Optimal Dividend Policy and Growth Option *

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Abstract

We analyse the interaction between dividend policy and investment decision in a growth opportunity of a liquidity constrained firm. This leads us to study a mixed singular control/optimal stopping problem for a diffusion that we solve quasi-explicitly establishing connections with two auxiliary optimal stopping problems. We characterize situations where it is optimal to postpone dividend distribution in order to invest at a subsequent date in the growth opportunity. We show that uncertainty and liquidity shocks have ambiguous effect on the investment decision.

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1 Introduction

Research on optimal dividend payouts for a cash constrained firm is based on the premise that the firm wants to pay some of its surplus to the shareholders as dividends and therefore follows a dividend policy that maximizes expected present value of all payouts until bankruptcy. This approach has been in particular used to determine the market value of a firm which, in line with Modigliani and Miller [19], is defined as the present value of the sum of future dividends. In diffusions models, the optimal dividend policy can be determined as the solution of a singular stochastic control problem. For instance, Jeanblanc and Shiryaev [15] and Radner and Shepp [22], assume that the firm exploit a technology defined by a cash generating process that follows a drifted Brownian motion. They show that the optimal dividend policy is characterized by a threshold so that whenever the cash reserve goes above this threshold, the excess is paid out as dividends.

A now large literature on optimal dividend payouts uses controlled diffusion techniques emphasizing for example, solvency restrictions imposed by a regulatory agency (Paulsen [21]), the interplay between dividend and risk policies (Højgaard and Taksar [14], Asmussen, Højgaard and Taksar [1], Choulli, Taksar and Zhou [3]), or the analysis of hedging and insurance decisions (Rochet and Villeneuve [23]).

Here, we consider a firm with a technology in place and a growth option. The growth option offers the firm the opportunity to invest in a new technology that increases its profit rate. The firm has no access to external funding and therefore finances the opportunity cost of the growth option on its cash reserve. Our objective is then to study the interactions between dividend policy and investment decisions. Such an objective leads us to consider a mixed singular control/optimal stopping problem that we solve quasi-explicitly.

Optimal stopping, singular stochastic controls or mixed singular/regular stochastic controls have been widely used in Mathematical Finance. Problems focusing both on singular control and optimal stopping are less usual and, to the best of our knowledge, only Guo and Pham [13] deal with such an issue. In Guo and Pham [13] a firm chooses the optimal time to activate production and then control it by buying or selling capital. This leads the authors to solve in a two-stage procedure, first a singular control problem and then an optimal stopping problem. In our setting the optimal dividend/investment policy cannot be stated as a two-step formulation of a singular stochastic control problem and an optimal stopping problem. However, we succeed in solving our mixed singular control/optimal stopping problem by establishing connections with two auxiliary stopping problems. The first one, that permits to characterize situations where it is optimal to postpone dividend distribution before investing, corresponds to the option value to invest in the growth opportunity when the manager decides to pay no dividend before exercising the growth option. The second one corresponds to the option value to choose among two alternative investment policies: (i) never invest in the growth option (and follow the associated optimal dividend policy), (ii) invest immediately in the growth option (and follow the associated optimal dividend policy). Our main theorem, founded on a verification procedure for singular control, proves that this latter optimal stopping problem is equivalent to our mixed singular control/optimal stopping problem. We show that the associated value function, that we compute quasi-explicitly, is piecewise C^2 and not necessarily concave as it is the case in standard singular control problems. Furthermore, from a detailed analysis based on properties of local time, we construct explicitly the optimal dividend/investment policy.

Our work bridges the gap between the literature on optimal dividend payouts and the now well established real option literature. The real option literature analyses optimal investment policy that can be mathematically determined as the solution of an optimal stopping problem. The original model is due to McDonald and Siegel [18] and has been extended in various ways by many authors[1]. An important assumption of these models is that the investment decision can be made independently of the financing decision. In contrast, in our paper, two inter-related features drive our investment problem. First, the firm is cash constrained and must finance the investment using its cash reserve. Second, the firm must decide its dividend distribution policy in view of its growth opportunity. From that perspective our paper can be related to Boyle and Guthrie [2] who analyse, in a numerical model, dynamic investment decision of a firm submitted to cash constraints. Two state variables drive their model: the cash process and a project value process for which the decision maker has to pay a fixed amount I. Boyle and Guthrie [2] do not consider dividend distribution policy.

Our model allows us therefore to study the following set of questions: When is it optimal to postpone dividend distribution, to accumulate cash and to invest at a subsequent date in the growth option? What are the effects of cash flow and uncertainty shocks on dividend policy and investment decision? What is the effect of financing constraints on dividend policy and investment decision with respect to a situation where the firm has unlimited cash?

The outline of the paper is as follows. Section 2 describes the model, analyses some useful benchmarks, provides a formulation of our problem based on the dynamic programming principle, and derives a sufficient condition for the growth option being worthless. Section 3 introduces two auxiliary stopping time problems that we solve quasi-explicitly. Section 4 contains our main result. We prove that our mixed singular control/optimal stopping problem can be reduced to an auxiliary optimal stopping problem studied in section 3 and we give an explicit construction of the optimal dividend/investment control. Section 5 interprets our mathematical results, proposes answers to the financial questions we raised in the introduction and concludes.

[1]See for instance Dixit and Pindyck [9] for an overview of this literature. Recent developments include for example the impact of asymmetric information in a duopoly model (see Lambrecht and Perraudin [17], Décamps and Mariotti [5]), regime switchs (Guo, Miao and Morellec [12]), learning (Décamps, Mariotti and Villeneuve [6]) or investment in alternative projects (Décamps, Mariotti and Villeneuve [7]).

2 The model

2.1 Formulation of the problem

We consider a firm whose activities generate a cash process. The firm faces liquidity constraints that cause bankruptcy as soon as the cash process reaches the threshold 0. The manager of the firm acts in the best interest of its shareholders and maximizes the expected present value of dividends up to bankruptcy. At any time the firm has the option to invest in a new technology that increases the drift of the cash generating process from μ_0 to $\mu_1 > \mu_0$ without affecting its volatility σ. This growth opportunity requires a fixed investment cost I that must be financed using the cash reserve. Our purpose is to study the optimal dividend/investment policy of such a firm.

The mathematical formulation of our problem is as follows. We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a Brownian Motion $W = (W_t)_{t\geq 0}$ with respect to $\mathcal{F}_t$. In the sequel, $\mathcal{Z}$ denotes the set of positive non-decreasing right continuous processes and $\mathcal{T}$, the set of $\mathcal{F}_t$-adapted stopping times. A control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$ modelizes a dividend/investment policy and is said to be admissible if Z_t^π belongs to $\mathcal{Z}$ and if τ^π belongs to $\mathcal{T}$. We denote the set of all admissible controls by Π. The control component Z_t^π therefore corresponds to the total amount of dividends paid out by the firm up to time t and the control component τ^π represents the investment time in the growth opportunity. A given control policy $(Z_t^\pi, \tau^\pi; t \geq 0)$ fully characterizes the associated investment process $(I_t^\pi)_{t\geq 0}$ which belongs to $\mathcal{Z}$ and is defined by relation $I_t = I 1\!\!1_{t\geq \tau^\pi}$. We denote by X_t^π the cash reserve of the firm at time t under a control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$. The dynamic of the cash process X_t^π satisfies

$$dX_t^\pi = (\mu_0 1\!\!1_{t<\tau^\pi} + \mu_1 1\!\!1_{t\geq\tau^\pi})dt + \sigma dW_t - dZ_t^\pi - dI_t^\pi, \qquad X_{0^-}^\pi = x.$$

Remark that, at the investment time τ^π, the cash process jumps for an amount of $(\Delta X^\pi)_{\tau^\pi} \equiv X_{\tau^\pi}^\pi - X_{\tau^\pi -}^\pi = -I - (Z_{\tau^\pi}^\pi - Z_{\tau^\pi -}^\pi)$. This reflects the fact that we do not exclude a priori strategies that distribute some dividend at the investment time τ^π. For a given control π, the time of bankruptcy is defined as

$$\tau_0^\pi = \inf\{t \geq 0 \ : \ X_t^\pi \leq 0\}.$$

The firm value V_π associated with a control π is therefore

$$V_\pi(x) = \mathbb{E}_x\left[\int_0^{\tau_0^\pi} e^{-rs} dZ_s^\pi\right].$$

The objective is to find the optimal return function which is defined as

$$V(x) = \sup_{\pi\in\Pi} V_\pi(x) \tag{2.1}$$

and the optimal policy $\pi^\star$ such that

$$V_{\pi^\star}(x) = V(x).$$

The problem we consider is thus a mixed singular control/optimal stopping problem that we are going to solve quasi explicitly using its connection with two auxiliary stopping time problems. We first study two useful benchmarks.

