially $r(0) > 0$, then $(\lambda_{Z0}, \lambda_{A0}, \bar{t}_r, \underline{t}^{RC})$ are defined by the following set of conditions:

$$\begin{aligned}
\bar{Z}e^{\alpha \underline{t}^{LC}} &= Z^0 + \int_0^{\bar{t}_r} x_d(\underline{\lambda}_{Z0}e^{(\rho+\alpha)t}dt + \int_{\bar{t}_r}^{\underline{t}^{RC}} x_d(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t})e^{\alpha t}dt \\
\bar{A} &= \int_0^{\bar{t}_r} a(0, r(\lambda_{A0}e^{\rho t})dt + \int_{\bar{t}_r}^{\underline{t}^C} a(x_c(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t}), r(\lambda_{Z0}e^{(\rho+\alpha)t}, \lambda_{A0}e^{\rho t}))dt \\
\bar{c}_s &= \lambda_{A0}a_c(0, r(\lambda_{A0}e^{\rho \bar{t}_t})e^{\rho \bar{t}_r} + \zeta \lambda_{Z0}e^{(\rho+\alpha)\bar{t}_r} \\
\underline{c}_s &= c_x + \zeta \lambda_{Z0}e^{(\rho+\alpha)\underline{t}^C}
\end{aligned}$$

Let $\underline{\lambda}_A^{RC} \equiv \lambda_{A0} e^{\rho \underline{t}^{RC}}$.

Pure learning then combined phases

Let $\underline{t}^{LC} = \underline{t}_Z = \bar{t}_A$ in a two phases scenario where the economy increases the know-how level only through learning during a first time phase $[0, \bar{t}_l)$ and then through a combination of learning and R&D during the phase $[\bar{t}_l, \underline{t}^{LC})$. Assume that initially $x_c(0) = 0$, then $(\lambda_{Z0}, \lambda_{A0}, \bar{t}_l, \underline{t}^{LC})$ are solutions of the following system of conditions:

$$\begin{aligned}
\bar{Z} e^{\alpha \underline{t}^{LC}} &= Z^0 + \int_0^{\bar{t}_l} x_d(\underline{\lambda}_{Z0} e^{(\rho+\alpha)t}, \lambda_{A0} e^{\rho t}))dt + \int_{\bar{t}_l}^{\underline{t}^{LC}} x_d(\lambda_{Z0} e^{(\rho+\alpha)t}, \lambda_{A0} e^{\rho t}) e^{\alpha t} dt \\
\bar{A} &= \int_0^{\bar{t}_l} a(x_c(\lambda_{Z0} e^{(\rho+\alpha)t}, \lambda_{A0} e^{\rho t}, 0)dt + \int_{\bar{t}_l}^{\underline{t}^{LC}} a(x_c(\lambda_{Z0} e^{(\rho+\alpha)t}, \lambda_{A0} e^{\rho t}), r(\lambda_{Z0} e^{(\rho+\alpha)t}, \lambda_{A0} e^{\rho t})) \\
\bar{c}_s &= \lambda_{A0} a_c(x_c(\lambda_{Z0} e^{(\rho+\alpha)\bar{t}_l}, \lambda_{A0} e^{\rho \bar{t}_l}), 0) e^{\rho \bar{t}_l} + \zeta \lambda_{Z0} e^{(\rho+\alpha)\bar{t}_l} \\
\underline{c}_s &= c_x + \zeta \lambda_{Z0} e^{(\rho+\alpha)\underline{t}^C}
\end{aligned}$$

Let $\underline{\lambda}_A^{LC} \equiv \lambda_{A0} e^{\rho \underline{t}^{LC}}$.

A.6 Appendix 6.

