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“A class of singular control problems with tipping points.”

Jean-Paul Décamps, Fabien Gensbittel, Thomas Mariotti, and Stéphane Villeneuve

Toulouse School of Economics

A class of singular control problems with tipping points.

Jean-Paul Décamps* Fabien Gensbittel† Thomas Mariotti‡ Stéphane Villeneuve §

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Abstract

Tipping points define situations where a system experiences sudden and irreversible changes and are generally associated with a random level of the system below which the changes materialize. In this paper, we study a singular stochastic control problem in which the performance criterion depends on the hitting time of a random level that is not a stopping time for the reference filtration. We establish a connection between the value of the problem and the value of a singular control problem involving a diffusion and its running minimum. We prove a verification theorem and apply our results to explicitly solve a resource extraction problem where the random evolution of the resource changes when it crosses a tipping point.

1 Introduction

Tipping points define situations where a dynamic system undergoes sudden and irreversible changes. They are generally associated with a random level of the system below (or above) which the changes materialize. There are numerous examples of tipping points in resource management and environmental sustainability. An increase in fires and grazing can exceed the forest's ability to regenerate, causing it to tip towards a savanna ecosystem. Excessive input of nutrients (like nitrates and phosphates) can lead to an explosive growth of algae, reducing oxygen in the water and creating dead zones.[1] The concept of tipping point has also entered in the financial literature since the 2008 crises where too much home-loan defaults caused a decrease in the value of collateralised debt obligations, leading to the insolvency of banks and insurers.[2] Other examples include public policy issues. For example, in a pandemic, lockdown policies aim to control the number of infected people, balancing the risks of economic and health system collapse.

*Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: jean-paul.decamps@tse-fr.eu.

†Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: fabien.gensbittel@tse-fr.eu.

‡Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France, CEPR, and CESifo. Email: thomas.mariotti@tse-fr.eu.

§Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. Email: stephane.villeneuve@tse-fr.eu.

[1] See Hirota et al. [16], Moore [23].

[2] See Battiston et al. [4]

A growing literature develops models where a tipping point occurs when a state variable exceeds a threshold whose exact value is unknown. Typically, a Decision Maker (DM) controls a state variable which models the stock of a resource. She faces a risk resulting from her own action and revises her belief about the likelihood of a tipping point each time a new minimum of the state variable is reached. The DM has to find the right balance so that the exploitation of the resource needed to create value does not lead to an irreversible effect caused by over-exploitation. To address this difficult control problem, the existing studies assume deterministic dynamics for the state variable.[3] However, almost all real dynamic systems (socio-ecological system, economic system) are subject to uncertainty independent of the action of the DM such as uncertainty due to weather variations (which, for example, impact the level of a resource or the number of people infected during an epidemic), uncertainty about the economic environment (changes in demand or interest rates), uncertainty generated by institutional settings. Adding uncertainty on the dynamics of the state variable subject to a tipping point is relevant and makes the DM problem very challenging.

To address this issue, this paper introduces a new class of stochastic control problems that combine singular control with a non-observable random threshold in a reflected Brownian diffusion setting. We propose a stylized model for a generic tipping point problem where a diffusion process X^L models the dynamics of a resource's reserve under an extraction policy L modeled as a non-decreasing process. The DM, whose objective is to maximize the expected sum of discounted extractions, controls the reserves X^L by the process L. There is a random level of reserves (tipping point) below which the dynamics of the reserves process will be irreversibly damaged. The DM only knows the distribution of the tipping point, which is modeled by a random variable Y. Below the tipping point Y, the DM is left with a continuation value function U which corresponds to the value function of a downgraded extraction problem, a problem that the DM hopes to face as late as possible. The classic trade-off between the costs and benefits of holding reserves is affected by the law of the random level Y that defines the tipping point and the characteristics of the downgraded extraction problem, making the DM control problem new and highly non-trivial.

Our first result shows that, because the time τ_Y at which the tipping point occurs is the hitting time by X^L of an unknown threshold Y, the Markovian formulation of the DM's problem leads to a two-dimensional control problem whose state variables are the process X^L and its running minimum process M^L. Intuitively, once the level of reserves reaches a minimum value without crossing the tipping point, the DM learns that as long as the level of reserves does not reach a new minimum level, the dynamics of the reserves process will be unchanged. This suggests that the relevant Markov state variables are indeed the controlled level of reserves and its controlled

[3] We will review the literature in a paragraph below.

minimum. This leads to a new bi-dimensional control problem in which the reward functional contains the expectation of an integral with respect to future increments of the minimum process M^L. A key feature of this problem is that a jump of the minimum process impacts differently the reward functional depending on whether the tipping point has been crossed after the jump. Our second result is to provide a verification theorem for this novel class of bi-dimensional control problems. Our third result is to solve the DM's problem by means of an explicit construction of its value function and to characterize the optimal extraction policy when it exists. In particular, when the level of reserves coincides with its historical minimum,we show that one of the three situations arises: 1) There is an optimal policy that allows reserves to grow to a free boundary, which is a function of the current minimum. The free boundary is defined as the solution to an ode which takes into account the law of the random threshold Y, the dynamics of the reserves before crossing the tipping point and the continuation value function U. 2) There is an optimal policy that extracts reserves up to an endogenous level in one go. 3) There is no optimal policy. This latter case corresponds to a situation in which extracting reserves up to an endogenous level in one go is optimal, provided this does not result in crossing the tipping point. The DM does not know the tipping point and cannot implement such a policy. We show that there exists an ϵ optimal policy, which enables the DM to identify the tipping point as quickly as possible. As far as we know, all our results, including the class of bi-dimensional control problem we study, are novel in the field of stochastic control and its applications.

Related literature Our paper is related to the large literature in environmental economics, in which a tipping point occurs when an underlying state variable with deterministic dynamics crosses an unknown threshold. This approach first considered in Kemp [20] has been developed and extended in several directions. For example, Tsur and Zemel [36] study saltwater intrusion in groundwater resources, Tsur and Zemel [37] and Lemoine and Traeger [21] explore climate tipping points, Naevdal [24] studies the eutrophication of lakes, Diekert [5] studies the cooperative or non-cooperative use of a resource under the threat of a disastrous threshold, Liski and Salanié [22] and Guillouet and Martimort [15] develop models with delay between the unobservable crossing of the tipping point and the observable occurrence of the catastrophe. We provide this literature with a rigorous mathematical framework that enables to account for a stochastic dynamics of the state variable.

Our paper is related to the literature on the optimal management of a resource in a reflected Brownian diffusion setting. The word resource is used here in a broad sense, it can represent corporate cash[4] (Décamps, Mariotti, Rochet and Villeneuve [7]), populations (Alvarez and Hening

[4]Jeanblanc and Shiryaev [18], Radner and Shepp [29] have set the benchmark case for the analysis of corporate cash management in continuous time. These contributions have been extended in a number of directions.

[1]), or any resource subject to a control problems of the storage or inventory type as in Shreve, Lehoczky and Gaver [33] which we will take as a natural benchmark for our model. These studies develop one-dimensional singular stochastic control problems, in which the optimal strategy is to consume the resource when its reserve exceeds an endogenous critical value. Other studies have incorporated stochastic profitability or stochastic interest rates into the standard optimal management model. These extensions lead to new and challenging control problems in a two-dimensional diffusion process setting.[5] Our paper deals with a different type of problem, where the DM controls the level of the resource and its running minimum. While there is an extensive literature on two-dimensional optimal stopping problems involving the running maximum[6] (or the running minimum) of a one-dimensional diffusion, control problems have not yet been considered. To the best of our knowledge, Ferrari and Rodosthenous [12], which became available during the writing of this paper, is the only other study that develops a two-dimensional optimal control problem involving the running minimum of a diffusion process.

Our paper and [12] complement each other and differ in several ways. In [12], the class of reward functionals is chosen such that a standard Neumann condition expresses the behavior of the value function at the diagonal where $X^L = M^L$. Using their vocabulary, the class of reward functionals is consistent with the HJB equation for the resulting two-dimensional singular control problem. This property is not satisfied in our setting where the reward functionnal is not set apriori but follows from the modelling of tipping point. Within our class of problems, jumps of the process (X^L, M^L) differently impact the reward functional, the construction of candidate value functions raises other difficulties and requires a new verification theorem. Accordingly, the example that we solve analytically in section 4 presents a number of unique features. In particular, it involves an integral with respect to the controlled minimum process. In the Markovian formulation of the control problem, this reflects the fact that the DM does not know at which value of the controlled minimum process an irreversible change will occur. Depending on the parameters of the model, our example admits optimal or only ϵ-optimal strategies, which we characterize analytically. Our results hold for a rich class of time-homogeneous diffusion processes.

The paper is organized as follows. Section 2 precisely describes the tipping point problem. Section 3 provides the Markovian formulation of the tipping point problem, presents the Hamilton-Jacobi-Bellman equation, and proves the required verification lemma. Section 4 provides a complete solution to a problem of optimal resource extraction with a tipping point.

[5] See for example, Reppen, Rochet and Soner [30], Décamps and Villeneuve [8], De Angelis [6], Bandini, De Angelis, Ferrari and Gozzi [3].

[6] See the seminal contributions of Shepp and Shiryaev [35], Dubins, Shepp, and Shiryaev [10], Graversen and Peskir [14], Peskir [27]. See also Guo and Zervos [17], Ott [26], Gapeev and Rodosthenous [13], Rodosthenous and Zervos [34], Décamps, Gensbittel and Mariotti [9] amongst many others.

2 The problem formulation

Let $(X_t)_{t\geq 0}$ be a one-dimensional time-homogeneous diffusion process defined over a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is a strong solution to:

$$dX_t = \mu(X_t)\, dt \ + \ \sigma(X_t)\, dB_t,\ X_0 = x,$$

where $B = (B_t)_{t\geq 0}$ is a one-dimensional Brownian motion defined on Ω. The state space for X is an interval $\mathcal{I} = (\alpha, +\infty)$ with $-\infty \leq \alpha < 0$ and:

Assumption 1. *μ and σ are Lipschitz on $\mathcal{I}$, $\sigma > 0$ on $\mathcal{I}$ and α and $+\infty$ are natural boundaries for X.*

Let $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ denote the augmented right-continuous filtration generated by B on Ω and $\mathbb{L}$ denote the set of $\mathbb{F}$-adapted processes that are $\mathbb{P}$ a.s. non-decreasing and right-continuous with $L_{0-} = 0$. We consider the controlled process X^L on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the following stochastic differential equation

$$dX^L_t = \mu(X^L_t)\, dt \ + \ \sigma(X^L_t)\, dB_t - dL_t,\ X^L_{0-} = x \in \mathcal{I}. \tag{1}$$

Thanks to Assumption 1, Equation (1) admits a unique strong solution for any process $L \in \mathbb{L}$ (see Protter [28], Theorem 7, page 253)[7]. Note that for $L \equiv 0$, the process X^L coincides with X.

The tipping point will be represented by a random variable Y with law $\mathbb{Q}$ taking values in $\mathcal{I}$ and independent of the Brownian motion B. Note that because X^L is $\mathbb{F}$-adapted for every $L \in \mathbb{L}$, X^L is independent of Y. For convenience, we will work on the product probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}}) \equiv (\Omega \times \mathcal{I}, \mathcal{F} \otimes \mathcal{B}(\mathcal{I}), \overline{\mathbb{P}} = \mathbb{P} \otimes \mathbb{Q})$, where $\mathcal{B}(\mathcal{I})$ is the Borel σ-field over $\mathcal{I}$, and assume that $Y(\omega, y) = y$ is defined on $\overline{\Omega}$. We denote by $\mathbb{E}$ and $\overline{\mathbb{E}}$ the expectation operators associated to $\mathbb{P}$ and $\overline{\mathbb{P}}$, respectively. Random variables defined on Ω are identified to variables defined on $\overline{\Omega}$. The random time τ_Y at which the tipping point is reached is defined by:

$$\tau_Y = \inf\{t \geq 0,\ X^L_t \leq Y\} \tag{2}$$

τ_Y depends on L but we omit this dependence in the notation to simplify the presentation.

To emphasize the dependence of X^L on x, we will use the slightly abusive notation $\mathbb{P}_x = \mathbb{P}(\cdot | X^L_{0-} = x)$ and $\mathbb{E}_x = \mathbb{E}[\cdot | X^L_{0-} = x]$, as well as $\overline{\mathbb{P}}_x, \overline{\mathbb{E}}_x$ defined similarly.

In what follows, for any random time $\tau : \overline{\Omega} \to [0, \infty]$, any measurable process H (not necessarily defined at time $+\infty$) and any process with finite variation A, we use the conventions

$$\int_{[0,\tau]} H_s dA_s := \int_{[0,\infty)} H_s dA_s \text{ and } e^{-r\tau} H_\tau := 0 \text{ on } \{\tau = +\infty\}.$$

[7]Precisely, one may assume that μ, σ are Lipschitz functions defined on $\mathbb{R}$ by taking any Lipschitz extension to obtain a solution X^L taking values in $\mathbb{R}$. We will only consider solutions taking values in $\mathcal{I}$ hereafter.

The tipping point problem We consider the problem of extracting a resource when there is a risk of reaching a tipping point, at which point the DM receives a continuation value. $U : \mathcal{I} \mapsto \mathbb{R}_+$.

As it is reasonable to think that the process X^L representing the resource cannot take negative values, we restrict the class of controls accordingly. Letting x^+ denote the positive part of $x \in \mathcal{I}$, the class of admissible controls with initial condition $X^L_{0-} = x$ is defined by:

$$\mathcal{A}(x) = \{L \in \mathbb{L},\, X^L_t \in \mathcal{I} \text{ and } (X^L_{t-})^+ - (\Delta L)_t \geq 0 \text{ for all } t\}.$$

Letting $\tau_y = \inf\{t \geq 0,\, X^L_t \leq y\}$ for $y \in \mathcal{I}$, for any $L \in \mathcal{A}(x)$, we have $X^L_t \geq 0$ for all $t \leq \tau_0$. We assume that the tipping point will be reached before the resource is depleted, that is that the random variable Y is positive, and also for technical reasons that Y admits a density:

Assumption 2. *The law $\mathbb{Q}$ of the random variable Y has a density f defined on $\mathcal{I}$ and Y is nonnegative. Moreover, there exists $\bar{y} > 0$ such that f is Lipschitz and positive on $[0, \bar{y}]$ and vanishes on $\mathcal{I} \setminus [0, \bar{y}]$.*

We denote by F the cumulative distribution function of Y.

The resource extraction problem with tipping point is then defined as

$$\overline{V}(x) = \sup_{L \in \mathcal{A}(x)} \overline{\mathbb{E}}_x \left[\int_{[0,\tau_Y]} e^{-rs} dL_s + e^{-r\tau_Y} U(X^L_{\tau_Y}) \right]. \tag{3}$$

In Section 3, we derive a Markovian formulation for the tipping point problem (3) and a verification lemma for constructing a candidate value function. In Section 4, under additional assumptions, we provide an explicit solution to problem (3).

3 Markovian formulation and HJB equation

In this section, we will show how to relate a tipping point problem to a Markovian stochastic control problem.

3.1 A Markovian formulation

Given a control $L \in \mathbb{L}$ and $m \in [0, x]$, let us define the minimum process $M^L \equiv (M^L_t)_{t \geq 0}$ by $M^L_t = m \wedge \inf_{s \leq t} X^L_s$. The pair (X^L, M^L) defines an $\mathbb{F}$-adapted process such that $(X^L_{0-}, M^L_{0-}) = (x, m)$. To emphasize the dependence on (x, m), we will use the notation $\mathbb{P}_{x,m} = \mathbb{P}(\cdot | (X^L_{0-}, M^L_{0-}) = (x, m))$ and $\mathbb{E}_{x,m} = \mathbb{E}[\cdot | (X^L_{0-}, M^L_{0-}) = (x, m)]$. As we only consider the process (X^L, M^L) up to the hitting time τ_0, the natural state space of our problem is

$$\mathcal{J} = \{(x, m) \in [0, \infty)^2,\, x \geq m\}.$$

The next result relates the tipping point problem to a singular stochastic control problem for a one-dimensional diffusion and its running minimum.

Proposition 1. *Let V be the value function of the singular control problem defined as*

$$V(x,m) = \sup_{L\in\mathcal{A}(x)} V(x,m;L) \tag{4}$$

with

$$\begin{aligned} V(x,m;L) &= \mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs}F(M^L_{s-})\,dL_s\right] - \mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs}U(M^L_s)f(M^L_s)\,dM^{L,c}_s\right] \\ &\quad + \mathbb{E}_{x,m}\left[\sum_{0\le s\le\tau_0} e^{-rs}U(M^L_s)(F(M^L_{s-})-F(M^L_s))\right] \end{aligned} \tag{5}$$

where $M^{L,c}$ is the continuous part of M. Then, we have

$$\overline{V}(x) = V(x,x) + U(x)(1-F(x)). \tag{6}$$

Proof of Proposition 1. First, we notice that $\tau_Y = 0$ on the set $\{Y \ge x\}$. Thus, we have for $x \in \mathcal{I}$ and $L \in \bar{\mathcal{A}}(x)$,

$$\overline{\mathbb{E}}_x\left[\int_{[0,\tau_Y]} e^{-rs}dL_s + e^{-r\tau_Y}U(X^L_{\tau_Y})\right] = J(x,L) + U(x)(1-F(x)),$$

where

$$J(x,L) = \overline{\mathbb{E}}_x\left[\left(\int_{[0,\tau_Y]} e^{-rs}dL_s + e^{-r\tau_Y}U(X^L_{\tau_Y})\right)\mathbb{1}_{Y<x}\right]$$

Then, considering the family of $\mathbb{F}$-stopping times $(\tau_y \equiv \inf\{t\ge 0 : X^L_t \le y\})_{y\in\mathcal{I}}$, and letting M^L be defined with initial condition $M^L_{0-} = x$, we have by using independence between B and Y,

$$\begin{aligned} J(x,L) &= \mathbb{E}_x\left[\int_0^x \left(\int_{[0,\tau_y]} e^{-rs}dL_s + e^{-r\tau_y}U(X^L_{\tau_y})\right) f(y)\,dy\right] \\ &= \mathbb{E}_x\left[\int_0^x \left(\int_{[0,\infty)} e^{-rs}\mathbb{1}_{s\le\tau_y}dL_s + e^{-r\tau_y}U(X^L_{\tau_y})\right) f(y)\,dy\right] \\ &= \mathbb{E}_x\left[\int_{[0,\infty)} e^{-rs}\left(\int_0^x \mathbb{1}_{s\le\tau_y}f(y)dy\right)dL_s + \int_0^x e^{-r\tau_y}U(X^L_{\tau_y})f(y)\,dy\right] \\ &= \mathbb{E}_{x,x}\left[\int_{[0,\infty)} \left(\int_0^{M^L_{s-}} f(y)\,dy\right)e^{-rs}\,dL_s + \int_0^x e^{-r\tau_y}U(X^L_{\tau_y})f(y)\,dy\right] \\ &= \mathbb{E}_{x,x}\left[\int_{[0,\infty)} F(M^L_{s-})e^{-rs}\,dL_s + \int_0^x e^{-r\tau_y}U(M^L_{\tau_y})f(y)\,dy\right]. \end{aligned} \tag{7}$$

