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A COMPLETE SOLUTION TO A CLASS OF PRINCIPAL–AGENT PROBLEMS WITH AN APPLICATION TO THE CONTROL OF A SELF-MANAGED FIRM

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1. Introduction

This paper considers a class of principal–agent problems which have the following features.

(1) There is adverse selection because the principal ignores the value of *one* parameter of the agent's true characteristics.

(2) Leaving aside the information parameter, the principal's welfare as well as the agent's welfare depend on two types of variables, observable to both of them. The first ones, possibly multidimensional, are called action variable(s), and the second one, which is one-dimensional, has in general the meaning of a money transfer.

(3) The principal is a Stackelberg leader of the two-person game. He can commit himself to decision rules which are admissible on informational grounds. He optimizes within the adequate class, taking into account, besides the agent's reaction, one constraint which has generally the meaning of an individual rationality constraint and sometimes of a feasibility constraint. The optimization is limited to the class of non-stochastic mechanisms.

(4) The problem can also receive an alternative interpretation: the principal faces a continuum of agents of unknown characteristics (the distribution being known, however).

Stylized principal agent problems of this type have often been considered in the economic literature. In particular, the reader will later be able to recognize that the standard income tax model à la Mirrlees (1971), the quality (or quantity) choice model of the monopolist à la Mussa and Rosen (1978) [or Maskin and Riley (1982)], and the model of government regulation of the private monopolist of Baron and Myerson (1982) all belong to

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the category we have just briefly defined. In fact, it is still another question which has awakened our interest for this class of models. We have been motivated both by the extension to the context of incomplete information of a previous work on public policy towards self-managed firms [Guesnerie and Laffont (1984a)], and also by a more general ongoing investigation of the question of government control of public firms under incomplete information [see Guesnerie and Laffont (1984b)]. Some of the assumptions as well as the general emphasis of the present study reflect the needs of our particular investigation and especially of the problem of public control of a self-managed firm (which is completely solved in section 4). However, we have attempted to extract the essence of the argument and to express it in a systematic way to cover the class of problems meeting the following conditions:

(1) The agent's preferences depend upon the action variables, the information parameter and the money transfer. They are monotone with respect to the money transfer.

(2) The principal's preferences are studied first in the polar case where the principal is only concerned with the level of the action variable. He is then indifferent to transfers. The results obtained in this case provide a basis for a comprehensive analysis of the case where the disutility of money transfers for the principal is linear.[1]

Most of the arguments and ingredients we use for the solution of the class of principal–agent problems under consideration are not new. Assumption (CS) can be traced back at least to Mirrlees (1971) or Spence (1974), the 'differential' technology has been popularized by Laffont and Maskin (1980), 'bunching' has already been analysed in the special case of Mussa and Rosen (1978), and existence and uniqueness questions have been precisely assessed in the model of Delpech and Lollivier (1982). Also, our corollary 3.1 has arguments, similar to the ones of Lollivier and Rochet (1983), and results closely reminiscent of Maskin and Riley (1982). However, with respect to the existing literature, our analysis can be differentiated by the following characteristics:

(1) It meets the specific needs for the analysis of government control of the self-managed firm which we solve in section 4. This problem has been considered by Thompson (1982) who provided interesting insights and a general conclusion of non-implementability of the first best which goes well in line with our analysis. Here, we provide a complete solution, which faces the difficulty of the intrinsic non-separability of the labour managed utility function and provides an analysis of independent interest. One of our results deserves to be stressed for its relevance to the general theory of incentives: the model of the self-managed firm in its simplistic multiplicative version

[1]This assumption rules out from our field of application the general version of the income tax problem but permits the complete treatment of some specific versions [see Lollivier and Rochet (1983)].

provides as a subproduct an example of a situation where incentives mechanisms are unable to extract information usable by the principal.

(2) The paper puts more emphasis on rigor than is usual in this segment of the incentives literature. It attempts to be precise and careful on assumptions and statements of results.[2] This observation concerns the implementability as well as the optimization questions. As one example among many, the analysis of sufficiency conditions for implementability when the agent has a non-separable utility function shows the need of a boundary assumption which has been overlooked in the few contributions which have faced this problem.

(3) The paper is much concerned by the systematization of the argument. Implementation and part of optimization are considered in the case where the action vector is multidimensional. Also, we show how various models can be put in our framework. The theorems are in fact directly applicable to most of the models mentioned before but also to a class of models of government control of public firms which are considered elsewhere [Laffont and Tirole (1984)] and other models can be easily adapted to fit our framework.

(4) The paper gives a converging algorithm allowing the computation of optimal solutions in cases where they cannot be solved analytically.

The rest of the paper is organized as follows. Subsection 2.1 presents the model. The incentives point of view is compared with the more traditional 'taxation' point of view. Definitions of implementability of action-profiles are recalled. Subsection 2.2 provides necessary conditions, and then necessary and sufficient conditions for implementability. Section 3 is devoted to the optimization problems under incentives constraints, under the assumption that the principal has type A preferences, and that the action variable is one-dimensional. An algorithm is presented in subsection 3.3. Section 4 is devoted to the study of the control of a self-managed firm. Section 5 extends the results of optimization to type B preferences and to the multidimensional case. A number of applications of the results are sketched.

2. Implementation

For the sake of simplicity, we will generally interpret the model presented here as involving one principal and one agent. However, the reader should remember that it is applicable in cases of a relationship between one principal and many agents, as explained in more detail below.

[2]However, things are relative and the paper is not intended to be a paper in mathematical economics, having often traded off rigor and generality on the one hand, simplicity of presentation on the other hand. Particularly, we limit our attention to piecewise differentiable mechanisms. It turns out that this is not a severe restriction in most of the cases we have treated here where it could be shown that the second-best optimal mechanism under incentives constraints belongs to this latter category. This is, however, an instance where generality has been sacrificed for simplicity of presentation.

2.1. Preliminaries: The model and the questions

In this model the relationship between the principal and the agent(s) involves only two types of variables. The first type is associated with a vector of L decision variables, denoted l, which are observable to both actors. With respect to the specific models we have in mind, it can be called an *action* vector; in section 4, it will be the one-dimensional quantity of labor allocated to the firm (the agent) by the planner (the principal); it may also be the quality of the product when the principal is a monopolist and the agents are buyers.

The variable of the second type, denoted t, is one-dimensional. It has generally the meaning of a money transfer from the principal to the agent. But it could be some other flow variable, under the condition that it affects monotonically the agent's preferences. The principal and the agent interacting through these two variables, the main features of the relationship, can be described as follows.

(1) There is a one-dimensional parameter θ which is known to the agent but unobservable to the principal. This parameter is assumed to belong to some closed *connected* set $\Theta \subset \mathbb{R}_+$. Without loss of generality we take $\Theta = [a, b]$. The principal has some a priori probability on Θ, which is associated with a continuous probability density function $v(\theta)$ s.t. $v(\theta) > 0$ for any θ in Θ. This function either reflects the principal's subjective assessment of the probability of θ when there is only one agent, or when there are many agents the objective distribution of their characteristics which is then assumed to be known by the principal.

(2) The agent's welfare is represented by a utility function depending upon l, t, the action vector and the transfer, and θ the unknown parameter. It is denoted $\mathscr{U}(l, t, \theta)$. It is defined on $X \times \Theta$, where X has the form $[\varepsilon, +\infty[^L \times \mathbb{R}$, with $\varepsilon > 0$. It is always assumed to be *strictly increasing in t*.

(3) Preferences of the principal are of two possible types.

Either (*type A*), they do not depend upon the transfer but only upon the action of the agent, l, and possibly the true value of the unknown parameter, θ. The corresponding utility function is denoted $W(l, \theta)$ and if the agent of characteristic θ takes the action $l(\theta)$, the expected value of the principal's utility is:

$$W = \int_{\Theta} W(l(\theta), \theta) v(\theta) \mathrm{d}\theta. \tag{2.1}$$

These type A preferences occur in particular in our framework when the principal is a planner or a government who has no concern for the transfers made to the firm, i.e. when the transfers are 'socially indifferent'.

Or (*type B*), they depend in addition linearly on transfers:

$$W = \int_{\Theta} W(l(\theta), \theta) v(\theta) \mathrm{d}\theta - \int_{\Theta} \lambda(\theta) t(\theta) \mathrm{d}\theta, \tag{2.2}$$

where $\lambda(\theta) \overset{\text{def}}{=} k(\theta)\nu(\theta)$. Whatever the preferences of the principal, he has to take into account constraints which generally concern the welfare of the agent. These constraints can take different forms. If θ is the value of a variable unknown to the agent when the binding contract between the principal and the agent is decided upon, the constraint is of the form: the expected value of the agent's welfare is greater than some a priori specified value. This constraint can be written:

$$\int_{\Theta} A(l(\theta), t(\theta), \theta)\nu(\theta)\,\mathrm{d}\theta \geqq \bar{u}, \tag{2.3a}$$

where $\bar{u}$ is the reservation level of utility and A some function which will fit the above interpretation as soon as it coincides with $\mathscr{U}$. [In fact, with this more general version, we leave open the possibility of considering a feasibility constraint rather than an individual rationality constraint (see section 5).]

In other problems, for instance the one we will consider in section 4, it is assumed that the contract is signed (or can be broken) after θ is known by the agent; the welfare constraint is then an individual rationality constraint for each value of θ:

$$\mathscr{U}(l(\theta), t(\theta), \theta) \geqq \bar{u}, \quad \forall \theta \in \Theta. \tag{2.3b}$$

It should be clear that with a principal's welfare function of type A, constraint (2.3b) has no obvious reason to be binding at the optimum. On the contrary, with type B preferences the constraint is normally binding for some θ since it is in the interest of the principal to decrease as much as possible the transfer to the agent.

We now focus attention on the abstract mechanisms which could be designed to govern the relationship between the principal and the agent. A mechanism can be viewed as a procedure giving the decision to the principal who commits himself to a decision rule relating the choice of l and t to messages sent by the agent. Mechanisms have the simultaneous purpose of extracting information and of making the decision. It is known from the incentives literature — this is the so-called revelation principle — that any mechanism is isomorphic to a revelation mechanism, called direct, by which the principal elicits truthful answers about θ. Therefore, without loss of generality, we restrict attention to direct mechanisms; such a mechanism is defined as a *vector function* $l, t: \theta \in \Theta \to \{l(\theta), t(\theta)\} \in X$ which relates the decision to the answer of the agent about his parameter.

Two definitions are now in order:

Definition 1. An action profile $l: \theta \in \Theta \to l(\theta)$ is *implementable via compensatory transfer* (for short, we will often say implementable) if there exists a transfer function $t(\theta)$ such that the revelation mechanism $(l(\theta), t(\theta))$ induces truthful revelation, i.e.

$$\mathcal{U}(l(\theta), t(\theta), \theta) \geqq \mathcal{U}(l(\theta'), t(\theta'), \theta), \quad \forall \theta, \theta' \in \Theta^2.$$

In other words, given **l**, **t**, the announcement of the truth is an optimal strategy for the agent, whatever the truth.

Definition 2. The revelation mechanism **l**, **t** is said to be *truthful* when

$$\mathcal{U}(l(\theta), t(\theta), \theta) \geqq \mathcal{U}(l(\theta'), t(\theta'), \theta), \quad \forall \theta, \theta' \in \Theta^2.$$

In other words, **l**, **t** is a truthful mechanism when **t** is a transfer function which makes **l** an action profile implementable via a compensatory transfer. Definition 2 puts the emphasis on the whole mechanism, while definition 1, which puts the emphasis only on the action profile, is better suited to the study of the case of type A preferences.

We next consider the following questions.

(1) What are the action profiles which are implementable via compensatory transfers or what are the truthful revelation mechanisms? This is the *implementability problem* (subsection 2.2).

(2) Among these truthful mechanisms or among these implementable action profiles, what are the best from the principal's view point? This is the *optimization problem* (section 3).

Before going into these problems two preliminary observations are worth making.

