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# A folk theorem for finitely repeated games with public monitoring

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**Abstract**

We adapt the methods from Abreu, Pearce and Stacchetti (1990) to finitely repeated games with imperfect public monitoring. Under a combination of (a slight strengthening of) the assumptions of Benoît and Krishna (1985) and those of Fudenberg, Levine and Maskin (1994), a folk theorem follows. Three counterexamples show that our assumptions are tight.

**Keywords:** Repeated games.

**JEL codes:** C72, C73

# 1 Introduction

The literature on finitely repeated games and discounted infinitely repeated games have proceeded somewhat independently in the last twenty years. Following Abreu, Pearce and Stacchetti (1990), tremendous progress has been accomplished in the analysis of infinitely repeated games with imperfect monitoring under discounting. Results in this literature have built on the fixed-point characterization that they give of the set of public perfect equilibrium payoffs, paving the way for a largely non-constructive characterization of the equilibrium payoff set, in particular, as discounting vanishes. See, among others, Fudenberg, Levine and Maskin (1994, hereafter FLM).

Clearly, no such fixed-point characterization exists in the case of finitely repeated games, as the (public perfect) equilibrium payoff set is not independent of the number of periods left.

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As a result, folk theorems for finitely repeated games have relied on explicit specifications of equilibrium strategies. This has led these authors to restrict attention to perfect monitoring. See Benoît and Krishna (1985), Gossner (1995) and Smith (1995), as well as Benoît and Krishna (1996) for a survey.[1]

Yet the (Bellman-Shapley) operator involved in the definition of the fixed point applies just as well to the case of a finite horizon, giving us an immediate link between the equilibrium payoff sets that obtain as the horizon length varies. Similarly, the main idea behind the proof of FLM applies as well. Suppose that the (average) equilibrium payoff set converges to a strict subset of the feasible, and individually rational payoff set $V$. Then there exists a direction in which this limit set has a boundary that is locally smooth (in an appropriately defined sense) yet bounded away from the extreme point of $V$ in that direction. Under the assumptions of FLM, then, a contradiction can be derived.

Some care must be taken in this argument, however. First, in the absence of discounting, the relative weights of current vs. continuation average payoffs in the definition of the operator are related to the number of periods left. Therefore, it is not exactly the same operator that is being applied repeatedly. As longer and longer horizons are considered, and as flow payoffs are assigned a vanishing weight, one must make sure that these weights do not decrease too fast in order for a contradiction to obtain.

More importantly, we show via example that combining the assumptions that are made for the folk theorem with an infinite horizon under imperfect public monitoring (FLM), with those made by Benoît and Krishna (1985) for finitely repeated games with perfect monitoring is not enough. Their assumption of distinct Nash payoffs in the stage game must be strengthened. Indeed, we strengthen this assumption by assuming that the convex hull of this set of vectors has non-empty interior. This assumption is added to the standard assumptions of pairwise full rank for some action profile, and admissibility of the minmax action profiles. Furthermore, we show that our assumptions cannot be relaxed, in the sense that we exhibit games that satisfy any pair of our three assumptions, yet for which the folk theorem does not hold.

While there has been few systematic analyses of finitely repeated games since Benoît and Krishna (1985) and Gossner (1995), very interesting examples have been produced by Mailath, Matthews and Sekiguchi (2002) to show how non-trivial equilibria can be constructed in finitely repeated with imperfect monitoring even as the stage game admits a unique Nash equilibrium.

[1] The equilibrium that Gossner defines is not entirely explicit, though, as his main innovation relies in applying approachability. However, the overall structure of the argument follows Benoît and Krishna.

Such equilibria involve private strategies, and so are not covered by our analysis. However, all that matters for our argument is that the finitely repeated game admits a set of equilibrium payoffs whose convex hull has non-empty interior for some horizon length. This length can then be treated as an "end-game" possibly involving private strategies, and treated as a "blackbox" when defining public strategies. Other related contributions involve Contou-Carrère and Tomala (2010) for the case of semi-standard monitoring, and Gonzàlez-Dìaz (2006) for the case of Nash equilibria. Standard results or definitions follow Mailath and Samuelson (2006).

