Irreversible Investment in Alternative Projects*

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Abstract

We study the problem of a risk-neutral decision-maker who has to choose among two alternative investment projects of different scales under output price uncertainty. We provide parameter restrictions under which the optimal investment strategy is not a trigger strategy and the optimal investment region is dichotomous. Whenever the decision-maker has the opportunity to switch from the smaller scale to the larger scale project, the dichotomy of the investment region can persist even when the volatility of the output price process becomes large.

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1 Introduction

The literature on irreversible investment under uncertainty pioneered by Arrow and Fisher [2] and Henry [12] is based on the premise that three factors mainly influence the investment decision. First, there is some uncertainty about the cash-flows that the investment project will generate. Next, investment is at least partially irreversible, in the sense that investment expenditures cannot be fully recovered. Last, there is some degree of flexibility in the timing of investment, which is valuable because it gives the decision-maker the option to wait for new information. The loss of this option value when investment occurs represents an additional opportunity cost of investment. As a result, investment options are exercised significantly above the point at which expected discounted cash-flows cover sunk capital costs, in contrast with the usual net present value rule.

In the benchmark case of a single indivisible project, the optimal investment policy can be mathematically determined as the solution of an optimal stopping problem. The prototype of this approach is the model of McDonald and Siegel [20], in which the underlying value of the investment project evolves as a geometric Brownian motion. Under this formulation, the optimal investment strategy is a trigger strategy. Specifically, the investment option should be exercised at the first time at which the value of the investment project reaches a critical threshold that can be explicitly computed using standard smooth-fit techniques (Dixit and Pindyck [8]).

The large and rapidly growing literature on real options has recently emphasized the sequential nature of investment decisions (Majd and Pindyck [19], Bar-Ilan and Strange [3]), the importance of incremental capacity choice (Pindyck [22], Kandel and Pearson [14]), and addressed issues like entry and exit decisions (Dixit [6]), costly reversibility (Abel and Eberly [1]), technology adoption (Farzin, Huisman and Kort [10]), and learning (Décamps, Mariotti and Villeneuve [4]).

In this paper, we leave aside these complex and meaningful extensions of the theory and revisit an old simple problem, namely the choice between mutually exclusive investment projects under uncertainty. Despite its simplicity, this is still a topical question, as illustrated for instance by Dias' [5] recent survey of real options in the petroleum industry.

The starting point of our analysis is the model of Dixit [7]. In his model, a decision-maker has to choose among alternative projects of different scales. The instantaneous cash-flow generated by any of these projects is a linear function of a single geometric Brownian motion that represents the dynamics of the output price, and of the constant output flow of the project. Projects with larger sunk capital costs are associated with larger output flows and, once installed, a project incurs no operating costs. When the decision-maker decides to invest, he naturally selects the project with the highest net present value. Therefore, his underlying payoff function is the upper envelope of the family of affine functions representing the net present values of investing in each project given the current output price.

Dixit's solution to this problem relies on a simple adaptation of the single project case studied by McDonald and Siegel [20]. Specifically, he argues that each project should be evaluated separately, and that the solution to the investment problem is simply to choose the project with the highest option value. As an illustration, suppose there are two projects, 1 and 2. One could for instance think of these two projects as alternative ways of producing electricity, by using a gas or a nuclear technology. Project 1 entails a lower sunk capital cost and generates a lower output flow than project 2. Separate evaluation of these two projects leads to two option values, and associated critical investment thresholds p_1 and p_2 for the output price, which is here the relevant state variable. According to Dixit, the solution of the investment problem is then twofold.

(i) If the initial output price is low enough, it is optimal to invest in the project that has the highest option value, at the first time at which the corresponding threshold p_1 or p_2 is reached by the output price.

(ii) For values of the initial output price greater than this optimal threshold, the project with the highest net present value is selected and investment is immediate.

While our solution coincides with Dixit's in case (i), we find that (ii) is questionable. Our counter-argument is as follows. Suppose that project 1 has a higher option value than project 2, which typically occurs if project 1 generates only a marginally lower output flow than project 2, but entails a significantly lower sunk capital cost. In particular, when the current output price is p_1, the decision-maker is strictly better off investing in project 1 than in project 2, so that p_1 is below the indifference point for which the net present values of the two projects are equal. We show that it is never optimal to invest in either project when the output price is at this indifference point, thus contradicting part (ii) of Dixit's solution. The reason is that the underlying payoff function of the decision-maker is not differentiable at the indifference point, where it exhibits an upward derivative jump. Using a local time argument, we show that this implies the optimality of delaying investment for values of the current output price in the neighborhood of this point.

As a result, when the option value of investing in project 1 is higher than that of investing in project 2, the optimal investment strategy is no longer a trigger strategy, and the optimal investment region in the state space is not connected. Instead, the optimal investment region is dichotomous. Specifically, there are two critical thresholds p_3 and p_4 around the indifference point such that if the current output price lies in $[p_1, p_3]$, it is optimal to invest in project 1, while if the current output price lies in $[p_4, \infty)$, it is optimal to invest in project 2. It should be noted that, for values of the output price below p_3, Dixit's solution remains correct. In particular, when the initial output price is below p_1, it is optimal to wait until the output price reaches p_1, and then to invest in project 1. By contrast, the intermediate region (p_3, p_4) is an inaction region in which the decision-maker waits to see in which project to invest. When the initial output price lies in (p_3, p_4), two scenarios can therefore occur. If the output price raises to p_4 before hitting p_3, the decision-maker will invest in project 2. By contrast, if the output price falls to p_3 before hitting p_4, the decision-maker will invest in project 1. Under these circumstances, the eventual project choice is therefore path-dependent.

The existence of the intermediate inaction region (p_3, p_4) implies that, in striking contrast with most standard real option models, it can be optimal to invest in a project even though the instantaneous profit flow associated to this project falls. Investing in project 1 when the output price falls down to p_3 is optimal because p_3 is higher than the output price threshold p_1 above which it would be optimal to invest in project 1 if that were the only investment option available, and because it would be too costly to wait until the threshold p_4 is reached to invest in project 2—in other terms, "a bird in the hand is worth two in the bush." Moreover, there is a region of the state space in which it is optimal to delay investment while it would be optimal to invest if only one project, 1 or 2, were available. This illustrates again the interaction between the two investment options, which is not taken into account by

Dixit's solution. Adding a new investment option to an existing one increases the demand for information and creates an additional incentive to delay investment, even if, when evaluated separately, the second option is dominated by the first.

As pointed out above, a sufficient condition for the dichotomy of the optimal investment region is that project 1 generates only a marginally lower output flow than project 2, but entails a significantly lower sunk capital cost. Alternatively, one could fix the output and cost parameters of the model, and investigate under which conditions on the underlying output price process the optimal investment region is dichotomous. It turns out that, if the volatility of the output price process is high enough, the optimal investment region is never dichotomous and, no matter the current output price, it is never optimal to invest in project 1. Thus, holding the output and cost parameters of the model fixed, the dichotomy of the optimal investment region requires a relatively low volatility of the output price process. More generally, greater volatility systematically leads to the adoption of larger projects. However, this result holds because in Dixit's original model, the decision-maker cannot switch from one project to the other. By contrast, we show that, whenever the decision-maker has the option to switch from project 1 to project 2 by incurring the corresponding sunk capital cost, there are output and cost parameters such that the dichotomy of the optimal investment region is preserved and investment in project 1 occurs with positive probability even for a very high volatility of the output price process. This suggests that greater volatility does not systematically lead to the adoption of larger projects when the decision-maker has the option to increase the scale of his operations as price conditions improve.

2 The basic model

2.1 The investment problem

Our basic setup and notation are directly in line with Dixit [7]. We simplify his model by assuming that only two alternative investment projects are available. Time is continuous, and labelled by $t \geq 0$. A single risk-neutral decision-maker can invest in one of two projects, $i \in \{1, 2\}$, of different scales. Project 2 generates a higher output flow than project 1, but entails a higher sunk capital cost. Formally, each project i generates an output flow $X_i > 0$, with $X_2 > X_1$, and entails a sunk capital cost $K_i > 0$, with $K_2 > K_1$. A project once installed incurs no operating costs. The instantaneous cash-flow generated by project i is $P_t X_i$, where $P = \{P_t; t \geq 0\}$ is a geometric Brownian motion with drift μ and volatility σ,

$$dP_t = \mu P_t dt + \sigma P_t dW_t; \quad t \geq 0,$$

that represents the dynamics of output price. The decision-maker discounts future revenues and costs at a constant rate $\rho > \mu$. A project once installed lasts for ever and, in the basic version of the model, there is no option to switch from the smaller scale project 1 to the larger scale project 2 once the former has been installed. Hence, for a current value p of the output price, the net expected discounted profit of investing in project i is given by:

$$V_i(p) = \frac{pX_i}{\rho - \mu} - K_i; \quad p \geq 0. \tag{1}$$

When contemplating investment, the decision-maker will select the project which generates the highest net expected discounted profit, given the current output price. So the value of investing is given by the upper envelope of V_1 and V_2:

$$V(p) = \max\{V_1(p), V_2(p)\}; \quad p \geq 0. \tag{2}$$

We denote by:

$$\tilde{p} = \frac{(\rho - \mu)(K_2 - K_1)}{X_2 - X_1} \tag{3}$$

the output price level at which the decision-maker is indifferent between investing in either project. In particular, $V_2(p) \geq V_1(p)$ if and only if $p \geq \tilde{p}$. Note that V is not differentiable at the indifference point $\tilde{p}$, with $D^+V(\tilde{p}) = X_2/(\rho - \mu) > X_1/(\rho - \mu) = D^-V(\tilde{p})$.

