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ADVANTAGEOUS REALLOCATIONS OF INITIAL RESOURCES

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This paper shows how the Debreu-Sonnenschein theorem can be used to construct economies where disadvantageous reallocations of resources take place.

1. INTRODUCTION

GIVEN THE INSTITUTIONAL FRAMEWORK of competitive economies, several authors have recently shown with examples that agents may find it advantageous to "cheat" in an undetectable way. Hurwicz [7] has shown in a two person economy that it might be in the interest of an agent to behave "competitively" with a wrong utility function, by nature undetectable. Gale [6] has given an example where a gift is advantageous for the giver, in the sense that, at the new competitive equilibrium, the giver is better off. Similarly Aumann and Peleg [1] have constructed examples in which the destruction of part of one's own goods is advantageous.[2]

We are particularly concerned in this note by this latter type of cheating which involves initial resources. We point out that besides the intuitive explanation of these "seemingly anomalous results" which puts the emphasis on the intricacies of income effects—for example, the controversy on the Leontieff paradox—there is now a powerful tool, the Debreu-Sonnenschein theorem (Debreu [3], Sonnenschein [11]), which sheds an original light on these phenomena by showing very clearly how and to which extent these results are non-pathological. More precisely the idea underlying the paper is to use this theorem for constructing infinities of economies where such results are observed. The interest of this approach is not mainly to be an automatic device for building examples, but rather to provide a better understanding of the phenomena by making clear, through the analysis of the parameters which can be manipulated in building the examples, the mechanisms underlying them. The paper is an application of the above idea to the problem of advantageous reallocation of initial endowments by a subgroup of an economy. However the method it describes is general and could easily be applied to the problem of advantageous gifts or a fortiori to the simpler problem of advantageous destructions.

In Section 2 we give some definitions. Preliminary lemmas and results are gathered in Section 3.

2. DEFINITIONS

We consider economies with l commodities, indexed by $k = 1, \ldots, l$. All the demand functions considered below are homogeneous of degree zero in prices so

[1] We are grateful to A. Mas-Colell for communicating to us unpublished material.

[2] Drèze-Gabsewicz-Postlewaite [5] have studied the relationship between disadvantageous monopolies and disadvantageous endowments.

that we can restrict the price space to the simplex of R^l. Let

$$S=\left\{p|p\in R^l, \sum_{k=1}^{l} p_k=1, p_k>0\right\},$$

$$\bar{S}=\left\{p|p\in R^l, \sum_{k=1}^{l} p_k=1, p_k\geqslant 0\right\}.$$

DEFINITION 1: A *primitive set* is a set of I consumers indexed by $i=1,\ldots,I$, with strictly convex, monotone, continuous preference preorderings $\geqslant_i$ defined on the consumption sets R^l_+ and represented by utility functions u^i and with initial endowments $w^i\in \mathring{R}^l_+$.

Let I denote also the set $(1,\ldots,I)$. Let $f^i(p, w^i)$ defined on $S\times\mathring{R}^l_+$ be the excess demand function of consumer i $(i=1,\ldots,I)$, associated with an endowment w^i. From Definition 1, $f^i(\cdot,\cdot)$ is continuous.

DEFINITION 2: An *adjoint set* is a set of J consumers indexed by $j=1,\ldots,J$, whose preference preorderings defined on the consumption sets R^l_+ are strictly convex, monotone, and continuous. Initial endowments belong to R^l_+.

It follows from Definition 2 that the excess demand functions $\phi^j(p)$, $j=1,\ldots,J$, defined on S are continuous.

DEFINITION 3: Given a primitive set I, an *I-economy* is an economy formed with the primitive set and an adjoint set.

We can now define what we mean by an advantageous reallocation of initial endowments.