## 2 A Simple Example

We start with an example in which the folk theorem holds with perfect monitoring and a finite horizon, as well as imperfect monitoring and an infinite horizon. Yet with imperfect monitoring and a finite horizon, the folk theorem fails. The game is illustrated in Figure **??**.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(1,1,0)_u$ | $(1,1,0)_u$ |
| $B$ | $(0,0,0)_v$ | $(-1,0,0)_w$ |

$W$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(1,1,0)_u$ | $(0,0,1)_u$ |
| $B$ | $(0,0,1)_v$ | $(0,0,1)_{u/2+v/2}$ |

$E$

Figure 1: An Example in which the Folk Theorem Fails.

Signals are denoted by $u$, $v$, $w$. Probabilities of signals are indicated in Figure **??**. For example, under $(B,R,E)$, signals $u$ and $v$ have both probability $1/2$.

Minmax payoffs are all 0, as is readily checked. The set of feasible and individually rational payoffs is

$$V := \text{conv}\{(0,0,0),(0,0,1),(1,1,0),(0,1/2,0)\},$$

a set with non-empty interior. The Nash equilibria of the stage game are $(B,R,E)$, $(T,L,W)$, $(T,L,E)$ and the convex combinations of $(T,L,W)$ and $(T,L,E)$. Hence the stage game has two Nash payoff vectors: $E_1 := \{(1,1,0),(0,0,1)\}$.

This implies that, for each player, there are two distinct Nash payoffs. It follows that the assumption of Benoît and Krishna (1985) is satisfied: under perfect monitoring, the equilibrium payoff would approach $V$ as the number of repetitions goes to infinity.

Let us return to imperfect monitoring, but consider an infinite horizon, and assume that players maximize the average discounted sum of rewards. Note that there exists a mixed action profile $\alpha$ that has pairwise full rank for each pair of players.[2] Hence, Condition 6.2 of FLM is satisfied. Furthermore, note that, for each player $i$, there exists a Nash equilibrium of the stage game achieving the minmax payoff for $i$. Hence, the minmax action profile is admissible. Hence, the standard assumptions for the infinite-horizon folk theorem to hold (Assumptions A2-A3 below) are satisfied, so that the set of average discounted (perfect public) equilibrium payoff vectors converges to $V$ as the discount factor tends to one.

Yet we claim that the folk theorem fails if we consider imperfect monitoring and a finite horizon. Denote by $E_n$ the set of public equilibrium payoffs of the $n$-stage game and by $\bar{E}$ the convex hull of $E_1$, that is,

$$\bar{E} = \{(\lambda, \lambda, 1-\lambda), \lambda \in [0,1]\}.$$

Indeed:

**Lemma 1** *It holds that*

$$\lim_n E_n = \overline{\bigcup_{n \geq 1} E_n} = \bar{E}.$$

**Proof.** We only need to show that $E_n \subset \bar{E}$ for each $n$. We proceed by induction.

Consider a perfect public equilibrium of the $n+1$-stage game, and denote by $x$ (resp., $y$, $z$) the probabilities that at the first stage player 1 (P1) plays $T$ (resp., player 2 (P2) plays $L$, player 3 (P3) plays $W$). Write $\alpha = (1-x)(1-y)z$, $\beta = (1-x)y$ and $\gamma = (1-x)(1-y)(1-z)/2$. The law of the public signal of stage 1 is $s = (x+\gamma)u + (\beta+\gamma)v + \alpha w$, and we have $x + \beta + 2\gamma + \alpha = 1$. By the induction hypothesis, the continuation payoffs are assumed to be in $\bar{E}$ and are denoted $(g(u), g(u), 1-g(u))$, $(g(v), g(v), 1-g(v))$ and $(g(w), g(w), 1-g(w))$.

