Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model

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Abstract

We investigate the long time behavior of the critical mass Patlak-Keller-Segel equation. This equation has a one parameter family of steady-state solutions ϱ_λ, $\lambda > 0$, with thick tails whose second moment is not bounded. We show that these steady state solutions are stable, and find basins of attraction for them using an entropy functional $\mathcal{H}_\lambda$ coming from the critical fast diffusion equation in $\mathbb{R}^2$. We construct solutions of Patlak-Keller-Segel equation satisfying an entropy-entropy dissipation inequality for $\mathcal{H}_\lambda$. While the entropy dissipation for $\mathcal{H}_\lambda$ is strictly positive, it turns out to be a difference of two terms, neither of which need to be small when the dissipation is small. We introduce a strategy of *controlled concentration* to deal with this issue, and then use the regularity obtained from the entropy-entropy dissipation inequality to prove the existence of basins of attraction for each stationary state composed by certain initial data converging towards ϱ_λ. In the present paper, we do not provide any estimate of the rate of convergence, but we discuss how this would result from a stability result for a certain sharp Gagliardo-Nirenberg-Sobolev inequality.

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1 Introduction

1.1 The PKS system and its critical mass

The Patlak-Keller-Segel system [31, 22] is one of the simplest models of *chemotaxis*, describing the evolution of the population density of a cell colony which is diffusing across a two dimensional surface. In addition to the diffusion, as the cells move across the surface, they continually emit a chemical attractant, which itself diffuses across the surface. The cells tend to move towards higher concentrations of the attractant, and this induces a drift term tending to concentrate the population, and countering the spreading effects of the diffusion. A model organism for this type of behavior is the *dictyostelium discoideum* which segregates *cyclic adenosine monophosphate*, another important example of chemotactic movement are endothelial cells who react to VEGF to form blood vessels.

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The model is mathematically interesting on account of this competition between the concentrating effects of the drift induced by the chemical attractant and the spreading effects of the diffusion, and there is a *critical value* to the total mass of the initial data, so that for masses above this value, the concentration wins, and the density collapses in a finite time. However, for masses below this critical mass, diffusion dominates, and the colony smoothly diffuses off to infinity. At the critical mass, there is a continuous family of stationary solutions, and this paper is concerned with determining their stability properties, and since they all turn out to be stable, basins of attraction for each of them. We begin by introducing the model and the critical mass associated with it. We refer to [19, 32] for recent reviews on chemotaxis models.

If ρ denotes the population density, and c the concentration of the chemical attractant, the system of equations is

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) = \mathrm{div}\left[\nabla \rho(t,x) - \rho(t,x)\nabla c(t,x)\right] & t>0\,,\ x\in\mathbb{R}^2\,,\\ c(t,x) = -\dfrac{1}{2\,\pi}\displaystyle\int_{\mathbb{R}^2}\log|x-y|\rho(t,y)\,\mathrm{d}y\,, & t>0\,,\ x\in\mathbb{R}^2\,,\\ \rho(0,x)=\rho_0(x)\geq 0 & x\in\mathbb{R}^2\,, \end{cases} \tag{1.1}$$

with an appropriate choices of units, so that all dimensional constants are unity.

In most of this paper, we consider initial data ρ_0 that belongs to $L^1(\mathbb{R}^2, \log(e+|x|^2)\,\mathrm{d}x)$, and such that $\rho_0\log\rho_0$ is integrable. The relevance of these conditions shall be explained shortly, but at the very least, they insure that $c(0,x)$ is well defined. It will sometimes be convenient to write the second equation in (1.1) in the compact form $c(t,x) = G*\rho(t,x)$ where $G(x) = -1/(2\,\pi)\log|x|$ is the Green's function for $-\Delta$ in $\mathbb{R}^2$. That is, $-\Delta c = \rho$.

Also throughout the paper, the term *density* shall always refer to a non-negative integrable function on $\mathbb{R}^2$, and we shall use the term *mass* to refer to the total integral of a density ρ. Because of the divergence form structure of the system, solutions formally satisfy the conservation of mass

$$\int_{\mathbb{R}^2}\rho(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2}\rho_0(x)\,\mathrm{d}x := M$$

for all $t\geq 0$; *i.e.*, the mass M is conserved in time.

The PKS system can be rewritten advantageously as follows: Introduce the *free energy functional* $\mathcal{F}_{\mathrm{PKS}}$

$$\mathcal{F}_{\mathrm{PKS}}[\rho] = \int_{\mathbb{R}^2}\rho(x)\log\rho(x)\,\mathrm{d}x + \frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\,.$$

The first integral is well defined if $\rho\log\rho$ is integrable, and the positive part of $\rho(x)\log|x-y|\rho(y)$ is integrable when ρ belongs to $L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)$, so that the second integral is at least well-defined under this condition.

Now suppose that the density ρ belongs to $L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)$, and moreover, $\rho\log\rho$ is integrable. Then a simple formal calculation shows that for all $u\in C_c^\infty(\mathbb{R}^2)$ with zero mean,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathcal{F}_{\mathrm{PKS}}[\rho+\epsilon u] - \mathcal{F}_{\mathrm{PKS}}[\rho]\right) = \int_{\mathbb{R}^2}\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta\rho}(x)\,u(x)\,\mathrm{d}x$$

where

$$\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta\rho}(x) := \log\rho(x) + \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|\rho(y)\,\mathrm{d}y = \log\rho(x) - G*\rho(x)\,.$$

It is then easy to see that the evolution equation in (1.1) can be rewritten as

$$\frac{\partial\rho}{\partial t}(t,x) = \mathrm{div}\left(\rho(t,x)\nabla\left[\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta\rho}(x)\right]\right)\,. \tag{1.2}$$

It follows that at least along well-behaved classical solutions (for which we may integrate by parts),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{\mathrm{PKS}}[\rho(t)] = -\int_{\mathbb{R}^2}\rho(t,x)\left|\nabla\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho(t)]}{\delta\rho}(x)\right|^2\,\mathrm{d}x\,. \tag{1.3}$$

In particular, along such solutions, $t\mapsto\mathcal{F}_{\mathrm{PKS}}[\rho(t)]$ is monotone non–increasing. The key to exploiting this monotonicity, as discovered in [17], is the sharp logarithmic Hardy–Littlewood–Sobolev (Log HLS) inequality [2, 9]:

1.1 LEMMA (Logarithmic Hardy-Littlewood-Sobolev inequality). *Let f be a non-negative measurable function in $\mathbb{R}^2$ such that $f\log f$ and $f\log(e+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. Let $M := \int_{\mathbb{R}^2} f\,\mathrm{d}x$. Then*

$$\int_{\mathbb{R}^2} f(x)\log f(x)\,\mathrm{d}x + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\log|x-y|f(y)\,\mathrm{d}x\,\mathrm{d}y \geq -\,C(M)\,, \tag{1.4}$$

with $C(M) := M\,(1+\log\pi-\log(M))$. There is equality if and only if $f(x)=\varrho_\lambda(x-x_0)$ for some $\lambda>0$ and some $x_0\in\mathbb{R}^2$, where

$$\varrho_\lambda(x) := \frac{M}{\pi}\frac{\lambda}{(\lambda+|x|^2)^2}$$

Following [17], one may apply sharp log HLS inequality (1.4) to deduce that

$$\begin{aligned}\mathcal{F}_{\mathrm{PKS}}[\rho] &= \frac{M}{8\pi}\left(\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x + \frac{2}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\log|x-y|\rho(y)\,\mathrm{d}x\,\mathrm{d}y\right)\\ &\quad+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x\\ &\geq -\frac{M}{8\pi}\,C(M)+\left(1-\frac{M}{8\pi}\right)\int_{\mathbb{R}^2}\rho(x)\,\log\rho(x)\,\mathrm{d}x\;.\end{aligned}\tag{1.5}$$

It follows from this and the monotonicity of $\mathcal{F}_{\mathrm{PKS}}[\rho(t)]$ that for solutions ρ of the PKS system for which $M<8\pi$,

$$\mathcal{E}[\rho(t)] := \int_{\mathbb{R}^2}\rho(t,x)\log\rho(t,x)\,\mathrm{d}x \leq \frac{8\pi\mathcal{F}[\rho_0]-M\,C(M)}{8\pi-M}\;.$$

Therefore, for $M<8\pi$, the entropy $\mathcal{E}[\rho(t)]$ stays bounded, uniformly in time. This precludes the collapse of mass into a point mass for such initial data. In [17, 5], this formal analysis is made rigorous, and the global existence of solutions below the critical value 8π and a number of their properties as well are established.

Previous work in this direction, by Jäger and Luckhaus [20], had shown that for initial data of sufficiently small mass, the entropy $\mathcal{E}[\rho(t)]$ stayed bounded uniformly in t. Their analysis used the Gagliardo–Nirenberg-Sobolev inequality for functions f in $\mathbb{R}^2$ that bounds $\|f\|_4$ in terms of $\|\nabla f\|_2$ and $\|f\|_2$, and not the Log HLS inequality, but their global existence result requires the mass to lie below a threshold that is strictly less than 8π.

That 8π is the actual critical value at which diffusive and concentrating effects are balanced, and not only a better lower bound, can be seen by computing moments: When the initial data has a finite second moment, and $M>8\pi$ such collapse, or "blow-up" does indeed occur in a finite time. To see this, we first note a weak formulation of our the PKS evolution equation that will useful to us later on. Let ψ be any test function. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\psi(x)\rho(t,x)\,\mathrm{d}x &= \int_{\mathbb{R}^2}\Delta\psi(x)\rho(t,x)\,\mathrm{d}x\\ &\quad-\frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(t,x)\frac{(\nabla\psi(x)-\nabla\psi(y))\cdot(x-y)}{|x-y|^2}\,\rho(y,t)\,\mathrm{d}x\,\mathrm{d}y\;.\end{aligned}\tag{1.6}$$

In addition to the usual integration by parts, we have *symmatrized* the second term on the right in x and y. Fixing any $a\in\mathbb{R}^2$ and taking $\psi(x)=a\cdot x$, we see from (1.6) that $\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_{\mathbb{R}^2}x\rho(t,x)\,\mathrm{d}x=0$; i.e., the center of mass is conserved. Due to the translational invariance, we henceforth assume zero center of mass. More interestingly, taking $\psi(x)=|x|^2$, so that $(\nabla\psi(x)-\nabla\psi(y))\cdot(x-y)=2$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|x|^2\rho(t,x)\,\mathrm{d}x = 4M-\frac{1}{2\pi}M^2 = 4M\left(1-\frac{M}{8\pi}\right)\;.$$

Thus, if $M>8\pi$, right hand side is strictly negative, and this shows that the second momentum of $\rho(t)$ reaches zero in a finite time if initially bounded, or else some sort of singularity develops that would invalidate the formal calculation we have just made.

Thus, the mass value $M = 8\pi$ is the *critical mass* for the PKS system: For $M < 8\pi$, one has global solutions for which diffusion dominates so that all of the mass tends to infinity as the time tends to infinity, see [5], while for $M > 8\pi$, solutions develop singularities, see [20].

Our focus in this paper is on the case $M = 8\pi$. Notice that for $M = 8\pi$, $\mathcal{F}_{\rm PKS}$ is exactly the functional that is on the left hand side in (1.4). Since the densities ϱ_λ are minimizers of $\mathcal{F}_{\rm PKS}$ for $M = 8\pi$, it follows that

$$\frac{\delta \mathcal{F}_{\rm PKS}[\varrho_\lambda]}{\delta \rho}(x) = 0 \ ,$$

and then from (1.2) that each ϱ_λ – and each of their translates – is a stationary solution of (1.2); *i.e.*, of (1.1). Of course, this can also be checked directly. *Our main goal in this paper is to determine the stability of these solutions, and to determine basins of attraction for them. In achieving this goal, we develop several novel functional inequalities, and a strategy of concentration control that may be useful elsewhere, and may be the main contribution of the paper.*

Each of the ϱ_λ has an infinite second moment, and so shall all of the functions in the basins of attraction that we find for them. This must be the case according to previous work [4] on the case $M = 8\,\pi$ for initial data with a finite second moment. The paper [4] proves the global existence of weak solutions with finite second moment that satisfy the free energy dissipation inequality

$$\mathcal{F}_{\rm PKS}[\rho(T)] + \int_0^T \left[\int_{\mathbb{R}^2} \rho(t,x) \left| \nabla \frac{\delta \mathcal{F}_{\rm PKS}[\rho(t)]}{\delta \rho}(x) \right|^2 \,{\rm d}x \right] \,{\rm d}t \leq \mathcal{F}_{\rm PKS}[\rho(0)] \ ,$$

which is what one would guess should hold from (1.3). Moreover, [4] proves that every such solution blows up at *infinite time*. That is, the $t \to \infty$ limit of any such solution is a Dirac mass $8\,\pi$ at the center of mass of the initial data. Furthermore, a point mass of mass M is a stationary measure-valued solution in the sense introduced in [18] if and only if $M \geq 8\pi$.

From this point of view, the solutions in the critical mass case with finite initial second moment are choosing in their large time asymptotics the only possible stationary state with a finite second moment.

Let us finally comment that the family of stationary solutions ϱ_λ play a role too in the conjectured profile of blow-up for any point singularity of the solutions for masses $M > 8\pi$. Velazquez has proved [34, 35] that the inner part of the matched-asymptotics expansion for the blow-up profile is given by these stationary solutions for the critical mass value.

1.2 The second Lyapunov functional

The essential tool in our construction and analysis of solutions of the critical mass PKS system is an interesting and somewhat surprising interplay between the PKS system and another evolution equation which also has the ϱ_λ as stationary solutions – the Fokker-Planck version of the fast diffusion equation in $\mathbb{R}^2$ with exponent $1/2$:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta \sqrt{u(t,x)} + 2\sqrt{\dfrac{\pi}{\lambda M}}\, {\rm div}(x\, u(t,x)) & t > 0\,,\ x \in \mathbb{R}^2\ , \\ u(0,x) = u_0(x) \geq 0 & x \in \mathbb{R}^2\ , \end{cases} \tag{1.7}$$

corresponding to the *fast diffusion equation* $\dfrac{\partial u}{\partial t} = \Delta \sqrt{u}$ by a self-similar change of variable, see [33]. In the interest of brevity we refer to (1.7) as the fast–diffusion equation.

This equation can also be written in a form analogous to (1.2): for $\lambda > 0$, define the functional $\mathcal{H}_\lambda$ on the non-negative functions in $L^1(\mathbb{R}^2)$ by

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \left(\sqrt{u(x)} - \sqrt{\varrho_\lambda(x)} \right)^2 {\varrho_\lambda}^{-1/2}(x) \,{\rm d}x$$

This functional is the relative entropy of the fast diffusion equation with respect to the stationary solution ϱ_λ. The unique minimizer of $\mathcal{H}_\lambda$ is ϱ_λ, and a simple formal computation yields

$$\frac{\delta \mathcal{H}_\lambda[u]}{\delta u} = \frac{1}{\sqrt{\varrho_\lambda}} - \frac{1}{\sqrt{u}} \;, \tag{1.8}$$

from which one sees that (1.7) can be rewritten as

$$\frac{\partial u}{\partial t}(t,x) = \mathrm{div}\left(u(t,x) \nabla \frac{\delta \mathcal{H}_\lambda[u(t)]}{\delta u}(x) \right) \;, \tag{1.9}$$

It follows that for classical solutions u of (1.7) for which one can integrate by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}_\lambda[u(t)] = -\int_{\mathbb{R}^2} u(t,x) \left| \nabla \frac{\delta \mathcal{H}_\lambda[u(t)]}{\delta u} \right|^2 \mathrm{d}x = -\int_{\mathbb{R}^2} \left| \frac{1}{2} \nabla \log u(t,x) + 2\sqrt{\frac{\pi}{\lambda M}}\, x \sqrt{u(t,x)} \right|^2 \mathrm{d}x \;.$$

As one sees from (1.8) and (1.9), the densities ϱ_λ are stationary solutions of the fast diffusion equation (1.7), as well as the PKS system (1.1). This is much more than a coincidence, and there are very close connections between the two evolution equations.

Unlike the functional $\mathcal{F}_{\mathrm{PKS}}$, the functional $\mathcal{H}_\lambda$ is convex. Making simple computations, one finds that

$$\mathcal{H}_\lambda[u] := \sqrt{\pi M \lambda} + \int_{\mathbb{R}^2} \left[\sqrt{\frac{\pi}{M\lambda}} |x|^2 u(x) - 2\sqrt{u(x)} + \sqrt{\varrho_\lambda(x)} \right] \mathrm{d}x \;. \tag{1.10}$$

Let us define the functionals

$$\mathcal{G}_1[u] := \int_{\mathbb{R}^2} |x|^2 u(x)\, \mathrm{d}x \qquad \text{and} \qquad \mathcal{G}_2[u] := -\int_{\mathbb{R}^2} \sqrt{u(x)}\, \mathrm{d}x \,.$$

Since $\mathcal{G}_1[u]$ is affine on its domain of definition, and since $\mathcal{G}_2[u]$ is convex on its domain of definition, one might formally conclude the convexity of $\mathcal{H}_\lambda$ on its domain of definition. In fact, there is a second notion of convexity, namely *displacement convexity*, that will play a basic role in our analysis. We shall explain the relevant aspects of displacement convexity in Section 3 of this paper, but for now we note that the functional $\mathcal{G}_1[u]$ is strictly displacement convex on its domain of definition, and the functional $\mathcal{G}_2[u]$ is displacement convex on its domain of definition, and hence one might formally conclude the strict displacement convexity of $\mathcal{H}_\lambda$ on its domain of definition.

Unfortunately, these arguments are only formal: Note that $\sqrt{\varrho_\lambda(x)}$ is not integrable, and thus if $\mathcal{H}_\lambda[u]$ is to be well defined, $\sqrt{u(x)}$ cannot be integrable either. Furthermore, since $|x|^2 \varrho_\lambda(x)$ is not integrable, it is clear that $|x|^2 u(x)$ also will not be integrable on the whole domain of definition of $\mathcal{H}_\lambda$. Thus, cancelations are crucial to the definition of $\mathcal{H}_\lambda$, and the integral in (1.10) cannot be split into a sum of three integrals to be analyzed separately.

As far as the convexity (in the usual sense) of $\mathcal{H}_\lambda$ is concerned, it is easy to give a rigorous proof: Indeed, $\mathcal{H}_\lambda[u]$ can be written as

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \left[\Phi(u(x)) - \Phi(\varrho_\lambda(x)) - \Phi'(\varrho_\lambda)(u(x) - \varrho_\lambda(x)) \right] \mathrm{d}x \tag{1.11}$$

with $\Phi(s) = -2\sqrt{s}$, which is a convex function. However, displacement convexity is essential to our strategy, and even after we have properly introduced the notion of displacement convexity, we shall have to work much harder to prove that $\mathcal{H}_\lambda$ is in fact strictly displacement convex on its domain of definition.

The convexity properties of $\mathcal{H}_\lambda$ are relevant to the analysis of the PKS system due to the perhaps surprising fact that $\mathcal{H}_\lambda$ is monotone decreasing also along solutions of the critical mass PKS system (1.1), and not only along solutions of the fast diffusion equation (1.7). *This gives us a second Lyapunov function for the critical mass PKS system.*

To see why this should be so, we make a formal calculation that we shall revisit in full rigor later on: Let ρ be a sufficiently nice solution of the PKS system. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] &= \int_{\mathbb{R}^2} \frac{\delta\mathcal{H}_\lambda[\rho]}{\delta\rho}\mathrm{div}\left(\rho(t,x)\nabla\left[\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta\rho}\right]\right)\,\mathrm{d}x = -\int_{\mathbb{R}^2}\rho\nabla\left[\frac{\delta\mathcal{H}_\lambda[\rho]}{\delta\rho}\right]\cdot\nabla\left[\frac{\delta\mathcal{F}_{\mathrm{PKS}}[\rho]}{\delta\rho}\right]\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^2}\rho\nabla\left[\frac{1}{\sqrt{\varrho_\lambda}}-\frac{1}{\sqrt{\rho}}\right]\cdot\nabla\left[\log\rho - G*\rho\right]\,\mathrm{d}x\\
&= -\int_{\mathbb{R}^2}\left[2\sqrt{\frac{\pi}{\lambda M}}\,x\,\rho+\nabla\sqrt{\rho}\right]\cdot\nabla\left[\log\rho - G*\rho\right]\,\mathrm{d}x \qquad (1.12)\end{aligned}$$

Integrating by parts once more on the term involving the Green's function,

$$\int_{\mathbb{R}^2}\nabla\sqrt{\rho}\cdot\nabla\left[\log\rho - G*\rho\right]\,\mathrm{d}x = \frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}+\int_{\mathbb{R}^2}\sqrt{\rho}\,\Delta G*\rho = \frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}-\int_{\mathbb{R}^2}\rho^{3/2}\,\mathrm{d}x\ .$$

Also, $\int_{\mathbb{R}^2} x\rho\cdot\nabla\log\rho\,\mathrm{d}x = -2M$ and, making the same symmetrization that led to (1.6),

$$\int_{\mathbb{R}^2}\rho(x)\,x\cdot\nabla G*\rho(x)\,\mathrm{d}x = \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\rho(t,x)\,(x-y)\cdot\frac{x-y}{|x-y|^2}\,\rho(t,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{M^2}{4\pi}\ . \qquad (1.13)$$

Using the last three calculations in (1.12), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[\rho(t)] = -\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}\,\mathrm{d}x+\int_{\mathbb{R}^2}\rho^{3/2}\,\mathrm{d}x+4\sqrt{\frac{M\,\pi}{\lambda}}\left(1-\frac{M}{8\pi}\right)\ .$$

Notice that the constant term vanishes in critical mass case $M = 8\pi$. Thus, in the critical mass case, formal calculation yields that for all $T > 0$,

$$\mathcal{H}_\lambda[\rho(T)]+\int_0^T\left[\frac{1}{2}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}(t,x)\,\mathrm{d}x-\int_{\mathbb{R}^2}\rho^{3/2}(t,x)\,\mathrm{d}x\right]\,\mathrm{d}t \le \mathcal{H}_\lambda[\rho_0]\ . \qquad (1.14)$$

In fact, the formal computation yields equality instead of merely inequality in (1.14), but it is this inequality that is useful to us, and this is what we shall actually prove for the solutions that we construct here.

The key to exploiting (1.14) is a particular case of the Gagliardo-Nirenberg-Sobolev (GNS) inequalities for which the sharp form was found by Del Pino and Dolbeault [16].

1.2 LEMMA (Gagliardo-Nirenberg-Sobolev inequality)**.** *For all functions f in $\mathbb{R}^2$ with a square integrable distributional gradient ∇f,*

$$\pi\int_{\mathbb{R}^2}|f|^6\,\mathrm{d}x \le \int_{\mathbb{R}^2}|\nabla f|^2\,\mathrm{d}x\int_{\mathbb{R}^2}|f|^4\,\mathrm{d}x\ ,$$

and there is equality if and only if f is a multiple of a translate of ${\varrho_\lambda}^{1/4}$ for some $\lambda > 0$.

To apply this, note that at least for strictly positive densities ρ,

$$\int_{\mathbb{R}^2}|\nabla\rho^{1/4}(x)|^2\,\mathrm{d}x = \frac{1}{16}\int_{\mathbb{R}^2}\frac{|\nabla\rho|^2}{\rho^{3/2}}(x)\,\mathrm{d}x\ .$$

Therefore, we define:

1.3 DEFINITION (Entropy dissipation functional)**.** For any density ρ of total mass 8π such that $\rho^{3/2}$ is integrable, we define the *entropy dissipation functional* $\mathcal{D}[\rho]$ by

$$\mathcal{D}[\rho] = 8\int_{\mathbb{R}^2}|\nabla\rho^{1/4}(x)|^2\,\mathrm{d}x-\int_{\mathbb{R}^2}\rho^{3/2}(x)\,\mathrm{d}x$$

where $\nabla\rho^{1/4}$ is the distributional gradient of $\rho^{1/4}$, which is of course locally integrable.

1.4 LEMMA (Dissipation of $\mathcal{H}_\lambda$)**.** *For all densities ρ of mass $M = 8\pi$ with $\rho^{3/2}$ integrable,*

$$\mathcal{D}[\rho] \geq 0 \ ,$$

and moreover, there is equality if and only ρ is a translate of ϱ_λ for some $\lambda > 0$.

Proof: Let $f = \rho^{1/4}$ and note that $\int_{\mathbb{R}^2} f^4(x)\,\mathrm{d}x = 8\pi$. Multiplying $\mathcal{D}[\rho]$ through by π, the claim follows directly from Lemma 1.2. ■

1.3 The main results on the PKS equation

The formal result (1.14) may now be written as

$$\mathcal{H}_\lambda[\rho(T)] + \int_0^T \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \mathcal{H}_\lambda[\rho_0] \ .$$

Since $\mathcal{H}_\lambda[\rho(T)] \geq 0$, this suggests at the very least that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\lambda[\rho_0] = 0 \ ,$$

and then Lemma 1.4 suggests that for all large t, $\rho(t)$ must be close to ϱ_μ for some $\mu > 0$. However, an easy calculation, see Remark 3.5, using the fact that $\int_{\mathbb{R}^2} |x|^2 \varrho_\lambda(x)\,\mathrm{d}x = \infty$, shows that $\mathcal{H}_\lambda(\varrho_\mu) = \infty$ for $\mu \neq \lambda$. Therefore, since $\mathcal{H}_\lambda[\rho(t)]$ is non-increasing, one expects that $\mu = \lambda$. In short, the formal calculations made so far suggest that for solutions ρ of the PKS system with initial data ρ_0 satisfying $\mathcal{H}_\lambda[\rho_0] < \infty$, $\lim_{t\to\infty} \rho(x,t) = \varrho_\lambda(x)$. As we shall see, this is essentially correct, though we will need to assume not only that $\mathcal{H}_\lambda[\rho_0] < \infty$, but that $\mathcal{H}_\lambda[\rho_0]$ is not too large to ensure the entropy dissipation inequality (1.14). We now make one more definition, and then state our main results on the PKS equation:

1.5 DEFINITION (Properly dissipative weak solutions of the PKS equation)**.** Let Let ρ_0 be any density on $\mathbb{R}^2$ with mass 8π, such that for some $\lambda > 0$, $\mathcal{H}_\lambda[\rho_0] < \infty$. Let $\rho : [0,\infty) \to L^1(\mathbb{R}^2)$ satisfy:

(1.5.1) For each $t \geq 0$, $\rho(t)$ is a continuous curve of densities of mass 8π in the sense that for each bounded and globally Lipschitz function ψ on $\mathbb{R}^2$:

$$t \mapsto \int_{\mathbb{R}^2} \psi(x)\,\rho(t,x)\,\mathrm{d}x$$

is continuous with $\rho(0) = \rho_0$.

(1.5.2) For each $T > S \geq 0$, and each smooth and compactly supported function ψ on $\mathbb{R}^2$,

$$\begin{aligned}\int_{\mathbb{R}^2} \psi(x)\rho(T,x)\,\mathrm{d}x = &\int_{\mathbb{R}^2} \psi(x)\rho(S,x)\,\mathrm{d}x + \int_S^T \int_{\mathbb{R}^2} \Delta\psi(x)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &- \frac{1}{4\pi}\int_S^T \int_{\mathbb{R}^2\times\mathbb{R}^2} \rho(t,x)\frac{(\nabla\psi(x) - \nabla\psi(y))\cdot(x-y)}{|x-y|^2}\,\rho(t,y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \ .\end{aligned}$$

(1.5.3) For each $t > 0$, $\displaystyle\int_{\mathbb{R}^2} \rho^{3/2}(t,x)\,\mathrm{d}x < \infty$ so that $\mathcal{D}[\rho(t)]$ is well defined.

(1.5.4) For each $T > 0$,

$$\mathcal{H}_\lambda[\rho(T)] + \int_0^T \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \mathcal{H}_\lambda[\rho_0] \ , \tag{1.15}$$

so that the ρ satisfies the *entropy–entropy dissipation* inequality expected of solution of the PKS equation.

Then ρ is a *properly dissipative weak solution* of the PKS equation (1.1) with initial data ρ_0.

1.6 THEOREM (Existence and regularity of properly dissipative weak solutions). *Let ρ_0 be any density on $\mathbb{R}^2$ with mass 8π, such that $\rho_0 \log \rho_0$ is integrable, and for some $\lambda > 0$,*

$$\mathcal{H}_\lambda[\rho_0] < \frac{\sqrt{\lambda}}{128\sqrt{2}\pi} \ . \tag{1.16}$$

Then there exists a properly dissipative solution of the PKS equation (1.1) *with initial data ρ_0.*

Moreover, the solutions we construct have additional regularity properties, including:

(1.6.1) *For any $S > 0$ and any p with $1 < p < \infty$, there is a constant C depending only on S, p, λ and $\mathcal{H}_\lambda[\rho_0]$ such that for all $t \geq S$, $\|\rho(t)\|_p \leq C$.*

(1.6.2) *The distributional gradient of $\rho^{1/4}$ is square integrable over $[0,\infty) \times \mathbb{R}^2$, and in fact,*

$$\int_0^\infty \int_{\mathbb{R}^2} |\nabla \rho^{1/4}(t,x)|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \mathcal{H}_\lambda(\rho_0) \ .$$

(1.6.3) *$\mathcal{F}_{\mathrm{PKS}}[\rho(t)$ is well defined for each t, and is monotone decreasing: $\mathcal{F}_{\mathrm{PKS}}[\rho(t)] \leq \mathcal{F}_{\mathrm{PKS}}[\rho(s)]$ for all $0 \leq s < t$,*

1.7 THEOREM (Basins of attraction). *Let ρ_0 be any density on $\mathbb{R}^2$ with mass 8π, such that for some $\lambda > 0$,* (1.16) *is satisfied. Let ρ be any properly dissipative weak solution of the PKS equation* (1.1) *with initial data ρ_0 satisfying the additional regularity properties (1.6.1), (1.6.1) and (1.6.3) of Theorem 1.6. Then*

$$\lim_{t\to\infty} \mathcal{F}_{\mathrm{PKS}}[\rho(t)] = \mathcal{F}_{\mathrm{PKS}}[\varrho_\lambda] \qquad \text{and} \qquad \lim_{t\to\infty} \|\rho(t) - \varrho_\lambda\|_1 = 0 \ .$$

Let $\mathcal{B}_\lambda$ denote the set of densities ρ_0 of mass 8π for which $\rho_0 \log \rho_0$ is integrable and (1.16) is satisfied. According to Theorem 1.7, $\mathcal{B}_\lambda$ is a basin of attraction for ϱ_λ under the PKS evolution in the sense that any properly dissipative weak solution with initial data in $\mathcal{B}_\lambda$, and the regularity produced here, converges strongly to ϱ_λ in $L^1(\mathbb{R}^2)$.

1.4 Controlled concentration inequalities

The proof of the additional regularity in Theorem 1.6, and then Theorem 1.7, might at first appear to be possible by a standard application of entropy-entropy dissipation methods, given the entropy-entropy dissipation inequality (1.15). However, this is not the case. The essential point is that $\mathcal{D}[\rho]$ is *not* a convex function of ρ, and even worse, it is a *difference* of two functionals of ρ that can each be arbitrarily large even when $\mathcal{D}[\rho]$ is very close to zero. Indeed, for $M = 8\pi$ and each $\lambda > 0$, $\mathcal{D}[\varrho_\lambda] = 0$ while

$$\lim_{\lambda\to 0} \|\varrho_\lambda\|_{3/2} = \infty \, , \quad \lim_{\lambda\to 0} \|\nabla {\varrho_\lambda}^{1/4}\|_2 = \infty \, , \quad \text{and} \quad \lim_{\lambda\to 0} \varrho_\lambda = 8\pi\delta_0 \, .$$

the point mass of 8π at 0. It follows that the level sets of $\mathcal{D}$ cannot be weakly compact in $L^1(\mathbb{R}^2)$.

Note that in this example of non-compactness, we have a family of functions in which there are members that concentrate *at least half of their total mass* on arbitrarily small sets. *We shall show here that this is essentially the only way compactness can fail for a family of densities of mass 8π on which $\mathcal{D}$ is uniformly bounded.* Let us emphasize the role of the uniform bound on $\mathcal{D}$ in this: A set of densities of fixed mass M will fail to be weakly compact in L^1 if it contains members that concentrate even a very small amount of mass on an arbitrarily small set. When we have a uniform bound on $\mathcal{D}$, we need only be concerned with a much worse behavior: functions that concentrate half (or any other substantial fraction) of their total mass on an small set – in fact, not even arbitrarily small.