DEFINITION 4: A reallocation of initial endowments in a primitive set I (or I-reallocation) is a vector $\delta w\in R^{Il}$ such that

$$w^i+\delta w^i\in\mathring{R}^l_+ \qquad (i=1,\ldots,I),$$

$$\sum_{i\in I}\delta w^i=0.$$

DEFINITION 5: A binary relation on $S\times\mathring{R}^{Il}_+$, denoted $>_I$, can be defined as follows (with $w=(w^1,\ldots,w^I)$ and $\tilde{w}=(\tilde{w}^1,\ldots,\tilde{w}^I)$): $(p, w)>_I(\tilde{p},\tilde{w})$ or (p, w) is I-preferred to $(\tilde{p},\tilde{w})$ iff

(a) $\sum_{i\in I} w^i=\sum_{i\in I}\tilde{w}^i$,

(b) $f^i(p, w^i)+w^i>_i f^i(\tilde{p},\tilde{w}^i)+\tilde{w}^i \qquad (i=1,\ldots,I).$

Let $A(p, w) = \{p' | p' \in S, \exists$ an I-reallocation δw such that $(p', w+\delta w) >_I (p, w)\}$ and let $W(p, p', w) = \{\delta w : \delta w$ is a reallocation such that $(p', w+\delta w) >_I (p, w)\}$.

DEFINITION 6: A pair $(\bar{p}, w) \in S \times \mathring{R}^{ll}$ is said to be a *competitive equilibrium in an I-economy*, iff:

$$\sum_{i=1}^{I} f^i(\bar{p}, w^i) + \sum_{j=1}^{J} \phi^j(\bar{p}) = 0.$$

DEFINITION 7: Given a competitive equilibrium $(\bar{p}, w)$ in an I-economy, an I-reallocation δw is *advantageous* (for I) iff there exists a competitive equilibrium $(\bar{\bar{p}}, w+\delta w)$, which is I-preferred to $(\bar{p}, w)$.[3]

3. RESULTS

We first prove two lemmas.

LEMMA 1: *Given a primitive set I, for any* $\bar{p} \in S$: (a) $A(\bar{p}, w)$ *has a non-empty relative interior.* (b) $W(\bar{p}; \bar{\bar{p}}, w)$ *has a non-empty relative interior,* $\forall \bar{\bar{p}} \in A(\bar{p}, w)$.

PROOF: (a) Let $X^i(\alpha) = \{x^i | x^i \in R^l_+, u^i(x^i) \geq \bar{u}^i + \alpha\}$ with $\bar{u}^i = u^i(f^i(\bar{p}, w^i) + w^i)$ and let $X^I(\alpha) = \sum_{i \in I} X^i(\alpha)$. Then $X^I(0)$ is the sum of the upper indifference contour sets associated with $(\bar{p}, w)$. Let $\psi(p) = \inf p \cdot X^I(0) \cdot \psi: R^l \to R$ is the support function of $X^I(0)$. It is a convex function (Rockafellar [**10**, Theorem 13-2]); therefore it is continuous relative to any open convex set of its effective domain (Rockafellar [**10**, Theorem 10.1]). Hence, since $X^I(0)$ is bounded from below, $\psi(\cdot)$ is continuous on S.

We first show that $A(\bar{p}, w) = S \backslash Q$ with $Q = \{p | p \in S, p \,.\, \sum_{i \in I} w^i \leq \psi(p)\}$. Clearly, $A(\bar{p}, w) \subset S \backslash Q$. Now, let $\bar{\bar{p}} \in S \backslash Q$. $\exists x' \in X^I(0)$ such that $\bar{\bar{p}} \cdot \sum_{i \in I} w^i > \bar{\bar{p}} \cdot x'$.[4] Therefore, $\exists \delta x \in R^l_+$ such that $\bar{\bar{p}} \cdot \sum_{i \in I} w^i = \bar{\bar{p}}(x' + \delta x)$. From monotonicity of preferences and continuity, $\exists \varepsilon > 0$ such that $x' + \delta x \in X^I(\varepsilon)$ and $\bar{\bar{x}}^i \in X^i(\varepsilon)$, $i = 1, \ldots, I$, such that $x' + \delta x = \sum_{i \in I} \bar{\bar{x}}^i$. Defining $R^i = \bar{\bar{p}} \cdot \bar{\bar{x}}^i$, $i = 1, \ldots, I$, $\exists(\tilde{w}^1, \ldots, \tilde{w}^I) \in \mathring{R}^{lI}_+$ such that $\bar{\bar{p}} \cdot \tilde{w}^i = R^i$, $i = 1, \ldots, I$ and $\sum_{i \in I} \tilde{w}^i = \sum_{i \in I} w^i$. (Take, for example, $\tilde{w}^i = \lambda^i \sum_{i \in I} w^i$ with $\lambda^i = R^i / \sum_{i' \in I} R^{i'}$.) Therefore $S \backslash Q = A(\bar{p}, w)$. (Let us note that $S \backslash Q$ is relatively open.)