The third equality follows from Fubini's theorem and the fourth equality uses the relation

$$\{M^L_{s-} > y\} \subset \{s \le \tau_y\} \subset \{M^L_{s-} \ge y\}.$$

We now show that for any $m \le x$:

$$\begin{aligned}
& \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(y) dy\right] \\
= & -\mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} U(M^L_s) f(M^L_s) dM^{L,c}_s\right] \\
+ & \mathbb{E}_{x,m}\left[\sum_{0\le s\le \tau_0} e^{-rs} U(M^L_s)(F(M^L_{s-}) - F(M^L_s))\right],
\end{aligned} \tag{8}$$

We have,

$$\begin{aligned}
& \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(y) dy\right] \\
= & \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(M^L_{\tau_y}) dy\right] + \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y})(f(y) - f(M^L_{\tau_y})) dy\right] \\
= & \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(M^L_{\tau_y}) dy\right] + \mathbb{E}_{x,m}\left[\sum_{0\le s\le\tau_0} \int_{M^L_s}^{M^L_{s-}} e^{-r\tau_y} U(M^L_{\tau_y})(f(y) - f(M^L_{\tau_y})) dy\right],
\end{aligned}$$

because $f(y) - f(M^L_{\tau_y})$ is different from 0 only at the jump times of M. For the first term of the latter equality, a standard change of variables formula for Stieljes integrals, (see Revuz and Yor [31], Chapter 0, Proposition 4.9), yields

$$\mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(M^L_{\tau_y}) dy\right] = -\mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} U(M^L_s) f(M^L_s) dM^L_s\right].$$

For the second term, we observe that for $y \in [M^L_s, M^L_{s-})$, we have $\tau_y = s$ and $M^L_{\tau_y} = M^L_s$. Therefore,

$$\begin{aligned}
& \mathbb{E}_{x,m}\left[\sum_{0\le s\le\tau_0} \int_{M^L_s}^{M^L_{s-}} e^{-r\tau_y} U(M^L_{\tau_y})(f(y) - f(M^L_{\tau_y})) dy\right] \\
= & \mathbb{E}_{x,m}\left[\sum_{0\le s\le\tau_0} e^{-rs} U(M^L_s) \int_{M^L_s}^{M^L_{s-}} (f(y) - f(M^L_s)) dy\right] \\
= & \mathbb{E}_{x,m}\left[\sum_{0\le s\le\tau_0} e^{-rs} U(M^L_s)[(F(M^L_{s-}) - F(M^L_s)) - f(M^L_s)(M^L_{s-} - M^L_s)]\right].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \mathbb{E}_{x,m}\left[\int_0^m e^{-r\tau_y} U(M^L_{\tau_y}) f(y) dy\right] = -\mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} U(M^L_s) f(M^L_s) dM^L_s\right] \\
& +\mathbb{E}_{x,m}\left[\sum_{0\le s\le \tau_0} e^{-rs} U(M^L_s)[(F(M^L_{s-}) - F(M^L_s)) - f(M^L_s)(M^L_{s-} - M^L_s)]\right] \\
= \; & -\mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} U(M^L_s) f(M^L_s) dM^{L,c}_s\right] - \mathbb{E}_{x,m}\left[\sum_{0\le s\le \tau_0} e^{-rs} U(M^L_s) f(M^L_s)(M^L_s - M^L_{s-})\right] \\
& +\mathbb{E}_{x,m}\left[\sum_{0\le s\le \tau_0} e^{-rs} U(M^L_s)[(F(M^L_{s-}) - F(M^L_s)) - f(M^L_s)(M^L_{s-} - M^L_s)]\right] \\
= \; & -\mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} U(M^L_s) f(M^L_s) dM^{L,c}_s\right] + \mathbb{E}_{x,m}\left[\sum_{0\le s\le \tau_0} e^{-rs} U(M^L_s)(F(M^L_{s-}) - F(M^L_s))\right],
\end{aligned}
$$

which proves (8). It follows from (7) and (8) with $m = x$ that, $J(x, L) = V(x, x, L)$. This concludes the proof of Proposition 1. □

To understand the intuition behind Proposition 1, note that $U(x)$ corresponds to the DM's value function conditional on the event that the tipping point is above x, the probability of which is $1-F(x)$. Thus, from (6), the ratio $V(x, x)/F(x)$ corresponds to the DM's value function conditional on the event that the tipping point is below x (the probability of which is $F(x)$). Proposition 1 shows that computing $V(x, x)$ involves solving problem (4), which is an optimal control problem for the two-dimensional process (X^L, M^L) over the state space $\mathcal{J}$. This is a different class of problem to the one studied in [12] because jumps of the process (X^L, M^L) differently impact the reward functional.[8] In particular, solving problem (4) requires a different verification lemma, which we present in the next section.

3.2 Hamilton-Jacobi-Bellman equation and the verification lemma

We introduce the infinitesimal generator $\mathcal{L}$ of the uncontrolled diffusion X, acting on functions defined on $\mathcal{J}$ which are twice differentiable with respect to x by

$$\mathcal{L}h(x, m) \equiv \mu(x)\, \frac{\partial h}{\partial x}\,(x, m) + \frac{1}{2}\, \sigma^2(x)\, \frac{\partial^2 h}{\partial x^2}\,(x, m), \quad (x, m) \in \mathcal{J}.$$

[8] Precisely, using the integral operators introduced in [12] Section 3, the expected reward functional (5) writes $\mathbb{E}_{x,m}\left[\int_0^T e^{-rs}(F(M^L_s) \diamond dL_s - U(M^L_s) f(M^L_s) \Box dM^L_s) + \sum_{0\le s\le T} e^{-rs}(U(M^L_s)(F(M^L_{s-}) - F(M^L_s)) - \int_{M^L_s}^{M^L_{s-}} (U(y) - y + M^L_s) f(y)\, dy\right]$. Except for case $U(y) = y$, it does not coincide with $\mathbb{E}_{x,m}\left[\int_0^T e^{-rs}(F(M^L_s) \diamond dL_s - U(M^L_s) f(M^L_s) \Box\, dM^L_s)\right]$, the form of functionals considered in [12].

By the dynamic programming principle, we expect the value function of (4) satisfies, in some sense, the Hamilton-Jacobi-Bellman equation

$$\max\Big((\mathcal{L}-r)V(x,m), F(m)-V_x(x,m)\Big)=0 \tag{9}$$

subject to the boundary conditions

$$V_m(m,m) \geq U(m)f(m), \tag{10}$$

$$V(m,m) \geq T[V](m), \tag{11}$$

where the operator T is defined on the set of continuous functions on $\mathcal{J}$ as

$$T[V](m) = \sup_{0\leq h\leq m}\Big(hF(m)+U(m-h)(F(m)-F(m-h))+V(m-h,m-h)\Big). \tag{12}$$

Condition (10) is a standard Neumann condition which states that, starting from a point (m,m) on the diagonal $\partial\mathcal{J}=\{(x,m)\in\mathcal{J},\, x=m\}$, a marginal increase in the second argument yields to the decision maker (DM) at least the expected payment $U(m)f(m)$, where $f(m)$ corresponds to the probability that the tipping point occurs in the infinitesimal interval $[m,m+dm]$. Condition (11) is a special feature of our class of control problems. Observe that the inequality $V(m,m)\leq T[V](m)$ always holds by definition of $T[V](m)$ (consider $h=0$ in the right-hand side of (11)). To grasp the intuition of (11), consider a couple (m,m) and choose a control policy with $L_0=h>0$. This policy must yield no more than the optimal policy. Using Proposition 1 together with a dynamic programming argument, we must have for every $h\in(0,m]$,

$$V(m,m)\geq V(m-h,m-h)+hF(m)+U(m-h)(F(m)-F(m-h)). \tag{13}$$

The HJB equation (9) and the Neumann condition (10) imply that (13) is satisfied at the limit when h tends to zero.[9] However, constructing a candidate solution for the singular control problem (4) also requires checking that (13) is satisfied for all positive h, thus introducing the non-local operator T in the boundary condition (11).

For further use, note that if $V_x(x,m)\geq F(m)$, we have for $(x,m)\in\mathcal{J}$ and $(x',m)\in\mathcal{J}$ with $m\leq x'\leq x$

$$V(x,m)\geq (x-x')F(m)+V(x',m). \tag{14}$$

We now present our verification lemma, which is based on applying Itô's formula for the processes (X_t^L, M_t^L) evolving on the *closed* set $\mathcal{J}$. Justifying Itô's formula on a closed set requires precisely defining some class of functions, which we subsequently denote by $\mathcal{R}(\mathcal{J})$, to which it

[9] To see this, simply note that (13) is equivalent to $\frac{V(m,m)-V(m-h,m-h)}{h}\geq F(m)+U(m-h)\frac{F(m)-F(m-h)}{h}$ and take the limit as h goes to 0. This yields $V_x(m,m)+V_m(m,m)\geq F(m)+U(m)f(m)$, which is indeed satisfied because, by (9) and (10), we have $V_x(m,m)\geq F(m)$ and $V_m(m,m)\geq U(m)f(m)$.

applies. To streamline the presentation, we refer the reader to Appendix 5.1 for the rigorous definition of this class.

Lemma 1. *Let $w \in \mathcal{R}(\mathcal{J})$ be a nonnegative solution to the HJB equation* (9) *with boundary conditions* (10), (11) *then,* $w \geq V$.

Proof of Lemma 1. For any given control policy $L \in \mathcal{A}(x)$, we use a standard localization procedure (see Karatzas and Shreve [19] page 34) by introducing the increasing sequence of stopping time

$$T_n = \inf\{t \geq 0, \int_0^{t\wedge\tau_0} e^{-2rs}\sigma^2(X_s^L)w_x^2(X_s^L, M_s^L)\, ds \geq n\} \tag{15}$$

to obtain that the process

$$\Big(\int_0^{t\wedge\tau_0\wedge T_n} e^{-rs}\sigma(X_s^L)w_x(X_s^L, M_s^L)\, dB_s\Big)_{t\geq 0}$$

is a martingale. Applying Itô's formula and taking expectations, we obtain

$$\begin{aligned}
w(x,m) &= \mathbb{E}_{x,m}[e^{-r(t\wedge\tau_0\wedge T_n)}w(X^L_{t\wedge\tau_0\wedge T_n}, M^L_{t\wedge\tau_0\wedge T_n})] \\
&- \mathbb{E}_{x,m}\left[\int_0^{t\wedge\tau_0\wedge T_n} e^{-rs}(\mathcal{L}-r)w(X_s^L, M_s^L)\, ds\right] \\
&- \mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}w_m(M_s^L, M_s^L)dM_s^{L,c}\right] + \mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}w_x(X_s^L, M_s^L)dL_s^c\right] \\
&- \mathbb{E}_{x,m}\left[\sum_{0\leq s\leq t\wedge\tau_0\wedge T_n} e^{-rs}(w(X_s^L, M_s^L) - w(X_{s-}^L, M_{s-}^L))\right],
\end{aligned}$$

where $M^{L,c}$ and L^c denote respectively the continuous parts of M^L and L. Using (9), (10) and (11), by observing that $-dM_s^{L,c}$ is a positive measure and because w is nonnegative, we have

$$\begin{aligned}
w(x,m) &\geq -\mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}U(M_s^L)f(M_s^L)dM_s^{L,c}\right] + \mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}F(M_s^L)dL_s^c\right] \\
&- \mathbb{E}_{x,m}\left[\sum_{0\leq s\leq t\wedge\tau_0\wedge T_n} e^{-rs}(w(X_s^L, M_s^L) - w(X_{s-}^L, M_{s-}^L))\right].
\end{aligned}$$

Let us focus on the last term of the above formula. We have

$$w(X_s^L, M_s^L) - w(X_{s-}^L, M_{s-}^L) = (w(X_s^L, M_s^L) - w(X_{s-}^L, M_{s-}^L))(\mathbb{1}_{M_s^L=M_{s-}^L} + \mathbb{1}_{M_s^L<M_{s-}^L}).$$

Using (14), we have

$$\begin{aligned}
(w(X_s^L, M_s^L) - w(X_{s-}^L, M_{s-}^L))\mathbb{1}_{M_s^L=M_{s-}^L} &= (w(X_s^L, M_{s-}^L) - w(X_{s-}^L, M_{s-}^L))\mathbb{1}_{M_s^L=M_{s-}^L} \\
&\leq -(X_{s-}^L - X_s^L)F(M_{s-}^L)\mathbb{1}_{M_s^L=M_{s-}^L}.
\end{aligned}$$

On the other hand, on the set $\{M^L_s < M^L_{s-}\}$, we have $X^L_s = M^L_s$ and thus by using (11) and (14), we have

$$
\begin{aligned}
&(w(X^L_s, M^L_s) - w(X^L_{s-}, M^L_{s-}))\mathbb{1}_{M^L_s<M^L_{s-}} \\
&\qquad = (w(M^L_s, M^L_s) - w(X^L_{s-}, M^L_{s-}))\mathbb{1}_{M^L_s<M^L_{s-}} \\
&\qquad = \Big(w(M^L_s, M^L_s) - w(M^L_{s-}, M^L_{s-}) + w(M^L_{s-}, M^L_{s-}) - w(X^L_{s-}, M^L_{s-})\Big)\mathbb{1}_{M^L_s<M^L_{s-}} \\
&\qquad \leq -\Big((M^L_{s-} - M^L_s)F(M^L_{s-}) + U(M^L_s)(F(M^L_{s-}) - F(M^L_s)) + (X^L_{s-} - M^L_{s-})F(M^L_{s-})\Big)\mathbb{1}_{M^L_s<M^L_{s-}} \\
&\qquad = -\Big((X^L_{s-} - X^L_s)F(M^L_{s-}) + U(M^L_s)(F(M^L_{s-}) - F(M^L_s))\Big)\mathbb{1}_{M^L_s<M^L_{s-}}.
\end{aligned}
$$

Finally, we obtain for every $L \in \mathcal{A}(x)$, observing that $\Delta L_s = -\Delta X^L_s$,

$$
\begin{aligned}
w(x,m) \quad &\geq \quad -\mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}U(M^L_s)f(M^L_s)dM^{L,c}_s\right] \\
&+ \quad \mathbb{E}_{x,m}\left[\int_{[0,t\wedge\tau_0\wedge T_n]} e^{-rs}F(M^L_{s-})dL_s\right] \\
&+ \quad \mathbb{E}_{x,m}\left[\sum_{0\leq s\leq t\wedge\tau_0\wedge T_n} e^{-rs}U(M^L_s)(F(M^L_{s-}) - F(M^L_s))\right]
\end{aligned}
$$

We let both n and t tend to $+\infty$ and use the monotone convergence theorem to obtain, for all $L \in \mathcal{A}(x)$,

$$
\begin{aligned}
w(x,m) \quad &\geq \quad \mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs}F(M^L_{s-})dL_s\right] - \mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs}U(M^L_s)F(M^L_s)dM^{L,c}_s\right] \\
&+ \quad \mathbb{E}_{x,m}\left[\sum_{0\leq s\leq\tau_0} e^{-rs}U(M^L_s)(F(M^L_{s-}) - F(M^L_s))\right].
\end{aligned}
$$

which ends the proof by arbitrariness of the control policy L in $\mathcal{A}(x)$.

□

4 A problem of optimal resource extraction with a tipping point

In this section we provide a solution to the two-dimensional stochastic control problem (4). This solution is explicit given a free boundary function b which is characterized as the solution to an ode. We proceed as follows. In Section 4.1 we study an auxiliary control problem in which the DM cannot extract below a fixed level $m \geq 0$ at which she receives a constant terminal payment $U(m)$. Formally, we will consider the one-dimensional control problem

$$
\forall x \in [m,\infty), \quad V^m(x) \equiv \sup_{L\in\mathcal{A}_m(x)} \mathbb{E}_x\left[\int_{[0,\tau_m]} e^{-rs}\, dL_s + e^{-r\tau_m}U(m)\right], \tag{16}
$$

where $\tau_m = \inf\{t \geq 0 : X^L_t \leq m\}$, and

$$\mathcal{A}_m(x) = \Big\{L \in \mathbb{L},\ (X^L_{t-} - m)^+ - (\Delta L)_t \geq m \text{ for every } t \geq 0\Big\}.$$

This problem, which was solved in [33] for the case $m = 0$, will prove useful to set up a free boundary problem whose solution gives a candidate value function for problem (4). We derive this free boundary problem in section 4.2 and characterize the associated free boundary function. Then, we solve the two-dimensional singular control problem (4) in section 4.3.

Before developing the analysis, we need to complete our set of assumptions. We work under Assumptions 1, 2 and the following ones.

Assumption 3. *μ and σ are C^1 with Lipschitz derivatives and satisfy:*

$$\mu(0) > rU(0) \quad \textit{and} \quad \sup_{x\geq 0} \mu'(x) < r.$$

Assumption 4. *The function U is non-decreasing and concave, twice continuously differentiable over $[0,\infty)$ with $U'(m) \geq 1$ for $m \leq u^*$, $U'(m) = 1$ for $m \geq u^*$, for some $u^* \geq 0$, and it satisfies the inequality*

$$\mathcal{L}U(m) - rU(m) > 0, \ \ \forall m \in [0, u^*). \tag{17}$$

Assumption 5. *The function μ is twice differentiable with $\mu''(x) \leq 0$ for $x \in [0,\infty)$. The cumulative distribution function F satisfies the monotone hazard rate property (MHRP): $\frac{f}{F}$ is decreasing on its support $[0, \overline{y}]$.*

Assumption 3 is of the type used in [33] and allows one to solve explicitly the auxiliary problem (16). Assumption 4 sets the features of the continuation value function U. It satisfies the properties of the value function of a standard extraction problem modelled as a singular control problem, as, for example, in [33]. In particular, the threshold u^* denotes the optimal extraction threshold associated with the extraction problem that the DM faces when the tipping point is crossed. Inequality (17) means that the later the tipping point is crossed on the interval $[0, u^*]$, the better it is for the DM. We will say that a function U that satisfies Assumption 4 is the value function of a downgraded extraction problem, as it is the case in example 1 below. Assumption 5 states that the marginal profitability of extracting the resource decreases with the level of reserves, and that the likelihood of reaching a tipping point, assuming it has not yet occurred, increases as the level of reserves decreases. These assumptions are standard in tipping point models.

