TSE working paper series

12-323

Research Group: Public Economics

July 2012

"Ambiguous Life Expectancy and the Demand for Annuities "

Hippolyte d'Albis and Emmanuel Thibault

Toulouse School of Economics (TSE) - Manufacture des Tabacs
Aile Jean-Jacques Laffont - 21, allée de Brienne - 31015 TOULOUSE Cedex 6
Tél : +33 (0) 5 61 12 85 89 - Fax : +33 (0) 5 61 12 86 37 - **www.tse-fr.eu** - contact@tse-fr.eu

Toulouse School of Economics

Ambiguous Life Expectancy and the Demand for Annuities*

Hippolyte d'ALBIS

Paris School of Economics, University Paris 1

Emmanuel THIBAULT

Toulouse School of Economics (University of Perpignan, CDED and IDEI)

Abstract: In this paper, ambiguity aversion to uncertain survival probabilities is introduced in a life-cycle model with a bequest motive to study the optimal demand for annuities. Provided that annuities return is sufficiently large, and notably when it is fair, positive annuitization is known to be the optimal strategy of ambiguity neutral individuals. Conversely, we show that the demand for annuities decreases with ambiguity aversion and that there exists a finite degree of aversion above which the demand is non positive: the optimal strategy is then to either sell annuities short or to hold zero annuities if the former option is not available. To conclude, ambiguity aversion appears as a relevant candidate for explaining the annuity puzzle.

Keywords: Demand for Annuities; Uncertain Survival Probabilities; Ambiguity Aversion.

JEL codes: D11, D81, G11, G22.

---

*We are grateful to Fabrice Collard, Firouz Gahvari, Johannes Gierlinger, Jérôme Glachant, Christian Gollier, André Masson, Pierre Pestieau, Svetlana Pashchenko, Jean-Marc Tallon and Nicolas Treich. We also thank the participants of seminars and conferences where earlier versions of this paper were presented. H. d'Albis thanks the European Research Council (ERC Starting Grant DU 283953) and E. Thibault thanks the Chair Fondation du Risque/SCOR "Marché du risque et création de valeurs" for their financial support.

1 Introduction

According to the life-cycle model of consumption with uncertain lifetime proposed by Yaari [46], full annuitization should be the optimal strategy followed by a rational individual without altruistic motives, provided that annuities are available. Since this theoretical prediction does not meet the facts (see Peijnenburg et al. [37]), the initial framework has hence been extended and many explanations of the so-called annuity market participation puzzle have been proposed (see Brown [6] or Sheshinski [42]). However, as shown by Davidoff et al. [9], positive annuitization still remains optimal under very general specifications and assumptions, including intergenerational altruism. According to the authors, the literature to date has failed to identify a sufficiently general explanation for consumers' aversion to annuities, suggesting that psychological or behavioral biases might be rather important in decisions involving uncertain longevity[1].

In this paper, we consider the possibility that a fully rational individual displaying some aversion toward ambiguous survival probabilities may be likely to exhibit a low demand for annuities. Indeed, in a life-cycle model with a bequest motive, we show that a zero annuitization strategy is optimal if the ambiguity aversion is sufficiently large. Instead, it is optimal to sell annuities short or, under some conditions (see Yaari [46] and Bernheim [3]), to purchase pure life insurance policies. For the sake of simplicity, we do not make any distinction between the two financial products and only deal with the demand for annuities.

Before detailing our results, let us first discuss the two main assumptions of the paper: the uncertainty on survival probabilities and the aversion toward this ambiguity.

[1]Most notably, Brown et al. [8] use the framing hypothesis and show that in an investment frame, individuals prefer non-annuitized products. This idea is developed in Benartzi et al. [2] who demonstrate the importance of the availability and framing of annuity options in the demand for annuity.

Despite all the available information displayed in Life Tables, we think that survival probabilities are nevertheless ambiguous for individuals.

First, there is a rather strong heterogeneity in the age at death. According to Edwards and Tuljapurkar [10], past age 10, its standard deviation is around 15 years in the US. After having controlled for sex and race differentials and for various socioeconomic statuses, they found a residual heterogeneity that remains significant. Heterogeneity is notably explained by biological differences that are not necessarily known to the individual. According to Post and Hanewald [40], this objective heterogeneity is positively linked to individuals' subjective estimate of survival probabilities. Second, the large increase in life expectancy experienced by populations over the last two centuries was characterized by changes in the distribution of survival probabilities at each age. This is referred to as the epidemiological transition and features an increase in the dispersion of heterogeneity computed in the later years. Moreover, opposite factors such as medical progress versus the emergence of new epidemic diseases cause some uncertainty in the dynamics of the distribution per age. Based on Life Tables, an individual belonging to a given cohort may, at best, only know the true distributions of past cohorts but remains uncertain about his/her own. Moreover, due to the small number of observations, data concerning the much later years are not reliable and there is no consensus among demographers about forecasts of the mean survival rate (see especially Oeppen and Vaupel [34]). The longevity risk, defined as the risk that people live longer than expected, is according to the IMF [25], systematically underestimated by governments and pension managers.

The assumption of an aversion toward the ambiguity of survival probabilities is also supported by a great deal of evidence. Initial intuitions concern health risks and can be found in the study by Keynes [26], in which he considers patients who must decide between two medical treatments. Keynes argues that most individuals would choose a treatment that has been extensively used in the past and has a well-known probability of success, rather than a new one, for which there is no information about its probability of success[2]. Moreover, the celebrated Ellsberg [13] experiment has also been applied to health and longevity risks in many studies using hypothetical scenarios. Among them, Viscusi et al. [45] show that individuals have a significant aversion to ambiguous information about the risk of lymphatic cancer. More recently, real case studies have confirmed that individuals are ambiguity averse. Riddel and Show [41] have found a negative relationship between the perceived uncertainty about the risks associated with nuclear waste transportation and the willingness to be exposed to such risks. Similarly, scientific disagreement about the efficiency of vaccination (see Meszaros et al. [34]) or screening mammography recommendations (see Han et al. [21], [20]) has been found to be negatively correlated with the perception of disease preventability and the decisions of preventive behaviors.

We consider a life-cycle model with consumption and bequest similar to Davidoff et al. [9]. We do not focus on market imperfections but instead assume, as in Yaari [46], some warm-glow altruism. Remark that a bequest motive is necessary to obtain some partial annuitization but it does not eliminate the advantage of annuities since they return more, in case of survival, than regular bonds. Our main departure from Yaari [46] and Davidoff et al. [9]'s works hinges on the assumption of ambiguity aversion toward uncertain survival probabilities. We apply the recent model for ambiguity aversion proposed by Klibanoff et al. [27] to a decision problem in a state-dependent utility framework yield by uncertain lifetimes. The representation functional is an expectation of an expectation: the inner expectation evaluate the expected utilities corresponding to possible first order probabilities while the outer expectation aggregates a transform of these expected utilities with respect to a second order prior[3].

We obtain the following results. Provided that annuities return is sufficiently larger

[2]This idea was formalized and developed by Manski [32].

[3]Gollier [17] has applied this framework to a standard portfolio problem and shown that ambiguity aversion does not necessarily reinforce risk aversion.

than bonds return, and notably when it is fair, the optimal share of annuities in the portfolio is positive if the individual is ambiguity neutral. This is not surprising as the individual maximizes a standard expected utility computed with a subjective mean survival probability. Conversely, as the index of ambiguity aversion of Klibanoff et al. [27] increases, the optimal demand for annuities decreases and there exists a finite threshold above which the demand is non positive. Our results are obtained under general specifications of both utility functions and survival probability distributions[4]. The aversion to ambiguous survival probabilities hence appears as a good candidate to explain the observed aversion to annuities. A numerical application of our model suggests that the impact of ambiguity aversion is likely to be quantitatively large.