2.2 Benchmarks

As a first benchmark, it is helpful to consider the case where the manager decides never to invest in the new technology. Under such a scenario, everything happens as if the growth opportunity did not exist and we are brought back to the standard model of optimal dividend policy developed in Jeanblanc and Shiryaev [15] or Radner and Shepp [22]. The cash process X satisfies

$$dX_t = \mu_0 dt + \sigma dW_t - dZ_t,$$

and the firm value V_t at time t is defined by the standard singular control problem:

$$V_t \;=\; \operatorname*{ess\,sup}_{Z \in \mathcal{Z}} \mathbb{E}_x \left[\int_{t\wedge\tau_0}^{\tau_0} e^{-r(s-t\wedge\tau_0)} dZ_s | \mathcal{F}_{t\wedge\tau_0} \right],$$

where $\tau_0 = \inf\{t \;:\; X_t \leq 0\}$ is the bankruptcy time. It is well known that the current value $X_{t\wedge\tau_0}$ is a sufficient statistic to compute the value of the firm. More precisely, it follows from Jeanblanc and Shiryaev [15]

Proposition 2.1 *The firm value satisfies $V_t = V_0(X_{t\wedge\tau_0})$ where*

$$V_0(x) \;=\; \sup_{Z \in \mathcal{Z}} \mathbb{E}_x \left[\int_0^{\tau_0} e^{-rt} dZ_t \right]. \tag{2.2}$$

Moreover, the value function V_0 can be characterized in terms of the free boundary problem:

$$\begin{cases} \mathcal{L}_0 V_0(x) - rV_0(x) = 0, \quad 0 \leq x \leq x_0, \\ V_0(0) = 0, \quad V_0'(x_0) = 1, \quad V_0''(x_0) = 0, \end{cases} \tag{2.3}$$

where $\mathcal{L}_0$ is the infinitesimal generator of the drifted Brownian motion $\mu_0 t + \sigma W_t$.

In order to define the optimal dividend policy solution to (2.2), let consider the process

$$(\mu_0 t + \sigma W_t - L_t^{x_0}(\mu_0, W))_{t\geq 0} \tag{2.4}$$

where

$$L_t^{x_0}(\mu_0, W) = \max\left[0, \max_{0\leq s\leq t}(\mu_0 s + \sigma W_s - x_0)\right].$$

It is well known[2] that (2.4) is the reflected drifted Brownian Motion at the boundary x_0. Furthermore, again following Jeanblanc and Shiryaev [15], the process (2.4) characterizes the optimal cash reserve process solution to problem (2.2). Equivalently, the optimal dividend

[2]See for instance Karatzas and Shreve [16], Proposition 3.6.16 page 211

policy solution of problem (2.2) is defined by the local time $L^{x_0}(\mu_0, W)$ determined by the process (2.4) at the boundary x_0. In words, whenever the cash reserve process goes above the threshold x_0, the excess is immediately paid out as dividends. Computations are explicit and give

$$V_0(x) = \mathbb{E}_x \left[\int_0^{\tau_0} e^{-rs} dL_s^{x_0}(\mu_0, W) \right] = \frac{f_0(x)}{f_0'(x_0)} \quad 0 \le x \le x_0, \tag{2.5}$$

with

$$f_0(x) = e^{\alpha_0^+ x} - e^{\alpha_0^- x} \quad \text{and} \quad x_0 = \frac{1}{\alpha_0^+ - \alpha_0^-} \ln \frac{\left(\alpha_0^-\right)^2}{\left(\alpha_0^+\right)^2}, \tag{2.6}$$

where $\alpha_0^- < 0 < \alpha_0^+$ are the roots of the characteristic equation

$$\mu_0 x + \frac{1}{2}\sigma^2 x^2 - r = 0.$$

Note that, if the firm starts with cash reserves x above x_0, the optimal dividend policy distributes immediately the amount $(x - x_0)$ as exceptional dividend and then follows the dividend policy characterized by the local time $L^{x_0}(\mu_0, W)$. Thus, for $x \ge x_0$ we have that

$$V_0(x) = x - x_0 + V_0(x_0), \tag{2.7}$$

where

$$V_0(x_0) = \mathbb{E}_{x_0} \left[\int_0^{\tau_0} e^{-rs} dL_s^{x_0}(\mu_0, W) \right] = \frac{\mu_0}{r}.$$

To sum up, we have that, for all positive x, $V_0(x) = V_{\pi^0}(x) \le V(x)$ where the control policy π^0 is defined by

$$\pi^0 = \left((x - x_0)_+ \mathbb{1}_{t=0} + L_t^{x_0}(\mu_0, W)\mathbb{1}_{t>0}\,,\,\infty\right).$$

Remark 2.1 *The function f_0 defined on $[0, \infty)$ is non negative, increasing, concave on $[0, x_0]$, convex on $[x_0, \infty)$ and satisfies $f_0' \ge 1$ on $[0, \infty)$ together with $\mathcal{L}_0 f_0 - r f_0 = 0$ on $[0, x_0]$. Remark also that V_0 is concave on $[0, x_0]$ and linear above x_0. We shall use repeatedly theses properties in the next sections.*

As a second benchmark, consider now that the manager invests immediately at date $t = 0$ in the new technology. The cash reserve process X therefore satisfies

$$dX_t = \mu_1 dt + \sigma dW_t - dZ_t, \;\text{ with }\; X_{0^-} = x \;\text{ and }\; X_0 = x - I.$$

Considering that $x - I \le 0$ leads to immediate bankruptcy, it easily follows from the previous results that the firm value $V_1(x - I)$ is defined by:

$$\begin{cases} V_1(x - I) = \max\left(0, \dfrac{f_1(x - I)}{f_1'(x_1)}\right), & 0 \le x \le x_1 + I, \\ V_1(x - I) = x - I - x_1 + \frac{\mu_1}{r} & x \ge x_1 + I, \end{cases} \tag{2.8}$$

with

$$f_1(x) = e^{\alpha_1^+ x} - e^{\alpha_1^- x} \quad \text{and} \quad x_1 = \frac{1}{\alpha_1^+ - \alpha_1^-} \ln \frac{\left(\alpha_1^-\right)^2}{\left(\alpha_1^+\right)^2}, \tag{2.9}$$

and where $\alpha_1^- < 0 < \alpha_1^+$ are the roots of the characteristic equation

$$\mu_1 x + \frac{1}{2}\sigma^2 x^2 - r = 0. \tag{2.10}$$

In line with Remark 2.1, note that, $V_1(. - I)$ is concave on $[I, x_1 + I]$, linear above $x_1 + I$, satisfies $V_1^{'}(. - I) \geq 1$ on $[I, \infty)$ and $\mathcal{L}_1 V_1(. - I) - rV_1(. - I) = 0$ on $[I, x_1 + I]$, where $\mathcal{L}_1$ is the infinitesimal generator of the drifted Brownian motion $\mu_1 t + \sigma W_t$. Note also that, for all positive x, $V_1(x - I) = V_{\pi^1}(x) \leq V(x)$ where the control policy π^1 is defined by

$$\pi^1 = ((x - I) - x_1)_+ \mathbb{1}_{t=0} + L_t^{x_1}(\mu_1, W)\mathbb{1}_{t>0}\,,\, 0)$$

with

$$L_t^{x_1}(\mu_1, W) = \max\left[0, \max_{0\leq s\leq t}\left(\mu_1 s + \sigma W_s - x\right)\right].$$

Finally remark that, following Proposition 2.1, a given admissible policy $\pi = (Z_t^\pi, \tau^\pi)$ leads to firm value V_{τ^π} at date τ^π that satisfies

$$V_{\tau^\pi} = V_1(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-} - I) = V_1(X^\pi_{\tau^\pi\wedge\tau_0^\pi}) = \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0^\pi} e^{-r(s-\tau^\pi\wedge\tau_0^\pi)} dZ_s | \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]. \tag{2.11}$$

2.3 Dynamic programming principle and first result

As a first use of our two previous benchmarks we now prove that the value function V satisfies the dynamic programming principle.