Lemma 2. *Under assumptions 3 and 4, there exists $\overline{x} > 0$ so that $\mu(x) - rU(x) > 0$ on $[0, \overline{x})$ and $\mu(x) - rU(x) < 0$ on $(\overline{x}, \infty)$, $\overline{x} \geq u^*$, and (17) is also satisfied on $[u^*, \overline{x})$.*

Proof. The map $x \to \mu(x) - rU(x)$ is differentiable and decreasing because $U' \geq 1$ and $\sup_{x\geq 0} \mu'(x) = \bar{\mu} < r$. Using that $\mu(0) - rU(0) > 0$, we shall deduce that there exists a unique point $\overline{x} > 0$ such that $\mu(x) - rU(x) > 0$ on $[0, \overline{x})$ and $\mu(x) - rU(x) < 0$ on $(\overline{x}, \infty)$. To prove the existence of such $\overline{x}$, it is sufficient to show that $\mu(x) - rU(x) < 0$ for large values of x. For large values of x, we have:

$$\mu(x) - r(x - u^* + U(x^*)) \leq \mu(0) + (\bar{\mu} - r)x - rU(u^*) < 0,$$

under the assumption $\sup_{x\geq 0} \mu'(x) = \bar{\mu} < r$, thus the result. Moreover, U is C^2 and satisfies $U'(u^*) = 1$ and $U''(u^*) = 0$, therefore:

$$(\mathcal{L} - r)U(u^*) = \mu(u^*)U'(u^*) + \frac{\sigma^2(u^*)}{2}U''(u^*) - rU(u^*) = \mu(u^*) - rU(u^*) \geq 0,$$

which implies $u^* \leq \overline{x}$. To prove the last assertion, note that $U'(x) = 1$ and $U''(x) = 0$ for all $x \geq u^*$, and thus for all $x \in [u^*, \overline{x})$

$$(\mathcal{L} - r)U(x) = \mu(x)U'(x) + \frac{\sigma^2(x)}{2}U''(x) - rU(x) = \mu(x) - rU(x) > 0.$$

□

As previously mentioned, the fact that inequality (17) is satisfied on $[0, \overline{x})$ means that the later the tipping point is crossed on the interval $[0, \overline{x}]$, the better it is for the DM, in the sense that for any $x < \overline{x}$, and any stopping time τ such that the stopped process $(X_t)_{t\leq\tau}$ stays in $[0, \overline{x}]$, we have by applying Itô's formula that $U(x) \leq \mathbb{E}_x[e^{-r\tau}U(X_\tau)]$ with a strict inequality if $\mathbb{P}(\tau > 0) > 0$.

Example 1. *Consider a function μ satisfying Assumption 3 except for $\mu(0) > rU(0)$. Let U be the value function of an extraction problem as in [33], whose uncontrolled dynamics of resource has the same volatility σ than X^0 and a drift $\underline{\mu}$ such that $\underline{\mu} < \mu$ on $[0, \infty)$ and with terminal payoff C. Assume that an optimal strategy corresponding to a threshold $u^* \geq 0$ exists (that is, Theorem 4.3 in [33] applies), then U is non-decreasing and concave, twice continuously differentiable on $[0, \infty)$, with $U(0) = C$, $U'(m) \geq 1$ for $m \leq u^*$, and $U'(m) = 1$ for $m \geq u^*$. Moreover, we have*

$$\forall x \in [0, u^*], \ \underline{\mu}(x)U'(x) + \frac{\sigma^2(x)}{2}U''(x) - rU(x) = 0,$$

which implies $(\mathcal{L} - r)U > 0$ since $U' \geq 1$ and $\underline{\mu} < \mu$. Thus, Assumption 4 is satisfied, and if $C < \frac{\mu(0)}{r}$, Assumption 3 is also satisfied.

4.1 The auxiliary problem (16)

Proposition 2 below solves the auxiliary problem (16) and is a direct consequence of Theorem 4.3 in [33]. We give the proof for the sake of completeness in the Appendix. Claim (i) of Proposition 2 states that if m is larger than the threshold $\overline{x}$ defined in Lemma 2, then a lump sum extraction is optimal. That is, the DM is better off extracting immediately up to m in order to get the terminal payment $U(m)$ right now. Claim (ii) of Proposition 2 states that, if m is below the threshold $\overline{x}$, the DM optimally allows the reserves of the resource to grow up to the extraction threshold $\eta(m)$ and extracts any reserve above $\eta(m)$. Claim (iii) of Proposition 2 provides useful and intuitive properties of the extracting threshold $\eta(m)$. Intuitively, the higher m is, where $m \in [0, \overline{x}]$, the higher the reserve requirement $\eta(m)$ before extraction. The equality $\eta(\overline{x}) = \overline{x}$ reflects the fact that, if the initial level of reserve is larger than $\overline{x}$, then the DM optimally extracts the surplus $x - \overline{x} > 0$.

Proposition 2. *The following holds.*

(i) If $m \geq \overline{x}$ then, the value function V^m of problem (16) satisfies

$$V^m(x) = x - m + U(m) \quad x \geq m, \tag{18}$$

and the policy of initially extracting $x - m$ is optimal.

(ii) If $m < \overline{x}$, then the value function V^m of problem (16) is concave, twice continuously differentiable on $[m, \infty)$, it satisfies $V^m(m) = U(m)$ and the HJB equation on $[m, \infty)$:

$$\max\{(\mathcal{L} - r)V^m(x), 1 - V^m_x(x)\} = 0. \tag{19}$$

Moreover,

$$\begin{cases} V^m(x) = A^m \psi(x) + B^m \phi(x), & m \leq x \leq \eta(m) \\ V^m(x) = x - \eta(m) + \frac{\mu(m)}{r}, & x \geq \eta(m) \end{cases} \tag{20}$$

where

$$A^m = \frac{\phi(m) - U(m)\phi'(\eta(m))}{D(\eta(m), m)}$$
$$B^m = \frac{U(m)\psi'(\eta(m)) - \psi(m)}{D(\eta(m), m)}$$

and where $\eta(m) \in (m, \overline{x})$ is defined as the unique solution $x \in [0, \infty)$ to $N(x, m) = 0$ where

$$N(x, m) = \phi(x)\psi(m) - \psi(x)\phi(m) + \frac{\mu(x)}{r} D(x, m) - D(x, x)U(m) \tag{21}$$

with $D(x, m) = \psi'(x)\phi(m) - \phi'(x)\psi(m) > 0$ for any $0 \leq m \leq x$. The policy of extracting any reserve above the threshold $\eta(m)$ is optimal. Precisely, the process $L^m \in \mathcal{A}_m(x)$ is an optimal extraction process where (X^{L^m}, L^m) is the unique solution of

$$\begin{cases} dX_t^{L^m} = \mu(X_t^{L^m})dt + \sigma(X_t^{L^m})dB_t - dL_t^m, \; X_{0-}^{L^m} = x \\ L_t^m = \sup_{0 \leq s \leq t}(X_s^{L^m} - \eta(m))^+ \end{cases} \tag{22}$$

(iii) Letting $x^0 := \eta(0)$, the map

$$\eta: \quad [0, \overline{x}] \longrightarrow [x^0, \overline{x}]$$
$$m \longrightarrow \eta(m)$$

is increasing, C^1, and satisfies $\eta(m) > m$ for all $m \in [0, \overline{x})$ and $\eta(\overline{x}) = \overline{x}$. Moreover, for any fixed $m \in [0, \overline{x}]$, $N(x, m) > 0$ for all $x < \eta(m)$ and, $N(x, m) < 0$ for all $x > \eta(m)$.

4.2 A free boundary problem

To solve the singular control problem (4), we will use a standard guess-and-verify approach. We guess that when the reserves of the resources is above a threshold which depends on its running minimum, it is optimal to extract. Furthermore, we guess from Proposition 2 that, if m is sufficiently large, let us say larger than a threshold $\overline{m}$, it should be optimal to extract immediately the surplus $x - \overline{m} > 0$. Therefore, we conjecture the existence of a free boundary function $b : [0, \overline{m}] \mapsto [0, \overline{m}]$ such that $b(m) \geq m$ that separates the waiting region $\mathcal{C}$ from the extraction region $\mathcal{E}$ where the control process L is activated. We set

$$\mathcal{C} = \{(x, m) \in \mathcal{J},\, 0 \leq m \leq \overline{m},\ m \leq x < b(m)\},$$

$$\mathcal{E} = \{(x, m) \in \mathcal{J},\, m > \overline{m} \text{ or } \ (m \in [0, \overline{m}] \text{ and } x \geq b(m))\}.$$

We should also have the inequality $b(m) \leq \eta(m)$ over $[0, \overline{m}]$. Intuitively, uncertainty on the tipping point should make the DM less cautious (and increases its value function) so that $\eta(m)$, the optimal extraction threshold when the DM's problem ends with certainty at the level m should be larger than $b(m)$. We thus expect that b, as η, is non-decreasing, satisfies $b(m) \geq m$ over $[0, \overline{m}]$ with $b(0) = x^0$ and $b(\overline{m}) = \overline{m} \leq \overline{x}$. Observe also that the DM learns nothing about the advent of a tipping point when the running minimum m is larger than $\overline{y}$, the upper bound of the distribution of Y. We thus expect that the function b is constant for $m \geq \overline{y}$.

Relying on the verification Lemma 1, we look for a solution to the HJB equation (9) with the boundary conditions (10), (11). More precisely, we aim at finding a pair of functions (W, b) with $W \in \mathcal{R}(\mathcal{J})$ and a threshold $\overline{m} > x^0$ such that the following free boundary problem is satisfied:

$$\begin{aligned}
b(0) &= x^0, && (23)\\
b(\overline{m}) &= \overline{m}, && (24)\\
\mathcal{L}W(x, m) - rW(x, m) &= 0,\ 0 \leq m < x < b(m) \text{ and } 0 \leq m \leq \overline{m}, && (25)\\
W_x(x, m) &= F(m),\ x \geq b(m) \ \text{ and } \ 0 < m \leq \overline{m} && (26)\\
W_m(m, m) &= U(m)f(m),\ \ 0 < m \leq \overline{m}, && (27)\\
W_m(m, m) &\geq U(m)f(m),\ \ \overline{m} < m && (28)\\
W(m, m) &\geq T[W](m),\ \ 0 < m && (29)\\
W_{xx}(b(m), m) &= 0,\ \ 0 < m \leq \overline{m}, && (30)
\end{aligned}$$

We show below that, if a solution to (23)-(30) exists, then the function b describing the free boundary satisfies an ODE. We first observe that a solution of (25) writes

$$W(x,m) = A(m)\psi(x) + B(m)\phi(x), \tag{31}$$

for some functions A, B. Then, (26) and (30) yield

$$A(m) = \frac{F(m)\phi''(b(m))}{\psi'(b(m))\phi''(b(m)) - \phi'(b(m))\psi''(b(m))}, \tag{32}$$

$$B(m) = -\frac{F(m)\psi''(b(m))}{\psi'(b(m))\phi''(b(m)) - \phi'(b(m))\psi''(b(m))}. \tag{33}$$

Notice that ϕ and ψ satisfy

$$u''(x) = \frac{2}{\sigma^2(x)}(ru(x) - \mu(x)u'(x)) \text{ on } [0,\infty). \tag{34}$$

Letting $D(x) = \psi'(x)\phi(x) - \phi'(x)\psi(x)$, a computation leads to

$$A(m) = \frac{F(m)}{D(b(m))}\left(\phi(b(m)) - \frac{\mu(b(m))}{r}\phi'(b(m))\right), \tag{35}$$

$$B(m) = -\frac{F(m)}{D(b(m))}\left(\psi(b(m)) - \frac{\mu(b(m))}{r}\psi'(b(m))\right). \tag{36}$$

Taking the derivatives for $m \in (0, \overline{m} \wedge \overline{y})$ yields,

$$\begin{aligned}
A'(m) &= \frac{b'(m)F(m)}{D(b(m))}\Big(\phi'(b(m)) - \frac{\mu(b(m))}{r}\phi''(b(m)) - \frac{\mu'(b(m))}{r}\phi'(b(m)) \\
&- \frac{D'(b(m))}{D(b(m))}(\phi(b(m)) - \frac{\mu(b(m))}{r}\phi'(b(m)))\Big) \\
&+ \frac{f(m)}{D(b(m))}\left(\phi(b(m)) - \frac{\mu(b(m))}{r}\phi'(b(m)))\right), \\
B'(m) &= -\frac{b'(m)F(m)}{D(b(m))}\Big(\psi'(b(m)) - \frac{\mu(b(m))}{r}\psi''(b(m)) - \frac{\mu'(b(m))}{r}\psi'(b(m)) \\
&- \frac{D'(b(m))}{D(b(m))}(\phi(b(m)) - \frac{\mu(b(m))}{r}\psi'(b(m)))\Big) \\
&+ \frac{f(m)}{D(b(m))}\left(\psi(b(m)) - \frac{\mu(b(m))}{r}\psi'(b(m)))\right).
\end{aligned}$$

Using again (34) and noticing the relation $\frac{D'(b(m))}{D(b(m))} = -\frac{2\mu(b(m))}{\sigma^2(b(m))}$, we obtain

$$\begin{aligned}
A'(m) &= \frac{1}{D(b(m))}\left(b'(m)F(m)(1 - \frac{\mu'(b(m))}{r})\phi'(b(m)) + f(m)(\phi(b(m)) - \frac{\mu(b(m))}{r}\phi'(b(m)))\right), \\
B'(m) &= \frac{1}{D(b(m))}\left(-b'(m)F(m)(1 - \frac{\mu'(b(m))}{r})\psi'(b(m)) - f(m)(\psi(b(m)) - \frac{\mu(b(m))}{r}\psi'(b(m)))\right).
\end{aligned}$$

Finally, the substitution of the last two equations into (27) and (23) yields the following ODE:

$$\forall m \in (0, \overline{m} \wedge \overline{y}),\ b'(m) = E(b(m), m) \text{ and } b(0) = x^0. \tag{37}$$

where the vector field E is defined over $(0,\infty)^2$ by the relation

$$E(x,m)=\frac{H(m)}{G(x)}\frac{N(x,m)}{D(x,m)}, \tag{38}$$

where

$$\begin{aligned}
N(x,m) &= \phi(x)\psi(m)-\psi(x)\phi(m)+D(x,m)\frac{\mu(x)}{r}-D(x,x)U(m),\\
D(x,m) &= \psi'(x)\phi(m)-\phi'(x)\psi(m),\\
H(m) &= \frac{f(m)}{F(m)},\\
G(x) &= 1-\frac{\mu'(x)}{r}.
\end{aligned}$$

Extending artificially the map b to $[0,\overline{x}]$ and taking into account the fact that b is constant above $\overline{y}$, we introduce the following Cauchy problem: find a map $b:[0,\overline{x}]\to[0,\overline{x}]$ which satisfies

$$\begin{cases} b\in C([0,\overline{x}])\cap C^1((0,\overline{x}\wedge\overline{y})) \\ b(0)=x_0 \\ b'(m)=E(b(m),m), & \forall m\in(0,\overline{x}\wedge\overline{y}) \\ b(m)=b(\overline{x}\wedge\overline{y}), & \forall m\in(\overline{x}\wedge\overline{y},\overline{x}\vee\overline{y}] \end{cases} \tag{39}$$

The vector field (38) is well defined on $(0,\infty)^2$ because $F>0$ over $(0,\infty)$, $D>0$ over $(0,\infty)^2$ and $G>0$ over $(0,\infty)$ by Assumption 3. Note also that E vanishes for $m>\overline{y}$. Because H is locally Lipschitz on $(0,\overline{y}]$ by Assumption 5, E is locally Lipschitz on the domain $(0,\infty)\times(0,\overline{y}]$. However, E cannot be continuously extended on $(0,\infty)\times[0,\overline{y}]$ as $\lim_{m\longrightarrow 0}H(m)=+\infty$, which implies from Proposition 2 (iii), that, for any $x>0$ with $x\neq x^0$, $\lim_{m\longrightarrow 0}E(x,m)=\infty$. Thus, the Cauchy problem (39) requires a specific analysis. We now state the main result of this section.

Proposition 3. *There exists a unique solution b to the Cauchy problem* (39). *This solution is non-decreasing and satisfies $b(m)\leq\eta(m)$ for all $m\in[0,\overline{x}]$. Moreover, the point $\overline{m}:=\min\{m\in[0,\overline{x}]|b(m)=m\}$ is well-defined, belongs to $(0,\overline{x})$ and is such that $b'(\overline{m})<1$.*

Proof of Proposition 3. Existence

At first, to avoid dealing with several cases depending on the value of $\overline{y}$, we consider an auxiliary well-posed ODE. Let $\hat{f}$ denote a Lipschitz positive map on $[0,\infty)$ which coincides with f on $[0,\overline{y}]$, and let $\hat{F}(m)=\int_0^m\hat{f}(t)dt$ and $\hat{H}(m)=\frac{\hat{f}(m)}{\hat{F}(m)}$ for $m>0$. Define then the map $\hat{E}$ on $(0,\infty)\times(0,\infty)$ by

$$\hat{E}(x,m)=\frac{\hat{H}(m)}{G(x)}\frac{N(x,m)}{D(x,m)},$$

which is locally Lipschitz on $(0,\infty)^2$ and coincides with E on $(0,\infty)\times(0,\overline{y}]$.

Note that from Proposition 2 (iii), because $\hat{H},D,G$ are positive, the sign of $\hat{E}$ coincides with the sign of N, and thus for all $m\in(0,\overline{x}]$, $\hat{E}(x,m)>0$ for $x\in(0,\eta(m))$, $\hat{E}(x,m)<0$ for $x\in(\eta(m),\infty)$

and $\hat{E}(\eta(m), m) = 0$. Therefore, for all $m_0 \in [0, \bar{x})$ and all initial condition $z < \eta(m_0)$, the maximal solution of the ODE $b'(m) = \hat{E}(b(m), m)$ such that $b(m_0) = z$ is well-defined and is increasing and strictly bounded from above by η on $[m_0, \overline{x})$.

By Lemma 4 (i) in the Appendix, together with the fact that $\hat{H} = H$ is decreasing on $(0, \overline{y}]$ by Assumption 5, the map $m \to \hat{E}(m, m)$ is decreasing on $(0, \overline{y}]$. Moreover, by Lemma 4 (ii) we have that for all $m \in (0, \overline{x}]$, $x \to \hat{E}(x, m)$ is decreasing on $[m, \eta(m)]$.