It is at this point that we begin to make actual use of the second Lyapunov functional $\mathcal{H}_\lambda$: If $\mathcal{H}_\lambda$ is uniformly bounded on a family of densities of mass 8π, then no member of this family may concentrate too much of its mass on a too small set:

1.8 LEMMA (Concentration control by $\mathcal{H}_\lambda$)**.** *For any density ρ with mass M and any $\beta > 0$. Define $A_\beta = \{x : \rho(x) \geq \beta\}$. Then*

$$\int_{A_\beta} \rho \, \mathrm{d}x \leq \frac{C_1}{\beta} + C_2\sqrt{\mathcal{H}_\lambda[\rho]} \ ,$$

with $C_1 := M^2/(\lambda\,\pi)$ and $C_2 := 2M^{3/4}(\lambda\pi)^{-1/4}$.

As a consequence, for any measurable set $E \subset \mathbb{R}^2$,

$$\int_E \rho \, \mathrm{d}x \leq \beta|E| + \frac{C_1}{\beta} + C_2\sqrt{\mathcal{H}_\lambda[\rho]},$$

and choosing $\beta = 1/\sqrt{|E|}$, we obtain an upper bound on the mass of ρ at E in terms of $|E|$ and $\mathcal{H}_\lambda[\rho]$. We stress that this lemma *does not* provide any uniform integrability; one can still concentrate a small amount of mass on a very, very small set.

We shall prove this lemma in Section 2. There we also prove that since $\mathcal{H}_\lambda$ controls concentration, a uniform bound on both $\mathcal{H}_\lambda$ and $\mathcal{D}$ does indeed provide compactness, justifying our claim about the only way compactness can fail when $\mathcal{D}$ is uniformly bounded. We shall prove:

1.9 THEOREM (Concentration control for $\mathcal{D}$)**.** *Let ρ be a any density in $L^{3/2}(\mathbb{R}^2)$ with mass 8π. For any $\gamma_1 \in (0, 4\,\pi)$, if*

$$\mathcal{H}_\lambda[\rho] < C_\lambda := \left(\frac{\sqrt{\lambda}}{128\sqrt{2}\pi}\right)\gamma_1^2 \ , \tag{1.17}$$

then there exist a finite positive constant $C_{\rm CCD}$, depending only on λ and γ_1

$$\gamma_1 \int_{\mathbb{R}^2} |\nabla\rho^{1/4}|^2 \, \mathrm{d}x \leq \pi\mathcal{D}[\rho] + C_{\rm CCD} \ .$$

The constant $C_{\rm CCD}$ is given in (2.13). Theorem 1.9 gives us the "vertical control" needed for a compactness result. The horizontal control is proved directly by $\mathcal{H}_\lambda$ The following lemma is also proved in Section 2:

1.10 LEMMA (Localization)**.** *For all densities ρ with mass M and all $\lambda > 0$,*

$$\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\,\rho(x)\,\mathrm{d}x \leq 2\sqrt{\lambda}\,M + 2M^{3/4}(\lambda/\pi)^{1/4}\sqrt{\mathcal{H}_\lambda[\rho]} \ .$$

Lemma 1.10 shows in particular that when $\mathcal{H}_\lambda[\rho] < \infty$, then $\rho\log(e + |x|^2) \in L^1(\mathbb{R}^2)$, so that the Newtonian potential of ρ is well defined. (A somewhat stronger localization result is proved in Section 3.)

The "compactness via controlled concentration" provided by $\mathcal{H}_\lambda$ and its dissipation $\mathcal{D}$ through Theorem 1.9 and Lemmas 1.8 and 1.10 is the core of our proof of Theorem 1.7. However this is not the only use we shall make of compactness via controlled concentration: It is absolutely essential to our construction of properly dissipative weak solutions.

Indeed, in many problems in which one seeks to prove an entropy-entropy-dissipation inequality such as (1.15), both the entropy functional $\mathcal{H}$ and its dissipation $\mathcal{D}$ would be weakly lower semicontinuous, often due to some convexity property. Then, if $\{\rho_n\}_n$ is a sequence of nice or approximate solutions of the evolution equation converging weakly to a weak solution ρ, one would have

$$\mathcal{H}[\rho(T)] \leq \lim_{n\to\infty} \mathcal{H}[\rho_n(T)] \qquad \text{and} \qquad \int_0^T \mathcal{D}[\rho(t)]\,\mathrm{d}t \leq \lim_{n\to\infty}\int_0^T \mathcal{D}[\rho_n(t)]\,\mathrm{d}t \ ,$$

which is very helpful if one is trying to prove something like (1.15).

While in our case $\mathcal{H}_\lambda$ is convex and lower semicontinuous, $\mathcal{D}$ is the *difference* of two non-comparable convex functions and has no lower semicontinuity. Therefore, we need new tools to prove (1.15), and as we shall see, it is once again the compactness via controlled concentration that does the trick.

In fact, we need one more "compactness via controlled concentration" result to prove Theorem 1.6, this time for $\mathcal{F}_{\rm PKS}$. For a density of mass 8π, an upper bound on $\mathcal{F}_{\rm PKS}[\rho]$ provides no upper bound on the entropy $\mathcal{E}[\rho] := \int_{\mathbb{R}^2} \rho \log \rho(x)\, {\rm d}x$. Indeed, $\mathcal{F}_{\rm PKS}[\rho]$ takes its minimum value for $\rho = \varrho_\lambda$ for each $\lambda > 0$, while $\lim_{\lambda\to 0} \mathcal{E}[\varrho_\lambda] = \infty$. However, an upper bound on both $\mathcal{H}_\lambda[\rho]$ and $\mathcal{F}_{\rm PKS}[\rho]$ *does* provide an upper bound on $\mathcal{E}[\rho]$. The following Theorem is also proved in Section 2:

1.11 THEOREM (Concentration control for $\mathcal{F}_{\rm PKS}$)**.** *Let ρ be any density with mass $M \geq 8\pi$, with $\rho\log\rho$ integrable and bounded in $L^1(\mathbb{R}^2, \log(e+|x|^2)\,{\rm d}x)$. Given $0 < \varepsilon_0 < 8\pi$, there exists $0 < \gamma_2 \leq 1$ depending only on M and ε_0, such that if*

$$\mathcal{H}_\lambda[\rho] < \frac{(8\,\pi - \varepsilon_0)^2}{4\,C_2^2} \tag{1.18}$$

with C_2 given in Lemma 1.8, then there exists a finite positive constant $C_{\rm CCF}$, depending only on M, λ and ε_0, such that

$$\gamma_2 \int_{\mathbb{R}^2} \rho \log_+ \rho \,{\rm d}x \leq \mathcal{F}_{\rm PKS}[\rho] + C_{\rm CCF}\,.$$

1.5 Further developments

One can build on the regularity results obtained here to prove additional regularity. Indeed, if ρ is one of the solutions we have constructed here, it is easy to prove that for any $a > 0$, $\nabla c(x,t)$ is bounded and continuous on $(a,\infty)\times\mathbb{R}^2$, only using the continuity properties on ρ in t, the uniform control on first moments, and the fact that $\rho(t)$ is uniformly bounded in both L^1 and L^3 for all $t > a$. Thus "freezing" $b := \nabla c$, ρ is seen to be a weak solution of the linear parabolic equation

$$\frac{\partial\rho}{\partial t} = \Delta\rho - {\rm div}(b\rho)\ ,$$

with b bounded and continuous. Parabolic regularity theory may now be applied. A further development that requires new tools is to bound the rate of convergence to the equilibrium ϱ_λ in our convergence theorem.

An interesting problem whose solution would lead to rate information is to characterize the stability of the GNS inequality that we have used. That is, we know that $\mathcal{D}[\rho] = 0$ if and only if ρ is a translate of ϱ_λ for some $\lambda > 0$, since, as we have seen, this is simply a restatement of a sharp GNS inequality of Del Pino and Dolbeault. A stability result for this inequality would be a result stating that, for any $\epsilon > 0$, if $\mathcal{D}[\rho]$ is sufficiently small, then the distance, in some metric, from ρ to some translate of some ϱ_λ, $\lambda > 0$, is no more than ϵ. It would also be useful to quantify the qualitative stability result for the Log-HLS inequality that we prove and use in Section 5. Work in this direction is underway.

1.6 Other equations with a second Lyapunov functional

The second Lyapunov functional $\mathcal{H}_\lambda$ is more useful to us than the primary Lyapunov functional $\mathcal{F}_{\rm PKS}$, which actually drives the evolution, because of its convexity properties, especially its displacement convexity, as explained in Section 3.

There is a "cannonical way" to produce gradient flow evolution equations that have a convex second Lyapunov functional that has been investigated in [25]. Indeed, both the PKS equation and the fast diffusion equation are gradient flow systems where the gradient is computed using the 2-Wasserstein metric, as we recall in Section 3. To keep things simple here, let us explain the mechanism studied in [25] in the finite dimensional Euclidean case.

Let V be a smooth convex function on $\mathbb{R}^n$. Let W be the smooth function on $\mathbb{R}^n$ defined by $W(x) = |\nabla V(x)|^2$. Now consider the evolution equation $\dfrac{\rm d}{{\rm d}t}x(t) = -\nabla W(x(t))$. Then of course, for any solution x,

$$\frac{\rm d}{{\rm d}t} W(x(t)) = -|\nabla W(x(t))|^2 \leq 0\ ,$$

and so W is monotone decreasing along the evolution. It is the primary Lyapunov function for this flow. Next, note that since $\nabla W = 2[\mathrm{Hess}V]\,\nabla V$,

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x(t)) = -\left[\nabla V\cdot\nabla W\right](x(t)) = -2\left\{\nabla V\cdot[\mathrm{Hess}V]\,\nabla V\right\}(x(t)) \leq 0\ ,$$

since the Hessian of V is positive. Thus, V is a second Lyapunov function for the gradient flow driven by W.

An example in [25] concerns a porous medium equation on the line, which is gradient flow in the 2-Wasserstein metric for a certain entropy functional. With this entropy functional playing the role of V, the gradient flow equation for the functional corresponding to W is a certain fourth order equation of thin-film type.

The fact that the entropy for the porous medium equations is a second Lyapunov functional for this fourth order thin film equation had been discovered earlier in [14] and exploited as the key to understanding the long time behavior of the latter equation. Again in this case, the second Lyapunov function is strictly and uniformly displacement convex, while the primary Lyapunov functional is not displacement convex at all.

In the case studied here, the second Lyapunov functional does not arise through the mechanism studied in [25], or any other evident natural mechanism, and we have no "explanation" of why one should expect $\mathcal{H}_\lambda$ to decrease along the PKS flow. However, as explained in [25], once one knows this, it is a consequence, formally at least, that $\mathcal{F}_{\rm PKS}$ decreases along the fast diffusion flow. This has interesting consequences that are investigated in [7].

The motivation for doing the computation to check the monotonicity is twofold: First, both evolution equations have the same steady states, which is certainly necessary, but not at all sufficient, for the computation to work out. Second, there are many sharp inequalities that have negative powers of $1+|x|^2$ as their cases of equality, so there are tools available to try to prove the positivity of the dissipation.

1.7 A brief outline of the rest of the paper

The rest of the paper is organized as follows. In Section 2 we prove the controlled concentration results that have been stated in the Introduction.

Section 3 begins with a brief summary of some results concerning the *2-Wasserstein metric* and gradient flows with respect to it. In particular, we recall a discrete variational scheme due to Jordan, Kinderlehrer and Otto [21] for constructing solutions of a class of equations including both the PKS equation and the critical fast diffusion equation. We also recall McCann's [27] notion of *displacement convexity*, and explain how this should, at least formally, lead to the entropy-entropy dissipation inequality that we seek. Making the formal calculation rigorous will then be reduced to rigorously proving certain consequences of displacement convexity for $\mathcal{H}_\lambda$, and this will be facilitated by the "robustness" of displacement convexity.

The latter half of Section 3 is more novel. As we have noted earlier, $\mathcal{H}_\lambda$ is formally a sum of displacement convex terms, however, for the densities that concern us, each of the terms is divergent. Thus, we are forced to introduce a *regularization* of $\mathcal{H}_\lambda$. While there are many tools available to regularize functions that are convex in the usual sense (e.g. infimal convolution), there is no general approach to regularizing functionals while preserving, or at least not severely damaging, their formal displacement convexity properties. The regularization developed in the second half of Section 3 is one of the cornerstones of the paper.

In Section 4, we lay the ground work for the proof of Theorem 1.6 on the existence of properly dissipative weak solutions. These will be constructed using a variant of the Jordan, Kinderlehrer and Otto [21] scheme, which constructs the evolutions by solving a sequence of variational problems, as in di Giorigi's "minimizing steps" method.

In this method, the Euler-Lagrange equation for the variational problem solved at each step often provides essential *a-priori* regularity on the minimizing density ρ. Once again, at this point in our problem, we encounter difficulties due to potential cancelation of infinities. To resolve these, we are forced to regularize $\mathcal{F}_{\rm PKS}$. The discrete scheme provides a very convenient framework in which to impose and control the regularization: We use a different degree of regularization at each discrete time step. Because of the regularization, we will at least know that at each time step, $\nabla\sqrt{\rho}$ is square integrable, but we shall have no useful quantitative bound on $\|\nabla\sqrt{\rho}\|_2$. Still, this gives us enough regularity to make some crucial integrations by parts, and then eventually through the use of Theorem 1.9, we shall obtain a useful quantitative bound on $\|\nabla\rho^{1/4}\|_2$.

In Section 5, we pass to the continuous time limit, and provide the proofs of Theorems 1.6 and 1.7.

2 Proof of the concentration controlled inequalities

2.1 Concentration control by $\mathcal{H}_\lambda$

We know from Lemma 1.1 that if some density ρ with mass 8π satisfies $\mathcal{F}_{\rm PKS}[\rho] = \mathcal{F}_{\rm PKS}[\varrho_\lambda]$, then, up to translation, $\rho = \varrho_\mu$ for some $\mu > 0$. But if we merely know that $\mathcal{F}_{\rm PKS}[\rho] - \mathcal{F}_{\rm PKS}[\varrho_\lambda]$ is *small*, we do not have any useful bound telling us that ρ is *close* to ϱ_μ for some $\mu > 0$.

The situation in this regard is much better concerning $\mathcal{H}_\lambda$ because of its convexity, as noted in the Introduction. The following Csiszàr-Kullback-Liebler-Pinsker type inequality for $\mathcal{H}_\lambda$ has already been used by Ledermann and Markowich [23] in their work on the critical fast diffusion (1.7); see [11] for general results in this direction. However, one can give a very simple proof that does not make any direct appeal to the convexity of $\mathcal{H}_\lambda$:

2.1 LEMMA (Csiszàr-Kullback-Liebler-Pinsker type inequality for $\mathcal{H}_\lambda$)**.** *For all densities ρ with mass M and all $\lambda > 0$,*

$$\int_{\mathbb{R}^2} |\rho(x) - \varrho_\lambda(x)| \sqrt{\lambda + |x|^2}\, {\rm d}x \le C_{\rm CKLP} \sqrt{\mathcal{H}_\lambda[\rho]}\ ,$$

with $C_{\rm CKLP} := 2\,(\lambda/\pi)^{1/4}\, M^{3/4}$.

Proof: By the Cauchy-Schwarz inequality

$$\left(\frac{\pi}{\lambda M}\right)^{1/4} \int_{\mathbb{R}^2} |\rho - \varrho_\lambda|\, \sqrt{\lambda + |x|^2}\, {\rm d}x = \int_{\mathbb{R}^2} \frac{|\rho - \varrho_\lambda|}{{\varrho_\lambda}^{1/4}}\, {\rm d}x = \int_{\mathbb{R}^2} \frac{\left|\sqrt{\rho} - \sqrt{\varrho_\lambda}\right|}{{\varrho_\lambda}^{1/4}} \left|\sqrt{\rho} + \sqrt{\varrho_\lambda}\right|\, {\rm d}x$$
$$\le \sqrt{\mathcal{H}_\lambda[u]} \sqrt{\int_{\mathbb{R}^2} \left|\sqrt{\rho} + \sqrt{\varrho_\lambda}\right|^2\, {\rm d}x} \le 2\sqrt{M}\sqrt{\mathcal{H}_\lambda[u]}\ .$$

■

Lemma 1.10, bounding the first moment of ρ in terms of $\mathcal{H}_\lambda[\rho]$, is now easily proved.

Proof of Lemma 1.10: We estimate:

$$\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\, \rho(x)\, {\rm d}x \le \int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\, \varrho_\lambda(x)\, {\rm d}x + \int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\, |\rho(x) - \varrho_\lambda(x)|\, {\rm d}x\ ,$$

and the result directly follows from Lemma 2.1 and a direct computation. ■

We next prove Lemma 1.8, which says that when ρ is a density of mass 8π and $\mathcal{H}_\lambda[\rho]$ is sufficiently small, then ρ cannot have too much of its mass concentrated on too small a set.

Proof of Lemma 1.8: Recall that $A_\beta := \{x\, :\, \rho(x) \ge \beta\}$. By Chebychev's inequality, $|A_\beta|$, the Lebesgue measure of A_β, satisfies $|A_\beta| \le M/\beta$. Then, since $\|\varrho_\lambda\|_\infty = M/(\lambda\,\pi)$,

$$\int_{A_\beta} \rho\, {\rm d}x \le \int_{A_\beta} \varrho_\lambda\, {\rm d}x + \int_{A_\beta} |\rho - \varrho_\lambda|\, {\rm d}x \le \frac{M}{\beta}\frac{M}{\lambda\pi} + \|\rho - \varrho_\lambda\|_1 \le \frac{M^2}{\beta\lambda\,\pi} + \frac{C_{\rm CKLP}}{\sqrt{\lambda}}\sqrt{\mathcal{H}_\lambda[\rho]},$$

where in the last inequality we have used Lemma 2.1. ■

2.2 Concentration control for $\mathcal{F}_{\rm PKS}$

To prepare the way for the proof of Theorem 1.11, it is useful to give an elementary demonstration of a crude form of the log HLS inequality, without sharp constants, but which would nonetheless provide bounds on $\mathcal{E}[\rho]$ for all $M < 8\pi$.

2.2 LEMMA (Bounds on the entropy). *Let f be a non-negative measurable function in $\mathbb{R}^2$ such that $f\log f$ and $f\log(e+|x|^2)$ is bounded in $L^1(\mathbb{R}^2)$. Then, for any $\alpha \le 1/(8\pi)$, there exists a constant $C_0 > 0$ only depending on M, α and λ such that*

$$\alpha\int_{\mathbb{R}^2}\rho\log\rho\;{\rm d}x - \frac{1}{2}(G)_+ * \rho \ge -C_0 - 4\,\alpha\,M\,\log\left(\frac{1}{M}\int_{\mathbb{R}^2}\sqrt{\lambda+|x|^2}\,\rho\,{\rm d}x\right)\;.$$

Proof: Recall the following Young type inequality: For all $s, t > 0$, $st \le s\log s + e^{t-1}$. Then, for any $\alpha > 0$, we have $st = \alpha\,[s\,(t/\alpha)] \le \alpha s\log s + \alpha e^{t/\alpha - 1}$. We shall apply this to

$$\frac{1}{2}(G)_+ * \rho(x) = \frac{1}{4\pi}\int_{\mathbb{R}^2}(\log|x-y|)_-\rho(y)\;{\rm d}y = \int_{|x-y|<1}\frac{1}{4\pi}(-\log|x-y|)\frac{\rho(y)}{\varrho_\lambda(y)}\varrho_\lambda(y)\;{\rm d}y\;,$$

under the integral sign with weight ϱ_λ, and with

$$s = \frac{\rho(y)}{\varrho_\lambda(y)} \qquad \text{and} \qquad t = -\frac{1}{4\pi}\log|x-y|\;.$$

Since ϱ_λ is bounded above by $M/(\lambda\,\pi)$, this yields

$$\frac{1}{2}(G)_+ * \rho(x) \le \alpha\int_{|y-x|<1}\left(\frac{\rho}{\varrho_\lambda}\right)\log\left(\frac{\rho}{\varrho_\lambda}\right)\varrho_\lambda\;{\rm d}y + \alpha\left(\frac{M}{\lambda\pi e}\int_{|z|\le 1}\frac{1}{|z|^{1/(4\pi\alpha)}}\;{\rm d}z\right)\;. \tag{2.1}$$

The second integral on the right converges as long as $\alpha > 1/8\pi$, in which case, doing the integral explicitly, we find

$$\frac{\alpha\,M}{\lambda\,\pi\,e}\int_{|z|\le 1}\frac{1}{|z|^{1/(4\pi\alpha)}}\;{\rm d}z = \frac{M}{\lambda e}\frac{8\pi\alpha^2}{8\pi\alpha - 1} \qquad \text{for } 8\pi\alpha > 1\;. \tag{2.2}$$

To relate the first integral to $\mathcal{E}[\rho]$, use the fact that $s \mapsto s\log s$ is bounded below by $-1/e$ to conclude that

$$\begin{aligned}\int_{|y-x|<1}\left(\frac{\rho}{\varrho_\lambda}\right)\log\left(\frac{\rho}{\varrho_\lambda}\right)\varrho_\lambda\;{\rm d}y &\le \int_{\mathbb{R}^2}\left(\frac{\rho}{\varrho_\lambda}\right)\log\left(\frac{\rho}{\varrho_\lambda}\right)\varrho_\lambda\;{\rm d}y + \frac{M}{e}\\ &\le \int_{\mathbb{R}^2}\rho\log\rho\;{\rm d}y - \int_{\mathbb{R}^2}\rho\log\varrho_\lambda\;{\rm d}y + \frac{M}{e}\end{aligned}$$

By Jensen's inequality for the concave function log in $L^1((\rho/M)\;{\rm d}x)$,

$$\begin{aligned}\int_{\mathbb{R}^2}\rho\log\varrho_\lambda\;{\rm d}x &= M\log\left(\frac{\lambda M}{\pi}\right) - 4\int_{\mathbb{R}^2}\log\left(\sqrt{\lambda+|x|^2}\right)\rho\;{\rm d}x\\ &\ge M\log\left(\frac{\lambda M}{\pi}\right) - 4M\log\left(\frac{1}{M}\int_{\mathbb{R}^2}\sqrt{\lambda+|x|^2}\rho\;{\rm d}x\right)\;.\end{aligned} \tag{2.3}$$

Using (2.2) and (2.3) in (2.1), we obtain

$$\begin{aligned}\frac{1}{2}(G)_+ * \rho(x) \le{}& \alpha\int_{\mathbb{R}^2}\rho\log\rho\;{\rm d}y + \frac{M}{\lambda e}\frac{8\pi\alpha^2}{8\pi\alpha-1}\\ &+ \alpha\,M\left[\frac{1}{e} - \log\left(\frac{\lambda M}{\pi}\right) + 4\log\left(\frac{1}{M}\int_{\mathbb{R}^2}\sqrt{\lambda+|x|^2}\,\rho\;{\rm d}x\right)\right]\;.\end{aligned}$$

The result follows with

$$C_0 := \alpha\,M\left[\frac{1}{\lambda e}\frac{8\pi\alpha}{8\pi\alpha-1} + \frac{1}{e} - \log\left(\frac{\lambda\,M}{\pi}\right)\right]_+\;.$$

■

2.3 Remark (Non-sharp version of the log-HLS inequality)**.** *As a consequence, the previous lemma provides us with a crude logarithmic HLS inequality, but one that would suffice for application to the sub-critical mass case. Since obviously*

$$-\int_{\mathbb{R}^2} \rho\, G * \rho \geq -\int_{\mathbb{R}^2} \rho\, (G)_+ * \rho,$$

we conclude

$$M\,\alpha \int_{\mathbb{R}^2} \rho \log \rho \;\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^2} (G) * \rho \geq -\left[C_0 + 4\,\alpha\, M\, \log\left(\frac{1}{M}\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\,\rho\,\mathrm{d}x\right)\right]$$

so that

$$\mathcal{F}_{\mathrm{PKS}}[\rho] \geq (1 - M\alpha)\int_{\mathbb{R}^2} \rho \log \rho\,\mathrm{d}x - \alpha\, M\, \left[C_0 + 4\,\alpha\, M\, \log\left(\frac{1}{M}\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2}\rho(x)\,\mathrm{d}x\right)\right] .$$

As long as $M < 8\pi$, we can choose α so that both $M\alpha < 1$ and $8\pi\alpha > 1$, in which case this elementary argument gives us an a priori *upper bound on $\mathcal{E}[\rho]$ in term of $\mathcal{F}_{\mathrm{PKS}}[\rho]$ and the first moment of ρ.*

The next lemma is a quite standard control of the negative contribution of the entropy in terms of the control at infinity of some moment of the distribution known as Carleman-type estimate.

2.4 LEMMA (Control on the negative part of the entropy)**.** *For any density $\rho \in L^1_+(\mathbb{R}^2)$, if the moment $\int_{\mathbb{R}^2} m(|x|)\,\rho(x)\,\mathrm{d}x$ is bounded with $e^{-m(|x|)} \in L^1(\mathbb{R}^2)$ and $m : \mathbb{R}^+_0 \longrightarrow \mathbb{R}^+_0$ non-decreasing function, then*

$$\int_{\mathbb{R}^2} \rho(x)\, \log_- \rho(x)\; \mathrm{d}x \leq \int_{\mathbb{R}^2} m(|x|)\rho(x)\; \mathrm{d}x + \frac{1}{e}\int_{\mathbb{R}^2} e^{-m(|x|)}\,\mathrm{d}x\;.$$

Proof: Let $\bar\rho := \rho\,\chi_{\{\rho\leq 1\}}$ and $\bar M = \int_{\mathbb{R}^2} \bar\rho(x)\,\mathrm{d}x \leq \int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x = M$. Then

$$\int_{\mathbb{R}^2} \bar\rho(x)\; (\log \bar\rho(x) + m(|x|))\; \mathrm{d}x = \int_{\mathbb{R}^2} [U(x) \log U(x)]\mu\,\mathrm{d}x - \bar M \log Z$$

where $U := \bar\rho/\mu$, $\mu(x) = e^{-m(|x|)}/Z$ with $Z = \int_{\mathbb{R}^2} e^{-m(|x|)}\,\mathrm{d}x$. The Jensen inequality yields

$$\int_{\mathbb{R}^2} [U(x)\, \log U(x)]\mu\,\mathrm{d}x \geq \left(\int_{\mathbb{R}^2} U(x)\mu\,\mathrm{d}x\right)\; \log\left(\int_{\mathbb{R}^2} U(x)\mu\,\mathrm{d}x\right) = \bar M\, \log \bar M$$

and

$$\begin{aligned} -\int_{\mathbb{R}^2} \rho(x)\, \log_- \rho(x)\; \mathrm{d}x = \int_{\mathbb{R}^2} \bar\rho(x)\, \log \bar\rho(x)\,\mathrm{d}x &\geq \bar M \log \bar M - \bar M \log Z - \int_{\mathbb{R}^2} m(|x|)\,\bar\rho(x)\,\mathrm{d}x \\ &\geq -\frac{Z}{e} - \int_{\mathbb{R}^d} m(|x|)\,\rho(x)\,\mathrm{d}x\;. \end{aligned}$$

■

We now turn to the critical mass case, and the proof of Theorem 1.11. As already noticed in (1.5), when $M < 8\pi$ not all the entropy $\int \rho \log \rho$ is "eaten" in the free energy $\mathcal{F}_{\mathrm{PKS}}[\rho]$ when we use the Log HLS inequality Lemma 1.1. Hence we obtain a control of the entropy. But in the critical case $M = 8\pi$, the free energy $\mathcal{F}_{\mathrm{PKS}}[\rho]$ is equal to the functional which appears in the Log HLS inequality so that there is no remainder term to obtain compactness. In [4, Lemma 3.1], the spreading of mass is enough to show that we can still have a remainder part of the entropy. The idea was to cut the function ρ is different parts: some parts of positive mass that we have to precisely control, and one main part of mass less than 8π where the Log HLS inequality leave a remainder term in $\int \rho \log \rho$. In the present lemma, the spreading of mass is quantified by the functional $\mathcal{H}_\lambda[\rho]$ and Lemma 1.8.

Proof of Theorem 1.11: The idea is to split the function ρ is two parts: Given $\beta > 0$, define $\rho_\beta(x) = \min\{\rho(x), \beta\}$. We will apply the Log HLS inequality to $\rho - \rho_\beta$ of mass less than 8π and control the rest by the functional $\mathcal{H}_\lambda[\rho]$ and Lemma 1.8.