Our results strongly differs from those obtained with an expected utility framework, may the survival probability be deterministic (as in Yaari [46]) or stochastic (Huang et al. [24], Post [39]). Applications of non-expected utility models to health and longevity uncertainties have been scarce. Among exceptions, Eeckhoudt and Jevela [12] study medical decisions and Treich [43] the value of a statistical life. Interestingly, Ponzetto [38] and Horneff et al. [23] apply Epstein and Zin [14]'s recursive utility framework to uncertain longevity. They show that the utility value of annuitization is decreasing in both risk aversion and elasticity of intertemporal substitution. Positive annuitization nevertheless remains optimal. Moreover, Groneck et al [19] apply the Bayesian learning model under uncertain survival developed by Ludwig and Zimper [31] to a life-cycle model without annuity markets. The dynamics of savings over the life-cycle is closer to the empirical observations. Similarly, Holler et al. [22] study a life-cycle model extended to incorporate an aversion to uncertain lifespans, as developed by Bommier [4].

Section 2 proposes a model of consumption and bequest with uncertain lifetimes and studies the annuitization decision under the assumption of ambiguity neutrality.

[4] Moreover, our results could also be derived within the framework proposed by Gajdos et al. [15].

Section 3 introduces the Klibanoff et al. [27] framework to analyze the impact of ambiguity aversion on optimal choices. It contains our main theoretical results and a numerical application. Concluding remarks are in Section 4 and proofs are gathered in the Appendix.

2 Annuitization in a benchmark expected utility framework

This section presents the model and studies the optimal demand for annuities in the expected utility case (EU, hereafter). This framework, which is used in most articles studying annuities, is a useful benchmark to be compared with the model with ambiguity aversion derived in Section 3.

We consider a static model of consumption and bequest under uncertain lifetime similar to Yaari [46] and Davidoff et al. [9]. The length of life is at most two periods with the second one being uncertain. The Decision Maker (DM, hereafter) derives utility from a bequest that might happen at the end of periods 1 and 2 and, upon survival, from consumption in period 2. At the first period, the DM is endowed with an initial positive income w that can be shared between bonds and annuities. Bonds return $R > 0$ units of consumption in period 2, whether the DM is alive or not, in exchange for each unit of the initial endowment. Conversely, annuities return $R_a \geq R$ in period 2 if the DM is alive and nothing if she is not alive. Due to the possibility of dying, the demand for bonds should be non negative and therefore annuities are the only way to borrow. The selling of annuities is here equivalent to the purchasing of pure life insurance policies (see Yaari [46] and Bernheim [3]). In the remaining of the paper, we will consider the selling of annuities as a zero annuitization strategy. If alive during the second period, the DM may allocate her financial wealth between consumption and bequest. Since death is certain at the end of period 2, the latter is exclusively a demand for bonds.

Denote by a, the demand for annuities and by $w-a$, the demand for bonds, decided in period 1. Moreover, let c and x denote the consumption and the bequest decided in period 2. The budget constraint writes:

$$c = c(a, x) = R_a a + R(w - a) - x, \tag{1}$$

and the non negativity constraints are:

$$c \geq 0,\ x \geq 0,\ a \leq w. \tag{2}$$

Following Davidoff et al. [9], we assume that whatever the length of the DM's life, bequests are received in period 3, involving additional interest: the bequest is therefore xR, if the DM is alive in period 2, while it is $(w-a)R^2$, if she is not.

We assume that the DM's utility is $u[c, xR]$ if alive in period 2 and $v[(w-a)R^2]$ if she is not alive. Functions u and v are supposed to satisfy the following assumptions:

Assumption 1. *The twice continuously differentiable function* $u : \mathbb{R}^2_+ \to \mathbb{R}_+$ *is strictly concave*[5] *and satisfies* $u'_1[c, \varrho] > 0$, $u'_2[c, \varrho] > 0$, $\lim\limits_{c \to 0} u'_2[c, \varrho] = +\infty$ *and* $\lim\limits_{\varrho \to 0} u'_2[c, \varrho] = +\infty$. *The twice continuously differentiable function* $v : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing, strictly concave and satisfies* $\lim\limits_{\varrho \to 0} v'[\varrho] = +\infty$. *Goods* c *and* x *are normal*[6].

Assuming infinite marginal utilities when consumption or bequest go to zero ensures that inequalities lying in (2) are strict at the optimum. This assumption, of course, has no impact on the sign of the optimal demand for annuities.

Assumption 2. $u[0, 0] = 0$ *and, for all non negative* c *and* ϱ, $u[c, \varrho] \geq v[\varrho]$ *and* $u'_2[c, \varrho] \geq v'[\varrho]$.

[5] The function u is strictly concave if and only if the Hessian of u is negative definite, or equivalently, if and only if $u''_{11} < 0$ and $u''_{11}u''_{22} - {u''_{12}}^2 > 0$ (see Theorems M.C.2 and M.D.2 in Mas-Colell et al [33]).

[6] Assuming the normality of c and x means assuming $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$ (see Appendix A).

Both the utility and the marginal utility of a given bequest is larger if the DM is alive than if she is not. These assumptions are commonly used in the literature on the economic valuation of risks to health and life (see e.g. Viscusi and Aldy [44]). They are obviously satisfied when the utility if alive is separable, i.e. for $u[c, xR] = h[c] + v[xR]$, as in Yaari [46] and Davidoff et al. [9].

Let us suppose that the survival probability, denoted $p \in (0, 1)$, is known by the DM. Equivalently, we may interpret p as a survival probability subjectively evaluated by the DM and thus the problem is the one of a subjective expected utility maximizer. Given p, the DM maximizes his expected utility, i.e. solves the EU problem, denoted $(\mathcal{P}_0)$:

$$\max_{a,x} \mathcal{U}(a, x, p)$$

$$\text{s.t.} \quad \begin{cases} \mathcal{U}(a, x, p) = pu[c(a, x), xR] + (1 - p)v[(w - a)R^2], \\ c = c(a, x) = R_a a + R(w - a) - x. \end{cases} \qquad (\mathcal{P}_0)$$

The solution of $(\mathcal{P}_0)$, denoted $(\bar{a}, \bar{x})$, satisfies the FOCs:

$$p(R_a - R)u'_1[c(\bar{a}, \bar{x}), xR] - (1 - p)R^2 v'[(w - \bar{a})R^2] = 0, \qquad (3)$$

$$-u'_1[c(\bar{a}, \bar{x}), \bar{x}R] + Ru'_2[c(\bar{a}, \bar{x}), \bar{x}R] = 0. \qquad (4)$$

We remark that the survival probability affects the optimal demand for annuities but not, as shown by equation (4), the optimal allocation of the financial wealth between consumption and bequest. It will be useful to use the condition (4) to define the application $\bar{x} = f(\bar{a})$, which satisfies $0 \leq f'(a) \leq R_a - R$. Hence, at the optimum, if the DM survives, her consumption $\widehat{c}(a) = c(a, f(a))$ and her bequest $f(a)$ increase with the demand for annuities and the optimal behavior is given in the following proposition:

Proposition 1 .

The optimal demand for annuities, denoted $\bar{a}$, which solves the EU problem $(\mathcal{P}_0)$:

(i) is lower than w but larger than $-Rw/(R_a - R)$.

(ii) is positive if and only if R_a is larger than a threshold $\widehat{R}_a \in (R, R/p)$.

(iii) is positive if and only if p is larger than a threshold $\widehat{p} \in (0,1)$.