Let $(m_n)_{n\geq 1}$ a decreasing sequence that converges to 0 with $m_1 < \overline{x}$. By the Cauchy-Lipschitz theorem, for each $n \geq 1$, there exists a unique maximal solution b_{x^0,m_n} to $b'(m) = \hat{E}(b(m), m)$ that satisfies $b_{x^0,m_n}(m_n) = x^0$. Because η is increasing and $\eta(0) = x^0$, $b_{x^0,m_n}(m_n) = x^0 < \eta(m_n)$ and thus b_{x^0,m_n} is well-defined on a neighborhood of $\mathcal{I}_n = [m_n, \overline{x}]$ and is increasing and strictly bounded from above by η on $[m_n, \overline{x})$. Let us consider the sequence of functions $(b_n)_{n\geq 1}$ defined by the relations

$$\begin{aligned} b_n(m) &= b_{x^0,m_n}(m) \; \forall m \in \mathcal{I}_n, \\ b_n(m) &= x^0 \quad \forall m \in [0, m_n]. \end{aligned}$$

The sequence $(b_n)_{n\geq 1}$ is a non-decreasing sequence of positive continuous non-decreasing functions bounded above by η. It admits a pointwise limit $\hat{b}$ such that $\hat{b}(0) = x^0$. By construction, the sequence $(b_n)_{n\geq 1}$ satisfies for any $m \geq 0$,

$$b_n(m) = x^0 + \int_0^m \hat{E}(b_n(s), s) I_{s\geq m_n} \, ds.$$

In particular, for any $0 < m' < m < \overline{x}$,

$$b_n(m) - b_n(m') = \int_{m'}^m \hat{E}(b_n(s), s) I_{s\geq m_n} \, ds.$$

Taking the limit in the above expression

$$\begin{aligned} \hat{b}(m) - \hat{b}(m') &= \lim_{n\longrightarrow\infty} \int_{m'}^m \hat{E}(b_n(s), s) I_{s\geq m_n} ds \\ &= \int_{m'}^m \lim_{n\longrightarrow\infty} \hat{E}(b_n(s), s) I_{s\geq m_n} ds \\ &= \int_{m'}^m \hat{E}(\hat{b}(s), s) ds. \end{aligned} \tag{40}$$

The second equality follows from bounded convergence. Indeed, $\hat{E}$ is continuous and we deduce from the properties of $(b_n)_{n\geq 1}$ that the sequence $(\hat{E}(b_n(s), s) I_{s\geq m_n})_{n\geq 1}$ is non-negative and bounded above by $\sup_{(x,m)\in[x^0,\overline{x}]\times[m',\overline{x}]} |\hat{E}(x, m)|$. We have $\hat{b}(m) \leq \eta(m)$ which implies that $\hat{b}(0+) \leq \eta(0) = x^0$, thus $\hat{b}$ is continuous at 0. Letting m' tend to 0 in (40), we get by monotone convergence the relation

$$\hat{b}(m) - \hat{b}(0) = \int_0^m \hat{E}(\hat{b}(s), s) \, ds < \infty, \tag{41}$$

which shows that function $\hat{b}$ satisfies the ode $\hat{b}'(m) = \hat{E}(b(m), m)$ on $(0, \overline{x})$.

Define then $b : [0, \overline{x}] \to [0, \overline{x}]$ by

$$b(m) = \begin{cases} \hat{b}(m) & \forall m \in [0, \overline{x} \wedge \overline{y}], \\ \hat{b}(\overline{x} \wedge \overline{y}) & \forall m \in (\overline{x} \wedge \overline{y}, \overline{x} \vee \overline{y}]. \end{cases}$$

By construction, b is a solution to the Cauchy problem (39).

Uniqueness We show that b is the unique solution to the Cauchy problem (39). Suppose the contrary, let b and $\tilde{b}$ be two different solutions of (39). Let $m \in (0, \overline{y} \wedge \overline{x})$ sufficiently small so that $b(m) > m$ and $\tilde{b}(m) > m$. It must be that $b(m) \neq \tilde{b}(m)$ and we can assume without loss of generality that $\tilde{b}(m) > b(m)$. It follows from (41) that,

$$\tilde{b}(m) - b(m) = \int_0^m \hat{E}(\tilde{b}(s), s) - \hat{E}(b(s), s)\, ds. \tag{42}$$

The left-hand side of (42) is strictly positive whereas, from Lemma 4 (ii), its right-hand side is negative, a contradiction.

Let $\overline{m} = \min\{m \in [0, \overline{x}], b(m) = m\}$. Note that this minimum exists by the intermediate value theorem, because $b(\overline{x}) \leq \eta(\overline{x}) = \overline{x}$ and $b(0) = x^0 > 0$. Let us prove that $\overline{m} \in (0, \overline{x})$, $b(\overline{m}) = \overline{m}$ and $b'(\overline{m}) < 1$.

If $\overline{m} > \overline{y}$ then it must be that $\overline{y} < \overline{x}$ and thus $\overline{m} = b(\overline{y}) \leq \eta(\overline{y}) < \overline{x}$ and $b'(\overline{m}) = 0 < 1$ which proves the result. If $\overline{m} \leq \overline{y}$, then because $b(m) \geq m$, it must be that $b'(\overline{m}) = \hat{E}(\overline{m}, \overline{m}) \leq 1$. Let us suppose, by way of contradiction, that $b'(\overline{m}) = 1$ and let us consider $\widetilde{b}$ the solution to the Cauchy problem

$$\widetilde{b}'(m) = \hat{E}(\widetilde{b}(m), m), \text{ and } \widetilde{b}(\widetilde{m}) = \widetilde{x},$$

with $\widetilde{m} \in (0, \overline{m})$ and $\widetilde{x} \in (\widetilde{m}, b(\widetilde{m}))$. Because $\tilde{b}(\widetilde{m}) < b(\widetilde{m})$, by the Cauchy-Lipschitz theorem, we have that $\widetilde{b} \leq b$ over $(\widetilde{m}, \overline{m})$. Also, for $m \in (\widetilde{m}, \overline{m})$, $\widetilde{b}(m) > m$, if not there exists $\widehat{m} \in (\widetilde{m}, \overline{m})$ such that $\tilde{b}(\widehat{m}) = \widehat{m}$ which implies that $E(\widehat{m}, \widehat{m}) \leq 1$, a contradiction with Lemma 4 (i). Therefore, we have that $m \leq \widetilde{b}(m) \leq b(m)$ over $[\widetilde{m}, \overline{m}]$ and thus, $\widetilde{b}(\overline{m}) = \overline{m} = b(\overline{m})$ which implies, by the Cauchy-Lipschitz theorem, that $\widetilde{b} = b$, a contradiction with $\widetilde{b}(\widetilde{m}) \neq b(\widetilde{m})$. Finally, it cannot be that $\overline{m} = \overline{x}$ because the solution of the ODE $b'(m) = \hat{E}(b(m), m)$ such that $b(\overline{x}) = \overline{x}$ must be non-increasing on $(0, \overline{x}]$ because $\hat{E}(x, m) \leq 0$ for $x \geq \overline{x}$ and $m \in (0, \overline{x}]$.

□

4.3 Solution to the DM problem

In this section, we characterise the value function of the singular control problem (4) and the associated optimal strategy, if any. The previous section suggests that: 1) the value function of the problem (4) satisfies the free boundary problem (23)-(30); 2) for any state (x, m) with $x \geq b(m)$

and $0 < m < \overline{m}$, it is optimal to extract $x - b(m)$; and 3) for any state (x, m) with $\overline{m} \leq m < x$, it is optimal to extract $x - m$. However, it does not say whether it is optimal to extract $m - \overline{m}$ of the reserves in a state (m, m) where $m > \overline{m}$. We will see that the answer depends on the relation between the thresholds $\overline{m}$ and u^*, the extraction threshold of the downgraded problem.

4.3.1 Candidate value function

The next proposition derives our candidate value function.

Proposition 4. *Let us assume the decision maker faces a tipping point problem with downgraded value function U and associated optimal extraction threshold $u^* \leq \overline{x}$. Let us consider $A(m)$ and $B(m)$ defined in (32) and (33) and, the solution b to the Cauchy problem* (39)*.*

Then the function $(x, m) \mapsto W(x, m)$ defined on $\mathcal{J}$ by

$$W(x,m) = A(m)\psi(x) + B(m)\phi(x), \quad m \leq x \leq b(m) \text{ and } 0 \leq m \leq \overline{m}, \tag{43}$$

$$W(x,m) = (x - b(m))F(m) + \frac{\mu(b(m))}{r}F(m),\ b(m) < x \text{ and } 0 \leq m \leq \overline{m}, \tag{44}$$

$$W(x,m) = \frac{\mu(\overline{m})}{r}F(\overline{m}) + (x-m)F(m) + \int_{\overline{m}}^{m} (U(s)f(s) + F(s))ds, \quad m > \overline{m} \tag{45}$$

satisfies $W \geq V$ where V is the value function of the singular control problem (4)*.*

Proof of Proposition 4. Consider the three domains

$$\mathcal{J}_1 = \{(x,m) : m \leq x \leq b(m) \text{ and } 0 < m \leq \overline{m}\},$$

$$\mathcal{J}_2 = \{(x,m) : b(m) \leq x \text{ and } 0 < m \leq \overline{m}\},$$

$$\mathcal{J}_3 = \{(x,m) \in \mathcal{J} : \ m \geq \overline{m}\}.$$

Lemma 6 in the Appendix shows that $W \in \mathcal{R}(\mathcal{J})$ and in particular is continuous over $\mathcal{J}$.

Using Lemma 1, it is sufficient to prove that W defined by (43), (45) is a solution of the HJB equation (9) with boundary conditions (10)-(11) on the three subdomains $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$.

Because $A(m)$ and $B(m)$ are defined by (32) and (33), and b is solution to the Cauchy problem (39), we deduce that W satisfies $\mathcal{L}W - rW = 0$ on $\mathcal{J}_1$ and that the two boundary conditions (26) and (27) are satisfied. This is obtained by direct computations mimicking those detailed in section 4.2 (recall that b is constant on $[\overline{y} \wedge \overline{m}, \overline{m}]$).

It follows from ([33], Lemma 4.2-c) that $x \mapsto W(x,m)$ is concave over $(m, b(m))$ for $m \leq \overline{m}$, which implies $W_x(x,m) \geq F(m)$ for $(x,m) \in \mathcal{J}_1$. In particular, $W_x(m,m) \geq F(m)$. On the other hand, take $m \le \overline{m}$ and $h \in (0, m]$, we have

$$\begin{aligned}
W(m,m) - W(m-h, m-h) &= \int_{m-h}^{m} (W_m(s,s) + W_x(s,s))\, ds \\
&\ge \int_{m-h}^{m} (U(s)f(s) + F(s))\, ds \\
&= U(m-h)(F(m) - F(m-h)) + \int_{m-h}^{m} ((U(s) - U(m-h))f(s) + F(s)) ds \\
&\ge U(m-h)(F(m) - F(m-h)) + \int_{m-h}^{m} (sf(s) + F(s))\, ds \\
&- (m-h)(F(m) - F(m-h)) \\
&= U(m-h)(F(m) - F(m-h)) + hF(m).
\end{aligned}$$

The first inequality follows from (26) and $W_x(m,m) \ge F(m)$. The second inequality follows from the inequality $U'(m) \ge 1$. The last equality follows from the relation $(mF(m))' = F(m) + mf(m)$. Then, we deduce $W(m,m) \ge T[W](m)$ for $m \le \overline{m}$ which shows that W satisfies (9)-(11) on $\mathcal{J}_1$.

On $\mathcal{J}_2$, W is affine in x with slope $F(m)$. Moreover, for all $(x,m) \in \mathcal{J}_2$, using Assumption 3, we have

$$\mathcal{L}W(x,m) - rW(x,m) = [(\mu(x) - rx) \; - \; (\mu(b(m) - rb(m))]F(m) \le 0. \tag{46}$$

Therefore, W satisfies (9) on $\mathcal{J}_2$.

On $\mathcal{J}_3$, W is affine in x with slope $F(m)$ and $W_m(m,m) = U(m)f(m)$. Therefore, the same computation as the one on $\mathcal{J}_1$ gives $W(m,m) \ge T[W](m)$ for any $m \ge 0$. It remains to show that the function W satisfies $\mathcal{L}W - rW \le 0$ on $\mathcal{J}_3$. For any $(x,m) \in \mathcal{J}_3$,

$$\begin{aligned}
\mathcal{L}W(x,m) - rW(x,m) &= F(m)(\mu(x) - rx) - F(\overline{m})(\mu(\overline{m}) - r\overline{m}) - r\overline{m}F(\overline{m}) - r\int_{\overline{m}}^{m} (U(s) - s)f(s)\, ds, \\
&\le l(m), \qquad (47)
\end{aligned}$$

with $l(m) \equiv F(m)(\mu(m) - rm) - F(\overline{m})(\mu(\overline{m}) - r\overline{m}) - r\int_{\overline{m}}^{m} (U(s) - s)f(s)\, ds$ where (47) follows from Assumption 3. We remark that $l(\overline{m}) = 0$ and a computation leads to

$$l'(m) = rF(m)\left((\frac{\mu'(m)}{r} - 1) + \frac{f(m)}{F(m)}(\frac{\mu(m)}{r} - U(m)) \right), \tag{48}$$

where the above expression for $m = \overline{y}$ stands for the left derivative of l whenever $\overline{y} \ge \overline{m}$. To establish our claim, we prove that $m \longrightarrow l(m)$ is non-increasing on $[\overline{m}, \infty)$. From Lemma 4 and Proposition 3 we have for $\overline{m} \le m$ (note that the left-hand side vanishes when $m > \overline{y}$)

$$E(m,m) = \frac{f(m)}{F(m)} \left(\frac{1}{1 - \frac{\mu'(m)}{r}} \right) \left(\frac{\mu(m)}{r} - U(m) \right) \le E(\overline{m}, \overline{m}) < 1,$$

and therefore

$$\frac{f(m)}{F(m)} \left(\frac{\mu(m)}{r} - U(m) \right) \le 1 - \frac{\mu'(m)}{r},$$

which implies that (48) is non-positive on $[\overline{m}, \infty)$, thus $\mathcal{L}W(x,m) - rW(x,m) \le 0$ on $\mathcal{J}_3$. □

4.3.2 Value function, optimal strategy and ϵ-optimal strategy

Some comments on our candidate value function W are in order. The expression of W on $\mathcal{J}_1$ follows from our analysis of (25) and from Proposition 3, and is therefore associated with a strategy that reflects the process X at the boundary b. Observe that, if $\overline{m} = b(\overline{y})$, then $A(m)$ and $B(m)$ are constant over $[\overline{y}, b(\overline{y})]$. The expression of W on $\mathcal{J}_2$ follows from (14), which we assume to be an equality for $m < x' < x$ with $x' = b(m)$ and corresponds to an initial extraction of $x - b(m)$. The equality $W(b(m), m) = \frac{\mu(b(m))}{r} F(m)$ follows from (25), (26) and (30).

In order to discuss the expression of W on $\mathcal{J}_3$, we distinguish the two cases $u^* \leq \overline{m}$ and $u^* > \overline{m}$[10]. Note at first that for $m > m' \geq u^*$, $U(m) = U(m') + (m - m')$. Therefore, if $u^* \leq \overline{m}$, then (45) can be written as

$$\begin{aligned}
W(x,m) &= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m)F(m) + \int_{\overline{m}}^{m} (U(s)f(s) + F(s))ds \\
&= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m)F(m) + \int_{\overline{m}}^{m} ((U(\overline{m}) + (s-\overline{m}))f(s) + F(s))ds \\
&= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m)F(m) + (U(\overline{m}) - \overline{m})(F(m) - F(\overline{m})) + \int_{\overline{m}}^{m} (sf(s) + F(s))ds \\
&= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m)F(m) + (U(\overline{m}) - \overline{m})(F(m) - F(\overline{m})) + mF(m) - \overline{m}F(\overline{m}) \\
&= W(\overline{m}, \overline{m}) + (x - \overline{m})F(m) + U(\overline{m})(F(m) - F(\overline{m})), \qquad (49)
\end{aligned}$$

where we used that $(mF(m))' = mf(m) + F(m)$. The expression (49) corresponds to a strategy that extracts $x - \overline{m}$ at time zero and, either $Y \in (\overline{m}, m)$, with the DM obtaining the continuation $U(\overline{m})$, or $Y \leq \overline{m}$, with the DM obtaining the payoff $W(\overline{m}, \overline{m})/F(\overline{m})$ associated with the strategy that reflects X on the boundary b, starting from $(\overline{m}, \overline{m})$. The intuition is as follows: when initially extracting the quantity $x - \overline{m}$, the DM faces the risk that $Y \in (\overline{m}, m)$. If the optimal level of reserves u^* associated with the downgraded value function U is less than $\overline{m}$, then the optimal strategy in the continuation problem consists of extracting immediately $z - u^*$ when starting at any level $z \geq u^*$ and is thus compatible with an extraction of $x - \overline{m}$ in the initial problem. This action is simultaneously optimal in both the initial and the continuation problem because the DM will always have the opportunity to extract immediately a larger quantity in the continuation problem with value U whenever the tipping point occurs in the interval $[\overline{m}, m]$.

In the case $u^* > \overline{m}$, we split $\mathcal{J}_3$ into the two subdomains

$$\mathcal{J}_{3'} = \{(x,m) \in \mathcal{J} \,:\, \overline{m} \leq m < u^*\},$$

$$\mathcal{J}_{3''} = \{(x,m) \in \mathcal{J} \,:\, m \geq u^*\}.$$

[10]Note that the vector field (38) depends on the function U and thus $\overline{m}$ depends on the threshold $u^* \leq \overline{x}$. It is therefore not obvious that the two cases $u^* \leq \overline{m}$ and $u^* \in (\overline{m}, \overline{x}]$ can occur. To see that the case $u^* < \overline{m}$ must be considered, simply set $U(x) = x$, so that $u^* = 0$. Then we have $u^* = 0 < x^0 = \eta(0) = b(0) < b(\overline{m}) = \overline{m}$. Lemma 5 in the appendix shows that the case $u^* > \overline{m}$ must also be considered.

As above, using that $U' = 1$ on $[u^*, \infty)$, we have for $(x,m) \in \mathcal{J}_{3''}$

$$W(x,m) = W(u^*,u^*) + (x-u^*)F(m) + U(u^*)(F(m) - F(u^*)), \tag{50}$$

which is associated with a strategy that extracts immediately $x - u^*$. To grasp the intuition of (45) on the set $\mathcal{J}_{3'}$, take $(x,m) \in \mathcal{J}$ with $m \in (\overline{m}, u^*)$ and consider the strategy that consists of extracting $x - \overline{m}$ at time 0. As in the previous case, the DM is facing the risk that $Y \in (\overline{m}, m)$. In this case, the DM ends up with a level of reserves strictly below Y in the continuation problem, whereas it would have been optimal to wait in the continuation problem starting from the tipping point Y because $Y \le m < u^*$. This strategy is therefore strictly suboptimal in the continuation problem. Assume now that the DM is informed of the value of the tipping point Y. Under this assumption, we guess that the optimal policy is to extract at time zero up to the tipping point if $Y \in (\overline{m}, m)$ and then to act optimally in the continuation problem, or to extract up to $\overline{m}$ if $Y \le \overline{m}$ and then apply the strategy which reflects x at the boundary b. The payment associated to this strategy is

$$\begin{aligned}
&\mathbb{E}_{x,m}[(U(Y) + x - Y)\mathbb{1}_{\{Y \in (\overline{m}, m)\}}] + (x - \overline{m})F(\overline{m}) + W(\overline{m}, \overline{m}) \\
&\quad = \int_{\overline{m}}^{m} (U(s) + x - s) f(s)\, ds + (x - \overline{m})F(\overline{m}) + W(\overline{m}, \overline{m}) \\
&\quad = (x - m)F(m) + \int_{\overline{m}}^{m} (U(s)f(s) + F(s)) ds + W(\overline{m}, \overline{m}),
\end{aligned}$$

which yields (45). This heuristic argument, developed under the additional assumption that the DM knows the exact value of Y raises the question of the existence of an optimal strategy. We will show below that the map W given in Proposition 4 does indeed correspond to the optimal value function of the DM problem, and that in case $u^* > \overline{m}$, there is no optimal strategy that achieves this value. Nevertheless, we will construct in the next section an ϵ-optimal strategy for every $\epsilon > 0$.

