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"Menu Auctions Under Asymmetric Information"

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Abstract We study menu auction games in which several principals influence the choice of a privately-informed agent by simultaneously offering action-contingent payments; the agent is free to accept any subset of the offers. Building on tools from non-smooth optimal control with type-dependent participation constraints, we provide necessary conditions for any equilibrium allocation as the (constrained) maximizer of an endogenous aggregate virtual-surplus program. The aggregate maximand includes an information-rent component that captures how the principals' rent-extraction motives combine. Although there is a large set of equilibria, including equilibrium allocations with discontinuities, we isolate one particular equilibrium allocation, the *maximal* allocation, which is the solution to an unconstrained maximization program. Under weak conditions, necessary conditions for a maximal allocation are also sufficient, and the corresponding equilibrium tariff offers are easily constructed.

1. Introduction

Economists have long been interested in strategic settings in which several interested parties (with either congruent or conflicting interests) attempt to influence a common agent through contribution schedules. In the almost four decades that have passed since the seminal strategic analysis by Bernheim and Whinston (1986), the *truthful* equilibrium of their complete-information model of menu auctions and influence games has become a workhorse in a wide range of settings. Applications include international trade (Grossman and Helpman, 1994), political economy (Grossman and Helpman, 1996), public finance (Persson and Tabellini, 2002), combinatorial auction design (Milgrom, 2007), industrial organization (Bernheim and Whinston, 1998), and environmental economics (Aidt, 1998), among others.

The menu auction game of Bernheim and Whinston (1986) owes its success, in part, to the simplicity and robustness of its equilibrium characterization, even in what may at first glance appear to be very complicated strategic settings. To review, the basic game consists of n principals and a single common agent. The agent chooses some action, $q \in \mathcal{Q}$, that has payoff consequences for each of the principals. Prior to taking an action, however, the principals may each offer the agent enforceable payment schedules – menus of promised payment-action pairs. After receiving a menu offer from each principal, the agent

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chooses which contracts to accept, selects an action to maximize his own utility, and the corresponding payments are enforced. Bernheim and Whinston (1986) show that there are a large number of equilibria to this influence game, but there is always an equilibrium in which the agent chooses an action which maximizes the collective surplus of the principals and the agent. Such a surplus-maximizing equilibrium can be supported with *"truthful"* menus in which each principal offers a transfer schedule whose margin is equal to the principal's marginal benefit of action. This focal property leads Bernheim and Whinston (1986) to argue that this collective surplus-maximizing allocation is a reasonable equilibrium to use for predicting outcomes in general menu auction games with complete information.

The novel contribution of this paper is to reconsider menu auction games under the assumption that the agent has private information. Our paper provides the first general analysis of this class of influence games. Here we are interested in which allocations are candidates for equilibria as well as in showing the existence of a profile of equilibrium schedules that are economically important. The first interest leads us to establish a set of general necessary conditions. The second leads us to a straightforward construction providing sufficient conditions.

Non-Smoothness. Moving towards a characterization of the set of equilibria in our asymmetric-information game of common agency raises a range of difficulties. The first one, although it appears purely technical, has deeper economic consequences. Most previous research in common agency under asymmetric information has imposed a refinement (often implicitly) that principals offer differentiable contribution schedules. Examples abound. In the scenario where common agency is *intrinsic* (i.e., the agent must choose between participating and accepting all principal offers, and not participating and rejecting all offers), several papers have made this refinement: Martimort (1992, 1996), Stole (1991) for models where different principals control different actions of the agent, Laffont and Tirole (1991) in a model of privatization, Mezetti (1997) and Olsen and Osmudsen (2003) for the taxation of multinationals, Laffont and Pouyet (2004) and Martimort and Stole (2009a) for models of regulation. In models where common agency is *delegated* (i.e., the agent is allowed to accept any subset of principal offers) and the principals contract on different actions, this implicit refinement has also been applied. Examples include Martimort and Semenov (2008) for lobbying, Martimort and Stole (2009b) for nonlinear pricing and Calzolari and De Nicolo (2013) for vertical contracting. In all of these papers, the differentiability refinement (although implicit) affords tractability and offers important economic insights for the applications under study. This paper relaxes the differentiability assumption to understand the less tractable, discontinuous equilibrium allocations and to what extent the economic insights of differentiable equilibria generalize.

Once we allow for the possibility that principal j may offer a discontinuous payment schedule, we introduce the technical difficulty that principal i's objective function (which includes the surplus of the agent as a function of her action) is discontinuous and non-differentiable. Standard textbook control-theoretic tools, however, assume the objective function is continuous and piecewise differentiable and, thus, cannot be applied to this setting. Fortunately, we are able to import results from our earlier work on non-smooth optimal control for contract theory (Martimort and Stole, 2022) to characterize each principal's best response function, providing the first step of our analysis (which culminates in Proposition 1). At a best response, the contract that each principal offers exhibits the familiar tradeoff between bilateral efficiency and information rent extraction. However, rent extraction is further limited by the possibility that the agent rejects a principal's offer and obtains a type-dependent reservation payoff by contracting with the remaining principals. The familiar textbook distortions from the screening literature must now be modified, taking into account the shadow value of the agent's type-dependent participation constraint.

Endogenous influence. A second difficulty introduced by asymmetric information is that an individual principal may only choose to actively influence a strict subset of types in equilibrium. By influence, we mean that the principal's offer induces the agent to choose an action that would not have chosen otherwise. The sets of types for which each principal is active must be determined in order to construct equilibrium tariffs, but the equilibrium tariffs, in turn, determine these regions of active influence. In short, the equilibrium activity sets must be jointly determined as part of a fixed point of the principals' best-response correspondences.

If the preferences of principals are linear in the agent's action, the agent's type-dependent participation constraint for principal i will always bind on a single interval of types (possibly degenerate), extending to an endpoint. We refer to this as the *monotonicity property* as it arises whenever a principal's payment schedule is monotone in the agent's action. For more general nonlinear principal preferences, we will focus on equilibria with this property so as to obtain a straightforward characterization of the relevant type-dependent participation constraint. To put this into a more economic context, principals are classified into two subsets: those principals who like more of the agent's action and those principals who like less. We demonstrate that the influence of a given principal can be summarized by means of a *virtual surplus* function that includes the preferences of the principal, an information rent term, and the impact of the agent's outside option when rejecting the principal's offer. Intuitively, whenever the marginal virtual surplus of a principal is positive (resp. negative), the principal pays the agent to increase (resp. decrease) his action relative to the outcome in which principal i does not make any offer. As a consequence, a principal who values more agent action offers non-negative marginal transfers and a principal who values less agent action offers non-positive marginal transfers.

Equilibrium Necessary Conditions. Because the agent's choice depends only upon the aggregate payment he receives from all principals, and each principal's preferences over strategy profiles can be reduced to preferences over the aggregate payment function and her own payments, the menu auction game is an *aggregate game*. Although our menu auction game has infinite-dimensional strategies and asymmetric information, it also has the convenient property that it is quasi-linear in strategies (i.e., payoffs are linear in payments). Equilibria in aggregate games with such quasi-linearity satisfy the *Aggregate Concurrence Principle* in Martimort and Stole (2012): if an equilibrium allocation is a best-response for each principal, it must also be a best response for an aggregate of the principals' objectives. Consequently, any equilibrium allocation necessarily maximizes the sum of the virtual valuations of all principals and the agent's objective over a properly chosen domain. We identify this objective with that of a surrogate principal whose choice replicates the non-cooperative decision process.

As in Bernheim and Whinston's (1986) complete information game, there are multiple equilibrium allocations in our incomplete information setting. We provide necessary conditions satisfied by every such equilibrium allocation. Our main theoretical contribution (Theorem 1) demonstrates that all equilibria exhibit the same kind of informational-rent distortion with differences in equilibria fully characterized by action domain restrictions in the surrogate maximization problem. In other words, the extant literature's conclusions about informational-rent distortions generalize beyond differentiable equilibria. Nevertheless, discontinuous equilibria generally exist and have additional economic properties (e.g., pooling is prevalent) that we explore.

Maximal Equilibrium and Sufficient Conditions. We define and fully characterize one particular equilibrium allocation – what we call the *maximal* allocation. The adjective *maximal* is used here because this allocation is the solution to the surrogate program with no constraints on the action domain (i.e., the domain is maximal). Under weak conditions, it alway exists (Theorem 2). The equilibrium allocation is easy to compute, continuous in type, and exhibits maximal separation across types compared to all equilibria. Although discontinuous equilibrium allocations exhibit bunching around discontinuities, *any* equilibrium allocation that is fully separating over an open interval of types must equal the maximal allocation over that interval (Corollary 1). This justifies our focus on maximal equilibrium allocations and makes precise our assertion above that all equilibria exhibit the same kind of informational rent distortion. Lastly, the maximal equilibrium allocation is implemented with continuously-differentiable schedules (Theorem 3) and corresponds to the differentiable equilibrium that was a priori selected by previous literature. These smooth equilibrium transfer functions are also reminiscent of the *truthful equilibria* found in complete information settings but now, the schedules account for informational distortions. Generalizing a well-known property of *truthful* schedules to a world of asymmetric information, the marginal *maximal contribution* of a given principal perfectly reflects her marginal *virtual* surplus.

Discontinuous Equilibria. Our general approach also sheds light on discontinuous equilibria. Because in any equilibrium of the game, the solution to the surrogate program is a subset of the actions induced by the maximal allocation, the characterization of the solution bears similarities with the allocations found in the mechanism design literature on delegation without transfers (Holmström, 1984; Melumad and Shibano, 1991; Martimort and Semenov, 2006; Alonso and Matouschek, 2008; Amador and Bagwell, 2013). We borrow from this literature techniques that allow us to provide sufficient conditions that ensure that discontinuous allocations satisfying these necessary conditions arise as equilibria (Theorem 4). Roughly, the ranges of these equilibria are obtained by introducing gaps in the range of the maximal equilibrium and having principals not paying if the agent were to choose actions in those gaps. Importantly, the agent achieves a higher payoff in the maximal equilibrium than in any such discontinuous equilibria.

Public Good Games. To illustrate our findings, we first consider a public good game where principals are contracting with an agent with privately-known marginal cost of providing a public good. The agent dislikes producing the public good, but always has the option to reject the principals' offers and choose an action of zero at no cost. For the application, we assume principals have linear surplus functions, possibly heterogeneous in marginal values. Indeed, we allow for the possibility that some principals may actually dislike the public good (e.g., there may be familiar NIMBY concerns). From a technical viewpoint, linearity brings a significant simplification of our analysis: virtual surplus functions become independent of the agent's outside option and the maximal allocation solves an unconstrained surrogate optimization problem.

In a world with a single principal, the public good setting is analogous to government regulation of a monopolist with unknown marginal cost– a workhorse model in regulatory economics since Baron and Myerson (1982). With multiple principals, however, we will see that there are additional effects that generates an allocation considerably different from either the first-best allocation or the Baron and Myerson (1982) optimal allocation. Instead, the maximal equilibrium allocation is a solution to a virtual version of the Lindahl-Samuelson conditions for public good provision.

The maximal allocation is simple to characterize and provides a number of interesting comparative statics. Among others, the equilibrium outcome features *non-neutrality*. In contrast with the scenario of complete information, the equilibrium allocation is now sensitive to *ex ante* redistributions of the marginal surplus across principals. Although a mean-preserving spread of the principal's marginal preferences does not affect the efficient amount of public provision, it leads to an increase the provision of the public good.[1]

Collective Action. Following Olson (1965), consider several individuals, with heterogenous (and possibly opposed) linear preferences for agent action, each of whom wants to influence the agent to take an action. Unlike the simple public goods setting in which the nonparticipating agent chooses to produce zero public good at no cost, our collective action setting allows for the possibility that the agent will choose a non-zero preferred action which will generally depend upon the agent's type. Under complete information, the truthful equilibrium allocation is efficient, maximizing the sum of the players' payoffs. Under asymmetric information, however, a version of the *"tragedy of the commons"* arises. Each principal maximizes the virtual bilateral surplus between the principal and the agent, given the other principals' offers. The principal introduces inefficiencies to extract the agent's information rents on the margin, ignoring the impact that harvesting the information rent has on the other principals. In the maximal equilibrium, there is over harvesting of the agent's rents leading to greater distortions relative to the setting in which all principals could cooperate. There may be an extreme form of inefficiency when the number of principals gets infinitely large, with no aggregate influence on the agent.

Organization. The basic menu auction game with asymmetric information is presented in Section 2. Section 3 analyzes the best-response of a given principal to other principals' offers. Necessary conditions that are satisfied by any equilibrium allocation are presented in Section 4. Section 5 focuses on the properties of the maximal allocation and provides sufficient conditions for such allocation to be an equilibrium. Section 6 also characterizes a class of discontinuous equilibria. Section 7 provides the analysis of the public good game and the collective action problem to highlight the new economic insights that

[1] This result is analogous to a result found in the public finance literature (Bergstrom, Blume and Varian, 1986), that a set of neutral taxes and subsidies on Cournot competitors will have a non-neutral aggregate price effect whenever the public intervention impacts the set of active firms. We extend this insight to a setting of incomplete information.

are now available. Finally, Section 8 concludes by comparing equilibria of delegated and intrinsic common agency games.

2. A Model of Menu Auctions with Incomplete Information

Preferences and Information. Our menu auction game consists of n principals (pronouns "she/they"), each simultaneously offering a non-negative payment schedule to influence a common agent (pronoun "he") for the choice of action, q. Each principal's preferences are linear in monetary transfers and monotone over the agent's choice of action. Given a transfer t_i to the agent and a choice of q by the agent, principal i's payoff is denoted

$$S_i(q) - t_i,$$

where S_i is an upper semi-continuous function on the set of feasible actions $q \in \mathcal{Q} \subset \Re$, the latter of which is a closed and bounded interval. Additionally, we assume that principals have strictly monotonic preferences (i.e., S_i is either strictly increasing or decreasing). We denote the former principals who prefer more action as $i \in \mathcal{A}$ and the latter principals who prefer less action $i \in \mathcal{B}$, where $\mathcal{A} \cup \mathcal{B} = \mathcal{N} = \{1, \ldots, n\}$. For some of our results below, we will assume additionally that each S_i is concave and differentiable. To obtain closed-form solutions in our applications, we further add the simplifying assumption that each principal's preferences are linear, $S_i(q) = s_i q$.

The agent has heterogeneous preferences over actions and monetary transfers, indexed by a type parameter $\theta \in \Theta = [\underline{\theta}, \overline{\theta}]$. Agent preferences are quasi-linear in the aggregate transfer t received from the principals and represented by

$$S(\theta, q) + t,$$

where $S(\theta, q) = S_0(q) - \theta q$ and S_0 is an upper-semicontinuous function of action $q \in \mathcal{Q}$. The assumption that S_0 is linear in θ is standard and for ease of presentation.[2]

The type parameter θ is drawn from a commonly-known distribution function, F, with an associated positive, atomless and differentiable density function f. To guarantee that the solutions to a relaxed optimization problem satisfy the standard monotonicity condition of screening models, we assume the familiar *Monotone Hazard Rate Condition* (hereafter *MHRC*), requiring that the distribution function, F, and its complement, $1 - F$, are log-concave. For future reference, the agent's stand-alone action $\overline{q}_0(\theta)$ and payoff $U_0(\theta)$ in the absence of any principal influence is defined as

$$\overline{q}_0(\theta) \equiv \arg\max_{q \in \mathcal{Q}} S(\theta, q) \text{ and } \overline{U}_0(\theta) \equiv \max_{q \in \mathcal{Q}} S(\theta, q), \tag{2.1}$$

respectively. In the public good setting, it is typical to take $\mathcal{Q} = [0, q_{max}]$ and $\overline{q}_0(\theta) = 0$ for all θ; in contrast, in the collective choice setting we entertain the possibility of non-constant $\overline{q}_0(\theta) \in \mathring{\mathcal{Q}}$ (where $\mathring{\mathcal{Q}}$ denotes the interior of $\mathcal{Q}$).

[2] It is straightforward to extend our analysis to preferences of the form $S_0(q) - \theta c(q)$, for some function c increasing and convex. Our arguments could also be easily extended to more general functions that are nonlinear in θ using the notion of C-convexity developed in Carlier (2001) but at the cost of added complexity in the characterization of incentive compatibility.

Contracts. Each principal i may offer the agent any upper semi-continuous function, $t_i : \mathcal{Q} \to \mathbb{R}_+$, as a contract; $\mathcal{T}$ denotes the set of such non-negative, upper semi-continuous functions on $\mathcal{Q}$. Requiring the schedules to be non-negative is without loss of generality if the agent has the option to reject any subset of the offered schedules, which is the scenario of *delegated common agency* and the setting of this paper.