Let $\mathcal{T}^P$ be the set of stopping times adapted to the filtration generated by the output price P. The decision-maker's investment problem can then be written as:

$$\mathcal{V}(p) = \sup_{\tau \in \mathcal{T}^P} \mathbb{E}\big[e^{-\rho\tau} V(P^p_\tau)\big]; \quad p \geq 0, \tag{4}$$

where the superscript p in P^p_t reflects the dependence of P on its initial value p. Let us denote by $\mathcal{S} = \{p > 0 \mid \mathcal{V}(p) = V(p)\}$ the stopping region for (4), and by $\tau_{\mathcal{S}} = \inf\{t \geq 0 \mid P^p_t \in \mathcal{S}\}$ the associated stopping time. Since $\rho > \mu$, the process $\{e^{-\rho t}\mathcal{V}(P^p_t); t \geq 0\}$ is uniformly integrable and converges almost surely to 0 (see Appendix). A sufficient condition for $\tau_{\mathcal{S}}$ to be an optimal stopping time for (4) is then that $\mathcal{S}$ be non-empty (El Karoui [9]). We have the following result.

Proposition 2.1 *The stopping region $\mathcal{S}$ for (4) is non-empty.*

Having shown the existence of a solution to (4), we now characterize the stopping region $\mathcal{S}$. Before that, it will be helpful to summarize Dixit's approach to this problem.

2.2 Dixit's solution

The solution proposed by Dixit to (4) is as follows. Relying on Itô's lemma, he argues that, on the continuation region $\mathbb{R}_+ \setminus \mathcal{S}$, $\mathcal{V}$ must be of the form Bp^β, where:

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}} > 1$$

is as usual the positive root of the quadratic equation:

$$q(\xi) = \frac{1}{2}\sigma^2\xi(\xi - 1) + \mu\xi - \rho = 0. \tag{5}$$

The optimal value of B is determined as the highest B such that the contact condition $V(p) = Bp^\beta$ is satisfied for some $p \geq 0$. Since V is an upper envelope of affine functions and Bp^β is a convex function of p as $\beta > 1$, the contact between $\mathcal{V}$ and V cannot occur at the indifference point $\tilde{p}$. Thus one needs only to compute the tangency points of functions of the form $B_i p^\beta$ with each of the affine functions V_i, and select the one with the highest B_i. That is, one solves, for each i:

$$B_i p_i^\beta = \frac{p_i X_i}{\rho - \mu} - K_i, \tag{6}$$

$$\beta B_i p_i^{\beta - 1} = \frac{X_i}{\rho - \mu}, \tag{7}$$

which leads to:

$$p_i = \frac{\beta}{\beta - 1} \frac{K_i(\rho - \mu)}{X_i}, \tag{8}$$

$$B_i = \left[\frac{X_i}{(\rho - \mu)\beta}\right]^{\beta} \left(\frac{\beta - 1}{K_i}\right)^{\beta - 1}. \tag{9}$$

Dixit's solution is then as follows. Let $\bar{\imath}$ be the project for which $X_i^{\beta}/K_i^{\beta-1}$ is the largest. Then, if the initial output price p is below $p_{\bar{\imath}}$, the decision-maker should wait until the output price reaches $p_{\bar{\imath}}$, and then invest in project $\bar{\imath}$. If the initial output price p is above $p_{\bar{\imath}}$, the decision-maker should immediately invest in the best available project. In other terms, the stopping region $\mathcal{S}$ is of the form $[p_{\bar{\imath}}, \infty)$, and for any $p \in \mathcal{S}$, the decision-maker should invest in the project for which $V_i(p)$ is the largest. In particular, the optimal investment strategy would then be a simple trigger strategy, as when a single investment project is available (McDonald and Siegel [20]).

Remark. There is a direct link between Dixit's solution and that of the problem of investing in a single project with output flow X_i and sunk capital cost K_i:

$$\mathcal{V}_i(p) = \sup_{\tau \in \mathcal{T}^P} \mathbb{E}\left[e^{-\rho\tau} V_i(P_\tau^p)\right]; \quad p \geq 0. \tag{10}$$

Standard computations (Dixit and Pindyck [8]) show that the solution to (10) consists to invest as soon as the output price reaches p_i, and that $\mathcal{V}_i(p) = B_i p^{\beta}$ for $p \leq p_i$. Equivalently,

$$\mathcal{V}_i(p) = \begin{cases} \left(\dfrac{p}{p_i}\right)^{\beta} \dfrac{K_i}{\beta - 1} & \text{if} \quad p \leq p_i, \\[2ex] \dfrac{pX_i}{\rho - \mu} - K_i & \text{if} \quad p > p_i. \end{cases} \tag{11}$$

Conditions (6) and (7) correspond respectively to the usual value-matching and smooth-pasting conditions for (10). Hence, Dixit's solution essentially consists to evaluate separately the two investment options on each project, and then to select the project with the highest option value.

2.3 A counter-argument

To rule out the relatively uninteresting case in which, no matter the current output price, it is never optimal to invest in project 1, and therefore $\tau = \inf\{t \geq 0 \mid P_t^p \geq p_2\}$ is an optimal stopping time for (4), let us from now on assume that:

$$\frac{X_1^{\beta}}{K_1^{\beta-1}} > \frac{X_2^{\beta}}{K_2^{\beta-1}}. \tag{12}$$

According to Dixit's solution, the stopping region $\mathcal{S}$ should then be $[p_1, \infty)$. In particular, $p_1 < \tilde{p}$ and thus the indifference point $\tilde{p}$ belongs to $\mathcal{S}$ (otherwise, it would never be optimal to invest in project 1). However, one has the following result.

Proposition 2.2 *The indifference point $\tilde{p}$ never belongs to the stopping region $\mathcal{S}$ for (4).*

The proof of this result is technical, and relies on a local time argument based on the Itô–Tanaka–Meyer formula (Karatzas and Shreve [15, Theorem 3.7.1]). The intuition is that, because the payoff function V defined by (2) is not differentiable at the indifference point $\tilde{p}$, with $D^+V(\tilde{p}) > D^-V(\tilde{p})$, the decision-maker is always better off delaying investment when the current output price is $\tilde{p}$ rather than investing in either project. As shown below, this implies that whenever (12) holds, the stopping region $\mathcal{S}$ is not connected, and the optimal investment strategy is not a trigger strategy, contrary to Dixit's solution.

Remark. An alternative way of deriving Proposition 2.2 is as follows. From optimal stopping theory, the process $\{e^{-\rho t}\mathcal{V}(P_t^p); t \geq 0\}$ is a supermartingale (El Karoui [9]). This implies that:

$$\mathcal{A}\mathcal{V} \leq 0$$

in the sense of distributions on $\mathbb{R}_{++}$,[1] where the operator $\mathcal{A}$ is defined by:

$$\mathcal{A}g = \frac{1}{2}\sigma^2 p^2 D^2 g + \mu p D g - \rho g \tag{13}$$

(Jaillet, Lamberton and Lapeyre [13]). In particular, on the interior of the stopping region $\mathcal{S}$, one has, in the sense of distributions:

$$\mathcal{A}V = \mathcal{A}\mathcal{V} \leq 0. \tag{14}$$

The distribution $\mathcal{A}V$ can be explicitly computed as follows:

$$\mathcal{A}V = (-pX_1 + \rho K_1)1_{[0,\tilde{p})} + (-pX_2 + \rho K_2)1_{(\tilde{p},\infty)} + \frac{X_2 - X_1}{\rho - \mu}\,\delta_{\tilde{p}},$$

where the Dirac mass $\delta_{\tilde{p}}$ reflects the derivative jump of V at $\tilde{p}$. Suppose now that $\mathcal{S} = [p_1, \infty)$, as prescribed by Dixit's solution under (12). Then $\tilde{p} > p_1$ and, for any non-negative function $\phi \in \mathcal{C}_K^\infty(\mathbb{R}_+)$ with support $[\tilde{p} - \varepsilon, \tilde{p} + \varepsilon] \subset (p_1, \infty)$,

$$\langle \mathcal{A}V, \phi \rangle = \int_{\tilde{p}-\varepsilon}^{\tilde{p}+\varepsilon} (-pX_2 + \rho K_2)\phi(p)\, dp + \frac{X_2 - X_1}{\rho - \mu}\,\phi(\tilde{p}).$$

Choosing ε small enough and $\phi(\tilde{p})$ large enough, it follows that $\langle \mathcal{A}V, \phi \rangle > 0$, which contradicts (14). It follows that the stopping region $\mathcal{S}$ cannot be of the prescribed form.

2.4 The optimal investment region

We are now ready to state and prove our main result.