Proposition 2.2 *The following holds.*

$$V(x) = \sup_{\pi\in\Pi} \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs} dZ_s^\pi \; + \; e^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-} - I)\right]. \tag{2.12}$$

Proof: Let us define

$$W(x) = \sup_{\pi\in\Pi} \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs} dZ_s^\pi \; + \; e^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-} - I)\right].$$

We start by proving the inequality $V(x) \leq W(x)$. For a given control policy π, we deduce from equation (2.11):

$$\begin{aligned} V_\pi(x) &= \mathbb{E}_x\left[\int_0^{\tau_0^\pi} e^{-rs} dZ_s^\pi\right] = \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs} dZ_s^\pi + \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0^\pi} e^{-rs} dZ_s^\pi | \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]\right] \\ &\leq \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs} dZ_s^\pi + e^{-r(\tau^\pi\wedge\tau_0^\pi)} \operatorname*{ess\,sup}_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_{\tau^\pi\wedge\tau_0^\pi}^{\tau_0} e^{-r(s-\tau^\pi\wedge\tau_0^\pi)} dZ_s | \mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]\right] \\ &\leq \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs} dZ_s^\pi + e^{-r(\tau^\pi\wedge\tau_0^\pi)} V_1(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-} - I)\right]. \end{aligned} \tag{2.13}$$

Taking the supremun over $\pi \in \Pi$ on both sides gives the desired inequality. The converse inequality relies on the fact that there is an optimal dividend policy solution to problem (2.11). Let us denote by Z^1 this optimal policy and consider the control $\pi = (Z_t^\pi \mathbb{1}_{t<\tau^\pi} + Z_t^1 \mathbb{1}_{t\geq\tau^\pi}, \tau^\pi)$ where Z_t^π and τ^π are arbitrarily chosen in $\mathcal{Z}$ and $\mathcal{T}$. We get

$$\begin{aligned} V(x) &\geq V_\pi(x) \\ &= \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}dZ_s^\pi + e^{-r(\tau^\pi\wedge\tau_0^\pi)}\operatorname*{ess\,sup}_{Z\in\mathcal{Z}}\mathbb{E}\left[\int_{\tau_0^\pi\wedge\tau^\pi}^{\tau_0^\pi} e^{-r(s-\tau^\pi\wedge\tau_0^\pi)}dZ_s^\pi|\mathcal{F}_{\tau^\pi\wedge\tau_0^\pi}\right]\right] \\ &= \mathbb{E}_x\left[\int_0^{(\tau^\pi\wedge\tau_0^\pi)^-} e^{-rs}dZ_s^\pi + e^{-r(\tau^\pi\wedge\tau_0^\pi)}V_1(X^\pi_{(\tau^\pi\wedge\tau_0^\pi)^-} - I)\right]. \end{aligned}$$

Taking the supremum over (Z^π, τ^π) on the right-hand side gives the result. $\diamond$

We now turn to our first result, namely a sufficient condition under which the growth opportunity is worthless. We show the following.

Proposition 2.3 *If* $\left(\frac{\mu_1 - \mu_0}{r}\right) \leq (x_1 + I) - x_0$*, then for all* $x \geq 0$, $\quad V(x) = V_0(x)$.

Remark 2.2 *Note that, under the assumption of Proposition 2.3, we have* $x_1 + I \geq x_0$. *We also recall that there is no obvious comparison between* x_0 *and* x_1 *(see for instance [23] Proposition 2).*

Proposition 2.3 relies on the lemma:

Lemma 2.1 *The following holds.*

$$V_0(x) \geq V_1(x-I) \ \text{ for all} x \geq 0 \ \text{ if and only if } \ \left(\frac{\mu_1 - \mu_0}{r}\right) \leq (x_1 + I) - x_0.$$

Proof of Lemma 2.1. The necessary condition is obvious since $V_0(x) \geq V_1(x - I)$ clearly implies for $x \geq \max\{x_0, x_1 + I\}$ the desired inequality.

Let us turn to the sufficient condition. First, if $x \in [0, I]$ then, $V_0(x) \geq 0 = V_1(x - I)$. Second, if $x \geq x_0$ then,

$$V_1(x-I) < x - x_1 + \frac{\mu_1}{r} < x - x_0 + \frac{\mu_0}{r} = V_0(x),$$

where the first inequality comes from the concavity of V_1, the second inequality is our assumption and the last equality follows from definition of V_0 for $x \geq x_0$. Finally, fix $x \in [I, x_0]$ and consider the function k defined on $[I, x_0]$ by the relation $k(x) = V_0(x) - V_1(x-I)$. We have already proved that $k(I) > 0$ and $k(x_0) > 0$. Note also that $k'(x_0) = 1 - V_1'(x_0 - I) \leq 0$ and $k''(x_0) \geq 0$. Now, suppose there exists $y \in (I, x_0)$ such that $k(y) = 0$. Since k is decreasing convex in a left neighbourhood of x_0, this implies that there exists $z \in (y, x_0)$ such that $k'(z) = 0$ with k concave in a neighbourhood centered in z. We thus deduce that

$$\mathcal{L}_0 k(z) - rk(z) = \frac{\sigma^2}{2}k''(z) - rk(z) < 0 \tag{2.14}$$

However, for all $x \in (I, x_0)$, we have that $\mathcal{L}_0 V_0(x) - rV_0(x) = 0$, which implies that

$$\mathcal{L}_0 k(x) - rk(x) = -\mathcal{L}_0 V_1(x - I) + rV_1(x - I). \tag{2.15}$$

Taking advantage of the equality $\mathcal{L}_1 V_1(x-I) - rV_1(x-I) = 0$ for $x \in (I, x_0)$ since $x_0 \leq x_1 + I$ by Remark 2.2, we deduce that for all $x \in (I, x_0)$,

$$\mathcal{L}_0 V_1(x - I) - rV_1(x - I) = (\mathcal{L}_0 - \mathcal{L}_1)V_1(x - I) = (\mu_0 - \mu_1)V_1'(x - I) < 0$$

where the inequality comes from $\mu_1 > \mu_0$ and from the increasness of $V_1(. - I)$. We thus have from (2.15) that $\mathcal{L}_0 k(z) - rk(z) > 0$ which contradicts (2.14). This concludes the proof of lemma 2.1.

Proof of Proposition 2.3 By equation (2.13), for all fixed $\pi = (Z_t^\pi, \tau^\pi; t \geq 0) \in \Pi$

$$\begin{aligned} V_\pi(x) &\leq \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rt} dZ_t^\pi + e^{-r(\tau^\pi \wedge \tau_0^\pi)} V_1(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)^-} - I) \right] \\ &\leq \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)} e^{-rt} dZ_t^\pi + e^{-r(\tau^\pi \wedge \tau_0^\pi)} V_0(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)^-}) \right] \\ &\leq V_0(x), \end{aligned}$$

where the second inequality comes from lemma 2.1 and the third from the dynamic programming principle applied to the value function V_0. It thus follows that $V(x) \leq V_0(x)$ which implies our result since the the converse inequality is always true.

Thereafter we rule out the relative uninteresting case where the growth option is worthless and we thus work under the condition

(H1) $$\frac{\mu_1 - \mu_0}{r} > (x_1 + I) - x_0.$$

Note that assumption (H1) ensures the existence and the uniqueness of a positive real number $\tilde{x}$ such that $V_0(x) \geq$ (resp. $\leq$) $V_1(x - I)$ for $x \leq$ (resp. $\geq$) $\tilde{x}$. This property will play a crucial role in the next section.

3 Auxiliary optimal stopping problems.

The main purpose of this paper is to examine the interactions between dividend policy and investment decision. We solve quasi explicitly the optimization problem (2.1) by exploiting its connections with two auxiliary optimal stopping problems that we now describe.

Let denote by $R = (R_t)_{t \geq 0}$ the cash reserve process generated by the activity in place in absence of dividend distribution:

$$dR_t = \mu_0 dt + \sigma dW_t,$$

with initial condition

$$R_0 = x.$$

We consider the two auxiliary optimal stopping time problems with value functions

$$\theta(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} V_1(R_{\tau \wedge \tau_0} - I) \right], \tag{3.16}$$

and

$$\phi(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau \wedge \tau_0)} \max(V_0(R_{\tau \wedge \tau_0}), V_1(R_{\tau \wedge \tau_0} - I)) \right] \tag{3.17}$$

where $\tau_0 = \inf\{t \geq 0 \ : \ R_t \leq 0\}$.

In this section we (quasi) explicitly determine the value function ϕ and explain its relation with value function θ. In the next section it will be proved that the value functions ϕ and V coincides.

We start the analysis with problem Θ. It follows from Dayanik and Karatzas [4] (Corollary 7.1) that the optimal value function θ is C^1 on $[0, \infty)$ furthermore, from Villeneuve [25] (Theorem 4.2. and Proposition 4.6) a threshold strategy is optimal and thus θ can be written in terms of the free boundary problem:

$$\begin{cases} \mathcal{L}_0 \theta(x) - r\theta(x) = 0, \ \ 0 \leq x \leq b, \ \ \text{and} \ \mathcal{L}_0 \theta(x) - r\theta(x) \leq 0, \ \ x \geq b, \\ \theta(b) = V_1(b - I), \ \ \theta'(b) = V_1'(b - I). \end{cases} \tag{3.18}$$

Standard computations lead to

$$\begin{cases} \theta(x) = \dfrac{f_0(x)}{f_0(b)} V_1(b - I) \quad x \leq b, \\ \theta(x) = V_1(x - I), \quad x \geq b, \end{cases} \tag{3.19}$$

where f_0 is defined in (2.6) and where $b > I$ is defined by the smooth-fit principle

$$\frac{V_1'(b - I)}{f_0'(b)} = \frac{V_1(b - I)}{f_0(b)}. \tag{3.20}$$

Remark 3.3 *The value function θ therefore represents the value of investing in the new project when the manager decides to pay no dividends before exercising the growth option. Note that, for all x, we have the following inequalities*

$$V_1(x - I) \leq \theta(x) = V_{\pi^\theta}(x) \leq V(x) \leq V_0(x) + \theta(x),$$

where the control π^θ is defined by

$$\pi^\theta = (((R_{\tau_b} - I) - x_1)_+ 1\!\!1_{t = \tau_b} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t > \tau_b} \, , \, \tau_b) \, ,$$

with $\tau_b = \inf\{t \ : \ R_t \geq b\}$. The inequality $V(x) \leq V_0(x) + \theta(x)$, which follows from proposition 2.2, ensures that problem (2.1) is indeed well defined.