Timing and Equilibrium Concept. Our delegated common agency game has three stages. First, nature chooses the agent's type. Second, each principal i chooses a transfer function, $t_i \in \mathcal{T}$. We will denote $T(q) \equiv \sum_{i=1}^n t_i(q)$ as the associated aggregate transfers of the principals from this stage and define $T_{-i}(q) \equiv \sum_{j \neq i} t_j(q)$ when this aggregate is taken over all principals except i. Third, the agent chooses an optimal action given the aggregate transfers offered in the second stage. We will denote such a best response as $\overline{q}(\theta \,|\, T)$ but sometimes omit the dependency on the aggregate T when obvious. Finally, payments are made by the principals in accord with the contracts.

Our solution concept is pure-strategy Perfect Bayesian equilibria. The strategy profile $\{\overline{q}, \overline{t}_1, \ldots, \overline{t}_n\}$ is an *equilibrium* of the menu auction game if and only if

$$\overline{q}(\theta \,|\, T) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + T(q) \quad \forall \theta \in \Theta, \, \forall T \in \mathcal{T}, \tag{2.2}$$

$$\overline{t}_i \in \arg\max_{t_i \in \mathcal{T}} \int_{\underline{\theta}}^{\overline{\theta}} \left(S_i(\overline{q}(\theta \,|\, \overline{T}_{-i} + t_i)) - t_i(\overline{q}(\theta \,|\, \overline{T}_{-i} + t_i)) \right) f(\theta) d\theta \quad \forall i \in \mathcal{N}. \tag{2.3}$$

Condition (2.2) is the agent's optimality condition. That $\overline{t}_i$ (and thus $\overline{T}$) is upper semi-continuous ensures existence of such best response $\overline{q}(\theta \,|\, T)$ for the agent's optimization problem over the compact set of actions $\mathcal{Q}$. Condition (2.3) is principal i's optimality condition given the agent's best response.

For any aggregate transfer function, $\overline{T}$, we will refer to the *allocation* $(\overline{U}, \overline{q})$, defined by $\overline{q}(\theta) \equiv \overline{q}(\theta \,|\, \overline{T})$ and $\overline{U}(\theta) = S(\theta, \overline{q}(\theta)) + \overline{T}(\overline{q}(\theta))$ for all $\theta \in \theta$. The equilibrium range of agent's choices is defined by $\overline{q}(\Theta) \equiv \{q \in \mathcal{Q} \,|\, \exists \theta \in \Theta \,\text{s.t.}\; q = \overline{q}(\theta)\}$ which we will sometimes refer to more succinctly as $\overline{\mathcal{Q}}$.

2.1 *Illustrations*

Example 1: Public Good Games. Consider the following public good game inspired by Bergstrom, Blume and Varian (1986), and its later developments in a complete information common agency context by Bernheim and Whinston (1986) and Laussel and Le Breton (2001). There are n principal-citizens and a privately-informed supplier of a public good. Principals may differ in terms of the intensities of their preferences. Principals have linear surplus functions for the public good of the form $S_i(q) = s_i q$. The n principals are ordered from highest to lowest marginal payoff, $s_1 \geq \cdots \geq s_n$ and we denote accordingly the marginal payoffs vector as $\mathbf{s} = (s_1, \ldots, s_n)$. If not all principals prefer the public good (i.e., the principals' payoffs are not congruent), then we denote $\mathcal{A} = \{1, \ldots, j\}$, and we define $\mathbf{s}_A = (s_1, \ldots, s_j)$ and $\mathbf{s}_B = (s_{j+1}, \ldots, s_n)$, allowing us to write the vector of marginal payoffs as $\mathbf{s} = (\mathbf{s}_A, \mathbf{s}_B)$, with a slight abuse of notation.

The agent's type is an unknown, positive marginal cost of production, $\theta \in \Theta$, and the domain of possible public goods is $\mathcal{Q} = [0, q_{\max}]$, with $q_{\max}$ being sufficiently large to avoid boundary solutions at $q = q_{\max}$. The agent's cost of production contains a known component, $C(q)$, which is continuously differentiable, increasing, convex, and satisfies $C(0) = 0$, and a linear private component depending upon type, θq. Thus, we have

$$S(\theta, q) = -\theta q - C(q).$$

The stand-alone action and payoffs are $\overline{q}_0(\theta) = \overline{U}_0(\theta) = 0$.

EXAMPLE 2: COLLECTIVE ACTION. In the spirit of Olson (1965), the principals are lobbyists with preferences that are linear in action, $S_i(q) = s_i q$. A decision-maker, the common agent, chooses a policy q from the domain $\mathcal{Q}$. In contrast with EXAMPLE 1, in the absence of any influence, the agent maximizes $S(\theta, q) = S_0(q) - \theta q$ and chooses a non-trivial stand-alone action $\overline{q}_0(\theta)$ that, for simplicity, is assumed to be interior (i.e., $\overline{q}_0(\theta) \in \mathring{\mathcal{Q}}$ for all $\theta \in \Theta$) and uniquely defined by the first-order condition,

$$S_0'(\overline{q}_0(\theta)) = \theta. \tag{2.4}$$

The corresponding stand-alone payoff is $\overline{U}_0(\theta) = S_0(\overline{q}_0(\theta)) - \theta \overline{q}_0(\theta)$.

3. PRELIMINARIES

3.1 *Statement of the Best-Response Problem*

We begin with a consideration of principal i's best response under the belief that the other principals offer the aggregate tariff schedule $\overline{T}_{-i}$. From principal i's vantage point, it is as if she is designing a contract for an agent with preferences given by

$$S(\theta, q) + \overline{T}_{-i}(q).$$

Absent principal i, the agent can secure a payoff $\overline{U}_{-i}(\theta)$ when contracting with the remaining $n-1$ principals and choosing an action $\overline{q}_{-i}(\theta)$ such that

$$\overline{U}_{-i}(\theta) \equiv \max_{q \in \mathcal{Q}} S(\theta, q) + \overline{T}_{-i}(q), \text{ and } \overline{q}_{-i}(\theta) \in \arg\max_{q \in \mathcal{Q}} S(\theta, q) + \overline{T}_{-i}(q).$$

Instead, when principal i offers a non-negative transfer schedule t_i, the agent obtains utility $U(\theta)$ and chooses an optimal action $q(\theta)$ such that

$$U(\theta) \equiv \max_{q \in \mathcal{Q}} S(\theta, q) + t_i(q) + \overline{T}_{-i}(q), \text{ and } q(\theta) \in \arg\max_{q \in \mathcal{Q}} S(\theta, q) + t_i(q) + \overline{T}_{-i}(q).$$

If the agent is offered a non-negative schedule by principal i, it necessarily follows that the agent's indirect utility of contracting with principal i weakly exceeds $\overline{U}_{-i}$. Similarly, if the agent's indirect utility exceeds $\overline{U}_{-i}$, then the agent must choose an action for which principal i has offered a positive payment. Hence, we can replace the requirement that $t_i \geq 0$ with the following individual rationality requirement for each principal i:

$$U(\theta) \geq \overline{U}_{-i}(\theta) \quad \forall \theta \in \Theta, \tag{3.1}$$

which naturally implies $U(\theta) \geq \overline{U}_0(\theta)$. Because the agent's preferences are bilinear in q and θ, $U(\theta)$ so defined is a maximum of linear functions of θ. Following Rochet (1987), incentive compatibility can be expressed as the tandem requirements

$$-q(\theta) \in \partial U(\theta), \tag{3.2}$$

$$U(\theta) \text{ convex.} \tag{3.3}$$

Condition (3.2) is a general statement of the agent's first-order envelope condition.[3] The convexity requirement (3.3) is equivalent to q being a non-increasing selection in the agent's best-response correspondence. Of course, conditions analogous to (3.2) and (3.3) apply to the allocation $(\overline{U}_{-i}, \overline{q}_{-i})$ absent principal i.

Framed in this manner, principal i's problem of choosing an optimal t_i can be reformulated as choosing an allocation (U, q) that is individually rational and incentive compatible for the agent relative to some outside option, $\overline{U}_{-i}$, and that solves the following program:

$$(\mathcal{P}_i)\text{:} \max_{(U,q)} \int_{\underline{\theta}}^{\overline{\theta}} \left\{ S_i(q(\theta)) + S(\theta, q(\theta)) + \overline{T}_{-i}(q(\theta)) - U(\theta) \right\} f(\theta) d\theta \text{ s.t. (3.1)-(3.2)-(3.3).}$$

If $\overline{T}_{-i}$ were known to be continuous and piecewise differentiable, and if the integrand were known to be concave, we could apply standard optimal control techniques to characterize the optimal contract. Assuming that $\overline{T}_{-i}$ is continuous and a.e. differentiable, however, imposes an equilibrium refinement that is worth explicit consideration. To provide a general solution to $(\mathcal{P}_i)$ that requires only that $\overline{T}_{-i}$ be upper semi-continuous, we utilize necessary and sufficient conditions for non-smooth control programs with type-dependent participation constraints developed in Martimort and Stole (2022). One can show that the solution to the program in which the objective function is replaced with its concavification is also a solution to the original program. The concavification, while continuous, is possibly non-differentiable at points, and so tools from non-smooth optimal control may be applied. These tools, fortunately, allow us to state necessary and sufficient conditions using a distribution of Lagrange multipliers that is reminiscent of the work of Jullien (2000) for the smooth scenario.[4]

3.2 *Monotonicity Property*

Because U and $\overline{U}_{-i}$ are two implementable rent profiles, they are both convex. The type-dependent participation constraint (3.1) thus amounts to comparing two convex functions

[3] Here, ∂U represents the sub-differential of a convex function, allowing for the possibility that, at a countable number of values of θ, U may fail to be differentiable. If U is differentiable at θ, then $\partial U(\theta) = \{\dot{U}(\theta)\}$ and thus $\dot{U}(\theta) = -q(\theta)$. At any point θ of non-differentiability, an incentive-compatible allocation q must nonetheless lie between the right and left derivatives of U at this point.

[4] Jullien (2000) provides necessary and sufficient conditions for control problems with pure type-dependent state constraints under the assumption that the objective function is continuous and piecewise differentiable. Martimort and Stole (2022) demonstrate that a variation of Jullien (2000)'s conditions can be applied to discontinuous models as well. It is worth noting that the simplicity of these conditions is a consequence of the assumption that the objective function is linear in the state variable. Because the preferences of the players are quasi-linear in money, this assumption is satisfied in the present setting.

– a scenario which might lead to a variety of patterns for the set Ω_i of types where the constraint binds for principal i.[5] As we will see below, when principal payoffs are linear in agent's action, in any equilibrium the binding set of types for principal i must be either $\Omega_i = [\hat{\theta}_i, \overline{\theta}]$ for principals who prefer higher action ($i \in \mathcal{A}$), or $\Omega_i = [\underline{\theta}, \hat{\theta}_i]$ for principals who prefer lower action ($i \in \mathcal{B}$). We refer to this as the monotonicity property.

DEFINITION 1. *Let $\{\overline{\Omega}_i\}_{i \in \mathcal{N}}$ be a collection of equilibrium inactive-type sets where $\overline{\Omega}_i = \{\theta \in \Theta \mid \overline{U}(\theta) = \overline{U}_{-i}(\theta)\}$. An equilibrium to the common agency game satisfies the* **monotonicity property** *(MP) (or equivalently, it is a* **monotone equilibrium***) if for each i there is a $\hat{\theta}_i \in [\underline{\theta}, \overline{\theta}]$ such that*

$$\overline{\Omega}_i = \begin{cases} [\hat{\theta}_i, \overline{\theta}] & \text{for } i \in \mathcal{A}, \\ [\underline{\theta}, \hat{\theta}_i] & \text{for } i \in \mathcal{B}. \end{cases}$$

More generally when principal preferences are nonlinear, it is reasonable to suppose that a principal $i \in \mathcal{A}$ who prefers greater agent actions would offer nondecreasing payments in q. With such monotonic payments, the set of inactive types for this principal is an upper interval (possibly degenerate), $[\hat{\theta}_i, \overline{\theta}]$, as in the linear case. A symmetric intuition holds for principals who prefer lower actions. In what follows, we restrict our attention to equilibria which satisfy this property.[6]

The monotonicity property implies that differences of payments across alternatives reflect the ranking of the alternatives for each principal. When MP holds, there is a clear segmentation of the principals' areas of influence and we can refer to $\overline{\Omega}_i^c$ (the complement of $\overline{\Omega}_i$) as the set of types for whom principal i is active and influences the agent's choice.

3.3 *Virtual Surplus: Definition, Properties*

While valuation functions are the object of interest to analyze how a principal influences the agent under complete information, virtual surplus turns out to play the same role under asymmetric information. When constructing best-responses for principal i given $\overline{T}_{-i}$, the principal's virtual surplus will largely determine the principal's best response. We are now prepared to define principal i's virtual bilateral surplus.

DEFINITION 2. *Principal i's* **virtual surplus** *relative to an allocation $(\overline{U}_{-i}, \overline{q}_{-i})$ is defined, for $\overline{q}_{-i}(\theta) \in \mathring{\mathcal{Q}}$,[7] as*

$$\mathcal{V}_i(\theta, q)[\overline{q}_{-i}] \equiv S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \max\left\{\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q. \tag{3.4}$$

where $\partial\overline{co}(S_i)$ is the sup-differential of the concave envelope of S_i.

[5]See the literature on countervailing incentives (Lewis and Sappington, 1989).

[6]It is an open question as to whether or not the imposition of the monotonicity property is an equilibrium refinement with nonlinear principal preferences that are independent of type. We have failed to find any example of equilibria which does not satisfy this property.

[7]If $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$, the precise statement of the virtual surplus needs to be amended. Lemma B.1 deals with the case where $0 \in \partial\overline{co}\left(S(\theta, \overline{q}_{-i}(\theta)) - \theta\overline{q}_{-i}(\theta)\right)$ and $\overline{q}_{-i}(\theta) \in \text{bd}\mathcal{Q}$.

Several remarks are in order. First, in many applications it is reasonable to assume that S_i is concave and continuously differentiable, in which case our expression for virtual surplus simplifies to

$$\mathcal{V}_i(\theta,q)[\overline{q}_{-i}] = S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \max\left\{S_i'(\overline{q}_{-i}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q$$

Differentiating with respect to q then yields

$$\mathcal{V}_{iq}(\theta,q)[\overline{q}_{-i}] = S_i'(q) - \min\left\{\frac{F(\theta)}{f(\theta)}, \max\left\{S_i'(\overline{q}_{-i}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}. \tag{3.5}$$

Observe that S_i being concave and $\overline{q}_{-i}$ non-increasing (due to incentive compatibility), the function $S_i'(\overline{q}_{-i}(\theta))$ is itself non-decreasing in θ. Moreover, $MHRC$ ensures that $\frac{F(\theta)}{f(\theta)}$ and $\frac{F(\theta)-1}{f(\theta)}$ are also non-decreasing in θ. Since the *min* and *max* operators preserve monotonicity, it immediately follows that $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ exhibits decreasing differences. That said, while decreasing differences is an important property to ensure monotonic actions at a best response, the difficulty remains that that principal i's virtual surplus depends on the action $\overline{q}_{-i}$ that is chosen in its absence, which in turn is an equilibrium object.

In some applications, it is reasonable to assume that S_i is a linear function of q, say $S_i(q) = s_i q$. In this special case, $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ is independent of $\overline{q}_{-i}$ and simplifies to

$$\mathcal{V}_i(\theta,q) = \begin{cases} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\} q, & \text{if } i \in \mathcal{A}, \\ \min\left\{s_i + \frac{1-F(\theta)}{f(\theta)}, 0\right\} q, & \text{if } i \in \mathcal{B}. \end{cases}$$

Here we can see that $MHRC$ implies principal $i \in \mathcal{A}$ will have non-increasing virtual surplus in θ, and therefore will choose to influence the agent for types in a lower interval; a symmetric argument applies to the case of $i \in \mathcal{B}$. This simple property guarantees that, with linear principal preferences, all equilibria must exhibit the monotonicity property.