(a) When the principal has complete information about θ, his optimization problem consists in maximizing his welfare under the constraints of type (2.3a) or (2.3b) but for a given known θ. This optimization determines the first-best optimum. For example, in cases of type A preferences and if W is differentiable and strictly concave in l, the first-best action profile $\bar{l}$ is defined by

$$\partial_l W(\bar{l}(\theta), \theta) = 0. \tag{2.4}$$

More generally, the first-best optimum is characterized by a function (more generally a correspondence) $\bar{l}, \bar{t}$ which associates to every θ a couple $\bar{l}(\theta)$, $\bar{t}(\theta)$ to solve: max W under (2.3a) or (2.3b), for given θ. Given the first-best action profile, one may wonder, following the Clarke–Groves approach in the problem of revelation of preferences for public goods, whether it is implementable via compensatory transfers in the sense of definition 1. If it is the case and if preferences are of type A (the principal is indifferent to transfers), it is straightforward that this first-best action profile will also be the solution to our optimization problem. However, this will not be the general case. The maximization of the expected value of the principal's welfare under the implementability constraints will determine a *second-best optimum.* The difference between the principal's welfare at the second-best optimum and his expected welfare (before he knows θ) in a first-best solution can be viewed as the cost associated with informational constraints. It is generally non-zero.

(b) The direct mechanisms described here can be viewed as decision rules to which the principal commits himself. Then the decision about the action is taken by the principal and the relationship can be interpreted as taking place in a 'semi-planning' context. Instead, one could imagine that the relationship between the principal and the agent is governed by market rules. Instead of designing a decision rule, the principal would leave the action under the agent's control but would relate the transfer to the action taken, through an adequate reward scheme.

Precisely, a *reward scheme* is defined as a function $\xi: l \to \xi(l)$ which associates a net payment $\xi(l)$ with the choice of the action l.

In many applications, and in particular in section 4, the reward scheme is better interpreted as a *non-linear tax schedule.*

Faced with a reward scheme (or tax schedule) the agent of characteristic θ solves the program $P_{\xi,\theta}$:

$$P_{\xi,\theta} \equiv \max_{(l,t)} \{\mathscr{U}(l,t,\theta): t \leqq \xi(l)\}.$$

It is important to understand that the 'market' approach that we have just sketched and the 'semi-planning' approach based on mechanisms are, in this model, basically equivalent. We will state this more precisely as proposition 1.

Proposition 1. A mechanism $\mathbf{l}, \mathbf{t}$ *is truthful if and only if there exists a tax schedule* ξ *such that,* $\forall \theta \in \Theta$, $l(\theta), t(\theta)$ *is a solution of the program* $P_{\xi,\theta}$.

The proof of the above assertion can be sketched as follows.

(i) A reward scheme can be considered as an indirect revelation mechanism where the message consists in the announcement of l, and which in the class of environments considered (given distribution ν) has a solution in 'dominant strategy'. Hence, according to the revelation principle there is a 'straightforward' (i.e. direct and with dominant strategy equilibria) mechanism which gives the same outcome.

(ii) Assume reciprocally that $\mathbf{l}$ is implementable and consider $Y = \cup_\theta [l(\theta), t(\theta)]$, where t is any compensatory transfer function associated with l. Define ξ as a function which associates with a given l either $-\infty$, if this l does not belong to the set $\cup_\theta l(\theta) \overset{\text{def}}{=} l(\Theta)$, or the unique t (because of the monotonicity in t of the principal's preferences) such that $(l,t) \in Y$ if l belongs to $l(\Theta)$.

We can immediately check that ξ is a tax schedule and that $l(\theta)$ is a solution of $P_{\xi,\theta}$. Q.E.D.

Hence, 'attainable actions and transfers' are identical for a principal relying on abstract revelation mechanisms and for a principal relying on market rules and using sophisticated reward schemes (or tax schedules). A fortiori, the maximization problems in both contexts are identical and lead to the same optimal level of social welfare.

The whole analysis will be conducted in terms of mechanisms, but it is worth reinterpreting the results, particularly in section 4, in terms of reward schemes or tax schedules.

2.2. Implementability

From now on, two assumptions will be made on the agent's utility function. On the 'relevant' domain $X \times \Theta$ the following hold:

(M) *monotonicity*: $\mathcal{U}$ is strictly increasing in t.

(D) *differentiability*: $\mathcal{U}$ is continuously differentiable of class C^2.

It follows from (M) and (D) that $\partial_l \mathcal{U}/\partial_t \mathcal{U}$, the vector of marginal rates of substitution between l and t, and $(\partial/\partial\theta)(\partial_l \mathcal{U}/\partial_t \mathcal{U})$, its partial derivative with respect to θ, exist.

We will limit our study to action profile functions $l:\theta \in \Theta \to l(\theta)$ which are piecewise continuously differentiable of class C^1 and strictly positive.[3] Although some of our results can be generalized to any implementable action profile, significant additional difficulties can be avoided with this assumption; we will show in the next section that the welfare optimal allocation of our problem actually belongs to the class of piecewise C^1 functions.

We can now state a necessary condition for implementation.

Theorem 1. Assume (M) *and* (D). *If a piecewise C^1 action profile is implementable via compensatory transfers, then necessarily*:

$$\left[\frac{\partial}{\partial\theta}\left(\frac{\partial_l \mathcal{U}}{\partial_t \mathcal{U}}\right)_{(l,t,\theta)}\right]\frac{\mathrm{d}l}{\mathrm{d}\theta}(\theta) \geqq 0, \tag{2.5}$$

or equivalently:

$$\sum_i \frac{\partial}{\partial\theta}\left(\frac{\partial_{l_i} \mathcal{U}}{\partial_t \mathcal{U}}\right)_{(l,t,\theta)}\frac{\mathrm{d}l_i}{\mathrm{d}\theta}(\theta) \geqq 0, \tag{2.5'}$$

for any l, t, θ such that $l = l(\theta)$, $t = t(\theta)$ and l is differentiable at θ.

[3]That is, almost everywhere a derivative exists; at points where the derivative does not exist, left-hand-side and right-hand-side derivatives exist. The positivity of l is unessential; it is assumed for technical convenience.

Proof. Consider first a piecewise C^2 allocation l *which is implementable via a piecewise* C^2 transfer function t. Let $\hat{\theta}$ denote the true value and consider the maximization problem of the agent:

$$\max_{\theta \in \Theta} \mathscr{U}(l(\theta), t(\theta), \hat{\theta}) \quad \text{and let} \quad \varphi(\theta, \hat{\theta}) \stackrel{\text{def}}{=} \mathscr{U}(l(\theta), t(\theta), \hat{\theta}).$$

At any θ where $\mathrm{d}l/\mathrm{d}\theta$ and $\mathrm{d}t/\mathrm{d}\theta$ exist, consider

$$\partial_1 \varphi(\theta, \hat{\theta}) = (\partial_l \mathscr{U})_{(l(\theta), t(\theta), \hat{\theta})} \cdot \left(\frac{\mathrm{d}l}{\mathrm{d}\theta}\right)_{(\theta)} + (\partial_t \mathscr{U})_{(l(\theta), t(\theta), \hat{\theta})} \left(\frac{\mathrm{d}t}{\mathrm{d}\theta}\right)_{(\theta)} \tag{2.6}$$

and $\partial_{11}\varphi(\theta, \hat{\theta})$ the derivative of $\partial_1 \varphi$ with respect to θ. Following a standard argument, we know that if the truth is an optimal response for the agent, we have necessarily:

$$\partial_1 \varphi(\theta, \theta) = 0 \quad \text{(first-order condition)}, \tag{2.7}$$

$$\partial_{11} \varphi(\theta, \theta) \leqq 0 \quad \text{(second-order condition)}. \tag{2.8}$$

Eqs. (2.7) and (2.8) are true for almost all θ, and since the function on the left-hand side of (2.7) is (almost everywhere) identical to zero, it has almost everywhere a zero derivative and for almost all θ:

$$\partial_{11} \varphi(\theta, \theta) + \partial_{12} \varphi(\theta, \theta) = 0. \tag{2.9}$$

So (2.8) is equivalent to:

$$\partial_{12} \varphi(\theta, \theta) \geqq 0 \tag{2.10}$$

or

$$(\partial^2_{l\theta} \mathscr{U})_{(\cdot)} \cdot \left(\frac{\mathrm{d}l}{\mathrm{d}\theta}\right)_{(\theta)} + (\partial^2_{t\theta} \mathscr{U})_{(\cdot)} \left(\frac{\mathrm{d}t}{\mathrm{d}\theta}\right)_{(\theta)} \geqq 0 \quad \text{with } (\cdot) = (l(\theta), t(\theta), \theta).$$

Using (2.7) and dividing by $\partial_t \mathscr{U}$ we obtain (always with vector notation):

$$\left(\frac{\mathrm{d}l}{\mathrm{d}\theta}\right)_{(\theta)} \cdot \left[\frac{(\partial^2_{l\theta} \mathscr{U})(\partial_t \mathscr{U}) - (\partial^2_{t\theta} \mathscr{U})(\partial_l \mathscr{U})}{\partial_t \mathscr{U}}\right] \geqq 0,$$

which implies (2.5).

We next prove that the transfer function associated with a piecewise C^2 allocation is necessarily piecewise C^2. Considering that $\varphi(\theta, \hat{\theta})$ is maximum for $\theta=\hat{\theta}$ and writing down a Taylor expansion, we can study a $\hat{\theta}$, where $dl/d\theta$ exists, the limit of $\Delta t/\Delta\theta$ when $\theta\to\hat{\theta}-0$ and $\theta\to\hat{\theta}+0$. We show then that $(dt/d\theta)$ exists for almost every θ and that it satisfies eq. (2.3); then, straightforwardly from (D), $d^2t/d\theta^2$ also exists.

Finally, since any piecewise C^1 function can be approximated by piecewise C^2 functions as closely as desired, property (2.5) is true by continuity for C^1 action profiles. Q.E.D.

An intuitive understanding of the monotonicity condition (2.5) can be obtained from fig. 1. An agent's utility is increasing in l and t; indifference curves for agent θ and agent $\theta+d\theta$ are represented under the assumption $(\partial/\partial\theta)(\partial_l\mathscr{U}/\partial_t\mathscr{U})>0$. Given that $l(\theta), t(\theta)$ is chosen by agent θ, when $l(\theta+d\theta)$ is larger than $l(\theta), t(\theta+d\theta)$ can be chosen so that the point $(l(\theta+d\theta), t(\theta+d\theta))$ is below agent θ's indifference curve and on agent $(\theta+d\theta)$'s indifference curve [through $l(\theta), t(\theta)$]. This incentive condition cannot be met if, on the contrary, $l(\theta+d\theta)$ is smaller than $l(\theta)$.

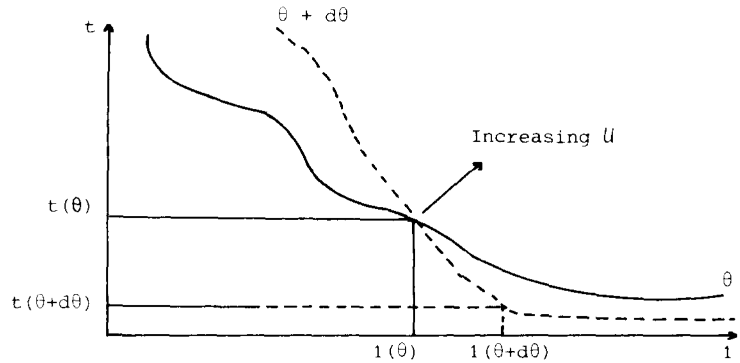

Fig. 1

Note also that condition (2.5) can be viewed as a condition of minimal compatibility between individual preferences and the candidate action profile.

Let us now investigate the problem of sufficient conditions for implementation. The reciprocal of theorem 1 does not hold even when l is one-dimensional. In that case, a function l, the derivative of which changes its sign over Θ, creates a problem. In fact, when $\partial_t\mathscr{U}$ is strictly positive, 'most' allocation functions with a changing sign derivative are not implementable. Furthermore, no simple sufficient condition can guarantee that a given function of this type belongs to the 'small' subset of implementable ones.[4] For that reason, we will restrict ourselves to consider the problem of sufficiency in the case where $(\partial/\partial\theta)(\partial_l\mathscr{U}/\partial_t\mathscr{U})$ has a constant sign (implying

[4]It is beyond the scope of this paper to give a precise meaning to this assertion.

that one-dimensional candidates for implementability can be limited from theorem 1 to functions l whose derivative has the same constant sign).