Theorem 2.1 *Suppose that (12) holds. Then:*

(i) The stopping region $\mathcal{S}$ for (4) is the union of two disjoint intervals $[p_1, p_3]$ and $[p_4, \infty)$, where p_1 is given by (8) and $p_3 < \tilde{p} < p_4$. If the current output price lies in $[p_1, p_3]$, it is optimal to invest in project 1, while if it lies in $[p_4, \infty)$, it is optimal to invest in project 2;

[1] Recall that a distribution T on an open set $\Omega \subset \mathbb{R}_{++}$ is a linear form on the space $\mathcal{C}_K^\infty(\Omega)$ of infinitely differentiable functions on Ω with compact support that satisfies the following property: for any sequence $\{\phi_n\}$ of functions in $\mathcal{C}_K^\infty(\Omega)$ whose supports are contained in a fixed compact subset of Ω, and such that the sequence $\{D^k\phi_n\}$ converges uniformly to 0 for every $k \in \mathbb{N}$, the sequence $\{\langle T, \phi_n \rangle\}$ converges to 0 (see for instance Ziemer [23, Definition 1.7.1]). The notation $T \leq 0$ means that $\langle T, \phi \rangle \leq 0$ for any non-negative $\phi \in \mathcal{C}_K^\infty(\Omega)$.

(ii) The value function $\mathcal{V}$ for (4) is continuously differentiable on $\mathbb{R}_{++}$ and satisfies the following variational inequalities:

$$\mathcal{A}\mathcal{V} \leq 0, \tag{15}$$

$$\mathcal{V} \geq V, \tag{16}$$

$$(\mathcal{V} - V)\,\mathcal{A}\mathcal{V} = 0, \tag{17}$$

where the operator $\mathcal{A}$ is given by (13). In particular, $\mathcal{V}$ coincides with $\mathcal{V}_1$ on $[0, p_3]$, and p_3 and p_4 are characterized by the following value-matching and smooth-pasting conditions:

$$\mathcal{V}(p_3) = V_1(p_3), \tag{18}$$

$$\mathcal{V}(p_4) = V_2(p_4), \tag{19}$$

$$D\mathcal{V}(p_3) = DV_1(p_3), \tag{20}$$

$$D\mathcal{V}(p_4) = DV_2(p_4). \tag{21}$$

The key insight of this result is that, whenever (12) holds, the optimal investment strategy is not a trigger strategy, and the optimal investment region is dichotomous. It should be noted that Dixit's solution remains valid on the segment $[0, p_1]$ of the state space. If the output price initially belongs to that region, it is optimal to wait until it reaches p_1 to invest in project 1. However, our solution departs from his in that there is a range of prices (p_3, p_4) around the indifference point $\tilde{p}$ in which it is optimal for the decision-maker to wait in order to decide in which project to invest. Note that if project 2 were not immediately available, it would be optimal to invest in project 1 in that region. Thus delay in the region (p_3, p_4) reflects the added opportunity to invest in project 2. Moreover, it is not difficult to check that $p_4 > p_2$ if (12) holds, so that for values of the output price in (p_2, p_4), the decision-maker chooses to delay investment although he would have invested immediately if only one project, 1 or 2, were available. This illustrates the interaction between the two investment options: the decision-maker is ready to delay further investment in project 2 because he knows that he will have the option to invest in project 1 if the output price deteriorates too much. Note that this remains true despite the fact that, on the interval $[0, \tilde{p})$, the option of investing in project 1 uniformly dominates that of investing in project 2 when these two options are evaluated separately, that is, $\mathcal{V}_1 > \mathcal{V}_2$ on this price range.

As shown on Figure 1, the value function $\mathcal{V}$ coincides with $\mathcal{V}_1$ on $[0, p_1]$, then takes off from V_1 on (p_3, p_4) and touches down V_2 at the point p_4. On the interval $[0, p_1]$, $\mathcal{V}$ is of the form $B_1 p^\beta$, where B_1 is given by (9). On the interval $[p_3, p_4]$, it follows from (15)–(17) that $\mathcal{V}$ is of the form $Ap^\alpha + Bp^\beta$, where:

$$\alpha = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\rho}{\sigma^2}}$$

is the negative root of the quadratic equation (5). The coefficients A and B, as well as the critical investment thresholds p_3 and p_4, can be found in principle by solving the value-matching and smooth-pasting conditions (18)–(21), although no analytic solution is available.

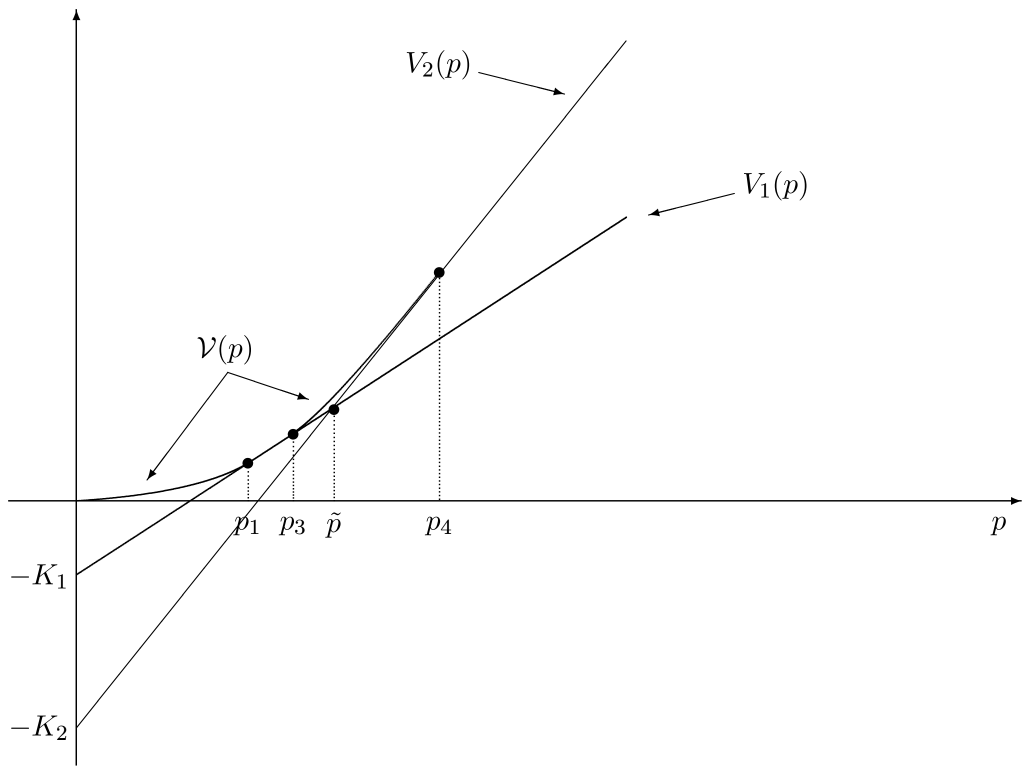

Figure 1. The value function $\mathcal{V}$

2.5 Discussion

Condition (12) is a necessary and sufficient condition for the optimal investment region to be dichotomous in Dixit's model when there are two alternative projects. For a fixed value of β, (12) holds if project 1 generates only a marginally lower output flow than project 2, but entails a significantly lower sunk capital cost. If (12) does not hold, then the option value of investing in project 2, $\mathcal{V}_2$, uniformly dominates the payoff of investing in project 1, V_1, so that it never pays to select project 1. It is worth noting that, whenever σ goes to infinity and thus β goes to 1, condition (12) is never satisfied as $X_1 < X_2$. Thus, holding the output and cost parameters of the model fixed, the dichotomy of the optimal investment region requires a relatively low volatility of the output price process. As we shall see, this is no longer the case once one allows the decision-maker to switch from project 1 to project 2.

Examples of investment policies more complex than a simple trigger strategy abound in the literature. The model of entry and exit by Dixit [6] generates a two-trigger strategy: the critical output price above which it is optimal to enter an industry is higher than the one below which it is optimal to exit from it. Similarly, Abel and Eberly [1] show that the investment policy of a firm under costly reversibility is characterized by a range of inaction in which it is optimal neither to purchase nor to sell capital. However, a common feature of these and related papers is that the lower boundary of the inaction region corresponds to a disinvestment decision. In our model, by contrast, both p_3 and p_4 correspond to an investment decision, albeit in different projects.

A striking feature of the optimal investment strategy is that it may be optimal to invest in project 1 after a drop in the output price if the output price initially lies in (p_3, p_4). Rational investment in a down market can also be triggered by other factors. Competition is one of them. Grenadier [11] develops a model of strategic exercise of investment options in which competitors can simultaneously invest when the value of the investment goes down in an attempt to avoid preemption. Décamps, Mariotti and Villeneuve [4] show that a similar phenomenon can occur in a single decision-maker context when the drift of the value process is not known ex-ante. The decision to invest then depends on both the observed current value of the project and the beliefs about the unknown drift. This generates path dependency in the optimal investment strategy, and it may be rational to invest after a drop in the current value of the investment project. As we have shown, this phenomenon can also result from the interaction between several investment options.

Finally, although we have derived our results for the case of two alternative projects, it is conceptually straightforward to extend them to an arbitrary number of projects with output flows $X_1, \ldots, X_N$ and sunk capital costs $K_1, \ldots, K_N$ ranked in increasing order. The analogue of Proposition 2.2 is that it is never optimal to invest when the current output price corresponds to a point of non-differentiability of the upper envelope of the net expected discounted profits of investing in each project. As soon as the option of investing in project N does not dominate all the other investment options, the optimal investment strategy will be characterized by several disconnected inaction regions, in which the decision-maker waits in order to determine in which project to invest.