Let $M_\beta := \int_{\mathbb{R}^2} \rho_\beta \, dx$. We apply the above proof of Lemma 2.2 to $\rho - \rho_\beta$ of mass $M - M_\beta$, to obtain

$$-\int_{\mathbb{R}^2} \rho \, G * \rho(x) \, dx \geq -\int_{\mathbb{R}^2} \rho \, (G)_+ * \rho(x) \, dx = -\int_{\mathbb{R}^2} \rho_\beta \, (G)_+ * \rho(x) \, dx - \int_{\mathbb{R}^2} (\rho - \rho_\beta) \, (G)_+ * \rho(x) \, dx \; .$$

By the point-wise bound on ρ_β, we obtain

$$\frac{1}{2} \int_{\mathbb{R}^2} \rho_\beta \, (G)_+ * \rho(x) \, dx \leq \beta \frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (\log |x-y|)_- \rho(y) \, dx \, dy = \frac{\beta \, M}{8} \; .$$

By the bound on the entropy, see Lemma 2.2, we deduce that, for any $\alpha > 1/8\pi$,

$$\begin{aligned}\frac{1}{2} \int_{\mathbb{R}^2} (\rho - \rho_\beta) \, (G)_+ * \rho(x) \, dx \leq & \, \alpha(M - M_\beta) \int_{\mathbb{R}^2} \rho \, \log \rho \, dy \\ & + (M - M_\beta) \left[C_0 + 4 \, \alpha \, M \, \log \left(\frac{1}{M} \int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2} \rho(x) \, dx \right) \right] \; .\end{aligned}$$

This provides, for any $\alpha > 1/8\pi$, the lower bound

$$\begin{aligned}\mathcal{F}_{\rm PKS}[\rho] \geq & \, [1 - (M - M_\beta)\alpha] \int_{\mathbb{R}^2} \rho \log \rho \, dx - (M - M_\beta) C_0 \\ & - 4 \, \alpha \, M \, (M - M_\beta) \, \log \left(\frac{1}{M} \int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2} \rho(x) \, dx \right) \; .\end{aligned}$$

Corollary 1.10 and assumption (1.18) imply that

$$\int_{\mathbb{R}^2} \sqrt{\lambda + |x|^2} \rho(x) \, dx \leq 2\sqrt{\lambda} M + C_{\rm CKLP} \sqrt{\mathcal{H}_\lambda[\rho]} \leq \sqrt{\lambda} \left(2M + \frac{8\pi - \varepsilon_0}{2} \right) \; . \tag{2.4}$$

Hence, for any $\alpha > 1/8\pi$,

$$\mathcal{F}_{\rm PKS}[\rho] \geq \, [1 - (M - M_\beta)\alpha] \int_{\mathbb{R}^2} \rho \log \rho \, dx - (M - M_\beta) \left[C_0 + 4 \, \alpha \, M \log \left(\sqrt{\lambda} \left(2 + \frac{8\pi - \varepsilon_0}{2 \, M} \right) \right) \right] \; .$$

To control the negative part of the entropy, we use Lemma 2.4 with $m(|x|) = \sqrt{\lambda + |x|^2}$ and (2.4):

$$\begin{aligned}-[1 - (M - M_\beta)\alpha] \int_{\mathbb{R}^2} \rho \log_- \rho \, dx \geq & -\int_{\mathbb{R}^2} \rho \log_- \rho \geq -\int_{\mathbb{R}^2} m(|x|) \rho(x) \; dx - \frac{1}{e} \int_{\mathbb{R}^2} e^{-m(|x|)} \, dx \\ \geq & -\sqrt{\lambda} \left(2M + \frac{8\pi - \varepsilon_0}{2} \right) - \frac{1}{e} \int_{\mathbb{R}^2} e^{-m(|x|)} \, dx \; .\end{aligned}$$

This gives the final control on the positive part of the entropy, for any $\alpha > 1/8\pi$:

$$\begin{aligned}\mathcal{F}_{\rm PKS}[\rho] \geq & \, [1 - (M - M_\beta)\alpha] \int_{\mathbb{R}^2} \rho \log_+ \rho \, dx - \frac{\beta \, M}{8} - \sqrt{\lambda} \left(2M + \frac{8\pi - \varepsilon_0}{2} \right) - \frac{1}{e} \int_{\mathbb{R}^2} e^{-m(|x|)} \, dx \\ & - (M - M_\beta) \left[C_0 + 4 \, \alpha \, M \log \left(\sqrt{\lambda} \left(2 + \frac{8\pi - \varepsilon_0}{2 \, M} \right) \right) \right] \; .\end{aligned} \tag{2.5}$$

By Lemma 1.8 for any given $\beta > 2C_1/(8\,\pi - \varepsilon_0)$ we can ensure

$$M - M_\beta \leq \frac{C_1}{\beta} + C_2 \sqrt{\mathcal{H}_\lambda[\rho]} \leq \frac{C_1}{\beta} + \frac{8\pi - \varepsilon_0}{2} < 8\,\pi - \varepsilon_0 \; .$$

By setting $\beta_1 := 4C_1/(8\,\pi - \varepsilon_0)$, we can choose $\gamma_2 := 1 - \alpha(M - M_{\beta_1}) \geq 1/4$ and $\alpha = (8\pi - \varepsilon_0)^{-1}$. The desired inequality follows immediately from (2.5) with

$$\begin{aligned}C_{\rm CCF} := & \frac{M^3}{2\lambda\pi(8\,\pi - \varepsilon_0)} + \sqrt{\lambda} \left(2M + \frac{8\pi - \varepsilon_0}{2} \right) + \frac{1}{e} \int_{\mathbb{R}^2} e^{-\sqrt{\lambda + |x|^2}} \, dx \\ & + M \left[\frac{8\pi}{\varepsilon_0 \lambda e} + \frac{1}{e} - \log \left(\frac{\lambda \, M}{\pi} \right) \right]_+ + 4 \, M \log \left(\sqrt{\lambda} \left(2 + \frac{8\pi - \varepsilon_0}{2 \, M} \right) \right) \; .\end{aligned}$$

∎

2.3 Concentration control for $\mathcal{D}$

Proof of Theorem 1.9. Let $f := \rho^{1/4}$. As in the previous proof, for $\beta > 0$, we split f in two parts by defining $f_\beta := \min\{f, \beta^{1/4}\}$ and $h_\beta := f - f_\beta$. We have

$$\pi\,\mathcal{D}[\rho] = 8\,\pi \int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x - \pi \int_{\mathbb{R}^2} f^6 \,\mathrm{d}x\,. \tag{2.6}$$

Defining $A_\beta = \{x\,:\, f(x) \ge \beta^{1/4}\} = \{x\,:\, \rho(x) \ge \beta\}$, we have

$$\int_{\mathbb{R}^2} f^6 \;\mathrm{d}x = \int_{\mathbb{R}^2\setminus A_\beta} f_\beta^6 \;\mathrm{d}x + \int_{A_\beta} (h_\beta + \beta^{1/4})^6 \;\mathrm{d}x = \int_{\mathbb{R}^2} f_\beta^6 \;\mathrm{d}x - \beta^{3/2}|A_\beta| + \int_{A_\beta} (h_\beta + \beta^{1/4})^6 \;\mathrm{d}x\,. \tag{2.7}$$

By the convexity of $x \mapsto x^6$, for any $\alpha \in (0,1)$

$$\int_{A_\beta} (h_\beta + \beta^{1/4})^6 \le \frac{\beta^{3/2}}{\alpha^5}|A_\beta| + \frac{1}{(1-\alpha)^5} \int_{A_\beta} h_\beta^6 \;\mathrm{d}x\,. \tag{2.8}$$

By the inequality $f_\beta^6 \le \sqrt{\beta} f^4$, and plugging (2.8) and (2.7) into (2.6), we obtain

$$\pi\,\mathcal{D}[\rho] \ge 8\,\pi \int_{\mathbb{R}^2} |\nabla f|^2 \;\mathrm{d}x - \frac{\pi}{(1-\alpha)^5} \int_{\mathbb{R}^2} h_\beta^6 \;\mathrm{d}x - 8\,\pi^2\sqrt{\beta} - \pi\,\beta^{3/2}\left(\frac{1}{\alpha^5} - 1\right)|A_\beta|\,. \tag{2.9}$$

Given any $\gamma > 1$ to be fixed later and using the GNS inequality, see Lemma 1.2, for h_β:

$$\pi \int_{\mathbb{R}^2} h_\beta^6 \,\mathrm{d}x - \int_{\mathbb{R}^2} |\nabla h_\beta|^2 \,\mathrm{d}x \int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x \ge 0\,. \tag{2.10}$$

By definition of f_β and h_β, $\nabla f_\beta = 0$ in the support of h_β so that

$$\int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x = \int_{\mathbb{R}^2} |\nabla f_\beta|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla h_\beta|^2 \,\mathrm{d}x\,. \tag{2.11}$$

By (2.10) and (2.11) we thus have

$$0 \le \pi \int_{\mathbb{R}^2} h_\beta^6 \,\mathrm{d}x - \int_{\mathbb{R}^2} |\nabla f_\beta|^2 \,\mathrm{d}x \int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x + \int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x \int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x\,.$$

Infering in (2.9), for any $\gamma > 0$ to be chosen later, we have

$$\begin{aligned} \pi\,\mathcal{D}[\rho] \ge & \left(8\,\pi - \gamma \int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x\right) \int_{\mathbb{R}^2} |\nabla f|^2 \;\mathrm{d}x + \gamma \int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x \int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x \\ & + \pi\left(\gamma - \frac{1}{(1-\alpha)^5}\right) \int_{\mathbb{R}^2} h_\beta^6 \;\mathrm{d}x - 8\,\pi^2\sqrt{\beta} - \pi\,\beta^{3/2}\left(\frac{1}{\alpha^5} - 1\right)|A_\beta|\,. \end{aligned} \tag{2.12}$$

Let γ be any real number in $(1,2)$. We set $\alpha := 1 - \gamma^{-1/5} \in (0,1)$ so that the last term in (2.12) vanishes. We will see that the result holds with the choice $\gamma_1 := 4\pi(2-\gamma)$. By the concentration control in Lemma 1.8 and Assumption (1.17), as long as $\beta := 4C_1/\gamma_1$, we have

$$\int_{\mathbb{R}^2} h_\beta^4 \,\mathrm{d}x = \int_{A_\beta} (f - \beta^{1/4})^4 \,\mathrm{d}x < \int_{A_\beta} f^4 \,\mathrm{d}x \le \frac{C_1}{\beta} + C_2\,\sqrt{\mathcal{H}_\lambda[\rho]} = \gamma_1\,.$$

Since $\gamma \in (1,2)$ and $\gamma_1 \in (0,4\pi)$, $8\pi - \gamma \int_{\mathbb{R}^2} h_\beta^4 \;\mathrm{d}x > 8\pi - \gamma\gamma_1 > 0$. And the result is proved with $\gamma_1 := 4\pi(2-\gamma) \in (0,4\pi)$ and

$$C_{\mathrm{CCD}} := \pi\,\beta^{3/2}\left(\frac{\gamma}{\left(\gamma^{1/5}-1\right)^5} - 1\right)|A_\beta| + 8\pi^2\sqrt{\beta} \qquad \text{where } \beta := \frac{64}{\lambda(2-\gamma)}. \tag{2.13}$$

■

Surprisingly, this proof can be adapted to solutions of mass strictly less than 16π but not above.

3 Displacement convexity and the PKS system

3.1 Gradient flows in the Wasserstein metric and displacement convexity

We recall some facts concerning the 2-Wasserstein metric that will be used here. We shall be brief, aiming mainly to establish terminology and notation. For more background, see [36] and [1].

Let $\mathcal{P}(\mathbb{R}^2)$ denote the set of probability measures in $\mathbb{R}^2$, and let $\mathcal{P}_2(\mathbb{R}^2)$ the subset of probability measures with finite second moments. Define the functional W_2 in $\mathcal{P}(\mathbb{R}^2) \times \mathcal{P}(\mathbb{R}^2)$ by

$$\mathrm{W}_2^2(\mu,\nu) = \inf_{\Pi\in\Gamma} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^2 \, \mathrm{d}\Pi(x,y) \ ,$$

where Π runs over the set of Γ of all *couplings* of the probability measures μ and ν; that is, the set of joint probability measures in $\mathbb{R}^2 \times \mathbb{R}^2$ with first marginal μ and second ν. For absolutely continuous probability measures $f\,\mathrm{d}x$ and $g\,\mathrm{d}x$ we will simply write $\mathrm{W}_2(f,g)$ in place of $\mathrm{W}_2(f\,\mathrm{d}x, g\,\mathrm{d}x)$. Clearly, W_2 is finite in $\mathcal{P}_2(\mathbb{R}^2) \times \mathcal{P}_2(\mathbb{R}^2)$, though it takes on the value $+\infty$ in certain pairs $(\mu,\nu) \in \mathcal{P}(\mathbb{R}^2) \times \mathcal{P}(\mathbb{R}^2)$ – for example if μ belongs to $\mathcal{P}_2(\mathbb{R}^2)$, but ν does not.

It is easy to see that W_2 is a metric on $\mathcal{P}_2(\mathbb{R}^2)$; it is called the *2-Wasserstein metric*, where the 2 refers to the exponent 2 on the distance $|x-y|$. More generally, given any $\nu \in \mathcal{P}(\mathbb{R}^2)$, W_2 is a metric on the subset of $\mathcal{P}(\mathbb{R}^2)$ given by $\left\{\mu \in \mathcal{P}(\mathbb{R}^2) \ : \ \mathrm{W}_2(\mu,\nu) < \infty \ \right\}$.

A result of Brenier [6] as extended by McCann [26], provides effective control over the minimization problem defining $\mathrm{W}_2(\mu,\nu)$. To recall this result, let T be a measurable map $\mathbb{R}^2 \to \mathbb{R}^2$. We say that T *transports* μ onto ν, if for any measurable set $B \subset \mathbb{R}^2$, $\nu(B) = \mu \circ T^{-1}(B)$. In this case we say that ν is the *push-forward* of μ by T, $\nu = T\#\mu$. An equivalent formulation is that $\nu = T\#\mu$ if

$$\int_{\mathbb{R}^2} \zeta(T(x))\,\mathrm{d}\mu(x) = \int_{\mathbb{R}^2} \zeta(y)\,\mathrm{d}\nu(y) \quad \forall \zeta \in \mathcal{C}_b^0(\mathbb{R}^2) \ . \tag{3.1}$$

By the Brenier-McCann Theorem [6, 26], for any two probability measures μ and ν on $\mathbb{R}^2$ not charging Hausdorff dimension 1 sets, there is an essentially unique convex function φ in $\mathbb{R}^2$ such that $\varphi\#\mu = \nu$ and

$$\mathrm{W}_2^2(\mu,\nu) = \int_{\mathbb{R}^2} |x - \nabla\varphi(x)|^2 \, \mathrm{d}\mu(x) \ . \tag{3.2}$$

The essential uniqueness is that if φ and $\widetilde{\varphi}$ are two such convex functions, then $\nabla\varphi = \nabla\widetilde{\varphi}$ almost everywhere with respect to μ. In this paper we will be concerned with densities whose mass is not necessarily one. If μ and ν are two positive measures of mass $M > 0$, we define $\mathrm{W}_2(\mu,\nu)$ in terms of the 2-Wasserstein distance between the probability measures μ/M and ν/M as follows:

$$\mathrm{W}_2^2(\mu,\nu) = M\mathrm{W}_2^2(\mu/M,\nu/M) \ . \tag{3.3}$$

This normalization convention has the advantage that if $\nabla\varphi\#(\mu/M) = (\nu/M)$, then (3.2) is still valid for arbitrary M. Note that if (3.1) holds for μ and ν, it also holds if we change μ and ν by multiplying them by a positive constant, i.e., $\nabla\varphi\#(\mu/M) = (\nu/M)$ if and only if $\nabla\varphi\#\mu = \nu$.

In Section 5 we shall also use the p-Wasserstein distance, $1 \le p < 2$, especially for $p = 1$, on account of a useful description of compact sets for this metric. For two probability measures μ and ν on $\mathbb{R}^2$, p-Wasserstein distance $\mathrm{W}_p(\mu,\nu)$ is defined by

$$\mathrm{W}_p^p(\mu,\nu) = \inf_{\Pi\in\Gamma} \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^p \, \mathrm{d}\Pi(x,y) \ , \tag{3.4}$$

with notations introduced above. For two positive measures of mass M, we define

$$\mathrm{W}_p(\mu,\nu) = \sqrt{M}\,\mathrm{W}_p(\mu/M,\nu/M) \ . \tag{3.5}$$

This normalization is chosen taking into account (3.3) to extend the standard ordering relation for the W_p-metrics on probability measures; that is, by Hölder's inequality, we have for any $1 \le p < 2$

$$W_p(\mu, \nu) \le W_2(\mu, \nu) \ . \tag{3.6}$$

A fundamental insight of Otto [28] is that the 2-Wasserstein metric is useful when considering any evolution equation on densities ρ that can be written in the form

$$\frac{\partial \rho}{\partial t}(t, x) = \mathrm{div} \left(\rho(t, x) \nabla \left[\frac{\delta \mathcal{G}[\rho(t)]}{\delta \rho}(t, x) \right] \right) \tag{3.7}$$

for some functional $\mathcal{G}$. The prime example of (3.7) considered in [21] is the Fokker-Planck equation for probability densities

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (x\, \rho) \ ,$$

for which

$$\mathcal{G}[\rho] = \int_{\mathbb{R}^2} \rho(x) \log \rho(x) \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho(x) \, \mathrm{d}x \ . \tag{3.8}$$

In [29], a more general class of evolution equations was considered, rescaled porous medium equations for which

$$\mathcal{G}[\rho] = \frac{1}{m-1} \int_{\mathbb{R}^2} [\rho^m(x) - 1] \, \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^2} |x|^2 \rho(x) \, \mathrm{d}x \tag{3.9}$$

for $m \neq 1$. Notice that the formal limit as $m \to 1$ of the functional in (3.9) is the functional in (3.8).

Otto's insight [28] is that the equation (3.7) is *gradient flow for the functional* $\mathcal{G}$ *with respect to the 2-Wasserstein metric*. This is true for a large class of equations of the form (3.7), see [1, 3, 8, 10, 12, 13, 36]. The "gradient flow in the 2-Wasserstein metric" point of view is useful to us here for two reasons:

- It provides the means for constructing well-behaved solutions of the equation in question through the solution of a sequence of variation problems; the *Jordan-Kinderlehrer-Otto* (JKO) *scheme*.
- It provides the means for studying the rate at which solutions of (3.7) converge to minimizers of the functional $\mathcal{G}$, at least when the functional $\mathcal{G}$ has a certain convexity property.

The convexity property referred to in the second point is McCann's notion of *displacement convexity* [27], which turns out to be convexity in the "Riemannian metric" associated to the 2-Wasserstein metric; see [29]. If the functional $\mathcal{G}$ is uniformly displacement convex, then there are automatically a family of functional inequalities that govern the convergence of solutions of (3.7) to minimizers of $\mathcal{G}$. In concrete terms, the functional $\mathcal{G}$ is said to be *displacement convex* in case the following is true: For any two densities ρ_0 and ρ_1 of the same mass M, let φ be the essentially unique convex function such that $\nabla\varphi \# \rho_0 = \rho_1$. For $0 < t < 1$, define

$$\varphi_t(x) = (1-t)\frac{|x|^2}{2} + t\varphi(x) \qquad \text{and} \qquad \rho_t = \nabla\varphi_t \# \rho_0 \ .$$

The *displacement interpolation* between ρ_0 and ρ_1 is the path of densities $t \mapsto \rho_t$, $0 \le t \le 1$.

Let γ be any real number. To say that $\mathcal{G}$ is γ*-displacement convex* means that for all such densities ρ_0 and ρ_1, and all $0 \le t \le 1$,

$$(1-t)\mathcal{G}[\rho_0] + t\mathcal{G}[\rho_1] - \mathcal{G}[\rho_t] \ge \gamma t(1-t) W_2^2(\rho_0, \rho_1) \ .$$

$\mathcal{G}$ is simply *displacement convex* is this is true for $\gamma = 0$, and $\mathcal{G}$ is *uniformly displacement convex* is this is true for some $\gamma > 0$. Let us recall the characterization of displacement convexity given by McCann in [27] for functionals of the form

$$\mathcal{G}_\Phi[\rho] := \int_{\mathbb{R}^d} \Phi(\rho(x)) \, \mathrm{d}x \, , \tag{3.10}$$

where $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$. Then McCann's Theorem says that if $s \mapsto s^d \Phi(s^{-d})$ is convex non-increasing on $(0, +\infty)$ then the functional $\mathcal{G}_\Phi$ is displacement convex, and this condition is essentially necessary.

A much simpler result, also from [27], is that if V is any real valued function on $\mathbb{R}^2$ such that for all $x_0, x_1 \in \mathbb{R}^2$ and all $0 \le t \le 1$, $(1-t)V(x_0) + tV(x_1) - V((1-t)x_0 + tx_1) \ge \gamma t(1-t)|x_0 - x_1|^2$, then the functional

$$V[\rho] = \int_{\mathbb{R}^2} V(x)\,\rho(x)\,\mathrm{d}x$$

is γ-displacement convex. Using these results, one readily checks that in the case of the Fokker-Planck equation, the functional (3.8) is indeed uniformly displacement convex. The consequent inequalities that govern the long time behavior of solutions are *Gross's logarithmic Sobolev inequality* and the *Talagrand inequality* for Gaussian measures. Our analysis of long time behavior for the PKS system falls outside the scope of previous work in this direction since the functional $\mathcal{F}_{\mathrm{PKS}}$ is *not* displacement convex. The key reason that it is useful to bring the second formal Lyapunov functional $\mathcal{H}_\lambda$ into the analysis of the PKS system is that it is displacement convex. In the next section we prove the displacement convexity of $\mathcal{H}_\lambda$, and study its consequences.

3.2 The critical fast diffusion equation as gradient flow of a uniformly displacement convex entropy

The equation

$$\frac{\partial u}{\partial t}(t,x) = \Delta u^m(t,x) + \kappa\,\mathrm{div}(x\,u(t,x))\ , \tag{3.11}$$

where κ is a non-negative constant and $m > 0$, is called the *porous medium equation* with $\kappa = 0$ and $m > 1$ while for $\kappa = 0$ and $0 < m < 1$ is called the fast diffusion equation. When $\kappa > 0$, there is a restoring drift. In case $m = 1$, (3.11) is of course the heat equation for $\kappa = 0$, and the linear Fokker-Planck equation for $\kappa > 0$.

Equation (3.11) can be written in the gradient flow form

$$\frac{\partial u}{\partial t}(t,x) = \mathrm{div}\left(u(t,x)\nabla\frac{\delta\mathcal{G}}{\delta u}\right) \quad \text{with} \quad \mathcal{G}[u] = \int_{\mathbb{R}^2}\left(\frac{1}{m-1}u^m(x) + \kappa\frac{|x|^2}{2}u(x)\right)\,\mathrm{d}x\ , \tag{3.12}$$

which shows that the evolution equation (3.11) is gradient flow for $\mathcal{G}$ with respect to the Wasserstein metric.

The value $m = 1/2$ for this equation in $\mathbb{R}^2$ is *critical* in the sense that the functional $\mathcal{G}$ in (3.12) is strictly displacement convex for $m \ge 1/2$, but is not displacement convex for $m < 1/2$. There are many of other "critical" values of m between 0 and 1/2 at which other things happen, see [15] for instance. But since displacement convexity plays a crucial role in our work, it is natural to refer to the $m = 1/2$ case as critical here. Indeed, by the criteria of McCann introduced above since $\mathcal{G}[u] = V[u] + \mathcal{G}_{\phi_m}[u]$ with $V(x) = \kappa|x|^2/2$ and $\phi_m(s) = s^m/(m-1)$, then $V[u]$ is uniformly displacement convex, and for $m \ge 1/2$, $\mathcal{G}_{\phi_m}[u]$ is displacement convex. As might be expected, some difficulties arise at the critical value $m = 1/2$.

Since (3.11) is gradient flow for $\mathcal{G}$, one might hope to find stable steady states by finding the minimizers $\bar{u}$ of $\mathcal{G}$. Computing the Euler-Lagrange equation we find $m/(m-1)\bar{u}^{m-1} + \kappa|x|^2/2 = C$, where C is a Lagrange multiplier for the constraint $M := \int_{\mathbb{R}^2} u(x)\,\mathrm{d}x$, which is conserved. In the case $m = 1/2$ and choosing

$$\kappa = \kappa_{M,\lambda} := 2\sqrt{\frac{\pi}{M\,\lambda}} \qquad \text{we find} \qquad \bar{u}(x) = \frac{M}{\pi}\frac{\lambda}{(\lambda + |x|^2)^2} = \varrho_\lambda(x)\ .$$

One readily checks that $\bar{u} = \varrho_\lambda$ is a steady state solution to (3.11) with $\kappa = \kappa_{M,\lambda}$ and so the family of stationary solutions of the PKS system which we are investigating are also stationary solutions of the critical fast diffusion equation for different drifts $\kappa = \kappa_{M,\lambda}$. However, as neither $\sqrt{\varrho_\lambda}(x)$ nor $|x|^2\varrho_\lambda$ is integrable, these functions are not in the domain of definition of $\mathcal{G}$, and so are *not* minimizers of

$$\mathcal{G}[u] = \int_{\mathbb{R}^2}\left(-2\sqrt{u(x)} + \kappa_{M,\lambda}\frac{|x|^2}{2}u(x)\right)\,\mathrm{d}x\ , \tag{3.13}$$

the $m = 1/2$ version of (3.11) with $\kappa = \kappa_{M,\lambda}$ as above.

The cure is a simple renormalization as introduced in [15, 23]: Consider instead the functional $\mathcal{H}_\lambda[u]$ defined by

$$u \mapsto \int_{\mathbb{R}^2} -2\left(\sqrt{u(x)} - \sqrt{\varrho_\lambda(x)}\right)\,\mathrm{d}x + \kappa_{M,\lambda}\int_{\mathbb{R}^2} \frac{|x|^2}{2}\,[u(x) - \varrho_\lambda(x)]\,\mathrm{d}x\,. \tag{3.14}$$

Then, at least as long as u has the same behavior at infinity as does ϱ_λ, the integrals will converge. The counter terms that we have subtracted off from our functional do not depend on u, and hence they do not affect $\delta\mathcal{G}/\delta u$. This is the key idea used in the improvements of rates of convergence for the fast diffusion equation, see [15, 23]. Since $\kappa_{M,\lambda}|x|^2 = 2/\sqrt{\varrho_\lambda} - \lambda\,\kappa_{M,\lambda}$, the functional in (3.14) can be written in the following simpler form, which we take to be the definition of the *critical fast diffusion entropy*:

$$\mathcal{H}_\lambda[u] := \int_{\mathbb{R}^2} \frac{\left(\sqrt{u} - \sqrt{\varrho_\lambda}\right)^2}{\sqrt{\varrho_\lambda}}\,\mathrm{d}x.$$

It is easy to check that for $m = 1/2$, (3.11) can be written

$$\frac{\partial}{\partial t}u(t,x) = \mathrm{div}\left(u(t,x)\nabla\frac{\delta\mathcal{H}_\lambda}{\delta u}\right)\,.$$

As noted above the displacement convexity of $\mathcal{H}_\lambda$ is *formally* obvious from the fact that were $\sqrt{u(x)}$, $\sqrt{\varrho_\lambda(x)}$ and $|x|^2u(x)$ are integrable, $\mathcal{H}_\lambda[u]$ would differ from the right hand side of (3.13) by a constant. We provide a rigorous proof in the next subsection.

3.3 Regularization of the critical fast diffusion entropy

To show that $u \mapsto \mathcal{H}_\lambda[u]$ is displacement convex, and more generally, to make rigorous computations involving critical fast diffusion entropy, $\mathcal{H}_\lambda[u]$, we introduce a regularized version of the critical fast diffusion entropy:

3.1 DEFINITION (Regularized fast-diffusion relative entropy functional)**.** For $\delta > 0$, and u a density with mass M, define $\mathcal{H}_{\lambda,\delta}[u]$ by

$$\mathcal{H}_{\lambda,\delta}[u] = \int_{\mathbb{R}^2} \frac{\left(\sqrt{u+\delta} - \sqrt{\varrho_\lambda+\delta}\right)^2}{\sqrt{\varrho_\lambda+\delta}}\,\mathrm{d}x$$

3.2 PROPOSITION (Displacement convexity of relative entropy functionals)**.** *For any density $u \in L^1_+(\mathbb{R}^2)$ of mass M, $\delta \mapsto \mathcal{H}_{\lambda,\delta}[u]$ is monotone increasing as δ decreases to zero, and*

$$\lim_{\delta\to 0} \mathcal{H}_{\lambda,\delta}[u] = \mathcal{H}_\lambda[u]\,. \tag{3.15}$$

Furthermore, let u_0 and u_1 belong to $L^1_+(\mathbb{R}^2)$ of total mass M such that $\mathrm{W}_2(u_0,u_1) < \infty$, and let u_t, $0 \le t \le 1$ be their displacement interpolation. Then for each $\delta > 0$,

$$(1-t)\,\mathcal{H}_{\lambda,\delta}[u_0] + t\,\mathcal{H}_{\lambda,\delta}[u_1] - \mathcal{H}_{\lambda,\delta}[u_t] \ge t\,(1-t)\,K_\delta(u_0,u_1)\,,$$

where $K_\delta(u_0,u_1)$ satisfies

$$\lim_{\delta\to 0} K_\delta(u_0,u_1) = \kappa_{M,\lambda}\,\mathrm{W}_2^2(u_0,u_1)\,, \tag{3.16}$$

and $K_\delta(u_0,u_1) \ge \gamma_\delta\,\mathrm{W}_2^2(u_0,u_1)$ with $\gamma_\delta < 0$. Consequently, the maps $u \mapsto \mathcal{H}_{\lambda,\delta}[u]$ are γ_δ-displacement convex and the map $u \mapsto \mathcal{H}_\lambda[u]$ is strictly uniformly displacement convex:

$$(1-t)\,\mathcal{H}_\lambda[u_0] + t\,\mathcal{H}_\lambda[u_1] - \mathcal{H}_\lambda[u_t] \ge \kappa_{M,\lambda}\,t\,(1-t)\,\mathrm{W}_2^2(u_0,u_1)\,.$$

Proof: As $\delta \mapsto \left(\sqrt{u+\delta} - \sqrt{\varrho_\lambda+\delta}\right)^2$ is non-increasing. Then, as δ decreases,

$$\frac{\left(\sqrt{u+\delta} - \sqrt{\varrho_\lambda+\delta}\right)^2}{\sqrt{\varrho_\lambda+\delta}} \quad \text{increases to} \quad \frac{\left(\sqrt{u} - \sqrt{\varrho_\lambda}\right)^2}{\sqrt{\varrho_\lambda}}\,.$$

By the monotone convergence theorem and (3.15), the monotonicity in δ follows. Next,

$$\frac{\left(\sqrt{u+\delta}-\sqrt{\varrho_\lambda+\delta}\right)^2}{\sqrt{\varrho_\lambda+\delta}} = \frac{u}{\sqrt{\varrho_\lambda+\delta}} - 2\left(\sqrt{u+\delta}-\sqrt{\delta}\right) + \frac{\delta}{\sqrt{\varrho_\lambda+\delta}} + \sqrt{\varrho_\lambda+\delta} - 2\sqrt{\delta}\ .$$

Where by the mean value theorem

$$\frac{u}{\sqrt{\varrho_\lambda+\delta)}} \le \frac{u}{\sqrt{\delta}}\,,\ \sqrt{u+\delta}-\sqrt{\delta} \le \frac{u}{2\sqrt{\delta}} \quad \text{and} \quad \left|\frac{\delta}{\sqrt{\varrho_\lambda+\delta}} + \sqrt{\varrho_\lambda+\delta} - 2\sqrt{\delta}\right| \le \frac{\varrho_\lambda}{\sqrt{\delta}}\,.$$

These three terms are integrable and

$$\mathcal{H}_{\lambda,\delta}[u] = (\mathrm{I}) + (\mathrm{II}) + \text{const.} \tag{3.17}$$

where

$$(\mathrm{I}) := \int_{\mathbb{R}^2} \frac{1}{\sqrt{\varrho_\lambda+\delta}}\, u\, \mathrm{d}x \quad \text{and} \quad (\mathrm{II}) := 2\int_{\mathbb{R}^2}\left(\sqrt{\delta}-\sqrt{u+\delta}\right)\, \mathrm{d}x$$

The criterion (3.10) are easily checked for $\Phi(u) = \sqrt{\delta}-\sqrt{u+\delta}$, and thus (II) is displacement convex.

The term (I) in (3.17) is unfortunately *not* displacement convex in general. In fact, we will show that is γ_δ-displacement convex with a explicit computable constant. In order to check the γ_δ-displacement convexity of the regularized functional, notice that $(\varrho_\lambda+\delta)^{-1/2}$ is a function of $|x|^2$. Thus, the functional (I) is of the general form:

$$u \mapsto \int_{\mathbb{R}^2} V_\delta(x)\, u(x)\ \mathrm{d}x$$

with $V_\delta(x) = (\varrho_\lambda+\delta)^{-1/2}$. The characterization of γ_δ-displacement convexity [1] ensures that this is implied by $D^2V_\delta \ge \gamma_\delta I_2$. Let us compute the hessian of the potential $V_\delta(x)$. Define the function f_δ on $[0,\infty)$ by

$$f_\delta(|x|^2) := V_\delta(x) = \frac{1}{\sqrt{\varrho_\lambda(x)+\delta}}\ .$$

We compute

$$f_\delta'(s) = \frac{A}{[A+\delta(\lambda+s)^2]^{3/2}} \quad \text{and} \quad f_\delta''(s) = -\frac{3\,A\,\delta(\lambda+s)}{[A+\delta(\lambda+s)^2]^{5/2}}\ ,$$

with $A = M\,\lambda/\pi$. Therefore

$$D^2V_\delta(x) = 2f_\delta'(|x|^2)\delta_{ij} + 4f_\delta''(|x|^2)(x\otimes x),$$

and taking into account that $f_\delta''(s)\le 0$, then

$$\xi\cdot D^2V_\delta(x)\cdot\xi^T \ge \left[2f_\delta'(|x|^2) + 4f_\delta''(|x|^2)|x|^2\right]|\xi|^2 := F_\delta(|x|^2)|\xi|^2 \tag{3.18}$$

for all $x,\xi\in\mathbb{R}^2$, where the function F_δ is given by

$$F_\delta(s) = 2f_\delta'(s) + 4sf_\delta''(s) = \frac{2A^2+2A\delta\lambda^2-8A\lambda\delta s-10A\delta s^2}{[A+\delta(\lambda+s)^2]^{5/2}}.$$

It is obvious that the function F_δ converges point-wise to the constant $\kappa_{M,\lambda}$ as $\delta\to 0$ in $[0,\infty)$. Moreover, since for each $\delta>0$, the function $F_\delta(s)\to 0$ as $s\to\infty$ and it is clear that is negative for s large enough since the denominator is positive and the numerator has a negative dominant term, then F_δ attains its maximum and minimum in $[0,\infty)$. Then, we can choose its minimum value as $\gamma_\delta<0$ and the γ_δ-displacement convexity is proved.