Proof – See Appendix A. □

The optimal demand for annuities can be positive or negative and the role of the annuities return is crucial in that matter. The share of annuities in the portfolio obviously increases with the return R_a. Similarly, a larger p reduces the distance between the return R_a and its fair value R/p. To be more precise, let us merge equations (3) and (4) to obtain:

$$(R_a - R)pu_2'[\widehat{c}(\bar{a}), \bar{x}R] - R(1-p)v'[(w-\bar{a})R^2] = 0. \tag{5}$$

Let us now examine two polar cases. Consider first that annuities and bonds returns are equal, i.e. suppose that $R_a = R$. The budget constraint (1) implies that the consumption is independent from the demand of annuities and, with equation (5), we conclude that the optimal behavior is to sell an infinite quantity of annuities to finance the purchase of an infinite quantity of bonds ($-\bar{a} = \bar{x} \to +\infty$). Conversely, for an actuarially fair return of annuity, such that $R_a = R/p$, equation (5) rewrites:

$$u_2'[\widehat{c}(\bar{a}), \bar{x}R] - v'[(w-\bar{a})R^2] = 0.$$

Using the fact that the marginal utility of a given bequest is larger in case of survival (Assumption 2), one concludes that $\bar{x} \geq (w-\bar{a})\,R$ and thus, with the budget constraint, that $\bar{c} = \widehat{c}(\bar{a}) \leq R_a\bar{a}$. As a consequence, the optimal demand for annuities is necessarily positive. Between these two cases, there exists a $R_a \in (0, R/p)$ such that the demand for annuities is zero.

The Proposition 1 is consistent with the findings of the literature on annuities. First, a bequest motive appears as a necessary condition for zero annuitization. Then, Davidoff et al. [9] shows that positive annuitization remains optimal if the return is fair. Conversely, Lockwood [30] shows that if the return is small enough, the demand is negative. In the next section, we consider the possibility of having a non positive demand for annuities for large returns, and most notably in the case of an actuarially fair annuity pricing.

3 Annuitization with ambiguity averse individuals

This section applies the recent rationale for ambiguity aversion proposed by Klibanoff et al. [27] to our life-cycle problem with uncertain lifetimes.

The first difference with the previous section's model is that we now assume that survival probabilities are uncertain. More precisely, the DM does not know her own probability distribution but only knows the set of possible distributions. There exist states of nature associated with given survival probabilities that may be interpreted as health types, to which the DM subjectively associates a probability to be in. Ambiguity is hence modeled as a second order probability over first order probability distributions. Let us denote the random (continuous or discrete) survival probability by $\widetilde{p}$ whose support is denoted $\mathrm{Supp}(\widetilde{p})$, and the survival expectancy as it is evaluated by the DM by $E(\widetilde{p}) = p$. It is supposed, since it does not affect the main results, that p corresponds to the mean survival probability. Denoting by p_m the lower bound of $\mathrm{Supp}(\widetilde{p})$, we consider some restrictions on the probability distribution of the survival probability $\widetilde{p}$.

Assumption 3. *$p_m < \widehat{p} < p$, where $\widehat{p}$ is defined in Proposition 1.*

Assumption 3 gives a lower bound for the mean survival probability or, equivalently, implies that the annuity return would be sufficiently large to induce a positive annuitization in the EU case. In particular, we still allow annuities to be fairly priced.

The expected utility, denoted $\mathcal{U}(a, x, \widetilde{p})$, is thus also a random variable that writes:

$$\mathcal{U}(a, x, \widetilde{p}) = \widetilde{p}u[c(a, x), xR] + (1 - \widetilde{p})v[(w - a)R^2], \tag{6}$$

The second difference with the previous section's model is that we also assume that the DM has smooth ambiguity preferences. Following Klibanoff et al [27], the aversion to ambiguity is introduced using a function ϕ whose concavity represents an index of this aversion. The utility function of the DM is then given by an expectation of an expectation. The inner expectations evaluate the expected utilities corresponding to possible first order probabilities while the outer expectation aggregates a transform of these expected utilities with respect to the second order prior. The utility function writes:

$$\phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p})))), \tag{7}$$

where ϕ satisfies the following restrictions:

Assumption 4. *The twice continuously differentiable function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave.*

Limit cases of (7) are interesting. First, when ϕ is linear, the DM is ambiguity neutral. Using (6), the utility function (7) rewrites:

$$\phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p}))))|_{\phi''=0} = \mathcal{U}(a, x, p), \tag{8}$$

which is linear with respect to the mean survival probability, p. Equation (8) corresponds to the EU case studied section 2. Alternatively, under some conditions[7] that will be assumed to be satisfied, the maxmin expected utility is obtained when the absolute ambiguity aversion, given by $-\phi''/\phi'$, is infinite. The DM then maximizes the expected utility for the worst possible realization of $\widetilde{p}$. Formally, this a maxmin problem that writes:

$$\lim_{-\frac{\phi''(.)}{\phi'(.)} \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p})))) = \max_{a,x} \min_{p_\varepsilon \in \mathrm{Supp}(\widetilde{p})} \mathcal{U}(a, x, p_\varepsilon). \tag{9}$$

This Gilboa and Schmeidler's [16] kind of problem has been studied by d'Albis and Thibault [1] in a framework with endogenous saving rates.

The ambiguity averse DM faces the following problem, denoted $(\mathcal{P}_\phi)$:

[7]See Proposition 3 page 1867 in Klibanoff et al. [27].

$$\max_{a,x} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p}))))$$

$$\text{s.t.} \begin{cases} \mathcal{U}(a,x,\widetilde{p}) = \widetilde{p}u[c(a,x),xR] + (1-\widetilde{p})v[(w-a)R^2], \\ c(a,x) = R_a a + R(w-a) - x. \end{cases} \tag{$\mathcal{P}_\phi$}$$

Then, the solution $(a^\star, x^\star)$ of $(\mathcal{P}_\phi)$ satisfies the FOCs:

$$E(\phi'(\mathcal{U}(a^\star,x^\star,\widetilde{p}))[\widetilde{p}(R_a-R)u_1'[c(a^\star,x^\star),x^\star R] - (1-\widetilde{p})R^2v'[(w-a^\star)R^2]]) = 0, \tag{10}$$

$$-u_1'[c(a^\star,x^\star),x^\star R] + Ru_2'[c(a^\star,x^\star),x^\star R] = 0. \tag{11}$$

Equation (11) is equivalent to (4). Since the trade-off between consumption and bequest if alive is not affected by the survival probability, it is also independent from ambiguity aversion. As in the previous section, we define $x^\star = f(a^\star)$. From (10), it is clear that ambiguity aversion however affects the demand for annuities. To evaluate this effect, we are going to consider two DM distinguished only by their ambiguity attitude. The ambiguity aversion of the first is characterized by the function ϕ, while the one of the second is given by an increasing and concave transformation of ϕ; the second DM being thus said more averse to ambiguity[8]. Let us turn to our main results.

Proposition 2 .

The optimal demand for annuities, denoted $a^\star$, which solves the problem $(\mathcal{P}_\phi)$:

(i) decreases with ambiguity aversion and is lower than $\bar{a}$.

(ii) is always negative when ambiguity aversion is infinite.

Then, there exists a finite degree of ambiguity aversion such that the zero annuitization strategy is optimal if and only if the DM's ambiguity aversion is larger than this threshold.

Proof – See Appendix B. □

Ambiguity aversion determines the optimal exposure to uncertainty. Obviously, a more ambiguity averse DM chooses to be less exposed, which means that she aims at

[8]See Theorem 2 page 1865 in Klibanoff et al. [27].

smoothing the expected utilities computed in each state of nature. Proposition 2 states this can be achieved by reducing the share of annuities in the portfolio. Moreover, the strategy to sell annuities short is optimal provided that ambiguity aversion is sufficiently large. These results hold for arbitrarily low bequest motive and arbitrarily large annuity returns, and notably when the latter are fair. Let us point out that our results may not be intuitive and that one may think an annuity as an insurance product whose demand should increase with the aversion to uncertain lifetime. Let us now explain why this is not the case.