3 Project switching

3.1 The investment problem

In this section, we modify Dixit's [7] original setup by allowing the decision-maker to switch from one project to another by incurring the corresponding sunk capital cost. Since project 2 generates a higher output flow than project 1, it is clear that it is never optimal to switch from project 2, once installed, to project 1. Thus, for a current value p of the output price, the net expected discounted profit of investing in project 2 is given by $V_2(p)$ as above. By contrast, once project 1 is installed, it is optimal to switch to project 2 if the output price becomes large enough, as we shall see below. The decision-maker's investment problem can then be written as:

$$\mathcal{V}^{switch}(p) = \sup_{\tau_1, \tau_2 \in \mathcal{T}^P, \tau_1 \leq \tau_2} \mathbb{E}\left[1_{\{\tau_1 < \tau_2\}}\left(\int_{\tau_1}^{\tau_2} e^{-\rho t} P_t^p X_1 \, dt - e^{-\rho \tau_1} K_1 + e^{-\rho \tau_2} V_2(P_{\tau_2}^p)\right)\right]$$
$$+ \mathbb{E}\left[1_{\{\tau_1 = \tau_2\}} e^{-\rho \tau_2} V_2(P_{\tau_2}^p)\right]; \quad p \geq 0. \tag{22}$$

Here, τ_1 denotes the first time at which the decision-maker invests, whichever project he selects, and $\tau_2 \geq \tau_1$ denotes the first time at which he invests in project 2. We shall solve (22) by dynamic programming techniques. As a first step, this requires computing the value of investing sequentially in projects 1 and 2.

3.2 The switching option

To determine the value of investing in project 1, we must first take into account the option of switching later from project 1 to project 2. Specifically, for a current value p of the output price, the net expected discounted profit of investing in project 1 is given by:

$$V_1^s(p) = \sup_{\tau \in \mathcal{T}^P} \mathbb{E}\left[\int_0^\tau e^{-\rho t} P_t^p X_1 \, dt + e^{-\rho \tau} V_2(P_\tau^p)\right] - K_1. \tag{23}$$

The optimal stopping problem in (23) is standard. In particular, $V_1^s + K_1$ is continuously differentiable on $\mathbb{R}_{++}$ and satisfies the following variational inequalities:

$$\mathcal{A}(V_1^s + K_1) + pX_1 \leq 0,$$

$$V_1^s + K_1 \geq V_2,$$

$$(V_1^s + K_1 - V_2)\,[\mathcal{A}(V_1^s + K_1) + pX_1] = 0,$$

where the operator $\mathcal{A}$ is given by (13). Simple computations then lead to the following explicit expression for V_1^s:

$$V_1^s(p) = \begin{cases} \dfrac{pX_1}{\rho - \mu} + \left(\dfrac{p}{p_{12}^s}\right)^\beta \dfrac{K_2}{\beta - 1} - K_1 & \text{if} \quad p \leq p_{12}^s, \\[2ex] \dfrac{pX_2}{\rho - \mu} - K_2 - K_1 & \text{if} \quad p > p_{12}^s, \end{cases} \tag{24}$$

where:

$$p_{12}^s = \frac{\beta}{\beta - 1} \frac{K_2(\rho - \mu)}{X_2 - X_1} \tag{25}$$

is the critical output price level above which it is optimal to switch from project 1 to project 2. By (8) and (25), $p_{12}^s > p_2$. Intuitively, this is because the expected discounted gain from switching from project 1 to project 2 is less than the expected discounted gain from investing in project 2 starting from a situation where no project is in place, while the cost to the decision-maker is the same.

From (1) and (24), $V_1^s(0) > V_2(0)$ and $V_1^s(p) < V_2(p)$ for any $p > p_{12}^s$. Since V_1^s is convex, with a slope that is everywhere smaller or equal than the constant slope of V_2, there exists a unique output price level $\tilde{p}^s < p_{12}^s$ such that $V_2(p) \geq V_1^s(p)$ if and only if $p \geq \tilde{p}^s$. Moreover, since the slopes of V_1^s and V_2 coincide only on $[p_{12}^s, \infty)$, $DV_2(\tilde{p}^s) > DV_1^s(\tilde{p}^s)$.

Remark. Coming back to problem (22), note that it is clearly suboptimal to invest in project 1 when the current output price is above p_{12}^s, since it would then be optimal to switch immediately to project 2, and thus investing directly in project 2 is a dominant strategy. In particular, since $p_{12}^s > p_2$, delaying investment in project 2 is pointless when the current output price is above p_{12}^s.

3.3 Constrained sequential investment

As a benchmark, it is useful to consider what would happen if the decision-maker were constrained to invest in project 1 before investing in project 2. His investment problem can then be written as:

$$\mathcal{V}^{seq}(p) = \sup_{\tau_1, \tau_2 \in \mathcal{T}^P, \, \tau_1 \leq \tau_2} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} e^{-\rho t} P_t^p X_1 \, dt - e^{-\rho \tau_1} K_1 + e^{-\rho \tau_2} V_2(P_{\tau_2}^p)\right]; \quad p \geq 0. \tag{26}$$

Here, τ_1 denotes the first time at which the decision-maker invests in project 2, and $\tau_2 \geq \tau_1$ denotes the first time at which he invests in project 2. We solve (26) by dynamic programming techniques. Given a current output price p, the value of investing in project 1, taking into account the option to switch later to project 2, is $V_1^s(p)$. Hence, in analogy with (10), it is natural to consider the following auxiliary optimal stopping problem, in which the payoff function is given by V_1^s instead of V_1:

$$\mathcal{V}_1^s(p) = \sup_{\tau \in \mathcal{T}^P} \mathbb{E}\left[e^{-\rho \tau} V_1^s(P_\tau^p)\right]; \quad p \geq 0. \tag{27}$$

We have the following result.

Lemma 3.1 *For any $p \geq 0$, $\mathcal{V}^{seq}(p) \leq \mathcal{V}_1^s(p)$.*

Our strategy for solving (26) is then as follows. First, we characterize the solution to (27). Next, we exhibit stopping times τ_1 and τ_2 which achieve the payoff $\mathcal{V}_1^s(p)$ in (26), and are thus optimal given Lemma 3.1.

Compared to (10), the difficulty in (27) is that V_1^s is not affine in the current output price, reflecting the fact that V_1^s is itself an option value. One can nevertheless proceed in a standard way by seeking a value function $\mathcal{V}_1^s$ for (27) that is continuously differentiable on $\mathbb{R}_{++}$ and satisfies the following variational inequalities:

$$\mathcal{A}\mathcal{V}_1^s \leq 0,$$

$$\mathcal{V}_1^s \geq V_1^s,$$

$$(\mathcal{V}_1^s - V_1^s)\,\mathcal{A}\mathcal{V}_1^s = 0,$$

where the operator $\mathcal{A}$ is given by (13). Simple computations show that the stopping region $\mathcal{S}_1^s = \{p > 0 \mid \mathcal{V}_1^s(p) = V_1^s(p)\}$ for (27) is of the form $[p_1^s, \infty)$, and that two cases can occur:

(i) If $X_2/X_1 < I_2/I_1 + 1$, then:

$$p_1^s = \frac{\beta}{\beta - 1} \frac{K_1(\rho - \mu)}{X_1} < p_{12}^s. \tag{28}$$

(ii) If $X_2/X_1 \geq I_2/I_1 + 1$, then:

$$p_1^s = \frac{\beta}{\beta - 1} \frac{(K_1 + K_2)(\rho - \mu)}{X_2} \geq p_{12}^s. \tag{29}$$

In both cases, the value function $\mathcal{V}_1^s$ can be written as:

$$\mathcal{V}_1^s(p) = \begin{cases} \left(\dfrac{p}{p_1^s}\right)^{\beta} V_1^s(p_1^s) & \text{if} \quad p \leq p_1^s, \\ \\ V_1^s(p) & \text{if} \quad p > p_1^s. \end{cases} \tag{30}$$

We can now exhibit stopping times that achieve the value $\mathcal{V}_1^s(p)$ in the sequential investment problem (26), thereby verifying that $\mathcal{V}^{seq}(p) = \mathcal{V}_1^s(p)$. In case (i), the optimal stopping times are $\tau_1 = \inf\{t \geq 0 \mid P_t^p \geq p_1^s\}$ and $\tau_2 = \inf\{t \geq 0 \mid P_t^p \geq p_{12}^s\}$. If $P_0 < p_{12}^s$, investment is sequential: it is optimal to invest in project 1 when the output price reaches p_1^s, and then to wait until the output price reaches p_{12}^s to invest in project 2. Note that, in this case, the optimal investment threshold p_1^s for project 1 is equal to p_1 as given in (8). Thus investment in project 1 is myopic, in the sense that it occurs at the same time than if the option of switching to project 2 were not present, as in Leahy [18]. In case (ii), the optimal stopping times are $\tau_1 = \tau_2 = \inf\{t \geq 0 \mid P_t^p \geq p_1^s\}$, and the two investments occur simultaneously.

3.4 The optimal investment strategy

We are now in position to solve problem (22). As for (26), we adopt a dynamic programming approach. Given a current output price p, the value of investing in project 1, taking into account the option to switch later to project 2, is $V_1^s(p)$. Hence, in analogy with (2) and (4), it is natural to consider the upper envelope of V_1^s and V_2:

$$V^s(p) = \max\{V_1^s(p), V_2(p)\}; \quad p \geq 0, \tag{31}$$

and the following auxiliary optimal stopping problem:

$$\mathcal{V}^s(p) = \sup_{\tau \in \mathcal{T}^P} \mathbb{E}\big[e^{-\rho\tau} V^s(P_\tau^p)\big]; \quad p \geq 0. \tag{32}$$

It should be noted that, in analogy with the basic model, V^s is not differentiable at $\tilde{p}^s$, with $D^+V^s(\tilde{p}^s) = DV_2(\tilde{p}^s) > DV_1^s(\tilde{p}^s) = D^-V^s(\tilde{p}^s)$. We have the following result.