In the remainder of this section, we construct for the case $u^* \le \overline{m}$ a control L^* which is optimal in the sense that the function W defined in Proposition 4 satisfies $W(x,m) = V(x,m,L^*)$ for all $(x,m) \in \mathcal{J}$, and for the case $u^* > \overline{m}$ a family of controls $L^{*\epsilon}$ indexed by $\epsilon > 0$ such that $W(x,m) = \lim_{\epsilon \longrightarrow 0} V(x,m,L^{*\epsilon})$ for all $(x,m) \in \mathcal{J}$, proving in both cases that $W = V$.

The proof relies on the following Proposition.

Proposition 5. *Let b be the solution of the Cauchy problem* (37) *given in Proposition 3. For any $(x,m) \in \mathcal{J}$ and any stopping time τ, there exists a unique solution (X,M,L) defined on $\{\tau < \infty\}$ on the time interval $[\tau, \tau_0] \cap [\tau, \infty)$ with $\tau_0 = \inf\{t \ge \tau, X_t \le 0\}$ to the reflected stochastic differential* *equation*

$$X_{\tau-} = x,\ M_{\tau-} = m,\ L_{\tau-} = 0, \tag{51}$$

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t - dL_t,\ \ M_t = m \wedge \inf_{\tau \le s \le t} X_s,\ \ t \ge \tau, \tag{52}$$

$$X_t \in [M_t, b(M_t)] \cap [0, \overline{m}]\ \ a.s\ ,\ t \ge \tau, \tag{53}$$

$$\int_{[\tau,t]} \mathbb{1}_{\{X_s < b(M_s)\}} dL_s = 0\ \ a.s\ for\ any\ t \ge \tau. \tag{54}$$

Proposition 5 is related to the existence and uniqueness of solutions of stochastic differential equation with reflecting boundary conditions for a domain that has a corner at which more than one oblique direction is allowed (see Dupuis and Ishii [11]). However, we have not been able to find a reference that exactly fits our setting, so we provide a proof in the Appendix.

We first consider the case $u^* \le \overline{m}$.

Proposition 6. *If $u^* \le \overline{m}$, then the function W defined by* (43)-(45) *identifies with the value function of the singular control problem* (4)*. Moreover, for any initial condition $(x, m) \in \mathcal{J}$, letting (X^{L^*}, M^{L^*}, L^*) denote the solution of (51-54) given in Proposition 5 with $\tau = 0$, then L^* is an optimal control.*

Proof of Proposition 6. Because $W \ge V$, it is sufficient to prove that W defined by (43)-(45) satisfies $W(x, m) = V(x, m, L^*)$ for all $(x, m) \in \mathcal{J}$.

Let (X^{L^*}, M^{L^*}, L^*) denote the solution of (51-54) given in Proposition 5 with $\tau = 0$. Note that (53) and (54) imply that

$$L_0^* = (x - \overline{m})\mathbb{1}_{(x,m)\in\mathcal{J}_3} + (x - b(m))\mathbb{1}_{(x,m)\in\mathcal{J}_2},$$

and that except this potential jump at time 0, the processes X^{L^*}, M^{L^*}, L^* are continuous and $(X^{L^*}, M^{L^*}) \in \mathcal{J}_1$. Because $W \in \mathcal{R}(\mathcal{J})$, as in Lemma 1, we can apply Itô's formula to the process $e^{-r(t\wedge T_n\wedge\tau_0)}W(X^{L^*}_{t\wedge T_n\wedge\tau_0}, M^{L^*}_{t\wedge T_n\wedge\tau_0})$ where T_n is defined as in (15) and $\tau_0 = \inf\{t \ge 0, X^{L^*}_t \le 0\}$. After taking expectations, we obtain

$$\begin{aligned}
W(x,m) &= \mathbb{E}_{x,m}[e^{-r(t\wedge T_n\wedge\tau_0)}W(X^{L^*}_{t\wedge T_n\wedge\tau_0}, M^{L^*}_{t\wedge T_n\wedge\tau_0})] \\
&- \mathbb{E}_{x,m}\left[\int_0^{t\wedge T_n\wedge\tau_0} e^{-rs}(\mathcal{L} - r)W(X^{L^*}_s, M^{L^*}_s)\,ds\right] \\
&- \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}W_m(M^{L^*}_s, M^{L^*}_s)dM^{L^*,c}_s\right] + \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}W_x(X^{L^*}_s, M^{L^*}_s)dL^{*,c}_s\right] \\
&- \mathbb{E}_{x,m}\left[\sum_{0\le s\le t\wedge T_n\wedge\tau_0} e^{-rs}(W(X^{L^*}_s, M^{L^*}_s) - W(X^{L^*}_{s-}, M^{L^*}_{s-}))\right].
\end{aligned}$$

Because $(X_s^{L^*}, M_s^{L^*})$ lies in $\mathcal{J}_1$ for $s \leq t \wedge T_n \wedge \tau_0$, we have $(\mathcal{L} - r)W(X_s^{L^*}, M_s^{L^*}) = 0$. Using that the only possible jump is at time zero, the last term is equal to

$$\begin{aligned}
&-\mathbb{E}_{x,m}\left[\sum_{0\leq s\leq t\wedge T_n\wedge\tau_0} e^{-rs}(W(X_s^{L^*}, M_s^{L^*}) - W(X_{s-}^{L^*}, M_{s-}^{L^*}))\right] \\
&\quad = [W(x,m) - W(\overline{m},\overline{m})]\mathbb{1}_{(x,m)\in\mathcal{J}_3} + [W(x,m) - W(m,b(m))]\mathbb{1}_{(x,m)\in\mathcal{J}_2} \\
&\quad = [(x-\overline{m})F(m) + U(\overline{m})(F(m) - F(\overline{m}))]\mathbb{1}_{(x,m)\in\mathcal{J}_3} + [(x-m)F(m)]\mathbb{1}_{(x,m)\in\mathcal{J}_2} \\
&\quad = F(M_{0-}^{L^*})\Delta L_0^* + U(M_0^{L^*})(F(M_{0-}^{L^*}) - F(M_0^{L^*})),
\end{aligned}$$

where the second equality follows from (49) and the fact that $W_x(y,p) = F(y)$ for $(y,p) \in \mathcal{J}_2$. Because the random measure $dM_s^{L^*,c}$ has support on the set $\{s \geq 0,\ X_s^{L^*} = M_s^{L^*}\}$, the boundary condition (10) gives

$$\mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}W_m(M_s^{L^*}, M_s^{L^*})dM_s^{L^*,c}\right] = \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}U(M_s^{L^*})f(M_s^{L^*})dM_s^{L^*,c}.\right]$$

Finally, conditions (53-54) together with the fact that $W_x(b(m),m) = F(m)$ for $m \in (0,\overline{m}]$ imply that

$$\mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}W_x(X_s^{L^*}, M_s^{L^*})dL_s^{*,c}\right] = \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}F(M_s^{L^*})dL_s^{*,c}\right].$$

Gathering the previous equalities, we obtain

$$\begin{aligned}
W(x,m) &= \mathbb{E}_{x,m}[e^{-r(t\wedge T_n\wedge\tau_0)}W(X_{t\wedge T_n\wedge\tau_0}^{L^*}, M_{t\wedge T_n\wedge\tau_0}^{L^*})] \\
&- \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}U(M_s^{L^*})f(M_s^{L^*})dM_s^{L^*,c}\right] + \mathbb{E}_{x,m}\left[\int_{[0,t\wedge T_n\wedge\tau_0]} e^{-rs}F(M_{s-}^{L^*})dL_s^*\right] \\
&+ \mathbb{E}_{x,m}\left[U(M_0^{L^*})(F(M_{0-}^{L^*}) - F(M_0^{L^*}))\right].
\end{aligned}$$

Letting n and then t go to $+\infty$, and using that W is bounded on $\mathcal{J}_1$, we conclude that $W(x,m) = V(x,m,L^*)$.

□

We now turn to the case $u^* > \overline{m}$. If $(x,m) \in \mathcal{J}_3$ and $\epsilon > 0$, we construct a process $(X^{L^{*\epsilon}}, M^{L^{*\epsilon}}, L^{*,\epsilon})$ which correspond to a policy which initially extracts up to u^*, then extracts according to $dL_t^{*\epsilon} = \frac{1}{\epsilon}dt$ until $X^{L^{*\epsilon}}$ reaches the level $\overline{m}$, and then reflects $X^{L^{*\epsilon}}$ on the boundary b.

Precisely, let

$$X_{0-}^{L^{*\epsilon}} = x,\ M_{0-}^{L^{*\epsilon}} = m,\ L_{0-}^{L^{*\epsilon}} = 0, \tag{55}$$

and

$$L_0^{L^{*\epsilon}} = (x - m\wedge u^*),\ X_0^{L^{*\epsilon}} = x - L_0^{L^{*\epsilon}} = m\wedge u^*,\ M_0^{L^{*\epsilon}} = m\wedge u^*. \tag{56}$$

Let $(X_t^\epsilon)_{t\geq 0}$ be the unique strong solution to

$$dX_t^\epsilon = (\mu(X_t^\epsilon) - \frac{1}{\epsilon})\, dt + \sigma(X_t^\epsilon) dB_t,\ \ X_0^\epsilon = X_0^{L^{*\epsilon}} = m \wedge u^* \in [\overline{m}, u^*] \tag{57}$$

and define the family of $\mathbb{F}$-stopping time $(\tau_y^\epsilon \equiv \inf\{t \geq 0 \,:\, X_t^\epsilon \leq y\})_{y\in[\overline{m}, m\wedge u^*]}$. We define

$$\forall t \in (0, \tau_{\overline{m}}^{\varepsilon}] \cap (0, \infty),\ \ L_t^{*\epsilon} = L_0^{*\epsilon} + \frac{1}{\epsilon} t,\ X_t^{L^{*\epsilon}} = X_t^\epsilon,\ M_t^{L^{*\epsilon}} = m \wedge \inf_{0\leq s\leq t} X_s^\epsilon. \tag{58}$$

Consider finally the solution $(\hat{X}^\epsilon, \hat{M}^\epsilon, \hat{L}^\epsilon)$ given in Proposition 5 with initial condition $(\overline{m}, \overline{m})$ at time $\tau_{\overline{m}}^\epsilon$, and define on the set $\{\tau_{\overline{m}}^\epsilon < \infty\}$:

$$\forall t \in (\tau_{\overline{m}}^\epsilon, \tau_0] \cap (\tau_{\overline{m}}^\epsilon, \infty),\ \ L_t^{*\epsilon} = L_{\tau_{\overline{m}}^\epsilon}^{*\epsilon} + \hat{L}_t^\epsilon,\ X_t^{L^{*\epsilon}} = \hat{X}_t^\epsilon,\ M_t^{L^{*\epsilon}} = \hat{M}_t^\epsilon, \tag{59}$$

where $\tau_0 = \inf\{t \geq \tau_{\overline{m}}^\epsilon, \hat{X}^\epsilon \leq 0\}$.

Proposition 7. *If $u^* > \overline{m}$, then the function W defined in Proposition 4 identifies with the value function of the singular control problem* (4)*. If $m \leq \overline{m}$, the control L^* defined in Proposition 6 is optimal. If $m > \overline{m}$, the family of controls $(L^{*\epsilon})_{\epsilon>0}$ defined by* (55),(56),(58) *and* (59) *satisfies $W(x, m) = \lim_{\epsilon\to 0} V(x, m, L^{*\epsilon})$.*

Proof of Proposition 7. In case $m \leq \overline{m}$ and $x \geq m$, the proof that $W(x, m) = V(x, m)$ and that the control L^* defined in Proposition 6 is optimal is similar to the proof of Proposition 6.

In case $m > \overline{m}$ and $x \geq m$, since $W \geq V$, it is sufficient to prove that $W(x, m) = \lim_{\epsilon\to 0} V(x, m, L^{*\epsilon})$, where the family of controls $(L^{*\epsilon})_{\epsilon>0}$ defined by (55),(56),(58) and (59). Using that $L^{*\epsilon}, X^{L^{*\epsilon}}, M^{L^{*\epsilon}}$ are continuous except possibly for a deterministic jump at time 0, we have

$$\begin{aligned}
&V(x, m; L^{*\epsilon})\\
&= \mathbb{E}_{x,m}\left[\int_{[0,\tau_0]} e^{-rs} F(M_{s-}^{L^{*\epsilon}})\, dL_s^{*\epsilon} - \int_{[0,\tau_0]} e^{-rs} U(M_s^{L^{*\epsilon}}) f(M_s^{L^{*\epsilon}})\, dM_s^{L^{*\epsilon},c}\right]\\
&\quad + \mathbb{E}_{x,m}\left[\sum_{0\leq s\leq \tau_0} e^{-rs} U(M_s^{L^{*\epsilon}})(F(M_{s-}^{L^{*\epsilon}}) - F(M_s^{L^{*\epsilon}}))\right]\\
&= \mathbb{E}_{x,m}\left[\int_{[0,\tau_{\overline{m}}^\epsilon]} e^{-rs} F(M_{s-}^{L^{*\epsilon}})\, dL_s^{*\epsilon} - \int_{[0,\tau_{\overline{m}}^\epsilon]} e^{-rs} U(M_s^{L^{*\epsilon}}) f(M_s^{L^{*\epsilon}})\, dM_s^{L^{*\epsilon},c}\right] + U(M_0^{L^{*\epsilon}})(F(M_{0-}^{L^{*\epsilon}}) - F(M_0^{L^{*\epsilon}}))\\
&\quad + \mathbb{E}_{x,m}\left[\int_{[\tau_{\overline{m}}^\epsilon,\tau_0]} e^{-rs} F(M_{s-}^{L^{*\epsilon}})\, dL_s^{*\epsilon} - \int_{[\tau_{\overline{m}}^\epsilon,\tau_0]} e^{-rs} U(M_s^{L^{*\epsilon}}) f(M_s^{L^{*\epsilon}})\, dM_s^{L^{*\epsilon},c}\right].
\end{aligned}$$

Recalling (59), using Itô's formula and arguing exactly as in the proof of Proposition 6, we can prove that on $\{\tau_{\overline{m}}^\epsilon < \infty\}$, we have almost surely

$$e^{-r\tau_{\overline{m}}^\epsilon} W(\overline{m}, \overline{m}) = \mathbb{E}_{x,m}\left[\int_{[\tau_{\overline{m}}^\epsilon,\tau_0]} e^{-rs} F(M_{s-}^{L^{*\epsilon}})\, dL_s^{*\epsilon} - \int_{[\tau_{\overline{m}}^\epsilon,\tau_0]} e^{-rs} U(M_s^{L^{*\epsilon}}) f(M_s^{L^{*\epsilon}})\, dM_s^{L^{*\epsilon},c} \middle| \mathcal{F}_{\tau_{\overline{m}}^\epsilon}\right].$$

Using the same arguments as in the proof of Proposition 1, we obtain

$$\mathbb{E}_{x,m}\left[-\int_{[\tau^{\epsilon}_{\overline{m}},\tau_0]} e^{-rs}U(M_s^{L^{*\epsilon}})f(M_s^{L^{*\epsilon}})\,dM_s^{L^{*\epsilon},c}\right] = \mathbb{E}_{x,m}\left[\int_{\overline{m}}^{m\wedge u^*} e^{-r\tau^{\epsilon}_y}U(y)f(y)dy\right]$$

Applying a standard change of variables formula for Stieljes integrals, we have almost surely

$$\begin{aligned}\int_{[0,\tau^{\epsilon}_{\overline{m}}]} e^{-rs}F(M_{s-}^{L^{*\epsilon}})\,dL_s^{*\epsilon} &= (x-m\wedge u^*)F(m) + \int_{[0,\tau^{\epsilon}_{\overline{m}}]} e^{-rs}F(M_{s-}^{L^{*\epsilon}})\,dL_s^{*\epsilon,c}\\ &= (x-m\wedge u^*)F(m) + \int_0^{\infty} e^{-ry\epsilon}F(M_{y\epsilon}^{L^{*\epsilon}})\mathbb{1}_{\{y\epsilon\leq\tau^{\epsilon}_{\overline{m}}\}}\,dy. \qquad (60)\end{aligned}$$

This follows from example from Proposition 4.9 (chapter 0) in [31]. Indeed, we can write

$$\int_{[0,\tau^{\epsilon}_{\overline{m}}]} e^{-rs}F(M_{s-}^{L^{*\epsilon}})\,dL_s^{*\epsilon,c} = \int_{[0,\infty)} g(s)dL_s^{*\epsilon,c},$$

with $g(s)\equiv e^{-rs}F(M_{s-}^{L^{*\epsilon}})\mathbb{1}_{\{s\leq\tau^{\epsilon}_{\overline{m}}\}}$ and using (58), $C^{\epsilon}_y \equiv \inf\{s\,:\,L_t^{*\epsilon,c}>y\} = y\epsilon$ for all $y\in(0,\epsilon\tau^{\epsilon}_{\overline{m}}]$.

Gathering the previous equalities, we obtain

$$\begin{aligned}V(x,m;L^{*\epsilon}) &= (x-m\wedge u^*)F(m) + U(m\wedge u^*)(F(m)-F(m\wedge u*)) \qquad (61)\\ &+ \mathbb{E}_{x,m}\left[\int_0^{\infty} e^{-ry\epsilon}F(M_{y\epsilon}^{L^{*\epsilon}})\mathbb{1}_{\{y\epsilon\leq\tau^{\epsilon}_{\overline{m}}\}}\,dy + \int_{\overline{m}}^{m} e^{-r\tau^{\epsilon}_y}U(y)f(y)\,dy + e^{-r\tau^{\epsilon}_{\overline{m}}}W(\overline{m},\overline{m})\right].\end{aligned}$$

To conclude the proof, we need the following lemma.