The study of the optimal stopping problem Φ is more involved. We show the following.

Theorem 3.1 *Assume condition* **(H1)** *holds then,*

(i) If $\theta(x_0) > V_0(x_0)$ *then, the value function* ϕ *satisfies for all* x, $\phi(x) = \theta(x)$.

(ii) If $\theta(x_0) \leq V_0(x_0)$ *then, the value function* ϕ *has the following structure.*

$$\phi(x) = \begin{cases} V_0(x) \quad x \leq a, \\ V_0(a)\mathbb{E}_x[e^{-r\tau_a} 1\!\!1_{\tau_a<\tau_c}] + V_1(c-I)\mathbb{E}_x[e^{-r\tau_c} 1\!\!1_{\tau_a>\tau_c}] = Ae^{\alpha_0^+ x} + Be^{\alpha_0^- x} \quad a \leq x \leq c, \\ V_1(x-I) \quad x \geq c, \end{cases}$$

where $\tau_a = \inf\{t \geq 0 \ : \ R_t \leq a\}$ *and* $\tau_c = \inf\{t \geq 0 \ : \ R_t \geq c\}$ *and where* A, B, a, c *are determined by the continuity and smooth-fit* C^1 *conditions at* a *and* c*:*

$$\begin{aligned} &\phi(a) = V_0(a), \\ &\phi(c) = V_1(c-I), \\ &\phi'(a) = V_0'(a), \\ &\phi'(c) = V_1'(c-I). \end{aligned}$$

Proof of Theorem 3.1 According to Optimal Stopping Theory (see El Karoui [10], Theorems 10.1.9 and 10.1.12 in Øksendal [20]), we introduce the stopping region

$$S = \{x > 0 \,|\, \phi(x) = \max(V_0(x), V_1(x-I))\}.$$

Now, from Proposition 5.13 and Corollary 7.1 by Dayanik-Karatzas [4], the hitting time $\tau_S = \inf\{t : R_t \in \mathcal{S}\}$ is optimal and the optimal value function is C^1 on $[0, \infty)$. Moreover, it follows from Lemma 4.3 from Villeneuve [25] that $\tilde{x}$, defined as the unique crossing point of the value functions $V_0(.)$ and $V_1(x-.)$, does not belong to S. Hence, the stopping region can be decomposed into two subregions $S = S_0 \cup S_1$ with

$$S_0 = \{0 < x < \tilde{x} \,|\, \phi(x) = V_0(x)\},$$

and

$$S_1 = \{x > \tilde{x} \,|\, \phi(x) = V_1(x-I)\}.$$

Assertion (i) of Theorem 3.1 is then obtained as a byproduct of the next Proposition.

Proposition 3.4 *The following assertions are equivalent:*

(i) $\theta(x_0) > V_0(x_0)$.

(ii) $\theta(x) > V_0(x)$ *for all* $x > 0$.

(iii) $S_0 = \emptyset$.

Proof of Proposition 3.4.

(i) $\Longrightarrow$ *(ii)*. We start with $x \in (0, x_0)$. Let us define $\tau_{x_0} = \inf\{t : R_t < x_0\} \in \mathcal{T}$. The inequality $\theta(x_0) > V_0(x_0)$ together with the initial conditions $\theta(0) = V_0(0) = 0$ implies

$$\mathbb{E}_x \left[e^{-r(\tau_{x_0} \wedge \tau_0)} \left(\theta(R_{\tau_{x_0} \wedge \tau_0}) - V_0(R_{\tau_{x_0} \wedge \tau_0}) \right) \right] > 0.$$

Itô's formula gives

$$\begin{aligned} 0 &< \mathbb{E}_x \left[e^{-r(\tau_{x_0} \wedge \tau_0)} \left(\theta(R_{\tau_{x_0} \wedge \tau_0}) - V_0(R_{\tau_{x_0} \wedge \tau_0}) \right) \right] \\ &= \theta(x) - V_0(x) + \mathbb{E}_x \left[\int_0^{\tau_{x_0} \wedge \tau_0} e^{-rt} \left(\mathcal{L}_0 \theta(R_t) - r\theta(R_t) \right) dt \right] \\ &\leq \theta(x) - V_0(x), \end{aligned}$$

where the last inequality follows from (3.18). Thus, $\theta(x) > V_0(x)$ for all $0 < x \leq x_0$. Assume now that $x > x_0$. Two cases have to be considered. If $b > x_0$, it follows from (2.5) and (3.19) that, $\theta(x) > V_0(x)$ for $x \leq x_0$ is equivalent to $\theta'(x_0) > 1$. Then, Remark 2.1 about the convexity properties of f_0 yields to $\theta'(x) > 1$, for all $x > 0$. If, on the contrary, $b \leq x_0$ then $\theta(x) = V_1(x - I)$ for all $x \geq x_0$. Since $V_1'(x - I) \geq 1$ for all $x \in [I, \infty)$, the smooth fit principle implies $\theta'(x) \geq 1$ for all $x \geq x_0$. Therefore, the function $\theta - V_0$ is increasing for $x \geq x_0$ which ends the proof.

(ii) $\Longrightarrow$ *(iii)*. Simply remark that equations (3.16) and (3.17) give $\phi \geq \theta$. Therefore, we have $\phi(x) \geq \theta(x) > V_0(x)$ for all $x > 0$ which implies the emptyness of S_0.

(iii) $\Longrightarrow$ *(i)*. Suppose $S_0 = \emptyset$ and let us show that $\theta = \phi$. This will clearly implies $\theta(x_0) = \phi(x_0) > V_0(x_0)$ and thus (i). From Optimal Stopping theory, the process $(e^{-r(t \wedge \tau_0 \wedge \tau_S)} \phi(X_{t \wedge \tau_0 \wedge \tau_S})_{t \geq 0}$ is a martingale. Moreover on the event $\{\tau_S < t\}$, we have $\phi(R_{\tau_S}) = V_1(R_{\tau_S} - I)$ *a.s.* It results that

$$\begin{aligned} \phi(x) &= \mathbb{E}_x \left[e^{-r(t \wedge \tau_S)} \phi(R_{t \wedge \tau_S}) \right] \\ &= \mathbb{E}_x \left[e^{-r\tau_S} V_1(R_{\tau_S} - I) \mathbb{1}_{\tau_S < t} \right] + \mathbb{E}_x \left[e^{-rt} \phi(R_t) \mathbb{1}_{t < \tau_S} \right] \\ &\leq \theta(x) + \mathbb{E}_x \left[e^{-rt} \phi(R_t) \right]. \end{aligned}$$

Now, it follows from (2.7), (2.8) that $\phi(x) \leq Cx$ for some positive constant C. This implies $\mathbb{E}_x \left[e^{-rt} \phi(R_t) \right]$ converges to 0 as t goes to infinity. We therefore deduce that $\phi \leq \theta$ and thus that $\phi = \theta$.

We now turn to Assertion (ii) of Theorem 3.1. We show the following.

Proposition 3.5 *Assume $\theta(x_0) \leq V_0(x_0)$ then, there are two positive real numbers $a \geq x_0$ and $c \leq x_1 + I$ such that*

$$S_0 =]0, a] \text{ and } S_1 = [c, +\infty[.$$

Proof of Proposition 3.5.