Lemma 3.2 *For any $p \geq 0$, $\mathcal{V}^{switch}(p) \leq \mathcal{V}^s(p)$.*

Our strategy for solving (22) is then as follows. First, we characterize the solution to (32). Next, we exhibit stopping times τ_1 and τ_2 which achieve the payoff $\mathcal{V}^s(p)$ in (22), and are thus optimal given Lemma 3.2.

Let us denote by $\mathcal{S}^s = \{p > 0 \mid \mathcal{V}^s(p) = V^s(p)\}$ the stopping region for (32), and by $\tau_{\mathcal{S}^s} = \inf\{t \geq 0 \mid P_t^p \in \mathcal{S}^s\}$ the associated stopping time. As for (4), $\tau_{\mathcal{S}^s}$ is an optimal stopping time for (32) if $\mathcal{S}^s$ is non-empty. We have the following result.

Proposition 3.1 *The stopping region $\mathcal{S}^s$ for (32) is non-empty.*

Proceeding as in Section 2, we first rule out the relatively uninteresting case in which, no matter the current output price, it is never optimal to invest in project 1 and 2 sequentially, and $\tau_1 = \tau_2 = \inf\{t \geq 0 \mid P_t^p \geq p_2\}$ are optimal stopping times for (22). A necessary and sufficient condition for this not to occur is that $V_1^s(p) > \mathcal{V}_2(p)$ for some $p \geq 0$. Using the explicit formulas (24) and (11) for V_1^s and $\mathcal{V}_2$, it is straightforward to verify that this condition is equivalent to:

$$\frac{X_1^\beta}{K_1^{\beta-1}} > \left[1-\left(1-\frac{X_1}{X_2}\right)^\beta\right]\frac{X_2^\beta}{K_2^{\beta-1}}, \tag{33}$$

which is weaker than (12). It should be noted that, because $\beta > 1$, (33) implies that $K_1/X_1 < K_2/X_2$, so that $p_1 < p_2$ by (8). In particular, one can check that (33) implies that $V_1^s(p_1) > \mathcal{V}_2(p_1) \geq V_2(p_1)$, so that $p_1 < \tilde{p}^s$ by definition of $\tilde{p}^s$. Finally, in terms of the sequential investment problem (26), we are in case (i), and thus $p_1^s = p_1$. We have the following result.

Theorem 3.1 *Suppose that (33) holds. Then*:

(i) The stopping region $\mathcal{S}^s$ *for (32) is the union of two disjoint intervals* $[p_1, p_3^s]$ *and* $[p_4^s, \infty)$, *where* p_1 *is given by (8) and* $p_3^s < \tilde{p}^s < p_4^s$;

(ii) The value function $\mathcal{V}^s$ *for (32) is continuously differentiable on* $\mathbb{R}_{++}$ *and satisfies the following variational inequalities*:

$$\mathcal{A}\mathcal{V}^s \leq 0, \tag{34}$$

$$\mathcal{V}^s \geq V^s, \tag{35}$$

$$(\mathcal{V}^s - V)\,\mathcal{A}\mathcal{V}^s = 0, \tag{36}$$

where the operator $\mathcal{A}$ *is given by (13). In particular,* $\mathcal{V}^s$ *coincides with* $\mathcal{V}^{seq}$ *on* $[0, p_3^s]$, *and* p_3^s *and* p_4^s *are characterized by the following value-matching and smooth-pasting conditions*:

$$\mathcal{V}^s(p_3^s) = V_1^s(p_3^s), \tag{37}$$

$$\mathcal{V}^s(p_4^s) = V_2(p_4^s), \tag{38}$$

$$D\mathcal{V}^s(p_3^s) = DV_1^s(p_3^s), \tag{39}$$

$$D\mathcal{V}^s(p_4^s) = DV_2(p_4^s). \tag{40}$$

(iii) $\mathcal{V}^{switch} = \mathcal{V}^s$ *and the optimal stopping times for (22) are*:

$$\tau_1 = \inf\{t \geq 0 \mid P_t^p \in \mathcal{S}^s\},$$

$$\tau_2 = \inf\left\{t \geq 0 \mid P_t^p \geq p_4^s,\ \min_{s\in[0,t]} P_s^p > p_3^s\right\} \wedge \inf\left\{t \geq 0 \mid P_t^p \geq p_{12}^s,\ \min_{s\in[0,t]} P_s^p \leq p_3^s\right\}.$$

In analogy with Proposition 2.2, the bulk of the proof consists to show that the indifference point $\tilde{p}^s$ never belongs to the optimal stopping region $\mathcal{S}^s$. The remaining arguments closely follow those used to prove Theorem 2.1.

As shown on Figure 2, the value function $\mathcal{V}^{switch}$ coincides with $\mathcal{V}^{seq}$ on $[0, p_1]$, then takes off from V_1^s on (p_3^s, p_4^s) and touches down V_2 at the point p_4^s. On the interval $[0, p_1]$, $\mathcal{V}^{switch}$ is of the form $B_1^s p^\beta$, where:

$$B_1^s = \frac{1}{p_1^\beta}\left[\frac{K_1}{\beta - 1} + \left(\frac{p_1}{p_{12}^s}\right)^\beta \frac{K_2}{\beta - 1}\right]$$

can be easily computed from (24) and (30). On the interval $[p_3^s, p_4^s]$, it follows from (34)–(36) that $\mathcal{V}^{switch}$ is of the form $A^s p^\alpha + B^s p^\beta$. The coefficients A^s and B^s, as well as the critical investment thresholds p_3^s and p_4^s, can be found in principle by solving the value matching and smooth-pasting conditions (37)–(40) although, as in the basic model, no analytic solution is available.

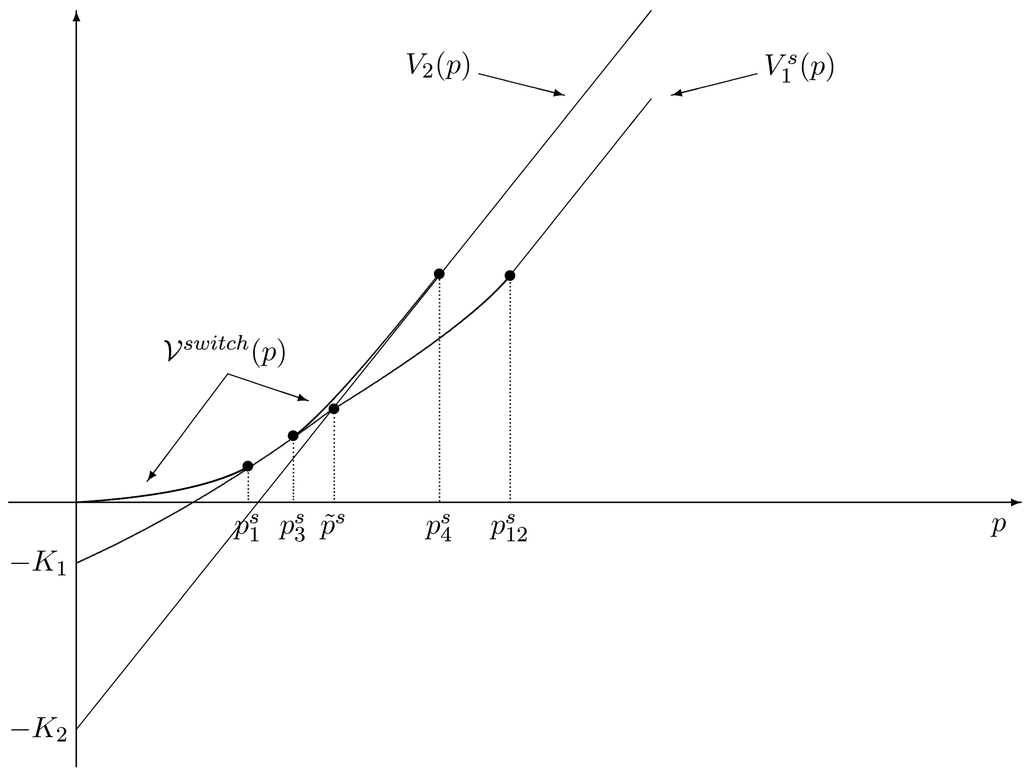

Figure 2. The value function $\mathcal{V}^{switch}$

3.5 Discussion

In the basic model, the dichotomy of the investment region is not robust to a high volatility of the output price process, as the option of investing in project 2 uniformly dominates the payoff of investing in project 1, and thus (12) is no longer satisfied. This is no longer the case when one allows switching from project 1 to project 2. Indeed, the necessary and sufficient condition (33) for a dichotomous investment region can be rewritten as:

$$\left(\frac{X_1}{X_2}\right)^\beta \left(\frac{K_1}{K_2}\right)^{1-\beta} + \left(1 - \frac{X_1}{X_2}\right)^\beta > 1. \quad (41)$$

Note that whenever σ goes to infinity and thus β goes to 1, (41) becomes an equality. Next, the derivative of the left-hand side of (41) with respect to β at $\beta = 1$ is $[\ln(X_1/X_2) - \ln(K_1/K_2)]X_1/X_2 + \ln(1 - X_1/X_2)(1 - X_1/X_2)$ which is positive if $K_1/X_1 < K_2/X_2$ and X_1 is close enough to X_2. Under those circumstances, (41) still holds for β close to 1, and the dichotomy of the investment region persists even if the output price process if highly volatile.