The assumption is formally written as condition CS.

(CS): *Constant sign of the marginal rate of substitution.* On the relevant domain of l, t, θ, the signs of the components of the vector $(\partial/\partial\theta)(\partial_l\mathcal{U}/\partial_t\mathcal{U})$ remain the same.

Without loss of generality, but possibly after changing some l_i in $-l_i$, we can assume that this sign is positive; a case that we stress by referring to condition (CS+). Before stating the sufficiency theorem, we must introduce one additional condition on the boundary behavior of the principal's utility function.

(B): *Boundary behavior of* $\mathcal{U}$. For any $(l,t,\theta)\in X\times\Theta, \exists K'>0$ such that for t large enough

$$\left\|\left(\frac{\partial_l\mathcal{U}}{\partial_t\mathcal{U}}\right)_{(l,t,\theta)}\right\| \leqq K'|t|, \quad \text{uniformly in } l,\theta.$$

This condition indicates that the marginal rates of substitution between l and t do not increase too fast when t increases. This condition is trivially satisfied with quasi-linear (additive and linear in t) $\mathcal{U}$ and also holds, for example, with a Cobb–Douglas utility function.

Since here $\partial_l\mathcal{U}/\partial_t\mathcal{U}$ is continuous, condition (B) implies that if l remains in a bounded cube, for any t, $\|(\partial_l\mathcal{U}/\partial_t\mathcal{U})\| \leqq K'|t| + K''$ for some well chosen K''. Note also that the condition is satisfied if $\partial_l\mathcal{U}/\partial_t\mathcal{U}$ is Lipschitzian in t on $X\times\theta$.

Let us now state theorem 2.

Theorem 2. Assume that the agent's utility function satisfies assumptions (M), (D), (CS+), *and* (B). *Then any piecewise C^1 action profile such that its derivative is non-negative,* $\mathrm{d}l/\mathrm{d}\theta \geqq 0$, *is implementable via compensatory transfers.*

Proof. Consider a candidate allocation $\bar{l}$ meeting the conditions of the theorem. Consider the following differential equation defined for every θ where $\mathrm{d}\bar{l}/\mathrm{d}\theta$ is defined, i.e. almost everywhere (a.e.):

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{(\partial_l\mathcal{U})_{(\bar{l}(\theta),t(\theta),\theta)}}{(\partial_t\mathcal{U})_{(\bar{l}(\theta),t(\theta),\theta)}}\left(\frac{\mathrm{d}\bar{l}}{\mathrm{d}\theta}\right)_{(\theta)}. \tag{2.11}$$

Thanks to (B), we have:

$$\left|\frac{\mathrm{d}t}{\mathrm{d}\theta}\right| \leqq K(K'|t|+K''), \quad \text{when } K \stackrel{\text{def}}{=} \sup_{\theta\in\Theta}\left\|\frac{\mathrm{d}\tilde{l}}{\mathrm{d}\theta}\right\|.$$

Consider $\bar{t}$ defined by

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}\theta} = K'K\bar{t} + K'',$$

i.e.

$$\bar{t}(\theta) = \left(\bar{t}(0) + \frac{K''}{K'K}\right)\mathrm{e}^{KK'\theta} - \frac{K''}{K'K}.$$

Clearly, $\bar{t}(1)$ is an 'a priori majoration' of t defined by (2.11), [with the initial conditions $t(0)=\bar{t}(0)$]. Hence, according to a theorem in the theory of differential equations [see Demazure (1979, p. 42)], (2.11) has a global solution on Θ ,$\forall t(0)$.

Let $\tilde{t}(\theta)$ be one solution of (2.11) (which depends on the initial point). We will show that an agent faced with the mechanism $\tilde{l}$, $\tilde{t}$ is induced to tell the truth.

For that, consider $\varphi(\theta,\hat{\theta})$ defined (as in the proof of theorem 1) by:

$$\varphi(\theta,\hat{\theta}) = \mathscr{U}(\tilde{l}(\theta),\tilde{t}(\theta),\hat{\theta})$$

and a.e.

$$\lambda(\theta,\hat{\theta}) = \left(\frac{\partial_l\mathscr{U}}{\partial_t\mathscr{U}}\right)_{(\tilde{l}(\theta),\tilde{t}(\theta),\hat{\theta})}\left(\frac{\mathrm{d}\tilde{l}}{\mathrm{d}\theta}\right)_{(\theta)} + \left(\frac{\mathrm{d}\tilde{t}}{\mathrm{d}\theta}\right)_{(\theta)}.$$

Clearly, for $\theta<\hat{\theta}$:

$$\varphi(\hat{\theta},\hat{\theta}) - \varphi(\theta,\hat{\theta}) = \int_\theta^{\hat{\theta}} \partial_1\varphi(\tau,\hat{\theta})\,\mathrm{d}\tau = \int_\theta^{\hat{\theta}} (\partial_t\mathscr{U})_{(\cdot)}\lambda(\tau,\hat{\theta})\,\mathrm{d}\tau. \tag{2.12}$$

By the very definition of $\tilde{t}$, $\lambda(\hat{\theta},\hat{\theta})=0$ a.e. In any $\theta<\hat{\theta}$, (CS+) implies:

$$\left(\frac{\partial_l\mathscr{U}}{\partial_t\mathscr{U}}\right)_{(\tilde{l}(\theta),\tilde{t}(\theta),\theta)} < \left(\frac{\partial_l\mathscr{U}}{\partial_t\mathscr{U}}\right)_{(\tilde{l}(\theta),\tilde{t}(\theta),\hat{\theta})}$$ (coordinate-by-coordinate inequality),

so that $\lambda(\theta,\theta)<\lambda(\theta,\hat{\theta})$. From $\lambda(\theta,\theta)=0$ a.e. it follows:

$$\lambda(\theta, \hat{\theta}) > 0 \text{ a.e.} \tag{2.13}$$

Hence, for $\theta < \hat{\theta}$, from (M) and (2.13), the integrand of the right-hand side of (2.13) is positive. Hence, $\varphi(\hat{\theta}, \hat{\theta}) > \varphi(\theta, \hat{\theta})$ a.e. A symmetric argument would prove the same equality for $\theta > \hat{\theta}$, a.e. and the argument extends by continuity to any $(\theta, \hat{\theta})$. Q.E.D.

In the proof, condition (B)[5] is crucial for global existence. The reason why it is required is now easy to understand. Incentive compatibility requires that the agent $\theta + d\theta$, in order to choose $l(\theta + d\theta)$, is compensated relatively to the agent θ with a positive dt; however, it may happen that this infinitesimal compensation increases so fast that the total transfer $t(\theta)$ tends to infinity before the upper bound of Θ is reached. Condition (B) rules out such a phenomenon. Although it is satisfied in the labor managed example considered in the next section, it should not be considered a technicality. Its failure is indeed possible and would have undoubtedly consequences in certain problems. An elegant implication of theorems 1 and 2 is the following:

Corollary 2.1. Assume that the agent's utility function satisfies (M), (D), (CS+) [*resp.* (CS−)] *and* (B). *Then a piecewise C^1 one-dimensional action profile l is implementable if and only if:*

$$dl/d\theta \geqq 0 (\textit{resp.} \leqq 0) \textit{ a.e.}$$

It is worth stressing, before closing this section, that the above results make neither any type of convexity assumption upon the utility function, nor a monotonicity assumption of utility with respect to l, two remarks which are crucial for the application to the self-managed case of section 4. Also, the extremely simple necessary and sufficient conditions of corollary 2.1 do not simply extend to the multidimensional case $L > 1$.

3. Optimization under incentives constraints: Type A preferences, one-dimensional action

The optimization problem of the principal consists in choosing the best implementable mechanism from among the ones which have been characterized in the preceding section. When the action is one-dimensional, we obtained through corollary 2.1 a very simple characterization of implementable mechanisms. Combining one-dimensional action and type A preferences will allow us under few additional assumptions to solve completely the optimization problem. This is the purpose of subsection 3.1. Subsection 3.2 proposes an algorithm for the computation of the solution. The results are extended to type B preferences and multi-dimensional actions in section 5.

[5]See appendix A for a different boundary condition.

3.1. Characterization of the optimum

With type A preferences and with an agent of characteristic θ, when the action variable is l, social welfare is $W(l,\theta)$; it depends upon the action taken and the unknown parameter θ, but is independent of transfers. Recall that the planner's a priori (subjective) probability distribution about the unknown parameter θ is characterized by a continuous density function $v(\theta)$ which is strictly positive on the support $\Theta=[a,b]$. Moreover, we make the following regularity assumption.

(R): *Regularity of W.* $W(l,\theta)$ is strictly concave in l and C^2, and the C^1 solution $l(\theta)$ of $(\partial W/\partial l)(l,\theta)=0$ is such that $\mathrm{d}l/\mathrm{d}\theta$ changes sign a finite number of times.

The expected social welfare associated with a labor allocation $l(\theta)$ is:

$$\int_a^b W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta.$$

Assume from now that the action profile l is one-dimensional $L=1$. Under the assumptions of corollary 2.1 with (CS+) and (B), a piecewise C^1 action profile is implementable if and only if $\mathrm{d}l/\mathrm{d}\theta \geqq 0$ almost everywhere. Restricted to the class of piecewise C^1 action profiles,[6] and with (B), the planner's choice is determined by the following program:

$$\max \int_a^b W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta$$

s.t.

$$\mathrm{d}l/\mathrm{d}\theta \geqq 0 \quad \text{a.e.} \tag{3.1}$$

The set of piecewise C^1 action profiles satisfying the implementability condition is convex and the objective function, as a function of $l(\cdot)$, is strictly concave: if the program has a solution, it is unique. Existence is not a trivial question, but since it is technical, it is relegated to appendix B. In fact we prove slightly more than existence.

[6]We do not consider here a broader set of functions, since we have characterized implementability only for piecewise C^1 profiles. In fact, on a broader set of functions, implementability is equivalent to the property of being weakly increasing. Over this 'universal domain', when (R) holds, the social optimum is indeed piecewise C^1. Therefore our approach does not involve any differentiability restriction but (R).

Theorem 3. Under (R), *program* (3.1) *has a unique solution which furthermore has the property of having a finite number of points of discontinuity for its derivative.*

Then, let us denote $\mu(\theta)=(\mathrm{d}l/\mathrm{d}\theta)(\theta)$. Using the control $\mu(\theta)$ restricted to the class of piecewise continuous and non-negative functions we write the following equivalent problem:

$$\max \int_a^b W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta$$

s.t.

$$\frac{\mathrm{d}l}{\mathrm{d}\theta}=\mu(\theta),^7$$

$$\mu(\theta)\geqq 0. \tag{3.2}$$

It is a classical non-autonomous control problem with free boundaries and an inequality constraint on the control; we can use the maximum principle to describe the solution.

The Hamiltonian is:

$$H(\theta,l,\mu,\delta)=W(l(\theta),\theta)\,v(\theta)+\delta(\theta)\mu(\theta).$$

The necessary and sufficient conditions derived from the Pontryagin principle [see theorems 4-3-1 and 5-7-1 in Hadley and Kemp (1971)] are:

$$\frac{\mathrm{d}\delta}{\mathrm{d}\theta}(\theta)=-\partial_l W(l(\theta),\theta)\,v(\theta), \tag{3.3}$$

$$\frac{\mathrm{d}l}{\mathrm{d}\theta}(\theta)=\mu(\theta), \tag{3.4}$$

$$\delta(a)=\delta(b)=0 \quad \text{(transversality conditions)}, \tag{3.5}$$

$$\mu(\theta) \quad \text{maximizes} \quad H(\theta,l,\mu,\delta) \quad \text{under the constraint} \quad \mu(\theta)\geqq 0. \tag{3.6}$$

Integrating the piecewise continuous [and therefore integrable over (a,b)] function $(\partial_l W)v$, and using the transversality conditions, we have:

$$\delta(b)-\delta(a)=0=-\int_a^b \partial_l W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta. \tag{3.7}$$

[7]At points of non-differentiability, we define the derivative of $l(\cdot)$ as the right derivative.