In order, to show the limiting uniform displacement convexity, we need to refine our arguments. For that, we come back to the definition of convexity. Let ψ be the essentially unique convex function such that $\nabla\psi\# u_0 = u_1$. For $0\le t\le 1$, define

$$\begin{aligned}\eta_\delta(t) &:= \int_{\mathbb{R}^2} V_\delta(x)\left[(1-t)\,u_0(x) + t\,u_1(x) - u_t(x)\right]\ \mathrm{d}x\\ &= \int_{\mathbb{R}^2}\left[(1-t)\,V_\delta(x) + t\,V_\delta(\nabla\psi(x)) - V_\delta(x+t(\nabla\psi(x)-x))\right]\,u_0(x)\,\mathrm{d}x\ .\end{aligned}$$

We seek a lower bound on η_δ of the form $\eta_\delta(t) \geq t\,(1-t)\,K_\delta(u_0,u_1)$. Since $\eta_\delta(0) = \eta_\delta(1) = 0$, it suffices for this purpose to show that $\eta_\delta''(t) \geq 2\,K_\delta(u_0,u_1)$ for all $0 \leq t \leq 1$. By denoting $y := \nabla\psi(x) - x$, we have

$$\eta_\delta''(t) = \int_{\mathbb{R}^2} y \cdot D^2 V_\delta(x+ty) \cdot y^T\, u_0(x)\ \mathrm{d}x\ .$$

Using (3.18), we readily obtain that

$$\eta_\delta''(t) \geq \int_{\mathbb{R}^2} F_\delta(|x+ty|^2)|y|^2\, u_0(x)\ \mathrm{d}x \geq K_\delta(u_0,u_1)\ ,$$

with

$$K_\delta(u_0,u_1) := \min_{0\leq t\leq 1} \int_{\mathbb{R}^2} F_\delta(|x+ty|^2)|y|^2\, u_0(x)\ \mathrm{d}x.$$

Now, let us observe that the function F_δ is bounded in $[0,\infty)$ uniformly in δ. For that, note that f_δ' is decreasing and thus $f_\delta'(s) \leq f_\delta'(0) \leq A^{-1/2}$. On the other hand, by the geometric-arithmetic mean inequality, we get

$$|s f_\delta''(s)| \leq 3\delta^{1/2} \frac{s}{A+\delta s^2} \leq \frac{3}{2\sqrt{A}}\ . \tag{3.19}$$

As a consequence, we get

$$\|F_\delta\|_{L^\infty(0,\infty)} \leq \frac{8}{\sqrt{A}},$$

and thus,

$$|F_\delta(|x+ty|^2)||y|^2\, u_0(x) \leq \frac{8}{\sqrt{A}} |\nabla\psi(x)|^2\, u_0(x) \in L^1(\mathbb{R}^2)\ ,$$

for all $0 \leq t \leq 1$. Thus, the dominated convergence theorem guarantees that

$$\lim_{\delta\to 0} \int_{\mathbb{R}^2} F_\delta(|x+ty|^2)|y|^2\, u_0(x)\ \mathrm{d}x = \kappa_{M,\lambda} \int_{\mathbb{R}^2} |y|^2\, u_0(x)\ \mathrm{d}x = \kappa_{M,\lambda} \mathrm{W}_2^2(u_0,u_1)\ ,$$

uniformly in $0 \leq t \leq 1$, which together with the definition of $K_\delta(u_0,u_1)$ implies the uniform displacement convexity of the limiting functional $\mathcal{H}_\lambda[u]$. ■

Continuing with the notation of Proposition 3.2, define the function h_δ on $[0,1]$ by

$$h_\delta(t) = \mathcal{H}_{\lambda,\delta}[u_t] - K_\delta(u_0,u_1)t^2\ .$$

Then by Proposition 3.2, $(1-t)\,h_\delta(0) + t\,h_\delta(1) - h_\delta(t) \geq 0$, so that h_δ is convex. Therefore, for all $t \in (0,1)$,

$$h_\delta(1) - h_\delta(0) \geq \frac{h_\delta(t) - h_\delta(0)}{t}\ .$$

This in turn implies that

$$\mathcal{H}_{\lambda,\delta}[u_1] - \mathcal{H}_{\lambda,\delta}[u_0] \geq \limsup_{t\to 0} \frac{\mathcal{H}_{\lambda,\delta}[u_t] - \mathcal{H}_{\lambda,\delta}[u_0]}{t} + K_\delta(u_0,u_1)\ .$$

To compute the lim sup of the right hand side, we treat the two non-constant terms (I) and (II) in (3.17) separately. As we have noted (II) is displacement convex, and by well known theorems on the sub-gradients of displacement convex functions [1, Chapter 10], this part contributes

$$\int_{\mathbb{R}^2} \frac{\nabla u_0(x)}{2(u_0(x)+\delta)^{3/2}} \cdot (\nabla\psi(x) - x)\, u_0(x)\ \mathrm{d}x\ ,$$

as long as the integrand satisfies mild regularity properties; in particular whenever u_0 is bounded below on every compact set by some strictly positive number, and $\sqrt{u_0}$ has a square integrable distributional gradient. We shall show that both of these conditions hold in our application. Given that they do, then by the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^2} \frac{|\nabla u_0|}{(u_0+\delta)^{3/2}} |\nabla\psi(x) - x|\, u_0 \leq \sqrt{\int_{\mathbb{R}^2} \frac{|\nabla u_0|^2}{(u_0+\delta)^3} u_0} \sqrt{\int_{\mathbb{R}^2} |\nabla\psi(x) - x|^2\, u_0}$$
$$\leq \sqrt{\frac{1}{3\delta}\int_{\mathbb{R}^2} \frac{|\nabla u_0|^2}{u_0}}\, \mathrm{W}_2(u_0,u_1) = \frac{2}{\sqrt{3\delta}}\sqrt{\int_{\mathbb{R}^2} |\nabla\sqrt{u_0}|^2}\, \mathrm{W}_2(u_0,u_1)\ .$$

The contribution of (I) in (3.17) can be treated by appealing to the general results in [1] since this functional is γ_δ-displacement convex, in the notation of [1], and thus, this part contributes

$$2\int_{\mathbb{R}^2} f_\delta'(|x|^2)\, x\cdot(\nabla\psi(x) - x)\, u_0\, \mathrm{d}x\ .$$

Which is finite because $s \mapsto \sqrt{s}\, f_\delta'(s)$ is a rational functional which tends to 0 when s goes to infinity. Thus we have:

3.3 LEMMA (First-order characterization of displacement convexity)**.** *Let u_0 and u_1 be two densities of total mass M such that $\mathrm{W}_2^2(u_0,u_1) < \infty$, and such that u_0 is uniformly bounded below on compact subsets of $\mathbb{R}^2$ by a strictly positive number, and that $\sqrt{u_0}$ has a square integrable distributional gradient. Let $\nabla\psi$ be the unique gradient of a convex function ψ in $\mathbb{R}^2$ so that $\nabla\psi\#u_0 = u_1$. Then*

$$\mathcal{H}_{\lambda,\delta}[u_1] - \mathcal{H}_{\lambda,\delta}[u_0] \geq \int_{\mathbb{R}^2}\left[\frac{2Ax}{[A+\delta(\lambda+|x|^2)^2]^{\frac{3}{2}}} + \frac{\nabla u_0}{2(u_0+\delta)^{\frac{3}{2}}}\right](\nabla\psi(x) - x)\, u_0\, \mathrm{d}x + K_\delta(u_0,u_1) \tag{3.20}$$

where $K_\delta(u_0,u_1)$ is defined in Proposition 3.2*, and the integrand in* (3.20) *is integrable.*

One might be tempted to take the limit $\delta \to 0$ at this stage and to conclude

$$\mathcal{H}_\lambda[u_0] \leq \mathcal{H}_\lambda[u_1] - \int_{\mathbb{R}^2}\left[\kappa_{M,\lambda}\, x + \frac{\nabla u_0}{2u_0^{3/2}}\right]\cdot(\nabla\psi(x) - x)\, u_0\ \mathrm{d}x - \kappa_{M,\lambda}\, \mathrm{W}_2^2(u_0,u_1)\ ,$$

but without further information about $\nabla\psi(x) - x$, it is not possible to do this, or to justify the convergence of the integral. In our applications, it will be simpler to use the specific information that we obtain on $\nabla\psi(x) - x$, then to do some integrations by parts, *and then* take the limit $\delta \to 0$.

Let us finally deduce as an application of the uniform displacement convexity of the functional $\mathcal{H}_\lambda[u]$, an interesting functional inequality of Talagrand type. Actually, generalized Log-Sobolev-type inequalities lead formally to generalized Talagrand-type inequalities for this functional by repeating arguments due to Otto and Villani [30, Theorem 1, Proposition 1] in the linear case and generalized in [12, Theorem 2.1]. Here, we are able to show it in full rigor by the previous approximation argument.

3.4 LEMMA (Talagrand's inequality)**.** *Whenever $u \in L^1_+(\mathbb{R}^2)$ of mass M with $\mathcal{H}_\lambda[u] < \infty$, then*

$$\mathrm{W}_2(u,\varrho_\lambda) \leq \sqrt{\frac{2\mathcal{H}_\lambda[u]}{\kappa_{M,\lambda}}}.$$

Proof: Using Lemma 3.3 with $u_0 = \varrho_\lambda$ and $u_1 = u$, we obtain that $\mathcal{H}_{\lambda,\delta}[u] \geq K_\delta(\varrho_\lambda,u)$. for all $\delta > 0$, since $\mathcal{H}_{\lambda,\delta}[\varrho_\lambda] = 0$ and

$$\frac{2Ax}{[A+\delta(\lambda+|x|^2)^2]^{3/2}} + \frac{\nabla\varrho_\lambda}{2(\varrho_\lambda+\delta)^{3/2}} = 0.$$

Thus, passing to the limit $\delta \to 0$ taking into account (3.16), the desired inequality is obtained. ∎

3.5 Remark (Basins of attraction)**.** *The previous result gives us a localization argument for the densities compared to Lemma* 2.1*. It also says in a weaker sense that the mass cannot be too spread from the stationary state* ϱ_λ *if the functional* $\mathcal{H}_\lambda[u]$ *is finite. Moreover, it tells us that the Wasserstein 2-distance of our initial data to the stationary state* ϱ_λ *is finite provided* $\mathcal{H}_\lambda[u_0] < \infty$*. Actually, we can observe that each of the equilibrium solutions* ϱ_λ *are infinitely far apart in the* W_2 *metric. We can easily check that with* $\varphi(x) = \sqrt{\lambda/\mu}|x|^2/2$*, one has* $\nabla\varphi\#\varrho_\mu = \varrho_\lambda$*. Thus, the uniqueness part of Brenier-McCann Theorem ensures*

$$\mathrm{W}_2^2(\varrho_\mu, \varrho_\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} \left|\sqrt{\frac{\lambda}{\mu}}x - x\right|^2 \varrho_\mu(x)\,\mathrm{d}x = +\infty$$

since the equilibrium densities ϱ_λ *all have infinite second moments. In particular,* $\mathcal{H}_\lambda[\varrho_\mu] = +\infty$ *for* $\mu \neq \lambda$*.*

4 Analysis of the discrete time variational scheme for the critical mass PKS system

From now on, we will assume that the mass is 8π.

4.1 The Jordan-Kinderlehrer-Otto scheme for the critical mass PKS system

The Jordan-Kinderlehrer-Otto (JKO) scheme for constructing solutions to (3.7), as described in Section 3, would be to fix a time step $\tau > 0$, and inductively define the sequence $\{\rho^k\}_{k\in\mathbb{N}}$ by setting ρ_0 to be the initial density, and then for $k \geq 0$,

$$\rho^{k+1} \in \mathrm{argmin}\left\{\frac{\mathrm{W}_2^2(\rho,\rho^k)}{2\tau} + \mathcal{G}[\rho]\right\} \ . \tag{4.1}$$

In other words, ρ^{k+1} is *some* minimizer of the functional $\rho \mapsto \mathrm{W}_2^2(\rho,\rho^k)/(2\tau) + \mathcal{G}[\rho]$. Only existence of the minimizer is an issue, and not uniqueness, although in many examples that have been investigated a strict convexity argument furnishes the uniqueness. The key point is existence of a minimizer, since that provides a solution of the Euler-Lagrange equation for the minimization problem in (4.1). Then, as shown in [21], the fact that each ρ^{k+1} satisfies this Euler-Lagrange equation means that, under certain conditions on $\mathcal{G}$, if one defines $\tilde\rho_\tau$ by

$$\tilde\rho_\tau(t,x) = \rho^k(x) \qquad \text{for} \quad k\tau \leq t < (k+1)\tau \ , \quad \text{with } \tilde\rho_\tau(0,x) = \rho_0 \ ,$$

there is a sequence of values of τ tending to zero along which $\tilde\rho_\tau$ tends to a solution of (3.7) in a suitable weak sense. This scheme of constructing weak solutions of the PKS system for $M < 8\pi$ was developed in [3]. However, for $M = 8\pi$ we can not proceed in a very direct manner. Our problem lies outside the scope of previous applications of the JKO scheme, since at the critical mass $M = 8\pi$, (1.5) provides no upper bound on $\mathcal{E}[\rho]$, and hence, it is not even clear that minimizers exist for the variational problem in (4.1) when $\mathcal{G} = \mathcal{F}_{\mathrm{PKS}}$ and $M = 8\pi$. To circumvent this difficulty, we introduce a regularized functional. In fact, for reasons that will become evident later on, we shall even be forced to choose a different degree of regularization at each time step.

4.2 Regularization of $\mathcal{F}_{\mathrm{PKS}}$

Let γ be a C^∞ probability density in $\mathbb{R}^2$ which is radially symmetric, and supported in the unit disc. Then, for all $\epsilon > 0$ define $\gamma_\epsilon(x) = \epsilon^{-2}\gamma(x/\epsilon)$. This is a radially symmetric, C^∞ probability density supported in the discs of radius ϵ. Finally, we define the regularized Green's function

$$G_\epsilon = \gamma_\epsilon * G * \gamma_\epsilon$$

where $*$ denotes convolution, and $G(x) = -1/(2\pi)\log|x|$.

4.1 LEMMA (First properties of G_ϵ)**.** *Let* G_ϵ *be defined as above then:*

(i) For all $x \in \mathbb{R}^2$, $G_\epsilon(x) \le G(x)$. Moreover, if $|x| > 2\epsilon$ then $G_\epsilon(x) = G(x)$.

(ii) There exists $C > 0$ such that for all $x \in \mathbb{R}^2$, $G_\epsilon(x) \le C\epsilon^{-2}$.

(iii) For all $(x, y) \in \mathbb{R}^4$,

$$G_\epsilon(x-y) \ge -\frac{1}{4\pi}\left[4 + \log\left(e + |x|^2\right) + \log\left(e + |y|^2\right)\right] \ .$$

Proof: (i) As γ is radially symmetric, G is harmonic in $\mathbb{R}^2\backslash\{(0,0)\}$, and subharmonic in $\mathbb{R}^2$ so that, by the mean value property the first item holds.
(ii) Since $\log_- |x|$ is locally integrable in $\mathbb{R}^2$, for any $x \in \mathbb{R}^2$

$$G * \gamma_\epsilon(x) = \int_{|y|\le\epsilon} G(x-y)\,\gamma_\epsilon(y)\ \mathrm{d}y \le \frac{1}{2\pi}\int_{|y|\le\epsilon} \log_- |x-y|\,\gamma_\epsilon(y)\ \mathrm{d}y \le \frac{C}{\epsilon^2}\ .$$

since γ is bounded. Thus, we get

$$G_\epsilon(w) \le \frac{C}{\epsilon^2}\int_{\mathbb{R}^2}\gamma_\epsilon(z)\ \mathrm{d}z = \frac{C}{\epsilon^2}\ .$$

(iii) From the elementary inequality $|z-w| \le |z| + |w| \le 2\max\{|z|\,,\,|w|\}$, we obtain

$$\log|z-w| \le \log 2 + \log|z| + \log|w|\ .$$

Therefore,

$$G(z-w) \ge -\frac{1}{2\pi}(2 + \log|z| + \log|w|)\ .$$

Integrating both sides against $\gamma_\epsilon(x-z)\,\gamma_\epsilon(y-w)$, and using Jensen's inequality, we find

$$\begin{aligned} G_\epsilon(x-y) &\ge -\frac{1}{2\pi}\left[2 + \log\left(\int_{\mathbb{R}^2}|z|\,\gamma_\epsilon(x-z)\,\mathrm{d}z\right) + \log\left(\int_{\mathbb{R}^2}|w|\,\gamma_\epsilon(y-w)\,\mathrm{d}w\right)\right] \\ &\ge -\frac{1}{2\pi}\left[2 + \log\left(|x| + \int_{\mathbb{R}^2}|z|\,\gamma_\epsilon(z)\,\mathrm{d}z\right) + \log\left(|y| + \int_{\mathbb{R}^2}|w|\,\gamma_\epsilon(w)\,\mathrm{d}w\right)\right] \\ &\ge -\frac{1}{4\pi}\left[4 + \log\left(e + |x|^2\right) + \log\left(e + |y|^2\right)\right]\ , \end{aligned}$$

at least for ϵ small enough so that $\int_{\mathbb{R}^2}|z|\gamma_\epsilon(z)\,\mathrm{d}z$ is small enough. ■

One of the main uses that we will make of the regularization of the self interaction functional is that it provides a regularized density for the chemical attractant: given a mass density ρ, we define the *regularized chemical attractant density* c_ϵ

$$c_\epsilon(x) = G_\epsilon * \rho(x)$$

4.2 LEMMA (Uniform estimate regularized chemoattractant)**.** *For all $\epsilon > 0$ and all densities ρ with mass 8π, the regularized chemical attractant density $c_\epsilon = G_\epsilon * \rho$ satisfies*

$$\|\nabla c_\epsilon\|_\infty \le \frac{4\,C_{\mathrm{HLS}}}{\epsilon}\,\|\gamma\|_{4/3}^2$$

and

$$\||x|\nabla c_\epsilon\|_\infty \le 8C_{\mathrm{HLS}}\,\|\gamma\|_{4/3}\||x|\gamma\|_{4/3} + 4 + \frac{C_{\mathrm{HLS}}}{2\pi\epsilon}\,\|\gamma\|_{4/3}^2\,\||x|\rho\|_1\ . \tag{4.2}$$

Here C_{HLS} denotes the constant of the sharp Hardy-Littlewood-Sobolev (HLS) inequality [24] for the special case $p = q = 4/3$:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x)\frac{1}{|x-y|}g(y)\,\mathrm{d}x\,\mathrm{d}y \le C_{\mathrm{HLS}}\|f\|_{4/3}\|g\|_{4/3}\ . \tag{4.3}$$

Though the explicit value of $C_{\rm HLS}$ is simple enough, see [24], our bounds and their proofs will perhaps be easier to read if leave $C_{\rm HLS}$ unevaluated in them, as a marker of the use of the HLS inequality.

Proof: By the Young inequality, we have $\|\nabla c_\epsilon\|_\infty = \|\nabla G_\epsilon * \rho\|_\infty \le 8\,\pi\,\|\nabla G_\epsilon\|_\infty$. And by the HLS inequality,

$$\|\nabla G_\epsilon\|_\infty \le \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \gamma_\epsilon(x-z)\,\frac{1}{|z-w|}\,\gamma_\epsilon(w-y)\,\mathrm{d}z\,\mathrm{d}w \le \frac{C_{\rm HLS}}{2\pi}\,\|\gamma_\epsilon\|_{4/3}^2 = \frac{C_{\rm HLS}}{2\pi\epsilon}\,\|\gamma\|_{4/3}^2\ .$$

Using the triangle inequality $|x| \le |x-z| + |z-w| + |w-y| + |y|$ we have

$$||x|\nabla c_\epsilon| \le 2\,|(|x|\gamma_\epsilon) * \nabla G * \gamma_\epsilon * \rho| + |\gamma_\epsilon * \rho| + |\gamma_\epsilon * \nabla G * \gamma_\epsilon * |x|\rho|$$

Using the Young and HLS inequalities, see (4.3), we obtain

$$\||x|\nabla c_\epsilon\|_\infty \le 8\,C_{\rm HLS}\,\||x|\gamma_\epsilon\|_{4/3}\,\|\gamma_\epsilon\|_{4/3} + 4 + \frac{C_{\rm HLS}}{2\pi}\,\|\gamma_\epsilon\|_{4/3}^2\,\||x|\rho\|_1\ ,$$

and the second part of the result is obtained by using $\|\gamma_\epsilon\|_{4/3} = \varepsilon^{-1/2}\,\|\gamma\|_{4/3}$. ■

Using the regularized Green's function G_ϵ, we introduce the *regularized self-interaction functional* $\mathcal{W}_\epsilon$:

$$\mathcal{W}_\epsilon(\rho) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,G_\epsilon(x-y)\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y\ .$$

4.3 LEMMA (Continuity of the regularized interaction energy)**.** *Let ρ_1 and ρ_2 be any two densities in $\mathbb{R}^2$ of mass 8π bounded in $L^1(\mathbb{R}^2, \log(e+|x|^2))$. Then, for all $0<\epsilon\le 1$,*

$$|\mathcal{W}_\epsilon[\rho_1] - \mathcal{W}_\epsilon[\rho_2]| \le \left[\frac{3}{\pi} + 2C\epsilon^{-2}\right]\|\rho_1\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)}\,\|\rho_1-\rho_2\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)} \tag{4.4}$$

Moreover, let $\{\rho_n\}_{n\ge0}$ be a sequence of densities all bounded in $L^1(\mathbb{R}^2, \log(e+|x|^2))$ uniformly in n. If $\{\rho_n\}_{n\ge0}$ converges weakly in $L^1(\mathbb{R}^2)$ to ρ, then for each $\epsilon>0$,

$$\mathcal{W}_\epsilon[\rho] \le \liminf_{n\to\infty}\,\mathcal{W}_\epsilon[\rho_n]\ .$$

Proof: By definition of G_ϵ:

$$\mathcal{W}_\epsilon[\rho] = \mathcal{W}_0[\rho*\gamma_\varepsilon] = \int_{\mathbb{R}^2} |\nabla G * \rho * \gamma_\varepsilon|^2(z)\,\mathrm{d}z \ge 0\ .$$

Using $\widetilde{\rho} := \rho_1 - \rho_2$ we write

$$\mathcal{W}_\epsilon[\rho_1] - \mathcal{W}_\epsilon[\rho_2] = \mathcal{W}_\epsilon[\widetilde{\rho}] + 2\int_{\mathbb{R}^2} \rho_1 * G_\epsilon(x)\,\widetilde{\rho}(x)\,\mathrm{d}x \ge 2\int_{\mathbb{R}^2} [\rho * G_\epsilon]\,\widetilde{\rho}\,\mathrm{d}x\ , \tag{4.5}$$

Then combining Lemma 4.1 and (4.5), we obtain

$$\begin{aligned}
\mathcal{W}_\epsilon[\rho_1] - \mathcal{W}_\epsilon[\rho_2] \ge\, & 2\iint_{\widetilde{\rho}>0} \rho(y)\,G_\epsilon(x-y)\,\widetilde{\rho}(x)\,\mathrm{d}y\,\mathrm{d}x + 2\iint_{\widetilde{\rho}\le0} \rho(y)\,G_\epsilon(x-y)\,\widetilde{\rho}(x)\,\mathrm{d}y\,\mathrm{d}x \\
\ge & -\frac{1}{2\pi}\iint_{\widetilde{\rho}>0} \rho(y)\left[4+\log\left(e+|x|^2\right)+\log\left(e+|y|^2\right)\right]\widetilde{\rho}(x)\,\mathrm{d}y\,\mathrm{d}x \\
& + 2\,\frac{C}{\epsilon^2}\iint_{\widetilde{\rho}\le0} \rho(y)\widetilde{\rho}(x)\,\mathrm{d}y\,\mathrm{d}x \\
\ge & -\left[\frac{3}{\pi} + 2\frac{C}{\epsilon^2}\right]\|\rho_1\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)}\,\|\widetilde{\rho}\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)}\ .
\end{aligned}$$

Now swapping the roles of ρ_1 and ρ_2, we obtain (4.4).

By Lemma 4.1 $\{\rho_n\}_{n\geq 0}$ bounded in $L^1(\mathbb{R}^2, \log(e+|x|^2)\,\mathrm{d}x)$ uniformly in n implies that $\rho_n * G_\epsilon$ is bounded in $L^\infty(\mathbb{R}^2)$ uniformly in n. Since $\{\rho_n\}_{n\geq 0}$ converges to ρ weakly in $L^1(\mathbb{R}^2)$, then $\rho_n * (\chi_R\, G_\epsilon) \to \rho * (\chi_R\, G_\epsilon)$ point-wise for given any cut-off function χ_R with support in $B(0,R)$ and thus $\rho * G_\epsilon \in L^\infty(\mathbb{R}^2)$. Therefore, applying (4.5) with $\rho_1 =: \rho$ and $\rho_2 := \rho_n$, we have

$$\liminf_{n\to\infty} (\mathcal{W}_\epsilon[\rho_n] - \mathcal{W}_\epsilon[\rho]) \geq \lim_{n\to\infty} 2 \int_{\mathbb{R}^2} [\rho * G_\epsilon]\,(\rho_n - \rho)\,\mathrm{d}x = 0 \ ,$$

where we have used the weak convergence on the right hand side. ■

We are now ready to introduce our regularized free energy functional.

4.4 DEFINITION (Regularized free energy functional)**.** For all $0 < \epsilon \leq 1$, define

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] := \int_{\mathbb{R}^2} \rho(x)\log\rho(x)\,\mathrm{d}x - \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\,G_\epsilon(x-y)\,\rho(y)\ \mathrm{d}x\,\mathrm{d}y\ .$$

on the set of densities ρ of mass 8π such that $\rho \in L^1(\mathbb{R}^2, \log(e+|x|^2)\,\mathrm{d}x)$ and $\rho\log\rho$ is integrable.

Note that by Lemma 4.1 (ii) and (iii), $\rho\,(G_\epsilon * \rho)$ is integrable for $\rho \in L^1(\mathbb{R}^2, \log(e+|x|^2)\,\mathrm{d}x)$. Moreover, by Lemma 4.1 (i)

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] \geq \mathcal{F}_{\mathrm{PKS}}[\rho]\ . \tag{4.6}$$

In particular, by the sharp log HLS inequality, see Lemma 1.1

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] \geq -C(8\pi) = -1 + \log 8\ . \tag{4.7}$$

By Lemma 4.1 (iii), we have the upper bound independent of ϵ:

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] \leq \int_{\mathbb{R}^2} \rho(x)\log\rho(x)\,\mathrm{d}x + 32\,\pi + 2\|\rho\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)}\ . \tag{4.8}$$

4.5 LEMMA (Error estimate for regularized free energy)**.** *For all $\rho \in L^1_+ \cap L^{3/2}(\mathbb{R}^2)$ with mass 8π, and all $\epsilon < (2\sqrt{e})^{-1}$,*

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] - \mathcal{F}_{\mathrm{PKS}}[\rho] \leq 4\sqrt{5\pi}\|\rho\|^{3/2}_{3/2}\,\epsilon|\log(2\epsilon)|\ .$$

Proof: We use Hölder's inequality and Young's inequality for convolutions to get

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] - \mathcal{F}_{\mathrm{PKS}}[\rho] \leq \|\rho\|^2_{4/3}\|G_\epsilon - G\|_2\ .$$

Hölder's inequality gives $\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] - \mathcal{F}_{\mathrm{PKS}}[\rho] \leq \sqrt{8\pi}\,\|\rho\|^{3/2}_{3/2}\|G_\epsilon - G\|_2$. By Lemma 4.1, the support of $G_\epsilon - G$ is in $B(0,2\epsilon)$ and $G_\epsilon \leq G$. We can thus directly compute

$$\|G_\epsilon - G\|_2^2 = \int_{|z|\leq 2\epsilon} |(G_\epsilon - G)(z)|^2\ \mathrm{d}z \leq 4\int_{|z|\leq 2\epsilon} |G(z)|^2\ \mathrm{d}z = 16\pi\,\epsilon^2\left(\frac{1}{2} - \log 2\epsilon + (\log 2\epsilon)^2\right).$$

Finally, simple computations show that when $|\log(2\epsilon)| > 1/2$, the term in parentheses on the right in no greater than $5|\log(2\epsilon)|^2$. ■

4.3 Existence and first properties of the JKO scheme minimizers

Let $\mathcal{S}$ denote the set of densities ρ of mass M such that $\mathrm{W}_2(\rho, \varrho_\lambda)$, $\mathcal{E}[\rho]$ and $\int_{\mathbb{R}^2}|x|\rho(x)\,\mathrm{d}x$ are all finite. By (4.7), the functional

$$\rho \mapsto \frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho],$$

is bounded from below on $\mathcal{S}$. The next lemma asserts that is has minimizers, and begins the task of their analysis. We state this lemma for a single step since we shall be changing the value of ϵ from step to step.

4.6 THEOREM (Existence of minimizers)**.** *Let $\lambda > 0$, $0 < \tau \le 1$ and $0 < \epsilon \le 1$. For all $\rho_0 \in \mathcal{S}$*

$$\arg\min_{\rho\in\mathcal{S}} \left\{ \frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] \right\} \tag{4.9}$$

is not empty, and each minimizer ρ belongs to $\mathcal{S}$.

Proof: Let $\{\rho^{(k)}\}_{k\in\mathbb{N}}$ be a minimizing sequence *i.e.* such that

$$\lim_{k\to\infty} \left(\frac{\mathrm{W}_2^2(\rho^{(k)},\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] \right) = \inf_{\rho\in\mathcal{S}} \left\{ \frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho] \right\} .$$

By what we have noted just above, the infimum on the right hand side is finite. The following observation is the starting point for obtaining all of the bounds we need: Considering the trial function $\rho = \rho_0$ itself, one sees that we may suppose

$$\frac{\mathrm{W}_2^2(\rho^{(k)},\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] \le \frac{\mathrm{W}_2^2(\rho_0,\rho_0)}{2\tau} + \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0] = \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0]$$

for all k. Consequently, for all k,

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] \le \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0] \quad \text{and} \quad \mathrm{W}_2^2(\rho^{(k)},\rho_0) \le 2\tau \left[\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0] - \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] \right] . \tag{4.10}$$

We first bound $\mathrm{W}_2(\rho^{(k)},\varrho_\lambda)$ uniformly in k. Since $\rho_0 \in \mathcal{S}$, (4.8) ensures that $\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0] < \infty$, and provides a bound depending only on $\mathcal{E}[\rho_0]$, $\int_{\mathbb{R}^2}|x|\rho_0$. Then (4.7) provides a universal lower bound on $\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}]$, and thus by (4.10), there is a finite constant K_1 depending only on $\mathcal{E}[\rho_0]$, $\int_{\mathbb{R}^2}|x|\rho_0$ such that for all k,

$$\mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] \le K_1 \quad \text{and} \quad \mathrm{W}_2^2(\rho^{(k)},\rho_0) \le K_1 . \tag{4.11}$$

In particular, by the triangle inequality, for all k, $\mathrm{W}_2(\rho^{(k)},\varrho_\lambda) \le \sqrt{K_1} + \mathrm{W}_2(\rho_0,\varrho_\lambda) < \infty$.

We next bound the first moments of $\rho^{(k)}$ uniformly in k. Let $\nabla\varphi$ be the optimal transportation plan $\nabla\varphi\#\rho^{(k)} = \varrho_\lambda$ as in the Brenier-McCann Theorem. Then since $|x| \le |x - \nabla\varphi(x)| + |\nabla\varphi(x)|$ for all x, integrating against $\rho^{(k)}$ and using the Cauchy-Schwarz inequality yields

$$\int_{\mathbb{R}^2} |x|\,\rho^{(k)}(x)\,\mathrm{d}x \le \sqrt{8\,\pi}\,\mathrm{W}_2(\rho^{(k)},\varrho_\lambda) + \int_{\mathbb{R}^2}|x|\,\varrho_\lambda(x)\,\mathrm{d}x \le \sqrt{8\,\pi}\,[\sqrt{K_1} + \mathrm{W}_2(\rho_0,\varrho_\lambda)] + \int_{\mathbb{R}^2}|x|\,\varrho_\lambda(x)\,\mathrm{d}x .$$

The right hand side is finite and independent of k.

We next bound $\mathcal{E}[\rho^{(k)}]$. By part *(ii)* of Lemma 4.1, there is a constant C such that

$$\int_{\mathbb{R}^2} \rho^{(k)}(x)\log\rho^{(k)}(x)\,\mathrm{d}x \le \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho^{(k)}] + \frac{C\,(8\pi)^2}{\epsilon^2} \le \mathcal{F}^\epsilon_{\mathrm{PKS}}[\rho_0] + \frac{C\,(8\pi)^2}{\epsilon^2} \le K_1 + \frac{C\,(8\pi)^2}{\epsilon^2} ,$$

where we have used (4.11) once more. Again the right side is finite and independent of k.

The last two uniform bounds show that $\{\rho^{(k)}\}_k$ is uniformly integrable. Hence, by the Dunford-Pettis theorem, there exist a weakly in L^1 convergent sub-sequence whose limit we shall denote by ρ.

By a standard weak lower semicontinuity argument (see e.g. [1] for the weak lower semicontinuity of W_2^2), ρ satisfies each of the three bound that we have proved uniformly for $\{\rho^{(k)}\}_k$, and thus $\rho \in \mathcal{S}$.

It remains to prove that the functional $\mathcal{F}^\epsilon_{\mathrm{PKS}}$ is lower semi-continuous on $L^1(\mathbb{R}^2)$. For the entropy part, this is standard. For the self interaction part, this follows from Lemma 4.3. It follows from this that the weak limit ρ is a minimizer. ■

4.7 PROPOSITION (Strict positivity of the minimizers)**.** *Let ρ_0 satisfies the conditions of* Theorem 4.6. *Then any minimizer ρ of* (4.9) *is uniformly bounded below on compact sets,* i.e. *for all $R > 0$, there exists $s > 0$ such that*

$$\rho(x) \ge s \ \text{ almost everywhere in } \ D_R := \{x : |x| \le R\} . \tag{4.12}$$

Moreover, s does not depend on the chosen minimizer of Problem (4.9) *in case of non-uniqueness.*

4.8 Remark (Idea of the proof)**.** *Let us try to quantify this simple statement: Entropy abhors a vacuum. The functional derivative of $\mathcal{E}[\rho]$ is $\log\rho$. On any set where ρ is very close to zero, $\log\rho$ is very negative, and we can lower $\mathcal{E}[\rho]$ by transporting some mass from where ρ is relatively large to this spot where it is very small. This will lower the entropy by a very large multiple of the transported mass. On the other hand, if we do not have to transport the mass too far, the effects on $\mathrm{W}_2^2(\rho,\rho_0)$ and $\mathcal{W}_\epsilon(\rho)$ will be relatively small.*

Proof: For any $s > 0$, let

$$A_R(s) := \{x \in D_R : \rho(x) \geq s\} \quad \text{and} \quad C_R := \int_{A_R(2/R^2)} \rho(x)\,\mathrm{d}x\,.$$

For any $s > 0$, let $\alpha_R(s) := \{x \in D_R : \rho(x) \leq s\}$, and let $|\alpha_R(s)|$ denote its Lebesgue measure. By Theorem 4.6, $|x|\rho(x)$ is integrable, and hence

$$\int_{D_R} \rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \rho(x)\,\mathrm{d}x - \int_{|x|>R} \rho(x)\,\mathrm{d}x \geq 8\,\pi - \int_{\mathbb{R}^2} \frac{|x|}{R}\rho(x)\,\mathrm{d}x \geq 4\,\pi\ ,$$

as long as $4\pi R \geq \int_{\mathbb{R}^2} |x|\rho(x)\,\mathrm{d}x$.