First, Proposition 2 states that the demand for annuities decreases with ambiguity aversion. Let us consider two states of nature, 1 and 2, for which the survival probabilities are respectively p_1 and p_2. Using equation (6), the optimal difference of expected utilities in both states is thus:

$$(p_1 - p_2) \left\{ u \left[c \left(a^\star, x^\star \right), x^\star R \right] - v \left[\left(w - a^\star \right) R^2 \right] \right\}, \tag{12}$$

which is proportional to the optimal difference between the utility if the DM lives for two periods and the utility if she lives for one period only. As the utility depends on the bequest, the sign of the latter difference is not a priori given. As an intermediary result, we proved that it is always positive. We notice this is not an immediate implication of Assumption 2 since the bequest depends on whether the DM lives for one or two periods. Then, we proved that reducing the demand for annuities reduces the difference (12). The idea is that upon survival, the utility increases with the share of annuities in the portfolio. As a consequence, to reduce the exposure to life uncertainty, the DM has to increase her demand for bonds and reduce her demand for annuities.

Second, Proposition 2 states that it is optimal to sell annuities short when the ambiguity aversion is infinite. The optimal behavior is thus to eliminate any exposure to the uncertainty, which is achieved by equalizing the utility derived from a two-period lifetime with the utility derived from a one-period lifetime. We showed that the optimal demand for annuities is in this case negative and finite. The utility benefits of a longer lifespan are then compensated by a reduction in consumption due to the reimbursement of the annuitized debt.

By combining these results with Assumption 3 and using a continuity argument, we conclude there exists a finite degree of ambiguity aversion such that the zero annuitization strategy is optimal if and only if the DM's ambiguity aversion is larger than this threshold.

Previous articles using an EU framework have not been able to show that a zero annuitization strategy can be optimal when annuities are fairly priced: a strong bequest motive, or a strong risk aversion reduce the demand for annuities but do not imply a negative demand. Of course, allowing for a low annuities return reinforces our results: the demand for annuities is then negative whatever the degree of ambiguity aversion.

To illustrated numerically our results, we consider a simple case where there are only two survival distributions which represent two types of respectively "good" and "bad" health. With subjective probability $q \in [0,1]$, the DM thinks she is of the "good" type for which the survival probability is $p_1 \in (0,1]$, while she is, with probability $1-q$, of the "bad" type for which the survival probability is $p_2 \in [0,p_1)$. We assume, moreover, that the bad health state does not imply additional costs and that the absolute ambiguity aversion is constant and is denoted $\alpha \geq 0$. The functional $\phi(.)$ is then an exponential function which satisfies:

$$\phi(x) = \begin{cases} \dfrac{1-e^{-\alpha x}}{\alpha} & \text{if } \alpha > 0, \\ \\ x & \text{if } \alpha = 0. \end{cases}$$

In this case, the objective function is:

$$\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \\ -\frac{\ln\left\{qe^{-\alpha(p_1-p_2)(u[c(a,x),xR]-v[(w-a)R^2])}\right\}+1-q}{\alpha}+\mathcal{U}(a,x,p_2)$$

Limit cases are derived using the Hôpital's rule. We obtain:

$$\lim_{\alpha\to 0}\phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \mathcal{U}(a,x,p_1 q+(1-q)p_2),$$

which is the EU case and where $p_1 q + (1-q)p_2$ is the subjective survival probability. Moreover[9]:

$$\lim_{\alpha \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a,x,\widetilde{p})))) = \min\{\mathcal{U}(a,x,p_1), \mathcal{U}(a,x,p_2)\}. \tag{13}$$

The calibration of the model has been done to reproduce a proportion of the initial endowment invested in annuities of about 70% in the EU case with fair annuities return. We have thus chosen the following functional forms: $u[c, xR] = c^{0.7} + (xR)^{0.3}$ and $v[(w-a)R^2] = [(w-a)R^2]^{0.3}$, and the following parameters: $w = 1$, $p_1 = q = 0.8$, $p_2 = 0.2$, $R = 2$ and $R_a = 2.9412$ which correspond to an actuarially fair annuity system. Since there is no consensus in the literature about the value of the ambiguity aversion coefficient except that it is likely to be heterogeneous among individual (see Borghans et al. [5]), we plotted in Figure 1 the optimal demand for annuities and the difference between the utility if alive in the second period and the utility if not as functions of the absolute ambiguity aversion.

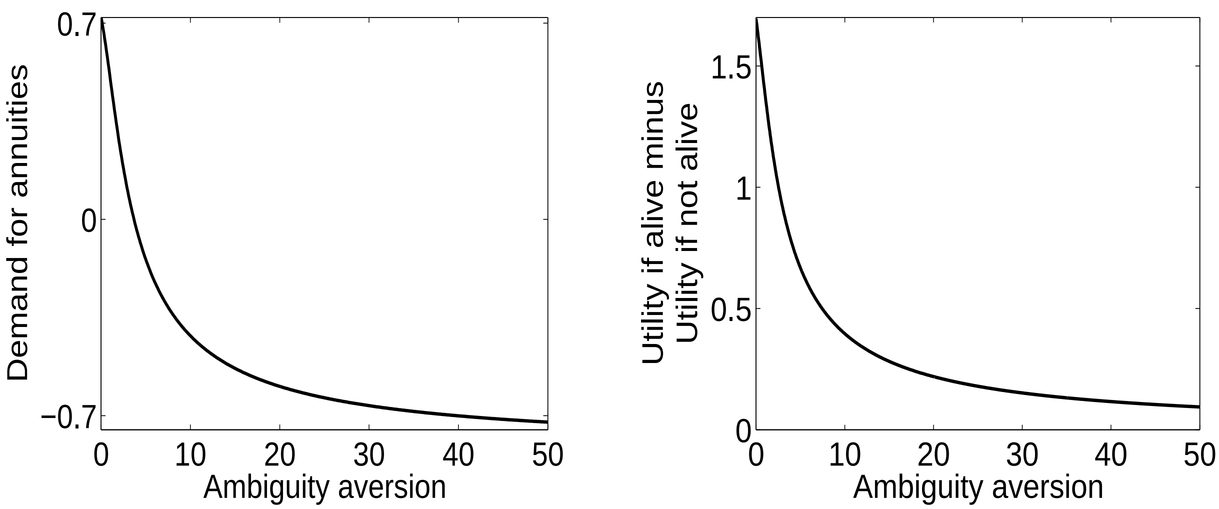

Figure 1: The impact of ambiguity aversion

The LHS shows that ambiguity aversion strongly affects the demand for annuities: the proportion of the initial endowment being invested in annuities goes from about

[9]See Appendix C for details.

70% in the EU case to 0% in the case such that absolute ambiguity aversion is about 6. The RHS shows that the difference in utilities monotonically reduces with the aversion.

4 Conclusion

In this paper, we have studied the optimal demand for annuities in a life-cycle model with a bequest motive. We claimed that ambiguity aversion to uncertain survival probabilities can explain the observed low demand for annuities. Indeed, purchasing annuities means participating to a lottery, which is very specific since it is a bet on longevity. Benartzi et al. [2] pointed out the fact that "an annuity should be viewed as a risk-reducing strategy, but it is instead often considered as a gamble". We showed that ambiguity aversion can be a rational for that attitude: a more ambiguity averse DM has been shown to be more reluctant to participate to the lottery and, thus, holds less annuities in her portfolio. Moreover, we proved that for sufficiently large ambiguity aversion, the demand for annuities is non positive.

Importantly, this result notably holds for arbitrarily low bequest motives and for fair annuity return. We therefore think that aversion to uncertain survival probabilities can contribute to explain, with other factors such as bequest motives or unfair pricing of annuities, the observed low demand for annuities. A natural extension of our work would be considering a multi-period life-cycle model. The dynamic framework proposed by Kreps and Porteus [29] and Klibanoff et al. [28], applied to life-cycle decisions as in Peijnenburg [36], could be a nice setting for that purpose.

Appendix

Appendix A – Proof of Proposition 1.

As a preliminary, let us establish that $u''_{11} + R^2 u''_{22} - 2Ru''_{12} < 0$, $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$.