The optimal vector $(\lambda_{Z0}, \lambda_{A0}, \underline{t}_Z, \bar{t}_A)$ in a combined policy where $\underline{t}_Z < \bar{t}_A$ is the solution of the following system of conditions:

$$\bar{Z} e^{\alpha \underline{t}_Z} = Z^0 + \zeta \int_0^{\underline{t}_Z} x_d(\lambda_Z(t), \lambda_A(t)) e^{\alpha t} dt \; ; \quad \text{(A.6.1)}$$

$$\bar{A} = \int_0^{\underline{t}_Z} a(\lambda_Z(t), \lambda_A(t)) dt + \int_{\underline{t}_Z}^{\bar{t}_A} a(\lambda_A(t)) dt \; ; \quad \text{(A.6.2)}$$

$$\zeta \lambda_{Z0} e^{(\rho+\alpha)\underline{t}_Z} = \bar{c}_s - \lambda_{A0} a_c(\underline{t}_Z) e^{\rho \underline{t}_Z} \; ; \quad \text{(A.6.3)}$$

$$\lambda_{A0} e^{\rho \bar{t}_A} = \bar{\lambda}_A(\bar{V}) \; . \quad \text{(A.6.4)}$$

Differentiating the ceiling attainment condition (A.6.1) for a given Z^0, we obtain:

$$d\bar{Z} e^{\alpha \underline{t}_Z} = \int_0^{\underline{t}_Z} \frac{\partial x_d(t)}{\partial \lambda_Z(t)} e^{(\rho+2\alpha)t} dt d\lambda_{Z0} + \int_0^{\underline{t}_Z} \frac{\partial x_d(t)}{\partial \lambda_A(t)} e^{(\rho+\alpha)t} dt d\lambda_{A0} \; .$$

Denote by:

$$
\begin{aligned}
I_Z^Z &= -\int_0^{\underline{t}_Z} \frac{\partial x_d(t)}{\partial \lambda_Z(t)} e^{(\rho+2\alpha)t} dt > 0 \ ; \\
I_Z^A &= -\int_0^{\underline{t}_Z} \frac{\partial x_d(t)}{\partial \lambda_A(t)} e^{(\rho+\alpha)t} dt > 0 \ .
\end{aligned}
$$

The signs of the integrals result from $\partial x_d/\partial\lambda_Z < 0$ and $\partial x_d/\partial\lambda_A < 0$.

Let $a_1(\lambda_Z(t), \lambda_A(t)) \equiv a(x_c(\lambda_A(t), \lambda_Z(t)), r(\lambda_A(t), \lambda_Z(t)))$ over the first time interval $[0, \underline{t}_Z)$. Denote similarly: $a_2(\lambda_A(t)) \equiv a(x_c(\lambda_A(t)), r(\lambda_A(t)))$ over the second time interval $[\underline{t}_Z, \bar{t}_A)$. Then differentiating the technological breakthrough condition, we get:

$$
\begin{aligned}
d\bar{A} &= a(\bar{t}_A^-)d\bar{t}_A + \int_0^{\underline{t}_Z} \frac{\partial a_1(\lambda_Z(t), \lambda_A(t))}{\partial \lambda_Z(t)} e^{(\rho+\alpha)t} dt d\lambda_{Z0} \\
&\quad + \left\{ \int_0^{\underline{t}_Z} \frac{\partial a_1(\lambda_Z(t), \lambda_A(t))}{\partial \lambda_A(t)} e^{\rho t} dt + \int_{\underline{t}_Z}^{\bar{t}_A} \frac{da_2(\lambda_A(t))}{d\lambda_A(t)} e^{\rho t} dt \right\} d\lambda_{A0} \ .
\end{aligned}
$$

Denote by:

$$
\begin{aligned}
I_A^Z &= \int_0^{\underline{t}_Z} \frac{\partial a_1(\lambda_Z(t), \lambda_A(t))}{\partial \lambda_Z(t)} e^{(\rho+\alpha)t} dt > 0 \ ; \\
I_A^1 &= \int_0^{\underline{t}_Z} \frac{\partial a_1(\lambda_Z(t), \lambda_A(t))}{\partial \lambda_A(t)} e^{\rho t} dt > 0 \ ; \\
I_A^2 &= \int_{\underline{t}_Z}^{\bar{t}_A} \frac{da_2(\lambda_A(t))}{d\lambda_A(t)} e^{\rho t} dt > 0 \ .
\end{aligned}
$$