If $|\alpha_R(s)| = 0$ for some $s > 0$, there is nothing to prove: ρ is bounded below uniformly by s on D_R. Therefore, suppose that $|\alpha_R(s)| > 0$ for all $s > 0$. Pick some small positive numbers δ and s, and define a new density $\widetilde{\rho}$ by transporting a mass $\delta\,C_R\,|\alpha_R(s)|$ from $A_R(2/R^2)$ to $\alpha_R(s)$, distributing it *uniformly* there, which raises the density there by $\delta\,C_R$. In formulas, choose $s < 2/R^2$ to have $\alpha_R(s) \cap A_R(2/R^2) = \emptyset$, and define a new density $\widetilde{\rho}$ by

$$\widetilde{\rho}(x) = \begin{cases} (1-\delta\,|\alpha_R(s)|)\,\rho(x) & x \in A_R(2/R^2)\ , \\ \rho(x) + \delta\,C_R & x \in \alpha_R(s)\ , \\ \rho(x) & \text{otherwise}\,. \end{cases}$$

In order to ensure positivity, we have to impose $\delta|\alpha_R(s)| \leq \delta\,\pi\,R^2 \leq 1/2$. In this way, it is easy to check that $\widetilde{\rho}$ is a density.

Note that $\|\widetilde{\rho}-\rho\|_1 \leq 2\,\delta\,|\alpha_R(s)|C_R$, and since all the modifications take place on D_R,

$$\|\widetilde{\rho}-\rho\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,\mathrm{d}x)} \leq \log(e+R^2)\,2\,\delta\,|\alpha_R(s)|\,C_R\ .$$

It now follows from the bounds on ρ derived Theorem 4.6 and from Lemma 4.3 that there is a constant K depending only on R, ϵ, $\mathcal{E}[\rho_0]$ and $\mathcal{H}_\lambda[\rho_0]$ such that

$$\mathcal{W}_\epsilon[\widetilde{\rho}] \leq \mathcal{W}_\epsilon[\rho] + \delta\,|\alpha_R(s)|\,K\ . \tag{4.13}$$

Using Taylor's expansion of $x \mapsto x\log x$, that $\log x$ is increasing and assuming $s \leq \delta\,C_R$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} [\widetilde{\rho}\log\widetilde{\rho} - \rho\log\rho]\,\mathrm{d}x &= \int_{A_R(2/R^2)} \{(1-\delta\,|\alpha_R(s)|)\,\rho\,\log\left[(1-\delta\,|\alpha_R(s)|)\,\rho\right] - \rho\log\rho\}\ \mathrm{d}x \\ &\quad + \int_{\alpha_R(s)} \left[(\rho+\delta\,C_R)\log\left(\rho+\delta\,C_R\right) - \rho\log\rho\right]\ \mathrm{d}x \\ &\leq -\delta\,|\alpha_R(s)| \int_{A_R(2/R^2)} \rho\,\log\left[(1-\delta\,|\alpha_R(s)|)\,\rho\right]\ \mathrm{d}x + \delta\,|\alpha_R(s)|\,C_R\log(2\delta\,C_R) \\ &\leq \delta\,|\alpha_R(s)|\,C_R\left[-\log\left(\frac{1}{R^2}\right) + \log(2\delta\,C_R)\right]\ ,\end{aligned} \tag{4.14}$$

where $\delta\,|\alpha_R(s)| \leq 1/2$ and $x \in A_R(2/R^2)$ were used in the last estimate.

To estimate the difference $\mathrm{W}_2^2(\widetilde{\rho},\rho_0) - \mathrm{W}_2^2(\rho,\rho_0)$, let Π denote the optimal coupling of ρ and ρ_0, and use it to define a non-optimal coupling $\widetilde{\Pi}$ of $\widetilde{\rho}$ and ρ_0. To do this, let μ be the measure supported on $A_R(2/R^2)$ with density

ρ, and hence total mass C_R. Let ν be the $\mathrm{d}x$–uniform distribution on $\alpha_R(s)$ with total mass C_R. Let $\nabla\psi$ be the optimal transportation plan with $\nabla\psi\#\mu = \nu$, and define the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x) = \begin{cases} \nabla\psi(x) & x \in A_R(2/R^2)\ , \\ x & \text{otherwise}\,. \end{cases}$$

Then $\widetilde{\Pi}$, given by $\widetilde{\Pi} = (1-\delta|\alpha_R(s)|)\Pi + \delta|\alpha_R(s)|(T\otimes \mathrm{Id})\#\Pi$ is a coupling of $\widetilde{\rho}$ and ρ_0, and hence

$$\begin{aligned} \mathrm{W}_2^2(\widetilde{\rho},\rho_0) &\le \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x-y|^2 \,\mathrm{d}\widetilde{\Pi}(x,y) \\ &= (1-\delta|\alpha_R(s)|)\,\mathrm{W}_2^2(\rho,\rho_0) + \delta|\alpha_R(s)| \iint_{\mathbb{R}^2\times\mathbb{R}^2} |T(x)-y|^2 \,\mathrm{d}\widetilde{\Pi}(x,y)\,. \end{aligned}$$

Then, since $|T(x)-y|^2 \le 2|T(x)-x|^2 + 2|x-y|^2$, and $|T(x)-x| \le 2R$, since all of the transportation induced by $\nabla\psi$ takes place inside D_R, it follows that

$$\mathrm{W}_2^2(\widetilde{\rho},\rho_0) \le (1+\delta|\alpha_R(s)|)\,\mathrm{W}_2^2(\rho,\rho_0) + \delta|\alpha_R(s)|(8\pi)^2\, 8R^2\ .$$

By bounds on ρ derived in the proof of Theorem 4.6, there is a constant $\tilde{K}$ depending only on $\mathcal{E}[\rho_0]$ and $\mathcal{H}_\lambda[\rho_0]$ such that $\mathrm{W}_2^2(\rho,\rho_0) \le \tilde{K}\tau$. Finally then, there is a constant depending only on R, τ, $\mathcal{E}[\rho_0]$ and $\mathcal{H}_\lambda[\rho_0]$ such that

$$\mathrm{W}_2^2(\widetilde{\rho},\rho_0) \le \mathrm{W}_2^2(\rho,\rho_0) + \delta|\alpha_R(s)|K\ . \tag{4.15}$$

Combining (4.14), (4.13) and (4.15) yields

$$\begin{aligned} \frac{\mathrm{W}_2^2(\widetilde{\rho},\rho_0)}{2\tau} + \mathcal{F}^{\epsilon}_{\mathrm{PKS}}[\rho] \le{}& \frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau} + \mathcal{F}^{\epsilon}_{\mathrm{PKS}}[\rho] \\ &+ \delta\,|\alpha_R(s)|\,C_R\left[-\log\left(\frac{1}{R^2}\right) + \log(2\delta C_R) + K'\right]\ , \end{aligned}$$

with a given constant K'. If $|\alpha_R(s)| > 0$ for all $s > 0$, then choosing δ small enough such that

$$-\log\left(\frac{1}{R^2}\right) + \log(2\delta\, C_R) + K' < 0$$

contradicts the optimality of ρ.

For instance, choosing $s_R = \delta/C_R$, all the above procedure can be done. Hence for some ρ is bounded below by s_R on D_R. This proves (4.12). Let us point out the the only information used about ρ is that it is a minimizer of the time-step variational problem (4.9). ■

We now continue the analysis of the minimizers ρ begun in Theorem 4.6. We obtained $\rho \in \mathcal{S}$ and the lower bound (4.12) directly from the variational principle, but to proceed, we need the Euler-Lagrange equation for the variational problem (4.9).

By the Brenier-McCann Theorem, there is a a lower semi-continuous convex function φ in $\mathbb{R}^2$ such that $\nabla\varphi\#\rho = \rho_0$, and $\nabla\varphi$ is uniquely determined on the support of ρ, which is all $\mathbb{R}^2$ by (4.12) . The Euler-Lagrange equation for (4.9) relates ρ, ρ_0 and $\nabla\varphi$:

4.9 LEMMA (Euler-Lagrange equation)**.** *Let ρ_0 satisfy the conditions of Theorem 4.6 and ρ be any minimizer for the variational problem in* (4.9)*, and let $\nabla\varphi$ be the unique gradient of a lower semi-continuous convex function such that $\nabla\varphi\#\rho = \rho_0$. Then the distributional gradient of ρ satisfies*

$$-\nabla\rho + \rho\nabla c_\epsilon = \frac{\mathrm{id} - \nabla\varphi}{\tau}\rho \tag{4.16}$$

where $c_\epsilon = G_\epsilon * \rho$. In particular, since c_ϵ is differentiable everywhere, and φ is differentiable almost everywhere, ρ is differentiable almost everywhere.

Proof: We proceed as in [21], and choose any smooth compactly supported vector-field $\xi : \mathbb{R}^2 \to \mathbb{R}^2$. Then for any $s > 0$, let T_s be the map from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by $T_s = \mathrm{id} + s\xi$. Since ξ is smooth with compact support, for all sufficiently small s, T_s is invertible. For all $s > 0$, we have

$$\frac{\mathrm{W}_2^2(\rho,\rho_0)}{2\tau} + \mathcal{F}^{\epsilon}_{\mathrm{PKS}}[\rho] \le \frac{\mathrm{W}_2^2(T_s\#\rho,\rho_0)}{2\tau} + \mathcal{F}^{\epsilon}_{\mathrm{PKS}}[T_s\#\rho] \ . \tag{4.17}$$

Exactly as shown in [21],

$$\int_{\mathbb{R}^2} [T_s\#\rho]\log[T_s\#\rho](y)\,\mathrm{d}y = \int_{\mathbb{R}^2} \rho(x)\log\rho(x)\,\mathrm{d}x - s\int_{\mathbb{R}^2} \mathrm{div}\xi(x)\,\rho(x)\,\mathrm{d}x + \mathcal{O}(s^2) \tag{4.18}$$

and

$$\mathrm{W}_2^2(T_s\#\rho,\rho_0) \le \mathrm{W}_2^2(\rho,\rho_0) + 2s\int_{\mathbb{R}^2}[x-\nabla\varphi(x)]\cdot\xi(x)\rho(x)\,\mathrm{d}x + \mathcal{O}(s^2) \ . \tag{4.19}$$

Next, by definition of the push-forward,

$$\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2}[T_s\#\rho](x)\,G_\epsilon(x-y)[T_s\#\rho](y)\,\mathrm{d}x\,\mathrm{d}y &= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,G_\epsilon(T_s(x)-T_s(y))\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y\\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,G_\epsilon([x-y]+s\,[\xi(x)-\xi(y)])\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y\\
&= s\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,[\xi(x)-\xi(y)]\cdot\nabla G_\epsilon(x-y)\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y\\
&+ \iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(x)\,G_\epsilon(x-y)\,\rho(y)\,\mathrm{d}x\,\mathrm{d}y + \mathcal{O}(s^2) \ .
\end{aligned} \tag{4.20}$$

Now using (4.18), (4.20) and (4.19) in (4.17), we obtain

$$\begin{aligned}
0 \le &-\int_{\mathbb{R}^2}\mathrm{div}\xi(x)\,\rho(x)\,\mathrm{d}x - \frac{1}{2}\iint_{\mathbb{R}^4}\rho(x)\,[\xi(x)-\xi(y)]\cdot\nabla G_\epsilon(x-y)\,\rho(y)\,\mathrm{d}y\\
&+\int_{\mathbb{R}^2}\frac{x-\nabla\varphi(x)}{\tau}\cdot\xi(x)\,\rho(x)\,\mathrm{d}x \ .
\end{aligned}$$

Arguing as in [21], since this holds for all smooth compactly supported ξ, it holds also for $-\xi$, and so it holds with equality. Therefore, for smooth and compactly supported ξ,

$$\int_{\mathbb{R}^2}\mathrm{div}\ \xi(x)\,\rho(x)\,\mathrm{d}x + \iint_{\mathbb{R}^2\times\mathbb{R}^2}\nabla G_\epsilon(x-y)\,\rho(y)\,\mathrm{d}y\cdot\xi(x)\,\rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^2}\rho(x)\frac{x-\nabla\varphi(x)}{\tau}\cdot\xi(x)\,\mathrm{d}x \ .$$

This implies (4.16), and the rest is clear. ■

4.10 LEMMA (Qualitative regularity estimates)**.** *Let ρ_0 satisfy the conditions of Theorem* 4.6*, and let ρ be any minimizer for the variational problem in* (4.9)*. Then $\sqrt{\rho}$ has a square integrable distributional gradient, and for any $1 < p < \infty$, ρ_0^p is integrable.*

Proof: By the positivity of ρ, see Proposition 4.6, we can divide both sides of (4.16) by $\sqrt{\rho}$, to obtain

$$2\nabla\sqrt{\rho} = \left(\nabla c_\epsilon - \frac{x-\nabla\varphi}{\tau}\right)\sqrt{\rho} \ ,$$

where $\nabla\varphi$ is such that $\nabla\varphi\#\rho = \rho_0$. By the triangle inequality,

$$\begin{aligned}
2\|\nabla\sqrt{\rho}\|_2 &\le \left(\int_{\mathbb{R}^2}|\nabla c_\epsilon(x)|^2\,\rho(x)\,\mathrm{d}x\right)^{1/2} + \frac{1}{\tau}\left(\int_{\mathbb{R}^2}|x-\nabla\varphi(x)|^2\rho(x)\,\mathrm{d}x\right)^{1/2}\\
&\le \left(\int_{\mathbb{R}^2}|\nabla c_\epsilon(x)|^2\,\rho(x)\,\mathrm{d}x\right)^{1/2} + \frac{1}{\tau}\mathrm{W}_2(\rho,\rho_0) \ .
\end{aligned} \tag{4.21}$$

By Lemma 4.2, $\|\nabla c_\epsilon\|_\infty$ is uniformly bounded, and so is the first term of (4.21). This proves that $\sqrt{\rho}$ has a square integrable distributional gradient. The integrability of ρ_0^p is then a consequence of the following classical version of the GNS inequality valid for functions on $\mathbb{R}^2$ with $p \in [2, \infty)$

$$\int_{\mathbb{R}^2} |v|^p \,\mathrm{d}x \le D_p \left[\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x \right]^{p/2-1} \int_{\mathbb{R}^2} |v|^2 \,\mathrm{d}x$$

applied to $v = \sqrt{\rho}$. ■

4.11 Remark. *Although the bounds in Lemma 4.10 are not quantitative, and would certainly be ϵ dependent if we were to extract quantitative bounds, we shall use them only to justify certain integrations by parts, and otherwise show that subtraction of infinities does not invalidate computations that follow. Thus, these qualitative estimates are all we require concerning $\nabla\sqrt{\rho}$ and ρ. However, they are absolutely crucial for their purpose, and their necessity is the main reason we have had to introduce the regularized Green's function G_ϵ, and along with it, the regularized chemical attractant. Without the regularization, we would only know that $2\nabla\sqrt{\rho} - \nabla c\sqrt{\rho}$ was square integrable – but the possible cancellation effects would not allow us to conclude that $\nabla\sqrt{\rho}$ was square integrable.*

4.4 A discrete form of the entropy-entropy dissipation inequality

Our main goal in this subsection is to prove a discrete version of the entropy-entropy dissipation inequality (1.15). The key idea is to use the κ_λ displacement convexity of $\mathcal{H}_\lambda$ and the "above the tangent" inequality for convex functions as follows: For given initial density ρ_0, let ρ be any minimizer for the variational problem in (4.9). Let u_t, $0 \le t \le 1$ denote the displacement interpolation between ρ and ρ_0 starting at ρ and ending at ρ_0. Then $u_0 = \rho$ and $u_1 = \rho_0$. Since $\mathcal{H}_\lambda$ is displacement convex, the "above the tangent" inequality for convex functions says that

$$\mathcal{H}_\lambda[\rho] + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[u_t]\bigg|_{t=0} + \kappa_\lambda \,\mathrm{W}_2^2(\rho,\rho_0) \le \mathcal{H}_\lambda[\rho_0] \ .$$

A formal computation of the second term on the left would give, for $\epsilon = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\lambda[u_t]\bigg|_{t=0} = \mathcal{D}[\rho] \ .$$

Indeed, assuming Lemma 3.3 holds for $\delta = \epsilon = 0$ applied to $u_0 = \rho$ and $u_1 = \rho_0$, we get

$$\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \frac{1}{2}\int_{\mathbb{R}^2} \left[\kappa_\lambda x + \frac{\nabla\rho}{\rho^{3/2}}\right] \cdot (\nabla\varphi(x) - x)\,\rho \,\mathrm{d}x - \kappa_\lambda \,\mathrm{W}_2^2(\rho,\rho_0) \ .$$

Using (4.16), *i.e.* $(\nabla\varphi(x) - x)\rho = \tau\,(\nabla\rho - \rho\nabla c)$ and expanding, we can rewrite this as

$$\begin{aligned}\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \frac{\tau}{2}&\left[\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{\rho^{3/2}}\,\mathrm{d}x - \int_{\mathbb{R}^2} \frac{\nabla c\cdot\nabla\rho}{\sqrt{\rho}}\,\mathrm{d}x - \kappa_\lambda\int_{\mathbb{R}^2} x\cdot\nabla c\,\rho\,\mathrm{d}x + \kappa_\lambda\int_{\mathbb{R}^2} x\cdot\nabla\rho\,\mathrm{d}x\right]\\ &- \kappa_\lambda\,\mathrm{W}_2^2(\rho,\rho_0)\\ :=\mathcal{H}_\lambda[\rho_0] - \frac{\tau}{2}&\,[(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV})] - \kappa_\lambda\,\mathrm{W}_2^2(\rho,\rho_0) \ .\end{aligned}$$

Using $-\Delta c = \rho$ we have

$$(\mathrm{II}) = -2\int_{\mathbb{R}^2} \nabla c\cdot\nabla\sqrt{\rho}\,\mathrm{d}x = -2\int_{\mathbb{R}^2} \rho^{3/2}\,\mathrm{d}x \ .$$

Using the symmetrization argument we obtain

$$(\mathrm{III}) = \kappa_\lambda\frac{1}{4\pi}\int_{\mathbb{R}^2} \rho\,\mathrm{d}x = 16\pi\kappa_\lambda \ .$$

And by integration by parts (IV) $= -16\pi\kappa_\lambda$, resulting into $\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \tau\mathcal{D}[\rho] - \kappa_\lambda\,\mathrm{W}_2^2(\rho,\rho_0)$.

However, to do the calculation in a rigorous manner we must take into account that $\epsilon > 0$, and we must use the regularized entropy functional $\mathcal{H}_{\lambda,\delta}$. Before proceding with this, we point out that no such estimate can be given for $\mathcal{F}_{\mathrm{PKS}}$ since this functional is not displacement convex.

4.12 LEMMA (Convexity estimates at the regularized level). *Let ρ_0 satisfy the conditions of Theorem 4.6, and let ρ be any minimizer for the variational problem in (4.9), then*

$$\mathcal{H}_{\lambda,\delta}[\rho] \le \mathcal{H}_{\lambda,\delta}[\rho_0] - \frac{\tau}{2}\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{(\rho+\delta)^{3/2}} + \tau \int_{\mathbb{R}^2} \rho^{3/2}\,dx + 16\,\pi\,\sqrt{\delta}\tau + 16\pi\sqrt{\delta}(J_\gamma + 1 + \hat{C}_\epsilon)\tau$$
$$- 16\pi\kappa_\lambda\tau + 2C_\epsilon \,\|\,|2f'_\delta - \kappa_\lambda|\,(1+|x|)\rho\|_1\tau + \frac{16\pi\, J_\gamma}{\sqrt{2\lambda}}\tau + 2\tau\int_{\mathbb{R}^2} \nabla\cdot[x\, f'_\delta(|x|^2)]\rho(x)\,dx$$
$$- K_\delta(\rho_0,\rho)\ ,$$

and

$$\mathcal{H}_{\lambda,\delta}[\rho] \le \mathcal{H}_{\lambda,\delta}[\rho_0] - \frac{\tau}{2}\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{(\rho+\delta)^{3/2}} + \tau \int_{\mathbb{R}^2} \rho^{3/2}\,dx + 16\,\pi\,\sqrt{\delta}\tau + 16\pi\sqrt{\delta}(J_\gamma + 1 + \hat{C}_\epsilon)\tau$$
$$- 16\pi\kappa_\lambda\tau + 2C_\epsilon \,\|\,|2f'_\delta - \kappa_\lambda|\,(1+|x|)\rho\|_1\tau + \frac{32\pi\, C_{\mathrm{HLS}}}{\sqrt{2\lambda}}\,\sqrt{\epsilon}\,\|\,|x|\,\gamma\|_{4/3}\,\|\rho\|_{4/3}\tau$$
$$+ 2\tau\int_{\mathbb{R}^2} \nabla\cdot[x\, f'_\delta(|x|^2)]\rho(x)\,dx - K_\delta(\rho_0,\rho)\ ,$$

where K_δ is defined in Proposition 3.2 *and the constants J_γ, $\hat{C}_\epsilon$, and C_ϵ are explicit constants.*

Proof: This is an elaborate calculation in which a number of integrations by parts operations must be carefully examined for boundary behavior. It is relegated to the Appendix. ■

As a consequence of this lemma, letting δ go to 0, we obtain the following result concerning the dissipation of $\mathcal{H}_\lambda$ in one discrete time step.

4.13 COROLLARY (Convexity estimates). *Let ρ_0 satisfy the conditions of Theorem 4.6. If ρ is any minimizer for the variational problem in (4.9) then*

$$\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \tau\,\mathcal{D}[\rho] + \tau\,A\,\|\gamma\|_{4/3} - \kappa_\lambda \mathrm{W}_2^2(\rho,\rho_0)\ , \tag{4.22}$$

and

$$\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \tau\,\mathcal{D}[\rho] + \tau\sqrt{\epsilon}\,A\,\|\rho\|_{4/3} - \kappa_\lambda \mathrm{W}_2^2(\rho,\rho_0)\ . \tag{4.23}$$

where $A := 32\pi(2\lambda)^{-1/2}C_{\mathrm{HLS}}\,\|\,|x|\,\gamma\|_{4/3}$.

Proof: Let us first observe that

$$\int_{\mathbb{R}^2} \nabla\cdot[x\, f'_\delta(|x|^2)]\rho(x)\,dx = \int_{\mathbb{R}^2} \left[2f'_\delta(|x|^2) + 2|x|^2 f''_\delta(|x|^2)\right]\rho(x)\,dx.$$

Let us recall from the proof of Proposition 3.2 that $2f'_\delta(s) \nearrow \kappa_\lambda$ and $f''_\delta(s) \to 0$ as $\delta \to 0$ for all $s \ge 0$. Moreover, we have that $sf''_\delta(s)$ is a bounded function uniformly in δ from (3.19). These properties together with the dominated convergence theorem leads easily to

$$\|\,|2f'_\delta - \kappa_\lambda|\,(1+|x|)\rho\|_1 \to 0 \qquad \text{and} \qquad \int_{\mathbb{R}^2} \nabla\cdot[x\, f'_\delta(|x|^2)]\rho(x)\,dx \to 8\kappa_\lambda\pi$$

as $\delta \to 0$, since $(1+|x|)\rho \in L^1(\mathbb{R}^2)$. By monotone convergence theorem, we obtain

$$\lim_{\delta\to 0}\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{(\rho+\delta)^{3/2}} = \int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{\rho^{3/2}}.$$

Putting together all these facts and Proposition 3.2, we can pass to the limit as $\delta \to 0$ in Lemma 4.12 to get the desired estimates (4.23) and (4.22). ■

4.5 One-step estimates

Neither of the one step dissipation estimates that we have so far, namely (4.23) and (4.22), are exactly what we need. The problem is the term $\tau A\|\gamma\|_{4/3}$ in the first of these, and the term $\tau A\|\rho\|_{4/3}$ in the second of these. These terms might be large compared to the other terms so that these estimates might even give only "negative dissipation".

In the first main result of this subsection, we use one and then the other of these inequalities in combination with the controlled concentration inequality of Theorem 1.9 to produce the kind of dissipation estimate that we really want. In the second main result, we show that L^p norms of the densities are essentially propagated along each step of the discrete variational scheme. Again, Theorem 1.9 play a crucial role in both proofs.

4.14 THEOREM (One-step theorem). *Let ρ_0 satisfy the conditions of Theorem* 4.6*, and let ρ be any minimizer for the variational problem in* (4.9)*. Moreover, let ρ_0 satisfy the bound*

$$\mathcal{H}_\lambda[\rho_0] < C_\lambda \tag{4.24}$$

with C_λ defined Theorem 1.11 *and* Lemma 1.8*. Define $Q_0 > 0$, $\tau_0^\star > 0$ by*

$$Q_0 := C_\lambda - \mathcal{H}_\lambda[\rho_0] \quad \text{and} \quad \tau_0^\star := \min\left\{ \frac{Q_0}{2\,A\|\gamma\|_{4/3}}\ ,\ 1 \right\}\ , \tag{4.25}$$

where A is the constant given in Corollary 4.13*. Finally, given Q_0 and $0 < \tau \le \tau_0^\star$, and also any positive integer ℓ, let ϵ_ℓ be given by*

$$\tau^{1/3}\sqrt{\epsilon_\ell}\left[8\,\pi^{1/3}\,A\,\gamma_1^{-2/3}\,(\pi\,C_\lambda + \tau_0^\star C_{\rm CCD})^{2/3}\right] = \frac{Q_0}{4}\tau^2 2^{-\ell}\ . \tag{4.26}$$

Then for all $\tau \le \tau_0^\star$ and all $\epsilon = \epsilon_\ell$, ρ satisfies

$$\mathcal{H}_\lambda[\rho] < C_\lambda$$

and

$$\mathcal{H}_\lambda[\rho] - \mathcal{H}_\lambda[\rho_0] \le -\,\tau\mathcal{D}[\rho] + \frac{Q_0}{4}\tau^2 2^{-\ell} - \kappa_\lambda {\rm W}_2^2(\rho_0,\rho)\ . \tag{4.27}$$

Proof: By (4.22) our choice of τ and of Q_0 in (4.25) imply that

$$\mathcal{H}_\lambda[\rho] \le \mathcal{H}_\lambda[\rho_0] - \tau\mathcal{D}[\rho] + \frac{Q_0}{2} = C_\lambda - Q_0 - \tau\mathcal{D}[\rho] + \frac{Q_0}{2} \le C_\lambda - \tau\mathcal{D}[\rho]\ . \tag{4.28}$$

On one hand, the GNS inequality, see Lemma (1.2), implies $\mathcal{D}[\rho] \ge 0$ so that (4.28) implies that ρ also satisfies (4.24). On the other hand, since $\mathcal{H}_\lambda[\rho]$ cannot be negative it implies

$$\mathcal{D}[\rho] \le \frac{C_\lambda}{\tau}\ . \tag{4.29}$$

We can thus apply the concentration controlled inequality, Theorem 1.9 which implies, using (4.29)

$$\int_{\mathbb{R}^2} \left|\nabla\rho^{1/4}\right|^2 {\rm d}x \le \frac{1}{\gamma_1}\left[\pi\mathcal{D}[\rho] + C_{\rm CCD}\right] \le \frac{1}{\tau}\frac{1}{\gamma_1}\left[\pi C_\lambda + \tau_0^\star C_{\rm CCD}\right]\ .$$

By the GNS inequality of Lemma 1.2, we have

$$\int_{\mathbb{R}^2} \rho^{3/2}\,{\rm d}x \le 8\int_{\mathbb{R}^2}\left|\nabla\rho^{1/4}\right|^2 {\rm d}x \le \frac{1}{\tau}\frac{8}{\gamma_1}\left[\pi C_\lambda + \tau_0^\star C_{\rm CCD}\right] := \frac{C_3}{\tau}\ , \tag{4.30}$$

Next, by Hölder's inequality,

$$\int_{\mathbb{R}^2}\rho^{4/3}\,{\rm d}x = \int_{\mathbb{R}^2}\rho^{1/3}\rho\,{\rm d}x \le (8\pi)^{1/3}\left(\int_{\mathbb{R}^2}\rho^{3/2}\,{\rm d}x\right)^{2/3} \le (8\pi)^{1/3}\left(\frac{C_3}{\tau}\right)^{2/3}\ .$$

Now using this bound in (4.23), we obtain

$$\mathcal{H}_\lambda[\rho] - \mathcal{H}_\lambda[\rho_0] \le -\tau\mathcal{D}[\rho] + \tau^{1/3}\sqrt{\epsilon}\left[A\,(8\pi)^{1/3}C_3^{2/3}\right] - \kappa_\lambda{\rm W}_2^2(\rho_0,\rho).$$

We thus obtain the stated result by choosing $\epsilon = \epsilon_\ell$ for any positive integer ℓ. ■

4.15 LEMMA (Propagation of the L^p-norm)**.** *Let ρ_0 satisfy the conditions of Theorem* 4.6 *together with the condition* (4.24)*. Assume additionally that $\rho_0 \in L^p(\mathbb{R}^2)$, $2 \le p < \infty$, and let ρ be any minimizer for the variational problem in* (4.9)*, then there exist $K_0 > 0$ such that for all $K \ge K_0$*

$$\int_{\mathbb{R}^2} (\rho - K)_+^p \,\mathrm{d}x \le \int_{\mathbb{R}^2} (\rho_0 - K)_+^p \,\mathrm{d}x + \tau A_1 + \tau A_2 \mathcal{D}[\rho] \ ,$$

where A_1 and A_2 are universal positive constants depending on K. Moreover, K_0 only depends on $\displaystyle\int_{\mathbb{R}^2} \rho |\log \rho| \,\mathrm{d}x$.

Proof: The displacement convexity of the functional

$$\rho \mapsto \int_{\mathbb{R}^2} (\rho - K)_+^p \,\mathrm{d}x$$

with $2 \le p < \infty$ and $K > 0$, is easy to check using the McCann's criterion [27], see Section 3. The Euler-Lagrange equation of the variational scheme $(x - \nabla\varphi)\,\rho = -\tau\nabla\rho + \tau\rho\nabla c_\epsilon$ together with the standard first-order displacement convexity characterization [36, 1] imply

$$\begin{aligned}
\int_{\mathbb{R}^2} (\rho - K)_+^p \,\mathrm{d}x - \int_{\mathbb{R}^2} (\rho_0 - K)_+^p \,\mathrm{d}x &\le -p \int_{\mathbb{R}^2} \nabla \left[(\rho - K)_+^{p-1}\right] (\nabla\varphi - x)\,\rho \,\mathrm{d}x \\
&\le -(p-1)p\tau \int_{\mathbb{R}^2} \left|\nabla(\rho - K)_+\right|^2 (\rho - K)_+^{p-2} \,\mathrm{d}x \\
&\quad + (p-1)\tau \int_{\mathbb{R}^2} \nabla\left[(\rho - K)_+^p\right] \nabla c_\epsilon \,\mathrm{d}x + p\tau K \int_{\mathbb{R}^2} \nabla\left[(\rho - K)_+^{p-1}\right] \nabla c_\epsilon \,\mathrm{d}x \\
&\le -\frac{4(p-1)}{p}\tau \int_{\mathbb{R}^2} \left|\nabla\left[(\rho - K)_+^{p/2}\right]\right|^2 \mathrm{d}x + (p-1)\tau \int_{\mathbb{R}^2} (\rho - K)_+^p (-\Delta c_\epsilon) \,\mathrm{d}x \\
&\quad + p\tau K \int_{\mathbb{R}^2} (\rho - K)_+^{p-1} (-\Delta c_\epsilon) \,\mathrm{d}x := \tau(I_1 + I_2 + I_3) \ . \qquad (4.31)
\end{aligned}$$

The last two integration by parts have to be justified for any given ϵ working as in the proof of Lemma 4.12 in the Appendix. Integrating by parts on the ball of radius R, we obtain for any $k \in \{p, p-1\}$

$$\begin{aligned}
\int_{|x|\le R} \nabla\left[(\rho - K)_+^k\right] \nabla c_\epsilon \,\mathrm{d}x &= \int_{|x|\le R} (\rho - K)_+^k (-\Delta c_\epsilon) \,\mathrm{d}x + \int_{|x|=R} (\rho - K)_+^k \nabla c_\epsilon \cdot n \,\mathrm{d}\sigma \\
&\le \int_{|x|\le R} (\rho - K)_+^k (-\Delta c_\epsilon) \,\mathrm{d}x + \|\nabla c_\epsilon\|_\infty \int_{|x|=R} \rho^k \,\mathrm{d}\sigma \ .
\end{aligned}$$

As above, it is enough to show by dominated convergence theorem that there exists a sequence of radii $\{R_j\}_{j\in\mathbb{N}}$ such that the boundary terms tend to zero as $j \to \infty$. Due to Lemma 4.10 with $p \ge 2$, for any given natural $N > 1$, we can write that

$$\sum_{N=1}^{\infty} \int_{N-1}^{N} \int_{|x|=r} \rho^k \,\mathrm{d}\sigma \,\mathrm{d}r = \int_{\mathbb{R}^2} \rho^k \,\mathrm{d}x < \infty \ , \qquad \text{implying that} \qquad \lim_{N\to\infty} \int_{N-1}^{N} \int_{|x|=r} \rho^k \,\mathrm{d}\sigma \,\mathrm{d}r = 0 \,,$$

for $k \in \{p, p-1\}$, and the two integration by parts for any given ϵ are justified.