Returning to our concave, differentiable S_i setting we can illustrate why (3.4) is the correct notion of virtual surplus for constructing best responses. Suppose that principal i values the agent's action (i.e., $i \in \mathcal{A}$, $S_i'(q) > 0$). Since $S_i'(\overline{q}_{-i}(\theta))$ is bounded below by some positive number, there always exists an interval of the form $[\underline{\theta}, \hat{\theta}_i)$, for which

$$\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = S_i'(\overline{q}_{-i}(\theta)) - \frac{F(\theta)}{f(\theta)} > 0. \tag{3.6}$$

In other words, principal i would like to expand the agent's output beyond $\overline{q}_{-i}(\theta)$ for those types. To foster intuition, suppose that principal i expands output above $\overline{q}_{-i}(\theta)$ over a small neighborhood $[\theta, \theta + d\theta]$ by a small amount dq. Principal i's expected marginal benefit of doing so would be $f(\theta)S_i'(\overline{q}_{-i}(\theta))dqd\theta$ while the expected extra information rent left to all inframarginal types so that they accept such deal would equal to $F(\theta)dqd\theta$. Condition (3.6) says that, for θ small enough, such a marginal change benefits principal i.[8] In

[8]Suppose now that principal i dislikes the agent's output, i.e., $S_i' < 0$. For θ in an interval of the form $\left[\hat{\theta}_i, \overline{\theta}\right]$, we thus have $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = S_i'(\overline{q}_{-i}(\theta)) + \frac{1-F(\theta)}{f(\theta)} < 0$. Principal i would like to reduce

contrast, suppose that $S_i'(\overline{q}_{-i}(\theta)) \leq \frac{F(\theta)}{f(\theta)}$. Inserting into (3.5) yields $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta))[\overline{q}_{-i}] = 0$ and principal i would not like to marginally increase the agent's output in that case.

The intuition is even simpler in the case of linear principal preferences, where the marginal valuation s_i of principal i is independent of which action is taken with the remaining $n-1$ other principals. To illustrate, consider the case $s_i > 0$. Then the existence of an interval of the form $\left[\hat{\theta}_i, \overline{\theta}\right]$ over which $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta)) = 0$ directly follows from $MHRC$. Because $MHRC$ holds, there is indeed a unique solution $\hat{\theta}_i$ to $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$ provided that $1 > s_i f(\overline{\theta})$ and, moreover, $(\hat{\theta}_i, \overline{\theta}] = \{\theta \mid F(\theta) - s_i f(\theta) > 0\}$. $MHRC$ also implies $f'(\theta)/f(\theta) \leq f(\theta)/F(\theta)$, from which it follows that $F(\theta) - s_i f(\theta)$ is increasing if $F(\theta)/f(\theta) > s_i$. Hence, $F(\theta) - s_i f(\theta)$ is strictly increasing on $(\hat{\theta}_i, \overline{\theta}]$ and $\mathcal{V}_{iq}(\theta, \overline{q}_{-i}(\theta)) = 0$ on that interval.

3.4 *Best-Responses: Characterization*

With multiple principals, an equilibrium depends upon the virtual surplus of every principal. A key contribution of this paper is determining the precise manner in which all $\mathcal{V}_i$s combine to determine such an equilibrium. We now present a key building block of our analysis which follows from applying the results in Martimort and Stole (2022). For completeness, we provide a self-contained proof in an online appendix.

Proposition 1. *In any monotone equilibrium, given the aggregate transfer function $\overline{T}_{-i}$ offered by other principals, and the agent's corresponding outside option $\overline{U}_{-i}$ and output $\overline{q}_{-i}$, the allocation $(\overline{U}, \overline{q})$ is a solution to principal i's program if and only if it satisfies (3.1)-(3.2)-(3.3), and*

$$\overline{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + \mathcal{V}_i(\theta, q)[\overline{q}_{-i}] + \overline{T}_{-i}(q), \;\; a.e. \tag{3.7}$$

where $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$ satisfies (3.4).

Moreover, if $\mathring{\Omega}_i \neq \emptyset$, the following property holds:

$$\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] = 0 \iff \overline{U}(\theta) = \overline{U}_{-i}(\theta) \text{ and } \overline{q}(\theta) = \overline{q}_{-i}(\theta), \quad \forall \theta \in \mathring{\Omega}_i. \tag{3.8}$$

If instead $\mathring{\Omega}_i = \emptyset$,

$$\begin{cases} \overline{t}_i(\overline{q}(\overline{\theta})) \geq 0 & \text{if } i \in \mathcal{A}, \\ \overline{t}_i(\overline{q}(\underline{\theta})) \geq 0 & \text{if } i \in \mathcal{B}. \end{cases} \tag{3.9}$$

Proposition 1 informs us that, for any type for which $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] = 0$, principal i finds it optimal not to influence the agent's choice and thus $\overline{q}(\theta) = \overline{q}_{-i}(\theta)$ for such a type. The transfer t_i which implements $(\overline{U}, \overline{q})$ above will have the property that $\overline{t}_i(\overline{q}(\theta)) = 0$ for all such θ. For these types, principal i is *inactive* and (3.1) is binding over an interval Ω_i with non-empty interior. Instead, for any θ for which principal i is active, $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}_{-i}] \neq 0$,

the agent's output below $\overline{q}_{-i}(\theta)$ for those types. Whether $i \in \mathcal{A}$ or $i \in \mathcal{B}$, principal i always influence the agent by increasing the latter's information rent $\overline{U}$ beyond his reservation payoff $\overline{U}_{-i}$ absent this principal and she does by rewarding the agent for changing his action in the direction she likes.

we have $\overline{U}(\theta) > \overline{U}_{-i}(\theta)$. Note that principal i can be active and offer a positive payment to the agent even when (3.1) is binding if it arises at extreme points.

Proposition 1 characterizes best-responses to any incentive compatible allocation $(\overline{U}_{-i}, \overline{q}_{-i})$. Its conclusions are particularly striking when we consider best-responses at equilibrium. Principal i will not induce a deviation away from such an equilibrium allocation $\overline{q}$, whenever $\mathcal{V}_{iq}(\theta, \overline{q}(\theta))[\overline{q}] = 0$ where the virtual surplus is now relative to this equilibrium allocation. This condition can be written as

$$\partial\overline{co}(S_i)(\overline{q}(\theta)) \leq \frac{F(\theta)}{f(\theta)} \text{for } i \in \mathcal{A} \text{ and } \partial\overline{co}(S_i)(\overline{q}(\theta)) \geq \frac{F(\theta)-1}{f(\theta)} \text{ for } i \in \mathcal{B}.^{9}$$

If, at equilibrium, principal i chooses to influence the agent, it must instead be that the reverse of the inequalities above hold. In such a case, $\mathcal{V}_i(\theta, \overline{q}(\theta))[\overline{q}_{-i}]$ is independent of $\overline{q}_{-i}$, and we may again consider virtual surplus relative to $\overline{q}$ instead of $\overline{q}_{-i}$. We may therefore use Proposition 1 to characterize equilibrium behavior where the virtual surplus is computed relative to the equilibrium allocation itself; for $\overline{q}(\theta)$ interior, the virtual surplus is

$$\overline{\mathcal{V}}_i(\theta, \overline{q}(\theta))[\overline{q}] = S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{\partial\overline{co}(S_i)(\overline{q}(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q \text{ a.e.}. \tag{3.10}$$

4. Equilibria: Necessary Conditions

Our menu auction game is an *aggregative game* since the agent's choice depends on the aggregate payment T he receives, and principal i's preferences over strategy profiles can be reduced to preferences over her own tariff t_i and this aggregate payment T. This allows us to aggregate the best-response conditions given in Proposition 1. The corresponding necessary conditions are obtained simply noting that $\overline{q}$ must solve (3.7) for each principal i. Hence, $\overline{q}$ must also maximize the sum of the objectives from these individual programs:

$$\overline{q}(\theta) \in \arg\max_{q \in \mathcal{Q}} \; S(\theta, q) + \overline{\mathcal{V}}(\theta, q)[\overline{q}] + (n-1)\left(S(\theta, q) + \overline{T}(q)\right), \tag{4.1}$$

where $\overline{T}$ implements $\overline{q}$ and $\overline{\mathcal{V}}(\theta, q)[\overline{q}] \equiv \sum_{i=1}^{n} \overline{\mathcal{V}}_i(\theta, q)[\overline{q}]$ is the aggregate virtual preferences of the principals relative to the equilibrium allocation itself.

Because $\overline{T}$ appears in the objective in (4.1) and it must also implement $\overline{q}$, this necessary condition is a fixed point. For a given equilibrium aggregate $\overline{T}$, there exists a $\overline{q}$, which in turn must be a solution to the program in (4.1). As we will demonstrate, there might be an infinite number of solutions (equilibria) to this self-referencing program. We provide necessary conditions satisfied by all such allocations.

Theorem 1. *Any equilibrium allocation $\overline{q}$ must satisfy the necessary conditions*

$$\overline{q}(\theta) \in \arg\max_{q \in \overline{Q}} \; S(\theta, q) + \overline{\mathcal{V}}(\theta, q)[\overline{q}], \quad \forall \theta \in \Theta \tag{4.2}$$

where $\overline{Q} = \overline{q}(\Theta) \subseteq \mathcal{Q}$ is the equilibrium range.

[9] Because $\partial\overline{co}(S_i)$ might be an interval, these inequalities should be understood as set inequalities.

Condition (4.2) represents a simplified, pointwise program that embeds the strategic interactions of the principals. The comparison of (4.1) and (4.2) shows in fact a remarkable simplification. The extra term $(n-1)\left(S(\theta,q)+\overline{T}(q)\right)$ that corresponds to $n-1$ times the agent's payoff has now disappeared in the final formulation (4.2). Intuitively, $\overline{q}(\theta)$ is also a maximizer for this last term since it has to be the agent's equilibrium choice. Although no assumption on differentiability of the aggregate tariff $\overline{T}(q)$ is made in the first place, an *Envelope Condition* can be used to simplify the optimality requirement.

Everything happens as if a *surrogate representative of the principals*, whose decisions reflect their non-cooperative behavior, is now optimizing on their behalf an objective function, namely $S(\theta,q)+\overline{\mathcal{V}}(\theta,q)[\overline{q}]$, which conflates the various influences of the principals. At any type θ, this surrogate principal should prefer to choose the equilibrium action $\overline{q}(\theta)$ rather than the action that would have been chosen by another type. This explains why in the maximand of (4.2), the maximization domain is over all possible actions that lie in the equilibrium range $\overline{Q}$. This maximization thus brings a set of incentive constraints for the surrogate principal that require a careful investigation provided in Theorem A.1 below.

In the surrogate's objective, each principal's surplus function $S_i(q)$ is now replaced by its virtual surplus $\mathcal{V}_i(\theta,q)[\overline{q}]$ relative to the equilibrium allocation itself. Therefore, this maximization problem still contains a fixed-point requirement because the aggregate virtual surplus is relative to the equilibrium allocation itself. This aggregate virtual surplus reflects in which direction principals would like collectively to push the agent's action. Depending on whether the marginal aggregate $\overline{\mathcal{V}}_q(\theta,\overline{q}(\theta))[\overline{q}]$ at that equilibrium allocation is positive or negative, the overall influence of principals pushes action up (resp. down) with respect to the stand-alone benchmark $\overline{q}_0(\theta)$.

5. Maximal Equilibria: Necessary and Sufficient Conditions

To make progress on the characterization of the equilibrium set, we first observe that a solution to Condition (4.2) stands out for special consideration. Define thus the *maximal allocation* $q^m(\theta)$ as a solution to

$$q^m(\theta)\in\arg\max_{q\in\mathcal{Q}}\; S(\theta,q)+\mathcal{V}(\theta,q)[q^m]. \tag{5.1}$$

The allocation $q^m(\theta)$ in (5.1) is said to be *maximal* because, in contrast with the more general Condition (4.2), the optimization domain $\mathcal{Q}$ is now left unrestricted. Note that q^m as defined is a fixed point; it is immediate to show that such a fixed point always exists.[10] Whilst there may be more than one solution in our most general setting, we now impose

[10] Consider the correspondence

$$\Psi(x)=\arg\max_{q\in\mathcal{Q}}\; S(\theta,q)+\sum_{i=1}^{n}S_i(q)-\min\left\{\frac{F(\theta)}{f(\theta)};\max\left\{\partial\overline{co}(S_i)(x),\frac{F(\theta)-1}{f(\theta)}\right\}\right\}q.$$

Because $\mathcal{Q}$ is compact and the above maximand is continuous in (q,x), we can apply Berge's Theorem of the Maximum which states that the correspondence Ψ is upper hemi-continuous, non-empty and compact. Because $\sum_{i=0}^{n}S_i(q)$ is concave, Ψ is convex-valued. By Kakutani's Fixed-Point Theorem, Ψ admits a fixed-point $q^m(\theta)$.

concavity and differentiability on our principals' and agent's preferences: preferences are continuously differentiable, S_i is concave for $i = 1, \dots, n$, and S_0 is strictly concave. Under these minimal restrictions which are satisfied in our applications, a unique solution exists and we can speak of *the* maximal equilibrium allocation.

THEOREM 2. *Suppose that $S_i(q)$ is concave and differentiable for all $i \in \mathcal{N}$, and $S_0(q)$ is strictly concave. At any interior point, the maximal allocation is uniquely defined by*

$$S_0'(q^m(\theta)) + \sum_{i \in \mathcal{A}} \max \left\{ S_i'(q^m(\theta)) - \frac{F(\theta)}{f(\theta)}; 0 \right\} + \sum_{i \in \mathcal{B}} \min \left\{ S_i'(q^m(\theta)) + \frac{1 - F(\theta)}{f(\theta)}; 0 \right\} = \theta. \quad (5.2)$$

Furthermore, $q^m(\theta)$ so defined is non-increasing and continuous.

For future reference, we define $\Omega_i^m \equiv \left\{ \theta \in \Theta \text{ s.t. } \frac{F(\theta)-1}{f(\theta)} \leq S_i'(q^m(\theta)) \leq \frac{F(\theta)}{f(\theta)} \right\}$. Ω_i^m is the subset of types for which principal i has no influence at the maximal allocation. In general, the influence area of principal i is thus determined by a joint condition on her own preferences and the equilibrium action. This difficulty renders the characterization of influence areas rather difficult. Section 7 nevertheless shows that this difficulty can be overcome in structured economic environments.

Theorem 2 highlights conditions that ensure existence of a maximal allocation. To complete our analysis, Theorem 3 below now provides sufficient conditions for existence of a maximal equilibrium. The important step on that route is to construct equilibrium tariffs. To this end, we must define the assignment correspondence $\vartheta^m(q) = \{\theta \in \Theta | q = q^m(\theta)\}$. Under the assumptions of Theorem 2, this correspondence is single-valued and continuous on $\mathcal{Q}^m \subseteq \mathcal{Q}$ since q^m is decreasing and continuous.[11]

THEOREM 3. *Suppose that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$. The maximal allocation $q^m(\theta)$ satisfying (5.2) is an equilibrium allocation induced by the following equilibrium maximal tariffs t_i^m:*

- *When $\mathring{\Omega}_i^m \neq \emptyset$,*

$$t_i^m(q) = \int_{\hat{q}_i}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x) dx \quad \forall q \in \mathcal{Q}^m, \quad \forall i \in \mathcal{N} \quad (5.3)$$

where $\hat{q}_i \in q^m(\Omega_i^m)$ is arbitrary.

- *When $\mathring{\Omega}_i^m = \emptyset$,*

$$t_i^m(q) = \begin{cases} t_i^m(q^m(\bar{\theta})) + \int_{q^m(\bar{\theta})}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x) dx & \forall i \in \mathcal{A}, \\ t_i^m(q^m(\underline{\theta})) + \int_{q^m(\underline{\theta})}^{q} \mathcal{V}_{iq}^m(\vartheta^m(x), x) dx & \forall i \in \mathcal{B}. \end{cases} \quad (5.4)$$

[11] The correspondence would not be single-valued whenever $q^m(\theta)$ is constant. Thanks to *MHRC*, this scenario might only arise when $q^m(\theta)$ lies on the boundaries of $\mathcal{Q}$. This possibility of a corner solution is ruled out in the sequel by making extra assumptions (explicit in the economic examples under scrutiny) and we shall thus focus on interior solutions.

where $t_i^m(q^m(\overline{\theta})) \geq 0$ *for* $i \in \mathcal{A}$ *and* $t_i^m(q^m(\underline{\theta})) \geq 0$ *for* $i \in \mathcal{B}$.

The expressions of the tariffs in (5.3)-(5.4) are reminiscent of the *truthful tariffs* proposed by Bernheim and Whinston (1986) under complete information. Remember that, in that setting, truthful tariffs are actually of the form

$$t_i(q) = \max\{S_i(q) - C_i; 0\}$$

for some constants C_i.[12] When positive, these schedules reflect the preferences of principals between alternatives. Under asymmetric information, informational distortions reduce (resp. increase) the marginal contribution of a principal $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$) below (resp. above) her marginal valuation. The *maximal contribution schedules* (5.3)-(5.4) reflect the virtual surplus of principals between alternatives.