(b) Continuity of preferences proves (b) immediately. *Q.E.D.*[5]

[3] Many other definitions of "advantageous" could be suggested. This one is really satisfactory only in a world of one-equilibrium economies. If we consider only small reallocations, Debreu's theorem [**4**] on the local uniqueness of equilibrium may also be used for justification. By using the McFadden et al. [**8**] version of Sonnenschein's theorem, it is possible to derive a result similar to our theorem in which *all* competitive equilibria of the new I-economy are preferred by the agents of the primitive set.

[4] This would not be true with complementary preferences but they are excluded by monotonicity and strong convexity assumed in Definition 1.

[5] We can make also the two obvious remarks: $\bar{p} \in \bar{A}(\bar{p}, w)$; $S \backslash A(\bar{p}, w)$ is convex.

ASSUMPTION 0:[6] For almost[7] all $p \in S$, for almost all endowments $(w^1, \dots, w^I) \in \mathring{R}^{II}_+$, there exist at least two consumers (i_1, i_2) in the primitive set I whose excess demand functions $f^i(p, w^i)$ are differentiable with respect to w^i and for which the matrices

$$\nabla_w f^{i_1} = \begin{pmatrix} & \vdots & \\ \dots & \dfrac{\partial f_k^{i_1}}{\partial w_j^{i_1}} & \dots \\ & \vdots & \end{pmatrix} \quad \text{and}$$

$$\nabla_w f^{i_2} = \begin{pmatrix} & \vdots & \\ \dots & \dfrac{\partial f_k^{i_2}}{\partial w_j^{i_2}} & \dots \\ & \vdots & \end{pmatrix} \quad \text{are different.}$$

Considering the demand function of individual i as a function of prices and income $g^i(p, R^i)$ and taking into account the fact that $g^i(p, pw^i) \equiv f^i(p, w^i) + w^i$. Assumption 0 can be interpreted as concerning the vector of income effects

$$\left(\frac{\partial g^i}{\partial R^i}\right)_{(p, pw^i)},^{8}$$

and stating that the vectors of "income effects" for individual i_1 and i_2 are different.

Assumption 0 is necessary to avoid the pathological cases where the primitive set is formed of consumers with "locally" the same homothetic preferences (see Chipman and Moore [**2**]). In such a case, $\sum_{i \in f} f^i(p, w^i + \delta w^i)$ is constant for all $\{\delta w^i\}$, $i \in I$, such that $p \sum_{i \in I} \delta w^i = 0$, and any "small" reallocation does not "locally" modify the excess demand function of set I.

Let $df^I(p, w, \delta w)$ be the variation of the excess demand function of the primitive set I, associated with the reallocation $\delta w = (\delta w^1, \dots, \delta w^I)$,

$$df^I(p, w, \delta w) = \sum_{i \in I} [f^i(p, w^i + \delta w^i) - f^i(p, w^i)].$$

LEMMA 2: *Under Assumption* 0, *for almost all* $\bar{p}, \bar{\bar{p}} \in A(\bar{p}, w)$, $\exists\ \delta w \in W(\bar{p}, \bar{\bar{p}}, w)$ *such that*: $df^I(\bar{p}, w, \delta w) \neq 0$ *and* $df^I(\bar{\bar{p}}, w, \delta w) \neq 0$.

[6] Using differential topology, it should be easy to show that part 2 of Assumption 0 is generic.