From the previous Proposition we know that the inequality $\theta(x_0) \leq V_0(x_0)$ implies $S_0 \neq \emptyset$. We start the proof with the shape of the subregion S_0. Take $x \in S_0$, we have to prove that any $y \leq x$ belongs to S_0. As a result, we will define $a = \sup\{x < \tilde{x} \,|\, x \in S_0\}$. Now, according to Proposition 5.13 by Dayanik and Karatzas [4], we have

$$\phi(y) = \mathbb{E}_y \left[e^{-r(\tau_S \wedge \tau_0)} \max(V_0(R_{\tau_S \wedge \tau_0}), V_1(R_{\tau_S \wedge \tau_0} - I)) \right].$$

Since $x \in S_0$, $x < \tilde{x}$ and thus $\tau_S = \tau_{S_0}$ $\mathbb{P}^y$ a.s. for all $y \leq x$. Hence,

$$\begin{aligned} \phi(y) &= \mathbb{E}_y \left[e^{-r(\tau_{S_0} \wedge \tau_0)} V_0(R_{\tau_{S_0} \wedge \tau_0}) \right] \\ &\leq V_0(y), \end{aligned}$$

where the last inequality follows from the supermartingale property of the process $(e^{-rt \wedge \tau_0} V_0(R_{t \wedge \tau_0}))_{t \geq 0}$. Now, assuming that $a < x_0$, (i.e. $\phi(x_0) > V_0(x_0)$) yields the contradiction

$$\begin{aligned} \phi(a) &= V_0(a) \\ &= \mathbb{E}_a \left[e^{-r\tau_{x_0}} 1\!\!1_{\tau_{x_0} < \tau_0} V_0(R_{\tau_{x_0}}) \right] \\ &\leq \mathbb{E}_a \left[e^{-r\tau_{x_0}} V_0(R_{\tau_{x_0}}) \right] \\ &< \mathbb{E}_a \left[e^{-r\tau_{x_0}} \phi(R_{\tau_{x_0}}) \right] \\ &\leq \phi(a), \end{aligned}$$

where the second equality follows from the martingale property of the process $(e^{-r(t \wedge \tau_{x_0} \wedge \tau_0)} V_0(R_{t \wedge \tau_{x_0} \wedge \tau_0}))_{t \geq 0}$ under $\mathbb{P}^a$ and the last inequality follows from the supermartingale property of the process $(e^{-rt \wedge \tau_0} \phi(R_{t \wedge \tau_0}))_{t \geq 0}$.

The shape of the subregion S_1 is a direct consequence of Lemma 4.4 by Villeneuve [25]. The only difficulty is to prove that $c \leq x_1 + I$. Let us consider $x \in (a, c)$, and let us introduce the stopping times $\tau_a = \inf\{t : R_t = a\}$, and $\tau_c = \inf\{t : R_t = c\}$, we have:

$$\begin{aligned} \phi(x) &= \mathbb{E}_x \left[e^{-r(\tau_a \wedge \tau_c)} \max(V_0(R_{\tau_a \wedge \tau_c}), V_1(R_{\tau_a \wedge \tau_c} - I)) \right] \\ &\leq \mathbb{E}_x \left[e^{-r(\tau_a \wedge \tau_c)} (R_{\tau_a \wedge \tau_c} - (x_1 + I) + \frac{\mu_1}{r}) \right] \\ &= x - (x_1 + I) + \frac{\mu_1}{r} + \mathbb{E}_x \left[\int_0^{\tau_a \wedge \tau_c} e^{-rs} (\mu_0 - r(R_s - (x_1 + I)) - \mu_1) \, ds \right]. \end{aligned}$$

Remark that, on the stochastic interval $[0, \tau_a \wedge \tau_c]$, $R_s \geq a \geq x_0$ $\mathbb{P}^x$ a.s. and thus

$$\mu_0 - r(R_s - (x_1 + I)) - \mu_1 \leq \mu_0 - r(x_0 - (x_1 + I)) - \mu_1 < 0,$$

by assumption (H1). Therefore, $\phi(x) \leq x - (x_1 + I) + \frac{\mu_1}{r}$ for $x \in (a, c)$. We conclude remarking that the inequality $c > x_1 + I$ yields to the contradiction

$$\frac{\mu_1}{r} = V_1(x_1) < \phi(x_1 + I) \leq \frac{\mu_1}{r}.$$

The structure of the value function ϕ in Theorem 3.1 is then a straightforward consequence of continuity and smooth-fit C^1 properties. Figures 1 and 2 illustrate Theorem 3.1.

Remark 3.4 *If $\theta(x_0) = V_0(x_0)$ then, we have that $a = x_0$, $c = b$ and the value functions ϕ and θ coincide. Indeed, using same argument than in the first part of the proof of Proposition 3.4, we easily deduce from $\theta(x_0) = V_0(x_0)$ that $\theta(x) = V_0(x) = \phi(x)$ for $x \leq x_0$. Furthermore, (2.5) and (3.19) imply that, $\theta(x_0) = V_0(x_0)$ is equivalent to $\theta'(x_0) = V'(x_0) = 1$, which implies that $a = x_0$. The equality $c = b$ follows then from relation (3.19) and (3.20). To summarize, if $\theta(x_0) = V_0(x_0)$ then, θ is the lowest supermartingale that majorizes $e^{-r(\tau\wedge\tau_0)}\max(V_0(R_{\tau\wedge\tau_0}), V_1(R_{\tau\wedge\tau_0} - I))$ from which it results that $\theta = \phi$.*

Remark 3.5 *Problem 3.1 corresponds to the option value to choose among two alternative investment policies: (i) never invest in the growth option (and follow the associated optimal dividend policy) or (ii), invest immediately in the growth option (and follow the associated optimal dividend policy). At this stage, the question to know whether or not there exists a dividend/investment policy that attains the value function ϕ is still unanswered. We show in the next section that such a policy exists and is actually the optimal policy solution to problem 2.1.*

4 Main Theorem

We are now in a position to state and prove the main Theorem of our paper:

Theorem 4.2 *Assume that* **(H1)** *holds then, $V = \phi$.*

We thus show in Theorem 4.2 that our mixed singular control/optimal stopping problem can be reduced to the optimal stopping problem Φ. The proof is based on a slightly modified standard verification procedure for singular control. One indeed expects from Proposition 2.2 that the value function V being solution of the Hamilton-Jacobi-Bellman (HJB) equation

$$\max(1 - v',\, \mathcal{L}_0 v - rv,\, V_1(. - I) - v) \;=\; 0. \tag{4.21}$$

This leads us to show in a first step that any piecewise function C^2 which is a supersolution to the HJB (4.21) is a majorant of the value functions V and ϕ. Second, we show that the value function ϕ is itself a supersolution to the HJB (4.21). Last we prove a verification Proposition, which shows that ϕ coincides with V. This proof constructs the optimal dividend/investment policy $\pi^\star$ solution to problem (2.1).

Proposition 4.6 *Suppose we can find a positive function $\tilde{V}$ piecewise C^2 on $(0, +\infty)$ with bounded first derivatives*[3] *and such that for all $x > 0$,*

(i) $\mathcal{L}_0\tilde{V} - r\tilde{V} \leq 0$ *in the sense of distributions,*

(ii) $\tilde{V}(x) \geq V_1(x - I)$,

(iii) $\tilde{V}'(x) \geq 1$,

[3] in the sense of Definition 4.8 page 271 in Karatzas and Schreve [16].

with the initial condition $\tilde{V}(0) = 0$ *then,* $\tilde{V}(x) \geq V(x)$ *for all* $x \in [0, \infty)$.

Proof of Proposition 4.6 We have to prove that for any control policy $\pi = (Z_t^\pi, \tau^\pi; t \geq 0)$, $\tilde{V}(x) \geq V_\pi(x)$ for all $x > 0$. Let us write the process $Z_t^\pi = Z_t^{\pi,c} + Z_t^{\pi,d}$ where $Z_t^{\pi,c}$ is the continuous part of Z_t^π and $Z_t^{\pi,d}$ is the pure discontinuous part of Z_t^π. Using a generalized Itô's formula (see Dellacherie and Meyer [8], Theorem VIII-25 and Remark c page 349), we can write

$$\begin{aligned} e^{-r(\tau^\pi \wedge \tau_0^\pi)} \tilde{V}(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-}) &= \tilde{V}(x) + \int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs}(\mathcal{L}_0 \tilde{V}(X_s^\pi) - r\tilde{V}(X_s^\pi))\, ds \\ &+ \int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs} \tilde{V}'(X_s^\pi)\, \sigma dW_t - \int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs} \tilde{V}'(X_s^\pi)\, dZ_s^c \\ &+ \sum_{s < \tau^\pi \wedge \tau_0^\pi} e^{-rs}(\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi)). \end{aligned}$$

Since $\tilde{V}$ satisfies (i), the second term of the right hand side is negative. On the other hand, the first derivative of $\tilde{V}$ being bounded, the third term is a square integrable martingale. Taking expectations, we get

$$\begin{aligned} \mathbb{E}_x \left[e^{-r(\tau^\pi \wedge \tau_0^\pi)} \tilde{V}(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-}) \right] &\leq \tilde{V}(x) - \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs} \tilde{V}'(X_s^\pi)\, dZ_s^{\pi,c} \right] \\ &+ \mathbb{E}_x \left[\sum_{s < \tau \wedge \tau_0} e^{-rs}(\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi)) \right]. \end{aligned}$$