First, let us provide some intuition. If the profile $(B, R, W)$, which has individual full rank but not pairwise full rank, is played at stage 1, the deviation of P3 induces the distribution of signals $(u+v)/2$, the deviation of P1 induces $u$ and the deviation of P2 induces $v$. Since $(u+v)/2$ lies in the convex hull of $u$ and $v$, the continuation strategies would have to punish both P3, and a fictitious player with payoff the average of P1 and P2's payoffs, and this turns out to be

[2] To see this, simply consider the mixed action profile where each player equally randomizes between his two actions. The induced law on signals is $s = \frac{9}{16}u + \frac{5}{16}v + \frac{2}{16}w$. With a unilateral deviation, player 1 induce laws over signals in the segment $[u, \frac{1}{8}u + \frac{5}{8}v + \frac{2}{8}w]$, player 2 in the segment $[\frac{1}{2}u + \frac{1}{2}v, \frac{5}{8}u + \frac{1}{8}v + \frac{2}{8}w]$, and player 3 in the segment $[\frac{1}{2}u + \frac{1}{4}v + \frac{1}{4}w, \frac{5}{8}u + \frac{3}{8}v]$. Since these segments only intersect at $s$, the mixed profile has pairwise full rank for any pair of players.

impossible.

We now formalize this idea. Assume that $\alpha > 0$, that is, at the first stage $(B, R, W)$ is played with positive probability. For the deviation by P1 playing $T$ at stage 1 to be unprofitable, it is necessary that $g(u) < g(s) \stackrel{def}{=} (x+\gamma)\ g(u) + (\beta+\gamma)\ g(v) + \alpha\ g(w)$. This implies

$$g(u) < \frac{(\beta+\gamma)g(v) + \alpha g(w)}{\beta+\gamma+\alpha}.$$

Similarly, the deviation by P3 playing $E$ at stage 1 should not be profitable, that is,

$$g(w) < (g(u) + g(v))/2.$$

And the deviation by P2 playing $L$ at stage 1 should not be profitable either, that is, it holds that $xg(u) + (1-x)g(v) \leq g(s)$, which gives

$$(\gamma+\alpha)g(v) \leq \gamma g(u) + \alpha g(w).$$

The three displayed inequalities are not compatible, hence we must have $\alpha = 0$.

Assume now that $(1-x)yz > 0$. If $y = 1$, P3 has a profitable (undetectable) deviation by playing $E$ at stage 1. So $y < 1$, and $\alpha > 0$. But as we have shown, this is impossible.

As a consequence, we obtain $(1-x)z = 0$, and the induced equilibrium payoff is in $\bar{E}$. ■

It follows from the example that for the folk theorem to hold with imperfect public monitoring and a finite horizon, it is not enough to combine the usual assumptions invoked for the folk theorem under both perfect monitoring and a finite horizon, and imperfect monitoring and infinite horizon. More precisely, combining the assumptions of Benoît and Krishna (1985) with those of FLM does not suffice.

However, we will show that if we strengthen the assumption of distinct payoffs (Benoît and Krishna, 1985) by requiring the convex hull of stage-game Nash equilibrium payoff vectors to have nonempty interior, the stronger assumptions that are usually made to guarantee the folk theorem under public monitoring –individual full rank and pairwise full rank for all pure action profiles, as well as the minmax profiles– do suffice to get the familiar conclusion.

# 3 Notation and Assumptions

We consider finitely repeated games between $I$ players. Notations mostly follow Mailath and Samuelson (2006). Actions sets are $A_i$, finite, with generic element $a_i$, and given action profile $a$, there is a public signal $y$ from a finite set $Y$ that is publicly observed. The distribution of signals given $a$ is denoted $\pi(\cdot|a)$. Rewards in the stage game are given by $g_i(a)$ for player $i$, given action profile $a$. The function $g(\cdot)$ is extended to $\Delta(A)$ in the obvious way. Action profiles are not observed, nor are realized payoffs. Let $F$ denote the set of feasible payoffs, $V$ the set of feasible and individually rational payoffs (where individual rationality is, as usual, defined with respect to the mixed minmax payoff, in which players $-i$ randomize independently).

We let $E_n \subset V$ denote the set of average equilibrium payoffs in the game that is repeated $n$ times (without discounting). We do not assume a public randomization device just yet. Throughout, equilibrium refers to public perfect equilibrium, or PPE (see, for instance, FLM). It is known that $E_n$ converges in the Hausdorff sense to the closure of $\cup_n E_n$ (see Renault and Tomala, 2011), although the rate of convergence is unknown. We denote by $E$ the limit of this set, which is convex (as opposed to $E_n$, which typically is not). We write $E_\delta$ for the set of $\delta$-discounted PPE payoff vectors in the infinitely repeated game.