6. Discontinuous Equilibria

We now demonstrate that there are (candidate) equilibrium allocations which satisfy the necessary conditions (4.2) but do not satisfy (5.1) when $\overline{Q} \subsetneq \mathcal{Q}^m$, and so condition (5.1) may implicitly refine the equilibrium set. To investigate the possibility of such equilibria and get a clear characterization, we now adopt the set of assumptions made in Theorem 2. Recall that under these circumstances, an interior maximal allocation is uniquely defined as (5.2), continuous and decreasing.

The restriction $\overline{Q} \subsetneq \mathcal{Q}^m$ on the equilibrium range of actions of course only matters when binding. When not so, the equilibrium action is necessarily the maximal allocation. Candidate equilibrium allocations are thus identical to the maximal allocation on $\mathring{\overline{Q}}$. Accordingly, we now rewrite (4.2) as

$$\overline{q}(\theta) \in \arg\max_{q \in \overline{Q}} \; S(\theta, q) + \mathcal{V}^m(\theta, q), \quad \forall \theta \in \Theta. \tag{6.1}$$

Condition (6.1) is a priori compatible with the existence of a countable number of downward discontinuities in the action profile when the range of equilibrium values $\overline{Q}$ is not connected. An important step on route to characterizing discontinuous equilibria, and much like in Theorem 1 above, consists in constructing tariffs that implement these allocations. The difficulty is that delegated common agency is a non-bijective aggregate game, using the language of Martimort and Stole (2012), and therefore not all solutions to (6.1) are equilibrium allocations. Of course, an easy way to handle discontinuities would be to have principals coordinate on large punishments if the agent would choose actions in any discontinuity gap. This coordination is feasible under intrinsic common agency as shown in Martimort, Semenov and Stole (2018). In a delegated common agency game, however, tariffs are non-negative for all principals which constrains what sort of discontinuities are sustainable in equilibrium. Our next theorem provides one such construction and exhibits

[12]The possible values of those constants are found at equilibrium from the binding participation constraints of the agent's in each principal's best-response problem. In contrast with the case of incomplete information analyzed here, those participation constraints are always binding.

an important class of equilibria whose tariffs are simply truncated version of maximal contribution schedules. For simplicity, we consider the case where all principals have congruent interests.

THEOREM 4. *Suppose that, $\mathcal{A} = \mathcal{N}$, and that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i \in \mathcal{N}$. Consider a triplet $(\theta_0, \theta_1, \theta_2) \in \Theta^3$ with $\theta_1 < \theta_0 < \theta_2$,*

$$[S(\theta_0, q) + \mathcal{V}^m(\theta_0, q)]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0, \tag{6.2}$$

$$\mathcal{V}_{iq}^m(\theta, q^m(\theta)) > 0 \quad \forall \theta \in [\theta_1, \theta_2], \forall i \in \mathcal{N}, \tag{6.3}$$

and

$$\frac{F(\theta_0)}{f(\theta_0)} = \frac{1}{q^m(\theta_1) - q^m(\theta_2)} \int_{q^m(\theta_2)}^{q^m(\theta_1)} \frac{F(\vartheta^m(x))}{f(\vartheta^m(x))} dx. \tag{6.4}$$

The allocation $\overline{q}$ defined as

$$\overline{q}(\theta) = \begin{cases} q^m(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_1] \bigcup [\theta_2, \overline{\theta}], \\ q^m(\theta_1) & \text{if } \theta \in [\theta_1, \theta_0], \\ q^m(\theta_2) & \text{if } \theta \in (\theta_0, \theta_2] \end{cases} \tag{6.5}$$

is an equilibrium allocation with θ_0 as a downward jump discontinuity. It is implemented by means of truncated maximal tariffs defined as

$$\bar{t}_i(q) = \begin{cases} t_i^m(q) & \text{if } q \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)], \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

EQUILIBRIUM SELECTION. Because equilibrium tariffs (6.6) are now truncated versions of maximal contribution schedules, the agent's possible choices are *de facto* restricted. The next proposition, whose proof is thus immediate, provides thus a strong reason to focus on maximal allocations nevertheless.

PROPOSITION 2. *Compared to the class of discontinuous equilibria characterized in Theorem 4, the agent's payoff is greater in the maximal equilibrium.*

More generally, when the maximal allocation is unique (as in the concave-differentiable setting), we have an immediate corollary to Theorem 2:

COROLLARY 1. *Let $\overline{q}(\cdot)$ be an equilibrium allocation that is fully separating over the open interval (θ_1, θ_2). Then*

$$\overline{q}(\theta) = q^m(\theta) \text{ for all } \theta \in (\theta_1, \theta_2).$$

Thus even discontinuous equilibrium allocations correspond to the maximal allocation over regions where there is full separation, suggesting the economic forces operating in the maximal equilibrium are more universally relevant.

We close this section with some further comments on equilibrium selection. Instead of selecting the most preferred allocation from the agents' viewpoint, we could have instead looked for the best equilibrium for the principals. There are two difficulties on that front. First, the notion of coalition-proofness that was used by Bernheim and Whinston (1986) in their complete information environment to select the truthful equilibrium is, to the best of our knowledge, not available in our asymmetric information context. Second, and in contrast with the scenario of intrinsic common agency studied in Martimort, Semenov and Stole (2018), the nature of the discontinuities in Theorem 4 is more restrictive.[13] Applying recursively the methodology of Theorem 4, one could construct equilibria with many such gaps.

7. Maximal Equilibria at Work

We now show how our characterization of maximal equilibria helps to derive important insights for structured economic environments that are of much interest for applications.

7.1 *Public Good Games*

Example 1 offers a particularly striking example of our general approach. Using (3.4), we first observe that principal i's virtual surplus at the maximal allocation is linear:

$$\mathcal{V}_i^m(\theta,q)=\begin{cases}\max\left\{s_i-\frac{F(\theta)}{f(\theta)};0\right\}q & \text{for } i\in\mathcal{A},\\ \min\left\{s_i+\frac{1-F(\theta)}{f(\theta)};0\right\}q & \text{for } i\in\mathcal{B},\end{cases} \tag{7.1}$$

and the marginal virtual surplus for a given principal does not depend on actions that might be taken in her absence. As a result, the surrogate principal's problem becomes a simple optimization problem. This property is the source of many sharp results in what follows. In particular, the influence area of each principal is now entirely determined by her own preferences.

Proposition 3. *Suppose that principals have linear surplus functions, i.e.,* $S_i(q)=s_iq$ *for all* $i\in\mathcal{N}$*, and that* $C(q)$ *is strictly convex with*

$$\sum_{i\in\mathcal{N}} s_i\geq\overline{\theta}+C'(0)+\frac{|\mathcal{A}|}{f(\overline{\theta})}. \tag{7.2}$$

Suppose also that

$$1\geq f(\overline{\theta})s_i \quad \text{if } i\in\mathcal{A} \text{ and } 1\leq -f(\underline{\theta})s_i \quad \text{if } i\in\mathcal{B}. \tag{7.3}$$

An interior maximal equilibrium exists and is unique. This maximal allocation q^m *solves*

$$\sum_{i\in\mathcal{A}}\max\left\{s_i-\frac{F(\theta)}{f(\theta)};0\right\}+\sum_{i\in\mathcal{B}}\min\left\{s_i+\frac{1-F(\theta)}{f(\theta)};0\right\}=\theta+C'(q^m(\theta)). \tag{7.4}$$

[13] Martimort, Semenov and Stole (2018) nevertheless show that, within a (restricted) class of equilibria that entail a floor on actions, principals may want to coordinate on a non-maximal equilibria.

Virtual Lindahl-Samuelson Conditions. Condition (7.4) is a virtual version of Lindahl-Samuelson conditions. The sum of the principals' marginal virtual surplus balances the agent's marginal cost of producing the public good at a maximal equilibrium allocation. A principal i who enjoys (resp. dislikes) the public good, $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$), influences the agent with a type $\theta \leq \hat{\theta}_i$ (resp. $\theta \geq \hat{\theta}_i$). Thanks to (7.3), the cut-off $\hat{\theta}_i$ is defined as $s_i = \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)}$ for $i \in \mathcal{A}$ (resp. $s_i = \frac{F(\hat{\theta}_i)-1}{f(\hat{\theta}_i)}$ for $i \in \mathcal{B}$).

Maximal Contributions. Using (5.3), the maximal allocation is implemented by means of the following maximal contributions:

$$t_i^m(q) = \begin{cases} \int_{q^m(\hat{\theta}_i)}^{q} \max\left\{s_i - \frac{F(\vartheta^m(x))}{f(\vartheta^m(x))}; 0\right\} dx & \text{for } i \in \mathcal{A}, \\ \int_{q^m(\hat{\theta}_i)}^{q} \min\left\{s_i + \frac{1-F(\vartheta^m(x))}{f(\vartheta^m(x))}; 0\right\} dx & \text{for } i \in \mathcal{B} \end{cases} \tag{7.5}$$

where $\vartheta^m(q)$ is the assignment rule for the maximal allocation. Confirming our earlier findings, it can be readily checked that a principal $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$) who enjoys (resp. dislikes) the public good wants to increase (resp. decrease) its level, and thus $t_i^{m'}(q) \geq 0$ (resp. $t_i^{m'}(q) \leq 0$) at all $q \in \mathcal{Q}$.

Comparative Statics. An interesting comparative static is to ask how a constant-sum redistribution of the principals' marginal payoffs impacts the maximal equilibrium allocation even though such a redistribution would have no impact on the efficient allocation.

Proposition 4. *Consider two configurations of principal preferences,* $\mathbf{s} = (\mathbf{s}_A, \mathbf{s}_B)$ *and* $\tilde{\mathbf{s}} = (\tilde{\mathbf{s}}_A, \tilde{\mathbf{s}}_B)$. *If* $\tilde{\mathbf{s}}_A$ *is a mean-preserving spread*[14] *of* $\mathbf{s}_A$ *and* $\tilde{\mathbf{s}}_B = \mathbf{s}_B$, *then the associated maximal allocations in each game have the property that for all* θ

$$\overline{q}^m_{\tilde{\mathbf{s}}}(\theta) \geq \overline{q}^m_{\mathbf{s}}(\theta),$$

with a strict inequality for some positive measure. Similarly, if $\tilde{\mathbf{s}}_A = \mathbf{s}_A$ *and* $\tilde{\mathbf{s}}_B$ *is a mean-preserving spread of* $\mathbf{s}_B$, *then*

$$\overline{q}^m_{\tilde{\mathbf{s}}}(\theta) \leq \overline{q}^m_{\mathbf{s}}(\theta),$$

with a strict inequality for some positive measure.

The result follows from noting that the lefthand side of (7.4) is convex in s_i for $i \in \mathcal{A}$ and concave in s_i for $i \in \mathcal{B}$. Applying Jensen's inequality finishes the proof.

Cooperative Principals. As a first illustration of Proposition 4, consider the case of cooperating principals in a setting where preferences are congruent, say $s_i > 0$ for all $i \in \mathcal{N}$. This is equivalent to one principal having now preferences $\tilde{s}_1 = \sum_{i=1}^n s_i > 0$ and the other $n-1$ principals having now preferences $\tilde{s}_j = 0$ (for $j \neq 1$) with contributions as defined in (7.5) identically zero. It follows that $\tilde{\mathbf{s}}$ is more dispersed than $\mathbf{s}$. From Proposition 4, the cooperative outcome entails a higher allocation in comparison with the non-cooperative scenario:

$$q^{coop}(\theta) \geq \overline{q}^m(\theta) \quad \forall \theta \in \Theta.$$

[14]Given two configurations $\mathbf{s}$ and $\tilde{\mathbf{s}}$ with same mean (i.e., $\sum_{i=1}^n s_i = \sum_{i=1}^n \tilde{s}_i$), we define the associated discrete distributions on the combined domain $\cup_i s_i \cup_j \tilde{s}_j$. If the distribution for $\mathbf{s}$ second-order stochastically dominates the distribution for $\tilde{\mathbf{s}}$, then we say that $\tilde{\mathbf{s}}$ is a mean-preserving spread of $\mathbf{s}$.

We emphasize that the source of this free-riding problem among principals is asymmetric information. If information were complete the maximal equilibrium leads to full efficiency. Each principal would offer the marginal truthful tariff $t_i'(q) = s_i$ as in Bernheim and Whinston's (1986). Thus, free riding need not arise in complete-information public good games if principals have the ability to offer nonlinear tariffs to a common agent rather than making direct, one-dimensional contributions to the public good. With asymmetric information, however, each principal has private incentives to distort the agent's output choice to extract additional information rent. Because each principal ignores the negative externality that doing so imposes on others, from a collective viewpoint, the principals inefficiently extract too much rent. The free-riding problem present in our setting more closely fits the narrative of a *"tragedy of the commons"* in which each principal over-harvests a common resource – here the agent's information rent.[15]

Non-Neutrality. As a second illustration of Proposition 4, consider now the case of two principals. A unit tax on principal 1's use of the public good that is exactly offset by a unit subsidy on principal 2's use could have a real impact on the equilibrium allocation of public goods if this policy changes the set of active principals for some types. The fact that mean-preserving variations in the principals' preferences can have real impacts in the final allocation is reminiscent of findings in the public finance literature on voluntary contribution games (see, e.g., Bergstrom, Blume and Varian, 1986). This literature, which has focused on complete information games in which players' strategies are scalar contributions (as opposed to nonlinear schedules under asymmetric information as considered here), demonstrates that neutrality arises in simple public goods games precisely when the set of contributors is unaffected by a variation in preferences or incomes. When the set of contributors changes, however, the level of public good provision is typically altered. Similarly, we find in our richer asymmetric-information setting that the key source of non-neutrality is that an underlying variation can impact the set of active principals.

Intensive and Extensive Margins. To further unveil the nature of distortions in delegated common agency games, we consider now the simple case $n = 2$ with both principals being congruent, with $s_1 > s_2 > 0$. The optimality condition (7.4) becomes

$$\sum_{i=1}^{2} \max\left\{ s_i - \frac{F(\theta)}{f(\theta)}; 0 \right\} = \theta + C'(q^m(\theta)). \tag{7.6}$$

Together with $MHRC$, condition (7.3) implies that there always exists a unique interior solution $\hat{\theta}_i$ to

$$s_i = \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)} \quad i = 1, 2.$$

Observe that $\hat{\theta}_1 > \hat{\theta}_2$, so that the principal with the stronger preference has a larger activity set than the other. (i.e., $\left[\underline{\theta}, \hat{\theta}_2\right] \subset \left[\underline{\theta}, \hat{\theta}_1\right]$).

[15] Similar findings arise in the private common agency settings analyzed in Stole (1991), Martimort (1992) and Martimort and Stole (2009b) where the former papers consider intrinsic common agency games while the later discusses also the scenario of delegated common agency. When different principals control different activities undertaken by their agent which are complements, each of them extracts too much of the agent's information rent; inducing excessively low levels of activities.

The optimality condition (7.6) clearly shows how distortions manifest themselves along two dimensions. First, because each active principal contributes less than her marginal valuation, inefficient provision arises at the intensive margin. The equilibrium action is lower than the cooperative solution and, eventually, features the same two-fold distortion that is present in intrinsic common-agency games. This is the case when both principals are active, i.e., for $\theta \in \left[\underline{\theta}, \hat{\theta}_2\right]$. A second distortion, novel to delegated common agency games, emerges from limited participation by the weaker principal 2. The agent's action is now also distorted at the extensive margin.

Exclusive Contracting. Another interpretation of the limited participation that may arise under asymmetric information is that some form of exclusive contracting emerges endogenously even if exclusivity clauses cannot be enforced at the outset. This is so even if both principals would otherwise have contracted with the agent under complete information. This finding is reminiscent of an important insight developed by Bernheim and Whinston (1998) in their study of vertical relationships between manufacturers and retailers. They showed that exclusive dealing in marketing practices arises when the agency costs of a common representation are too large compared with those under exclusive dealing. There is, however, an important difference between their result and ours. They assume that the possibility of exclusive representation arises *ex ante*, i.e., before the realization of uncertainty. Although their general contracting model is thus consistent with hidden actions or hidden information, it cannot account for the possibility of exclusivity arising for some realizations of shocks and not for others. In this regard, our model, where contracting takes place *ex post*, i.e., once the agent is already informed, generates richer patterns of behavior.

Discontinuous Equilibria. To exhibit discontinuous equilibria in a simple form, consider a scenario with two principals having the same preferences, i.e., $s_1 = s_2 = s$ with $s > 1$. We take a quadratic form for the cost function, $C(q) = \frac{q^2}{2}$, and assume that θ is uniformly distributed over $[0, 1]$. Inserting into (7.4), the maximal allocation satisfies

$$q^m(\theta) = 2s - 3\theta. \tag{7.7}$$

Fix now $\theta_0 \in [0, 1]$ and take Δ such that $\theta_2 = \theta_0 + \frac{\Delta}{6} < \frac{2s}{3}$ and $\theta_1 = \theta_0 - \frac{\Delta}{6} > 0$. It is straightforward to check that conditions (6.2), (6.3) and (6.4) hold altogether for any such Δ. In other words, there are a continuum of discontinuous equilibria with a discontinuity at θ_0 and the downward jump discontinuity $\Delta = q^m(\theta_1) - q^m(\theta_2)$.