Integrating (3.3) between a and θ, we get:

$$\delta(\theta) = -\int_a^\theta \partial_l W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta. \tag{3.8}$$

At the solution, from (3.6) we know that $\delta(\theta) \leqq 0$, or:

$$\int_a^\theta \partial_l W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta \geqq 0, \quad \forall \theta \in [a,b]. \tag{3.9}$$

Moreover, if $\delta(\theta) < 0$, then from (3.6) $\mu(\theta) = 0$; then

$$\frac{\mathrm{d}l}{\mathrm{d}\theta}(\theta)\left[\int_a^\theta \partial_l W(l(\theta),\theta)\,v(\theta)\,\mathrm{d}\theta\right] = 0, \quad \forall \theta \in [a,b]. \tag{3.10}$$

A more concrete description of the solution can be obtained from (3.7), (3.9) and (3.10). Let us call $l^*(\theta)$ the solution.

If $\mathrm{d}l^*/\mathrm{d}\theta(\theta) > 0$ in an interval, then $l^*(\theta) = \bar{l}(\theta)$. Indeed, if $\mu(\theta) > 0$, $\delta(\theta) = 0$ in this interval; then $(\mathrm{d}\delta/\mathrm{d}\theta)(\theta) = 0$ in this interval, and from (3.3) $\partial_l W(l^*(\theta),\theta) = 0$, implying $l^*(\theta) = \bar{l}(\theta)$.

The optimal solution coincides with $\bar{l}(\theta)$ or is constant. We show next how to obtain the finite number of intervals where it is constant. Let (θ'_0, θ'_1) be such an interval; from above:

$$\delta(\theta) = 0 \quad \text{for } \theta < \theta'_0 \text{ close enough to } \theta'_0,$$

$$\delta(\theta) = 0 \quad \text{for } \theta > \theta'_1 \text{ close enough to } \theta'_1.$$

Since the multiplier is continuous, we have:

$$\delta(\theta'_0) = \delta(\theta'_1) = 0.$$

To characterize exactly what happens in the intervals where $l^*(\theta)$ is constant, we must distinguish three cases.

Case (*a*). The first potential interval starts at $\theta_0 = a$ (fig. 2). Integrating (3.3) between a and θ'_1 we have:

$$\int_a^{\theta'_1} \partial_l W(l^1,\theta)\,v(\theta)\,\mathrm{d}\theta = 0 \quad \text{and} \quad l^1 = \bar{l}(\theta'_1). \tag{3.11}$$

Now, if we assume that $l^1 > \bar{l}(a)$, then $l^1 > \bar{l}(\theta)$ for any θ close enough to a, and then $\delta(\theta) > 0$, contradicting (3.6).

Case (*b*). An interior interval (fig. 3). Integrating (3.3) between θ'_0 and θ'_1

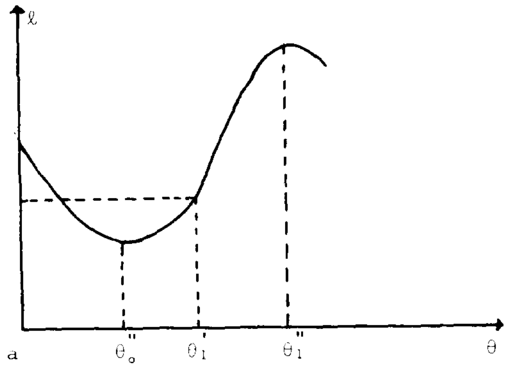

Fig. 2

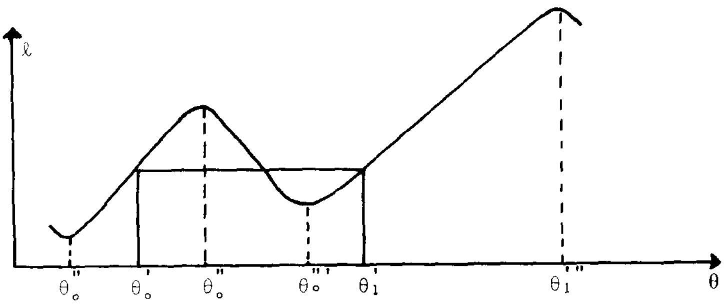

Fig. 3

we have:

$$\int_{\theta'_0}^{\theta'_1} \partial_l W(l^*,\theta)v(\theta)\mathrm{d}\theta=0 \quad \text{and} \quad \bar{l}(\theta'_0)=l^*, \bar{l}(\theta'_1)=l^*. \tag{3.12}$$

Case (*c*). The potential interval ends at $\theta=b$ (fig. 4). As in case (a) the interval, if it exists, is associated with the system of two equations:

$$\int_{\theta'_0}^{b} \partial_l W(l^K,\theta)v(\theta)\mathrm{d}\theta=0, \qquad l^K=\bar{l}(\theta'_0). \tag{3.13}$$

With the same argument as in (a), we show that $l^K \geqq \bar{l}(b)$.

Since $\bar{l}(\theta)$ changes sign a finite number of times, and since we know that there exists a solution, the end result can be represented as in fig. 5.

We gather the results in the next theorem.

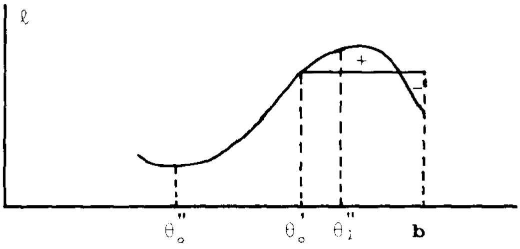

Fig. 4

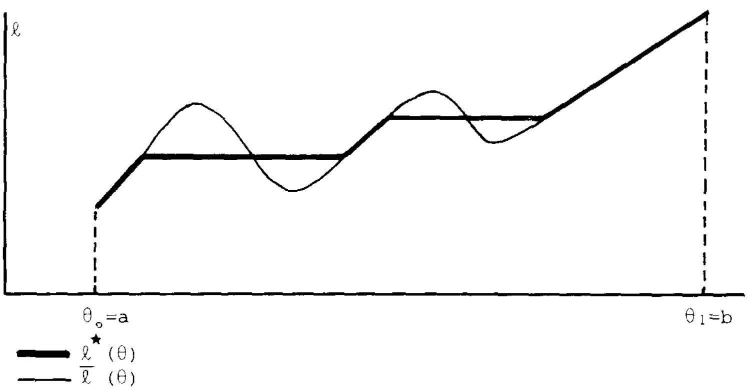

Fig. 5

Theorem 4. Optimization in the case $L=1$. *Under* (M), (D), (CS+), (R) *and* (B), *the unique optimal solution is characterized by a piecewise* C^1 *weakly increasing function* $l^*(\theta)$ *such that*:

(i) $l^*(\theta)$ *concides with* $\bar{l}(\theta)$ *except on a finite number* K *of disjoint intervals* $I_k=(\theta_0^k,\theta_1^k), k=1,\dots,K$, θ_0^k *increasing with* k, *where* $l^*(\theta)=l^k\,\forall\theta\in I^k$.

(ii) *In each interior interval* $(\theta_0^k\neq a, \theta_1^k\neq b)$ *we have*:

$$\int_{\theta_0^k}^{\theta_1^k}\partial_l W(l^k,\theta)\nu(\theta)\,\mathrm{d}\theta=0,$$

$$\bar{l}(\theta_0^k)=\bar{l}(\theta_1^k)=l^k.$$

(iii) *If* I_1 *starts at* $\theta=a$, *we have*:

$$\int_{a}^{\theta_1^1}\partial_l W(l^1,\theta)\nu(\theta)\,\mathrm{d}\theta=0,\qquad \bar{l}(\theta_1^1)=l^1.$$

Furthermore $l^1\leqq\bar{l}(a)$.

(iv) *If I_K ends at $\theta=b$, we have*:

$$\int_{\theta_0^K}^{b} \partial_l W(l^K,\theta)v(\theta)\,\mathrm{d}\theta=0, \qquad \bar{l}(\theta_0^K)=l^K.$$

Furthermore $l^K \geqq \bar{l}(b)$.

(v) *For any* $\theta\in[a,b]$:

$$\int_{a}^{\theta} \partial_l W(l^*(\theta),\theta)v(\theta)\,\mathrm{d}\theta \geqq 0.$$

Note that the results of this section, stated with (CS+), also hold with straightforward modifications when the constant sign condition (CS+) is reversed in (CS−):

$$\frac{\partial}{\partial\theta}\frac{\partial_l \mathscr{U}}{\partial_t \mathscr{U}}<0.$$

3.2. *A constructive algorithm* (*under* (*CS*+))

The fact that, from (R), we know that $\bar{l}(\theta)$ has a finite number of peaks leads to a constructive algorithm that we sketch below (by convention, here $a=0$, $b=1$).

Note first that if $\bar{l}(\theta)$ decreases at $\theta=0$, we consider that $\theta=0$ yields a peak; similarly, if $\bar{l}(\theta)$ decreases at $\theta=1$, $\theta=1$ yields a peak. From now on we neglect the slight specificity (exhibited in theorem 4) of these extreme peaks.

If $\bar{l}(\theta)$ is weakly increasing it is the solution.

Consider first the case where $\bar{l}(\theta)$ has one peak, as in fig. 6. Note that $\partial_l W$ is strictly positive below $\bar{l}(\theta)$, strictly negative above from the strict concavity of W and the definition of $\bar{l}(\theta)$.

The conditions of theorem 4 are necessary and sufficient. Moreover, a constant piece of $l^*(\theta)$ must join two increasing pieces of $\bar{l}(\theta)$. Therefore, with one peak, there is a unique constant piece.

The constant piece must be between AB and CD of fig. 6. Clearly:

$$\int_{\theta_0'}^{\theta_1'} \partial_l W(l',\theta)v(\theta)\,\mathrm{d}\theta>0, \tag{3.14}$$

$$\int_{\theta_0''}^{\theta_1''} \partial_l W(l'',\theta)v(\theta)\,\mathrm{d}\theta<0. \tag{3.15}$$

Consider an intermediary constant piece joining AC to DB with $\bar{l}(\theta)=\bar{l}$:

$$\int_{\theta_0}^{\theta_1} \partial_l W(\bar{l},\theta)v(\theta)\,\mathrm{d}\theta=0, \tag{3.16}$$

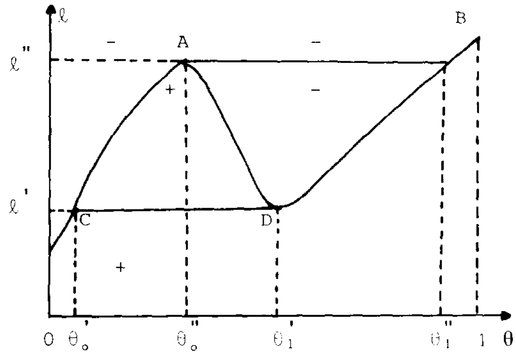

Fig. 6

Then,

$$\bar{l}(\theta_1)=\tilde{l},\tag{3.17}$$

$$\bar{l}(\theta_2)=\tilde{l}.\tag{3.18}$$

Inverting (3.17) and (3.18), and substituting into (3.16) we have:

$$\Phi(\tilde{l})=\int_{\varphi^1(\tilde{l})}^{\varphi^2(\tilde{l})} \partial_l W(\tilde{l},\theta)\nu(\theta)\mathrm{d}\theta$$

and

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\tilde{l}}=\frac{\mathrm{d}\varphi^2}{\mathrm{d}\tilde{l}}(\tilde{l})\partial_l W(\tilde{l},\varphi^2(\tilde{l}))\nu(\varphi^2(\tilde{l}))$$

$$-\frac{\mathrm{d}\varphi^1}{\mathrm{d}\tilde{l}}(\tilde{l})\partial_l W(\tilde{l},\varphi^1(\tilde{l}))\nu(\varphi^1(\tilde{l}))$$

$$+\int_{\varphi^1(\tilde{l})}^{\varphi^2(\tilde{l})}\frac{\partial^2 W}{\partial l^2}(\tilde{l},\theta)\nu(\theta)\mathrm{d}\theta.$$

Since $\partial_l W(\tilde{l},\varphi^1(\tilde{l}))=\partial_l W(\tilde{l},\varphi^2(\tilde{l}))=0$, and since W is strictly concave, $\mathrm{d}\Phi/\mathrm{d}\tilde{l}<0$. Then, from (3.14) and (3.15), the continuity of Φ, and the intermediate value theorem, there exists a unique $\tilde{l}$ such that $\int_{\theta_0}^{\theta_1}\partial_l W(\tilde{l},\theta)\nu(\theta)\mathrm{d}\theta=0$ satisfying the necessary and sufficient conditions (since $\delta(\theta)$ is always non-positive).