The signs of the integrals result from $a_c > 0$, $a_r > 0$, and $\partial x_c/\partial\lambda_Z > 0$, $\partial x_c/\partial\lambda_A > 0$, $\partial r/\partial\lambda_Z > 0$, $\partial r/\partial\lambda_A$ during the time interval $[0, \underline{t}_Z)$, together with $dx_c/d\lambda_A > 0$ and $dr/d\lambda_A > 0$ during the second time interval $[\underline{t}_Z, \bar{t}_A)$. Now remark that during the first time interval $[0, \underline{t}_Z)$:

$$
\dot{a}_1(t) = (\rho+\alpha)\lambda_{Z0} \frac{\partial a_1(t)}{\partial \lambda_Z(t)} e^{(\rho+\alpha)t} + \rho\lambda_{A0} \frac{\partial a_1(t)}{\partial \lambda_A(t)} e^{\rho t}
$$

Denote by $a^0 \equiv a_1(0)$ and $a^Z \equiv a(x_c(\underline{t}_Z), r(\underline{t}_Z))$. Integrating over $[0, \underline{t}_Z)$ we obtain:

$$
\begin{aligned}
a^Z - a^0 &= (\rho+\alpha)\lambda_{Z0} \int_0^{\underline{t}_Z} \frac{\partial a_1(t)}{\partial \lambda_Z(t)} e^{(\rho+\alpha)t} dt + \rho\lambda_{A0} \int_0^{\underline{t}_Z} \frac{\partial a_1(t)}{\partial \lambda_Z(t)} e^{\rho t} dt \\
&= (\rho+\alpha)\lambda_{Z0} I_A^Z + \rho\lambda_{A0} I_A^1 \ . \qquad \text{(A.6.5)}
\end{aligned}
$$

During the second time interval $[\underline{t}_Z, \bar{t}_A)$:

$$\dot{a}_2(t) \quad = \quad \rho\lambda_{A0}\frac{da_2(t)}{d\lambda_A(t)}e^{\rho t}$$

Denote by $a^A \equiv a(x_c(\bar{t}_A), r(\bar{t}_A))$. Integrating over $[\underline{t}_Z, \bar{t}_A)$, we get:

$$a^A - a^Z \quad = \quad \rho\lambda_{A0}\int_{\underline{t}_Z}^{\bar{t}_A}\frac{da_2(t)}{d\lambda_A(t)}e^{\rho t}dt = \rho\lambda_{A0}I_A^2 \ . \qquad \text{(A.6.6)}$$

Next differentiating the continuity condition at the ceiling beginning time for a given $\bar{c}_s$, we obtain:

$$\begin{aligned} & \left[\zeta e^{(\rho+\alpha)\underline{t}_Z}a_{cc}\lambda_{A0}e^{(2\rho+\alpha)\underline{t}_Z}\frac{\partial x_c}{\partial\lambda_Z} + a_{cr}\lambda_{A0}e^{(2\rho+\alpha)\underline{t}_Z}\frac{\partial r}{\partial\lambda_A}\right]d\lambda_{Z0} \\ & + \left[a_c e^{\rho\underline{t}_Z} + a_{cc}\lambda_{A0}e^{2\rho\underline{t}_Z}\frac{\partial x_c}{\partial\lambda_A} + a_{cr}\lambda_{A0}e^{2\rho\underline{t}_Z}\frac{\partial r}{\partial\lambda_A}\right]d\lambda_{A0} \\ & + \left[\zeta(\rho+\alpha)\lambda_{Z0}e^{(\rho+\alpha)\underline{t}_Z} + \rho a_c\lambda_{A0}e^{\rho\underline{t}_Z}\right]d\underline{t}_Z \\ = \ & 0 \end{aligned}$$

Making use of (5.9)-(5.12), this simplifies to:

$$\left[\zeta(\rho+\alpha)\lambda_{Z0}e^{\alpha\underline{t}_Z} + \rho a_c\lambda_{A0}\right]d\underline{t}_Z \quad \equiv \quad \pi_Z d\underline{t}_Z \ = \ 0 \ .$$

Last the differentiation of the terminal condition results in:

$$d\lambda_{A0} + \rho\lambda_{A0}d\bar{t}_A = \frac{d\bar{\lambda}_A}{d\bar{V}}e^{-\rho\bar{t}_A}d\bar{V} \ .$$