7.2 *Collective Action*

The analysis for Example 2 follows *mutatis mutandis* from Proposition 3. The only change being that the maximal allocation $q^m(\theta)$, when interior, now solves

$$S_0'(q^m(\theta)) + \sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i \in \mathcal{B}} \min\left\{s_i + \frac{1 - F(\theta)}{f(\theta)}; 0\right\} = \theta.^{16} \tag{7.8}$$

Several sharp economic insights emerge for this specific political economy context.

[16]Remember that, for the sake of Example 2, we assumed that S_0 is strictly concave, differentiable and that the stand-alone action $\overline{q}_0(\theta)$ satisfies (2.4).

The Free-Riding Problem. Consider the case in which n symmetric principals have the same marginal benefit $s_i = S/n$ $(i = 1, .., n)$ so that the aggregate principals' benefit Sq taken as a group is fixed independent of n. Using (7.5), it follows that an increase in the number of principals, holding S fixed, reduces collective action. Specifically, we have

$$S_0'(q_n^m(\theta)) + n \max\left\{\frac{S}{n} - \frac{F(\theta)}{f(\theta)}; 0\right\} = \theta.$$

For $n \to \infty$, $q_n^m(\theta)$ now converges pointwise towards the stand-alone action $\overline{q}_0(\theta)$. This asymptotic inefficiency result is reminiscent of a related result found in public good games by Mailath and Postlewaite (1990) but its source is quite different. In their setting, the agent's cost function is common knowledge, the agent's decision is binary, and each contributing principal has private information about his own willingness to pay. Their result follows because the probability that any contributing principal is pivotal goes to zero as the number of players increases. In contrast, in our setting inefficiency arises because each (uninformed) principal attempts to extract the (privately informed) agent's marginal rent, ignoring the externality she exerts on others when doing so. The per capita benefit $\frac{S}{n}$ of doing so vanishes as n increases; making it worthless for any principal to pay the agency cost necessary to influence the agent.

Excessive Polarization. Consider the following bare-bone model of budget allocation. The common agent (a legislator) allocates one unit of budget between two principals. Let $q \in \mathcal{Q} = [0, 1]$ denote the fraction captured by principal 1. Those principals have the same constant marginal benefit for money, $b > 1$ and their preferences are respectively expressed as $S_1(q) = bq$ and $S_2(q) = b(1 - q)$. Let now suppose that $S_0(q) = \frac{1}{2}q(1 - q)$. The parameter θ is uniformly distributed on $\Theta = \left[-\frac{1}{2}, \frac{1}{2}\right]$. A positive (resp. negative) value of θ thus means a bias towards principal 2 (resp. 1). To illustrate, the complete information collective surplus maximizing budget share for principal 1, which is also the agent's stand-alone action here, would be given by

$$q_0(\theta) = \frac{1}{2} - \theta.$$

The optimality condition (7.8) yields the following expression of the maximal allocation

$$q^m(\theta) = \min\left\{1; \max\left\{\frac{1}{2} - 3\theta; 0\right\}\right\}. \tag{7.9}$$

While the complete information benchmark offers a rather balanced distribution of budget across groups, the maximal equilibrium above is more sensitive to the decision-maker's preferences and may end up in extreme allocations with one group getting the entire budget while the other obtains nothing. For this much polarized allocation, each principal wants to influence types who are more inclined to grant favors while, for incentive compatibility reasons, she also eschews contributions to types less willing to do so.

8. Intrinsic versus Delegated Common Agency

When the agent must either accept of reject the entire set of the n offers, common agency is *intrinsic*. The set of equilibria for this simpler setting is explored in Martimort, Semenov and Stole (2018). Intrinsic common agency is the appropriate setting if the principals have some control of the agent's choice as in the case of public regulation by different government agencies. When common agency is intrinsic, the principals' activity sets always coincide, so the equilibrium analysis of these games avoids the difficulties in the present paper. Nonetheless, intrinsic common agency with public contracts provides an interesting comparison for the influence games in the current paper.

There are similarities but also quite noticeable differences between the intrinsic and delegate scenarios and these differences significantly complicate the analysis of the present paper. First, under intrinsic common agency, all principals consider the same participation constraint for the agent with the latter's sole outside option being now his stand-alone payoff:

$$U(\theta) \geq \overline{U}_0(\theta). \tag{8.1}$$

This feature of the problem simplifies by a tall order the analysis in comparison with the present paper. Because all principals consider the same participation constraint, they all agree on the identity of the worst type. All informational distortions due to their non-cooperative behavior thus go in the same direction. Provided that $S_i(q)$ is concave and differentiable for all $i \in \mathcal{N}$ and $\overline{q}_0(\theta)$ is interior, it can be shown that the aggregate virtual surplus can now be expressed as

$$\mathcal{V}^I(\theta, q) = \sum_{i=1}^{n} S_i(q) - \sum_{i=1}^{n} \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(\overline{q}_0(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q. \tag{8.2}$$

Martimort, Semenov and Stole (2018) have shown that all equilibria of intrinsic common agency games can also be expressed as optimization problems for a surrogate principal. This surrogate principal's optimization problem no longer has a fixed-point flavor because virtual valuations of each principal no longer depend on the equilibrium action but instead on a common stand-alone action.

Under intrinsic common agency, the surrogate principal's optimization problem no longer has a fixed-point flavor because virtual valuations of each principal no longer depend on the equilibrium action but instead on a common stand-alone action. In comparison with the cooperative scenario, this expression leads to an overall n-fold informational distortion whose consequences are studied in more details in Martimort, Semenov and Stole (2018).

Observe that, when evaluated at the stand-alone action, the marginal aggregate virtual surplus under intrinsic agency is

$$\mathcal{V}_q^I(\theta, \overline{q}_0(\theta)) = \sum_{i=1}^{n} \max\left\{S_i'(\overline{q}_0(\theta)) - \frac{F(\theta)}{f(\theta)}; \min\left\{S_i'(\overline{q}_0(\theta)) + \frac{1-F(\theta)}{f(\theta)}; 0\right\}\right\},$$

while, under delegated common agency, the marginal aggregate virtual surplus evaluated at the same point is

$$\overline{\mathcal{V}}_q(\theta, \overline{q}_0(\theta)) = \sum_{i=1}^{n} \max\left\{S_i'(\overline{q}_0(\theta)) - \frac{F(\theta)}{f(\theta)};\right.$$

$$\min\left\{S_i'(\overline{q}_0(\theta))+\frac{1-F(\theta)}{f(\theta)};S_i'(\overline{q}_0(\theta))-S_i'(\overline{q}(\theta))\right\}\Bigg\}.$$

Whenever the collective action of principals pushes the equilibrium action $\overline{q}(\theta)$ above the stand-alone action $\overline{q}_0(\theta)$, we have by concavity of S_i, $S_i'(\overline{q}_0(\theta)) \geq S_i'(\overline{q}(\theta))$. Hence, $\overline{\mathcal{V}}_q(\theta,\overline{q}_0(\theta)) \geq \mathcal{V}_q^I(\theta,\overline{q}_0(\theta))$ and all principals have greater incentives to expand output beyond that stand-alone action under delegated than under intrinsic common agency. The reverse holds when the equilibrium action $\overline{q}(\theta)$ lies below $\overline{q}_0(\theta)$.

Finally, and because all principals consider the same set of incentive-feasible allocations, intrinsic common agency games are bijective aggregate games in the vocabulary of Martimort and Stole (2012). Any incentive-compatible allocation can be achieved by a given principal provided that she undoes the aggregate offers made by her rivals, possibly with negative payments. This property aligns the preferences of principals who all achieve the same equilibrium net payoff. Under these circumstances, it is straightforward to demonstrate that the necessary conditions that pertain to a solution to the surrogate principal's problem are also sufficient. Our delegated common agency game is not bijective for the simple reason that a given principal might not be able to undo others' offers when restricted to offer positive payments. This also leads to a more careful construction of the kind of discontinuities that may arise in equilibrium.

Appendix: Proofs of Main Results

Proof of Theorem 1. Proposition 1 must hold for any equilibrium allocation. Adding up (3.7) across all n principals, we obtain the following condition:

$$\overline{q}(\theta)\in\arg\max_{q\in\mathcal{Q}}\ S(\theta,q)+\overline{\mathcal{V}}(\theta,q)+(n-1)(S(\theta,q)+\overline{T}(q)),\ \text{ a.e. }\theta \tag{A.1}$$

where $\overline{T}$ implements $(\overline{U},\overline{q})$. Simple revealed preference arguments show that $\overline{q}(\theta)$ is necessarily non-decreasing since $\overline{\mathcal{V}}(\theta,q)$ and $S(\theta,q)$ both have decreasing differences.

Define the value function for the above program as

$$\overline{V}(\theta)\equiv\max_{q\in\mathcal{Q}}\ S(\theta,q)+\overline{\mathcal{V}}(\theta,q)+(n-1)(S(\theta,q)+\overline{T}(q)). \tag{A.2}$$

Remember that $\overline{M}_i$, as a distribution function, has bounded variation. Therefore, $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ and thus $\overline{\mathcal{V}}(\theta,q)$ have also bounded variation. From that, and the fact that the above maximand is upper semi-continuous in q and $\mathcal{Q}$ is compact, it follows that $\overline{V}$ is itself absolutely continuous.[17] Given that $(\overline{U},\overline{q})$ is an incentive-compatible allocation which solves this program, we have

$$\overline{V}(\theta)=S(\theta,\overline{q}(\theta))+\overline{\mathcal{V}}(\theta,\overline{q}(\theta))+(n-1)\overline{U}(\theta). \tag{A.3}$$

Because $\overline{V}$ is absolutely continuous, it is a.e. differentiable. Applying the Envelope Theorem (Milgrom and Segal, 2002), we get

$$\dot{\overline{V}}(\theta)=\overline{\mathcal{V}}_\theta(\theta,\overline{q}(\theta))-n\overline{q}(\theta),\qquad\text{a.e.}$$

From absolute continuity, we then deduce the integral representation

$$\overline{V}(\theta)-\overline{V}(\theta')=\int_{\theta'}^{\theta}(\overline{\mathcal{V}}_\theta(x,\overline{q}(x))-n\overline{q}(x))dx\quad\forall(\theta,\theta')\in\Theta^2.$$

[17] Milgrom and Segal (2002).

Because $\overline{U}$ is also absolutely continuous, we thus have for any pair (θ, θ')

$$\overline{U}(\theta) - \overline{U}(\theta') = -\int_{\theta'}^{\theta} \overline{q}(x)dx$$

Note that

$$\left[S_0(\tilde{\theta}, \overline{q}(\tilde{\theta})) + \overline{\mathcal{V}}(\tilde{\theta}, \overline{q}(\tilde{\theta}))\right]_{\theta'}^{\theta} = \left[\overline{V}(\tilde{\theta}) - (n-1)\overline{U}(\tilde{\theta})\right]_{\theta'}^{\theta} = \int_{\theta'}^{\theta} (\mathcal{V}_\theta(x, \overline{q}(x)) - \overline{q}(x))dx. \tag{A.4}$$

Using the relationship

$$\left[S_0(\tilde{\theta}, \overline{q}(\theta')) + \mathcal{V}(\tilde{\theta}, \overline{q}(\theta'))\right]_{\theta'}^{\theta} = \int_{\theta'}^{\theta} (\mathcal{V}_\theta(x, \overline{q}(\theta')) - \overline{q}(\theta'))dx;$$

that $\mathcal{V}_i(\theta, q)[\overline{q}_{-i}]$ has decreasing differences and that $\overline{q}$ is non-increasing, we obtain:

$$\left[S_0(\theta, \overline{q}(\tilde{\theta}) + \mathcal{V}(\theta, \overline{q}(\tilde{\theta}))\right]_{\theta'}^{\theta} = \int_{\theta'}^{\theta} \int_{\overline{q}(\theta')}^{\overline{q}(x)} \left(\mathcal{V}_{\theta q}(x, \tilde{q}) - 1\right) d\tilde{q}dx \geq 0.$$

Because any $q' \in \overline{Q}(\Theta)$ can be identified with some $\theta' \in \theta$ such that $q' = \overline{q}(\theta')$, the inequality implies that $\overline{q}(\theta)$ satisfies (4.2) pointwise in θ.

By definition, the maximal allocation $q^m(\theta)$ defined as (5.1) also satisfies (4.2). Moreover, any putative equilibrium with range $\overline{q} = \overline{q}(\Theta)$ is such that $\overline{q} \subseteq q^m(\Theta)$. □

Proof of Theorem 2. When S_i is concave and differentiable and the maximal allocation is interior, the virtual surplus as defined $\mathcal{V}(\theta, q)[q^m]$ can be expressed as

$$\mathcal{V}_i^m(\theta, q) = S_i(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\} q. \tag{A.5}$$

The maximand in (5.1) is strictly concave when S_0 is strictly so and any interior solution $q^m(\theta)$ is thus given by the first-order condition

$$\sum_{i=0}^{n} S_i'(q^m(\theta)) = \theta + \sum_{i=1}^{n} \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}. \tag{A.6}$$

Decomposing for $i \in \mathcal{A}$ and $i \in \mathcal{B}$ yields (5.2). Because of strict concavity of S_0 and concavity of S_i,

$$S_0'(q) + \sum_{i \in \mathcal{A}} \max\left\{S_i'(q) - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i \in \mathcal{B}} \min\left\{S_i'(q) + \frac{1-F(\theta)}{f(\theta)}; 0\right\} \tag{A.7}$$

is a decreasing function of q and thus $q^m(\theta)$ as defined in (5.2) is unique. Because $MHRC$ holds, (A.7) is non-increasing in θ. Thus, $q^m(\theta)$ is itself non-increasing. Hence, q^m is a.e. differentiable and thus a.e. continuous. Moreover, it cannot have a jump discontinuity at any point since then (5.2) would have two solutions at this point. Hence, q^m is continuous. □

Proof of Theorem 3. To prove that the necessary conditions (5.1) satisfied by q^m are also sufficient, we construct individual tariffs that implement this allocation at equilibrium.

Preliminaries. For a maximal allocation, the marginal virtual surplus (3.5) relative to that allocation itself now writes as

$$\mathcal{V}_{iq}^m(\theta, q) = S_i'(q) - \min\left\{\frac{F(\theta)}{f(\theta)}; \max\left\{S_i'(q^m(\theta)), \frac{F(\theta)-1}{f(\theta)}\right\}\right\}. \tag{A.8}$$

Since $\mathcal{V}_{iq}^m(\theta,q)$ so defined is uniformly bounded in θ, $t_i^m(q)$ as defined in (5.3) or (5.4) is absolutely continuous and, in fact, differentiable at all $q \in \mathring{\mathcal{Q}}^m$ since $\vartheta(q)$ is single-valued under the conditions of the Theorem. Its derivative is

$$t_i^{m'}(q) = \mathcal{V}_{iq}^m(\vartheta^m(q), q). \tag{A.9}$$

Suppose now that $\mathring{\Omega}_i^m \neq \emptyset$. The definition (5.3) is independent of the choice $\hat{q}_i \in q_{-i}^m(\Omega_i^m)$. This comes from applying Proposition 1 for the allocation $\overline{q}_{-i} = q^m$ itself, together with the fact that $\mathcal{V}_{iq}^m(\theta, q^m(\theta)) = 0$ for all $\theta \in \Omega_i^m$. This condition can also be written as $\mathcal{V}_{iq}^m(\vartheta^m(q), q) = 0$ for all $q \in q^m(\Omega_i^m)$. Hence, for $(\hat{q}_i, \hat{q}_i') \in q^m(\Omega_i) \times q^m(\Omega_i)$, we have $\int_{\hat{q}_i}^{\hat{q}_i'} \mathcal{V}_{iq}^m(\vartheta^m(x), x)dx = 0$; so the result. Suppose instead that $\mathring{\Omega}_i^m = \emptyset$. The definition (5.4) again follows from Proposition 1 and, more specially, (3.9).

Non-Negative Tariffs. Observe that $\mathcal{V}_{iq}^m(\theta, q^m(\theta)) \geq 0$ (resp. $\leq$) for $\theta \in \Omega_i^{mc}$ and $i \in \mathcal{A}$ (resp. $i \in \mathcal{B}$).From this, it follows that $\mathcal{V}_{iq}^m(\vartheta^m(q), q) \geq 0$ (resp. $\leq$) for $q \geq \hat{q}_i$ (resp. $q \leq \hat{q}_i$) and thus, using (5.3) (resp. (5.4)) and the fact that $t_i^m(\hat{q}_i) = 0$ (resp. either $t_i^m(q^m(\overline{\theta})) \geq 0$ or $t_i^m(q^m(\overline{\theta})) \geq 0$) immediately gives us that $t_i^m(q)$ is non-negative.