Suppose that $\bar{l}(\theta)$ has two peaks. Either it is possible *with an increasing function* to deal as above with each peak separately, or it is not possible.

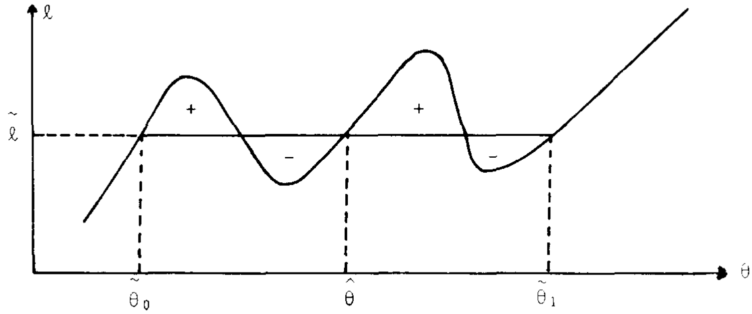

Fig. 7

Then, by a continuity argument similar to the one above there exists $\tilde{\theta}_0$, $\tilde{\theta}_1$, $\tilde{l}$ such that (see fig. 7):

$$\int_{\tilde{\theta}_0}^{\tilde{\theta}_1} \partial_l W(\tilde{l}, \theta) v(\theta) \mathrm{d}\theta = 0,$$

where $[\tilde{\theta}_0, \tilde{\theta}_1]$ contains both peaks. It remains to show that $\delta(\theta) \leqq 0$ for any θ.

$\delta(\theta)$ takes its largest value at $\hat{\theta}$. If $\delta(\hat{\theta})$ were positive, it would mean that:

(1) by continuity, there exists $[\theta_0^1, \theta_1^1]$, l^1 with $\theta_1^1 < \hat{\theta}$ such that

$$\int_{\theta_0^1}^{\theta_1^1} \partial_l W(l^1, \theta) v(\theta) \mathrm{d}\theta = 0; \tag{3.19}$$

(2) since $\delta(\tilde{\theta}_1) = 0$ and $\delta(\hat{\theta}) > 0$,

$$\int_{\hat{\theta}}^{\tilde{\theta}_1} \partial_l W(l(\theta), \theta) v(\theta) \mathrm{d}\theta > 0,$$

and by continuity there exists $[\theta_0^2, \theta_1^2]$, l^2 with $\theta_0^2 > \hat{\theta}$ such that

$$\int_{\theta_0^2}^{\theta_1^2} \partial_l W(l^2, \theta) v(\theta) \mathrm{d}\theta = 0. \tag{3.20}$$

But (3.19) and (3.20) would contradict the fact that it was not possible to deal with both peaks separately with an increasing function. Therefore $\delta(\theta) \leqq 0$ for any $\theta \in [0, 1]$ and the solution with one constant piece satisfies the necessary and sufficient conditions for optimality.

Clearly, the above procedure can be extended to any finite number of peaks.

4. Application to the public control of a self-managed firm

The example presented in this section is in fact the heart of our work on the class of principal–agent problems considered above. We will first explain briefly its motivation, and then place it in the context of more general research on the government control of public firms.

Standard normative theory of the last thirty years most often considered that firms with legal public status were totally in the hands of the government. It was, for example, implicitly assumed that their production sets were known to the center, which was also in a position to dictate their production plans with as much detail as desired. This view of the world does not fit the experience of practitioners, and theorists have now become increasingly aware of the distance between real public firms and ideal 'controlled' firms of second-best models.

For a more realistic modelling of public firms it seems that two important facts should be singled out: first, public firms in general have specific objectives different from those of the 'planner'; secondly, the so-called planner has incomplete information on the inside working and hence on the production possibilities of the firm. It should be understood that the interesting problems of public control of a firm follow from the coexistence of these two features. Under complete information, specific objectives for the firm are irrelevant: the government is able to impose the decisions considered as optimal whatever the reluctance of the firm to implement them. Also, with a total identity of objectives between the center and the public firm, incomplete information would present no problem at all. Also, both ingredients have to be introduced together.

A reasonable formulation on the firm's objectives is likely to be the most difficult issue for the theoretical modelling of the problem. As a complex organization, a public firm is a place where different actors interact (management, unions, etc.) and the outcome of the decision process reflects basic characteristics of the firm which affect the bargaining power of different parties [see, for example, Rees (1982) for a discussion of the subject]. Our analysis started from the simple case of a firm employing homogeneous workers who have no conflict of interest. We assumed, and this is at least a polar case of special relevance, that the management of the firm only reflected the common interest of its existing workers. In that case, everything goes in our original model as if the public firm were labor-managed in the sense of Vanek (1970). The reader who finds it surprising to have public and labor-management associated, should remember the already ancient argument by Ward (1958) concerning the relevance of the self-managed hypothesis for the study of the behavior of nationalized industries.

4.1. The model

We consider a two-commodity economy, with labor and a consumption good. There exists a public firm in an otherwise competitive economy where the marginal productivity of labor is λ. The total quantity of labor in the economy is $\bar{L}$ and all agents have a linear utility function in consumption taken to be $U(x)=x, x>0.$, and feel no disutility from working their whole one unit endowment of labor.

The technology of the public firm requires a fixed cost of k (in good units) and then provides $f(l,\hat{\theta})$ units of good for l units of labor: $\hat{\theta}$ represents a positive real-valued parameter which affects in a continuous way the production possibilities of the firm, and which is known only by the agents working in that firm. We will treat l as a continuous variable and ignore the problems that might arise from the need of a marginal worker who can work only part-time in the L-M firm. We assume that $f(l,\hat{\theta})$ is C^2 in $(l,\hat{\theta})$ and strictly concave in l. The subset of $\mathbb{R}$, where $\hat{\theta}$ can be a priori, is a closed interval Θ. We assume, without loss of generality, that $\Theta=[0,1]$. The planner has incomplete information on the parameter θ; he knows the set Θ but does not know where $\hat{\theta}$ exactly lies in this set.

The firm sells its product in a competitive market, by convention at a unit price. Production decisions in the public firm are made by workers who are assumed to agree on *maximizing per capita value-added.* This criterion assures to the homogeneous workers maximal earnings (which, unless specific mention, will be assumed to be superior to earnings associated with the market wage). The public firm behaves as the prototype labor-managed firm described by Vanek (1970).[8]

We first consider the problem of a planner who has to decide in an authoritarian way on the allocation of labor to the firm.[9] The heading of *semi-planning* is particularly adequate here to characterize this context: allocation of labor is made on the basis of quantity decisions while the firm's output is sold in the market. To fulfill his task in a satisfactory way, the planner needs information on $\hat{\theta}$, but in many cases the labor-managed firm which maximizes per capita value-added will wish to misrepresent its information; it will be the case in particular when the planner attempts to equate the (unknown) marginal productivity of labor in the firm to its social value.

We assume that the planner is utilitarian; if the number of workers employed in the labor-managed firm of characteristic θ is l, given that the transfer t is paid from the rest of society, and given our assumption on the productivity of labor, social welfare equals:

$$\left(\frac{f(l,\theta)-k+t}{l}\right)l-t+\lambda(\bar{L}-l)=f(l,\theta)-k+\lambda(\bar{L}-l).$$

[8]As discussed in the literature, the validity of this criterion may depend upon the number of 'initial' workers in the firm. In other words, it is more satisfactory in the context of the growth of the firm than in the context of contraction.

[9]See Thomson (1982) for a first formal study of this problem.

Under incomplete information, the planner will compare two different mechanisms by looking at the expected values of the above expression. These values depend on a priori expectations on θ which are described by a density function $v(\theta)$ associated with the (subjective or objective)[10] probability distribution of the planner on θ. The problem of the planner is then:

$$\max \int_{\Theta} [f(l(\theta),\theta) - \lambda l(\theta)]\, v(\theta)\, \mathrm{d}\theta$$

under the constraint that $l:\theta \to l(\theta)$ is an implementable labor allocation.

Consider the function $\bar{l}$ defined by

$$(\partial_l f)_{(\bar{l}(\theta,\theta))} = \lambda.$$

Given the objectives of the planner, if he had complete information on θ, he would allocate $\bar{l}(\theta)$ to a firm of characteristic θ; $\bar{l}(\cdot)$ is what we called *the first-best labor allocation.*

4.2. Implementability in the self-managed case

It is clear that the model of the preceding subsection is a particular case of the model of section 2. In other words, we have to specify the general results for

$$\mathscr{U}(l,t,\theta) = \frac{f(l,\theta) - k + t}{l}.$$

In the space (l,t) indifference curves have the shape suggested by fig. 8. The marginal rate of substitution $\partial_l \mathscr{U} / \partial_t \mathscr{U}$ is nothing else than:

$$\partial_l f(l,\theta) - \frac{f(l,\theta) - k + t}{l},$$

so that condition (CS −) takes the specific form of condition C:

Condition C:

$$\frac{\partial}{\partial \theta}\left(\partial_l f(l,\theta) - \frac{f(l,\theta)}{l} \right) \leq 0. \tag{4.1}$$

[10]We have considered here a unique L-M firm. Alternatively, the formulation applies when the L-M sector, although small with respect to the rest of the economy, consists of an infinity of firms, $v(\theta)$ being the density associated with the true distribution.

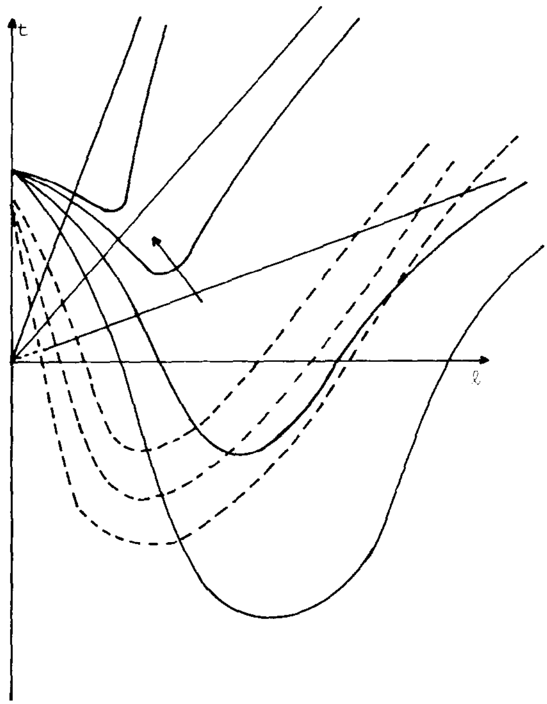

Fig. 8

Condition C asserts that for a fixed l, the derivative with respect to θ of the difference between the marginal and the average product decreases. In general, the interpretation of this condition is difficult; however, note that when θ is a parameter of productivity which operates in a multiplicative way ($f(\theta, l) = \theta f(l)$), condition C reduces to:

$$\partial_l f - \frac{f(l)}{l} \leqq 0,$$

and is automatically fulfilled with a concave f: the marginal product is always smaller than the average one. Then, it is only when f 'substantially' differs from the multiplicative form that the condition may not be verified.

From now on we assume that condition C holds, which is reasonable in view of our interpretation of θ as a productivity parameter. We check that all the conditions of corollary 2.1 hold; (M) as well as (D) are satisfied. From the expression of $\partial_l \mathscr{U}/\partial_t \mathscr{U}$, and the assumption on the domain we can show that (B) holds.