We now estimate I_2 and I_3, showing in particular that they are finite. Starting with I_2, using $-\Delta c_\epsilon = \rho_\epsilon$ where $\rho_\epsilon := \gamma_\epsilon * \rho * \gamma_\epsilon$, so that by Hölder's inequality and Young's inequality for convolutions, obtain

$$\int_{\mathbb{R}^2} (\rho - K)_+^p (-\Delta c_\epsilon) \,\mathrm{d}x = \int_{\mathbb{R}^2} (\rho - K)_+^p \rho_\epsilon \,\mathrm{d}x \le \|(\rho - K)_+\|_{p+1}^p \|\rho_\epsilon\|_{p+1} \le \|(\rho - K)_+\|_{p+1}^p \|\rho\|_{p+1} \qquad (4.32)$$

Likewise for I_3, we use the fact that on the support of $(\rho - K)_+$, $K \le \rho$. Therefore

$$K \int_{\mathbb{R}^2} (\rho - K)_+^{p-1} (-\Delta c_\epsilon) \,\mathrm{d}x = K \int_{\mathbb{R}^2} (\rho - K)_+^{p-1} \rho_\epsilon \,\mathrm{d}x \le \int_{\mathbb{R}^2} (\rho - K)_+^{p-1} \rho_\epsilon \rho \,\mathrm{d}x \ .$$

Therefore, by Hölder's inequality and Young's inequality for convolutions,

$$K\int_{\mathbb{R}^2}(\rho-K)_+^{p-1}(-\Delta c_\epsilon)\,\mathrm{d}x \le \|(\rho-K)_+\|_{p+1}^{p-1}\|\rho\|_{p+1}^2\ . \tag{4.33}$$

Applying the arithmetic-geometric mean inequality to the right side of (4.32), we have that for any $\nu>0$,

$$\|(\rho-K)_+\|_{p+1}^{p}\|\rho\|_{p+1} \le \frac{p}{p+1}\nu^{-(p+1)/p}\|(\rho-K)_+\|_{p+1}^{p+1}+\frac{1}{p+1}\nu^{p+1}\|\rho\|_{p+1}^{p+1}\ .$$

Making a similar estimate for the right hand side of (4.33), and combining results, we have that

$$I_2+I_3 \le F_1(\nu)\|(\rho-K)_+\|_{p+1}^{p+1}+F_2(\nu)\|\rho\|_{p+1}^{p+1} \tag{4.34}$$

where $F_1(\nu)$ is a positive linear combination of negative powers of ν, and $F_2(\nu)$ is a positive linear combination of positive powers of ν.

By Lemma 4.10, ρ^p is integrable for any $1\le p<\infty$, and so the right side of (4.34) is finite. Then by (4.31),

$$\int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x<\infty\,.$$

From here we show that

$$\int_{\mathbb{R}^2}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x \le \int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x+\frac{16K^{(2p-1)/2}}{\gamma_1}\left[\pi\mathcal{D}[\rho]+C_{\mathrm{CCD}}\right]\ . \tag{4.35}$$

Indeed,

$$\int_{\mathbb{R}^2}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x=\int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x-\int_{\{\rho<K\}}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x\ ,$$

and

$$\int_{\{\rho<K\}}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x\le 16K^{(2p-1)/2}\int_{\mathbb{R}^2}\left|\nabla\rho^{1/4}\right|^2\,\mathrm{d}x\le\frac{16K^{(2p-1)/2}}{\gamma_1}\left[\pi\mathcal{D}[\rho]+C_{\mathrm{CCD}}\right]\ ,$$

where we applied again the concentration controlled inequality, Theorem 1.9 using the hypothesis (4.24).

Following an idea of Jäger and Luckhaus [20], we use the GNS inequality

$$\int_{\mathbb{R}^2}v^{p+1}\,\mathrm{d}x\le D_p\left(\int_{\mathbb{R}^2}|\nabla v^{p/2}|^2\,\mathrm{d}x\right)\left(\int_{\mathbb{R}^2}v\,\mathrm{d}x\right)\ , \tag{4.36}$$

which is a consequence of the Sobolev embedding inequality $\|u\|_{L^2(\mathbb{R}^2)}\le C\|\nabla u\|_{L^1(\mathbb{R}^2)}$ applied to $u=v^{(p+1)/2}$ and Cauchy-Schwartz inequality since $\nabla u=\frac{p+1}{p}v^{1/2}\nabla v^{p/2}$. Applying (4.36) to $v=(\rho-K)_+$, we get

$$\int_{\mathbb{R}^2}(\rho-K)_+^{p+1}\,\mathrm{d}x\le M(K)\int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x\qquad\text{where } M(K):=\int_{\mathbb{R}^2}(\rho-K)_+\,\mathrm{d}x\ .$$

Then (4.34) becomes

$$I_2+I_3\le F_1(\nu)M(K)\int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x+F_2(\nu)\,8\pi\int_{\mathbb{R}^2}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x\,.$$

We finally work with I_1 to estimate it using (4.35) as

$$\frac{p}{2(p-1)}I_1\le-\int_{\mathbb{R}^2}\left|\nabla\left[(\rho-K)_+^{p/2}\right]\right|^2\,\mathrm{d}x-\int_{\mathbb{R}^2}\left|\nabla\rho^{p/2}\right|^2\,\mathrm{d}x+\frac{16K^{(2p-1)/2}}{\gamma_1}\left[\pi\mathcal{D}[\rho]+C_{\mathrm{CCD}}\right]\ .$$

Now choose $\nu_0>0$ small enough such that $8\pi F_2(\nu_0)<2(p-1)/p$, and then $K_0<\infty$ large enough such that $M(K)F_1(\nu_0)<2(p-1)/p$. This choice of K_0 only depends on ν_0 and the bound on $\int_{\mathbb{R}^2}\rho|\log\rho|\,\mathrm{d}x$ since

$$M(K)=\int_{\mathbb{R}^2}(\rho-K)_+\,\mathrm{d}x\le\int_{\rho>K}\rho\,\mathrm{d}x\le\frac{1}{\log K}\int_{\rho>K}\rho\log\rho\,\mathrm{d}x\le\frac{1}{\log K}\int_{\mathbb{R}^2}\rho\log^+\rho\,\mathrm{d}x\ .$$

We find

$$\int_{\mathbb{R}^2}(\rho - K)_+^p\,\mathrm{d}x - \int_{\mathbb{R}^2}(\rho_0 - K)_+^p\,\mathrm{d}x \le \tau\frac{32K^{(2p-1)/2}(p-1)}{p\gamma_1}\left[\pi\mathcal{D}[\rho] + C_{\mathrm{CCD}}\right] ,$$

for all $K \ge K_0$. The desired result follows with

$$A_1 = \frac{32K^{(2p-1)/2}(p-1)}{p\gamma_1}C_{\mathrm{CCD}} \qquad \text{and} \qquad A_2 = \frac{32K^{(2p-1)/2}(p-1)}{p\gamma_1}\pi\,.$$

■

5 Proof of the main results

5.1 Approximate solutions

We now combine the single step operations described in the previous section to inductively define infinite sequences $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ giving a discrete-time approximation to the PKS evolution. *For the rest of this section*, fix any $\lambda > 0$, and any density ρ_0 in $\mathbb{R}^2$ with total mass 8π such that $\rho_0 \log \rho_0$ and such that

$$\mathcal{H}_\lambda[\rho] \le \frac{\sqrt{\lambda}}{128\sqrt{2}\pi} = C_\lambda\ . \tag{5.1}$$

It then follows from Lemma 1.10 that $|x|\rho_0$ is integrable, and from Lemma 3.4 that $\mathrm{W}_2(\rho_0, \varrho_\lambda) < \infty$. Thus, ρ_0 satisfies the conditions of Theorem 4.6 on the existence of minimizers for our single step variational problem.

Then, since (5.1) is satisfied, we may choose $0 < \gamma_1 < 1$ so that $\mathcal{H}_\lambda[\rho] \le C_\lambda$, the constant in the condition for applicability of Theorem 1.9, the concentration control theorem for $\mathcal{D}[\rho]$. Thus, ρ_0 satisfies the conditions of Theorem 4.14, which provides us our single step estimates.

Fixing an arbitrarily small parameter $\tau > 0$, we now inductively define the sequence of densities $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ with $\rho_\tau^0 := \rho_0$ by solving the sequence of variational problems

$$\rho_\tau^k \in \arg\min_{\rho\in\mathcal{S}}\left\{\frac{\mathrm{W}_2^2(\rho,\rho_\tau^{k-1})}{2\tau} + \mathcal{F}_{\mathrm{PKS}}^{\epsilon_k}[\rho]\right\} \tag{5.2}$$

for a sequence of regularization parameters $\{\epsilon_k\}_{k\in\mathbb{N}}$ to be specified now. By Thereom 4.6, the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ is well defined no matter how we choose $0 < \tau < 1$ and $\{\epsilon_k\}_{k\in\mathbb{N}}$. However, we must make these choices carefully to ensure

$$Q_k := C_\lambda - \mathcal{H}_\lambda[\rho_\tau^k] > 0 \qquad \text{for each } k\,. \tag{5.3}$$

5.1 LEMMA (Good step sizes)**.** *Let* $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ *be the sequence of minimizers defined inductively using* (5.2) *starting from* $\rho_\tau = \rho_0$*. With* Q_k *defined as in* (5.3)*, let* A *be the constant given in* Corollary 4.13*, and let* Λ *be defined by*

$$\Lambda := \prod_{m=1}^{\infty}\left(1 - \frac{2^{-m}}{4}\right) ,$$

and note that $1 > \Lambda > 0$*. Choose any* $\tau > 0$ *satisfying*

$$\tau < \min\left\{\frac{\Lambda Q_0}{2\,A\|\gamma\|_{4/3}}, 1\right\} := \tau^\star\ , \tag{5.4}$$

and define ϵ_k *by*

$$\tau^{1/3}\sqrt{\epsilon_k}\left[8\,\pi^{1/3}\,A\,\gamma_1^{-2/3}\,(\pi\,C_\lambda + C_{\mathrm{CCD}})^{2/3}\right] = \frac{Q_0}{4}\tau^2 2^{-k}\,. \tag{5.5}$$

Then for all k*,* $Q_k > \Lambda Q_0 > 0$*, and in particular,* $\mathcal{H}_\lambda[\rho_\tau^k] \le C_\lambda$*. Note that for some constant* Z*,* $\epsilon_k := Z\,\tau^{10/3}\,4^{-k}$*.*

Proof: We shall show by induction that each for each positive integer j

$$Q_j \geq \prod_{m=1}^{j} \left(1 - \frac{2^{-m}}{4}\right) Q_0 \ , \tag{5.6}$$

which is somewhat more than we need since the right hand side is larger than ΛQ_0.

We now make the inductive hypothesis that for some positive integer k, (5.6) is true for all positive integers $j < k$. Since $\Lambda < 1$, we may apply Theorem 4.14 with ρ_τ^{k-1} in place of ρ_0, and ρ_τ^k in place of ρ and with τ and ϵ_k specified as above. Then the conclusion (4.27) can be simplified and rewritten as

$$\mathcal{H}_\lambda[\rho_\tau^k] \leq \mathcal{H}_\lambda[\rho_\tau^{k-1}] + \tau^2 \frac{Q_{k-1}}{4} 2^{-k} \ . \tag{5.7}$$

Since $\tau < 1$, this means that

$$Q_k \geq Q_{k-1} \left(1 - \frac{2^{-k}}{4}\right) \ .$$

By the inductive hypothesis, we obtain (5.6) for $j = k$. The proof that (5.6) is valid for $j = 1$ is a direct application of Theorem 4.14, in the same way, since $\Lambda < 1$. ■

5.2 The passage to continuous time

Throughout the rest of this section, we assume that $0 < \tau < \tau^\star$*, where* $\tau^\star$ *is defined in* (5.4)*, and that* ϵ_k *is defined by* (5.5)*, and then that* $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ *is a corresponding sequence of minimizers of* (5.2).

We now interpolate between the terms of the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ to produce a function from $[0,\infty)$ to $L^1(\mathbb{R}^2)$ that we shall show to be, for sufficiently small τ, an approximate solution of the PKS system. For technical reasons, we shall need two distinct, but closely related, interpolations.

- *The Lipschitz interpolation:* For each positive integer k, let $\nabla\varphi^k$ be the optimal transportation plan with $\nabla\varphi^k \# \rho_\tau^k = \rho_\tau^{k-1}$. Then for $(k-1)\tau \leq t \leq k\tau$ we define

$$\rho_\tau(t) = \left(\frac{t-(k-1)\tau}{\tau}\mathrm{Id} + \frac{k\tau - t}{\tau}\nabla\varphi^k\right)\#\rho_\tau^k \ .$$

- *The piecewise constant interpolation:* For each t and each positive integer k with $(k-1)\tau \leq t < k\tau$ we define $\tilde{\rho}_\tau(t) = \rho_\tau^{k-1}$, with $\tilde{\rho}_\tau(0) = \rho_0$.

For displacement convex functionals of ρ, such as $\mathcal{H}_\lambda[\rho]$, $\mathcal{E}[\rho]$, or the absolute first moment, any uniform bounds on the functional along the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ extend to $\rho(t)$ for all t, since if $\mathcal{G}$ is such a functional, then for $(k-1)\tau < t < k\tau$,

$$\mathcal{G}[\rho_\tau(t)] \leq \frac{t-(k-1)\tau}{\tau}\mathcal{G}[\rho_\tau^{k-1}] + \frac{k\tau - t}{\tau}\mathcal{G}[\rho_\tau^k] \ .$$

Of course it is evident that for *any* sort of functional $\mathcal{G}[\rho]$, displacement convex or not, a uniform bound on $\mathcal{G}[\rho]$ along the sequence $\{\rho_\tau^k\}_{k\in\mathbb{N}}$ extends to $\tilde{\rho}(t)$ for all t. Some of the functionals with which we work, such as $\mathcal{D}[\rho]$, are not displacement convex, and this is the reason we need the second interpolation.

The uniform equicontinuity properties that we prove next explain the utility of the first interpolation, and also why we can use the two different interpolations at once. Since ρ_τ^k is a minimizer for (5.2), using ρ_τ^{k-1} as trial function yields

$$\mathcal{F}_{\mathrm{PKS}}^{\epsilon_k}[\rho_\tau^k] + \frac{1}{2\,\tau}\mathrm{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1}) \leq \mathcal{F}_{\mathrm{PKS}}^{\epsilon_k}[\rho_\tau^{k-1}] \ ,$$

and hence,

$$\mathrm{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1}) \leq 2\tau\left[\mathcal{F}_{\mathrm{PKS}}^{\epsilon_k}[\rho_\tau^{k-1}] - \mathcal{F}_{\mathrm{PKS}}^{\epsilon_k}[\rho_\tau^k]\right] \ . \tag{5.8}$$

In standard applications of the JKO scheme, in which the functional in the variational problem does not change from step to step, one would sum both sides in (5.8) over a range of values of k, and then the sum of the terms on the right would telescope. This is not so in our case. However, for small ϵ, $\mathcal{F}^{\epsilon}_{\rm PKS} \approx \mathcal{F}_{\rm PKS}$ and we recover the telescoping sum in a useful approximate sense. The precise version of $\mathcal{F}^{\epsilon}_{\rm PKS} \approx \mathcal{F}_{\rm PKS}$ follows from (4.6), Lemma 4.5, and (5.5), which says that $\epsilon_k = Z\tau^{10/3}4^{-k}$ to get

$$\begin{aligned}\mathcal{F}^{\epsilon_k}_{\rm PKS}[\rho^{k-1}_\tau] - \mathcal{F}_{\rm PKS}[\rho^{k-1}_\tau] &\le 16\sqrt{5\pi}Z\|\rho^{k-1}_\tau\|^{3/2}_{3/2}\tau^{10/3}4^{-k}\left(|\log(2Z)| + \frac{10}{3}|\log(\tau)| + k\log 4\right)\\ &\le \tilde{Z}\|\rho^{k-1}_\tau\|^{3/2}_{3/2}\tau^3 2^{-k}\end{aligned} \tag{5.9}$$

for $\tau < \tilde{\tau}^\star := \min(\tau^\star, (2Z\sqrt{e})^{-3/10})$ according to Lemma 4.5 and since $\tau|\log\tau|$ and $k\,2^{-k}$ are bounded for $\tau < 1$ and k positive integer. We thus deduce

$$\mathrm{W}_2^2(\rho^k_\tau, \rho^{k-1}_\tau) \le 2\tau\left[\mathcal{F}_{\rm PKS}[\rho^{k-1}_\tau] - \mathcal{F}_{\rm PKS}[\rho^k_\tau]\right] + 2\tilde{Z}\|\rho^{k-1}_\tau\|^{3/2}_{3/2}\tau^4 2^{-k}\,.$$

Using (4.30) as in the proof of Theorem 4.14 where the concentration control inequality (1.9) is crucial, we deduce that $\|\rho^{k-1}_\tau\|^{3/2}_{3/2} \le C_3/\tau$ to conclude that

$$\mathrm{W}_2^2(\rho^k_\tau, \rho^{k-1}_\tau) \le 2\tau\left[\mathcal{F}_{\rm PKS}[\rho^{k-1}_\tau] - \mathcal{F}_{\rm PKS}[\rho^k_\tau]\right] + 2\tilde{Z}C_3\tau^3 2^{-k}\,. \tag{5.10}$$

We are almost in a position to obtain a crucial *a-priori* Hölder continuity estimate, but there is still one more consequence of our step dependent regularization to deal with: If for each k we had been using the functional $\mathcal{F}_{\rm PKS}$ instead of the functional $\mathcal{F}^{\epsilon_k}_{\rm PKS}$, it would be immediate that $k \mapsto \mathcal{F}_{\rm PKS}[\rho^k_\tau]$ would be decreasing. Since by the Log-HLS inequality, $\mathcal{F}_{\rm PKS}$ is bounded below, this would give an immediate upper bound on the sum of the right hand side of (5.10) over any range of k.

However, we have used our freedom to choose the sequence $\{\epsilon_k\}_{k\in\mathbb{N}}$ of regularization parameters to converge to zero as rapidly as we may require, and hence easily obtain:

5.2 LEMMA (The free energy $\mathcal{F}_{\rm PKS}$ is almost decreasing along $\{\rho^k_\tau\}_{k\in\mathbb{N}}$)**.** *There are positive constants $\bar{\mathcal{F}}_0$, $\bar{\mathcal{F}}_1$ depending only on the initial data and the regularization mollifier γ such that for each $\tau < \tilde{\tau}^\star$ and each $k \in \mathbb{N}$,*

$$\mathcal{F}_{\rm PKS}(\rho^k_\tau) \le \bar{\mathcal{F}}_0 + \bar{\mathcal{F}}_1\tau^2.$$

Proof: Directly from the variational problem (5.2) we have $\mathcal{F}^{\epsilon_k}_{\rm PKS}[\rho^k_\tau] \le \mathcal{F}^{\epsilon_k}_{\rm PKS}[\rho^{k-1}_\tau]$. Then, as above from (4.6), (5.9), and (4.30) we get

$$\mathcal{F}^{\epsilon_k}_{\rm PKS}[\rho^{k-1}_\tau] \le \mathcal{F}^{\epsilon_{k-1}}_{\rm PKS}[\rho^{k-1}_\tau] + \tilde{Z}C_3\tau^2 2^{-k}\,. \tag{5.11}$$

A telescoping sum argument yields $\mathcal{F}^{\epsilon_k}_{\rm PKS}[\rho^k_\tau] \le \mathcal{F}^{\epsilon_0}_{\rm PKS}[\rho_0] + \tilde{Z}C_3\tau^2$, and then one more application of (4.6) gives

$$\mathcal{F}_{\rm PKS}(\rho^k_\tau) \le \mathcal{F}^{\epsilon_0}_{\rm PKS}[\rho_0] + \tilde{Z}C_3\tau^2 \le \mathcal{E}[\rho_0] + 32\,\pi + 2\|\rho_0\|_{L^1(\mathbb{R}^2,\log(e+|x|^2)\,dx)} + \tilde{Z}C_3\tau^2 := \bar{\mathcal{F}}_0 + \bar{\mathcal{F}}_1\tau^2\,,$$

where (4.8) was used. ■

We are now ready to prove the Hölder continuity estimate.

5.3 LEMMA (Hölder continuity)**.** *There is a positive constant $\bar{\mathcal{F}}_2$ depending only on the initial data and the regularization mollifier γ such that for each $\tau < \tilde{\tau}^\star$ and each $k \in \mathbb{N}$, such that for all $t > s \ge 0$,*

$$\mathrm{W}_2(\rho_\tau(t), \rho_\tau(s)) \le \bar{\mathcal{F}}_2\,(t-s)^{1/2}\quad.$$

Proof: Let j be such that $(j-1)\tau \le s \le j\tau$ and let ℓ be such that $\ell\tau \le t \le (\ell+1)\tau$. By the geodesic property of McCann's displacement interpolation,

$$\mathrm{W}_2(\rho_\tau(s), \rho^j_\tau) = \frac{j\tau - s}{\tau}\mathrm{W}_2(\rho^{j-1}_\tau, \rho^j_\tau) \qquad\text{and}\qquad \mathrm{W}_2(\rho_\tau(t), \rho^\ell_\tau) = \frac{t-\ell\tau}{\tau}\mathrm{W}_2(\rho^\ell_\tau, \rho^{\ell+1}_\tau)\,.$$

By Lemma 5.2 and the Log-HLS inequality (4.7), $\mathcal{F}_{\rm PKS}[\rho_\tau^k] - \mathcal{F}_{\rm PKS}[\rho_\tau^{k-1}] \le \bar{\mathcal{F}}_0 + \bar{\mathcal{F}}_1\tau^2 - (\log 8 - 1)$, and thus, by plugging into (5.10), we get

$$\mathrm{W}_2^2(\rho_\tau^k, \rho_\tau^{k-1}) \le 2\tau\left[\bar{\mathcal{F}}_0 + \bar{\mathcal{F}}_1\tau^2 - (\log 8 - 1)\right] + 2\tilde{Z}C_3\tau^3 2^{-k} \le \tau\left[2\bar{\mathcal{F}}_0 + 2\bar{\mathcal{F}}_1 - 2(\log 8 - 1) + 2\tilde{Z}C_3\right] := \bar{\mathcal{F}}_2^2\tau \quad (5.12)$$

since $\tau < 1$ and $k \in \mathbb{N}$. Therefore, we deduce

$$\mathrm{W}_2(\rho_\tau(s), \rho_\tau^j) \le \frac{j\tau - s}{\tau}\bar{\mathcal{F}}_2\sqrt{\tau} \qquad \text{and} \qquad \mathrm{W}_2(\rho_\tau(t), \rho_\tau^\ell) \le \frac{t - \ell\tau}{\tau}\bar{\mathcal{F}}_2\sqrt{\tau}\,.$$

Adding these two estimates and using the concavity of square root,

$$\mathrm{W}_2(\rho_\tau(s), \rho_\tau^j) + \mathrm{W}_2(\rho_\tau(t), \rho_\tau^\ell) \le \bar{\mathcal{F}}_2\sqrt{(t-s) - (\ell - j)\tau}\,. \quad (5.13)$$

Next, by the triangle inequality, the Cauchy-Schwartz inequality, and (5.10) and proceeding as in (5.12), we finally conclude

$$\mathrm{W}_2(\rho_\tau^\ell, \rho_\tau^{j-1}) \le \sum_{k=j}^{\ell}\mathrm{W}_2(\rho_\tau^k, \rho_\tau^{k-1}) \le (\ell - j)^{1/2}\left(\sum_{k=j}^{\ell}\left(2\tau\left[\mathcal{F}_{\rm PKS}[\rho_\tau^{k-1}] - \mathcal{F}_{\rm PKS}[\rho_\tau^k]\right] + 2\tilde{Z}C_3\tau^3 2^{-k}\right)\right)^{1/2}$$
$$\le [(\ell - j)\tau]^{1/2}\left(2\left[\mathcal{F}_{\rm PKS}[\rho_\tau^{j-1}] - \mathcal{F}_{\rm PKS}[\rho_\tau^k]\right] + 2\tilde{Z}C_3\tau^2\right)^{1/2} \le \bar{\mathcal{F}}_2\,[(\ell - j)\tau]^{1/2}.$$

Adding this to the estimate in (5.13), and using the subadditivity of the square root concludes the proof. ■

5.3 Compactness

In this subsection, we will show the compactness of the sequence of interpolating curves. We cannot proceed as usually done, for instance in [1], since we want to show that the limiting curves are not only measures but rather densities for each time and also since our densities have infinite second moment. The next lemma shows a compactness property of the sets $\{\rho_\tau(t) \ : \ 0 < \tau < \tilde{\tau}^\star\}$ for each fixed t.

5.4 LEMMA (Uniform integrability at fixed t)**.** *There is a finite and computable constant $\bar{\mathcal{F}}_3$ depending only on ρ_0 and for any fixed $1 \le p < 2$ so that for all $\tau < \tilde{\tau}^\star$*

$$\mathcal{E}[\rho_\tau(t)] = \int_{\mathbb{R}^2}\rho_\tau(t,x)\log\rho_\tau(t,x)\,\mathrm{d}x \le \bar{\mathcal{F}}_3 \qquad \textit{and} \qquad \int_{\mathbb{R}^2}|x|^p\rho_\tau(t,x)\,\mathrm{d}x \le \bar{\mathcal{F}}_3(1 + t^{p/2})\,.$$

Proof: By the uniform control of $\mathcal{H}_\lambda$ in (5.3), Lemma 5.2, and by the first concentration control Theorem 1.11 we conclude that

$$\gamma_2\int_{\mathbb{R}^2}\rho_\tau^k\log_+\rho_\tau^k\,\mathrm{d}x \le \bar{\mathcal{F}}_0 + \bar{\mathcal{F}}_1\tau^2 + C_{\rm CCF} \quad (5.14)$$

where $0 < \gamma_2 \le 1$, uniformly in k.

Next, using the triangle inequality for W_p defined in (3.4) and (3.5), we deduce

$$\mathrm{W}_p^p(\rho_\tau(t), \delta_0) = \int_{\mathbb{R}^2}|x|^p\rho_\tau(t,x)\,dx \le 2^p\left(\mathrm{W}_p^p(\varrho_\lambda, \delta_0) + \mathrm{W}_p^p(\rho_\tau(t), \varrho_\lambda)\right) = 2^p\int_{\mathbb{R}^2}|y|^p\varrho_\lambda(y)\,\mathrm{d}y + 2^p\,\mathrm{W}_p^p(\rho_\tau(t), \varrho_\lambda)\,.$$

Then by (3.6), the last term on the right is bounded by $\mathrm{W}_2^p(\rho_\tau(t), \varrho_\lambda)$. Since $\mathrm{W}_2(\rho_0, \varrho_\lambda)$ is finite, and since by the previous lemma, $\mathrm{W}_2(\rho_\tau(t), \rho_0) \le \bar{\mathcal{F}}_2\sqrt{t}$, the proof of the moment bound is completed by simple computations.

Finally, using the bound of the absolute first moment together with (5.14) and Lemma 2.4, we conclude that $\mathcal{E}[\rho_\tau^k]$ is bounded uniformly in k. Then, by the displacement convexity of $\mathcal{E}$, this bound extends to $\rho_\tau(t)$ for all $t > 0$, as explained at the beginning of this subsection. ■

It follows immediately from (3.6) that Lemma 5.3 remains true if W_2 there is replaced by any of the weaker metrics W_p, $1 \le p < 2$. The following characterization of the convergence in W_p metrics in [36, Chapter 9]: convergence of the absolute p-moment plus the weak-* convergence as measures of a sequence of densities $\{\rho_n\}_{n\in\mathbb{N}}$ towards ρ is equivalent to W_p convergence; implies the following compactness result.

5.5 LEMMA (Compactness for the W_p metric)**.** *For any $M > 0$, let K be a subset of the set of densities ρ of mass M that is uniformly integrable, and such that $\{|x|^p\rho(x) \ : \ \rho \in K\}$ is also uniformly integrable. Suppose also that K is closed in the L^1-weak topology. Then K is compact in the W_p metric.*

Proof: Let $\{\rho_n\}_{n\in\mathbb{N}}$ be any sequence in K. Since K is uniformly integrable and weakly closed in L^1, the Dunford-Pettis Theorem provides us with a $\rho \in K$ and a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \rho_{n_k} = \rho$ weakly in L^1 and thus, weakly-* as measures. It is trivial to check that weak-L^1 convergence plus the uniform integrability of $\{|x|^p\rho(x) \ : \ \rho \in K\}$ implies that

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} |x|^p \rho_{n_k}(x)\, \mathrm{d}x = \int_{\mathbb{R}^2} |x|^p \rho(x)\, \mathrm{d}x\,.$$

The characterization of W_p convergence mentioned above then implies $\lim_{k\to\infty} \mathrm{W}_p(\rho_{n_k}, \rho) = 0$. ■

5.6 THEOREM (Convergence as $\tau \to 0$)**.** *Given $T > 0$ and any $1 \le p < 2$, define $(\mathcal{M}_T^\delta, \mathrm{W}_p)$ to be the metric space in which $\mathcal{M}_T^\delta$ is the set of densities on $\mathbb{R}^2$ satisfying $\mathcal{E}[\rho] \le \bar{\mathcal{F}}_3$ and*

$$\int_{\mathbb{R}^2} |x|^{p+\delta} \rho(x)\, \mathrm{d}x \le \bar{\mathcal{F}}_3 (1 + T^{(p+\delta)/2})\,,$$

with $p < p + \delta < 2$. Then there is a function ρ on $[0, \infty)$ with values in the set of densities of mass 8π such that for all $T > 0$, the restriction of ρ to $[0, T]$ is continuous in $(\mathcal{M}_T^\delta, \mathrm{W}_p)$, and there is a sequence $\{\tau^n\}_{n\in\mathbb{N}}$ such that for all $T > 0$,

$$\lim_{n\to\infty} \left[\max_{0\le t\le T} \mathrm{W}_p(\rho_{\tau^n}(t), \rho(t)) \right] = \lim_{n\to\infty} \left[\max_{0\le t\le T} \mathrm{W}_p(\tilde{\rho}_{\tau^n}(t), \rho(t)) \right] = 0\ . \tag{5.15}$$

Moreover the sequence $\{\tau^n\}_{n\in\mathbb{N}}$ can be chosen independently of p, i.e., such that the convergence property (5.15) *holds for all $1 \le p < 2$. Furthermore, as a consequence for all $t > s \ge 0$ and all $1 \le p \le 2$:*

$$\mathrm{W}_p(\rho(t), \rho(s)) \le \bar{\mathcal{F}}_2\, (t - s)^{1/2}\ . \tag{5.16}$$

Proof: For each $T > 0$, $(\mathcal{M}_T^\delta, \mathrm{W}_p)$ is a compact metric space as a consequence of Lemma 5.5. By Lemma 5.4, for each $t \le T$, and each $0 < \tau < \tilde{\tau}_\star$, the restriction of ρ_τ to $[0, T]$ takes values in $(\mathcal{M}_T^\delta, \mathrm{W}_p)$. Next, by (3.6) and Lemmas 5.3, the set of these functions for $0 < \tau < \tilde{\tau}_\star$ is uniformly equicontinous into $(\mathcal{M}_T^\delta, \mathrm{W}_p)$. Thus by the the Arzela-Ascoli Theorem, we can select a uniformly convergent sequence.