First, as the Hessian of u is negative definiteness, it is straightforward that $u''_{11} + R^2 u''_{22} - 2Ru''_{12} < 0$. Second, consider the static program $\max_{c,x} u[c, xR]$ subject to the constraint $\Omega = c + x$ where Ω is the life-cycle income of an agent. This program is equivalent to $\max_c u[c, (\Omega - c)R]$ or $\max_x u[\Omega - x, xR]$. Then, using the FOC of these programs and the implicit function theorem, there exists two functions A and B such that $c = A(\Omega)$ and $x = B(\Omega)$. It is straightforward that $A'(\Omega)$ has the sign of $u''_{12} - Ru''_{22}$ while $B'(\Omega)$ has the one of $Ru''_{12} - u''_{11}$. By definition, c and x are normal goods if and only if $\partial c/\partial\Omega$ and $\partial x/\partial\Omega$ are positive. Then, Assumption 1 implies $u''_{12} - Ru''_{22} > 0$ and $Ru''_{12} - u''_{11} > 0$.

A.1 – Proof of statement (i).

Using (4), let us define $F(a, x) = -u'_1[c(a, x), xR] + Ru'_2[c(a, x), xR] = 0$. Under Assumption 1, $F'_1(a, x) = (R_a - R)(Ru''_{12} - u''_{11})$ is non negative whereas $F'_2(a, x) = u''_{11} + R^2 u''_{22} - 2Ru''_{12}$ is negative. Then, there exists a continuous and differentiable function f such that $x = f(a)$ and, under Assumption 1, $0 \leq f'(a) = -F'_1(a, x)/F'_2(a, x) \leq R_a - R$. Replacing $x = f(a)$ and (4) in (3) allows us to define the solution $(\bar{a}, f(\bar{a}))$ as a pair satisfying:

$$G(a, R_a, p) = p(R_a - R)u'_2[\widehat{c}(a), f(a)R] - (1 - p)Rv'[(w - a)R^2] = 0.$$

Then, an optimal demand for annuities $\bar{a}$ is a real root of $G(a, R_a, p) = (R_a - R)p\varphi(a) - R(1 - p)\psi(a)$ where $\varphi(a) = u'_2[\widehat{c}(a), f(a)R]$ and $\psi(a) = v'[(w - a)R^2]$. After computations, $\varphi'(a) = -R(R_a - R)({u''_{12}}^2 - u''_{11}u''_{22})/F'_2(a, x)$ and $\psi'(a) = -R^2 v''[(w - a)R^2]$. Then, under Assumption 1, φ is a decreasing function whereas ψ is an increasing one, implying $G'_1(a, R_a, p) < 0$. Moreover $\varphi(-Rw/(R_a - R)) = +\infty$, $0 < \varphi(w) < +\infty$, $0 < \psi(-Rw/(R_a - R)) < +\infty$ and $\psi(w) = +\infty$. Then, as $G'_1(a, R_a, p) < 0$, $G(-Rw/(R_a - R), R_a, p) > 0$ and $G(w, R_a, p) < 0$, there exists a unique optimal pair $(\bar{a}, \bar{x})$. This pair is such that $\bar{a} \in (-Rw/(R_a - R), w)$ and $0 < \bar{x} = f(\bar{a}) < w$.

A.2 – Proof of statement (ii).

Since $G(\bar{a}, R_a, p) = 0$ and $G'_1(a, R_a, p) < 0$, $\bar{a}$ is positive if and only if $G(0, R_a, p)$ is positive. Importantly $f(0)$ is independent of R_a because (4) is independent of R_a when $a = 0$. Consequently, $\varphi(0) = u'_2[Rw - f(0), f(0)R]$ and $\psi(0) = v'[wR^2]$ are independent of R_a and we have $G'_2(0, R_a, p) = p\varphi(0) > 0$. Note that $G(0, R/p, p) = R(1-p)(u'_2[Rw - f(0), f(0)R] - v'[wR^2])$. As $f(0) < wR$ and $u'_2[c, y] \geq v'[y]$, we have $G(0, R/p, p) > 0$. Since $G(0, R, p) < 0$ and $G'_2(0, R_a, p) > 0$, there exists a unique $\widehat{R}_a \in (R, R/p)$ such that $G(0, \widehat{R}_a, p) = 0$. Consequently, $\bar{a}$ is positive if and only if R_a is larger than $\widehat{R}_a$. $\square$

A.3 – Proof of statement (iii).

According to Appendix A.2, $\bar{a} \geq 0$ if and only if $G(0, R_a, p) \geq 0$. As $G'_3(0, R_a, p) > 0$, $G(0, R_a, 0) = -Rv'[wR^2] < 0$ and $G(0, R_a, 1) = (R_a - R)u'_2[\widehat{c}(0), f(0)R] > 0$, there exists a unique $\widehat{p} \in (0, 1)$ such that $\bar{a}$ is positive if and only if p is larger than $\widehat{p}$. $\square$

Appendix B – Proof of Proposition 2.

As a preliminary, it would be useful to establish the following lemma:

Lemma 1 *If (4) is satisfied (i.e., if $\bar{x} = f(\bar{a})$) then the difference between $u[.]$ the utility if alive and $v[.]$ the utility if not is positive if and only if p is larger than a threshold $\check{p} \in (0, \widehat{p})$. Moreover, the demand for annuities $\underline{a}$ equalizing the utilities in the two states of nature is unique and negative.*

Proof. Let $\tau(a) = u[\widehat{c}(a), f(a)R]$ be the utility if alive. As $0 \leq f'(a) \leq R_a - R$ and $\tau'(a) = [(R_a - R) - f'(a)]u'_1[\widehat{c}(a), f(a)R] + Rf'(a)u'_2[\widehat{c}(a), f(a)R]$, the utility if alive increases with the demand for annuity while the utility if not, namely $v[(w-a)R^2]$, decreases with a. Hence, the difference between $u[.]$ the utility if alive and $v[.]$ the utility if not:

$$\xi(a) = u[\widehat{c}(a), f(a)R] - v[(w-a)R^2]$$

increases with respect to a.

Let us define $\underline{a}$, the demand for annuities equalizing the utilities in the two states of nature (i.e. such that $\xi(\underline{a}) = 0$). Under Assumption 2 $\xi(-wR/(R_a - R)) = u[0,0] - v[R^2 R_a w/(R_a - R)] < 0$ and $\xi(0) = u[\widehat{c}(0), f(0)R] - v[wR^2] > 0$. As $\xi'(a) > 0$, the demand for annuities $\underline{a}$ is unique and negative.

As $\xi(\underline{a}) = 0$ and $\xi'(a) > 0$, $\xi(\bar{a}) > 0$ if and only if $\bar{a} > \underline{a}$, i.e. if and only if $G(\underline{a}, R_a, p) > 0$. As $G'_3(\underline{a}, R_a, p) > 0$, $G(\underline{a}, R_a, 0) = -Rv'[(w-\underline{a})R^2] < 0$ and $G(\underline{a}, R_a, 1) = (R_a - R)u'_2[\widehat{c}(\underline{a}), f(\underline{a})R] > 0$, there exists a unique $\check{p} \in (0,1)$ such that $\xi(\bar{a})$ is positive if and only if p is larger than $\check{p}$. As $G(\underline{a}, R_a, \check{p}) = G(\bar{a}, R_a, \widehat{p}) = 0$, $\underline{a} < 0$, $G'_1(a, R_a, p) < 0$ and $G'_3(\underline{a}, R_a, p) > 0$ we have $\check{p} < \widehat{p}$. □

We now consider the problem $(\mathcal{P}_\phi)$ and denote by $(a^\star, x^\star)$ its solution which is the solution of (10) and (11).

B.1 – Proof of statement (i).