Denote now the aggregate by $T^m = \sum_{i \in \mathcal{N}} t_i^m$ where $t_i^m(q)$ satisfies (5.3) or (5.4). What remains to be shown is (i) T^m induces the agent with type θ to choose $q^m(\theta)$, and (ii) each principal i, facing the rivals' aggregate T_{-i}^m, finds it optimal to implement $q^m(\theta)$ as well.

Incentive Compatibility. Consider the agent's problem when facing the aggregate payment T^m so constructed. Note that T^m is differentiable and its derivative is

$$T^{m'}(q) = \mathcal{V}_q^m(\vartheta^m(q), q). \tag{A.10}$$

Incentive compatibility requires

$$q^m(\theta) \in \arg\max_{q \in \mathcal{Q}} S(\theta, q) + T^m(q) \quad \forall \theta \in \Theta. \tag{A.11}$$

The necessary and sufficient condition for an interior maximum is thus

$$S_0'(q^m(\theta)) + \mathcal{V}_q^m(\vartheta^m(q^m(\theta)), q^m(\theta)) = \theta. \tag{A.12}$$

Observe that, for $q^m(\theta) \in \mathring{\mathcal{Q}}$, $\mathcal{V}_q^m(\vartheta^m(q^m(\theta)), q^m(\theta)) = \mathcal{V}_q^m(\theta, q^m(\theta))$ and (A.12) writes as (5.2). We now check that the necessary conditions (A.12) that define $q^m(\theta)$ are also sufficient for incentive compatibility. Consider now the rent profile

$$U^m(\theta) = \max_{q \in \mathcal{Q}^m} S(\theta, q) + T^m(q).$$

It is routine to prove that U^m so defined is absolutely continuous and admits the following integral representation

$$U^m(\theta) - U^m(\hat{\theta}) = \int_\theta^{\hat{\theta}} q^m(\tilde{\theta}) d\tilde{\theta} \tag{A.13}$$

where $q^m(\theta)$ satisfies (5.2). We thus rewrite the incentive compatibility conditions (A.11) as

$$U^m(\theta) \geq U^m(\hat{\theta}) + S(\theta, q^m(\hat{\theta})) - S(\hat{\theta}, q^m(\hat{\theta})) \quad \forall (\theta, \hat{\theta}) \in \Theta^2. \tag{A.14}$$

From the fact that $q^m(\theta)$ is non-increasing, (A.13) implies

$$U^m(\theta) - U^m(\hat{\theta}) \geq (\hat{\theta} - \theta) q^m(\hat{\theta}) = S_0(\theta, q^m(\hat{\theta})) - S_0(\hat{\theta}, q^m(\hat{\theta}))$$

and the incentive compatibility conditions (A.14) hold.

Principals' Optimality. Consider principal i's program. In light of Theorem 1 and Lemma B.1, we need to check that $q^m(\theta)$ is a best response allocation for principal i, i.e., it satisfies

$$q^m(\theta) \in \arg\max_{q\in\mathcal{Q}} S(\theta,q) + \mathcal{V}_i^m(\theta,q) + T_{-i}^m(q), \quad a.e. \tag{A.15}$$

where again T_{-i}^m is the aggregate for all principal except i obtained from individual tariffs satisfying (5.3)/(5.4). Mimicking what we did above for the agent's incentive compatibility problem, we write the corresponding necessary conditions for optimality as

$$S_0'(q^m(\theta)) + \mathcal{V}_{iq}^m(\theta, q^m(\theta)) + T_{-i}^{m'}(q^m(\theta)) = \theta. \tag{A.16}$$

Proceeding again as for the agent's incentive compatibility problem, it is straightforward to check that (A.16) again boils down to (5.2) as requested.

Turning now to sufficiency for the principals' optimality problem, let us define

$$V_i^m(\theta) = \max_{q\in\mathcal{Q}^m} S(\theta,q) + \mathcal{V}_i^m(\theta,q) + T_{-i}^m(q).$$

It is routine to prove that V_i^m is absolutely continuous with the following integral representation

$$V_i^m(\theta) - V_i^m(\hat\theta) = \int_\theta^{\hat\theta} (q^m(\tilde\theta) - \mathcal{V}_{i\theta}^m(\tilde\theta, q^m(\tilde\theta)))d\tilde\theta \quad (\theta,\hat\theta)\in\Theta^2. \tag{A.17}$$

where $q^m(\theta)$ satisfies (5.2). We may rewrite the incentive compatibility conditions (A.15) as

$$V_i^m(\theta) \geq V_i^m(\hat\theta) + S(\theta, q^m(\hat\theta)) + \mathcal{V}_i^m(\theta, q^m(\hat\theta)) - S(\hat\theta, q^m(\hat\theta)) - \mathcal{V}_i^m(\hat\theta, q^m(\hat\theta)) \quad \forall(\theta,\hat\theta)\in\Theta^2. \tag{A.18}$$

The fact that $q^m(\theta)$ is non-increasing and $\mathcal{V}_i^m$ has decreasing differences implies that

$$V_i^m(\theta) - V_i^m(\hat\theta) \geq (\theta-\hat\theta)q^m(\hat\theta) - (\mathcal{V}_i^m(\hat\theta, q^m(\hat\theta)) - \mathcal{V}_i^m(\theta, q^m(\hat\theta))).$$

It follows that the principal's optimality conditions (A.15) necessarily hold.

□

On our route to characterize discontinuous equilibria, next Theorem offers further necessary conditions for equilibrium allocations. As already noticed in the intrinsic common agency scenario by Martimort, Semenov and Stole (2018), the characterization of equilibrium allocations by means of the surrogate principal's incentive constraints (4.2) bears strong similarities with the characterization of implementable allocations found in the mechanism design literature on delegation as in Holmström (1984), Melumad and Shibano (1991), Martimort and Semenov (2006), Alonso and Matouschek (2008) and Amador and Bagwell (2013). Borrowing techniques that were developed in the aforementioned literature, the following theorem provides a sharp requirement for all equilibrium allocations.

Theorem A.1. *Suppose that $S_i(q)$ (resp. $S_0(q)$) is (resp. strictly) concave and differentiable for all $i\in\mathcal{N}$.*

1. *Any candidate equilibrium allocation $\overline{q}$ satisfying (6.1) is non-increasing.*
2. *At any point of differentiability, the following condition holds:*

$$\dot{\overline{q}}(\theta)\left(S_0'(\overline{q}(\theta)) - \theta + \mathcal{V}_q^m(\theta, \overline{q}(\theta))\right) = 0. \tag{A.19}$$

3. *At any isolated point of discontinuity, $\theta_0 \in (\underline{\theta}, \overline{\theta})$, bunching arises on both sides of θ_0 with*

$$\overline{q}(\theta) = \overline{q}^m(\theta_1) \quad \forall \theta \in [\theta_1, \theta_0) \text{ and } \overline{q}(\theta) = \overline{q}^m(\theta_2) \quad \forall \theta \in (\theta_0, \theta_2]$$

for some θ_1 and θ_2 such that $\theta_1 < \theta_0 < \theta_2$.[18] *The surrogate surplus is continuous at θ_0:*

$$\left[S_0(\theta_0, q) + \mathcal{V}^m(\theta_0, q)\right]_{\overline{q}^m(\theta_2)}^{\overline{q}^m(\theta_1)} = 0. \tag{A.20}$$

Proof of Theorem A.1 . That $\overline{q}$ is non-increasing follows from observing that both S_0 and $\mathcal{V}^m$ have decreasing differences. Thus, $\overline{q}$ is a.e. differentiable with a countable number of downward-jump discontinuities. Observing that, for any $q \in \overline{Q}$ there exists $\hat{\theta}$ such that $q = \overline{q}(\hat{\theta})$; the surrogate principal's incentive problem can be written as

$$\theta \in \arg\max_{\hat{\theta} \in \Theta} \; S(\theta, \overline{q}(\hat{\theta})) + \mathcal{V}^m(\theta, \overline{q}(\hat{\theta})) \quad \forall \theta \in \Theta.$$

The corresponding first-order necessary condition for optimality with respect to $\hat{\theta}$, at any point θ where $\overline{q}$ is differentiable, writes as (A.19).

Consider now the value function $\overline{V}$ as defined in (A.2). We already know that $\overline{V}$ is absolutely continuous. At any point of discontinuity θ_0 for $\overline{q}$, continuity of $\overline{V}$ still implies:

$$\lim_{\theta \to \theta_0^-} \overline{V}(\theta) = \lim_{\theta \to \theta_0^+} \overline{V}(\theta). \tag{A.21}$$

Consider a discontinuity at θ_0 which is isolated. On the right- and the left-neighborhoods of θ_0, (A.19) thus applies and either $\dot{\overline{q}}(\theta) = 0$ or $\overline{q}(\theta) = \overline{q}^m(\theta)$ defined as (5.2). Moreover, at a point at which $\overline{q}$ is continuous but not differentiable, it must be that either the right- or the left-derivative is zero. Taking stock of those remarks, we are now proving that bunching arises both on a right- and a left-neighborhood of θ_0. We proceed by contradiction. To this end, suppose first that bunching arises on the left-neighborhood only and call thus $\overline{q}(\theta_0^-) = \lim_{\theta \to \theta_0^-} \overline{q}(\theta)$ with $\overline{q}(\theta_0^-) > \overline{q}^m(\theta_0)$ because θ_0 must be a downward-jump discontinuity. Observe that $\overline{q}(\theta_0^-) = \overline{q}^m(\theta_1)$ for some type $\theta_1 < \theta_0$ such that $\theta_1 = \max\{\theta \text{ s.t. } q^m(\theta) \geq \overline{q}(\theta_0^-)\}$ and that $\overline{q}(\theta) = \overline{q}^m(\theta_1)$ for all $\theta \in [\theta_1, \theta_0)$.

Because the agent's information rent $\overline{U}$ is also absolutely continuous at θ_0, we also have:

$$\overline{U}(\theta_0) = \lim_{\theta \to \theta_0^-} S(\theta, \overline{q}(\theta)) + \overline{T}(\overline{q}(\theta)) = S(\theta_0, \overline{q}(\theta_0^-)) + \overline{T}(\overline{q}(\theta_0^-))$$

and

$$\overline{U}(\theta_0) = \lim_{\theta \to \theta_0^+} S(\theta, \overline{q}(\theta) + \overline{T}(\overline{q}(\theta)) = S(\theta_0, \overline{q}^m(\theta_0)) + \overline{T}(q^m(\theta_0)).$$

Therefore, we get:

$$S(\theta_0, \overline{q}(\theta_0^-)) + \overline{T}(\overline{q}(\theta_0^-)) = S(\theta_0, \overline{q}^m(\theta_0)) + \overline{T}(\overline{q}^m(\theta_0)).$$

Inserting this equality into (A.21), taking into account the definition (A.3) and simplifying yields:

$$\lim_{\theta \to \theta_0^-} S(\theta_0, \overline{q}(\theta)) + \mathcal{V}(\theta_0, \overline{q}(\theta)) = \lim_{\theta \to \theta_0^+} S(\theta_0, q^m(\theta)) + \mathcal{V}(\theta_0, q^m(\theta)).$$

[18] $\overline{q}$ can be made either right-continuous ($\overline{q}(\theta_0) = \overline{q}^m(\theta_2)$) or left-continuous ($\overline{q}(\theta_0) = \overline{q}^m(\theta_1)$) with, of course, no consequences on payoffs for any player.

Expressing those right- and left-hand side limits gives us:

$$S(\theta_0,\overline{q}(\theta_0^-))+\mathcal{V}^m(\theta_0,\overline{q}(\theta_0^-))=S(\theta_0,q^m(\theta_0))+\mathcal{V}^m(\theta_0,q^m(\theta_0)). \tag{A.22}$$

Because $S(\theta_0,q)+\mathcal{V}^m(\theta_0,q)$ is strictly concave in q, it has a unique maximizer $\overline{q}^m(\theta_0)$ that is supposed to be interior. Therefore, (A.22) necessarily implies that $\overline{q}(\theta_0^-)=\overline{q}^m(\theta_0)$. A contradiction with our starting premise that $\overline{q}(\theta_0^-)>\overline{q}^m(\theta_0)$ at the discontinuity θ_0. Similarly, we could also rule out the case where bunching only arises on the right-neighborhood of θ_0 at a value $\overline{q}(\theta_0^+)=\lim_{\theta\to\theta_0^+}\overline{q}(\theta)$. Taking stock of these findings, we necessarily have $\overline{q}(\theta_0^-)>\overline{q}(\theta_0^+)$ at a discontinuity point θ_0. Moreover, bunching arises on both sides of θ_0 which means $\overline{q}(\theta)=\overline{q}(\theta_0^-)$ (resp. $\overline{q}(\theta)=\overline{q}(\theta_0^+)$) for θ on this left- (resp. right-) neighborhood. Because $q^m(\theta)$ is strictly decreasing, there thus exist $\theta_1<\theta_0<\theta_2$ such that $\overline{q}(\theta_0^-)=\overline{q}^m(\theta_1)$ and $\overline{q}(\theta_0^+)=\overline{q}^m(\theta_2)$. In fact $\overline{q}(\theta)=\overline{q}^m(\theta_1)$ for all $\theta\in[\theta_1,\theta_0)$. Suppose not. Then, $\overline{q}$ would have a downward discontinuity at some $\theta_0'\in(\theta_1,\theta_0)$. The same argument as above shows that at any such putative discontinuity, we should have $\overline{q}(\theta_0^{'-})>q^m(\theta_0^{'})>\overline{q}(\theta_0^{'+})$ and $\overline{q}(\theta_0^{'+})\geq\overline{q}^m(\theta_0)$. Since $q^m(\theta)$ is decreasing, this contradicts the definition of $\theta_0^{'}$. Because the agent's rent $\overline{U}$ is continuous at θ_0, we also have:

$$\overline{U}(\theta_0)=\lim_{\theta\to\theta_0^-}S(\theta,\overline{q}(\theta))+\overline{T}(\overline{q}(\theta))=S(\theta_0,\overline{q}^m(\theta_1))+\overline{T}(\overline{q}^m(\theta_1))$$

and

$$\overline{U}(\theta_0)=\lim_{\theta\to\theta_0^+}S(\theta,\overline{q}(\theta))+\overline{T}(\overline{q}(\theta))=S(\theta_0,\overline{q}^m(\theta_2))+\overline{T}(\overline{q}^m(\theta_2)).$$

It follows that:

$$S(\theta_0,\overline{q}^m(\theta_1))+\overline{T}(\overline{q}^m(\theta_1))=S(\theta_0,\overline{q}^m(\theta_2))+\overline{T}(\overline{q}^m(\theta_2)).$$

Inserting into (A.21), taking into account (A.3) and simplifying yields:

$$\lim_{\theta\to\theta_0^-}S(\theta,\overline{q}(\theta))+\mathcal{V}(\theta,\overline{q}(\theta))=\lim_{\theta\to\theta_0^+}S(\theta,\overline{q}(\theta))+\mathcal{V}(\theta,\overline{q}(\theta))$$

or, expressing those right- and left-hand side limits,

$$S(\theta_0,\overline{q}^m(\theta_1))+\mathcal{V}(\theta_0,\overline{q}^m(\theta_1))=S(\theta_0,\overline{q}^m(\theta_2))+\mathcal{V}(\theta_0,\overline{q}^m(\theta_2))$$

which is (A.20). □

Proof of Theorem 4 . Observe that $\overline{q}$ as defined in (6.3) satisfies the necessary conditions (A.19) and (A.20). To prove that $\overline{q}$ is actually an equilibrium allocation, we construct individual tariffs that implement this allocation at equilibrium.

Preliminaries. Because $\mathcal{A}=\mathcal{N}$, the marginal virtual surplus (3.5) relative to the maximal allocation writes as

$$\mathcal{V}^m_{iq}(\theta,q)=S_i'(q)-\min\left\{\frac{F(\theta)}{f(\theta)};S_i'(q^m(\theta))\right\}\quad\forall i\in\mathcal{N} \tag{A.23}$$

When evaluated at the maximal allocation itself, this marginal virtual surplus becomes

$$\mathcal{V}^m_{iq}(\theta,q^m(\theta))=\max\left\{S_i'(q^m(\theta))-\frac{F(\theta)}{f(\theta)};0\right\}\quad\forall i\in\mathcal{N}. \tag{A.24}$$

An interior maximal allocation q^m is then defined as

$$S_0'(q^m(\theta)) + \sum_{i\in\mathcal{N}} \max\left\{S_i'(q^m(\theta)) - \frac{F(\theta)}{f(\theta)}; 0\right\} = \theta. \tag{A.25}$$

From Theorem 3, this maximal allocation is actually an equilibrium sustained with the non-negative tariffs $\bar{t}_i^m(q)$ as defined in (5.3) and/or (5.4).