[7] A more precise statement is: $\exists a$ subset $N \subset S$ of Lebesgue measure zero such that $\forall p \in S \backslash N$, $\exists N'_p \subset \mathring{R}^{II}_+$ of Lebesgue measure zero such that $\forall w \in \mathring{R}^{II}_+ \backslash N'_p \dots$

[8] One has:

$$p_k\left(\frac{\partial g_j^i}{\partial R^i}\right) = \frac{\partial f_j^i}{\partial w_k^i} + \delta_{jk}$$

where δ_{jk} is the Kronecker symbol.

PROOF: From Assumption 0, for almost all $(\bar{p}, \bar{\bar{p}}) \in S \times A(\bar{p}, w)$, for almost all $\delta w \in W(\bar{p}, \bar{\bar{p}}, w)$, there exist consumers i_1 and i_2 and good k on the one hand, and there exist consumers i_1' and i_2' and good k' on the other hand, such that

$$\frac{\partial f^{i_1}}{\partial w_k^{i_1}}(\bar{p}, w^{i_1}+\delta w^{i_1}) \neq \frac{\partial f^{i_2}}{\partial w_k^{i_2}}(\bar{p}, w^{i_2}+\delta w^{i_2}),$$

$$\frac{\partial f^{i_1'}}{\partial w_{k'}^{i_1'}}(\bar{\bar{p}}, w^{i_1'}+\delta w^{i_1'}) \neq \frac{\partial f^{i_2'}}{\partial w_{k'}^{i_2'}}(\bar{\bar{p}}, w^{i_2'}+\delta w^{i_2'}).$$

Take such a $\delta w'$ and let us assume that either $df^I(\bar{p}, w, \delta w')=0$ or $df^I(\bar{\bar{p}}, w, \delta w')=0$ (or both). From the continuity of $f^i(\cdot, \cdot)$ in w^i, and from Lemma 1(b) one can find $\delta w''$ through a reallocation of good k between i_1 and i_2 or through a reallocation of good k' between i_1' and i_2' or both) such that $df^I(\bar{p}, w, \delta w'') \neq 0$ and $df^I(\bar{\bar{p}}, w, \delta w'') \neq 0$. *Q.E.D.*

THEOREM: *Under Assumption 0 to any primitive set I and to almost any $(\bar{p}, \bar{\bar{p}}) \in S \times A(\bar{p}, w)$ and for an infinity of $\delta w \in W(\bar{p}, \bar{\bar{p}}, w)$, we can associate an infinity of adjoint sets such that in the so formed I-economies*: (a) *δw is an advantageous (to the primitive set) I-reallocation of initial resources.* (b) *$\bar{p}$ is a price equilibrium before reallocation (and $\bar{\bar{p}}$ is not); $\bar{\bar{p}}$ is a price equilibrium after reallocation (and $\bar{p}$ is not).*

PROOF: Let $\bar{p}, \bar{\bar{p}}$ such that $\bar{\bar{p}} \in A(\bar{p}, w)$ and let $\delta w \in W(\bar{p}, \bar{\bar{p}}, w)$, as in Lemma 2.

The correspondence $\Phi: p \to \{x \mid px=0, x \in R^l\}$ from S into R^l is l.h.c. and convex-valued. Let $\tilde{\Phi}$ be the correspondence derived from Φ as follows:

$$\tilde{\Phi}(\bar{p}) = -\sum_{i \in I} f^i(\bar{p}, w^i),$$

$$\tilde{\Phi}(\bar{\bar{p}}) = -\sum_{i \in I} f^i(\bar{\bar{p}}, w^i) - df^I(\bar{\bar{p}}, w, \delta w),$$

$$\tilde{\Phi}(p) = \Phi(p), \qquad \forall p \in S, p \neq \bar{p}, p \neq \bar{\bar{p}}.$$

$\tilde{\Phi}(\cdot): S \to R^l$ is convex valued and l.h.c.