Since $\tilde{V}'(x) \geq 1$ for all $x > 0$, we have $\tilde{V}(X_s^\pi) - \tilde{V}(X_{s-}^\pi) \leq X_s^\pi - X_{s-}^\pi$. Therefore, using the equality $X_s^\pi - X_{s-}^\pi = -(Z_s^\pi - Z_{s-}^\pi)$ for $s < \tau^\pi \wedge \tau_0^\pi$, we finally get

$$\begin{aligned} \tilde{V}(x) &\geq \mathbb{E}_x \left[e^{-r(\tau^\pi \wedge \tau_0^\pi)} \tilde{V}(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)-}) \right] + \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs} \tilde{V}'(X_s^\pi)\, dZ_s^{\pi,c} \right] \\ &\quad + \mathbb{E}_x \left[\sum_{s < \tau \wedge \tau_0} e^{-rs}(Z_s^\pi - Z_{s-}^\pi) \right] \\ &\geq \mathbb{E}_x \left[e^{-r(\tau^\pi \wedge \tau_0^\pi)} V_1(X^\pi_{(\tau^\pi \wedge \tau_0^\pi)^-} - I) \right] + \mathbb{E}_x \left[\int_0^{(\tau^\pi \wedge \tau_0^\pi)^-} e^{-rs}\, dZ_s^\pi \right] \\ &= V_\pi(x), \end{aligned}$$

where assumptions (ii) and (iii) have been used for the second inequality.

We call thereafter supersolution to HJB (4.21) any solution $\tilde{V}$ satisfying Proposition 4.6. It follows from proposition 4.6 that the process $(e^{-rt \wedge \tau_0^\pi} \tilde{V}(X^\pi_{t \wedge \tau_0^\pi}))_{t \geq 0}$ is a supermartingale which dominates $\max(V_0, V_1(. - I))$. Therefore, according to optimal stopping theory, any supersolution $\tilde{V}$ satisfies $\tilde{V} \geq \phi$. We now turn to the second step of our proof and prove that ϕ is a supersolution.

Proposition 4.7 *ϕ is a supersolution.*

Proof of Proposition 4.7 Two cases have to be considered.

i) $\theta(x_0) > V_0(x_0)$.
In this case, $\phi = \theta$ according to part (i) of Theorem 3.1. It remains to check that the function θ satisfies the assumptions of Proposition 4.6. But according to optimal stopping theory, $\theta \in C^2[(0,\infty) \setminus b)]$, $\mathcal{L}_0\theta - r\theta \leq 0$ and obviously $\theta \geq V_1(. - I)$. Moreover, it is shown in the first part of the proof of Proposition 3.4 that $\theta^{'}(x) \geq 1$ for all $x > 0$. It remains to check that $\theta^{'}$ is bounded above in the neighborhood of zero. Clearly,

$$\theta(x) \leq \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r(\tau\wedge\tau_0)} V_1(R_{\tau\wedge\tau_0}) \right].$$

On the other hand, the process $(e^{-r(t\wedge\tau_0)}V_1(R_{t\wedge\tau_0}))_{t\geq 0}$ is a supermartingale since $\mu_1 > \mu_0$. Therefore, $\theta \leq V_1$ which gives the boundedness of the first derivative of θ by Equation (2.8).

ii) $\theta(x_0) \leq V_0(x_0)$.
In this case, the function ϕ is characterized by part (ii) of Theorem 3.1. Thus, $\phi = V_0$ on $(0, a)$, $\phi = V(. - I)$ on $(c, +\infty)$ and $\phi(x) = Ae^{\alpha_0^+ x} + Be^{\alpha_0^- x}$ on (a, c). Hence, ϕ will be a supersolution if we prove that $\phi^{'}(x) \geq 1$ for all $x > 0$. In fact, it is enough to prove that $\phi^{'}(x) \geq 1$ for $x \in (a, c)$ since $V_0^{'} \geq 1$ and $V_1^{'}(. - I) \geq 1$. The smooth fit principle gives $\phi^{'}(a) = V_0^{'}(a) \geq 1$ and $\phi^{'}(c) = V_1^{'}(c - I) \geq 1$. Clearly, ϕ is convex in a right neighbourhood of a. Therefore, if ϕ is convex on (a, c), the proof is over. If not, the second derivative of ϕ given by $A(\alpha_0^+)^2 e^{\alpha_0^+ x} + B(\alpha_0^-)^2 e^{\alpha_0^- x}$ vanishes at most one time on (a, c), say in d. Therefore,

$$1 \leq \phi^{'}(a) \leq \phi^{'}(x) \leq \phi^{'}(d) \text{ for } x \in (a, d),$$

and

$$1 \leq \phi^{'}(c) \leq \phi^{'}(x) \leq \phi^{'}(d) \text{ for } x \in (d, c),$$

which completes the proof.

Remark 4.6 *It follows from Theorem 3.1 and Proposition 4.7 that the supersolution ϕ is indeed a solution of HJB equation (4.21).*

To prove Theorem 4.2 it remains to show that ϕ is attainable, that is, there exist $\pi \in \Pi$ such that $\phi = V_\pi$. We show the following

Proposition 4.8 *Assume condition* (**H1**) *holds then,*

(i) If $\theta(x_0) > V_0(x_0)$ then, the policy $\pi^* = ((Z_t^{\pi^\star}), \tau^{\pi^\star})$ defined by the increasing right-continuous process

$$Z_t^{\pi^\star} = ((R_{\tau_b} - I) - x_1)_+ 1\!\!1_{t=\tau_b} + L_t^{x_1}(\mu_1, W) 1\!\!1_{t>\tau_b},$$

and by the stopping time

$$\tau^{\pi^\star} = \tau_b$$

satisfies the relation $\phi(x) = V_{\pi^*}(x)$ *for* $x > 0$.

(ii) If $\theta(x_0) \leq V_0(x_0)$ *then, the policy* $\pi^* = ((Z_t^{\pi^\star}), \tau^{\pi^\star})$ *defined by the increasing right-continuous process*

$$\begin{aligned} Z_t^{\pi^\star} = &\left[(R_{\tau_a} - x_0)_+ \mathbb{1}_{t=\tau_a} + (L_t^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W))\mathbb{1}_{t>\tau_a}\right] \mathbb{1}_{\tau_a<\tau_c} \\ &+ \left[((R_{\tau_c} - I) - x_1)_+ \mathbb{1}_{t=\tau_c} + L_t^{x_1}(\mu_1, W)\mathbb{1}_{t>\tau_c}\right] \mathbb{1}_{\tau_c<\tau_a}, \end{aligned}$$

and by the stopping time

$$\tau^{\pi^\star} = \begin{cases} \tau_c \text{ if } \tau_c < \tau_a \\ \infty \text{ if } \tau_c > \tau_a \end{cases}$$

satisfies the relation $\phi(x) = V_{\pi^*}(x)$ *for* $x > 0$.

Proof of Proposition 4.8 Part (i) is immediate from Theorem 3.1 and Remark 3.3. We start the proof of part (ii) by some helpful remarks on the considered policy $\pi^\star$. On the event $\{\tau_a < \tau_c\}$, the investment time $\tau^{\pi^\star}$ is infinite *a.s.* Moreover, denoting by $X^{\pi^\star}$ the cash process generated by the policy π^*, we have that $X_{\tau_a}^{\pi^\star} = x_0$ *a.s* and for $t \geq 0$, we have

$$X_{\tau_a+t}^{\pi^\star} = x_0 + \mu_0 t + \sigma(W_{\tau_a+t} - W_{\tau_a}) - (L_{\tau_a+t}^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W)). \tag{4.22}$$

Now, introduce the process $B_t^{(a)} = W_{\tau_a+t} - W_{\tau_a}$. We know that $B^{(a)}$ is a Brownian motion independent of $\mathcal{F}_{\tau_a}$ (Theorem 6.16 in Karatzas and Shreve [16]) and from the unicity of the Skorohod equation (Ch IX, Exercise 2.14 in Revuz and Yor [24]) it follows from (4.22) the identity in law

$$L_{\tau_a+t}^{x_0}(\mu_0, W) - L_{\tau_a}^{x_0}(\mu_0, W) \overset{law}{=} L_t^{x_0}(\mu_0, B^{(a)}). \tag{4.23}$$

We now turn to the proof of (ii). According to the structure of value function ϕ in Theorem 3.1, three cases have to be considered.