Taking stock, we get the following linearized system in matrix form:

$$\begin{bmatrix} -I_Z^Z & -I_Z^A & x_c^Z & 0 \\ I_A^Z & I_A^1 + I_A^2 & 0 & a^A \\ 0 & 0 & \pi_Z & 0 \\ 0 & 1 & 0 & \rho\lambda_{A0} \end{bmatrix}\begin{bmatrix} d\lambda_{Z0} \\ d\lambda_{A0} \\ d\underline{t}_Z \\ d\bar{t}_A \end{bmatrix} = \begin{bmatrix} e^{\alpha\underline{t}_Z} \\ \Phi_0 \\ 0 \\ \Phi_1 \end{bmatrix}d\bar{Z} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}d\bar{A} \ .$$

Φ_0 and Φ_1 will be computed later. Denote by Δ the determinant of the linearized system.

$$\begin{aligned}
\Delta &= \pi_Z \begin{vmatrix} -I_Z^Z & -I_Z^A & 0 \\ I_A^Z & I_A^1 + I_A^2 & a^A \\ 0 & 1 & \rho\lambda_{A0} \end{vmatrix} \\
&= \pi_z \left\{ -I_Z^Z \left[\rho\lambda_{A0}(I_A^1 + I_A^2) - a^A \right] + \rho\lambda_{A0} I_A^Z I_Z^A \right\} \\
&= \pi_Z \left[-\rho\lambda_{A0} I_Z^Z I_A^1 + I_Z^Z (a^A - \rho\lambda_{A0} I_A^Z) + \rho\lambda_{A0} I_A^Z I_Z^A \right] \ .
\end{aligned}$$

Taking (A.6.6) into account: $a^Z = a^A - \rho\lambda_{A0} I_A^2$, thus:

$$\Delta = \pi_z \left[-\rho\lambda_{A0} I_Z^Z I_A^1 + I_Z^Z a^Z + \rho\lambda_{A0} I_A^Z I_Z^A \right] \ .$$

Taking (A.6.5) into account:

$$\rho\lambda_{A0} I_A^1 = a^Z - a^0 - (\rho + \alpha)\lambda_{Z0} I_A^Z \ .$$

Thus:

$$\begin{aligned}
\Delta &= \pi_Z \left[-a^Z I_Z^Z + \left(a^0 + (\rho + \alpha)\lambda_{Z0} I_A^Z \right) I_Z^Z + I_Z^Z a^Z + \rho\lambda_{A0} I_A^Z I_Z^A \right] \\
&= \pi_Z \left[\left(a^0 + (\rho + \alpha)\lambda_{Z0} I_A^Z \right) I_Z^Z + \rho\lambda_{A0} I_A^Z I_Z^A \right] \\
&\equiv \pi_Z \Delta_0 \ > 0 \ .
\end{aligned}$$

Now, we compute the expressions of Φ_0 and Φ_1. It is easily checked that during the time interval $[\underline{t}_Z, \bar{t}_A)$:

$$\begin{aligned}
\frac{\partial x_c}{\partial \bar{Z}} &= -\frac{\alpha u''}{\zeta \Delta_2} (\lambda_A a_{rr} - c_r') < 0 \ ; \\
\frac{\partial r}{\partial \bar{Z}} &= \frac{\alpha u''}{\zeta \Delta_2} \lambda_A a_{cr} < 0 \ .
\end{aligned}$$

Hence:

$$\Phi_0 = -\int_{\underline{t}_Z}^{\bar{t}_A} \left[a_c \frac{\partial x_c}{\partial \bar{Z}} + a_r \frac{\partial r}{\partial \bar{Z}} \right] dt > 0 \ .$$

Secondly:

$$\frac{\partial \bar{V}}{\partial \bar{Z}} = \frac{\alpha \underline{c}_s}{\zeta} \ .$$

Hence:

$$\Phi_1 = \frac{\alpha \underline{c}_s}{\zeta} \frac{d\bar{\lambda}_A}{d\bar{V}} > 0$$