Lemma 3. *$\forall\,\overline{m}\leq y<m\wedge u^*$, $\tau^{\epsilon}_y \xrightarrow{\mathbb{P}} 0$ as ϵ tends to 0.*

Proof of Lemma 3. Let consider the process X^{ϵ} defined by (57). From the time change formula for Itô integrals[11], the process $\tilde{X}^{\epsilon}$ defined by $\tilde{X}^{\epsilon}_t \equiv X^{\epsilon}_{t\epsilon}$, $t\geq 0$ satisfies the stochastic differential equation,

$$d\tilde{X}^{\epsilon}_t = -dt + \epsilon\mu(\tilde{X}^{\epsilon}_t)dt + \sqrt{\epsilon}\sigma(\tilde{X}^{\epsilon}_t)d\tilde{B}_t,\ \ \tilde{X}^{\epsilon}_0 = m\wedge u^* \in (\overline{m},u^*],$$

where $\tilde{B}$ is an $(\mathcal{F}_{t\epsilon})$-Brownian motion. We show that, for any $T>m\wedge u^*-\overline{m}$,

$$\mathbb{E}\left[\sup_{s\in[0,T]}|\tilde{X}^{\epsilon}_s-(m\wedge u^*-s)|^2\right]\leq \epsilon(K_1\epsilon+K_2), \qquad (62)$$

where K_1 and K_2 are two positive constants. Indeed,

$$\mathbb{E}\left[\sup_{s\in[0,T]}|\tilde{X}^{\epsilon}_s-(m\wedge u^*-s)|^2\right]\leq 2\mathbb{E}\left[\sup_{s\in[0,T]}|\int_0^s \epsilon\mu(\tilde{X}^{\epsilon}_s)\,dt|^2\right] + 2\mathbb{E}\left[\sup_{s\in[0,T]}|\int_0^s \sqrt{\epsilon}\sigma(\tilde{X}^{\epsilon}_s)\,d\tilde{B}_s|^2\right],$$

and,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,T]}|\int_0^s \sqrt{\epsilon}\sigma(\tilde{X}^{\epsilon}_s)\,d\tilde{B}_s|^2\right] &\leq C_1\mathbb{E}\left[\left(\int_0^T \epsilon\sigma^2(\tilde{X}^{\epsilon}_s)\,ds\right)\right]\\ &\leq \epsilon C_1 T\mathbb{E}\left[K\left(1+\sup_{s\in[0,T]}|\tilde{X}^{\epsilon}_s|^2\right)\right]\\ &\leq \epsilon C_1C_2T(1+m^2)e^{C_2T}, \qquad (63)\end{aligned}$$

[11] See, for instance, Oksendal [25], Theorem 8.5.7.

for some positive constants K, C_1, C_2. The first inequality follows from the Burkholder-Davis-Gundy inequality (see [19] page 166). The second inequality holds because σ is Lipschitz over $(0,\infty)$. The last inequality follows from [19], inequality (3.18) page 306. Using the same arguments, there exists positive constants K, C_3 such that

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[0,T]}|\int_0^s \epsilon\mu(\tilde{X}_s^\epsilon)\,dt|^2\right] &\leq \epsilon^2T^2\mathbb{E}\left[K\left(1+\sup_{s\in[0,T]}|\tilde{X}_s^\epsilon|^2\right)\right]\\ &\leq \epsilon^2T^2C_3(1+m^2)e^{C_3T}.\end{aligned} \tag{64}$$

Then, the inequality (62) follows from (63) and (64). We deduce from (62) that, for any $\delta \in (0, T-(m\wedge u^* - \overline{m}))$,

$$\mathbb{P}(T\wedge\tilde{\tau}_y^\epsilon - (m\wedge u^* - y) > \delta) \leq \mathbb{P}(\sup_{s\in(0,T]}|\tilde{X}_s^\epsilon - (m\wedge u^* - s)| > \delta) \leq \frac{\epsilon(K_1\epsilon + K_2)}{\delta^2}, \tag{65}$$

where $\tilde{\tau}_y^\epsilon \equiv \inf\{t \geq 0\,|\,\tilde{X}_t^\epsilon \leq y\}$. The first inequality in (65) holds because the inequality $\sup_{s\in(0,T]}|\tilde{X}_s^\epsilon - (m\wedge u^* - s)| \leq \delta$ implies almost surely that $T\wedge\tilde{\tau}_y^\epsilon \in [m\wedge u^* - y - \delta, m\wedge u^* - y + \delta]$. The second inequality in (65) follows from (62) and the use of the Chebyshev's inequality.

It follows from (65) that $\tilde{\tau}_y^\epsilon \xrightarrow{\mathbb{P}} m\wedge u^* - y$ as ϵ goes to 0, and, in turn, that $\epsilon\tilde{\tau}_y^\epsilon \xrightarrow{\mathbb{P}} 0$ as ϵ goes to 0. The proof of Lemma 3 is complete noting that, $\tilde{\tau}_y^\epsilon = \frac{\tau_y^\epsilon}{\epsilon}$ $\mathbb{P}$-a.s. □

Let us return to the proof of Proposition 7. With the help of Lemma 3 we analyze the convergence of (61) as ϵ goes to zero.

Applying Fubini Theorem and the dominated convergence Theorem, we have

$$\begin{aligned}\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[\int_{\overline{m}}^{m\wedge u^*} e^{-r\tau_y^\epsilon}U(y)f(y)\,dy\right] &= \lim_{\epsilon\longrightarrow 0}\int_{\overline{m}}^{m\wedge u^*}\mathbb{E}_{x,m}\left[e^{-r\tau_y^\epsilon}\right]U(y)f(y)\,dy\\ &= \int_{\overline{m}}^{m\wedge u^*}\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[e^{-r\tau_y^\epsilon}\right]U(y)f(y)\,dy = \int_{\overline{m}}^{m\wedge u^*}U(y)f(y)\,dy.\end{aligned} \tag{66}$$

The last equality holds because the random variable $e^{-r\tau_y^\epsilon}$ is bounded and, from Lemma 3, converges in probability to 1 as ϵ goes to 0 so that $\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}[e^{-r\tau_y^\epsilon}] = 1$. Using the same argument, we have

$$\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[e^{-r\tau_{\overline{m}}^\epsilon}W(\overline{m},\overline{m})\right] = W(\overline{m},\overline{m}). \tag{67}$$

We now prove that

$$\liminf_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[\int_0^\infty e^{-ry\epsilon}F(M_{y\epsilon}^{L^{*\epsilon}})\mathbb{1}_{\{y\epsilon\leq\tau_{\overline{m}}^\epsilon\}}\,dy\right] \geq \int_{\overline{m}}^{m\wedge u^*}F(y)\,dy. \tag{68}$$

From the proof of Lemma 3 we have, $\tilde{\tau}_{\overline{m}}^\epsilon = \frac{\tau_{\overline{m}}^\epsilon}{\epsilon}$ a.s. It follows that

$$\begin{aligned}\mathbb{E}_{x,m}\left[\int_0^\infty e^{-ry\epsilon}F(M_{y\epsilon}^{L^{*\epsilon}})\mathbb{1}_{\{y\epsilon\leq\tau_{\overline{m}}^\epsilon\}}\,dy\right] &= \mathbb{E}_{x,m}\left[\int_0^{\tilde{\tau}_{\overline{m}}^\epsilon} e^{-r\epsilon y}F(M_{y\epsilon}^{L^{*\epsilon}})\,dy\right]\\ &\geq \mathbb{E}_{x,m}\left[\int_0^{\tilde{\tau}_{\overline{m}}^\epsilon\wedge(m\wedge u^*-\overline{m})} e^{-r\epsilon y}F(M_{y\epsilon}^{L^{*\epsilon}})\,dy\right].\end{aligned} \tag{69}$$

The right-hand side of (69) can be decomposed as follows

$$\mathbb{E}_{x,m}\left[\int_0^{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})} e^{-r\epsilon y}F(M^{L^{*\epsilon}}_{y\epsilon})\,dy\right] = \int_0^{m\wedge u^*-\overline{m}} e^{-r\epsilon y}F(m\wedge u^*-y)\,dy$$
$$-\mathbb{E}_{x,m}\left[\int_{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})}^{m\wedge u^*-\overline{m}} F(m\wedge u^*-y)\,dy\right]$$
$$+\,\mathbb{E}_{x,m}\left[\int_0^{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})} e^{-r\epsilon y}(F(M^{L^{*\epsilon}}_{y\epsilon})-F(m\wedge u^*-y))\,dy\right].$$

Let us examine the three terms in the last expression separately. For the first one, we have

$$\lim_{\epsilon\longrightarrow 0}\int_0^{m\wedge u^*-\overline{m}} e^{-r\epsilon y}F(m\wedge u^*-y)\,dy = \int_{\overline{m}}^{m\wedge u^*} F(y)\,dy. \tag{70}$$

For the second one, we have

$$\left|\int_{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})}^{m\wedge u^*-\overline{m}} F(m\wedge u^*-y)\,dy\right| \le \left|(m\wedge u^*-\overline{m}-\tilde{\tau}^\epsilon_{\overline{m}})\mathbb{1}_{\tilde{\tau}^\epsilon_{\overline{m}}<m\wedge u^*-\overline{m}}\right|.$$

From the proof of Lemma 3, the right-hand-side of the latter inequality is a bounded random variable which converges in probability to 0 as ϵ goes to 0. Thus, its expectation tends to 0 as ϵ goes to 0 which implies that,

$$\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[\int_{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})}^{m\wedge u^*-\overline{m}} F(m\wedge u^*-y)\,dy\right] = 0. \tag{71}$$

For the third one, we have for some positive constant C,

$$\begin{aligned}
&\mathbb{E}_{x,m}\left[\left|\int_0^{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})} e^{-r\epsilon y}(F(M^{L^{*\epsilon}}_{y\epsilon})-F(m\wedge u^*-y))\,dy\right|\right]\\
&\le C\mathbb{E}_{x,m}\left[\int_0^{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})} \left|\inf_{t\in[0,y]}\tilde{X}^\epsilon_t-\inf_{t\in[0,y]}(m\wedge u^*-t)\right|dy\right]\\
&\le C(m\wedge u^*-\overline{m})\mathbb{E}_{x,m}\left[\sup_{y\in[0,\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})]}\left|\inf_{t\in[0,y]}\tilde{X}^\epsilon_t-\inf_{t\in[0,y]}(m\wedge u^*-t)\right|\right]\\
&\le C(m\wedge u^*-\overline{m})\mathbb{E}_{x,m}\left[\sup_{y\in[0,T]}\left|\tilde{X}^\epsilon_y-(m\wedge u^*-y)\right|\right]\\
&\le C(m\wedge u^*-\overline{m})\sqrt{\mathbb{E}_{x,m}\left[\sup_{y\in[0,T]}\left|\tilde{X}^\epsilon_y-(m\wedge u^*-y)\right|^2\right]}.
\end{aligned} \tag{72}$$

The first inequality holds because F is Lipschitz and uses that $M^{L^{*\epsilon}}_{y\epsilon} = \inf_{t\le y\epsilon} X^{L^{*\epsilon}}_t = \inf_{t\le y}\tilde{X}^{L^{*\epsilon}}_t$. It then follows from (62) and (72) that

$$\lim_{\epsilon\longrightarrow 0}\mathbb{E}_{x,m}\left[\left|\int_0^{\tilde{\tau}^\epsilon_{\overline{m}}\wedge(m\wedge u^*-\overline{m})} e^{-r\epsilon y}(F(M^{L^{*\epsilon}}_{y\epsilon})-F(m\wedge u^*-y))\,dy\right|\right] = 0. \tag{73}$$

(68) follows therefore from (70), (71), (73). To complete the proof, note that using (45), we have

$$
\begin{aligned}
W(x,m) &= W(\overline{m},\overline{m}) + (x-m)F(m) + \int_{\overline{m}}^{m} (U(s)f(s)+F(s))ds \\
&= W(\overline{m},\overline{m}) + (x-m\wedge u^*)F(m) + U(m\wedge u^*)(F(m)-F(m\wedge u^*)) + \int_{\overline{m}}^{m\wedge u^*} (U(s)f(s)+F(s))ds,
\end{aligned}
$$

where the second equality is obtained in the same way as (49). Using (66), (67) and (68), we have

$$\liminf_{\epsilon\longrightarrow 0} V(x,m,L^{*\epsilon}) \geq W(x,m).$$

From Proposition 4, we have $W(x,m) \geq V(x,m,L^{*\epsilon})$ for all $\epsilon > 0$, which proves that

$$\lim_{\epsilon\longrightarrow 0} V(x,m,L^{*\epsilon}) = W(x,m).$$

□

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5 Appendix

5.1 A class of smooth functions on $\mathcal{J}$

We define $\mathcal{R}(\mathcal{J})$ as the set of functions $w \in C^0(\mathcal{J})$ such that there exists a finite sequence $0 < m_1 < .... < m_k$ with $k \geq 1$ and for all $i = 1, ..., k+1$, $w \in C^{2,1}(A_i)$ where

$$A_1 = \{(x,m) \in \mathcal{J} | 0 < m \leq m_1\},\ A_{k+1} = \{(x,m) \in \mathcal{J} | m_k \leq m\},$$

$$\text{and for } i = 2, ..., k \quad A_i = \{(x,m) \in \mathcal{J} | m_{i-1} \leq m \leq m_i\}$$

and where, for an arbitrary set $A \subset (0,\infty)^2$, $w \in C^{2,1}(A)$ if there exists a $C^{2,1}$ function defined on a neighbourhood of A in $(0,\infty)^2$, which coincides with w on A. For every $w \in \mathcal{R}(\mathcal{J})$, the partial derivatives w_m, w_x, w_{xx} are well-defined for $(x,m) \in \mathcal{J} \cap \{m > 0\}$ except maybe for $m = m_i$, $i = 1, ..., k$. However, on $\mathcal{J} \cap \{m > 0\}$, the limits $w_m(x, m-), w_x(x, m-), w_{xx}(x, m-)$ are always well-defined, and we use the convention below that the quantities w_m, w_x, w_{xx} correspond to these limits whenever $m = m_i$ for some $i = 1, ..., k$. Consequently, we can justify the application of the Itô formula for functions of class $\mathcal{R}(\mathcal{J})$ by applying the Itô-Meyer formula on each of the A_i as follows. Let σ_i denote the first exit time of A_i by the process (X^L, M^L) for $i = 1, , ..., k+1$ for some $L \in \mathcal{A}(x)$. Note that $\sigma_1 = \tau_0$ and that $\sigma_i \leq \sigma_{i+1}$ because the process M^L is non-increasing. Because $w \in \mathcal{R}(\mathcal{J})$, we can apply Itô-Meyer's formula for semimartingales to the process $e^{-r(t\wedge\tau_0\wedge T_n)} w(X^L_{t\wedge\tau_0\wedge T_n}, M^L_{t\wedge\tau_0\wedge T_n})$ where (15) defines the stopping time T_n. Indeed, if $(X^L_0, M^L_0) \in A_{i_0}$, then the formula applies on $[0, t\wedge\sigma_{i_0}\wedge T_n)$ and extends to $t\wedge\sigma_{i_0}\wedge T_n$ by adding a potential jump at $t\wedge\sigma_{i_0}\wedge T_n$. As the process M^L is non-increasing, (X^L, M^L) will exit only finitely many sets $(A_j)_{j=1,...,i_0}$ before $\sigma_1 = \tau_0$, and the general formula is obtained by concatenation.

5.2 Proofs

Proof of Proposition 2. We borrow several arguments from the proof of Theorem 4.3 in [33].

If $m \geq \overline{x}$, Lemma 2 implies that $\mu(m) \leq rU(m)$. Letting $W^m(x) = x - m + U(m)$, we have $W^m(m) = m$, $(\mathcal{L} - r)W^m \leq 0$, $(W^m)' = 1$ and $(W^m)'' = 0$ on $[m, \infty)$, where the inequality follows from the fact that $\mu'(x) < r$ (Assumption 3). A standard verification argument (see Lemma 3.1 in [33]) implies that $W^m \geq V^m$. As W^m is the payoff associated to any strategy such that $L_0 = x - m$, we conclude that $V^m = W^m$ which proves Proposition 2 (i).

If $m < \overline{x}$, then we have by Lemma 2 $\mu(m) > rU(m)$.

The proof of $(ii)-(iii)$ in Proposition 2 is divided in 5 steps.

Step 1: *Let $m \in [0,\overline{x})$, let us prove that: if there exists $\eta(m) > m$ such that $N(\eta(m), m) = 0$, then V^m is given by formula* (20) *and the process L^m defined by* (22) *is optimal.*

Let W^m be given by (20). By construction, we have $W^m(m) = U(m)$, $(\mathcal{L} - r)W^m = 0$ on $[0, \eta(m))$, and $W^m_x = 1$ and $W^m_{xx} = 0$ on $(\eta(m), \infty)$. The constants A^m, B^m are constructed so that $W^m_x(\eta(m)^-) = 1$ so that W^m is C^1. Notice that ϕ and ψ satisfy

$$u''(x) = \frac{2}{\sigma^2(x)}(ru(x) - \mu(x)u'(x)) \text{ on } [0,\infty). \tag{74}$$

Using (74), a direct computation shows that $N(\eta(m), m) = 0$ implies that

$$A^m\psi''(\eta(m)) + B^m\phi''(\eta(m)) = 0,$$

so that W^m is C^2 on $[m,\infty)$ with $W^m_{xx}(\eta(m)) = 0$. Finally, the properties of the solutions of the differential equation $\mathcal{L}h - rh = 0$ when the Assumption 3 is satisfied, developed in Lemma 4.2 in [33], imply that W^m is increasing and concave on $[m, \eta(m)]$.

We conclude by verification (see Lemma 3.1 of [33]) that $W^m \geq V^m$. Existence of the process L^m solution of (22) is well-known and follows for example as a special case of Lemma 7 below. We verify that W^m coincides with the payoff associated to L^m by applying Itô formula (see e.g. the discussion before Theorem 4.3 in [33]) which concludes the proof that $V^m = W^m$.

The next three steps prove the existence and uniqueness of $\eta(m)$ and the regularity properties of the map η.

Step 2: *Let us prove that: If $m \in [0,\overline{x}]$, $N(x,m) = 0$ admits at most one solution $x \in [0,\infty]$ denoted $\eta(m)$.*

Note that $N(m,m) = (\frac{\mu(m)}{r} - U(m))D(m,m) > 0$ for $m \in [0,\overline{x})$ and that $N(\overline{x},\overline{x}) = 0$.