From Michael's theorem **[9]**, there exists a continuous selection $Z(\cdot)$ in $\Phi(\cdot)$ such that

(i) $\quad p \cdot Z(p) = 0, \qquad \forall p \in S,$

(ii) $\quad Z(\bar{p}) + \sum_{i \in I} f^i(\bar{p}, w^i) = 0,$

(iii) $\quad Z(\bar{\bar{p}}) + \sum_{i \in I} f^i(\bar{\bar{p}}, w^i) + df^I(\bar{\bar{p}}, w, \delta w) = 0.$

We then apply Debreu's theorem **[3]**, to construct an adjoint set with excess demand function $Z(p)$. *Q.E.D.*

This Theorem exhibits reallocations which are advantageous when the initial equilibrium price $\bar{p}$, is replaced by $\bar{\bar{p}}$. Thanks to Assumption 0, we ensure that after one of these reallocations, $\bar{p}$ is destroyed as an equilibrium price. However, we did not prove that $\bar{\bar{p}}$ was the only equilibrium price after reallocation. On this ground, the term advantageous may be challenged. However, by considering adjoint sets with individual demand functions meeting only the weak axiom of revealed preferences, and therefore possibly discontinuous (see footnote 3), we could guarantee the uniqueness of the equilibrium price before and after reallocation. In such a case, the role of Assumption 0, in enabling us to upset $\bar{p}$ would be crucial.

CONCLUSION

As noticed in the introduction, the principles of the analysis developed here can apply in similar problems (advantageous gifts or destructions which are indeed simpler than the one treated here). On the other hand, some results of the paper could be refined; for example intermediate notions of "advantage" could be considered (for which weakened versions of the theorem could probably be obtained).

However, even if additional or complementary studies could be pursued, the question raised in the paper: "considering *a group of agents* with given characteristics, can they belong *to an economy* in which they could *find profitable to reallocate their initial endowments*?" receives an unambiguous answer. This answer is positive: "Nearly" any group of agents may be imbedded in an economy (and even in infinities of economies) in which it will find (infinities of) possibilities of advantageous "cheating" on initial endowments.

In addition, the general approach taken in this paper yields a better understanding of this problem than what is obtained from the construction of specific examples. First, it makes clear that almost no assumption on the characteristics of agents can preclude the possibility of advantageous reallocations of initial endowments. Second, it proves that, for monotonic preferences, the extent of the advantage may be as large as desired. This is an important additional insight which can only be gained through the abstract approach allowed by the Sonnenschein theorem.

Such are, briefly summarized, the main conclusions of this note intended to shed light on the mechanisms of advantageous reallocation of resources. However the method used here has its limits and unsolved problems in this field (when, in a given economy, are advantageous reallocations feasible?, etc.) certainly require other types of approaches.

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REFERENCES

[1] AUMANN, R., AND B. PELEG: "A Note on Gale's Example," *Journal of Mathematical Economics*, 1 (1974), 209–211.

[2] CHIPMAN, J. S., AND J. C. MOORE: "Aggregate Demand, Real National Income, and the Compensation Principle," *International Economic Review*, 14 (1973) 153–181.

[3] ———: "Excess Demand Functions," *Journal of Mathematical Economics*, 1 (1974), 15–21.

[4] ———: "Economies with a Finite Set of Equilibria," *Econometrica*, 38 (1970), 387–392.

[5] DRÈZE, J., J. GABSEWICZ, AND A. POSTLEWAITE: "Disadvantageous Monopolies and Disadvantageous Endowments," CORE discussion paper 7419.

[6] GALE, D.: "Exchange Equilibrium and Coalitions: an Example," *Journal of Mathematical Economics*, 1 (1974), 63–66.

[7] HURWICZ, L.: "On Informationally Decentralized Systems," in *Decision and Organisation*, ed. by R. Radner and C. McGuire. Amsterdam: North-Holland, 1972.

[8] MCFADDEN, D., R. MANTEL, A. MAS-COLELL, AND M. K. RICHTER: "A Characterization of Community Excess Demand Functions," *Journal of Economic Theory*, 9 (1974), 361–374.

[9] MICHAEL, E.: "Continuous Selections. I," *Annals of Mathematics*, 63 (1956), 361–382.

[10] ROCKAFELLAR, T.: *Convex Analysis*. Princeton, New-Jersey: Princeton University Press, 1970.

[11] SONNENSCHEIN, H.: "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?" *Journal of Economic Theory*, 6 (1973), 345–354.