Appendix

Proof of Proposition 2.1: Since $\rho > \mu$, the process $\{e^{-\rho t} P_t^p; t \geq 0\}$ is a positive and continuous supermartingale. In particular, it has a last element, namely 0. Hence, by the optional sampling theorem (Karatzas and Shreve [15, Theorem 1.3.22]), $\mathbb{E}\,[e^{-\rho\tau} P_\tau^p] \leq p$ for any stopping time $\tau \in \mathcal{T}^P$. Noting that there exist positive constants C_1 and C_2 such that $V(p) \leq C_1 p + C_2$ for any $p \geq 0$, we therefore have:

$$\mathbb{E}\big[e^{-\rho\tau} V(P_\tau^p)\big] \leq C_1 \mathbb{E}\big[e^{-\rho\tau} P_\tau^p\big] + C_2 \leq C_1 p + C_2$$

for any stopping time $\tau \in \mathcal{T}^P$, and thus $\mathcal{V}(p) \leq C_1 p + C_2$ for any $p \geq 0$. It follows in particular that $\mathcal{V}(p)$ is well-defined and finite and, since $\rho > \mu$, that the process $\{e^{-\rho t}\mathcal{V}(P_t^p); t \geq 0\}$ is uniformly integrable and converges almost surely to 0. According to optimal stopping theory, the process $\{e^{-\rho(t \wedge \tau_S)}\mathcal{V}(P_{t\wedge\tau_S}^p); t \geq 0\}$ is a martingale (El Karoui [9]). Now, suppose that $\mathcal{S} = \emptyset$. Then the process $\{e^{-\rho t}\mathcal{V}(P_t^p); t \geq 0\}$ is a martingale. Therefore, for any $t \geq 0$,

$$\mathcal{V}(p) = \mathbb{E}\big[e^{-\rho t}\mathcal{V}(P_t^p)\big] \leq C_1 p\, e^{-(\rho-\mu)t} + C_2 e^{-\rho t}.$$

Since t is arbitrary and $\rho > \mu$, it follows that $\mathcal{V}$ is identically equal to 0, which is absurd as $\mathcal{V} \geq \max\{0, V_1, V_2\}$. Thus $\mathcal{S} \neq \emptyset$, as claimed. □

Proof of Proposition 2.2: Let us define the operator $\mathcal{A}$ by (13) and let $f = V_2 - V_1$. By construction, f is a difference of two affine functions which satisfies $f(\tilde{p}) = 0$ and $Df(\tilde{p}) > 0$, and we have $V = V_1 + \max\{f, 0\}$. Since V_1 is of class $\mathcal{C}^2$ on $\mathbb{R}_{++}$, it follows from Itô's lemma and the Itô–Tanaka–Meyer formula (Karatzas and Shreve, [15, Theorem 3.7.1]) that, for any $t \geq 0$,

$$\begin{aligned}\mathbb{E}\Big[e^{-\rho t} V(P_t^{\tilde{p}})\Big] &= V(\tilde{p}) + \mathbb{E}\left[\int_0^t e^{-\rho s}\mathcal{A}V_1(P_s^{\tilde{p}})\, ds\right] \\ &\quad + \mathbb{E}\left[\int_0^t e^{-\rho s}\mathcal{A}f(P_s^{\tilde{p}}) 1_{\{P_s^{\tilde{p}} \geq \tilde{p}\}}\, ds\right] + \frac{1}{2}\, Df(\tilde{p})\, \mathbb{E}\Big[e^{-rt} L_t^{\tilde{p}}\Big],\end{aligned}$$

where $\{L_t^{\tilde{p}}; t \geq 0\}$ is the local time for the continuous semimartingale P at $\tilde{p}$. We treat each term on the right-hand side of this equation separately. For the second term, one has $\mathcal{A}V_1(p) = -pX_1 + \rho K_1$. Hence, for any $t \geq 0$,

$$\begin{aligned}\left|\mathbb{E}\left[\int_0^t e^{-\rho s}\mathcal{A}V_1(P_s^{\tilde{p}})\, ds\right]\right| &\leq \mathbb{E}\left[\int_0^t e^{-\rho s}\left|\mathcal{A}V_1(P_s^{\tilde{p}})\right| ds\right] \\ &\leq \mathbb{E}\left[\int_0^t e^{-\rho s}(P_s^{\tilde{p}} X_1 + \rho K_1)\, ds\right] \\ &= \frac{\tilde{p} X_1}{\rho - \mu}\,[1 - e^{-(\rho-\mu)t}] + K_1(1 - e^{-\rho t}).\end{aligned}$$

Since $\rho > \mu$, it follows that:

$$\mathbb{E}\Big[\int_0^t e^{-rs}\mathcal{A}V_1(P_s^{\tilde{p}})\,ds\Big] = o(\sqrt{t}).$$

For the third term, note that there exists positive constants C_1 and C_2 such that $|\mathcal{A}f(p)| \leq C_1 p + C_2$ for $p \geq \tilde{p}$. Hence, for every $t \geq 0$,

$$\begin{aligned}
\Big|\mathbb{E}\Big[\int_0^t e^{-\rho s}\mathcal{A}f(P_s^{\tilde{p}})1_{\{P_s^{\tilde{p}}\geq\tilde{p}\}}\,ds\Big]\Big| &\leq \mathbb{E}\Big[\int_0^t e^{-\rho s}\,\Big|\mathcal{A}f(P_s^{\tilde{p}})1_{\{P_s^{\tilde{p}}\geq\tilde{p}\}}\Big|\,ds\Big] \\
&\leq \mathbb{E}\Big[\int_0^t e^{-\rho s}(C_1 P_s^{\tilde{p}} + C_2)\,ds\Big] \\
&= \frac{C_1\tilde{p}}{\rho-\mu}\,[1-e^{-(\rho-\mu)t}] + \frac{C_2}{\rho}\,(1-e^{-\rho t}).
\end{aligned}$$

Since $\rho > \mu$, it follows that:

$$\mathbb{E}\Big[\int_0^t e^{-\rho s}\mathcal{A}f(P_s^{\tilde{p}})1_{\{P_s^{\tilde{p}}\geq\tilde{p}\}}\,ds\Big] = o(\sqrt{t}).$$

For the fourth term, note that by the Itô–Tanaka–Meyer formula,

$$\mathbb{E}\Big[e^{-\rho t}(P_t^{\tilde{p}}-\tilde{p})^+\Big] = \mu\,\mathbb{E}\Big[\int_0^t e^{-\rho s}P_s^{\tilde{p}}1_{\{P_s^{\tilde{p}}\geq\tilde{p}\}}\,ds\Big] - \rho\,\mathbb{E}\Big[\int_0^t e^{-\rho s}(P_s^{\tilde{p}}-\tilde{p})^+\,ds\Big] + \frac{1}{2}\,\mathbb{E}\Big[e^{-\rho t}L_t^{\tilde{p}}\Big].$$

The first two terms on the right-hand side of this equality can be shown to be on the order of $o(\sqrt{t})$ by the same reasoning as above. Hence:

$$\begin{aligned}
\frac{1}{2}\,\mathbb{E}\Big[e^{-\rho t}L_t^{\tilde{p}}\Big] &= \mathbb{E}\Big[e^{-\rho t}(P_t^{\tilde{p}}-\tilde{p})^+\Big] + o(\sqrt{t}) \\
&= e^{-\rho t}\tilde{p}\,\Big\{e^{\mu t}\,\mathbb{P}\Big[W_1 \leq \frac{\mu+\sigma^2/2}{\sigma}\,\sqrt{t}\Big] - \mathbb{P}\Big[W_1 \leq \frac{\mu-\sigma^2/2}{\sigma}\,\sqrt{t}\Big]\Big\} + o(\sqrt{t}) \\
&= \tilde{p}\,\sigma\sqrt{\frac{t}{2\pi}} + o(\sqrt{t}),
\end{aligned}$$

where the second and third equalities follow from a direct computation. Summing up, we therefore obtain:

$$\mathbb{E}\Big[e^{-\rho t}V(P_t^{\tilde{p}})\Big] = V(\tilde{p}) + \tilde{p}\,\sigma Df(\tilde{p})\sqrt{\frac{t}{2\pi}} + o(\sqrt{t}).$$

Since $Df(\tilde{p}) > 0$, letting t tend to 0 yields that $\sup_{\tau\in\mathcal{T}^P}\mathbb{E}\,[e^{-\rho\tau}V(P_\tau^{\tilde{p}})] > V(\tilde{p})$. Hence $\tilde{p}$ does not belong to the stopping region $\mathcal{S}$. □