Assume that there exists $\eta(m)$ such that $N(\eta(m), m) = 0$. To prove uniqueness, note that

$$\frac{\partial N}{\partial x}(x,m) = (\frac{\mu'(x)}{r} - 1)D(x,m) + \frac{\mu(x)}{r}(\phi(m)\psi''(x) - \psi(m)\phi''(x)) - U(m)(\phi(x)\psi''(x) - \psi(x)\phi''(x)).$$

Using that $N(\eta(m), m) = 0$ together with (74), we find that

$$\frac{\partial N}{\partial x}(\eta(m), m) = (\frac{\mu'(\eta(m))}{r} - 1)D(\eta(m), m) < 0, \tag{75}$$

which proves that there exists at most one solution $\eta(m)$ and that whenever it exists, for all $x \in [0,\infty)$, $x < \eta(m) \Rightarrow N(x,m) > 0$ and $x > \eta(m) \Rightarrow N(x,m) < 0$.

step 3: *If $m_1 < m_2 \leq \overline{x}$ are such that $\eta(m_2)$ exists and $x \geq m_2$ then, $V^{m_1}(x) > V^{m_2}(x)$.*

Let us define a control $\tilde{L}$ as follows $\tilde{L} = L^{m_2}$ on $[0, \tau_{m_2})$ where $\tau_{m_2} = \inf\{t \geq 0,\ X^{L_{m_2}}_t \leq m_2\}$, and $\tilde{L} = L^*$ on $[\tau_{m_2}, \infty)$ where L^* reflects the process X at $\overline{x}$ starting at time τ_{m_2}, that is we consider the unique solution for $t \geq \tau_{m_2}$ of (existence follows for example from Lemma 7)

$$dX_t^{L^*} = \mu(X_t^{L^*})dt + \sigma(X_t^{L^*})dB_t - dL_t^*, \; X_{\tau_{m_2}-}^{L^*} = m_2,$$

$$L_t^* = L_{\tau_{m_2}}^{m_2} + \sup_{\tau_{m_2} \leq s \leq t} (X_s^{L^*} - \overline{x})^+.$$

Note that $\tilde{L}$ is continuous except maybe at time 0. Letting $\tau_{m_1} = \inf\{t \geq 0, X_t^{\tilde{L}} \leq m_1\}$, we have that

$$\begin{aligned} V^{m_1}(x) &\geq \mathbb{E}_x\left[\int_{[0,\tau_{m_1}]} e^{-rs}\, d\tilde{L}_s \,+\, e^{-r\tau_{m_1}} U(m_1)\right] \\ &= \mathbb{E}_x\left[\int_{[0,\tau_{m_2}]} e^{-rs}\, dL_s^{m_2} \,+ \int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}\, dL_s^* \,+\, e^{-r\tau_{m_1}} U(m_1)\right] \\ &> \mathbb{E}_x\left[\int_{[0,\tau_{m_2}]} e^{-rs}\, dL_s^{m_2} \,+\, e^{-r\tau_{m_2}} U(m_2)\right] = V^{m_2}(x), \end{aligned}$$

where the first inequality holds because $\tilde{L}$ is an admissible process for problem (16) when $m = m_1$, and the equality holds because $\tau_{m_2} < \tau_{m_1}$ $a.s$ on $\{\tau_{m_2} < \infty\}$. The last inequality uses the inequality (17). Indeed, notice that $X^{\tilde{L}} \in [0, \overline{x}]$ for $t \in [\tau_{m_2}, \tau_{m_1}] \cap [0, \infty)$ and is continuous on that time interval, and that $X_{\tau_{m_i}}^{\tilde{L}} = m_i$ on $\{\tau_{m_i} < \infty\}$ for $i = 1, 2$. Therefore we have

$$\begin{aligned} \mathbb{E}_x\left[\int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}\, dL_s^* + e^{-r\tau_{m_1}} U(m_1)\right] &= \mathbb{E}_x\left[\int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}\, dL_s^* + e^{-r\tau_{m_1}} U(X_{\tau_{m_1}}^{\tilde{L}})\right] \\ &= \mathbb{E}_x\left[e^{-r\tau_{m_2}} U(X_{\tau_{m_2}}^{\tilde{L}})\right] \,+\, \mathbb{E}_x\left[\int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}(\mathcal{L}U - rU)(X_s^{\tilde{L}})\, ds\right] \\ &\quad -\mathbb{E}_x\left[\int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}(1 - U'(X_s^{\tilde{L}}))\, dL_s^*\right] \\ &= \mathbb{E}_x\left[e^{-r\tau_{m_2}} U(m_2)\right] \,+\, \mathbb{E}_x\left[\int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}(\mathcal{L}U - rU)(X_s^{\tilde{L}})\, ds\right] \\ &\quad -\mathbb{E}_x\left[(1 - U'(\overline{x})) \int_{(\tau_{m_2},\tau_{m_1}]} e^{-rs}\, dL_s^*\right] \\ &> \mathbb{E}_x\left[e^{-r\tau_{m_2}} U(m_2)\right], \end{aligned}$$

where the second equality follows from Itô's formula, the third equality from the definition of L^*. The inequality follows from the fact that $\overline{x} \geq u^*$ so that $U'(\overline{x}) = 1$, and from the fact that $\mathbb{P}_x(\tau_{m_2} < \tau_{m_1}) > 0$ together with inequality (17).

The next step concludes the proofs of Proposition 2 (ii)-(iii).

step 4: *The map η is well-defined, increasing and C^1 on $[0, \overline{x})$. It satisfies $\eta(m) > m$ for all $m \in [0, \overline{x})$ and $\lim_{m \to \overline{x}} \eta(m) = \overline{x}$.*

As $N(\overline{x}, \overline{x}) = 0$, we have $\eta(\overline{x}) = \overline{x}$, and using (75), $N(x, \overline{x}) > 0$ for all $x \in (\overline{x}, \infty)$.

By continuity of N, the set

$$H := \{m \in [0, \overline{x}] \,|\, \exists x > m, N(x,m) < 0\},$$

is open in $[0, \overline{x}]$ and contains $\overline{x}$. Let $\delta := \inf\{y \in [0, \overline{x}) \,|\, [y, \overline{x}] \subset H\}$ so that $\delta < \overline{x}$ and there exists a unique solution $\eta(m)$ for all $m \in (\delta, \overline{x}]$. The map N being C^1 on $[0,\infty)^2$, we deduce from (75) together with the implicit function theorem that η is C^1 on $(\delta, \overline{x}]$.

Let us define the function $h(x,m) = V^m(x) - \frac{\mu(x)}{r}$ for m$\in (\delta, \overline{x})$ and $x \geq m$. By step 1, (20) holds for $m \in (\delta, \overline{x}]$, and thus $h(\eta(m), m) = 0$. As η is C^1, we deduce from (20) that h is C^1. Differentiating the previous equality, we obtain

$$\eta'(m) h_x(\eta(m), m) + h_m(\eta(m), m) = 0. \tag{76}$$

Observe that

$$h_x(x,m) = V_x^m(x) - \frac{\mu'(x)}{r} \geq 1 - \frac{\mu'(x)}{r} > 0.$$

On the other hand, the mapping $m \longrightarrow V^m(x)$ is decreasing on $(\delta, \overline{x})$ from step 3, so that $h_m(\eta(m), m) \leq 0$ and η is non-decreasing on $(\delta, \overline{x})$. Moreover, η must be increasing on $(\delta, \overline{x})$. Otherwise, if $\eta(m) = x$ for all m on some non-degenerate interval, it would imply that $m \to V^m(x)$ is also constant on that interval, a contradiction.

Define $\underline{\eta} := \lim_{m \to \delta} \eta(m)$, which exists by monotonicity. By continuity of N we have $N(\underline{\eta}, \delta) = 0$. Together with (75), we deduce that $\delta \in H$, which implies that $\delta = 0$ as H is open. From (76), we deduce that η' remains bounded at 0, so that η is C^1 and increasing on $[0, \overline{x})$. That $\eta(m) > m$ follows from the fact that $N(m,m) > 0$ for $m \in [0, \overline{x})$. Using continuity of N, we have $N(\lim_{m \to \overline{x}} \eta(m), \overline{x}) = 0$, and thus $\lim_{m \to \overline{x}} \eta(m) = \overline{x}$. □

Lemma 4. *Let consider the map η defined in Proposition 2 and the maps G, N, D defined in* (38). *The following holds.*

(i) The map $m \longrightarrow \frac{N(m,m)}{G(m)D(m,m)}$ is decreasing over $(0, \overline{x}]$,

(ii) For any $m \in (0, \overline{x}]$, the mapping $x \longrightarrow \frac{N(x,m)}{G(x)D(x,m)}$ is decreasing over $[m, \eta(m)]$.

Proof. Let us prove (i). Note that

$$\frac{N(m,m)}{G(m)D(m,m)} = \frac{\frac{\mu(m)}{r} - U(m)}{G(m)}.$$

The derivative with respect to m has therefore the same sign as

$$G(m)(\frac{\mu'(m)}{r} - U'(m)) - G'(m)(\frac{\mu(m)}{r} - U(m)),$$

which is negative on $(0, \overline{x}]$ because $G'(m) = -\frac{\mu''(m)}{r} \geq 0$ by Assumption 5, $\frac{\mu(m)}{r} - U(m) \geq 0$ on $[0, \overline{x}]$ by Lemma 2, and $\mu'(m) < r$ and $U'(m) \geq 1$ by Assumptions 3 and 4.

We now prove (ii). The derivative with respect to $x \longrightarrow \frac{N(x,m)}{G(x)D(x,m)}$ has the same sign as

$$G(x)D(x,m)N_x(x,m) - N(x,m)(G'(x)D(x,m) + G(x)D_x(x,m)).$$

We deduce from Proposition 2 and Assumption 5 that $-N(x,m)G'(m)D(x,m) \leq 0$. As $G(x) > 0$, it is therefore sufficient to prove that

$$D(x,m)N_x(x,m) - N(x,m)D_x(x,m) < 0. \tag{77}$$

Letting $D(x) = D(x,x)$, we have

$$N(x,m) = \phi(x)\psi(m) - \psi(x)\phi(m) + \frac{\mu(x)}{r}D(x,m) - D(x,x)U(m),$$

$$N_x(x,m) = (\frac{\mu'(x)}{r} - 1)D(x,m) + \frac{\mu(x)}{r}D_x(x,m) - D'(x)U(m),$$

$$D'(x) = -\frac{2}{\sigma^2(x)}\mu(x)D(x) \text{ and } D_x(x,m) = \frac{2r}{\sigma^2(x)}(\psi(x)\phi(m) - \phi(x)\psi(m) - \frac{\mu(x)}{r}D(x,m)).$$

Substituting these equalities in (77), we obtain after simplifying and rearranging terms

$$D(x,m)N_x(x,m) - N(x,m)D_x(x,m) = D^2(x,m)(\frac{\mu'(x)}{r} - 1) - \frac{2r}{\sigma^2(x)}(\psi(x)\phi(m) - \phi(x)\psi(m))N(x,m)$$

which is negative because $D(x,m) > 0$, $\mu'(x) < r$, $N(x,m) > 0$ and $\psi(x)\phi(m) - \phi(x)\psi(m) \geq 0$ as ψ is increasing, ϕ is decreasing and $x \geq m$. □

The next lemma shows that there are instances of the model satisfying our assumptions such that $u^* > \overline{m}$. Let us first give an example where $x^0 < u^*$. Assume that X is an arithmetic Brownian motion with drift $\mu > 0$ and volatility $\sigma > 0$, and that U is, as in Example 1, the value function of a downgraded extraction problem with a smaller drift $\underline{\mu} \in (0, \mu)$ and the same volatility σ. Precisely, using similar notation as for our main problem, for $z \geq 0$ define the controlled process Z^L as the unique strong solution of

$$dZ_t^L = \underline{\mu}dt + \sigma dB_t - dL_t, \ Z_{0-} = z$$

and $U(z) = \sup_{L \in \mathcal{A}'(0)} \mathbb{E}_z[\int_{[0,\tau_0']} e^{-rt} dL_t]$ with $\mathcal{A}'(0) = \{L \in \mathbb{L}, (Z_t^L)^+ - (\Delta L)_t \geq 0 \text{ for all } t\}$ and $\tau_0' = \inf\{t \geq 0, Z_t = 0\}$. Then, as explained in Example 1, U satisfies Assumption 4 and Assumption 3 as $rU(0) = 0 < \mu$. It is known (see e.g. chapter 2.5.2 in [32]) that

$$u^* = \frac{2}{\underline{\beta}^+ - \underline{\beta}^-} \log\left(\frac{-\underline{\beta}^-}{\underline{\beta}^+}\right),$$

where $\underline{\beta}^- < 0 < \underline{\beta}^+$ are the two roots of the quadratic equation $\frac{\sigma^2}{2}\beta^2 + \underline{\mu}\beta - r = 0$. Because $U(0) = 0$, the function V^0 is solution of a similar control problem as U with a larger drift μ and therefore the threshold $x^0 = \eta(0)$ is given by the same formula

$$x^0 = \frac{2}{\beta^+ - \beta^-} \log\left(\frac{-\beta^-}{\beta^+}\right),$$

where $\beta^- < 0 < \beta^+$ are the two roots of the quadratic equation $\frac{\sigma^2}{2}\beta^2 + \mu\beta - r = 0$. In particular, the parameters σ, r being fixed, one can prove that by choosing $\underline{\mu} < \mu$ sufficiently large, we have $u^* > x^0$. Finally, the map η being continuous and increasing on $[0, \overline{x}]$, we can choose $c > 0$ such that $x^0 < \eta(c) < u^*$.

Lemma 5. *Assume that $0 < x^0 < u^*$ and that $c > 0$ is such that $\eta(c) < u^*$. For $\epsilon \in (0, 2(c \wedge (\overline{x} - c))$, define the density function f_ϵ the over $[0, \overline{x}]$ by*

$$\begin{cases} f_\epsilon(x) = d_\epsilon, \ \ \forall x \in [0, c-\epsilon], \\ f_\epsilon(x) = (1 - \frac{d_\epsilon}{\epsilon})x + c(\frac{d_\epsilon}{\epsilon} - 1) + \epsilon, \ \ \forall x \in [c-\epsilon, c], \\ f_\epsilon(x) = \epsilon, \ \ \forall x \in [c, \overline{x}], \end{cases} \tag{78}$$

with $d_\epsilon = \frac{1 - \epsilon(\overline{x} - c - \frac{\epsilon}{2})}{c - \frac{\epsilon}{2}}$. Assume that the density f_ϵ defines the law of Y and let denote b_ϵ the solution to the Cauchy problem (37)*. Then, there exists $\epsilon > 0$ such that $\overline{m} < u^*$.*

Proof of Lemma 5. Recall that $u^* \le \overline{x}$ and that $\eta(u^*) \ge u^*$, so that $\eta(c) < u^* \le \eta(u^*)$ implies $c < u^*$ as η is increasing. Because the mapping $(x, m) \longrightarrow E_\epsilon(x, m)$ with $E_\epsilon(x, m) = \frac{f_\epsilon(m)}{F_\epsilon(m)} \frac{N(x,m)}{G(x) D(x,m)}$ is continuous over $\mathcal{K} = \{(x, m) : m \le x \le \overline{x}; \, c \le m \le \overline{x}\}$, and $\frac{f_\epsilon(m)}{F_\epsilon(m)} = \frac{\epsilon}{1 - \epsilon(\overline{x} - m)}$ over $[c, \overline{x}]$, we can chose ϵ sufficiently small so that

$$M_\epsilon = \left(\sup_{(x,m) \in \mathcal{K}} E_\epsilon(x, m) \right) < \frac{u^* - \eta(c)}{u^* - c}. \tag{79}$$

From Proposition 3, we have $b_\epsilon(c) < \eta(c)$ and, by assumption, $\eta(c) < u^*$. Because b_ϵ is solution to the Cauchy problem (37) and $\overline{y} = \overline{x}$, we have

$$b_\epsilon(u^*) = b_\epsilon(c) + \int_c^{u^*} E_\epsilon(b_\epsilon(s), s) ds \le \eta(c) + M_\epsilon(u^* - c) < u^*.$$

Recalling that $\overline{m} = \inf\{m \in [0, \overline{x}], b_\epsilon(m) = m\}$, because $b_\epsilon(0) = x^0 > 0$ and b_ϵ is continuous, we conclude from the intermediate value theorem that $\overline{m} < u^*$. □

Lemma 6. *$W \in \mathcal{R}(\mathcal{J})$.*

Proof of Lemma 6. We recall the definition of W given by the equations (43)-(45). It can be easily verified that $W \in \mathcal{C}^0(\mathcal{J})$. Let us consider that $\overline{m} < \overline{y}$, the other cases ($\overline{m} = \overline{y}$ and $\overline{m} > \overline{y}$) can be handled similarly without difficulty. We will show in turn that $W \in C^{2,1}(A_3)$, $W \in C^{2,1}(A_2)$ and $W \in C^{2,1}(A_1)$ where $A_3 = \{(x, m) \in \mathcal{J} \,|\, \overline{y} \le m\}$, $A_2 = \{(x, m) \in \mathcal{J} \,|\, \overline{m} \le m \le \overline{y}\}$ and, $A_1 = \{(x, m) \in \mathcal{J} \,|\, 0 < m \le \overline{m}\}$.

1. For $(x,m) \in A_3$, because $f(s) = 0$ for $s > \overline{y}$ and $F(s) = 1$ for $s \geq \overline{y}$, we have

$$\begin{aligned} W(x,m) &= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m)F(m) + \int_{\overline{m}}^{m} (U(s)f(s) + F(s))ds \\ &= \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m) + \int_{\overline{m}}^{\overline{y}} (U(s)f(s) + F(s))ds + (m - \overline{y}) \end{aligned}$$

The function $(x,m) \to \frac{\mu(\overline{m})}{r} F(\overline{m}) + (x-m) + \int_{\overline{m}}^{\overline{y}} (U(s)f(s) + F(s))ds + (m - \overline{y})$ defined on $(0,\infty)^2$ coincides with W on A_3 and is $C^{2,1}$ on $(0,\infty)^2$, and thus on any neighborhood of A_3 in $(0,\infty)^2$, that is, $W \in C^{2,1}(A_3)$.

2. The set A_2 presents a more delicate situation due to W's insufficient regularity across $\overline{y}$. As for the proof of Proposition 3, let $\hat{f}$ denote a Lipschitz positive map on $[0,\infty)$ which coincides with f on $[0,\overline{y}]$, and let $\hat{F}(m) = \int_0^m \hat{f}(t)dt$. The function $(x,m) \longrightarrow \frac{\mu(\overline{m})}{r} \hat{F}(\overline{m}) + (x-m)\hat{F}(m) + \int_{\overline{m}}^{m} (U(s)\hat{f}(s) + \hat{F}(s))ds$ defined on $(0,\infty)^2$ coincides with W on A_2 and is $C^{2,1}$ on any neighborhood of A_2 in $(0,\infty)^2$, that is, $W \in C^{2,1}(A_2)$.

3. Finally, we address the case of the set A_1. We first prove that W is $\mathcal{C}^{2,1}$ on the open set $\{(x,m) : 0 < m < \overline{m} \text{ and } m < x\}$. As ψ and ϕ are $\mathcal{C}^2$ on $(0,\infty)$, F and b are $\mathcal{C}^1$ on $(0,\overline{m})$, the functions (43)-(44) are $\mathcal{C}^{2,1}$ over the domains $int(\mathcal{J}_1)$, $int(\mathcal{J}_2)$, respectively. The function obtained by pasting together (43)-(44) is $\mathcal{C}^{2,1}$ on $\{(x,m) : 0 < m < \overline{m} \text{ and } m < x\}$. To prove this, we simply need to verify the assertion on a neighborhood of $(m, b(m))$ for any $m \in (0,\overline{m})$. The result follows from two observations. First, by construction,

$$W_x(b(m)^+, m) = F(m) = W_x(b(m)^-, m), \text{ and } W_{xx}(b(m)^+, m) = 0 = W_{xx}(b(m)^-, m). \tag{80}$$

Second, because W is continuous at $(m, b(m))$, for any $m \in (0,\overline{m})$, we have

$$\frac{\mu(b(m))}{r} F(m) = A(m)\psi(b(m)) + B(m)\phi(b(m)).$$

Differentiating this latter expression and using (80) yields

$$\begin{aligned} [\frac{\mu(b(m))}{r} F(m)]' &= (A(m)\psi'(b(m)) + B(m)\phi'(b(m)))b'(m) + A'(m)\psi(b(m)) + B'(m)\phi(b(m)) \\ &= F(m)b'(m) + A'(m)\psi(b(m)) + B'(m)\phi(b(m)). \end{aligned} \tag{81}$$

We then deduce from (81) that

$$W_m(b(m), m^+) = -b'(m)F(m) + [\frac{\mu(b(m))}{r} F(m)]' = W_m(b(m), m^-).$$

Thus, W is $\mathcal{C}^{2,1}$ on the open set $\{(x,m) : 0 < m < \overline{m} \text{ and } x < m\}$. By Proposition 3, the solution b to the Cauchy problem (39) is defined over $[0,\overline{x}]$ and is C^1 on $(0,\overline{y} \wedge \overline{x})$

with $\overline{m} < \overline{y} \wedge \overline{x}$ thanks to our assumption. It follows that the expressions (43)-(44) can be extended to a function which is $C^{2,1}$ on some neighborhood of A_1 and which coincides with W on $\{(x,m) : 0 < m \leq \overline{m} \text{ and } m \leq x\}$, that is, $W \in C^{2,1}(A_1)$.