We first show that $a^\star$ is lower than $\bar{a}$. Let us define the random variable $\widehat{\mathcal{U}}(a, \widetilde{p}) = \mathcal{U}(a, f(a), \widetilde{p})$ where the application $x = f(a)$ is derived from (11). Then, $\widehat{\mathcal{U}}(a, \widetilde{p})$ represents the expected utility when the budget constraint (1) and the consumption-bequest optimal allocation (11) are satisfied. It is a function of the random variable $\widetilde{p}$. Moreover, let $\widehat{\mathcal{U}}(a, p_\varepsilon)$ denotes the expected utility associated to p_ε, a realization of $\widetilde{p}$. Then, the system of equations (10) and (11) rewrites as a single equation in a:

$$\eta_\phi(a) = E(\phi'(\widehat{\mathcal{U}}(a, \widetilde{p}))\{\widetilde{p}(R_a - R)u'_1[\widehat{c}(a), f(a)R] - (1-\widetilde{p})R^2 v'[(w-a)R^2]\}) = 0.$$

We are going to prove that $\eta_\phi(\underline{a}) > 0$, $\eta_\phi(\bar{a}) < 0$ and $\eta'_\phi(a) < 0$ for $a \in [\underline{a}, \bar{a}]$.

As $\widehat{\mathcal{U}}(\underline{a},\widetilde{p})$ is independent of $\widetilde{p}$, $\eta_{\phi}(\underline{a})$ has the sign of:

$$E(\widetilde{p}(R_a - R)u_1'[\widehat{c}(\underline{a}), f(\underline{a})R] - (1-\widetilde{p})R^2v'[(w-\underline{a})R^2])$$

i.e., since $E\widetilde{p} = p$ and (11), the one of $G(\underline{a}, R_a, p)$. Under Assumption 3 and according to Lemma 1 we have $G(\underline{a}, R_a, p) > 0$ and, consequently, $\eta_{\phi}(\underline{a}) > 0$. Since $G(\bar{a}, R_a, p) = 0$, using (11) we have $\eta_{\phi}(\bar{a}) = E(\phi'(\widehat{\mathcal{U}}(\bar{a},\widetilde{p}))RG(\bar{a}, R_a, \widetilde{p})) = Cov(\phi'(\widehat{\mathcal{U}}(\bar{a},\widetilde{p})), RG(\bar{a}, R_a, \widetilde{p})) < 0$. Since ϕ is concave and $G_1'(a, R_a, \widetilde{p}) < 0$, $\eta_{\phi}'(a) < 0$. Consequently, η_{ϕ} has a unique real root that shall be denoted $a^\star$ and that belongs to $(\underline{a}, \bar{a})$.

Importantly the derivative of function $\widehat{\mathcal{U}}(a, p_\varepsilon)$ with respect to a realization p_ε is:

$$\widehat{\mathcal{U}}_2'(a, p_\varepsilon) = u[\widehat{c}(a), f(a)R] - v[(w-a)R^2] \equiv \xi(a).$$

Then, $\xi(a) = \widehat{\mathcal{U}}_2'(a, p_\varepsilon)$ is independent of any realizations of $\widetilde{p}$. Since $\xi'(a) > 0$, $\xi(\underline{a}) = 0$ and $a^\star > \underline{a}$ we have $\xi(a^\star) \geq 0$.

We now show that $a^\star$ decreases with ambiguity aversion. The strategy is to consider two independent DM. The problem of the first DM is given by $(\mathcal{P}_\phi)$ and her optimal demand for annuities is denoted $a^\star$. The second DM faces the similar problem $(\mathcal{P}_\psi)$ with $\psi \equiv T \circ \phi$ and where T is an increasing and concave function. Her optimal demand, denoted $a^{\star\star}$, is solution of:

$$\eta_{T\circ\phi}(a) = E(\psi'(\widehat{\mathcal{U}}(a,\widetilde{p}))\{\widetilde{p}(R_a - R)u_1'[\widehat{c}(a), f(a)R] - (1-\widetilde{p})R^2v'[(w-a)R^2]\}) = 0,$$

where $\psi'(\widehat{\mathcal{U}}(a,\widetilde{p})) = T'(\phi(\widehat{\mathcal{U}}(a,\widetilde{p})))\phi'(\widehat{\mathcal{U}}(a,\widetilde{p}))$. Note that $\eta_{T\circ\phi}(a)$ is a decreasing function of a.

From Step 1 to Step 4 it supposed that $\widetilde{p}$ has only two realizations, p_1 and p_2, satisfying $1 \geq p_1 > p_2 \geq 0$, and whose occurrence probabilities are respectively q and $1-q$. The result obtained is generalized Step 5 for any distribution of $\widetilde{p}$.

Step 1 – $\widehat{\mathcal{U}}(a^\star, p_1) \geq \widehat{\mathcal{U}}(a^\star, p_2)$ and $\widehat{\mathcal{U}}(a^{\star\star}, p_1) \geq \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

Let $a^\sharp$ an optimum (i.e. $a^\sharp = a^\star$ or $a^{\star\star}$). Then, we have $\widehat{\mathcal{U}}(a^\sharp, p_1) - \widehat{\mathcal{U}}(a^\sharp, p_2) =$

$(p_1 - p_2)\xi(a^\sharp)$. As we have proved that $\xi(a^\sharp) \geq 0$, we have $\widehat{\mathcal{U}}(a^\sharp, p_1) \geq \widehat{\mathcal{U}}(a^\sharp, p_2)$ for $a^\sharp = a^\star$ and $a^\sharp = a^{\star\star}$.

Step 2 – $\widehat{\mathcal{U}}(a^\star, p_1) - \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and $\widehat{\mathcal{U}}(a^\star, p_2) - \widehat{\mathcal{U}}(a^{\star\star}, p_2)$ have opposite signs.

Proceed by contradiction. Suppose first they are both positive. This implies that $E(\psi(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))) > E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$, which is not possible since $E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$ is a maximum. Similarly, they can not be both negative because this would implies that $E(\phi(\widehat{\mathcal{U}}(a^\star, \widetilde{p})))$ is not a maximum.

Step 3 – $\widehat{\mathcal{U}}(a^\star, p_1) > \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and $\widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

As a preliminary, the definition of a maximum yields $E(\phi(\widehat{\mathcal{U}}(a^\star, \widetilde{p}))) > E(\phi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p})))$, which rewrites:

$$q[\phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \phi(\widehat{\mathcal{U}}(a^\star, p_1))] < (1-q)[\phi(\widehat{\mathcal{U}}(a^\star, p_2)) - \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))]. \tag{14}$$

Equivalently, $E(\psi(\widehat{\mathcal{U}}(a^{\star\star}, \widetilde{p}))) > E(\psi(\widehat{\mathcal{U}}(a^\star, \widetilde{p})))$ rewrites:

$$q[\psi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \psi(\widehat{\mathcal{U}}(a^\star, p_1))] > (1-q)[\psi(\widehat{\mathcal{U}}(a^\star, p_2)) - \psi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))]. \tag{15}$$

Now proceed by contradiction by supposing that $\widehat{\mathcal{U}}(a^\star, p_1) < \widehat{\mathcal{U}}(a^{\star\star}, p_1)$. Using Step 1 and 2, this implies that $\widehat{\mathcal{U}}(a^{\star\star}, p_2) < \widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^\star, p_1) < \widehat{\mathcal{U}}(a^{\star\star}, p_1)$. Since ϕ and ψ are both increasing, dividing (15) by (14) yields the following inequalities:

$$\frac{\psi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \psi(\widehat{\mathcal{U}}(a^\star, p_1))}{\phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1)) - \phi(\widehat{\mathcal{U}}(a^\star, p_1))} > \frac{\psi(\widehat{\mathcal{U}}(a^\star, p_2)) - \psi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))}{\phi(\widehat{\mathcal{U}}(a^\star, p_2)) - \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))} > 0. \tag{16}$$

Denote $y_1 = \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_2))$, $y_2 = \phi(\widehat{\mathcal{U}}(a^\star, p_2))$, $y_3 = \phi(\widehat{\mathcal{U}}(a^\star, p_1))$, and $y_4 = \phi(\widehat{\mathcal{U}}(a^{\star\star}, p_1))$. Hence $y_1 < y_2 < y_3 < y_4$ and (16) rewrites as follows:

$$\frac{T(y_4) - T(y_3)}{y_4 - y_3} > \frac{T(y_2) - T(y_1)}{y_2 - y_1}. \tag{17}$$

This latter inequality is true if and only if T is convex. Since T is concave, conclude that $\widehat{\mathcal{U}}(a^\star, p_1) > \widehat{\mathcal{U}}(a^{\star\star}, p_1)$ and, using Step 2, that $\widehat{\mathcal{U}}(a^\star, p_2) < \widehat{\mathcal{U}}(a^{\star\star}, p_2)$.