Proof of Theorem 2.1: (i) Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be the option values for each project as defined by (10). It is easy to check from (8) that (12) and $K_1 < K_2$ imply that $p_1 < p_2$ and $V_1(p_1) > B_2 p_1^\beta = \mathcal{V}_2(p_1)$. It follows that $\mathcal{V} \neq \mathcal{V}_2$, and thus $\mathcal{S}\cap[0,\tilde{p}] \neq \emptyset$. Moreover, by Proposition 2.2, $\inf\mathcal{S} < \tilde{p}$. It is clear that for $p < p_1$, $\mathcal{V}(p) \geq \mathcal{V}_1(p) > V_1(p)$, so that $\inf\mathcal{S} \geq p_1$. We now prove the reverse inequality, establishing that $\inf\mathcal{S} = p_1$. We have:

$$\mathcal{V}(p_1) = \mathbb{E}\big[e^{-\rho\tau_{\mathcal{S}}}V_1(P_{\tau_{\mathcal{S}}}^{p_1})\big] \leq \sup_{\tau\in\mathcal{T}^P}\mathbb{E}\big[e^{-\rho\tau}V_1(P_\tau^{p_1})\big] = V_1(p_1),$$

where the first equality follows from the fact that $\inf \mathcal{S} < \tilde{p}$, and the second from the definition of p_1. Since $\mathcal{V}(p_1) \geq V_1(p_1)$, these quantities are in fact equal, and it follows that $p_1 \geq \inf \mathcal{S}$, as claimed. A similar argument establishes that $\mathcal{S} \cap [0, \tilde{p}]$ is an interval $[p_1, p_3]$, and by Proposition 2.2, $p_3 < \tilde{p}$. We now prove that $\mathcal{S} \cap [\tilde{p}, \infty) \neq \emptyset$. If this were not the case, then $\mathcal{S} = [p_1, p_3]$, and for any $p > p_3$, we would have:

$$\mathcal{V}(p) = \mathbb{E}\big[e^{-\rho\tau_{\mathcal{S}}} V_1(P^p_{\tau_{\mathcal{S}}})\big] > V_2(p),$$

or, equivalently:

$$\left(\frac{p}{p_3}\right)^{\alpha}\left(\frac{p_3 X_1}{\rho-\mu} - K_1\right) > \frac{pX_2}{\rho-\mu} - K_2, \tag{42}$$

where α is the negative root of the quadratic equation (5). For p large enough, (42) is violated, and we obtain a contradiction. Let $p_4 = \inf \mathcal{S} \cap [\tilde{p}, \infty)$. By Proposition 2.2, $p_4 > \tilde{p}$. Moreover, it is immediate that $p_4 \geq p_2$, since otherwise waiting until the output price exceeds p_2 to invest in project 2 would secure the decision-maker a strictly greater payoff than investing in project 2 when the current value of the output price is p_4. It then follows by an argument similar to the one used to prove that $p_1 = \inf \mathcal{S}$ that $\mathcal{S} \cap [\tilde{p}, \infty) = [p_4, \infty)$.

(ii) The variational inequalities (15)–(17) characterize $\mathcal{V}$ if $\mathcal{V}$ is differentiable (Øksendal [21, Theorem 10.4.1]). We only need to prove this at the investment triggers p_1, p_3 and p_4. On $[0, p_1]$, we clearly have $\mathcal{V} = \mathcal{V}_1$, and the differentiability of $\mathcal{V}$ at p_1 follows from a standard smooth-pasting argument for problem (10). We now prove that $D\mathcal{V}(p_3) = DV_1(p_3)$. Since $\mathcal{V} \geq V_1$ and $\mathcal{V}(p_3) = V_1(p_3)$, we have, for any $\varepsilon > 0$,

$$\frac{\mathcal{V}(p_3+\varepsilon) - \mathcal{V}(p_3)}{\varepsilon} = \frac{\mathcal{V}(p_3+\varepsilon) - V_1(p_3)}{\varepsilon} \geq \frac{V_1(p_3+\varepsilon) - V_1(p_3)}{\varepsilon}. \tag{43}$$

Following Karatzas and Shreve [16, Lemma 7.8], let us introduce the stopping times $T^1_\varepsilon = \inf\{t \geq 0 \mid (p_3+\varepsilon)H_t \leq p_3\}$ and $T^2_\varepsilon = \inf\{t \geq 0 \mid (p_3+\varepsilon)H_t \geq p_4\}$, where $H_t = \exp((\mu - \sigma^2/2)t + \sigma W_t)$ for each $t \geq 0$. The stopping time $T_\varepsilon = T^1_\varepsilon \wedge T^2_\varepsilon$ is optimal whenever the initial value of P is $p_3 + \varepsilon$. Therefore:

$$\mathcal{V}(p_3+\varepsilon) = \mathbb{E}\big[e^{-\rho T_\varepsilon} V((p_3+\varepsilon)H_{T_\varepsilon})\big] \leq \mathcal{V}(p_3) + \mathbb{E}\big[e^{-\rho T_\varepsilon}[V((p_3+\varepsilon)H_{T_\varepsilon}) - V(p_3 H_{T_\varepsilon})]\big],$$

where the inequality follows from the fact that T_ε is not an optimal stopping time whenever the initial value of P is p_3. Up to the term $\mathcal{V}(p_3)$, the right-hand side of this inequality can be written as:

$$\left[V_1(p_3) - V_1\left(\frac{p_3^2}{p_3+\varepsilon}\right)\right]\mathbb{E}\big[e^{-\rho T_\varepsilon} 1_{\{T_\varepsilon = T^1_\varepsilon\}}\big] + \left[V_2(p_4) - V_2\left(\frac{p_3 p_4}{p_3+\varepsilon}\right)\right]\mathbb{E}\big[e^{-\rho T_\varepsilon} 1_{\{T_\varepsilon = T^2_\varepsilon\}}\big].$$

Using standard results on the Laplace transform of the exit time of a Brownian motion on a finite interval (Karatzas and Shreve [15, §2.8.C]), together with the continuity of V_2, we obtain that:

$$\left[V_2(p_4) - V_2\left(\frac{p_3 p_4}{p_3+\varepsilon}\right)\right]\mathbb{E}\big[e^{-\rho T_\varepsilon} 1_{\{T_\varepsilon = T^2_\varepsilon\}}\big] = o(\varepsilon).$$

It follows that:

$$\frac{\mathcal{V}(p_3+\varepsilon) - \mathcal{V}(p_3)}{\varepsilon} \leq \frac{V_1(p_3) - V_1\left(\dfrac{p_3^2}{p_3+\varepsilon}\right)}{\varepsilon} + o(1). \tag{44}$$

Using (43)–(44) and letting ε go to 0, we obtain that $D^+\mathcal{V}(p_3) = D^+V_1(p_3)$, which implies the result. The proof of the differentiability of $\mathcal{V}$ at p_4 is similar, and therefore omitted. $\square$

Proof of Lemma 3.1: Consider two stopping times $\tau_1, \tau_2 \in \mathcal{T}^P$ such that $\tau_1 \leq \tau_2$. Then, using the strong Markov property and the definition of V_1^s, we obtain that:

$$\begin{aligned}
&\mathbb{E}\Big[\int_{\tau_1}^{\tau_2} e^{-\rho t} P_t^p X_1\, dt - e^{-\rho\tau_1} K_1 + e^{-\rho\tau_2} V_2(P_{\tau_2}^p)\Big] \\
&= \mathbb{E}\Big[e^{-\rho\tau_1}\mathbb{E}_{\tau_1}\Big[\int_{\tau_1}^{\tau_2} e^{-\rho(t-\tau_1)} P_t^p X_1\, dt + e^{-\rho(\tau_2-\tau_1)} V_2(P_{\tau_2}^p) - K_1\Big]\Big] \\
&= \mathbb{E}\Big[e^{-\rho\tau_1}\mathbb{E}_{\tau_1}\Big[\int_0^{\tau_2-\tau_1} e^{-\rho t} P_t^{P_{\tau_1}^p} X_1\, dt + e^{-\rho(\tau_2-\tau_1)} V_2(P_{\tau_2}^{P_{\tau_1}^p}) - K_1\Big]\Big] \\
&\leq \mathbb{E}\big[e^{-\rho\tau_1} V_1^s(P_{\tau_1}^p)\big].
\end{aligned}$$

The result follows then directly from comparing the objective functions in (26) and (27). $\square$

Proof of Lemma 3.2: Consider two stopping times $\tau_1, \tau_2 \in \mathcal{T}^P$ such that $\tau_1 \leq \tau_2$. Then, proceeding as in the proof of Lemma 3.1, we obtain that:

$$\mathbb{E}\Big[1_{\{\tau_1<\tau_2\}}\Big(\int_{\tau_1}^{\tau_2} e^{-\rho t} P_t^p X_1\, dt - e^{-\rho\tau_1} K_1 + e^{-\rho\tau_2} V_2(P_{\tau_2}^p)\Big)\Big] \leq \mathbb{E}\big[1_{\{\tau_1<\tau_2\}} e^{-\rho\tau_1} V_1^s(P_{\tau_1}^p)\big].$$

Hence, the objective function in (22) is bounded above by:

$$\begin{aligned}
&\mathbb{E}\big[1_{\{\tau_1<\tau_2\}} e^{-\rho\tau_1} V_1^s(P_{\tau_1}^p) + 1_{\{\tau_1=\tau_2\}} e^{-\rho\tau_2} V_2(P_{\tau_2}^p)\big] \\
&\qquad = \mathbb{E}\big[1_{\{\tau_1<\tau_2\}} e^{-\rho\tau_1} V_1^s(P_{\tau_1}^p) + 1_{\{\tau_1=\tau_2\}} e^{-\rho\tau_1} V_2(P_{\tau_1}^p)\big] \\
&\qquad \leq \mathbb{E}\big[e^{-\rho\tau_1} \max\{V_1^s(P_{\tau_1}^p), V_2(P_{\tau_1}^p)\}\big] \\
&\qquad = \mathbb{E}\big[e^{-\rho\tau_1} V^s(P_{\tau_1}^p)\big],
\end{aligned}$$

from which the result follows by (32). $\square$

Proof of Proposition 3.1: The proof mimics that of Proposition 2.1, with V_1^s, V^s and $\mathcal{V}^s$ instead of V_1, V and $\mathcal{V}$. $\square$