α) If $x \leq a$ then, $\tau_a = 0$, $\tau^{\pi^\star} = \infty$ *a.s* and

$$Z_t^{\pi^\star} = (x - x_0)_+ \mathbb{1}_{t=0} + L_t^{x_0}(\mu_0, W)\mathbb{1}_{t>0}.$$

We get

$$\begin{aligned} V_{\pi^\star}(x) = \mathbb{E}_x\left[\int_0^{\tau_0^{\pi^\star}} e^{-rs} dZ_s^\star\right] &= (x - x_0)_+ + \mathbb{E}_{\min(x,x_0)}\left[\int_0^{\tau_0^{\pi^\star}} e^{-rs} dL_s^{x_0}(\mu_0, W)\right] \\ &= V_0(x) \\ &= \phi(x). \end{aligned}$$

β) If $x \geq c$ then $\tau^{\pi^\star} = \tau_c = 0$ *a.s*, $Z_t^{\pi^\star} = ((x - I) - x_1))_+ \mathbb{1}_{t=0} + L_t^{x_1}(\mu_1, W)\mathbb{1}_{t>0}$ and $X_{\tau_c}^{\pi^\star} = x - I$ *a.s.* Thus, $V_{\pi^\star}(x) = V_1(x - I) = \phi(x)$.

γ) Last, assume that $a < x < c$. We have

$$V_{\pi^\star}(x) = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0^{\pi^\star}} e^{-rs}dZ_s^{\pi^\star}\right] + \mathbb{E}_x\left[\mathbb{1}_{\tau_a>\tau_c}e^{-r\tau_c}V_1(c-I)\right].$$

Now,

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0{}^{\pi^\star}} e^{-rs}dZ_s^\star\right] &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\left(e^{-r\tau_a}(a-x_0)+\int \mathbb{1}_{]\tau_a,\tau_0^{\pi^\star}]}(s)e^{-rs}dL_s^{x_0}(\mu_0,W)\right)\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}e^{-r\tau_a}(a-x_0)\right]+A \qquad (4.24)\end{aligned}$$

On the other hand, note that we have on the event $\tau_a < \tau_c$,

$$\tau_0^{\pi^\star} \equiv \inf\{s\ :\ X_s^{\pi^\star} \le 0\} = \tau_a + \inf\{s\ :\ X_{s+\tau_a}^{\pi^\star} \le 0\} \quad a.s$$

It then follows from (4.22) and (4.23) that

$$\tau_0^{\pi^\star} - \tau_a \overset{law}{=} T_0 \equiv \inf\{s \ge 0\ :\ x_0 + \mu_0 s + \sigma B_s^{(a)} - L_s^{x_0}(\mu_0, B^{(a)}) \le 0\}.$$

Coming back to (4.24) we thus obtain,

$$\begin{aligned}A &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathbb{E}\left(\int \mathbb{1}_{]\tau_a,\tau_0^{\pi^\star}]}(s)e^{-rs}dL_s^{x_0}(\mu_0,W)|\mathcal{F}_{\tau_a}\right)\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\mathbb{E}\left(\int \mathbb{1}_{]0,\tau_0^{\pi^\star}-\tau_a]}(u)e^{-r(u+\tau_a)}dL_{u+\tau_a}^{x_0}(\mu_0,W)|\mathcal{F}_{\tau_a}\right)\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}e^{-r\tau_a}\mathbb{E}_{x_0}\left[\int \mathbb{1}_{]0,T_0]}(u)e^{-ru}dL_u^{x_0}(\mu_0,B^{(a)})\right]\right]\\ &= \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}e^{-r\tau_a}V_0(x_0)\right]\end{aligned}$$

where the third equality follows from the independence of $B^{(a)}$ with respect to $\mathcal{F}_{\tau_a}$, (4.23) together with the fact that $L^{x_0}(\mu_0, B^{(a)})$ is an additive functional.
Hence,

$$\mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}\int_0^{\tau_0^{\pi^\star}} e^{-rs}dZ_s^{\pi^\star}\right] = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}e^{-r\tau_a}V_0(a)\right]$$

Finally,

$$V_{\pi^\star}(x) = \mathbb{E}_x\left[\mathbb{1}_{\tau_a<\tau_c}e^{-r\tau_a}V_0(a)\right] + \mathbb{E}_x\left[\mathbb{1}_{\tau_a>\tau_c}e^{-r\tau_c}V_1(c-I)\right] = \phi(x).$$

5 Discussion and concluding remarks.

Our Mathematical analysis is rich enough to address several important questions and we describe in this section how dividend and investment policies interact. We start by characterizing situations where it is optimal to postpone dividend distribution in order to invest later in the growth opportunity. We then investigate the effect of liquidity shock on the optimal dividend/investment policy. In particular, we show that it can result from a liquidity shock an inaction region in which the manager waits to see whether or not the growth opportunity is valuable. In a third step we analyse the effect of positive uncertainty shock. In stark difference with the standard real option literature, we explain why a sufficiently large positive uncertainty shock can make worthless the option value to invest in a growth opportunity. Last we identify situations where a cash constrained firm may want to accumulate cash in order to invest in the growth opportunity whereas an unconstrained firm will definitively decide not to invest.

When to postpone dividend distribution? A natural intuition is that delaying dividend distribution is optimal when the growth option is "sufficiently" valuable. Our model allows to precise this point. Assume that the current value x of the cash reserve is lower than the threshold level x_0 that triggers distribution of dividends when the firm is run under the initial technology then, the optimal dividend/investment policy is as follows. If, evaluated at the threshold x_0, the net expected discounted profit of investing in the new project is larger than the value of the firm run under the technology in place (that is $\theta(x_0) > V_0(x_0)$) then, the manager postpones dividend distribution in order to accumulate cash and to invest in the new technology at threshold b. Any surplus above x_1 will be then distributed as dividends. If, on the contrary, $\theta(x_0) < V_0(x_0)$ then, the manager optimally ignores the growth option, runs the firm under the technology in place and pays out any surplus above x_0 as dividends.

The effect of liquidity shock. Our model emphasizes the value of cash for optimal dividend/investment timing. Consider indeed the case where the current value of the cash reserve x is lower than the threshold x_0 and where $\theta(x_0) \leq V_0(x_0)$. Assume that an exogenous positive shock on the cash reserve occurs so that the current value x is now larger than x_0. Three cases must be considered. First, if $x > c$, then, according to theorem 4.2, the manager optimally invests immediately in the new project (and pays out any surplus above $I + x_1$ as dividends). Second, if x lies in (x_0, a), then the manager pays out $x - x_0$ as "exceptional dividends", never invest in the new technology and pays out any surplus above x_0 as dividends. Last, if x lies in (a, c), then two scenarii can occur. If the cash reserve raises to c before hitting a, the manager invests in the new project (and pays out any surplus above x_1 as dividends). By contrast, if the cash reserve falls to a, before hitting c, the manager pays $a - x_0$ as "exceptional dividends", never invest in the new technology and pays out any surplus above x_0 as dividends. The region (a, c) is therefore an inaction region where the manager has no enough information to decide whether or not the growth option is valuable. He therefore chooses neither to distribute dividends nor to invest in the new technology. His final decision depends on which bounds a or c will be first reached by the cash flow process. As a result, our model suggest that, a given cash injection does not always provokes or accelerates investment decision.

The effect of uncertainty shock. In the standard real option literature as well as in the optimal dividend policy literature, increasing the volatility of the cash process has an unambiguous effect: Greater uncertainty increases both the option value to invest (see McDonald and Siegel [18]), and the threshold that triggers distribution of dividends (see Rochet and Villeneuve [23]). In our setting, because the dividend and the investment policies are inter-related, the effect of uncertainty shock is ambiguous.

Consider for instance a situation where, initially, $\theta(x_0) < V(x_0)$ with a current value x of the cash reserve lower than x_0 and assume that a positive shock on the volatility of the cash process occurs. The volatility shock increases the trigger x_0 but does not affect $V(x_0)$ which is by construction equal to $\frac{\mu_0}{r}$. A volatility shock however increases $\theta(x_0)$, the option value to invest in the new project, and therefore it can happen that the inequality $\theta(x_0) < V(x_0)$ being reversed. In this case, the manager who initially ignores the growth opportunity, will decide, after a positive shock on uncertainty, to accumulate cash and to exercise the growth opportunity at threshold b. Here, in line with standard literature, a positive volatility shock makes worthy the growth option. An interesting feature of our model is that the opposite can also occur, precisely a sudden increase of the volatility can kill the growth option. The crucial remark is that the difference $x_1 - x_0$ considered as a function of the volatility σ tends to $\frac{\mu_1-\mu_0}{r}$ when σ tends to infinity. This implies that for large volatility, condition (**H1**) is never satisfied and thus that the growth opportunity is worthless. As a matter of fact, think to an initial situation where $\theta(x_0) > V(x_0)$ (and thus condition (**H1**) holds) and consider a shock on the volatility such that (**H1**) is no more satisfied. In such a scenario, before the shock, the optimal strategy is to postpone dividends and to invest in the new technology at threshold b whereas after the uncertainty shock, the growth option is worthless and will be thus no more considered by the manager.