Step 4 – $a^\star > a^{\star\star}$ *for a binary distribution.*

For an optimum a, we have $\widehat{\mathcal{U}}(a,p_1)-\widehat{\mathcal{U}}(a,p_2)=(p_1-p_2)\xi(a)$ where $\xi(a)>0$ and $\xi'(a)>0$. Then, $\widehat{\mathcal{U}}(a,p_1)-\widehat{\mathcal{U}}(a,p_2)$ is an increasing function of a. According to Step 1 and 2, $\widehat{\mathcal{U}}(a^\star,p_1)-\widehat{\mathcal{U}}(a^\star,p_2)>\widehat{\mathcal{U}}(a^{\star\star},p_1)-\widehat{\mathcal{U}}(a^{\star\star},p_2)\geq 0$. Consequently, $a^\star > a^{\star\star}$.

Step 5 – $a^\star > a^{\star\star}$ *for any distribution of* $\widetilde{p}$.

We have previously shown that $\eta_{_\phi}(a)$ and $\eta_{_{T\circ\phi}}(a)$ are decreasing functions of a. Consequently, $(a^\star > a^{\star\star})$ if and only if $(E(g(\widetilde{p}))=0 \Rightarrow E(h(\widetilde{p}))<0)$ where $g(\widetilde{p})=\phi'(\widehat{\mathcal{U}}(a^\star,\widetilde{p}))\{\widetilde{p}(R_a-R)u_1'[\widehat{c}(a^\star),f(a^\star)R]-(1-\widetilde{p})R^2v'[(w-a^\star)R^2]\}$ and $h(\widetilde{p})=T'(\phi(\widehat{\mathcal{U}}(a^\star,\widetilde{p})))\phi'(\widehat{\mathcal{U}}(a^\star,\widetilde{p}))\{\widetilde{p}(R_a-R)u_1'[\widehat{c}(a^\star),f(a^\star)R]-(1-\widetilde{p})R^2v'[(w-a^\star)R^2]\}$.

The diffidence theorem then applies (see Lemma 1 page 84 in Gollier [17]) and, thus, the result $a^\star > a^{\star\star}$ holds for any distribution of $\widetilde{p}$. Then, the demand for annuities decreases with ambiguity aversion.

B.2 – Proof of statement (ii).

According to (9) the following maxmin problem, denoted $(\mathcal{P}_\infty)$, is obtained when the absolute ambiguity aversion, given by $-\phi''/\phi'$, is infinite:

$$\max_{a,x}\ \min_{p_\varepsilon\in\mathrm{Supp}(\widetilde{p})}\ \mathcal{U}(a,x,p_\varepsilon)$$

$$\text{s.t.}\quad \begin{cases} \mathcal{U}(a,x,p_\varepsilon)=p_\varepsilon u[c(a,x),xR]+(1-p_\varepsilon)v[(w-a)R^2], \\ c=c(a,x)=R_a a+R(w-a)-x. \end{cases} \qquad (\mathcal{P}_\infty)$$

First observe that the optimum $(a^\sharp,x^\sharp)$ is such that $x^\sharp=f(a^\sharp)$. Indeed, for any optimal value $p_\varepsilon^\sharp$, $(a^\sharp,x^\sharp)$ is determined by maximizing $\mathcal{U}(a,x,p_\varepsilon^\sharp)$ subject to (1) and (2). The FOC (4) of the expected utility problem $(\mathcal{P}_0)$ (where $p=p_\varepsilon^\sharp$) is independent of $p_\varepsilon^\sharp$ and implies that $x^\sharp=f(a^\sharp)$.

Consequently, there are only three candidates for the optimum $a^\sharp$ according to the sign of $\xi(a^\sharp)$.

The first candidate corresponds to the case where $\xi(a^\sharp)=0$. Then it corresponds to the (unique) pair $(\underline{a}, f(\underline{a}))$ exhibited in Lemma 1. The maxmin utility attainable is then $v[(w-\underline{a})R^2]$.

The second candidate $\breve{a}$ corresponds to the case where $\xi(a^\sharp) < 0$. As $\widehat{\mathcal{U}}(\breve{a}, p_\varepsilon) = p_\varepsilon \xi(\breve{a}) + v[(w-\breve{a})R^2] < v[(w-\breve{a})R^2]$, the maxmin utility attainable is lower than $v[(w-\breve{a})R^2]$. Importantly $\breve{a}$ is obviously the solution of the expected utility problem $(\mathcal{P}_0)$ where $p = p_M$ is the upper bound of $\mathrm{Supp}(\widetilde{p})$, the support of $\widetilde{p}$. Then, according to Assumption 3 and Lemma 1, we have $\underline{a} < \breve{a}$ and, consequently, $v[(w-\breve{a})R^2] < v[(w-\underline{a})R^2]$. Since the maxmin utility attainable with $\breve{a}$ is strictly lower than the one attainable with $\underline{a}$, we have necessarily $\xi(a^\sharp) \geq 0$ at the optimum.

The third candidate $\breve{a}$ corresponds to the case where $\xi(a^\sharp) > 0$. Importantly $\breve{a}$ is the solution of the expected utility problem $(\mathcal{P}_0)$ where $p = p_m$ is the lower bound of $\mathrm{Supp}(\widetilde{p})$. Then, according to Assumption 3 and Lemma 1, we have $\underline{a} < \breve{a}$. We now show that this third candidate cannot solves the problem $(\mathcal{P}_\infty)$ because if $\breve{a}$ solves the problem $(\mathcal{P}_\infty)$ we have necessarily $\underline{a} \geq \breve{a}$. Indeed, assume that $\breve{a}$ solves the problem $(\mathcal{P}_\infty)$. Then we have $\widehat{\mathcal{U}}(\underline{a}, p_m) \leq \widehat{\mathcal{U}}(\breve{a}, p_m)$. As $0 = \xi(\underline{a}) = \widehat{\mathcal{U}}_2'(\underline{a}, p_\varepsilon) < \xi(\breve{a}) = \widehat{\mathcal{U}}_2'(\breve{a}, p_\varepsilon)$, we obtain $\widehat{\mathcal{U}}(\underline{a}, 1) \leq \widehat{\mathcal{U}}(\breve{a}, 1)$, i.e. $u[\widehat{c}(\underline{a}), f(\underline{a})R] \leq u[\widehat{c}(\breve{a}), f(\breve{a})R]$. As, $u[\widehat{c}(a), f(a)R]$ is increasing function of a, we have $\underline{a} \geq \breve{a}$.

Consequently, we have $\xi(a^\sharp) = 0$ at the optimum and the optimal pair $(a^\sharp, x^\sharp)$ which solves the problem $(\mathcal{P}_\infty)$ is the pair $(\underline{a}, f(\underline{a}))$ exhibited Lemma 1. Then, according to Lemma 1, the demand for annuities is always negative when ambiguity aversion is infinite. $\square$

Appendix C – Derivation of equation (13).

There are two cases.