Tariffs. Observe that the tariff $\bar{t}_i(q)$ as defined in (6.6) is non-negative, so is the aggregate payment $\overline{T}(q) = \sum_{i\in\mathcal{N}} \bar{t}_i(q)$.

By construction, we have

$$\left[\bar{t}_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[t_i^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \int_{q^m(\theta_2)}^{q^m(\theta_1)} \mathcal{V}_{iq}(\vartheta^m(q), q)dq.$$

Because (6.3) holds, (A.24) implies that

$$\left[t_i^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \int_{q^m(\theta_2)}^{q^m(\theta_1)} \left(S_i'(q) - \frac{F(\vartheta^m(q))}{f(\vartheta^m(q))}\right) dq = \left[S_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} - \int_{q^m(\theta_2)}^{q^m(\theta_1)} \frac{F(\vartheta^m(q))}{f(\vartheta^m(q))} dq$$

$$= \left[S_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} - \frac{F(\theta_0)}{f(\theta_0)}(q^m(\theta_1) - q^m(\theta_2))$$

where the last equality follows from (6.4).

Summing over $i \in \mathcal{N}$, we get

$$\left[\overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[\sum_{i\in\mathcal{N}} S_i(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} - n\frac{F(\theta_0)}{f(\theta_0)}(q^m(\theta_1) - q^m(\theta_2)). \tag{A.26}$$

Incentive Compatibility. Incentive compatibility can be expressed as

$$\bar{q}(\theta) \in \arg\max_{q\in\mathcal{Q}} S(\theta, q) + \overline{T}(q) = \arg \max_{q\in\mathcal{Q}^m/[q^m(\theta_2), q^m(\theta_1)]} S(\theta, q) + T^m(q) \quad \forall \theta \in \Theta. \tag{A.27}$$

For $\theta \in [\underline{\theta}, \theta_1] \bigcup \left[\theta_2, \overline{\theta}\right]$, the arg max above is of course achieved for $q^m(\theta)$ since, for such θ, we have $q^m(\theta) \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]$.

Consider now θ_0. Using (A.26), we observe that

$$\left[S(\theta_0, q) + \overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[S(\theta_0, q) + \mathcal{V}^m(\theta_0, q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0 \tag{A.28}$$

where the last equality follows from (6.2); which proves that an agent with type θ_0 is indifferent between choosing $q^m(\theta_2)$ or $q^m(\theta_1)$.

Consider now $\theta \in [\theta_1, \theta_0)$ (resp. $\theta \in (\theta_0, \theta_2]$). Because of increasing differences and (A.28), we thus have

$$\left[S(\theta, q) + \overline{T}(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} \geq 0. \tag{A.29}$$

Because $q^m(\theta)$ satisfies (A.25), S_i and S_0 are concave, we necessarily have

$$\left[S(\theta, q) + \mathcal{V}^m(\theta, q)\right]_{q^m(\theta_1)}^{q} \leq 0 \quad \forall q \geq q^m(\theta_1) \text{ and } \left[S(\theta, q) + \mathcal{V}^m(\theta, q)\right]_{q}^{q^m(\theta_2)} \leq 0 \quad \forall q \leq q^m(\theta_2). \tag{A.30}$$

Gathering (A.29) and (A.30) yields that $\bar{q}(\theta) = q^m(\theta_1)$ for $\theta \in [\theta_1, \theta_0)$.

The case $\theta \in (\theta_0, \theta_2]$ can be treated similarly to obtain that $\overline{q}(\theta) = q^m(\theta_2)$ for such θ. Finally, the output $\overline{q}(\theta)$ so obtained satisfies in (6.5).

By construction, the action profile $\overline{q}(\theta)$ is non-increasing. Following the same steps as in the Proof of Theorem 3, we can prove sufficiency.

PRINCIPALS' OPTIMALITY. Consider principal i's program. From Theorem 1 and Lemma B.1, we need to check that $\overline{q}(\theta)$ is a best response allocation for principal i, i.e., it satisfies

$$\overline{q}(\theta) \in \arg \max_{q \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]} S(\theta, q) + \mathcal{V}_i^m(\theta, q) + \overline{T}_{-i}(q), \quad a.e.$$

where again $\overline{T}_{-i}$ is the aggregate for all principals except i obtained from individual tariffs satisfying (6.6). Hence, we rewrite this optimality condition as

$$\overline{q}(\theta) \in \arg \max_{q \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]} S(\theta, q) + \mathcal{V}_i^m(\theta, q) + T_{-i}^m(q), \quad a.e. \tag{A.31}$$

For $\theta \in [\underline{\theta}, \theta_1] \bigcup \left[\theta_2, \overline{\theta}\right]$, the arg max above is of course achieved for $q^m(\theta)$ since, for such θ, we have $q^m(\theta) \in \mathcal{Q}^m / [q^m(\theta_2), q^m(\theta_1)]$.

Consider now θ_0. Using (A.26), we observe that

$$\left[S(\theta_0, q) + \mathcal{V}_i^m(\theta_0, q) + T_{-i}^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[S(\theta_0, q) + \mathcal{V}^m(\theta_0, q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = 0 \tag{A.32}$$

where the last equality again follows from (6.2); which proves that, at θ_0, principal i is indifferent between choosing $q^m(\theta_2)$ or $q^m(\theta_1)$.

Consider now $\theta \in [\theta_1, \theta_0)$ (resp. $\theta \in (\theta_0, \theta_2]$). Because of increasing differences and (A.32), we thus have

$$\left[S(\theta, q) + \mathcal{V}_i^m(\theta_0, q) + T_{-i}^m(q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} = \left[S(\theta_0, q) + \mathcal{V}^m(\theta_0, q)\right]_{q^m(\theta_2)}^{q^m(\theta_1)} \geq 0. \tag{A.33}$$

Because $q^m(\theta)$ satisfies (A.25), S_i and S_0 are concave, we necessarily have

$$\left[S(\theta, q) + \mathcal{V}^m(\theta, q)\right]_{q^m(\theta_1)}^{q} \leq 0 \quad \forall q \geq q^m(\theta_1) \text{ and } \left[S(\theta, q) + \mathcal{V}^m(\theta, q)\right]_{q}^{q^m(\theta_2)} \leq 0 \quad \forall q \leq q^m(\theta_2). \tag{A.34}$$

Gathering (A.33), and (A.34) yields that $\overline{q}(\theta) = q^m(\theta_1)$ for $\theta \in [\theta_1, \theta_0)$. The case $\theta \in (\theta_0, \theta_2]$ can be treated similarly to obtain that $\overline{q}(\theta) = q^m(\theta_2)$ for such θ. Finally, sufficiency for the principals' optimality problem can be proved as in the Proof of Theorem 3. □

PROOF OF PROPOSITION 3. PRELIMINARIES. From (7.3) and $MHRC$, there always exists a unique interior solution $\hat{\theta}_i$ to $s_i = \begin{cases} \frac{F(\hat{\theta}_i)}{f(\hat{\theta}_i)} & \text{if } i \in \mathcal{A}, \\ \frac{F(\hat{\theta}_i)-1}{f(\hat{\theta}_i)} & \text{if } i \in \mathcal{B}. \end{cases}$ From there, it follows that principal i's inactivity sets are non-empty and of the form $\Omega_i^m = \begin{cases} \left[\hat{\theta}_i, \overline{\theta}\right] & \text{if } i \in \mathcal{A}, \\ \left[\underline{\theta}, \hat{\theta}_i\right] & \text{if } i \in \mathcal{B}. \end{cases}$

CHARACTERIZATION. Inserting the expression of the virtual surplus (7.1) into (5.2), $q^m(\theta)$, when interior, satisfies (7.4). The condition for an interior solution (i.e., $q_m(\theta) \geq 0$) is that

$$\sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}; 0\right\} + \sum_{i \in \mathcal{B}} \min\left\{s_i + \frac{1 - F(\theta)}{f(\theta)}; 0\right\} \geq \theta + C'(0) \quad \forall \theta \in \Theta.$$

Because of $MHRC$, this condition holds when

$$\sum_{i \in \mathcal{A}} \max\left\{s_i - \frac{1}{f(\overline{\theta})}; 0\right\} + \sum_{i \in \mathcal{B}} s_i \geq \overline{\theta} + C'(0)$$

and a sufficient condition is then (7.2). Following the same steps as in Theorem 3, we thus define a set of transfers $t_i^m(q)$ as in (7.5). □

Proof of Proposition 4. Recall from (7.4) that

$$q^m(\theta) = \arg\max_{q \in \mathcal{Q}} S_0(q) - \theta q + \left(\sum_{i=1}^{n} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}\right) q$$

Because $\sum_{i=1}^{n} \max\left\{s_i - \frac{F(\theta)}{f(\theta)}, 0\right\}$ is convex in s_i, it is weakly higher under $\tilde{\mathbf{s}}$ compared to $\mathbf{s}$. Define $\hat{\theta}_i$ by $s_i f(\hat{\theta}_i) = F(\hat{\theta}_i)$ and define $\tilde{\theta}_i$ by $\tilde{s}_i f(\tilde{\theta}_i) = F(\tilde{\theta}_i)$. Choose i such that $s_i < \tilde{s}_i$, and thus $\hat{\theta}_i < \tilde{\theta}_i$. Then for any $\theta \in (\hat{\theta}_i, \tilde{\theta}_i)$, the *argmax* above is strictly higher under $\tilde{\mathbf{s}}$ compared to $\mathbf{s}$. Hence, the maximal allocation under $\tilde{\mathbf{s}}$ is weakly higher than that under $\mathbf{s}$ (and it is strictly higher for some types). □

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Online Appendix: Proof of Proposition 1

The proof of Proposition 1 proceeds in three steps. First, using a result in Martimort and Stole (2022), we provide a set of conditions that are necessary and sufficient for the solution to principal i's relaxed program (ignoring the convexity constraint on U). Second, we demonstrate the adjoint equations in these conditions can be further simplified given that the principal's preferences are linear in q. Third, we show that the solution to the relaxed and simplified program is a solution to the original program.

Step 0: Statement of the Problem. For the sake of completeness, we now briefly present Theorem 1 in Martimort and Stole (2022). This latter paper considers general control problems (beyond the class of principal-agent models) in which the state variable, u, is restricted to be an absolutely continuous function on the interval $\Theta = [\underline{\theta}, \overline{\theta}]$. Let $AC(\Theta, \mathbb{R})$ denote the set of such functions. In the present context, the state variable is the agent's information rent as a function of his type, absolute continuity then follows from incentive compatibility.[19] Martimort and Stole (2022) focus attention on problems in which that state variable must satisfy a non-negativity participation constraint constraint:

$$u(\theta) \geq 0 \quad \forall \theta \in \Theta. \tag{B.1}$$

When the state variable u is both absolutely continuous and non-negative, it is said *admissible*. We are interested in the following pure-state control program:

$$(\mathcal{P}) : \text{Maximize}_{u \in AC(\Theta, \mathbb{R})} \mathbb{R} \int_{\underline{\theta}}^{\overline{\theta}} \left(s(\theta, -\dot{u}(\theta)) - u(\theta)\right) f(\theta) d\theta \text{ s.t. (B.1).}$$

Step 1 below shows how this general formalism applies to our common agency context. Readers already familiar with the work of Jullien (2001) have certainly recognized the well-known framework developed with type-dependent participation constraints. The key novelty in Martimort and Stole (2022) is that similar results are obtained with substantially weaker assumptions on the primitive function s. In particular, $s(\theta, v)$ is not necessarily concave nor continuously differentiable. Accordingly, let $\overline{co}(s)(\theta, v)$ denote the v-concave envelope of $s(\theta, v)$. We denote the sup-differential of $\overline{co}(s)$ as $\partial_v \overline{co}(s)(\theta, v)$[20] Because $\overline{co}(s)$ is concave, it is a.e. differentiable (Rockafellar, 1997, Theorem 25.5.). Henceforth, the correspondence $\partial_v \overline{co}(s)$ is a.e. single-valued.

Theorem 1 in Martimort and Stole (2022) is the main result for this class of problems. Necessary and sufficient conditions are stated in terms of a probability measure which serves to express a complementary slackness condition (B.2) and a first-order optimality condition (B.4).

Theorem B.1. *(Martimort and Stole, 2022): $\overline{u}$ is a solution to program $(\mathcal{P})$ if and only if $\overline{u}$ is admissible and there exists a probability measure μ defined over the Borel subsets of Θ with an associated adjoint function, $\overline{M} : \Theta \to [0,1]$, defined by $\overline{M}(\underline{\theta}) = 0$ and*

$$\overline{M}(\theta) = \int_{[\underline{\theta}, \theta)} \mu(d\tilde{\theta}) \;\; \textit{for}\, \theta > \underline{\theta},$$

such that the following conditions are satisfied:

$$\int_{\underline{\theta}}^{\overline{\theta}} \overline{u}(\tilde{\theta}) \mu(d\tilde{\theta}) = 0, \tag{B.2}$$

[19] See Milgrom and Segal (2002).

[20] Remember that $\partial_v \overline{co}(s)(\theta, v) = \{p \text{ s.t. } \overline{co}(s)(\theta, w) \leq \overline{co}(s)(\theta, v) + p(w - v) \quad \forall w\}$.

$$\overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) = s(\theta, -\dot{\overline{u}}(\theta)) \; for \; a.e. \; \theta \in \Theta, \tag{B.3}$$

$$\overline{M}(\theta) \in F(\theta) - f(\theta)\partial_v \overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) \; for \; a.e. \; \theta \in \Theta. \tag{B.4}$$

Heuristic Proof. Before proceeding, it is useful to give an heuristic proof of this Theorem.[21] First, observe that the cone $u \geq \overline{u}$ with $v \in AC(\Theta, \mathbb{R})$ and $\dot{u}(\theta) = -v(\theta)$ a.e. defines a set of allocations which are admissible deviations. Second, for any such deviation, we necessarily have

$$\int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta)) d\mu(\theta) \geq 0$$

since μ is positive. Integrating by parts the left-hand side yields

$$0 \leq \left[(u(\theta) - \overline{u}(\theta))\overline{M}(\theta)\right]_{\underline{\theta}}^{\overline{\theta}} + \int_{\underline{\theta}}^{\overline{\theta}} \overline{M}(\theta)(v(\theta) - \overline{v}(\theta))d\theta = u(\overline{\theta}) - \overline{u}(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} M(\theta)(v(\theta) - \overline{v}(\theta))d\theta$$

where the last equality follows from $\overline{M}(\underline{\theta}) = 0$ and $\overline{M}(\overline{\theta}) = 1$. Similarly, another integration by parts yields

$$\int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta)) f(\theta) d\theta = u(\overline{\theta}) - \overline{u}(\overline{\theta}) + \int_{\underline{\theta}}^{\overline{\theta}} F(\theta)(v(\theta) - \overline{v}(\theta))d\theta$$

Inserting above yields

$$0 \leq \int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta)) f(\theta) d\theta + \int_{\underline{\theta}}^{\overline{\theta}} (\overline{M}(\theta) - F(\theta))(v(\theta) - \overline{v}(\theta))d\theta$$

It immediately follows from a simple convexity argument that, if $\overline{M}(\theta)$ satisfies (B.4),

$$0 \leq \int_{\underline{\theta}}^{\overline{\theta}} (u(\theta) - \overline{u}(\theta)) f(\theta) d\theta + \int_{\underline{\theta}}^{\overline{\theta}} f(\theta)(\overline{co}(s)(\theta, -\dot{\overline{u}}(\theta)) - \overline{co}(s)(\theta, -\dot{u}(\theta)))d\theta$$

or

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\overline{co}(s)(\theta, \overline{v}(\theta)) - \overline{u}(\theta)\right) f(\theta) d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \left(\overline{co}(s)(\theta, v(\theta)) - u(\theta)\right) f(\theta) d\theta$$

Using that $\overline{co}(s) \geq s$, the right-hand side above is greater than

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(s(\theta, v(\theta)) - u(\theta)\right) f(\theta) d\theta$$

for any admissible pair (u, v). Using (B.3) then yields

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(s(\theta, \overline{v}(\theta)) - \overline{u}(\theta)\right) f(\theta) d\theta \geq \int_{\underline{\theta}}^{\overline{\theta}} \left(s(\theta, v(\theta)) - u(\theta)\right) f(\theta) d\theta$$

i.e., $(\overline{u}, \overline{q})$ is an optimal allocation, as requested. □

To prepare for the rest of our analysis, it is also useful to consider the case where admissible profiles $u(\theta)$ are either monotonically increasing or decreasing respectively, and draw from this