Now a simple diagonal sequence argument concludes the construction of ρ and proof of (5.15). Apply the above for $T = 1$ to get the initial sequence. Now take $T = 2$ and choose a subsequence of the first sequence, and so forth. For the piecewise-interpolation sequence, note that $\mathrm{W}_2(\tilde{\rho}_\tau(t), \rho_\tau(t)) \le C\sqrt{\tau}$ by standard properties of displacement interpolation and (5.12) in Lemma 5.3.

Another simple diagonal sequence argument shows that the sequence can be made independent of p. Take a sequence of increasing exponents $\{p_n\}_{n\in\mathbb{N}} \nearrow 2$ and $\{\delta_n\}_{n\in\mathbb{N}} \searrow 0$ with $1 \le p_n < p_n + \delta_n < 2$ and apply the diagonal sequence argument to the constructed sequences for each p_n. Also, take into account that the sequence of distances $\mathrm{W}_p(\rho_{\tau^n}(t), \rho(t))$ is increasing in p.

The last part of the claim (5.16) follows directly from (5.15) and Lemma 5.3 for all $1 \le p < 2$. Since the constant $\bar{\mathcal{F}}_2$ obtained in Lemma 5.3 does not depend on $1 \le p < 2$, then we conclude (5.16) for $p = 2$. ■

5.4 L^p regularity

Our goal in this section is to prove:

5.7 THEOREM (L^p-regularity)**.** *For each finite $a > 0$ and $p > 1$, there is a finite and computable constant C_p depending only on a, p and ρ_0 such that whenever $\tau < a$,*

$$\int_{\mathbb{R}^2} \rho_\tau^p(t, x)\, \mathrm{d}x \le \tilde{C}_p \qquad \textit{for all } t \ge a\,.$$

Proof: From (4.27), we deduce that for each m, $\mathcal{H}_\lambda[\rho_\tau^m] + \mathcal{D}[\rho_\tau^m] \leq \mathcal{H}_\lambda[\rho_\tau^{m-1}] + \frac{Q_m}{4}2^{-m}\tau^2$, proceeding in the same way that we did in deducing (5.7), except this time we do not discard the dissipation term. Let $n \geq k$ be positive integers. Since $Q_m \leq C_\lambda$ for all m, summing from $m = n-k$ to n yields

$$\mathcal{H}_\lambda[\rho_\tau^n] + \sum_{m=n-k}^{n} \tau\mathcal{D}[\rho_\tau^m] \leq \mathcal{H}_\lambda[\rho_\tau^{n-k-1}] + \frac{C_\lambda}{4}\tau^2 \,. \tag{5.17}$$

Then since $0 \leq \mathcal{H}_\lambda[\rho_\tau^k] \leq C_\lambda$ for all k, using $\tau < 1$ and dividing by $k\tau$, we may simplify this to

$$\frac{1}{k}\sum_{m=n-k}^{n} \mathcal{D}[\rho_\tau^m] \leq \frac{2C_\lambda}{k\tau} \,. \tag{5.18}$$

We now choose k to be the greatest integer less than or equal to a/τ, and of course suppose that $n > k$. Since $k\tau \leq a < (k+1)\tau$, and $k \geq 1$, $a/2 \leq k\tau$, and then the fact that averages dominate minima yields the conclusion that for some positive integer m with $\tau \leq a$,

$$\mathcal{D}[\rho_\tau^m] \leq \frac{4C_\lambda}{a} \,.$$

Then since $\mathcal{H}_\lambda[\rho_\tau^m] \leq C_\lambda$ we have from Theorem 1.9 that

$$\int_{\mathbb{R}^2} |\nabla(\rho_\tau^m)^{1/4}|^2 \,\mathrm{d}x \leq \frac{4\pi C_\lambda}{a\gamma_2} + \frac{C_{\mathrm{CCD}}}{\gamma_1} \,.$$

Recall the GNS inequality

$$\|f\|_q^q \leq B_q \|\nabla f\|_2^{q-4}\|f\|_4^4 \,,$$

valid for locally integrable functions f on $\mathbb{R}^2$ and $q > 4$. Applying this with $q = 4p$ and $f = (\rho_\tau^m)^{1/4}$, we obtain

$$\|\rho_\tau^m\|_p^p \leq \left[B_{4p}\left(\frac{4\pi C_\lambda}{a\gamma_2} + \frac{C_{\mathrm{CCD}}}{\gamma_1}\right)^{2(p-1)} 8\pi \right]^{1/4} \,. \tag{5.19}$$

Thus we have an *a-priori* bound on $\|\rho_\tau^m\|_p^p$ for *some* m with $n-k \leq m \leq n$. We now apply Lemma 4.15, in which the constant A_1 and A_2 in (5.20) are defined, to conclude that

$$\int_{\mathbb{R}^2} (\rho_\tau^n - K)_+^p \,\mathrm{d}x \leq \int_{\mathbb{R}^2} (\rho_\tau^m - K)_+^p \,\mathrm{d}x + A_1 k\tau + A_2 \sum_{j=m}^{n} \tau\mathcal{D}[\rho_\tau^j] \,. \tag{5.20}$$

We bound the first term on the right in (5.20) using (5.19), the second using the fact that $k\tau \leq a$, and the third using (5.18). The result,

$$\int_{\mathbb{R}^2} (\rho_\tau^n - K)_+^p \,\mathrm{d}x \leq \left[B_{4p}\left(\frac{4\pi C_\lambda}{a\gamma_2} + \frac{C_{\mathrm{CCD}}}{\gamma_1}\right)^{2(p-1)} 8\pi \right]^{1/4} + A_1 a + 2A_2 C_\lambda$$

uniformly for all n such that $n\tau \geq a$. Note that the bound depends only on a, p. Since $\|\rho_\tau^n\|_p \leq \|(\rho_\tau^n - K)_+\|_p + K^{(p-1)/p}(8\pi)^{1/p}$, we have the same type of bound on $\|\rho_\tau^n\|_p$, uniformly for all n such that $n\tau \geq a$. By the displacement convexity of

$$\rho \mapsto \int_{\mathbb{R}^2} \rho^p(x) \,\mathrm{d}x$$

for $p > 1$, this bound immediately extends to $\rho_\tau(t)$ for all $t \geq a$. ■

5.5 Verification that $\rho = \lim_{\tau\to 0}\rho_\tau$ is a solution of the PKS system

Let τ^n, ρ_{τ^n} and ρ be given as in Theorem 5.6. Our goal in this subsection is to prove that ρ is a weak solution on the PKS system as given in item (1.5.2) of the Definition 1.5.

5.8 LEMMA (ρ is a weak solution of the PKS system). *Let τ^n, ρ_{τ^n} and ρ be given as in Theorem 5.6. Then for all smooth and compactly supported test functions η and all $t_2 > t_1 \geq 0$,*

$$\int_{\mathbb{R}^d}\zeta(x)\left[\rho(t_2,x)-\rho(t_1,x)\right]\,\mathrm{d}x = -\frac{1}{4\pi}\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(s,x)\,\rho(s,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}y\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Delta\zeta(x)\,\rho(s,x)\,\mathrm{d}x\,\mathrm{d}s\,,$$

In order to prove Lemma 5.8, we first prove an analog for the functions ρ_τ:

5.9 LEMMA (Approximate weak solutions of the PKS system). *For $0 < \tau < 1$, define $\epsilon(t,\tau) = \epsilon_k$ for $t \in ((k-1)\tau, k\tau]$, and all integers $k \geq 1$. Then for all smooth and compactly supported test functions η and all $t_2 > t_1 \geq 0$,*

$$\int_{\mathbb{R}^d}\zeta(x)\left[\rho_\tau(t_2,x)-\rho_\tau(t_1,x)\right]\mathrm{d}x = \frac{1}{2}\int_{t_1}^{t_2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho_\tau(s,x)\,\rho_\tau(s,y)\,\nabla G_{\epsilon(t,\tau)}(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))\,\mathrm{d}y\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Delta\zeta(x)\,\rho_\tau(s,x)\,\mathrm{d}x\,\mathrm{d}s + O(\tau^{1/2})\,.$$

Proof: By Lemma 4.9 for any ζ smooth and compactly supported $\nabla\zeta$, we have

$$\int_{\mathbb{R}^2}\rho_\tau^{k+1}(x)\frac{[x-\nabla\varphi^k(x)]}{\tau}\cdot\nabla\zeta(x)\,\mathrm{d}x = \int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho_\tau^{k+1}(x)\,\mathrm{d}x + \iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho_\tau^{k+1}(x)\,\rho_\tau^{k+1}(y)\,\nabla G_{\epsilon_k}(x-y)\cdot\nabla\zeta(x)\,\mathrm{d}y\,\mathrm{d}x\ . \tag{5.21}$$

Using the Taylor's expansion $\zeta(x)-\zeta\left[\nabla\varphi^k(x)\right] = \left[x-\nabla\varphi^k(x)\right]\cdot\nabla\zeta(x)+O\left[\left|x-\nabla\varphi^k(x)\right|^2\right]$, we can recast τ times the left-hand side of the Euler-Lagrange equation (5.21) as

$$\int_{\mathbb{R}^d}\zeta(x)\,\rho_\tau^{k+1}(x)\,\mathrm{d}x - \int_{\mathbb{R}^d}\zeta\left[\nabla\varphi^k(x)\right]\,\rho_\tau^{k+1}(x)\,\mathrm{d}x + O\left[\int_{\mathbb{R}^d}\left|x-\nabla\varphi^k(x)\right|^2\rho_\tau^{k+1}(t)\,\mathrm{d}x\right] = \int_{\mathbb{R}^d}\zeta(x)\,\rho_\tau^{k+1}(x)\,\mathrm{d}x - \int_{\mathbb{R}^d}\zeta(x)\,\rho_\tau^{k}(x)\,\mathrm{d}x + O\left[\mathrm{W}_2^2\left(\rho_\tau^k,\rho_\tau^{k+1}\right)\right]\ .$$

We multiply (5.26) by τ and eventually obtain

$$\int_{\mathbb{R}^d}\zeta(x)\left[\rho_\tau^{k+1}(x)-\rho_\tau^k(x)\right]\,\mathrm{d}x + O\left[\mathrm{W}_2^2\left(\rho_\tau^k,\rho_\tau^{k+1}\right)\right] = \tau\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho_\tau^{k+1}(x)\,\mathrm{d}x + \tau\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho_\tau^{k+1}(x)\,\rho_\tau^{k+1}(y)\,\nabla G_{\epsilon_k}(x-y)\cdot\nabla\zeta(x)\,\mathrm{d}y\,\mathrm{d}x\ . \tag{5.22}$$

Let $0 \leq t_1 < t_2$ be fixed times, $m = [t_1/\tau]+1$ and $n = [t_2/\tau]$. By summing equation (5.22) and symmetrizing the convolution term,

$$\int_{\mathbb{R}^d}\zeta(x)\left[\rho_\tau^n(x)-\rho_\tau^m(x)\right]\,\mathrm{d}x = \frac{\tau}{2}\sum_{k=m}^{n-1}\iint_{\mathbb{R}^4}\rho_\tau^{k+1}(x)\,\rho_\tau^{k+1}(y)\,\nabla G_{\epsilon_k}(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))\,\mathrm{d}y\,\mathrm{d}x + \tau\sum_{k=m}^{n-1}\int_{\mathbb{R}^d}\Delta\zeta(x)\,\rho_\tau^{k+1}(x)\,\mathrm{d}x + O(\tau)\ . \tag{5.23}$$

Now add and subtract to conclude

$$\rho_\tau(t_2,x)-\rho_\tau(t_1,x)=[\rho_\tau(t_2,x)-\rho_\tau^n(x)]+[\rho_\tau^n(x)-\rho_\tau^m(x)]+[\rho_\tau^m(x)-\rho_\tau(t_1,x)]\ . \tag{5.24}$$

By the Kantorovich-Rubinstein duality Theorem, for any bound Lipschiz function ζ with $\|\zeta\|_{\mathrm{Lip}}=L$,

$$\left|\int_{\mathbb{R}^d}\zeta(x)\,\mathrm{d}\mu-\int_{\mathbb{R}^d}\zeta(x)\,\mathrm{d}\nu\right|\le L\ \mathrm{W}_2(\mu,\nu). \tag{5.25}$$

Using (5.25), we control the integrals against the third term on the right in (5.24):

$$\int_{\mathbb{R}^d}\zeta(x)\,|\rho_\tau^m(x)-\rho_\tau(t_1,x)|\ \mathrm{d}x\le C\,\mathrm{W}_2(\rho_\tau^m,\rho_\tau(t_1,x))\le C\,\mathrm{W}_2(\rho_\tau^m,\rho_\tau^{m-1})\le O(\tau^{1/2})\ ,$$

and likewise with the first. In the same manner, we obtain that for all k in $[m,n-1]$

$$\tau\int_{\mathbb{R}^d}\psi(x)\rho_\tau^{k+1}(x)\,\mathrm{d}x=\int_{k\tau}^{(k+1)\tau}\int_{\mathbb{R}^d}\psi(x)\rho_\tau(s,x)\,\mathrm{d}s+O(\tau^{3/2})\ ,$$

where ψ denotes any Lipschitz test function. This observation applies to the first term in (5.23) with $\psi(x)=\Delta\zeta(x)$ and to the second term in (5.23) in the product space $\mathbb{R}^{2d}$ applied to the function $\psi(x,y)=\nabla G_{\epsilon_k}(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))$ which is compactly supported and Lipschitz in each time interval $(k\tau,(k+1)\tau)$, $m\le k\le n-1$. Finally, the test contributions are bounded in $L^\infty((0,T)\times\mathbb{R}^d)$ so that the bordering time integrands in $(t_1,m\tau)$ and $(n\tau,t_2)$ are negligible of order τ. Hence we can transform the discrete in time sum (5.23) into a continuous time integration. Collecting all the terms we conclude the proof. ■

Proof of Lemma 5.8: As $\tau\to 0$ along the sequence $\{\tau^n\}_n$, $\lim_{n\to\infty}\mathrm{W}_1[\rho_{\tau^n}(t),\rho(t)]=0$ uniformly on $[0,T]$ for any finite T. Hence by the Kantorovich-Rubinstein Theorem,

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho_{\tau^n}(t,x)\,\mathrm{d}x=\int_{\mathbb{R}^2}\Delta\zeta(x)\,\rho(t,x)\,\mathrm{d}x\ ,$$

uniformly on $[0,T]$. The interaction term can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left[\nabla G_{\epsilon(t,\tau^n)}\ *\ \rho_{\tau^n}(t)\right](x)\cdot\nabla\zeta(x)\,\rho_{\tau^n}(t,x)\ \mathrm{d}x\\
&\qquad=\int_{\mathbb{R}^2}\left[\nabla G*\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t)\right](x)\cdot\left[\gamma_{\epsilon(t,\tau^n)}*(\rho_{\tau^n}(t)\,\nabla\zeta)\right](x)\ \mathrm{d}x\\
&\qquad=-\frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}(\gamma_\epsilon*\rho_{\tau^n})(t,x)\,(\gamma_\epsilon*\rho_{\tau^n})(t,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad\quad+\int_{\mathbb{R}^2}\left[\nabla G*\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t)\right](x)\cdot\left[\gamma_{\epsilon(t,\tau^n)}*(\rho_{\tau^n}(t)\,\nabla\zeta)-(\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t))\nabla\zeta\right](x)\ \mathrm{d}x\\
&\qquad:=I_1+I_2.
\end{aligned} \tag{5.26}$$

As $\{\rho_{\tau^n}(t)\}_n$ converges weakly in $L^1(\mathbb{R}^2)$ towards $\rho(t)$ as $n\to\infty$, so does $\{\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t)\}_n$. We then deduce that $\{(\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t))\otimes(\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t))\}_n$ converges weakly in $L^1(\mathbb{R}^2\times\mathbb{R}^2)$ towards $\rho(t)\otimes\rho(t)$ when $n\to\infty$, see [3, Lemma 2.3]. As a consequence we can pass to the limit in the first term in the right-hand-side of (5.26) to obtain

$$\lim_{n\to\infty}I_1=-\frac{1}{4\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\rho(t,x)\,\rho(t,y)\,\frac{(x-y)\cdot(\nabla\zeta(x)-\nabla\zeta(y))}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\ .$$

We must now show that I_2 disappears in the limit. We can estimate I_2 using

$$\begin{aligned}
|\gamma_{\epsilon(t,\tau^n)}*(\rho_{\tau^n}(t)\,\nabla\zeta)-(\gamma_{\epsilon(t,\tau^n)}*\rho_{\tau^n}(t))\nabla\zeta|(x)&\le\int_{\mathbb{R}^2}\gamma_{\epsilon(t,\tau^n)}(x-y)\,|\nabla\zeta(y)-\nabla\zeta(x)|\,\rho_{\tau^n}(t,y)\ \mathrm{d}y\\
&\le C_\zeta\int_{\mathbb{R}^2}\gamma_{\epsilon(t,\tau^n)}(x-y)\,|x-y|\,\rho_{\tau^n}(t,y)\ \mathrm{d}y=C_\zeta\,((\gamma_{\epsilon(t,\tau^n)}|x|)*\rho_{\tau^n}(t))(x)\,.
\end{aligned}$$

By the HLS inequality, (4.3),

$$|I_2| \le \frac{C_{\rm HLS}}{2\pi} \|\gamma_{\epsilon(t,\tau^n)} * \rho_{\tau^n}(t)\|_{4/3} \, \|\gamma_{\epsilon(t,\tau^n)} * (\rho_{\tau^n}(t)\, \nabla\zeta) - (\gamma_{\epsilon(t,\tau^n)} * \rho_{\tau^n}(t))\nabla\zeta\|_{4/3}.$$

Then by similar arguments similar to those used to prove Lemma 4.12,

$$|I_2| \le 4\, C_{\rm HLS}\, C_\zeta \|\rho_{\tau^n}(t)\|_{4/3} \, \|\gamma|x|\|_{4/3} \sqrt{\epsilon(t,\tau^n)}.$$

In case $t_1 > 0$, estimating $\|\rho_{\tau^n}(t)\|_{4/3}$ using Theorem 5.7, we obtain the result. If $t_1 = 0$ we can use instead (4.30) and Hölder's inequality to obtain

$$|I_2| \le 4\, C_{\rm HLS}\, C_\zeta\, (8\pi)^{1/4} \, \sqrt{2\, C_3}\, \|\gamma|x|\|_{4/3} \sqrt{\frac{\epsilon(t,\tau^n)}{\tau}} = O((\tau^n)^{7/6})\,,$$

where (5.5) was used. ■

5.6 Entropy Dissipation

At this point we have shown that the limit $\rho_\tau = \lim_{n\to\infty} \rho_{\tau^n}$ posesses the properties (1.5.1), (1.5.2), and (1.5.3) in Definition 1.5 of properly dissipative weak solutions. In this subsection, we show that (1.5.4) is also satisfied. This will complete the proof of the existence of properly dissipative solutions in Theorem 1.6. Choosing $n = k$ in (5.17) we obtain, for all $k \in \mathbb{N}$

$$\mathcal{H}_\lambda[\rho_\tau^k] + \sum_{m=1}^{k} \tau \mathcal{D}[\rho_\tau^m] \le \mathcal{H}_\lambda[\rho_0] + \frac{C_\lambda}{4}\tau^2\,.$$

Thus, using the discrete time interpolation $\tilde\rho_\tau$, we have that for any $T > 0$ and the positive integer N such that $N\tau \le T \le (N+1)T$,

$$\mathcal{H}_\lambda[\tilde\rho_\tau(T)] + \int_0^{N\tau} \mathcal{D}[\tilde\rho_\tau(t)]\, {\rm d}t \le \mathcal{H}_\lambda[\rho_0]\ . \tag{5.27}$$

We emphasize that the use of the piecewise constant interpolation is essential at this point since the functional $\mathcal{D}[\rho]$ is not displacement convex.

Note that the L^p bounds deduced in Theorem 5.7 apply to $\tilde\rho_\tau$ as well as to ρ_τ. To make full use of these bounds, we choose any fixed $a > 0$, and then for all $\tau < a$, we weaken the bound in (5.27) by increasing the lower limit of integration in t to a. Also writing $b := N\tau$, this yields

$$\mathcal{H}_\lambda[\tilde\rho_\tau(T)] + 8\int_a^b \int_{\mathbb{R}^2} |\nabla \tilde\rho_\tau^{1/4}(t,x)|^2\, {\rm d}t - \int_a^b \int_{\mathbb{R}^2} \tilde\rho_\tau^{3/2}(t,x)\, {\rm d}x\, {\rm d}t \le \mathcal{H}_\lambda[\rho_0]\ .$$

It is legitimate to express $\mathcal{D}[\tilde\rho_\tau]$ as the difference of two integrals since Theorem 5.7 tells us the $(\tilde\rho_\tau)^{3/2}$ is integrable over $[a,T]\times\mathbb{R}^2$. We now show that passing to a further subsequence of $\{\tau^n\}_{n\in\mathbb{N}}$, we may arrange that for all $0 < a < b < \infty$, along this subsequence,

$$\lim_{n\to\infty} \int_a^b \int_{\mathbb{R}^2} |\tilde\rho_\tau(t,x) - \rho(t,x)|^{3/2}\, {\rm d}x\, {\rm d}t = 0 \tag{5.28}$$

and

$$\lim_{n\to\infty} \tilde\rho_\tau(t,x) = \rho(t,x) \tag{5.29}$$

for almost every $(t,x) \in [a,b]\times\mathbb{R}^2$. The following lemma is the key:

5.10 LEMMA (Uniform integrability for $\rho_\tau^{3/2}(x,t)$)**.** *For each integer $N > 1$, the set of function $\{(\tilde\rho_{\tau^n})^{3/2}(t,x), n \in \mathbb{N}\}$ is uniformly integrable on $[1/N, N]\times\mathbb{R}^2$.*

Proof: First, note that for each $\tau = \tau^n$, $\tilde{\rho}_\tau^{3/2} = \tilde{\rho}_\tau^{2/3}\tilde{\rho}_\tau^{5/6}$. Therefore, by Hölder's inequality

$$\int_{\mathbb{R}^2} |x|^{2/3}\tilde{\rho}_\tau^{3/2}(t,x)\,\mathrm{d}x \le \left(\int_{\mathbb{R}^2} |x|\tilde{\rho}_\tau(t,x)\,\mathrm{d}x\right)^{2/3}\left(\int_{\mathbb{R}^2} \tilde{\rho}_\tau^{5/2}(t,x)\,\mathrm{d}x\right)^{1/3} .$$

Recall the first moment is controlled by $\mathcal{H}_\lambda$ in Lemma 1.10, and that $\mathcal{H}_\lambda[\tilde{\rho}_\tau(t)] \le C_\lambda$. Also, Theorem 5.7 give us a bound on $\|\tilde{\rho}_\tau(t)\|_{5/3}$ uniformly in $t \in [1/N, N]$ for all sufficiently small τ. Thus, there is a constant C depending only on N so that for all sufficiently small τ,

$$\int_{1/N}^N \int_{\mathbb{R}^2} |x|^{2/3}\tilde{\rho}_\tau^{3/2}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le C .$$

Even more simply, by Theorem 5.7 we have a constant C depending only on N so that for all sufficiently small τ, such that

$$\int_{1/N}^N \int_{\mathbb{R}^2} \tilde{\rho}_\tau^{3/2}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le C \qquad \text{and} \qquad \int_{1/N}^N \int_{\mathbb{R}^2} \tilde{\rho}_\tau^{2}(t,x)\,\mathrm{d}x\,\mathrm{d}t \le C .$$

The uniform integrability is an immediate consequence of these estimates. ■

By Lemma 5.10 and the Dunford-Pettis Theorem, we may select a subsequence along which

$$\lim_{n\to\infty}\int_a^b \int_{\mathbb{R}^2} \tilde{\rho}_{\tau^n}^{3/2}(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_a^b \int_{\mathbb{R}^2} \rho^{3/2}(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

Since bounded sets in $L^{3/2}$ are weakly compact, we may select a further subsequence along which $\{\tilde{\rho}_{\tau^n}\}_n$ is weakly convergent in $L^{3/2}([a,b]\times\mathbb{R}^2)$. By a classical lemma, a weakly convergent sequence of functions in $L^{3/2}$ whose norms converge is strongly convergence. Hence for *some* function $\sigma \in L^{3/2}([a,b]\times\mathbb{R}^2)$, we have that $\lim_{n\to\infty}\|\tilde{\rho}_{\tau^n} - \sigma\|_{3/2} = 0$. We many now pass to a further subsequence along which $\lim_{n\to\infty}\tilde{\rho}_{\tau^n}(t,x) = \sigma(x,t)$ almost everywhere.

We claim that $\sigma(x,t) = \rho(x,t)$ almost everywhere. For this purpose, let φ by any smooth, compactly supported φ on $[a,b]\times\mathbb{R}^2$. By Fubini's Theorem, the Dominated Convergence Theorem and by Theorem 5.6,

$$\lim_{n\to\infty}\int_a^b \int_{\mathbb{R}^2} \varphi(t,x)\tilde{\rho}_{\tau^n}(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_a^b \left(\int_{\mathbb{R}^2} \lim_{n\to\infty}\varphi(t,x)\tilde{\rho}_{\tau^n}(t,x)\,\mathrm{d}x\right)\mathrm{d}t = \int_a^b \int_{\mathbb{R}^2} \varphi(t,x)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t .$$

On the other hand, by the definition of σ as a weak limit of $\{\tilde{\rho}_{\tau^n}\}_n$,

$$\lim_{n\to\infty}\int_a^b \int_{\mathbb{R}^2} \varphi(t,x)\tilde{\rho}_{\tau^n}(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_a^b \int_{\mathbb{R}^2} \varphi(t,x)\sigma(t,x)\,\mathrm{d}x\,\mathrm{d}t .$$

This proves $\sigma = \rho$, and thus, that along the sequence we have chosen, $\lim_{n\to\infty}\tilde{\rho}_{\tau^n}(t,x) = \rho(x,t)$ almost everywhere. Thus, we have seen how to choose a subsequence $\{\tau^n\}_{n\in\mathbb{N}}$ along which (5.28) and (5.29) both hold. Of course, the construction depends on a and b. In the next Lemma, we use a diagonal sequence argument to make one final choice of the subsequence.

5.11 LEMMA (Further subsequence)**.** *There is a subsequence of the sequence $\{\tau^n\}_{n\in\mathbb{N}}$, denoted with the same index, such that along this subsequence,* (5.28) *is valid for each $0 < a < T < \infty$ and*

$$\lim_{n\to\infty}\tilde{\rho}_{\tau^n}(t,x) = \rho(t,x) \qquad \text{almost everywhere on} \quad [0,T]\times\mathbb{R}^2 .$$

Proof: For $N = 2$ choose, using the argument just given, a subsequence of $\{\tau^n\}_{n\in\mathbb{N}}$ along which $\{\rho_{\tau^n}\}_{n\in\mathbb{N}}$ converges to ρ strongly in $L^{3/2}([1/2,2]\times\mathbb{R}^2)$ and almost surely on $[1/2,2]\times\mathbb{R}^2$. Next, for $N = 3$, choose a subsequence of $\{\tau^n\}$ of the $N = 2$ subsequence along which $\{\rho_{\tau^n}^{3/2}\}_{n\in\mathbb{N}}$ converges to ρ strongly in $L^{3/2}([1/3,3]\times\mathbb{R}^2)$ and almost surely on $[1/3,3]\times\mathbb{R}^2$. We finish by an inductive argument. ■

5.12 THEOREM (Entropy-entropy dissipation). *For each $T > 0$ the weak solution ρ of the PKS system that we have constructed for the initial data ρ_0 satisfies*

$$\mathcal{H}_\lambda[\rho(T)] + \int_0^T \mathcal{D}[\rho(t)]\,\mathrm{d}t \le \mathcal{H}_\lambda[\rho_0]\ . \tag{5.30}$$

Proof: By previous lemmas, it suffices to show that

$$\mathcal{H}_\lambda[\rho(T)] \le \liminf_{n\to\infty} \mathcal{H}_\lambda[\tilde\rho_{\tau^n}(T)]\ ,$$

and that for any $0 < a < b < T < \infty$,

$$\int_a^b \int_{\mathbb{R}^2} \left|\nabla \rho^{1/4}\right|^2 \,\mathrm{d}x\ \mathrm{d}t \le \liminf_{n\to\infty} \int_a^b \int_{\mathbb{R}^2} \left|\nabla \tilde\rho_{\tau^n}^{1/4}\right|^2 \,\mathrm{d}x\ \mathrm{d}t\,, \tag{5.31}$$

for a suitable sequence $\{\tau^n\}_{n\in\mathbb{N}}$, since the rest easily follows by a monotone convergence argument for taking a to 0 and b to T.

The first of these follows from the fact that $\mathcal{H}_\lambda[\rho]$ is a lower semi-continuous function on L^1 with respect to the W_1 metric just by using the expression of $\mathcal{H}_\lambda[\rho]$ in (1.11). To see the second, denote $f_n = \tilde\rho_{\tau^n}^{1/4}$ and $f = \rho^{1/4}$, then the sequence of functions $\{f_n\}_{n\in\mathbb{N}} \to f$ in $L^4 \cap L^6((a,T)\times\mathbb{R}^2)$ from Lemma 5.11. From (5.27), we have that the sequence $\{\nabla f_n\}_{n\in\mathbb{N}}$ is bounded in $L^2((a,T)\times\mathbb{R}^2)$, therefore it has a weakly convergent subsequence denoted with the same index such that $\{\nabla f_n\}_{n\in\mathbb{N}} \rightharpoonup \sigma$ weakly in $L^2((a,T)\times\mathbb{R}^2)$. Due to the strong convergence of the sequence $\{f_n\}_{n\in\mathbb{N}} \to f$ in $L^4 \cap L^6((a,T)\times\mathbb{R}^2)$, it is simple to identify the weak limit as $\sigma = \nabla f$. By standard properties of L^2-weak convergence, we deduce that

$$\int_a^b \int_{\mathbb{R}^2} |\nabla f|^2 \,\mathrm{d}x\ \mathrm{d}t \le \liminf_{n\to\infty} \int_a^b \int_{\mathbb{R}^2} |\nabla f_n|^2 \,\mathrm{d}x\ \mathrm{d}t$$

which shows (5.31). ■

Proof of Theorems 1.6: As noted above, Theorem 5.12 provides the final step in the construction of the properly dissipative weak solutions. Theorems 5.7 and 5.12 provide the additional regularity properties (1.6.1) and (1.6.2). It remains to prove (1.6.3), the dissipation of $\mathcal{F}_{\mathrm{PKS}}$.

We now show that $\mathcal{F}_{\mathrm{PKS}}[\rho(t)] \le \mathcal{F}_{\mathrm{PKS}}[\rho(s)]$ for all $0 \le s < t$. Take τ to be any element of the sequence $\{\tau^n\}_{n\in\mathbb{N}}$ whose corresponding approximated solutions $\{\tilde\rho_{\tau^n}\}_{n\in\mathbb{N}}$ converges to the constructed properly dissipative weak solution ρ. Let j be such that $(j-1)\tau \le s \le j\tau$ and let ℓ be such that $\ell\tau \le t \le (\ell+1)\tau$. Using (5.11) in Lemma 5.2 and (4.6), we deduce

$$\mathcal{F}_{\mathrm{PKS}}[\tilde\rho_\tau(t)] \le \mathcal{F}_{\mathrm{PKS}}^{\epsilon_\ell}[\rho_\tau^\ell] \le \mathcal{F}_{\mathrm{PKS}}^{\epsilon_{\ell-1}}[\rho_\tau^{\ell-1}] + \tilde Z C_3 \tau^2 2^{-\ell} \le \mathcal{F}_{\mathrm{PKS}}^{\epsilon_{j-1}}[\rho_\tau^{j-1}] + \tilde Z C_3 \tau^2 \sum_{k=j}^{\ell} 2^{-k} \le \mathcal{F}_{\mathrm{PKS}}^{\epsilon_{j-1}}[\rho_\tau^{j-1}] + \tilde Z C_3 \tau^2\ .$$

Using (5.9) and the Lemma 5.7, we can control the error term in the right-hand side by

$$\mathcal{F}_{\mathrm{PKS}}[\tilde\rho_\tau(t)] \le \mathcal{F}_{\mathrm{PKS}}[\tilde\rho_\tau(s)] + \tilde Z \|\tilde\rho_\tau(s)\|_{3/2}^{3/2} \tau^3 2^{-j+1} + \tilde Z C_3 \tau^2 \le \mathcal{F}_{\mathrm{PKS}}[\tilde\rho_\tau(s)] + \tilde Z (2\tilde C_{3/2}^{3/2} + C_3)\tau^2$$

Finally, the *a-priori* bounds uniform in τ due to Lemmas 5.6 and 5.7 together with Lemma 5.14 allow us to pass to the limit $\tau \to 0$ leading to our claim. ■

5.7 Large-Time Asymptotics

We start by identifying the large-time asymptotics of the solutions in an time average sense.