The effect of liquidity constraints. As a last implication of our model, we now investigate the role of liquidity constraints. In absence of liquidity constraints, the manager has unlimited cash holdings. The firm is never in bankruptcy, the manager injects money whenever needed and distribute any cash surplus in the form of dividends. In this setting, for a current cash reserve x, we thus have that $V_0(x) = x + \frac{\mu_0}{r}$ while $V_1(x - I) = x + \frac{\mu_1}{r} - I$. It follows that the manager invests in the growth option if and only if $\frac{\mu_1-\mu_0}{r} > I$, furthermore this decision is immediate. We point out here that liquidity constraints have an ambiguous effect on the decision to exercise the growth opportunity. Indeed it can happen that, in absence of liquidity constraints exercising the growth option is optimal (that is $\frac{\mu_1-\mu_0}{r} > I$), whereas it is never the case when there are liquidity constraints because condition (**H1**) does not hold. On the contrary, the growth opportunity can be worthless in absence of liquidity constraints whereas this is not the case with liquidity constraints. Such a situation occurs when $\frac{\mu_1-\mu_0}{r} < I$, Condition (**H1**) holds and $\theta(x_0) > V_0(x_0)$ (that is[4] $r(x_1 + I - x_0) < \mu_1 - \mu_0 < rI$

[4]These conditions are indeed compatible. Keeping in mind that the threshold x_0 is a single peaked function of μ_0 (see Rochet and Villeneuve [23]), consider μ_0 large, I small and μ_1 in a left neighbourhood

and $\theta(x_0) > \frac{\mu_0}{r}$). This surprising result highlights the fact that, under liquidity constraints, the investment decision in our model is dynamic and reflects the value of cash on the value of the growth option to invest.

In this paper, we consider the implications of liquidity for the dividend/investment policy of a firm that owns the perpetual right to invest in a new technology that increases its profit rate. The mathematical formulation of our problem leads to a mixed singular control/optimal stopping problem that we solve quasi explicitly using connections with two auxiliary stopping problems. A detailed analysis based on the properties of local time gives the precise optimal dividend/investment policy. This type of problem is non standard and does not seem to have attracted much attention in the Mathematical Finance literature. Our analysis follows the line of stochastic control and relies on the choice of a drifted Brownian motion for the cash reserve process in absence of dividend distribution. This modelling assumption guarantees the quasi explicit nature of value function ϕ. We use for instance this feature in Proposition 4.7 where we show that ϕ is a supersolution. Furthermore, the property of independent increments for Brownian motion plays a central role for proving that ϕ is attainable (Proposition 4.8). Clearly, the robustness of our results to more general diffusions than a drifted Brownian motion remains an open question. This and related questions must await for future work.

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of $rI + \mu_0$. It then follows that $r(x_1 + I - x_0) < 0 < \mu_1 - \mu_0 < rI$ and $\tilde{x} < x_1 + I$ which implies $\theta(x_0) \geq V_1(x_0 - I) > V_0(x_0) = \frac{\mu_0}{r}$.

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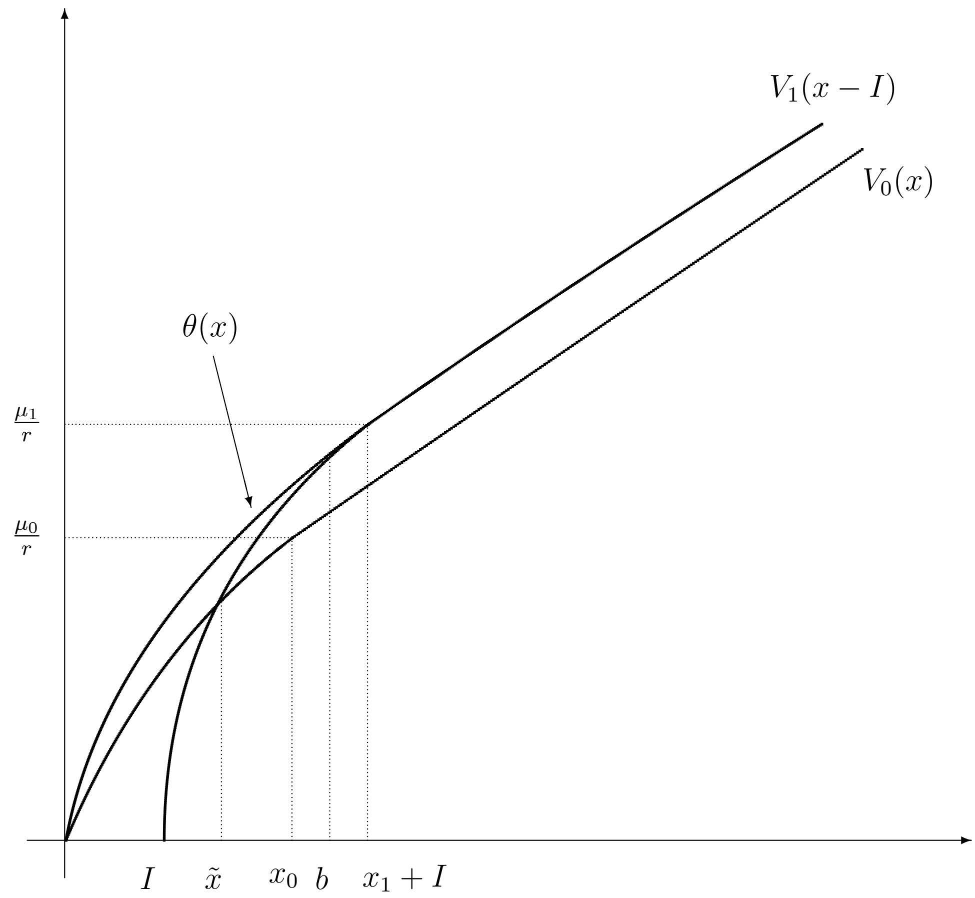

Figure 1: $\theta(x_0) > V_0(x_0)$

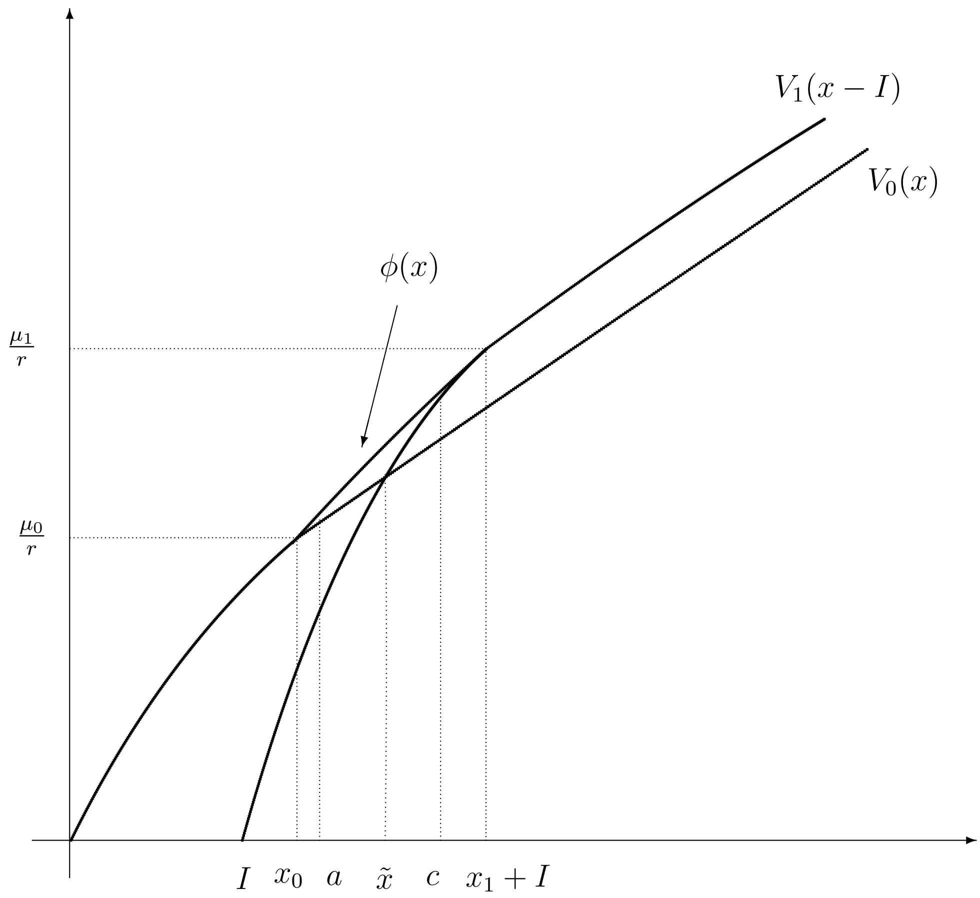

Figure 2: $\theta(x_0) < V_0(x_0)$