Proof of Theorem 3.1: (i) The first step of the proof consists to show that $\tilde{p}^s \notin \mathcal{S}^s$. We proceed as in the proof of Proposition 2.2. Let $f^s = V_2 - V_1^s$. By construction, f^s is a difference of two convex functions which satisfies $f^s(\tilde{p}^s) = 0$ and $Df^s(\tilde{p}^s) > 0$, and we have $V^s = V_1^s + \max\{f^s, 0\}$. Since V_1^s is of class $\mathcal{C}^2$ on $\mathbb{R}_{++} \setminus \{p_{12}^s\}$, and its derivative is absolutely continuous, it follows from the generalized Itô formula (Krylov [17, §2.10]) and the Itô–Tanaka–Meyer formula (Karatzas and Shreve [15, Theorem 3.7.1]) that, for any $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\Big[e^{-\rho t} V^s(P_t^{\tilde{p}^s})\Big] &= V^s(\tilde{p}^s) + \mathbb{E}\Big[\int_0^t e^{-\rho u} \mathcal{A} V_1^s(P_u^{\tilde{p}^s})\, du\Big] \\
&\quad + \mathbb{E}\Big[\int_0^t e^{-\rho u} \mathcal{A} f^s(P_u^{\tilde{p}^s}) 1_{\{P_u^{\tilde{p}^s} \geq \tilde{p}^s\}}\, du\Big] + \frac{1}{2} Df(\tilde{p}^s)\, \mathbb{E}\Big[e^{-rt} L_t^{\tilde{p}^s}\Big].
\end{aligned}$$

We treat each term on the right-hand side of this equation separately. For the second term, since V_1^s is of class $\mathcal{C}^2$ on $(0, p_{12}^s)$, there exists a positive constant C_1 such that $|\mathcal{A}V_1^s(p)| \leq C_1$ for any $p \in (0, p_{12}^s)$. For $p \in (p_{12}^s, \infty)$, $\mathcal{A}V_1^s(p) = -pX_2 + \rho(K_1 + K_2)$. Hence, for every $t \geq 0$,

$$\begin{aligned}\left|\mathbb{E}\left[\int_0^t e^{-\rho u}\mathcal{A}V_1^s(P_u^{\tilde{p}^s})\,du\right]\right| &\leq \mathbb{E}\left[\int_0^t e^{-\rho u}\left|\mathcal{A}V_1^s(P_u^{\tilde{p}^s})\right|du\right] \\ &\leq \mathbb{E}\left[\int_0^t e^{-\rho u}[P_u^{\tilde{p}^s}X_2 + \rho(K_1+K_2) + C_1]\,du\right] \\ &= \frac{\tilde{p}^s X_2}{\rho-\mu}\,[1 - e^{-(\rho-\mu)t}] + \left(K_1 + K_2 + \frac{C_1}{\rho}\right)(1 - e^{-\rho t}).\end{aligned}$$

Since $\rho > \mu$, it follows that:

$$\mathbb{E}\left[\int_0^t e^{-\rho u}\mathcal{A}V_1^s(P_u^{\tilde{p}^s})\,du\right] = o(\sqrt{t}).$$

The third and fourth terms can be treated as in the proof of Proposition 2.2, and thus:

$$\mathbb{E}\left[e^{-\rho t}V^s(P_t^{\tilde{p}^s})\right] = V^s(\tilde{p}^s) + \tilde{p}^s\sigma Df^s(\tilde{p}^s)\sqrt{\frac{t}{2\pi}} + o(\sqrt{t}).$$

Since $Df^s(\tilde{p}^s) > 0$, letting t tend to 0 yields that $\sup_{\tau\in\mathcal{T}^P}\mathbb{E}\left[e^{-\rho\tau}V^s(P_\tau^{\tilde{p}^s})\right] > V^s(\tilde{p}^s)$. Hence $\tilde{p}^s$ does not belong to the stopping region $\mathcal{S}^s$. Next, as mentioned in the text, (33) implies that $V_1^s(p_1) > \mathcal{V}_2(p_1)$. It follows that $\mathcal{V}^s \neq \mathcal{V}_2$, and thus $\mathcal{S}^s \cap [0, \tilde{p}^s] \neq \emptyset$. The remaining of the proof mimics that of Theorem 2.1(i), with V_1^s, $\mathcal{V}_1^s$, $\mathcal{V}^s$, $\mathcal{S}^s$, $\tilde{p}^s$, p_3^s and p_4^s instead of V_1, $\mathcal{V}_1$, $\mathcal{V}$, $\mathcal{S}$, $\tilde{p}$, p_3 and p_4. (Note that $p_1^s = p_1$ under (33).) The only point that requires modification is (42), which becomes:

$$\left(\frac{p}{p_3^s}\right)^\alpha\left[\frac{p_3^s X_1}{\rho-\mu} + \left(\frac{p_3^s}{p_{12}^s}\right)^\beta\frac{K_2}{\beta-1} - K_1\right] > \frac{pX_2}{\rho-\mu} - K_2. \tag{45}$$

Just as (42), (45) is violated for p large enough, which implies the desired result.

(ii) Noting that $\mathcal{V}_s^1 = \mathcal{V}^{seq}$, the proof mimics that of Theorem 2.1(ii), with V_1^s, $\mathcal{V}_1^s$, $\mathcal{V}^s$, $\mathcal{S}^s$, $\tilde{p}^s$, p_3^s and p_4^s instead of V_1, $\mathcal{V}_1$, $\mathcal{V}$, $\mathcal{S}$, $\tilde{p}$, p_3 and p_4.

(iii) It is immediate to check that the stopping times τ_1 and τ_2 yield the value $\mathcal{V}^s(p)$ in (22). Since $\mathcal{V}^{switch}(p) \leq \mathcal{V}^s(p)$ for any $p \geq 0$ by Lemma 3.2, the result follows. □

References

1. Abel, A.B., Eberly, J.C.: Optimal investment with costly reversibility. Review of Economic Studies **63**, 581–593 (1996)

2. Arrow, K., Fisher, A.C.: Environmental preservation, uncertainty, and irreversibility. Quarterly Journal of Economics **88**, 312–319 (1974)

3. Bar-Ilan, A., Strange, W.C.: A model of sequential investment. Journal of Economic Dynamics and Control **22**, 437–463 (1998)

4. Décamps, J.-P., Mariotti, T., Villeneuve, S.: Investment timing under incomplete information. Mathematics of Operations Research (forthcoming)

5. Dias, M.A.G.: Real options in upstream petroleum: overview of models and applications. Working paper, Petrobras and Department of Industrial Engineering, PUC-Rio (2003)

6. Dixit, A.K.: Entry and exit decisions under uncertainty. Journal of Political Economy **97**, 620–638 (1989)

7. Dixit, A.K.: Choosing among alternative discrete investment projects under uncertainty. Economics Letters **41**, 265–268 (1993)

8. Dixit, A.K., Pindyck, R.S.: Investment under uncertainty. Princeton, NJ: Princeton University Press 1994

9. El Karoui, N.: Les aspects probabilistes du contrôle stochastique. In: Bickel, J.-P. (ed.) Ecole d'Eté de Probabilités de Saint-Flour, IX, 1979, Lecture Notes in Mathematics 876, pp. 74–239. Berlin: Springer-Verlag 1981

10. Farzin, Y.H., Huisman, K.J.M., Kort, P.M.: Optimal timing of technology adoption. Journal of Economic Dynamics and Control **22**, 779–799 (1998)

11. Grenadier, S.: The strategic exercise of options: development cascades and overbuilding in real estate markets. Journal of Finance **51**, 1653–1679 (1996)

12. Henry, C.: Investment decisions under uncertainty: the irreversibility effect. American Economic Review **64**, 89–104 (1974)

13. Jaillet, P., Lamberton, D., Lapeyre, B.: Variational inequalities and the pricing of American options. Acta Applicandae Mathematicae **21**, 263–289 (1990)

14. Kandel, E., Pearson, N.D.: Option value, uncertainty, and the investment decision. Journal of Financial and Quantitative Analysis **37**, 341–374 (2002)

15. Karatzas, I., Shreve, S.: Brownian motion and stochastic calculus. Berlin: Springer-Verlag 1991

16. Karatzas, I., Shreve, S.: Methods of mathematical finance. Berlin: Springer-Verlag 1998

17. Krylov, N.V.: Controlled diffusion processes. Berlin: Springer-Verlag 1980

18. Leahy, J.: Investment in competitive equilibrium: the optimality of myopic behavior. Quarterly Journal of Economics **104**, 1105–1133 (1993).

19. Majd, S., Pindyck, R.S.: Time to build, option value, and investment decisions. Journal of Financial Economics **18**, 7–27 (1987)

20. McDonald, R., Siegel, D.: The value of waiting to invest. Quarterly Journal of Economics **101**, 707–727 (1986)

21. Øksendal, B.: Stochastic differential equations. Berlin: Springer-Verlag 2000

22. Pindyck, R.S.: Irreversible investment, capacity choice, and the value of the firm. American Economic Review **78**, 969–985 (1988)

23. Ziemer, W.P.: Weakly differentiable functions. Berlin: Springer-Verlag 1989