Theorem 5. In the case of the labor-managed public firm assume that f is C^2 and concave and that condition C holds. Then a necessary and sufficient condition for a piecewise C^1 allocation to be implementable is $(\mathrm{d}l/\mathrm{d}\theta)(\theta)\leqq 0$ *a.e. on Θ; and the transfer function satisfies:*

$$\frac{\mathrm{d}t}{\mathrm{d}\theta}(\theta)+\frac{\mathrm{d}l}{\mathrm{d}\theta}(\theta)\left[\partial_l f(l(\theta),\theta)-\frac{f(l(\theta),\theta)-k+t(\theta)}{l(\theta)}\right]=0.$$

Theorem 5 shows that only labor allocations which give a quantity of labor decreasing with the parameter θ are implementable. With the productivity interpretation, intuition suggests that, as a first-best objective, one would like to allocate increased labor for increased θ so that the implementation and first-best requirements are opposite.

Coming back to the maximization problem defined in section 2, we can make precise the above idea as follows.

Corollary 5.1. Under condition C, when $f(\cdot)$ is strictly concave in l, the optimal labor allocation for the first-best problem is implementable if and only if:

$$\partial^2_{l\theta} f(l,\theta)\leqq 0,\quad \forall l\geqq 0, \forall\theta. \tag{4.2}$$

Proof. We know above that the first-best allocation $\bar{l}(\theta)$ is a solution of:

$$(\partial_l f)(\bar{l}(\theta),\theta)=\lambda.$$

Hence, by differentiation,

$$\partial^2_{l\theta} f+\partial^2_{ll} f\,\frac{\mathrm{d}\bar{l}}{\mathrm{d}\theta}=0.$$

It follows that $\mathrm{d}\bar{l}/\mathrm{d}\theta$ has the same sign as $\partial^2_{l\theta} f$. Q.E.D.

Note, in particular, that *if $f(l,\theta)=\theta f(l)$, the first-best allocation is not implementable.* Indeed, condition (4.2.) then reduces to $\partial f/\partial l\leqq 0 \forall l, \forall\theta$, which contradicts the assumption of strictly positive marginal productivity of labor.

Finally, to illustrate the results of this section, we will consider the following example: $f(l,\theta)=\theta\sqrt{l}$. The first-best labor allocation function is $l(\theta)=\theta/4\lambda^2$; it is increasing in θ and therefore not implementable. For a decreasing $l(\cdot)$ function, $t(\theta)$ must be a solution of:

$$\frac{\mathrm{d}t}{\mathrm{d}\theta}-\frac{\mathrm{d}l}{\mathrm{d}\theta}\frac{t(\theta)}{l(\theta)}=\frac{\mathrm{d}l}{\mathrm{d}\theta}\left[\frac{\theta}{2\sqrt{l(\theta)}}-\sqrt{l(\theta)}-\frac{k}{l(\theta)}\right],$$

i.e.

$$t(\theta)=l(\theta)\left[\int_0^\theta \frac{\mathrm{d}l}{\mathrm{d}\tau}(\tau)[\tau l(\tau)^{-\frac{3}{2}}-kl(\tau)^{-\frac{1}{2}}]\mathrm{d}\tau+A\right],$$

where A is an arbitrary constant.

One might wonder whether the fact that the implementability condition is $\mathrm{d}l/\mathrm{d}\theta \leqq 0$ (and then that this first best is not implementable) and the Ward pathology (i.e. the fact that the labor-managed firm wants to produce less when it is more productive) are connected.

In the special case of multiplicative uncertainty, when the objective function takes the more general form $\mathscr{U}(l,t)$, the following connection can be made between the famous Ward pathology, i.e. $\mathrm{d}l/\mathrm{d}p<0$, where p is the price of the commodity, and the possibility of implementing the first best.

The behavior of the labor supply is obtained from maximizing

$$\mathscr{U}\left(l,\frac{p\theta f(l)-k}{l}\right),$$

yielding the first-order condition:

$$\mathscr{U}_1+\mathscr{U}_2\cdot\left\{\frac{p\theta f'(l)}{l}-\frac{(p\theta f(l)-k)}{l^2}\right\}=0.$$

Hence,

$$\frac{\mathrm{d}l}{\mathrm{d}p}=\frac{\frac{\theta}{l}\left(f'(l)-\frac{f(l)}{l}\right)\mathscr{U}_2+\left[\mathscr{U}_{12}+\mathscr{U}_{22}\cdot\left(\frac{-\mathscr{U}_1}{\mathscr{U}_2}\right)\right]\frac{\theta f(l)}{l}}{\left[-\mathscr{U}_{11}+2\mathscr{U}_{21}\left\{\frac{p\theta f'(l)}{l}-\frac{(p\theta f(l)-k)}{l^2}\right\}-\mathscr{U}_{22}\left\{\frac{p\theta f'(l)}{l}-\frac{(p\theta f(l)-k)}{l^2}\right\}^2\right]}.$$

If $\mathscr{U}$ is concave,

$$\operatorname{sign}\frac{\mathrm{d}l}{\mathrm{d}p}=\operatorname{sign}\left\{\left(f'(l)-\frac{f(l)}{l}\right)\mathscr{U}_2+\left[\mathscr{U}_{12}+\mathscr{U}_{22}\left(\frac{-\mathscr{U}_1}{\mathscr{U}_2}\right)\right]f(l)\right\}.$$

In the mechanism framework, we consider

$$\mathscr{U}\left(l,\frac{p\theta f(l)-k+t}{l}\right).$$

The first best is implementable if

$$\frac{\partial}{\partial\theta}\left(\frac{\mathscr{U}_1+\mathscr{U}_2\left\{\frac{p\theta f'(l)}{l}-\frac{(p\theta f(l)-k+t)}{l^2}\right\}}{\frac{\mathscr{U}_2}{l}}\right)>0,$$

or

$$\left(f'(l)-\frac{f(l)}{l}\right)\mathscr{U}_2+\left[\mathscr{U}_{12}+\mathscr{U}_{22}\left(-\frac{\mathscr{U}_1}{\mathscr{U}_2}\right)\right]f(l)>0.$$

So, we obtain that the first best is implementable if and only if the Ward pathology is not present. However, this equivalence does not hold for more general utility functions or more general technologies.

4.3. *Welfare optimum under incentives constraints when transfers are socially indifferent*

As we have just made clear, first-best allocations will not be in general implementable. The planner's problem is then to choose the best allocation under the implementability constraints, i.e. the second-best allocation.

Given the planner's utilitarian objective function defined in subsection 4.1, his expected social welfare associated with $l(\cdot)$, is, up to a constant:

$$\int_0^1 [f(l(\theta),\theta)-\lambda l(\theta)]\,v(\theta)\,\mathrm{d}\theta.$$

Under the assumptions of subsections 4.1 and 4.2 (f is C^2 and strictly concave, condition (C)), implementable (piecewise C^1) labor allocations are characterized by

$$\frac{\mathrm{d}l}{\mathrm{d}\theta}\leqq 0 \text{ a.e.} \quad \text{(Theorem 5).}$$

Furthermore, if f is C^2, the regularity condition (R) obtains. So the planner's problem falls in the class of problems solved in the previous section (subsection 3.1) [with (CS$-$)]. The results of theorem 3 and theorem 4 then apply.

We summarize them as follows.

(1) There exists a unique piecewise C^1 labor allocation which maximizes social welfare under implementability constraints.

(2) Recalling that the first-best labor allocation $\bar{l}(\cdot)$ is defined by $\partial_l f(\bar{l}(\theta),\theta)=\lambda$, we can consider a certain number of cases.

(i) *$\bar{l}(\theta)$ is a decreasing function of θ.* Then $l^*=\bar{l}$, the first-best labor allocation, is implementable. However, as noted above, this case occurs when $\partial^2_{l\theta} f \leqq 0$, a condition which contradicts the productivity interpretation of θ.

(ii) $\bar{l}(\theta)$ *is an increasing function of* θ. This case is in line with the productivity interpretation of θ and occurs with the multiplicative specification $f(\theta, l) = \theta f(l)$.

Then the optimal solution is of the form $l^*(\theta) = l^{**}$, where l^{**} is a number between $l(0)$ and $l(1)$ and such that

$$\int_0^1 [\partial_l f(l^{**}, \theta) - \lambda] \nu(\theta) \mathrm{d}\theta = 0.$$

Hence, l^{**} is the optimal quantity of labor to be allocated in the absence of any information. In that case, the use of mechanisms is particularly disappointing since the *optimal mechanism does not extract any usable information.* The planner's ability to regulate the self-managed firm is severely limited.

(iii) In an *intermediate case* such as that of fig. 9, the optimal solution coincides with $\bar{l}$ on some part of Θ and is constant on the complement.

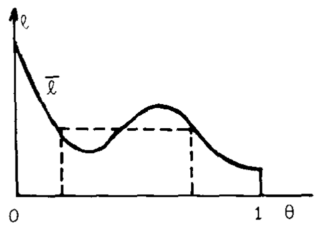

Fig. 9

We now return to the equivalence between what we called the semi-planning and the market contexts. We showed in section 2 that the optimal labor allocation described above could be considered as the outcome of an optimal tax schedule in a market context. We will exhibit here such optimal schedules, in the three typical cases. Note, however, that in each case there is not one but a family of tax schedules with one second-best optimal labor allocation[11] and that we only describe one of them.

Consider the space t, l. In case (i) a possible optimal tax schedule [which implements $\bar{l}(\theta)$] has the shape shown in fig. 10. The subsidy decreases (resp. the tax increases) with the quantity of labor demanded.

In case (ii) the optimal tax schedule takes the extreme form shown in fig. 11. The tax is extremely high unless $l = l^{**}$, where it may be a subsidy.

[11]In the implementation problem, the corresponding multiplicity relates to the arbitrariness of the value of $t(0)$ in the solution of the differential equation. This arbitrariness could be somewhat attenuated by introducing an 'individual rationality' constraint implying that the per capita income within the labor-managed firm is higher than the competitive wage.

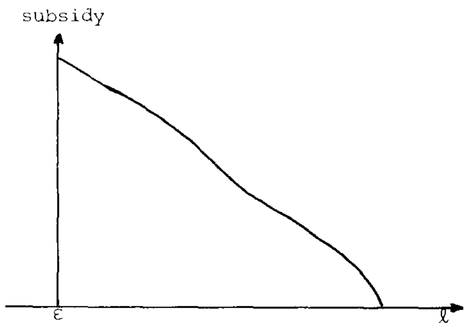

Fig. 10

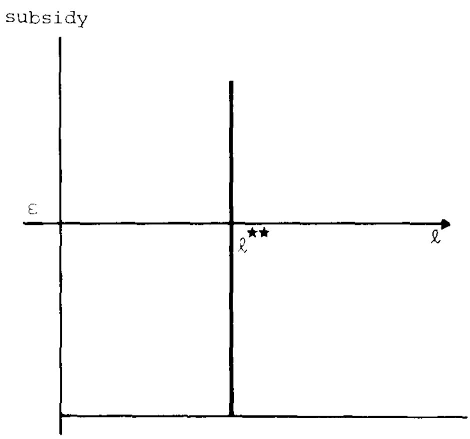

Fig. 11

In case (iii) we may observe a tax schedule as in fig. 12. Subsidies decrease with l. But the constancy of the labor allocation has a counterpart in fig. 12 at point A, where what is called in the optimal taxation literature, *bunching*, obtains (a set of non-zero measure of 'firms' choose the same quantity of labor).

The conditions for the occurrence of *bunching* in this optimal taxation problem are hence rigorously exhibited; any property of the constant part of the labor allocation can be interpreted as a property of the optimal tax schedule [particularly conditions (ii) in theorem 4].