5.13 LEMMA (Time-averaged Strong Convergence). *Let ρ be the properly dissipative weak solution of the PKS system that we have constructed. Then*

$$\lim_{T\to\infty}\left(\int_T^{T+1}\int_{\mathbb{R}^2}|\rho(t,x)-\varrho_\lambda(x)|\,\mathrm{d}x\,\mathrm{d}t\right)=0\ . \tag{5.32}$$

Proof: This follows by a standard entropy dissipation argument. Let $\{t_n\}_{n\in\mathbb{N}}\nearrow+\infty$ be an increasing diverging sequence of times and consider $\sigma_n(t,x)=\rho(t+t_n,x)$, for $0\le t\le 1$. By using the entropy dissipation inequality (5.30) which is true for all $T>0$, we deduce that

$$\int_0^\infty \mathcal{D}[\rho(t)]\,\mathrm{d}t<\infty\,,\qquad\text{and thus,}\quad \lim_{n\to\infty}\int_0^1\mathcal{D}[\sigma_n(t)]\,\mathrm{d}t=0\,. \tag{5.33}$$

Now, again using the uniform bounds (5.30) for the solution in time $\mathcal{H}_\lambda[\sigma_n(t)]\le\mathcal{H}_\lambda[\rho_0]\le C_\lambda$, the concentration control inequality in Lemma 1.10 and the Gagliardo-Nirenberg-Sobolev inequality in Lemma 1.2, we deduce

$$\frac{1}{8}\int_0^1\int_{\mathbb{R}^2}\sigma_n^{3/2}\,\mathrm{d}x\,\mathrm{d}t\le\int_0^1\int_{\mathbb{R}^2}\left|\nabla[\sigma_n^{1/4}]\right|^2\,\mathrm{d}x\,\mathrm{d}t\le\frac{\pi}{\gamma_1}\int_0^1\mathcal{D}[\sigma_n]\,\mathrm{d}t+\frac{C_{\mathrm{CCD}}}{\gamma_1}\le\frac{\pi}{\gamma_1}C_\lambda+\frac{C_{\mathrm{CCD}}}{\gamma_1}\,. \tag{5.34}$$

Moreover, due to Theorem 5.7, we deduce

$$\sup_{n\in\mathbb{N}}\sup_{0\le t\le 1}\int_{\mathbb{R}^2}\sigma_n^p(t,x)\,\mathrm{d}x\le C_p\,, \tag{5.35}$$

for all $1\le p<\infty$. Note that the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ satisfies the equicontinuity property (5.16) in Theorem 5.6.

Summarizing, the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ has the same properties (5.34) and (5.35) as the sequence of approximate solutions we used in previous sections to construct the solution in Theorem 1.6. Proceeding as in Subsections 5.3-5.6, we deduce the existence of a subsequence, denoted with the same index, such that $\{\sigma_n\}_{n\in\mathbb{N}}$ converges towards ρ_∞ with the same convergence properties as in previous subsections 5.3-5.6. Here, ρ_∞ is a weak solution of (1.1) on the time interval $(0,1)$ in the sense of (1.5.2) and satisfying (1.5.3) in Definition 1.5. In particular, $\{\sigma_n\}_{n\in\mathbb{N}}$ converges to ρ_∞ in the metric space $(\mathcal{M}_1^{1/2},\mathrm{W}_1)$, with the notation of Theorem 5.6, giving

$$\lim_{n\to\infty}\left[\max_{0\le t\le 1}\mathrm{W}_1(\sigma_n(t),\rho_\infty(t))\right]=0\ . \tag{5.36}$$

Moreover, repeating the arguments in Theorem 5.12 and 5.11, we get

$$\sup_{0\le t\le 1}\mathcal{H}_\lambda[\rho_\infty(t)]\le C_\lambda\qquad\text{and}\qquad\liminf_{n\to\infty}\int_0^1\int_{\mathbb{R}^2}\left|\nabla(\sigma_n)^{1/4}\right|^2\,\mathrm{d}x\,\mathrm{d}t\ge\int_0^1\int_{\mathbb{R}^2}\left|\nabla\rho_\infty^{1/4}\right|^2\,\mathrm{d}x\,\mathrm{d}t\,, \tag{5.37}$$

and

$$\lim_{n\to\infty}\int_0^1\int_{\mathbb{R}^2}\sigma_n^{3/2}(t,x)\,\mathrm{d}x\ \mathrm{d}t=\int_0^1\int_{\mathbb{R}^2}\rho_\infty^{3/2}(t,x)\,\mathrm{d}x\ \mathrm{d}t\,.$$

Furthermore, Theorem 5.11 implies the almost everywhere convergence in $(0,1)\times\mathbb{R}^2$ of $\{\sigma_n\}_{n\in\mathbb{N}}$ towards ρ_∞, that together with (5.36) implies that

$$\lim_{n\to\infty}\left(\int_0^1\int_{\mathbb{R}^2}|\sigma_n(t,x)-\rho_\infty(t,x)|\,\mathrm{d}x\,\mathrm{d}t\right)=0\ . \tag{5.38}$$

Now, let us identify the limit function ρ_∞, passing to the limit using (5.33), we obtain

$$\int_0^1\int_{\mathbb{R}^2}\left(8\left|\nabla\rho_\infty^{1/4}\right|^2-\rho_\infty^{3/2}\right)\,\mathrm{d}x\,\mathrm{d}t=0\ ,$$

which means that $\rho_\infty(t)$ is a minimizer to the Gagliardo-Nirenberg-Sobolev inequality for al $t\in(0,1)$, see Lemma 1.2, and thus that there exists $\bar\lambda(t)$ such that $\rho_\infty(t)=\varrho_{\bar\lambda(t)}$ where ϱ_λ is the family of the minimizers of the Gagliardo-Nirenberg-Sobolev inequality, see Lemma 1.2. Due to (5.37) then $\mathcal{H}_\lambda[\varrho_{\bar\lambda(t)}] < \infty$, we conclude that $\bar\lambda(t) = \lambda$ since $\mathcal{H}_\lambda[\varrho_\mu] = +\infty$ for $\mu \neq \lambda$. Therefore, $\rho_\infty(t) = \varrho_\lambda$ that together with (5.38) implies (5.32). ■

We now will take advantage of the other Lyapunov functional, we shall prove that $\lim_{t\to\infty} \mathcal{F}_{\rm PKS}[\rho(t)] = \mathcal{F}_{\rm PKS}[\varrho_\lambda]$. In doing this, we shall make essential use of the monotonicity of $\mathcal{F}_{\rm PKS}[\rho(t)]$. Let us introduce for any $C > 0$ and $\delta > 0$ the set

$$\mathcal{S}_{C,\delta} := \left\{ \rho \in L^1_+(\mathbb{R}^2) \,:\, \int_{\mathbb{R}^2} \rho(x)\,{\rm d}x = 8\pi\,,\ \int_{\mathbb{R}^2} |x|^\delta \rho(x)\,{\rm d}x \le C \ \text{ and } \int_{\mathbb{R}^2} \rho^{1+\delta}(x)\,{\rm d}x \le C \right\}$$

5.14 LEMMA (Convergence for $\mathcal{F}_{\rm PKS}$)**.** *Given any sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}_{C,\delta}$ there is a $\rho \in \mathcal{S}_{C,\delta}$ and a subsequence $\{\rho_{n_k}\}_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty} \|\rho_{n_k} - \rho\|_1 = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \mathcal{F}_{\rm PKS}[\rho_{n_k}] = \mathcal{F}_{\rm PKS}[\rho]\ .$$

Proof: Choose any $0 < \delta' < \min(\delta, 1)$ so that $2\delta'/(1-\delta') \le \delta$. By uniform integrability arguments such as we have made above, see Lemma 5.10, we can find a subsequence (denoted with the same index) along which $\{\rho_n\}_{n\in\mathbb{N}}$ is weakly convergent in $L^{1+\delta'}(\mathbb{R}^2)$ and along which $\{\rho_n^{1+\delta'}\}_{n\in\mathbb{N}}$ is weakly convergent in L^1. It follows as in Subsection 5.6 that $\{\rho_n\}_{n\in\mathbb{N}}$ is strongly convergent in $L^{1+\delta'}(\mathbb{R}^2)$, and passing to a further subsequence, we may suppose it is also almost everywhere convergent, and strongly convergent in $L^1(\mathbb{R}^2)$. Let ρ denote the limit. By Fatou's Lemma, $\rho \in \mathcal{S}_{C,\delta}$.

Since for $t \ge 1$, $t\log t \le (t^\delta - 1)/\delta'$, we have for $\rho \ge 1$, $\rho\log\rho \le (1/\delta')\rho^{1+\delta'}$ and for $\rho < 1$, $\rho\log(1/\rho) \le (1/\delta')\rho^{1-\delta'}$. Since for $\epsilon = 2\delta'$,

$$\begin{aligned}\int_{\mathbb{R}^2} \rho^{1-\delta'}(1+|x|^2)^\epsilon\,{\rm d}x &= \int_{\mathbb{R}^2} \rho^{1-\delta'}(1+|x|^2)^{2\epsilon}(1+|x|^2)^{-\epsilon}\,{\rm d}x \\ &\le \left(\int_{\mathbb{R}^2} \rho(1+|x|^2)^{2\epsilon/(1-\delta')}\,{\rm d}x\right)^{1-\delta'} \left(\int_{\mathbb{R}^2} (1+|x|^2)^{-2}\,{\rm d}x\right)^{\delta'}\ ,\end{aligned}$$

our choice of δ' gives the uniform integrability of $\{\rho_n^{1-\delta'}\}_{n\in\mathbb{N}}$. Then, by what we have said above, $\{\rho_n\log\rho_n\}_{n\in\mathbb{N}}$ is uniformly integrable, and hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \rho_n \log\rho_n\,{\rm d}x = \int_{\mathbb{R}^2} \rho\log\rho\,{\rm d}x\ .$$

The convergence of the positive part of the interaction potential is straightforward, due to the uniform bound of $\{|x|^\delta \rho_n\}_{n\in\mathbb{N}}$ in $L^1(\mathbb{R}^2)$ and a dominated convergence argument. Concerning the negative part, it follows by Young's inequality for convolutions using the convergence of $\{\rho_n\}_{n\in\mathbb{N}} \to \rho$ in $L^{1+\delta'}(\mathbb{R}^2)$ and the fact that $\log_- |x| \in L^p(\mathbb{R}^2)$ for all $1 \le p < \infty$. ■

5.15 LEMMA (Qualitative Stability got $\mathcal{F}_{\rm PKS}$)**.** *For any $\epsilon > 0$ and $C > 0$, there exists $\delta(\epsilon, C) > 0$ so that if $\rho \in \mathcal{S}_{C,\delta}$, then*

$$\mathcal{F}_{\rm PKS}[\rho] \le (-1 + \log(8\pi)) + \delta(\epsilon, C) \quad \Rightarrow \quad \|\rho - \varrho_\mu\|_1 \le \epsilon \qquad \text{for some } \mu > 0\ ,$$

and for any $R > 0$, there exists $\delta(\epsilon, C, R) > 0$ such that

$$\mathcal{F}_{\rm PKS}[\rho] \le (-1 + \log(8\pi)) + \delta(\epsilon, C, R) \quad \Rightarrow \quad \left(\int_{\{|x|\le R\}} |(\sqrt{\rho} - \sqrt{\varrho_\mu})(x)|^2\,{\rm d}x\right)^{1/2} \le \epsilon \qquad \text{for some } \mu > 0\ .$$

Proof: Given $C, R > 0$ fixed, suppose not. Then for some $\epsilon > 0$, there is a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}_{C,\delta}$ such that $\lim_{n\to\infty} \mathcal{F}_{\rm PKS}[\rho_n] = -1 + \log(8\pi)$ but

$$\inf_{n,\mu} \|\rho_n - \varrho_\mu\|_1 \ge \epsilon\ .$$

However, by Lemma 5.14, there is a subsequence, still indexed by n, converging strongly in $L^1(\mathbb{R}^2)$ to $\rho \in \mathcal{S}_{C,\delta}$, such that

$$-1+\log(8\pi) = \lim_{n\to\infty} \mathcal{F}_{\rm PKS}[\rho_n] = \mathcal{F}_{\rm PKS}[\rho] \ .$$

By the cases of equality in the Log-HLS inequality, $\rho = \rho_\mu$ for some μ. This is a contradiction. The second part is proved the same way, using the uniform integrability of the $\sqrt{\rho}$, $\rho \in \mathcal{S}_{C,\delta}$ on $\{|x| \le R\}$. ■

Proof of Theorems 1.7: Recall that $\mathcal{F}_{\rm PKS}[\rho(t)] \le \mathcal{F}_{\rm PKS}[\rho(s)]$ for all $0 \le s < t$. We now apply this monotonicity to improve our large time asymptotic result.

By (5.32) in Lemma 5.13, there is a sequence of times $\{t_n\}_{n\in\mathbb{N}} \nearrow \infty$ such that $\lim_{n\to\infty} \|\rho(t_n) - \varrho_\lambda\|_1 = 0$. By our regularity results in Lemmas 5.6 and 5.7, $\{\rho(t_n)\}_{n\in\mathbb{N}} \subset \mathcal{S}_{C,\delta}$ for some $0 < C,\delta < \infty$. Then by Lemma 5.14, there is a subsequence, still indexed by n, such that $\lim_{n\to\infty} \mathcal{F}_{\rm PKS}[\rho(t_n)] = \mathcal{F}_{\rm PKS}[\varrho_\lambda]$. By the monotonicity of $\mathcal{F}_{\rm PKS}[\rho(t)]$ it follows that

$$\lim_{t\to\infty} \mathcal{F}_{\rm PKS}[\rho(t)] = \mathcal{F}_{\rm PKS}[\varrho_\lambda] = -1+\log(8\pi) \ .$$

Then by Lemma 5.15 it follows that given $R > 0$ there exists $\mu > 0$ such that for all sufficiently large t,

$$\|\rho(t) - \varrho_\mu\|_1 \le \epsilon \qquad \text{and} \qquad \|\sqrt{\varrho_\mu} - \sqrt{\rho}\|_{2,R} := \left(\int_{\{|x|\le R\}} |(\sqrt{\rho} - \sqrt{\varrho_\mu})(x)|^2 \,{\rm d}x\right)^{1/2} \le \epsilon.$$

However, for any $R > 0$, by Minkowskii's inequality and (5.30),

$$\begin{aligned}\left(\int_{\{|x|\le R\}} |\sqrt{\varrho_\mu} - \sqrt{\varrho_\lambda}|^2 {\varrho_\lambda}^{-1/2} \,{\rm d}x\right)^{1/2} &\le \left(\int_{\{|x|\le R\}} |\sqrt{\varrho_\mu} - \sqrt{\rho}|^2 {\varrho_\lambda}^{-1/2} \,{\rm d}x\right)^{1/2} + \sqrt{\mathcal{H}_\lambda[\rho]} \\ &\le \sqrt{8\lambda}(\lambda + R^2)\|\sqrt{\varrho_\mu} - \sqrt{\rho}\|_{2,R} + \sqrt{C_\lambda} \ .\end{aligned}$$

Since the left hand side diverges as R increases, uniformly on for $|\mu - \lambda| > \delta > 0$, we readily conclude that $\mu = \lambda$ and $\lim_{t\to\infty} \|\rho(t) - \varrho_\lambda\|_1 = 0$. ■

Appendix: proof of Lemma 4.12

By Lemma 3.3 applied to $u_0 = \rho$ and $u_1 = \rho_0$

$$\mathcal{H}_{\lambda,\delta}[\rho] \le \mathcal{H}_{\lambda,\delta}[\rho_0] - \int_{\mathbb{R}^2} \left[2\,x\,f_\delta'(|x|^2) + \frac{1}{2}\frac{\nabla\rho}{(\rho+\delta)^{3/2}}\right] \cdot (\nabla\varphi(x) - x)\,\rho \,{\rm d}x - K_\delta(\rho,\rho_0)$$

with $f_\delta'(s) = 8\lambda\left[8\lambda + \delta(\lambda+s)^2\right]^{-3/2}$. Using (4.16), *i.e.* $x - \nabla\varphi(x) = \tau\,(\nabla c_\epsilon - \nabla\rho/\rho)$ and expanding we can rewrite it as

$$\mathcal{H}_{\lambda,\delta}[\rho] \le \mathcal{H}_{\lambda,\delta}[\rho_0] + \tau\left(\frac{1}{2}({\rm I}) + \frac{1}{2}\,({\rm II}) + 2\,({\rm III}) + 2\,({\rm IV})\right) - K_\delta(\rho,\rho_0) \ ,$$

where

$$({\rm I}) := -\int_{\mathbb{R}^2} \frac{|\nabla\rho|^2}{(\rho+\delta)^{3/2}} \,{\rm d}x, \qquad ({\rm II}) := \int_{\mathbb{R}^2} \frac{\rho\nabla c_\epsilon\cdot\nabla\rho}{(\rho+\delta)^{3/2}} \,{\rm d}x$$

$$({\rm III}) := \int_{\mathbb{R}^2} f_\delta'(|x|^2)\,x\cdot\nabla c_\epsilon\,\rho \,{\rm d}x \quad \text{and} \quad ({\rm IV}) := -\int_{\mathbb{R}^2} f_\delta'(|x|^2)\,x\cdot\nabla\rho \,{\rm d}x \ ,$$

We will keep the term (I) and we need to perform some integration-by-parts in the other terms:
Control of (II)**:** We can rewrite this term as

$$({\rm II}) = 2\int_{\mathbb{R}^2} \nabla\left(\frac{\rho+2\delta}{\sqrt{\rho+\delta}}\right)\cdot\nabla c_\epsilon \,{\rm d}x \ .$$

By integrating by parts on the ball of radius R and noticing that $-\Delta c_\epsilon = \gamma_\epsilon * \rho * \gamma_\epsilon$, we obtain

$$
\begin{aligned}
2\int_{|x|\le R} \nabla\left(\frac{\rho+2\delta}{\sqrt{\rho+\delta}}\right)\cdot\nabla c_\epsilon\,\mathrm{d}x =& 2\int_{|x|\le R}\left(\frac{\rho+2\delta}{\sqrt{\rho+\delta}}\right)\,(\gamma_\epsilon*\rho*\gamma_\epsilon)\,\mathrm{d}x \\
&+2\int_{|x|=R}\frac{\rho+2\delta}{\sqrt{\rho+\delta}}\nabla c_\epsilon\cdot n\,\mathrm{d}\sigma \\
\le& 2\int_{|x|\le R}\sqrt{\rho}\,(\gamma_\epsilon*\rho*\gamma_\epsilon)\,\mathrm{d}x + 4\sqrt{\delta}\int_{|x|\le R}\gamma_\epsilon*\rho*\gamma_\epsilon\,\mathrm{d}x \\
&+2\int_{|x|=R}\frac{\rho+2\delta}{\sqrt{\rho+\delta}}\,\left|\nabla c_\epsilon\right|\,\mathrm{d}\sigma \\
\le& 2\int_{|x|\le R}\sqrt{\rho}\,(\gamma_\epsilon*\rho*\gamma_\epsilon)\,\mathrm{d}x + 32\,\pi\,\sqrt{\delta} \\
&+2\int_{|x|=R}(\sqrt{\rho}+2\sqrt{\delta})\,\left|\nabla c_\epsilon\right|\,\mathrm{d}\sigma\;, \qquad (5.39)
\end{aligned}
$$

where we used twice the estimate

$$\frac{\rho+2\delta}{\sqrt{\rho+\delta}} \le \sqrt{\rho}+2\sqrt{\delta}\;.$$

Let us deal first with the second boundary term. By (4.2) in Lemma 4.2, we have

$$
\begin{aligned}
4\sqrt{\delta}\int_{|x|=R}|\nabla c_\epsilon|\,\mathrm{d}\sigma \le& 8\pi\sqrt{\delta}\left(8C_{\mathrm{HLS}}\,\|\gamma\|_{4/3}\||x|\gamma\|_{4/3}+4+\frac{C_{\mathrm{HLS}}}{2\pi\epsilon}\,\|\gamma\|^2_{4/3}\,\||x|\rho\|_1\right) \qquad (5.40)\\
:=& 16\pi\sqrt{\delta}(J_\gamma+1+\hat{C}_\epsilon)\;,
\end{aligned}
$$

for all $R>0$. To cope with the first boundary term, we observe that taking any natural $N>1$, and consider

$$
\begin{aligned}
\left(\int_{N-1}^{N}\int_{|x|=r}\sqrt{\rho}\,|\nabla c_\epsilon|\,\mathrm{d}\sigma\,\mathrm{d}r\right)^2 \le& \pi\left(N^2-(N-1)^2\right)\,\|\nabla c_\epsilon\|^2_\infty\int_{N-1\le|x|\le N}\rho\,\mathrm{d}x \\
\le& \pi\,\|\nabla c_\epsilon\|^2_\infty\,\frac{2\,N-1}{N-1}\int_{N-1\le|x|\le N}|x|\,\rho\,\mathrm{d}x \\
\le& \widetilde{C}^2_\epsilon\int_{N-1\le|x|\le N}|x|\,\rho\,\mathrm{d}x
\end{aligned}
$$

where $\widetilde{C}_\epsilon := \sqrt{2\,\pi}\,\|\nabla c_\epsilon\|_\infty$. Since

$$\sum_{N=3}^{\infty}\int_{N-1}^{N}\int_{|x|=r}\sqrt{\rho}\,|\nabla c_\epsilon|\,\mathrm{d}\sigma\,\mathrm{d}r \le \widetilde{C}_\epsilon\,\left(\int_{\mathbb{R}^2}|x|\,\rho\,\mathrm{d}x\right)^{1/2}\;,$$

it follows that

$$\lim_{N\to\infty}\int_{N-1}^{N}\int_{|x|=r}\sqrt{\rho}\,|\nabla c_\epsilon|\,\mathrm{d}\sigma\,\mathrm{d}r = 0\;,$$

and thus, there exists a sequence $\{R_j\}\nearrow\infty$ such that

$$\lim_{j\to\infty}\int_{|x|=R_j}\sqrt{\rho}\,|\nabla c_\epsilon|\,\mathrm{d}\sigma = 0. \qquad (5.41)$$

Plugging (5.40) and (5.41) into (5.39), we get

$$(\mathrm{II}) \le 2\int_{\mathbb{R}^2}\sqrt{\rho}\,(\gamma_\epsilon*\rho*\gamma_\epsilon)\,\mathrm{d}x + 32\,\pi\,\sqrt{\delta}+4\sqrt{\delta}(J_\gamma+2\pi)\;.$$

Finally, a simple application of Hölder's inequality gives

$$\int_{\mathbb{R}^2} \sqrt{\rho}\, (\gamma_\epsilon * \rho * \gamma_\epsilon)\, \mathrm{d}x \le \|\sqrt{\rho}\|_3\, \|\gamma_\epsilon * \rho * \gamma_\epsilon\|_{3/2} \le \int_{\mathbb{R}^2} \rho^{3/2}\, \mathrm{d}x \ ,$$

to conclude

$$(\mathrm{II}) \le 2 \int_{\mathbb{R}^2} \rho^{3/2}\, \mathrm{d}x + 32\,\pi\,\sqrt{\delta} + 4\sqrt{\delta}(J_\gamma + 2\pi)\ . \tag{5.42}$$

Control of (III)**:** Remind that $f_\delta'(s) \le (8\lambda)^{-1/2} := \kappa_\lambda/2$ and that $2f_\delta'(s) \to \kappa_\lambda$ as $\delta \to 0$, see Proposition 3.2. By definition of c_ϵ and G_ϵ and by symmetry of γ

$$\begin{aligned}(\mathrm{III}) &= \int_{\mathbb{R}^2} \rho(x)\, f_\delta'(x)\, x \cdot (\nabla G_\epsilon * \rho)\,(x)\, \mathrm{d}x = \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho(x)\, f_\delta'(x)\, x\, \gamma_\epsilon(x-z)\ (\nabla G * \gamma_\epsilon * \rho)\,(z)\, \mathrm{d}z\, \mathrm{d}x \\ &= \int_{\mathbb{R}^2} (\rho\, f_\delta'\, \mathrm{id} * \gamma_\epsilon)(z)\ (\nabla G * \gamma_\epsilon * \rho)\,(z)\, \mathrm{d}z\ .\end{aligned}$$

By definition of the convolution, we have

$$\begin{aligned}\rho\, f_\delta'\, \mathrm{id} * \gamma_\epsilon(z) &= \int_{\mathbb{R}^2} \rho(z-x)\, f_\delta'(z-x)\,(z-x)\,\gamma_\epsilon(x)\, \mathrm{d}x \\ &= z \int_{\mathbb{R}^2} \rho(z-x)\, f_\delta'(z-x)\,\gamma_\epsilon(x)\, \mathrm{d}x - \int_{\mathbb{R}^2} \rho(z-x)\, f_\delta'(z-x)\, x\,\gamma_\epsilon(x)\, \mathrm{d}x \\ &= z\,(\rho f_\delta' * \gamma_\epsilon)(z) - (\rho f_\delta' * \mathrm{id}\,\gamma_\epsilon)(z)\ .\end{aligned}$$

As a consequence, we infer

$$(\mathrm{III}) = \int_{\mathbb{R}^2} \left[z\,(\rho f_\delta' * \gamma_\epsilon)(z) - (\rho f_\delta' * \mathrm{id}\,\gamma_\epsilon)(z)\right] \cdot (\nabla G * \gamma_\epsilon * \rho)\,(z)\, \mathrm{d}z := (\mathrm{III})_1 - (\mathrm{III})_2\ . \tag{5.43}$$

By the symmetrization argument just as in (1.13), the first term of the right hand side of (5.43) reads

$$\begin{aligned}(\mathrm{III})_1 &= \frac{1}{2}\int_{\mathbb{R}^2} z\,(\rho(2f_\delta' - \kappa_\lambda) * \gamma_\epsilon)(z) \cdot (\nabla G * \gamma_\epsilon * \rho)\,(z)\, \mathrm{d}z - 8\pi\kappa_\lambda \\ &:= (\mathrm{III})_{11} - 8\pi\kappa_\lambda\ .\end{aligned}$$

We now control $(\mathrm{III})_{11}$ using the HLS and Young inequalities, see (4.3) to obtain

$$\begin{aligned}|(\mathrm{III})_{11}| &\le \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |z|\,|(\rho(2f_\delta' - \kappa_\lambda) * \gamma_\epsilon)(z)|\frac{1}{|x-z|}|(\gamma_\epsilon * \rho)(x)|\, \mathrm{d}z\, \mathrm{d}x \\ &\le \frac{C_{\mathrm{HLS}}}{4\pi}\, \||z|\,|(\rho|2f_\delta' - \kappa_\lambda| * \gamma_\epsilon)\|_{4/3}\, \|\rho * \gamma_\epsilon\|_{4/3} \\ &\le \frac{C_{\mathrm{HLS}}}{4\pi}\, \left(\|(|x|\,\rho|2f_\delta' - \kappa_\lambda|) * \gamma_\epsilon\|_{4/3} + \|(\rho|2f_\delta' - \kappa_\lambda|) * (|x|\,\gamma_\epsilon)\|_{4/3}\right)\, \|\rho * \gamma_\epsilon\|_{4/3} \\ &\le 2C_{\mathrm{HLS}}\, \left(\||x|\,\rho|2f_\delta' - \kappa_\lambda|\|_1\, \|\gamma_\epsilon\|_{4/3} + \|\rho|2f_\delta' - \kappa_\lambda|\|_1\, \||x|\,\gamma_\epsilon\|_{4/3}\right)\, \|\gamma_\epsilon\|_{4/3}\,,\end{aligned}$$

from which

$$(\mathrm{III})_1 \le -8\pi\kappa_\lambda + C_\epsilon\, \|\,|2f_\delta' - \kappa_\lambda|\,(1+|x|)\rho\|_1\ . \tag{5.44}$$

To estimate the second term of the right hand side of (5.43), we make again use of the HLS inequality, see (4.3):

$$\begin{aligned}|(\mathrm{III})_2| &\le \frac{1}{4\sqrt{2\lambda}\,\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} (\rho * |z|\,\gamma_\epsilon)(z)\,\frac{1}{|z-y|}\,(\gamma_\epsilon * \rho)(y)\, \mathrm{d}y\, \mathrm{d}z \\ &\le \frac{C_{\mathrm{HLS}}}{4\sqrt{2\lambda}\,\pi}\, \|\rho * (|z|\,\gamma_\epsilon)\|_{4/3}\, \|\gamma_\epsilon * \rho\|_{4/3}\ .\end{aligned}$$

By the Young inequality, and a direct calculation,

$$\|\rho * (|z|\,\gamma_\epsilon)\|_{4/3} \le 8\pi\, \||x|\,\gamma_\epsilon\|_{4/3} = 8\pi\,\sqrt{\epsilon}\,\||x|\,\gamma\|_{4/3}\ .$$

and in the same way

$$\|\gamma_\epsilon * \rho\|_{4/3} \le 8\pi\,\|\gamma_\epsilon\|_{4/3} = 8\pi\,\frac{1}{\sqrt{\epsilon}}\,\|\gamma\|_{4/3}\ .$$

The positive and negative powers of ϵ cancel, and using (5.44), we conclude

$$\begin{aligned}(\mathrm{III}) &\le -8\pi\kappa_\lambda + C_\epsilon\,\|\,|2f_\delta' - \kappa_\lambda|\,(1+|x|)\rho\|_1 + \frac{16\pi\,C_{\mathrm{HLS}}}{\sqrt{2\lambda}}\,\||x|\,\gamma\|_{4/3}\,\|\gamma\|_{4/3}\\ &= -8\pi\kappa_\lambda + C_\epsilon\,\|\,|2f_\delta' - \kappa_\lambda|\,(1+|x|)\rho\|_1 + \frac{8\pi\,J_\gamma}{\sqrt{2\lambda}}. \end{aligned} \tag{5.45}$$

Let us estimate this third term in a different way that will be useful later on. Using again the Young inequality, but this time eliminating γ instead of ρ, *i.e.* $\|\gamma_\epsilon * \rho\|_{4/3} \le \|\rho\|_{4/3}$, we get

$$\|\rho * (|z|\,\gamma_\epsilon)\|_{4/3}\,\|\gamma_\epsilon * \rho\|_{4/3} \le 64\pi^2\sqrt{\epsilon}\,\||x|\,\gamma\|_{4/3}\,\|\rho\|_{4/3}\ .$$

As a consequence, we get this other control on (III) by

$$(\mathrm{III}) \le -8\pi\kappa_\lambda + C_\epsilon\,\|\,|2f_\delta' - \kappa_\lambda|\,(1+|x|)\rho\|_1 + \frac{16\pi\,C_{\mathrm{HLS}}}{\sqrt{2\lambda}}\,\sqrt{\epsilon}\,\||x|\,\gamma\|_{4/3}\,\|\rho\|_{4/3}\ . \tag{5.46}$$

Control of (IV)**:** By integrating by parts for any $R>0$, we have

$$\int_{|x|\le R} x\,f_\delta'(|x|^2)\cdot\nabla\rho(x)\,\mathrm{d}x = \int_{|x|=R}\rho(x)\,f_\delta'(|x|^2)\,x\cdot n\,\mathrm{d}\sigma - \int_{|x|\le R}\nabla\cdot[x\,f_\delta'(|x|^2)]\,\rho(x)\,\mathrm{d}x\ ,$$

where n denotes the outward normal to the disk D_R. Taking into account that

$$\sum_{N=3}^\infty\int_{N-1}^N\int_{|x|=r}|x|\,|f_\delta'(|x|^2)|\,\rho(x)\,\mathrm{d}\sigma\,\mathrm{d}r \le \frac{1}{2\sqrt{2\lambda}}\sum_{N=3}^\infty\int_{N-1\le|x|\le N}|x|\,\rho\,\mathrm{d}x \le \frac{1}{2\sqrt{2\lambda}}\int_{\mathbb{R}^2}|x|\,\rho\,\mathrm{d}x < \infty$$

we have

$$\lim_{N\to\infty}\int_{N-1}^N\int_{|x|=r}\rho(x)\,|f_\delta'(|x|^2)|\,|x|\,\mathrm{d}\sigma\,\mathrm{d}r = 0\ .$$

As a consequence, there exists a sequence $\{R_j\}\nearrow\infty$ such that

$$\lim_{j\to\infty}\int_{|x|=R_j}\rho(x)\,|f_\delta'(|x|^2)|\,|x|\,\mathrm{d}\sigma = 0,$$

and thus, we conclude

$$(\mathrm{IV}) = \int_{\mathbb{R}^2}\nabla\cdot[x\,f_\delta'(|x|^2)]\rho(x)\,\mathrm{d}x\ . \tag{5.47}$$

The desired estimates are obtained by putting together estimates (5.42), (5.45), (5.46) and (5.47). ■

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