□

Proof of Proposition 5. We will assume in the following that μ and σ are defined and globally Lipschitz on $\mathbb{R}$. It does not affect the result of Proposition 5 as the solutions are considered only up to the first hitting time of 0. We start by proving the following existence result.

Lemma 7. *Let l a differentiable function defined on $(-\infty, \overline{m}]$ with $|l'(x)| < c < 1$ for any $x \in (-\infty, \overline{m}]$ and $l(\overline{m}) = \overline{m}$. For every initial condition $(x,m) \in \mathbb{R}^2$ with $x \geq m$, there exists a unique strong solution (X, M, L) to the reflected stochastic differential equation*

$$X_{0-} = x,\ M_{0-} = m,\ L_0- = 0 \tag{82}$$

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t - dL_t,\ \ M_t = m \wedge \inf_{s\leq t} X_s,\ \ t \geq 0, \tag{83}$$

$$\forall t \geq 0,\ X_t \in [M_t, l(M_t)] \cap (-\infty, \overline{m}]\ \ a.s\ , \tag{84}$$

$$\forall t \geq 0,\ \int_{[0,t]} \mathbb{1}_{\{X_s < l(M_s^X)\}} dL_s = 0\ \ a.s\ . \tag{85}$$

Proof of Lemma 7. At first, note that l being Lipschitz with constant $c < 1$, we have $l(x) > x$ for all $x \in (-\infty, \overline{m}]$.

Conditions (84) and (85) imply that for any solution

$$L_0 = (x - \overline{m})\mathbb{1}_{m > \overline{m}} + (x - l(m))\mathbb{1}_{\{m \leq \overline{m} \text{ and } x \geq l(m)\}},$$

$$X_0 = \overline{m}\mathbb{1}_{m > \overline{m}} + l(m)\mathbb{1}_{\{m \leq \overline{m} \text{ and } x \geq l(m)\}},\ M_0 = \overline{m}\mathbb{1}_{m > \overline{m}} + m\mathbb{1}_{m \leq \overline{m}},$$

and (X, M, L) are continuous except for a possible deterministic jump at time zero.

Therefore, without loss of generality, we may assume that the initial condition is $(x,m) = (l(m), m)$ for some $m \in (-\infty, \overline{m}]$, as the general solution can be obtained by adding the initial deterministic jump. To simplify notation, we will only write the proof in the case $x = m = \overline{m}$, the other cases being similar.

The proof follows usual arguments. We first show that, for T small enough there exists over $[0,T]$ a unique continuous solution to (83), (84), (85) with $X_0 = M_0 = \overline{m}$.

Let $T > 0$ and let consider

$$\mathcal{E} = \left\{ (X_t)_{0 \leq t \leq T},\ \mathcal{F}_t\text{-adapted and continuous process such that } \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t|^2\right] < +\infty \right\},$$

which, endowed with the norm $||X|| = \sqrt{\mathbb{E}[\sup_{0\le t\le T} |X_t|^2]}$ is a Banach space. We will apply the contraction mapping theorem to the mapping Φ defined on $\mathcal{E}$ by

$$\forall t \in [0,T],\ \Phi(X)_t = \overline{m} + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s - L(X)_t,$$

with

$$L(X)_t = \sup_{0\le s\le t} \left(\overline{m} + \int_0^s \mu(X_u)\,du + \int_0^s \sigma(X_u)\,dB_u - l\left(M_s^X\right)\right)^+ .$$

where $M_s^X = \inf_{u\in[0,s]} X_u$. Note that the process $\Phi(X)$ is well-defined, $\mathcal{F}_t$ adapted and continuous.

Let us show that Φ takes values in $\mathcal{E}$ and is Lipschitz.

Consider X and Y two elements of $\mathcal{E}$. We use that μ and σ are Lipschitz and Doob's inequality to obtain that

$$\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t (\mu(X_s)-\mu(Y_s))\,ds\right|^2\right]} + \sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t (\sigma(X_s)-\sigma(Y_s))\,dB_s\right|^2\right]}$$
$$\le (KT + 2K\sqrt{T})\sqrt{\mathbb{E}\left[\sup_{0\le s\le T}|X_t - Y_t|^2\right]} < \infty, \qquad (86)$$

for some positive constant K. Letting $M_s^X = \inf_{u\in[0,s]} X_u$ and $M_s^Y = \inf_{u\in[0,s]} Y_u$, one can easily check that for $t\in[0,T]$

$$|L(X)_t - L(Y)_t| \le \sup_{s\le t}\Bigg[\left(\overline{m} + \int_0^s \mu(X_u)\,du + \int_0^s \sigma(X_u)\,dB_u - l\left(M_s^X\right)\right)^+$$
$$- \left(\overline{m} + \int_0^s \mu(Y_u)\,du + \int_0^s \sigma(Y_u)\,dB_u - l\left(M_s^Y\right)\right)^+\Bigg],$$

which, because $x\to x^+$ is 1-Lipschitz, leads to

$$|L(X)_t - L(Y)_t| \le \sup_{0\le s\le t}\left(\left|\int_0^s (\mu(X_u)-\mu(Y_u))du\right| + \left|\int_0^s (\sigma(X_u)-\sigma(Y_u))dB_u\right| + \left|l(M_s^X) - l(M_s^Y)\right|\right).$$

We deduce that,

$$\begin{aligned} ||L(X) - L(Y)|| \ \le\ & \sqrt{\mathbb{E}[\sup_{0\le t\le T}|\int_0^t (\mu(X_s)-\mu(Y_s))\,ds|^2]} \\ & + \sqrt{\mathbb{E}[\sup_{0\le t\le T}|\int_0^t (\sigma(X_s)-\sigma(Y_s))\,dB_s|^2]} \\ & + \sqrt{\mathbb{E}[\sup_{0\le t\le T}|l(M_t^X) - l(M_t^y)|^2]}. \end{aligned} \qquad (87)$$

Because l is Lipschitz with constant $c<1$, we have,

$$\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left|l(M_t^X) - l(M_t^Y)\right|^2\right]} \le c\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left|M_t^X - M_t^Y\right|^2\right]} \le c\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}|X_t - Y_t|^2\right]} < \infty. \qquad (88)$$

Using the triangle inequality, it follows from (86)-(88) that,

$$\begin{aligned}
||\Phi(X)-\Phi(Y)|| &\leq \sqrt{\mathbb{E}[\sup_{0\leq t\leq T}|\int_0^t(\mu(X_s)-\mu(Y_s))\,ds|^2]}\\
&\quad+\sqrt{\mathbb{E}[\sup_{0\leq t\leq T}|\int_0^t(\sigma(X_s)-\sigma(Y_s))\,dB_s|^2]}\\
&\quad+||L(X)-L(Y)||.\\
&\leq (2(KT+2K\sqrt{T})+c)\,||X-Y||<\infty. \qquad (89)
\end{aligned}$$

The proof that $||\Phi(X)||<\infty$ for $x\in\mathcal{E}$ follows along the same lines as the proof of (89) and is thus omitted. Therefore Φ takes values in $\mathcal{E}$ and is Lipschitz with coefficient $2(KT+2K\sqrt{T})+c$. For T small enough, we have $2(KT+2K\sqrt{T})+c<1$ and Φ is a contraction.

Thus, for T small enough, by the contraction mapping theorem, we get the existence and the uniqueness in $\mathcal{E}$ of a fixed point X of Φ. From the definition of Φ, the triple $(X, L(X), M^X)$ satisfies all the conditions of the system (83), (84), (85) on $[0,T]$ with initial condition $X_0=M_0=\overline{m}$.

Finally, we show that any strong solution to (83), (84), (85) on $[0,T]$ is necessarily in $\mathcal{E}$. The proof again follows usual arguments. We consider $f^n(t)=\mathbb{E}[\sup_{0\leq s\leq t\wedge T_n}|X_s|^2]$ with $T_n=\inf\{u\geq 0,\,|X_u|>n\}$ and, proceeding analogously as above, we find two constants, c_1 and c_2, independent of n such that for $t\leq T$, we have $f^n(t)\leq c_1+c_2\int_0^t f^n(s)\,ds$. Using the Gronwall Lemma, we get that $\mathbb{E}[\sup_{0\leq t\leq T\wedge T_n}|X_s|^2]\leq C$, for some constant C independent of n. Letting n tend to ∞, we conclude that X lies in $\mathcal{E}$.

To conclude for any $T>0$, we consider a subdivision of $[0,T]$ in n subintervals where n is large enough to ensure the existence and uniqueness of the solution to (83), (84), (85) on the intervals $[kT/n,(k+1)T/n]$. This requires to extend the previous proof for a random initial condition, which can be done directly by adapting the notation. □

We now turn to the proof of Proposition 5. We only consider the case $\tau=0$ as the proof, including Lemma 7 can be easily extended to an arbitrary stopping time τ. The function b is defined on $[0,\overline{x}]$ with $b(0)=x^0$. We extend b to the set of nonpositive real numbers by defining $b(m)=x^0-m$ for $m<0$ to maintain the ratio $\frac{b(m)+m}{2}$ constant for $m\leq 0$. According to Proposition 3, the function b is continuously differentiable on $(0,\overline{m}]$ with $b'(\overline{m})<1$. Therefore, there is some $m_0\in(0,\overline{m}]$ such that $b'(m)<1$ for $m\in[m_0,\overline{m}]$. Moreover, we observe that the function $m\mapsto b(m)-m$ is uniformly bounded below by some constant $\delta>0$ on the interval $(-\infty,m_0]$. Let $(x,m)\in\mathcal{J}$. If $m\geq m_0$, using Lemma 7 applied to some function l which coincides with b on $[m_0, \overline{m}]$ we consider a triple $(\overline{X}_t, \overline{M}_t, \overline{L}_t)$ such that

$$\overline{X}_{0-} = x,\ \overline{M}_{0-} = m \tag{90}$$

$$d\overline{X}_t = \mu(\overline{X}_t)\, dt + \sigma(\overline{X}_t)\, dB_t - d\overline{L}_t,\ \ \overline{M}_t = m \wedge \inf_{s\le t} \overline{X}_s,\ 0 \le t \le \gamma_0, \tag{91}$$

$$\overline{X}_t \in [\overline{M}_t, b(\overline{M}_t)] \cap [m_0, \overline{m}] \text{ a.s },\ \ 0 \le t \le \gamma_0, \tag{92}$$

$$\int_{[0,t]} \mathbb{1}_{\{\overline{X}_s < b(\overline{M}_s)\}} d\overline{L}_s = 0 \text{ a.s for any } 0 \le t \le \gamma_0,\ \text{where} \tag{93}$$

$$\gamma_0 = \inf\{t \ge 0,\ \overline{X}_t \le m_0\}. \tag{94}$$

If $m < m_0$, we simply define $\gamma_0 = 0$, $\overline{X}_0 = x \wedge b(m)$, $\overline{M}_0 = m$, and $\overline{L}_0 = (x - x \wedge b(m))$. We set $b_0 = b(m_0)$ and we consider the sequence of pair $(X_t^{(k)}, L_t^{(k)})$ defined by $X_t^{(0)} = \overline{X}_t$, $L_t^{(0)} = \overline{L}_t$ on the random interval $[0, \gamma_0]$ and,

$$\begin{aligned} X_t^{(k)} &= X_{\gamma_k}^{(k-1)} + \int_{\gamma_k}^t \mu(X_s^{(k)})\, ds\, +\, \int_{\gamma_k}^t \sigma(X_s^{(k)})\, dB_s\, -\, L_t^{(k)}, \\ L_t^{(k)} &= \sup_{\gamma_k \le s \le t} \left(X_{\gamma_k}^{(k-1)} + \int_{\gamma_k}^s \mu(X_u^{(k)})\, du\, +\, \int_{\gamma_k}^s \sigma(X_u^{(k)})\, dB_u - b_k \right)^+, \end{aligned}$$

on the random interval $[\gamma_k, \gamma_{k+1}]$ where

$$\gamma_{k+1} = \inf\left\{ t \ge \gamma_k,\ M_t^{(k)} < m_k \text{ and } X_t^{(k)} = \frac{b(M_t^{(k)}) + M_t^{(k)}}{2} \right\},$$

with $m_k = M_{\gamma_k}^{(k-1)}$, $b_k = b(m_k)$, $M_t^{(k)} = m_k \wedge \inf_{s\le t} X_s^{(k)}$. Thus, by definition, $(X_t^{(k)})_{t\ge 0}$ is the solution to a reflected SDE at the fixed level b_k. For $k = 0$, we set $(X_t, L_t) = (X_t^{(0)}, L_t^{(0)})$ on $[0, \gamma_0]$ and, for each $k \ge 1$, $(X_t, L_t) = (X_t^{(k)}, L_{\gamma_k} + L_t^{(k)})$ on the interval $(\gamma_k, \gamma_{k+1}]$. By construction the pair (X_t, L_t) is defined on $[0, \lim_{k\longrightarrow\infty} \gamma_k)$ and satisfies (52), (53), (54). To conclude the proof it remains to show that the sequence $(\gamma_k)_{k\ge 0}$ diverges to $+\infty$. We consider the sequence of random variables

$$Z_k = \frac{b_k + m_k}{2} \mathbb{1}_{\gamma_k < \infty}.$$

Note that $X_{\gamma_k} = Z_k$ on $\{\gamma_k < \infty\}$. The sequence $(Z_k)_{k\ge 0}$ is non-increasing and bounded below by 0, and we have $Z_k \ge \frac{x^0}{2}$ on $\{\gamma_k < \infty\}$ by construction of the extension of the function b. Therefore, the sequence $(Z_k)_{k\ge 0}$ almost surely converges to a non-negative random variable Z which satisfies $Z \ge \frac{x^0}{2} > \frac{\delta}{2}$ on $\{\lim_{k\to\infty} \gamma_k < \infty\}$. For a positive constant c, we have

$$\mathbb{P}\left[\gamma_{k+1} - \gamma_k > c | \mathcal{F}_{\gamma_k}\right] \mathbb{1}_{\gamma_k < \infty} \ge \mathbb{P}[T_k(\delta) > c | \mathcal{F}_{\gamma_k}] \mathbb{1}_{\gamma_k < \infty},$$

where

$$T_k(\delta) = \inf\left\{ t \ge 0, X_{\gamma_k + t} \notin \left[Z_k - \frac{\delta}{4}, Z_k + \frac{\delta}{4} \right] \right\} \mathbb{1}_{\gamma_k < \infty}.$$

By construction, on $\{\gamma_k < \infty\}$, the process X is a solution on the random time-interval $[\gamma_k, T_k(\delta)]$ of the uncontrolled SDE

$$dX_t = \mu(X_t) dt + \sigma(X_t) dB_t. \tag{95}$$

Therefore, we have using the strong Markov property

$$\mathbb{P}[T_k(\delta) > c|\mathcal{F}_{\gamma_k}]\mathbb{1}_{\gamma_k<\infty} = \widehat{\mathbb{P}}_{Z_k}[T(\delta) > c]\mathbb{1}_{\gamma_k<\infty}$$

where $\widehat{\mathbb{P}}_x$ denotes the distribution of the unique solution X of (95) with initial condition x at time 0 and $T(\delta) = \inf\{t \geq 0, X_t \notin [X_0 - \frac{\delta}{4}, X_0 + \frac{\delta}{4}]\}$. We have

$$\begin{aligned}\widehat{\mathbb{P}}_{Z_k}[T(\delta) > c]\mathbb{1}_{\gamma_k<\infty} &= (1 - \widehat{\mathbb{P}}_{Z_k}[T(\delta) \leq c])\mathbb{1}_{\gamma_k<\infty} \\ &\geq \left(1 - e^{rc}\widehat{\mathbb{E}}_{Z_k}\left[e^{-rT(\delta)}\right]\right)\mathbb{1}_{\gamma_k<\infty}.\end{aligned}$$

Now, we consider the standard exit-time problem for a one-dimensional SDE

$$u(x, a, b) = \widehat{\mathbb{E}}_x\left[e^{-rT_{a,b}}\right], \text{ with } T_{a,b} = \inf\left\{t \geq 0, X_t \notin [a, b]\right\}.$$

From Baldi ([2] Ex 10.2 page 312), we know that the function u is jointly continuous for every interval $[a, b]$ strictly included in the state space $\mathbb{R}$ (following our initial convention in this proof). Therefore, we have

$$\widehat{\mathbb{E}}_{Z_k}\left[e^{-rT(\delta)}\right] = u(Z_k, Z_k - \frac{\delta}{4}, Z_k + \frac{\delta}{4}).$$

For all $k \geq 0$, on $\{\gamma_k < \infty\}$, we have $Z_k \in [\frac{x^0}{2}, b_0]$ and thus

$$\widehat{\mathbb{E}}_{Z_k}\left[e^{-rT_k(\delta)}\right] \leq \sup_{z\in[\frac{x^0}{2}, b_0]} u(z, z - \frac{\delta}{4}, z + \frac{\delta}{4}) := \bar{u} < 1.$$

Choosing $c \in \left[0, \frac{1}{r}\ln\left(\frac{1}{\bar{u}}\right)\right]$ to conclude that on $\{\gamma_k < \infty\}$ we have

$$\mathbb{P}\left[\gamma_{k+1} - \gamma_k > c|\mathcal{F}_{\gamma_k}\right] \geq 1 - e^{rc}\bar{u} > 0.$$

Consequently, the series of general term $\mathbb{P}\left[\gamma_{k+1} - \gamma_k > c|\mathcal{F}_{\gamma_k}\right]$ diverges on the set $\{\lim_{k\to\infty}\gamma_k < \infty\}$. By an extension of the Borel Cantelli lemma (see Williams [38] Theorem 12.15 p.124), we conclude that $\lim_{k\to+\infty}\gamma_k = +\infty$ which ends the proof. $\square$