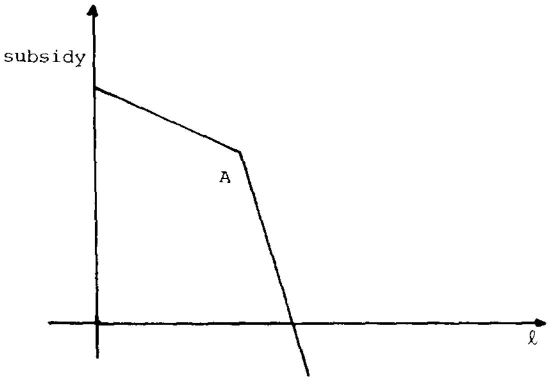

Fig. 12

5. Extensions

We will show how the reasonings we have presented can be adapted to deal with the case of type B preferences and of multidimensional l. A number of applications of the extensions so obtained will be briefly considered.

5.1. Optimization with type B preferences: The case L=1

In the case where preferences are of type B, further assumptions on the agent's utility function will allow us to rely on previous theorems to provide an explicit solution to the maximization problem of the principal. Let us state now these assumptions.

(S): *Separability.* $\mathscr{U}(l,t,\theta)=V(l,\theta)+t.$

Recall here that (S) does imply (B) of section 2.2. We also assume:

$$(\mathrm{CS}++)\text{: } \partial^2_{l\theta}V \geqq 0, \partial_\theta V \geqq 0 \text{ or } (\mathrm{CS}--)\text{: } \partial^2_{l\theta}V \leqq 0, \partial_\theta V \leqq 0.$$

(CS+ +) (resp. CS− −) has two parts. The first is only a specification of (CS +) (resp. CS−) to the separable function $\mathscr{U}$. The second part says that the value of θ influences in an unambiguous way the value of V. With (CS+ +), it is straightforward, from the incentive compatibility differentiable condition, that for any differentiable implementable (l,t) the function $\vartheta(\theta) \stackrel{\text{def}}{=} V(l(\theta),\theta) + t(\theta)$ is an increasing function of θ.[12]

It follows that the individual rationality constraint (2.3b) reduces to:

$$V(l(a),a)+t(a)\geqq \bar{u}.$$

[12] $\mathrm{d}\vartheta/\mathrm{d}\theta=\partial_l V(\mathrm{d}l/\mathrm{d}\theta)+\partial_\theta V+(\mathrm{d}t/\mathrm{d}\theta)=\partial_\theta V.$

Limiting ourselves to piecewise differentiable truthfully implementable action profiles, we know that under the assumptions of corollary 2.1, they have to satisfy $\mathrm{d}l/\mathrm{d}\theta \geqq 0$ a.e. It follows that with (CS++) we can write down the principal's maximization problem as:

$$\max \int_a^b \left[W(l(\theta),\theta) - k(\theta)t(\theta)\right] v(\theta)\,\mathrm{d}\theta \tag{5.1}$$

over the set of functions $l(\cdot), t(\cdot)$ satisfying $\mathrm{d}l/\mathrm{d}\theta \geqq 0$ a.e.

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\partial_l V(l(\theta),\theta)\frac{\mathrm{d}l}{\mathrm{d}\theta}, \qquad V(l(a),a) + t(a) \geqq \bar{u}.$$

We will show:

Lemma. If l, t satisfies the constraint of the program (5.1), *then the maximand can be written*:

$$\int_a^b \left[W(l(\theta),\theta)v(\theta) - \psi(\theta)\partial_\theta V(l(\theta),\theta) + \lambda(\theta)V(l(\theta),\theta)\right]\mathrm{d}\theta - C,$$

with

$$C = \psi(a)[t(a) + V(l(a),a)], \qquad \lambda(\theta) = k(\theta)v(\theta), \qquad \psi(\theta) = \int_\theta^b \lambda(\tau)\,\mathrm{d}\tau.$$

Noting that the incentive compatibility constraint can be written

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = -\frac{\mathrm{d}V}{\mathrm{d}\theta} + \partial_\theta V,$$

we get:

$$\int_a^b k(\theta)t(\theta)v(\theta)\,\mathrm{d}\theta = \int_a^b (k(\theta)v(\theta))\left[t(a) - \int_a^\theta \frac{\mathrm{d}V}{\mathrm{d}\mu}[l(\mu),\mu]\,\mathrm{d}\mu + \int_a^\theta \partial_\mu V(l(\mu),\mu)\,\mathrm{d}\mu\right]\mathrm{d}\theta$$

$$= \int_a^b [t(a) - V(l(\theta),\theta) + V(l(a),a)]\lambda(\theta)\,\mathrm{d}\theta + \int_a^b \left(\lambda(\theta)\int_a^\theta \partial_\mu V\,\mathrm{d}\mu\right)\mathrm{d}\theta.$$

Inverting the order of integration in the last integral (Fubini's theorem), we obtain the expression of the lemma.

Let us define:

$$\tilde{W}(l,\theta) = W(l,\theta)v(\theta) - \psi(\theta)\partial_\theta V(l,\theta) + \lambda(\theta)V(l,\theta). \tag{5.2}$$

The problem (5.1) is equivalent to the following problem:

$$\max \int_a^b \tilde{W}(l(\theta), \theta)\,\mathrm{d}\theta \tag{5.3}$$

over the set of piecewise C^1 functions $l(\cdot)$ such that

$\mathrm{d}l/\mathrm{d}\theta \geqq 0$ a.e.

and

$$\text{choose } t(a) \text{ such that } V(l(a), a) + t(a) = \bar{u}. \tag{5.4}$$

In other words, program (5.3) has the same structure as the optimization program of subsection 3.1, where $\tilde{W}$ can be viewed as a surrogate welfare function (with surrogate uniform expectations). Hence, if we assume:

(CRS): *Concavity and regularity of the surrogate social welfare function.*

(i) The function $\tilde{W}$ defined in (5.2) is strictly concave in $l, \forall \theta$.

(ii) The function $\bar{l}(\theta)$ defined by $\partial_l \tilde{W}(l, \theta) = 0$ is C^1 and has at most a finite number of peaks.

Theorem 3 applies to program (5.3) and we get:

Corollary 3.1. Assume that the constraint of the principal's optimization problem is of type 3b *and he has preferences of type* B *with* $k(\theta) \geqq 0$. *Assume that the agent's preferences satisfy* (M), (D), (S) *and* (CS++). *Then, the optimal action profile over the set of piecewise* C^1 *action profiles is necessarily a solution of:*

$$\max \int_a^b \tilde{W}(l, \theta) \text{ over the set of functions } l \,|\, \mathrm{d}l/\mathrm{d}\theta \geqq 0,$$

with

$$\tilde{W}(l, \theta) = W(l, \theta)\nu(\theta) - \psi(\theta)\partial_\theta V(l, \theta) + \lambda(\theta) V(l, \theta),$$

with

$$\lambda(\theta) = k(\theta)\nu(\theta), \qquad \psi(\theta) = \int_\theta^b \lambda(\mu)\,\mathrm{d}\mu.$$

If, in addition, $\tilde{W}$ *satisfies* (CRS), *the optimal action profile exists and is the unique solution characterized, with* $\tilde{W}$ *instead of* W, *in theorem 4, and computable from the algorithm of subsection* 3.3.

Note that (CRS) is implied by V strictly concave and $\partial_\theta V$ strictly convex, but these conditions are clearly not necessary.

Consider now the case of a constraint of type (2.3a) and assume:

(A): *Additivity of constraint* (2.3a). (2.3a) takes the form:

$$\int_a^b \tilde{V}(l(\theta),\theta)\,\tilde{v}(\theta)\,\mathrm{d}\theta + \int_a^b t(\theta)\,\tilde{v}(\theta)\,\mathrm{d}\theta \geqq \bar{u},$$

where $\tilde{V}$ is a C^2 function and $\tilde{v}$ a C^1 density function.

The computation of the preceding page can be repeated and implies that:

$$\int_a^b t(\theta)\,\tilde{v}(\theta)\,\mathrm{d}\theta = t(a) + V(l(a),a) - \int_a^b V(l(\theta),\theta)\,\tilde{v}(\theta)\,\mathrm{d}\theta + \int_a^b \tilde{\psi}(\theta)\partial_\theta V(l(\theta),\theta)\,\mathrm{d}\theta$$

with

$$\tilde{\psi}(\theta) = \int_\theta^b \tilde{v}(\theta)\,\mathrm{d}\theta.$$

Combining the binding constraint (2.3a) with the previous equality, we can express the maximand as (with straightforward notation):

$$= \int_a^b \tilde{W}\,\mathrm{d}\theta - \psi(a)(t(a) + V(l(a),a))$$

$$= \int_a^b \tilde{W}\,\mathrm{d}\theta - \psi(a)\left[\int_a^b (V - \tilde{V})\tilde{v}\,\mathrm{d}\theta - \int_a^b \tilde{\psi}\partial_\theta V\,\mathrm{d}\theta\right] - \psi(a)\bar{u} \text{ or } \int_a^b \tilde{\tilde{W}}\,\mathrm{d}\theta, \text{ with}$$

$$\tilde{\tilde{W}} = Wv - (\psi - \bar{k}\tilde{\psi})\partial_\theta V + \lambda V - \bar{k}(V - \tilde{V})\tilde{v}, \qquad \bar{k} = \psi(a), \tag{5.5}$$

where, as above, $\tilde{\tilde{W}}$ only depends upon $(l(\theta),\theta)$ and not upon $t(\theta)$. This can be expressed as:

Corollary 3.2. If the constraint is of the form (2.3a) *and satisfies* (A) *and if the agent's preferences are the same as in corollary* 3.1, *then the optimal action profile over the set of piecewise C^1 action profiles is a solution of*

$$\max \int_a^b \tilde{\tilde{W}}(l(\theta),\theta)\,\mathrm{d}\theta \textit{ over the set of } l \,|\, \mathrm{d}l/\mathrm{d}\theta \geqq 0 \textit{ a.e.},$$

with $\tilde{\tilde{W}}(l,\theta)$ given by (5.5).

If, furthermore, $\tilde{\tilde{W}}$ *satisfies* (CRS), *then theorem 4 applies to the characterization of the optimal action profile which can also be computed from the algorithm of subsection* 3.3.

Although complex, corollary 3.2 deserves to be stated since it gives as a particular case some of the few explicit expressions of the optimal income tax in a (small) class of problems (see subsection 5.3).

5.2. Optimization with type B preferences: The case $L>1$

Up to now, our analysis has focused on the case where the action l is one-dimensional. It must be understood that what is specific to the one-dimensional case is *not* the argument developed in section 3 but the fact that in the multidimensional case the set of implementable action profiles contains, but is not identical to, the set of functions $l|\mathrm{d}l/\mathrm{d}\theta \geqq 0$. In other words, the results of theorem 3 and its corollaries are easily extended to the case where l is multidimensional (this is left to the reader) but they *characterize the solutions of program* (3.1) *and not necessarily the best incentives mechanisms from the planner's point of view.* There is, however, a full characterization theorem of the optimal multidimensional solution in a restricted subset of problems which obtains as a byproduct of the analysis of this section and of section 2.

Theorem 6. Optimization in the multidimensional case. Assume that the agent's utility function satisfies (S), (D) *and* (CS++) *with an individual rationality constraint* (2.3b).[13] *Consider the surrogate social welfare function of corollary* 3.1:

$$\tilde{W}(l,\theta)=W(l,\theta)\,v(\theta)-\psi(\theta)\,\partial_\theta V(l,\theta)+\lambda(\theta)\,V(l,\theta).$$

If the solution $\bar{l}(\theta)$ *of* $\max_l \tilde{W}(l,\theta)$ *is* C^1 *and satisfies* $\mathrm{d}\bar{l}/\mathrm{d}\theta \geqq 0$, *it is the optimal action profile for the principal.*

The *proof* obtains as a combination of the following remarks.

(1) The argument of corollary 3.1 transforming the initial optimization problem into two problems is independent of the dimensionality of l. Hence, the problem can be transformed into (i) max $\tilde{W}(l,\theta)$ over the implementable l, and (ii) choose $t(a)$ such that $V(l(a),a)+t(a)=\bar{u}$.

(2) If the program max $\tilde{W}(l,\theta)$ has a solution $\bar{l}(\theta)$ which satisfies the above conditions, then according to theorem 2, and thanks to (CS++), $\bar{l}$ is truthfully implementable and is necessarily a solution of program (i). Q.E.D.