[21] The complete proof is omitted for the sake of keeping the present paper at reasonable length.

assumption further properties for the adjoint function $\overline{M}(\theta)$. First, notice that the support of the probability measure μ, i.e., the set of points θ such that $\overline{u}(\theta)=0$, is necessarily non-empty, and closed, and thus either of the form $[\hat{\theta},\overline{\theta}]$ or of the form $[\underline{\theta},\hat{\theta}]$. From this observation, we get, for $\overline{u}$ non-increasing, that

$$\overline{M}(\theta)\begin{cases}\in F(\theta)-f(\theta)\partial_v\overline{co}(s)(\theta,0) & \text{a.e., if } \theta\in[\hat{\theta},\overline{\theta}],\\ =0 & \text{if } \theta\in[\underline{\theta},\hat{\theta}).\end{cases} \tag{B.5}$$

In the case where $\overline{u}$ is non-decreasing, we instead have

$$\overline{M}(\theta)\begin{cases}\in F(\theta)-f(\theta)\partial_v\overline{co}(s)(\theta,0) & \text{a.e, if } \theta\in[\underline{\theta},\hat{\theta})\\ =1 & \text{if } \theta\in[\hat{\theta},\overline{\theta}].\end{cases} \tag{B.6}$$

Some remarks are in order. First, remember that $\partial_v\overline{co}(s)(\theta,0)$ is a.e. single-valued so that $\overline{M}$ is a.e. defined without any ambiguity. At a point θ where $\partial_v\overline{co}(s)(\theta,0)$ is multivalued, $\overline{M}(\theta)$ is a selection within the correspondence $F(\theta)-f(\theta)\partial_v\overline{co}(s)(\theta,0)$. Second, $\overline{M}(\theta)$ must necessarily be non-decreasing. From this, we deduce that $F(\theta)-f(\theta)\partial_v\overline{co}(s)(\theta,0)$ is non-decreasing on $\mathit{supp}\{\mu\}$. This condition implicitly puts some restriction on the support $\mathit{supp}\{\mu\}$ where $\overline{u}(\theta)=0$. Third, when $\overline{u}$ is non-increasing and $\mathit{supp}\{\mu\}=\{\overline{\theta}\}$, μ puts mass one at $\overline{\theta}$. Similarly, when $\overline{u}$ is non-decreasing and $\mathit{supp}\{\mu\}=\{\underline{\theta}\}$, μ puts mass one at $\underline{\theta}$.

Step 1: The relaxed program. Let us now come back to our more specific optimization program $(\mathcal{P}_i^r)$. Because the domain of (U,q) is the set of incentive compatible, individually rational allocations, U is convex on a compact set and q is non-increasing. It follows that q is measurable, U is absolutely continuous and thus a.e. differentiable. The same applies to the pair $(\overline{U}_{-i},\overline{q}_{-i})$. As such, we may focus our attention on the domain $\mathcal{D}=\{(U,q)$ satisfying (3.1)-(3.2) with q non-increasing $\}$. Consider thus the relaxed program $(\mathcal{P}_i^r)$ taken over this set of admissible allocations, but that ignores the convexity constraint (3.3):

$$(\mathcal{P}_i^r):\quad \max_{(U,q)\in\mathcal{D}}\int_{\underline{\theta}}^{\overline{\theta}}\big(S_i(\theta,q(\theta))+S(\theta,q(\theta))+\overline{T}_{-i}(q(\theta))-U(\theta)\big)f(\theta)d\theta \text{ s.t. (3.1)-(3.2)}.$$

Note that, in the above description of $(\mathcal{P}_i^r)$, we have implicitly allowed principal i to resolve the agent's indifference in her favor if the agent's best-response set is multi-valued. Because incentive compatibility requires that the agent's indirect utility function is convex, and because a convex function has at most a countable number of kinks, the set of types who do not have a unique optimal choice is necessarily of measure zero. Thus, we may arbitrarily assign the agent's choice in case of indifference (i.e., we may take any selection satisfying (2.2)) without any impact on the best-responses of the players in (2.3). By the same token, (3.2) can be replaced by the requirement

$$\dot{U}(\theta)=-q(\theta),\qquad \text{a.e.} \tag{B.7}$$

without changing the value of the program $(\mathcal{P}_i^r)$.

We now rewrite $(\mathcal{P}_i^r)$ using a change of variables in order to get a more useful format amenable to applying Theorem B.1 above. Specifically, define the net utility that principal i's contract provides to the agent as $u_i=U-\overline{U}_{-i}$. We use u_i as the state variable and $v_i=q-\overline{q}_{-i}$ as the control variable in our new optimal control problem. It follows that (3.1) rewrites in this context as

$$u_i(\theta)\geq 0. \tag{B.8}$$

It also follows from (B.7) that $u_i(\theta)$ is absolutely continuous and a.e. differentiable with

$$\dot{u}_i(\theta) = -v_i(\theta) \quad \text{a.e..} \tag{B.9}$$

Now define principal i's incremental surplus as

$$s_i(\theta, v_i) = S_i(v_i + \overline{q}_{-i}(\theta)) - S_i(\overline{q}_{-i}(\theta)) + S(\theta, v_i + \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(v_i + \overline{q}_{-i}(\theta)) - \overline{U}_{-i}(\theta)$$

or, using the definition of $\overline{U}_{-i}(\theta)$ as $\overline{U}_{-i}(\theta) = S(\theta, \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(\overline{q}_{-i}(\theta))$,

$$s_i(\theta, v_i) = \left[S_i(x + \overline{q}_{-i}(\theta)) + S(\theta, x + \overline{q}_{-i}(\theta)) + \overline{T}_{-i}(x + \overline{q}_{-i}(\theta))\right]_0^{v_i}.^{22} \tag{B.10}$$

Using (B.9), we can now state principal i's *relaxed* program in terms of net payoffs in a form which is comparable to the generic form $(\mathcal{P})$ above as

$$(\mathcal{P}_i^r): \quad \max_{u_i \in AC(\Theta, \mathbb{R})} \int_{\underline{\theta}}^{\overline{\theta}} \left(s_i(\theta, -\dot{u}_i(\theta)) - u_i(\theta)\right) f(\theta) d\theta \text{ s.t. (B.8).}$$

We now apply Theorem B.1 and conclude that for any transfer $\overline{T}_{-i}$ offered by rival principals, the rent-output profile $(\overline{U}, \overline{q})$ is a solution to $(\mathcal{P}_i^r)$ if and only if $(\overline{u}_i, \overline{v}_i)$ satisfies (B.8) and (B.9) and there exists a probability measure μ_i defined over the Borel subsets of θ with an associated adjoint function, $\overline{M}_i : \theta \to [0,1]$, defined by $\overline{M}_i(\underline{\theta}) = 0$ and for $\theta > \underline{\theta}$,

$$\overline{M}_i(\theta) \equiv \int_{[\underline{\theta}, \theta)} \mu_i(d\theta),$$

such that the following two conditions are satisfied:

$$supp\{\mu_i\} \subseteq \{\theta \,|\, \overline{u}_i(\theta) = 0\}, \tag{B.11}$$

$$\overline{co}(s_i)(\theta, \overline{v}_i(\theta)) = s_i(\theta, \overline{v}_i(\theta)) \text{ for a.e. } \theta \in \Theta, \tag{B.12}$$

$$\overline{M}_i(\theta) \in F(\theta) - f(\theta)\partial_{v_i}\overline{co}(s_i)(\theta, \overline{v}_i(\theta)) \text{ for a.e. } \theta \in \Theta. \tag{B.13}$$

Step 2: Characterization of the adjoint function, $\overline{M}_i$, for monotonic common-agency games. We prove the following simplifying lemma that characterizes adjoint functions such that the optimality conditions (B.11)-(B.12)-(B.13) hold.

Lemma B.1. *Consider a monotonic common-agency game, and let $(\overline{q}, \overline{U})$ be an equilibrium allocation which solves $(\mathcal{P}_i^r)$ for each principal i. The adjoint function $\overline{M}_i$ for this problem satisfies first $\overline{M}_i(\underline{\theta}) = 0$ and second the following properties.*

1. *When $\overline{q}_{-i}(\theta)$ is interior*

$$\overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))\right\} & a.e. for\ i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta))\right\} & a.e.\ or\ i \in \mathcal{B}. \end{cases} \tag{B.14}$$

In particular, when S_i is concave and differentiable, (B.14) becomes

$$\overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)S_i'(\overline{q}_{-i}(\theta))\right\} & a.e.\ for\ i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)S_i'(\overline{q}_{-i}(\theta))\right\} & a.e.\ for\ i \in \mathcal{B}. \end{cases} \tag{B.15}$$

[22] Here, we use the notation $[f(x)]_{x_2}^{x_1} = f(x_1) - f(x_2)$.

2. *When $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$,*

$$\overline{M}_i(\theta) = \begin{cases} \max\left\{0, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta)\right\} & a.e. \text{ for } i \in \mathcal{A}, \\ \min\left\{1, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta)\right\} & a.e. \text{ for } i \in \mathcal{B}. \end{cases} \tag{B.16}$$

In particular, if $\overline{q}_{-i}(\theta) \equiv \overline{q}_0(\theta)$ on the boundary of $\mathcal{Q}$, (B.16) becomes

$$\overline{M}_i(\theta) = \begin{cases} \max\{0, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_0(\theta)) + \partial\overline{co}(S_0)(\overline{q}_0(\theta)) - \theta)\} & a.e. \text{ for } i \in \mathcal{A}, \\ \min\{1, F(\theta) - f(\theta)(\partial\overline{co}(S_i)(\overline{q}_0(\theta)) + \partial\overline{co}(S_0)(\overline{q}_0(\theta)) - \theta)\} & a.e. \text{ for } i \in \mathcal{B}. \end{cases} \tag{B.17}$$

Proof of Lemma B.1. For a monotonic common-agency game, $u_i = U - \overline{U}_{-i}$ is decreasing (resp. increasing) when $i \in \mathcal{A}$ (resp. when $i \in \mathcal{B}$) and the characterization of $\overline{M}_i$ follows from (B.5) and (B.6) respectively. We obtain, for $i \in \mathcal{A}$

$$\overline{M}_i(\theta) \begin{cases} \in F(\theta) - f(\theta)\partial_v\overline{co}(s_i)(\theta, 0) & \text{a.e., if } \theta \in supp\{\mu_i\} = [\hat{\theta}_i, \overline{\theta}], \\ = 0 \text{ if } \theta \in [\underline{\theta}, \hat{\theta}_i) \end{cases} \tag{B.18}$$

and, for $i \in \mathcal{B}$,

$$\overline{M}_i(\theta) \begin{cases} = 1 \text{ if } \theta \in (\hat{\theta}_i, \overline{\theta}], \\ \in F(\theta) - f(\theta)\partial_v\overline{co}(s_i)(\theta, 0) & \text{a.e., if } \theta \in supp\{\mu_i\} = [\underline{\theta}, \hat{\theta}_i]. \end{cases} \tag{B.19}$$

We can further refine this characterization. Denote the set of perturbations v_i which are admissible in the relaxed problem $(\mathcal{P}_i^r)$ as $\mathcal{D}_i = \left\{v_i \text{ s.t. } v_i + \overline{q}_{-i} \in \mathcal{Q}\right\}$. Observe that both sides of (B.10) are equal for $v_i = 0$, i.e., $s_i(\theta, 0) = 0$. Taking concave envelopes of both sides of (B.10) and using the fact that the concavification operator is subadditive yields, for all $v_i \in \mathcal{D}_i$,

$$\overline{co}(s_i)(\theta, v_i) \leq \overline{co}(S_i(v_i + \overline{q}_{-i}(\theta))) - S_i(\overline{q}_{-i}(\theta)) \tag{B.20}$$

$$+\overline{co}(\overline{T}_{-i}(v_i + \overline{q}_{-i}(\theta)) - \overline{T}_{-i}(\overline{q}_{-i}(\theta)) + S_0(v_i + \overline{q}_{-i}(\theta)) - S_0(\overline{q}_{-i}(\theta)) - \theta v_i).$$

Since, $s_i(\theta, 0) = \overline{co}(s_i)(\theta, 0) = 0$, we immediately deduce from the inequality (B.20) between two concave functions that take the same value at $v_i = 0$ the following inclusion for their sup-differentials,

$$\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) + \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta \subseteq \partial_{v_i}\overline{co}(s_i)(\theta, 0). \tag{B.21}$$

By definition, we have

$$\overline{q}_{-i}(\theta) \in \arg\max_{q \in \mathcal{Q}} \overline{T}_{-i}(q) + S_0(q) - \theta q. \tag{B.22}$$

We know distinguish two cases:

1. When $\overline{q}_{-i}(\theta)$ is an interior solution to the agent's problem, we have

$$0 \in \partial\overline{co}(\overline{T}_{-i} + S_0)(\overline{q}_{-i}(\theta)) - \theta. \tag{B.23}$$

Inserting into (B.21) implies that

$$\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) \subseteq \partial_{v_i}\overline{co}(s_i)(\theta, 0), \text{ a.e.} \tag{B.24}$$

Because $\partial_{v_i}\overline{co}(s_i)(\theta,0)$ is a.e. single valued, we thus have

$$\partial\overline{co}(S_i)(\overline{q}_{-i}(\theta)) = \partial_{v_i}\overline{co}(s_i)(\theta,0) \text{ a.e..} \tag{B.25}$$

Inserting into (B.18) (resp. (B.19)) gives (B.14).

Suppose now that S_i is concave and differentiable. Concavity implies $\overline{co}(S_i)=S_i$. Differentiability thus implies $\partial\overline{co}(S_i)=S_i'$. Observe that $\overline{q}_{-i}(\theta)$ is non-decreasing and thus a.e. differentiable and continuous. Therefore, $S_i'(\overline{q}_{-i}(\theta))$ is defined a.e.. [23] Then, (B.14) becomes (B.15).

2. When $\overline{q}_{-i}(\theta)$ lies on the boundary of $\mathcal{Q}$, i.e., $\overline{q}_{-i}(\theta)=q_{min}$ or $\overline{q}_{-i}(\theta)=q_{max}$, we directly insert (B.21) into (B.18) to get (B.16). In particular, if $\overline{q}_{-i}(\theta)\equiv\overline{q}_0(\theta)$ on the boundary of $\mathcal{Q}$, $\overline{T}_{-i}=0$ and (B.16) writes as (B.17).

□

Step 3: Transformation by means of $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$. From (B.13), we deduce

$$\overline{v}_i(\theta)\in\arg\max_v \overline{co}(s_i)(\theta,v)-\frac{F(\theta)-\overline{M}_i(\theta)}{f(\theta)}v.$$

From (B.12) and the fact that $\overline{co}(s_i)\geq s_i$, we can rewrite

$$\overline{v}_i(\theta)\in\arg\max_v s_i(\theta,v)-\frac{F(\theta)-\overline{M}_i(\theta)}{f(\theta)}v$$

or, expressed in terms of q,

$$\overline{q}(\theta)\in\arg\max_{q\in\mathcal{Q}}\left[S_i(x)+S_0(\theta,x)+\overline{T}_{-i}(x)-\frac{F(\theta)-\overline{M}_i(\theta)}{f(\theta)}x\right]_{\overline{q}_{-i}(\theta)}^{q}. \tag{B.26}$$

Consider thus $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ defined as

$$\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]=S_i(q)-\frac{F(\theta)-\overline{M}_i(\theta)}{f(\theta)}q \tag{B.27}$$

where $\overline{M}_i(\theta)$ is given by (B.14). The optimality condition (B.26) finally rewrites as (3.7).

Since $\overline{U}(\theta)=\overline{U}_{-i}(\theta)$ for $\theta\in\Omega_i$ and both $\overline{U}(\theta)$ and $\overline{U}_{-i}(\theta)$ are a.e. differentiable, we have $\overline{q}(\theta)=\overline{q}_{-i}(\theta))$ a.e. $\theta\in\text{int}\Omega_i$. Therefore, (3.8) follows.

Step 4: The solution to the relaxed program $(\mathcal{P}_i^r)$ is convex. Simple revealed preference arguments show that $\overline{q}(\theta)$ is necessarily non-decreasing since $\mathcal{V}_i(\theta,q)[\overline{q}_{-i}]$ and $S(\theta,q)$ both have decreasing differences. □

[23] At a point θ where $\overline{q}_{-i}(\theta)$ has a downward-jump discontinuity, we take the convention that $\overline{q}_{-i}(\theta)=\lim_{\tilde{\theta}\to\theta^+}\overline{q}_{-i}(\tilde{\theta})$.