Suppose first that $u[c, xR] \geq v[(w-a)R^2]$. Then, it is straightforward that:

$$\lim_{\alpha \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p})))) = \mathcal{U}(a, x, p_2).$$

Suppose now that $u[c, xR] < v[(w-a)R^2]$ and let us denote $\alpha = 1/\gamma$ and $\sigma =$

$v[(w-a)R^2] - u[c, xR]$. Then:

$$\lim_{\alpha \to +\infty} \phi^{-1}(E(\phi(\mathcal{U}(a, x, \widetilde{p})))) = \lim_{\gamma \to 0} [\mathcal{U}(a, x, p_2) - \gamma \ln(q e^{\frac{(p_1 - p_2)\sigma}{\gamma}} + 1 - q)] = \mathcal{U}(a, x, p_1),$$

which implies (13). $\square$

References

[1] H. d'Albis and E. Thibault (2012), Optimal annuitization, uncertain survival probabilities and maxmin preferences. Economics Letters 115, 295-299.

[2] S. Benartzi, A. Previtero and R. H. Thaler (2011), Annuitization puzzles. Journal of Economic Perspectives 25, 143-164.

[3] B. D. Bernheim (1991), How strong are bequest motives? Evidence based on estimates of the demand for live insurance and annuities. Journal of Political Economy 99, 899-927.

[4] A. Bommier (2006), Uncertain lifetime and intertemporal choice: risk aversion as a rationale for time discounting. International Economic Review 47, 1223-1246.

[5] L. Borghans, B. Golsteyn, J. Heckman and H. Meijers (2009), Gender differences in risk aversion and ambiguity aversion. Journal of the European Economic Association 7, 649-658.

[6] J. R. Brown (2007), Rational and behavioral perspectives on the role of annuities in retirement planning. NBER Working Paper 13537.

[7] J. R. Brown, M. D. Casey and O. S. Mitchell (2008), Who values the social security annuity? New Evidence on the Annuity Puzzle. NBER Working Paper 13800.

[8] J. R. Brown, J. R. Kling, S. Mullainathan and M. V. Wrobel (2008), Why don't people insure late-life consumption? A framing explanation of the under-annuitization puzzle. American Economic Review: Papers and Proceedings 98, 304-309.

[9] T. Davidoff, J. R. Brown and P. A. Diamond (2005), Annuities and individual welfare. American Economic Review 95, 1573-1590.

[10] R. Edwards and S. Tuljapurkar (2005), Inequality in life spans and a new perspective on mortality convergence across industrialized countries. Population and Development Review 31, 645-675.

[11] L. Eeckhoudt and C. Gollier (1995), Demand for risky assets and the monotone probability ratio order. Journal of Risk and Uncertainty 11, 113-122.

[12] L. Eeckhoudt and M. Jeleva (2004), Décision médicale et probabilities imprécises. Revue Economique 55, 869-882.

[13] D. Ellsberg (1961), Risk, ambiguity and the Savage axioms. Quarterly Journal of Economics 75, 643-669.

[14] L. G. Epstein and S. E. Zin (1989), Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework. Econometrica 57, 937-969.

[15] T. Gajdos, J.M. Tallon, T. Hayashi, and J.C. Vergnaud (2008), Attitude toward imprecise information. Journal of Economic Theory 140, 23-56.

[16] I. Gilboa, D. Schmeidler (1989), Maxmin expected utility with non-unique prior. Journal of Mathematical Economics 18, 141-153.

[17] C. Gollier (2001), The Economics of risk and time. The MIT Press. Cambridge MA.

[18] C. Gollier (2011), Portfolio choices and asset prices: the comparative statics of ambiguity aversion. Review of Economic Studies 78, 1329-1344.

[19] M. Groneck, A. Ludwig and A. Zimper (2011), A life-cycle consumption model with ambiguous survival beliefs. Netspar Discussion Paper 03/2011-044.

[20] P. Han, S. Kobrin, W. Klein, W. Davis, M. Stefanek and S. Taplin (2007), Perceived ambiguity about screening mammography recommendations: association with future mammography uptake and perceptions. Cancer Epidemiology, Biomarkers and Prevention 16, 458-466.

[21] P. Han, R. Moser and W. Klein (2006), Perceived ambiguity about cancer prevention recommendations: relationship to perceptions of cancer preventability, risk, and worry. Journal of Health Communication 11, 51-69.

[22] J. Holler, M. Kuhn and A. Prskawetz (2010), Savings and health investment under risk aversion towards length of life. LEPAS Working Paper 2010-15.

[23] W. J. Horneff, R. H. Maurer and M. Z. Stamos (2008), Life-cycle asset allocation with annuity markets. Journal of Economic Dynamics and Control 32, 3590-3612.

[24] H. Huang, M.A. Milevsky and T.S. Salisbury (2011), Consumption under a stochastic force of mortality, IFID Working Paper.

[25] International Monetary Fund (2012), The financial impact of longevity risk. Global Financial Stability Report, Chapter 4.

[26] J. M. Keynes (1921), A Treatise on probability. Macmillan and Co.

[27] P. Klibanoff, M. Marinacci, S. Mukerji (2005), A smooth model of decision making under ambiguity. Econometrica 73, 1849-1892.

[28] P. Klibanoff, M. Marinacci, S. Mukerji (2009), Recursive smooth ambiguity preferences. Journal of Economic Theory 144, 930-976.

[29] D. Kreps and E. Porteus (1978), Temporal resolution of uncertainty and dynamic choice theory. Econometrica 46, 185-200.

[30] L.M. Lockwood (2012), Bequest motives and the annuity puzzle. Review of Economic Dynamics 15, 226-243.

[31] A. Ludwig and A. Zimper (2007), A parsimonious model of subjective life expectancy. MEA Discussion Paper 2007-154.

[32] C. Manski (2009), The 2009 Lawrence R. Klein Lecture: Diversified treatment under ambiguity. International Economic Review 50, 1013-1041.

[33] A. Mass-Colell, M. Whinston and J. Green (1995), Microeconomic theory. New York and Oxford: Oxford University Press.

[34] J. R. Meszaros, D. A. Asch, J. Baron, J. C. Hershey, H. Kunreuther and J. Schwartz-Buzaglo (1996), Cognitive processes and the decisions of some parents to forego pertussis vaccination for their children. Journal of Clinical Epidemiology 49, 697-703.

[35] J. Oeppen and J. W. Vaupel (2002), Broken limits to life expectancy. Science 296, 1029-1031.

[36] K. Peijnenburg (2011), Life-cycle asset allocation with ambiguity aversion and learning. Mimeo

[37] K. Peijnenburg, T. Nijman and B. Werker (2011), The annuity puzzle remains a puzzle. Available at SSRN: http://ssrn.com/abstract=1341674

[38] G. A. Ponzetto (2003), Risk aversion and the utility of annuities. CeRP Working Paper No. 31/03.

[39] T. Post (2012), Individual welfare gains from deferred life-annuities under stochastic mortality, Asia-Pacific Journal of Risk and Insurance 6 (2), Article 2.

[40] T. Post and K. Hanewald (2010), Stochastic mortality, subjective survival expectations, and individual saving behavior. SBF 649 Discussion Paper 2010-040.

[41] M. Riddel and W. D. Shaw (2006), A theoretically-consistent empirical model of non-expected utility: An application to nuclear-waste transport. Journal of Risk and Uncertainty 32, 131-50.

[42] E. Sheshinski (2008), The economic theory of annuities. Princeton University Press.

[43] N. Treich (2010), The value of a statistical life under ambiguity aversion. Journal of Environmental Economics and Management 59, 15-26.

[44] W. K. Viscusi, W. Kip and J. E. Aldy (2003), The value of a statistical life: a critical review of market estimates throughout the world. Journal of Risk and Uncertainty 27, 5-76.

[45] W. K. Viscusi, W. A. Magat and J. Huber (1991), Communication of ambiguous risk information. Theory and Decision 31, 159-173.

[46] M. E. Yaari (1965), Uncertain lifetime, life insurance, and the theory of the consumer. Review of Economic Studies 32, 137-160.