[13]Note that both assumptions (S) and (CS++) do not refer specifically to the one-dimensional case.

Note that the theorem remains true with straightforward changes with corollary 3.2 instead of corollary 3.1 in the statement. Note also that the theorem (and its variant) although less powerful than theorem 4 for the one-dimensional problem, is likely to apply in a significant class of problems.

5.3. Applications

5.3.1. The quality choice problem

l is the quality of a product, θ is the parameter taste of the agent (there is here a continuum of agents). The principal is a monopolist who tries to maximize his profit. The (marginal and average) cost of producing a unit of quality l is $C(l)$. Take $\mathscr{U}(l,t,\theta)=t+\theta l$ (Mussa and Rosen's formulation). The objective of the principal is $\max\int_{\Theta}(-t-C(l))v(\theta)\,\mathrm{d}\theta$ [so that $k(\theta)=1$ and $W(l,\theta)=-C(l)$]. The surrogate social welfare function is

$$\tilde{W}=-\psi(\theta)l+v(\theta)(\theta l-C(l)) \quad \text{with } \psi(\theta)=\int_{\theta}^{b} v(\tau)\,\mathrm{d}\tau.$$

With $C(l)=\alpha+(\beta/2)l^2$ and a uniform distribution, it is straightforward that

$$\bar{l}(\theta)=\frac{2\theta}{\beta}-\frac{b+\alpha}{\beta}$$

[see also Mussa and Rosen (1978, p. 312)] and the optimum is $l^*(\theta)=\bar{l}(\theta)$ for θ bigger than some 'cut off' point. When the distribution is non uniform

$$\bar{l}(\theta)=\frac{\theta-\alpha}{\beta}+\frac{F(\theta)-1}{\beta v(\theta)},$$

where F is the cumulative distribution function associated with v. Clearly, $\mathrm{d}\bar{l}/\mathrm{d}\theta$ may be negative for some non-uniform distribution (for example with a 'high peak') and the way $l^*(\theta)$ obtains from $\bar{l}(\theta)$ conforms the principles of theorem 4.

In Maskin and Riley (1982), l is the quantity of the product instead of its quality. The formulation can however be made very similar to Mussa and Rosen's. With $W=\theta v(l)$, $C(l)=cl$, the above calculation is hardly modified and one obtains:

$$\bar{l}(\theta)=v'^{-1}\left[\frac{c}{\theta-\dfrac{\psi(\theta)}{v(\theta)}}\right],$$

a formula similar to the ones on which is based the (slightly more complex) analysis of the quoted article.

5.3.2. The government control of a private monopolist

The agent is a firm with a known fixed cost C_0 and an unknown marginal productivity θ so that total cost as a function of the quantity put on the market, which is the action variable l, is $C=C_0+l/\theta$. The inverse demand function is $D(l)$ and the principal's objective function is:

$$\int_0^l D(x)\,\mathrm{d}x - C(\theta,l) - kt.$$

The agent's utility function is $-C(\theta,l)+t$ and satisfies (CS+). With $D(l)=\alpha-\beta l$, the surrogate social welfare function is:

$$\tilde{W}=v(\theta)\left[-\frac{\beta l^2}{2}+l\left(\alpha-\frac{1}{\theta}\right)-k\frac{(1-F(\theta)}{\theta^2 v(\theta)}-\frac{k}{\theta}\right]-C_0.$$

With a uniform distribution,

$$\bar{l}(\theta)=\frac{1}{\beta}\left[\alpha-\frac{1}{\theta}(1+k)-k\frac{(1-F(\theta))}{\theta^2 v(\theta)}\right]$$

is actually implementable and hence optimal. The corresponding optimal market price is hence ($b-a=1$):

$$p^*(\theta)=\frac{1}{\theta}\left(1+k\frac{b}{\theta}\right),$$

a formula which has the same flavour as the ones derived in Baron and Myerson (1982).

5.3.3. The income tax problem with a quasi-linear utility function

Take t as the complement of effective labour ('effective' leasure), l as consumption and assume $\mathscr{U}(l,t,\theta)=t+\theta v(l)$, when θ is the usual productivity parameter.

The feasibility constraint is $\int_a^b t\,\mathrm{d}\theta=\int_a^b l\,\mathrm{d}\theta$ which identifies with (2.3a) as written in assumption A when $\tilde{V}$ is the identity $\tilde{v}=1, b-a=1$.

The principal's objective is:

$$\max\int_a^b \mu(\theta)\mathscr{U}(l,t,\theta)v(\theta)\,\mathrm{d}\theta=\int_a^b v(\theta)(\mu(\theta)\theta v(l)+\mu(\theta)t)\,\mathrm{d}\theta.$$

Assuming $\int_a^b \mu(\theta)\mathrm{d}\theta = 1$ (normalization) and $v(\theta)=1$ (uniform distribution), we note that $k(\theta) = -\mu(\theta)$, $\psi(\theta) = 1 - \int_a^\theta \mu(\theta)\mathrm{d}\theta$, $\psi(a) = -1$ and $\tilde{\psi}(\theta) = 2-\theta$, so that the surrogate social welfare function may be written:

$$\tilde{\tilde{W}}(l,\theta) = v(l)\left[2\theta - 1 - \int_a^\theta \mu(\theta)\mathrm{d}\theta\right] - l.$$

One easily derives $\bar{l}$ and checks, after Lollivier and Rochet (1983), that $[\mathrm{d}\bar{l}/\mathrm{d}\theta]_{\theta=1} < 0$ if $\mu(\theta) > 2$, so that bunching obtains at the bottom of the optimal tax schedule.

5.3.4. The self-managed firm which trades off between unemployment and per capita value added

Here $\mathscr{U}(l,t,\theta) = [(\theta f(l) - K + t)/l]\, l^\beta$ and $f(l) = l^\alpha$.

We immediately check, following Guesnerie and Laffont (1984b), that when $\beta > 1-\alpha$, i.e. when the employment concern is sufficient, (CS+) holds and, contrarily to what happens in section 4, the first-best allocation is implementable through compensatory transfers.

5.3.5. The planner can observe production and (marginal) cost but cannot monitor an effort variable

The problem can be formalized as follows. (Apparent) productivity is written θe (e effort), which we call l_2. Call l_1 the production of a commodity which is assumed to be a pure public good.

The firm maximizes $\mathscr{U} = t - (l_1/l_2) - \varphi(l_2/\theta)$, where φ is the disutility of effort. Take $\varphi\,(x) = x^2/2$.

As above, assume $W(l,\theta) = \alpha l_1 - (\beta l_1^2/2) - (l_1/l_2)$ and $v(\theta) = 1\,(b-a=1)$. The surrogate social welfare function is:

$$\alpha l_1 - \frac{\beta l_1^2}{2} - \frac{l_1}{l_2} - \frac{2(b-\theta)}{\theta^3} k l_2^2 + k\left(-\frac{l_1}{l_2} - \left(\frac{l_2}{\theta}\right)^2\right). \tag{5.6}$$

It is actually quasi-concave and one obtains $\bar{l}_1(\theta)$ and $\bar{l}_2(\theta)$ (cf. theorem 6) by differentiating (5.6):

$$\alpha - \beta \bar{l}_1(\theta) = \frac{1+k}{\bar{l}_2(\theta)},$$

$$\bar{l}_2(\theta) = \sqrt[3]{\frac{1+k}{2k}}\frac{\theta}{\sqrt[3]{2b-\theta}},$$

and the results have the same flavor as the ones obtained by Laffont and Tirole (1984).

6. Conclusion

This paper attempted to achieve two goals. First, in a class of incentives problems, which covers a variety of cases considered in the literature, it provided a complete characterization of implementable mechanisms as well as a characterization and determination of a solution for the welfare problem. Secondly, the application of this machinery to a simple problem of control of a public self-managed firm attempted to illustrate the relevance of the incomplete information approach for a renewed understanding of the public firms control. Each of these goals deserves a final comment.

The general analysis conducted here relies on the use of mechanisms rather than on a non-linear taxation approach. In spite of the equivalence of non-linear taxes and mechanisms emphasized in section 2, mechanisms appear as particularly adequate tools for the analysis in the class of problems of this paper. When the assumptions of this paper are not fulfilled, it is not clear, at this stage, how far one can go with a mechanism approach; the corresponding mechanisms will be generally badly behaved and there is no simple way to meaningfully restrict the analysis to a manageable class of mechanisms. On the other hand, one can think that, on the contrary, the non-linear prices which can be used are easily restricted to simple classes of well-behaved functions without too much loss. In fact, the complexity reappears in the analysis of the first-order conditions through the possibility of corner solutions and the question of sufficiency of these first-order conditions to characterize an optimum. It is unclear at this stage which robust qualitative results of the principal–agent problem of sufficient generality and precision can be obtained, when one leaves the class of problems considered here.

Appendix A: Integrability of the differential equation

Instead of condition (B), another condition, (BR), easier to satisfy in some contexts, could have been introduced.

(BR): A piecewise C^2 allocation l satisfies (BR) if given $K \stackrel{\text{def}}{=} \sup[\mathrm{d}l/\mathrm{d}\theta]$, $\exists$ two numbers $K''>0$ and t_0 such that $\forall t \in [-KK'+t_0, KK'+t_0]$, $l \in l(\Phi)$:

$$\left|\frac{\partial_l \mathscr{U}}{\partial_t U}\right| \leqq K'.$$

Hence (BR) implies that the trajectory of the differential equation remains below (resp. above) the diagonal of the upper rectangle (resp. lower); see fig.

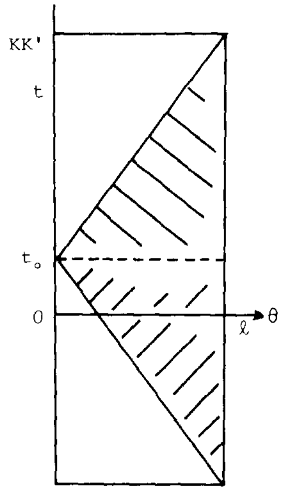

Fig. 13

13. It then gives an a priori majoration which assures existence along the same lines as in the proof of theorem 2.

Since (B),(BR) implies that the marginal rate of substitution does not increase too fast when t increases, the condition relates not only to the boundary behavior of $\partial_l \mathcal{U}/\partial_t U$ but also to the diameter of Θ.

For example, with a Cobb–Douglas utility function (BR) is satisfied when the diameter $l(\Theta)$ is smaller than 1. Also, (BR) is a condition involving t_0, so that it only implies existence for trajectories starting from well chosen t_0. The theory would have to take this point into account to replace (B) by (BR).

Appendix B: Existence proof

Proof of proposition 2. First, we note from program (3.1) that if an optimal solution $l^*(\theta)$ exists, on any interval where it differs (a.e.) from $\bar{l}(\theta)$, it is constant. Otherwise any increasing piece of $l^*(\theta)$ over an interval $[\theta'_0, \theta'_1]$ could be replaced by an increasing piecewise differentiable piece $\tilde{l}(\theta)$ with the same terminal points and uniformly closer to $\bar{l}(\theta)$ over the interval. Since $W(l, \theta)$ is strictly concave in l, this would contradict the optimality of $\bar{l}(\theta)$.

Consider the space of continuous functions on $[a, b]$ endowed with the uniform topology. Let $G(l(\cdot)) = \int_a^b W(l(\theta), \theta)\nu(\theta)\mathrm{d}\theta$. $G(\cdot)$ is continuous in $l(\cdot)$ under (R).

Consider the subset A of continuous functions obtained from $\bar{l}(\theta)$ by piecing together, with constants, increasing portions of $\bar{l}(\theta)$. Since $\bar{l}$ is C^1, A forms a family of equicontinuous functions. From Ascoli's theorem, A is compact; $G(\cdot)$ achieves its maximum on A. The maximum is unique because W is strictly concave in l, and from above, is also the maximum of the program (3.1).

Finally, since from (R), $\bar{l}(\cdot)$ has a finite number of increasing portions and since any constant piece in $l^*(\cdot)$ must join two different increasing pieces, the unique optimum has a finite number of points of non-differentiability. Q.E.D.

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