We recall the following elementary result, which establishes that any equilibrium payoff of the finitely repeated game, no matter how long the horizon, is necessarily an equilibrium payoff of the infinitely repeated game with low enough discounting (and the same monitoring structure). As the proof makes clear, this result holds whether or not a public randomization device is assumed.

**Lemma 2** *For every $n$, it holds that*

$$E_n \subseteq \lim_{\delta \to 1} E_\delta.$$

**Proof.** As is well known, $\lim_{\delta\to 1} E_\delta$ is the same whether or not a public randomization device is assumed (see Fudenberg, Levine and Takahashi, 2007), and for all $\delta' < 1$, $E_{\delta'} \subseteq \lim_{\delta\to 1} E_\delta$. On the other hand, $E_n$ is weakly larger with a public randomization device than without, hence the result holds if it can be shown by assuming a public randomization device. In that case, $E_n$ is convex. The result then follows from the monotonicity of the Bellman-Shapley operator $B_\delta$ in $\delta$ (see Abreu, Pearce and Stacchetti, 1990, Theorem 6). The proof is by induction. Clearly, $E_1 = E_\delta|_{\delta=0} \subset \lim_{\delta\to 1} E_\delta$. Fix $\delta' > \frac{n}{n+1}$ such that $E_n \subseteq E_{\delta'}$. Then $E_{n+1} = B_{\frac{n}{n+1}}(E_n) \subseteq B_{\frac{n}{n+1}}(E_{\delta'}) \subseteq B_{\delta'}(E_{\delta'}) = E_{\delta'}$ and so $E_{n+1} \subseteq \lim_{\delta\to 1} E_\delta$. ■

The proof of Lemma **??** already takes advantage of the recursive structure of $E_n$. Namely, following Abreu, Pearce and Stacchetti (1990), define the operator $B$ as follows. Let $\mathcal{B}(\mathbb{R}^I)$ denote the set of subsets of $\mathbb{R}^I$, and, given some payoff function $u : A \to \mathbb{R}^I$, let $NE(u) \subset \mathbb{R}^I$ denote the set of Nash equilibrium payoff vectors of the game with payoff function $u$. We then define

$$\begin{aligned} B \colon \mathcal{B}(\mathbb{R}^I) &\to \mathcal{B}(\mathbb{R}^I) \\ W &\mapsto \left\{ NE\left( g + \sum\nolimits_{y \in Y} \pi(y \mid a) x(y) \right), \text{ some } x : Y \to W \right\}. \end{aligned}$$

It follows that, $E_1 = B(\{0\})$ and, more generally,

$$(n+1)E_{n+1} = B(nE_n).$$

This recursive characterization of the sets $E_n$ is extensively used in the sequel. An immediate by-product (using Kandori, 1992) is that the convex hull of $E_n$ is contained in the convex hull of the set of subgame-perfect Nash equilibria of the $n$-times repeated game under perfect monitoring. Hence, the conditions for the folk theorem (in PPE) to hold for finitely repeated games with imperfect monitoring are necessarily at least as strong as those for finitely repeated games with perfect monitoring, as well as those for the infinitely repeated game with imperfect public monitoring and vanishing discounting (by Lemma **??**).

We now turn to these assumptions. Clearly, $E_1 \subset E$. Hence, a sufficient condition that guarantees that $E$ has non-empty interior is to assume that the convex hull of $E_1$ has non-empty interior, *i.e.*, there exists distinct Nash equilibrium payoff vectors whose convex hull has non-empty interior. This is a strengthening of the assumption of Benoît and Krishna (1985). Our proofs rely on it.

**Assumption 1 (A1)** *The convex hull of the set $E_1$ has non-empty interior.*

As discussed in the introduction, this can be weakened to the assumption that the convex hull of the set $E_n$ has non-empty interior, for some $n$ (encompassing thereby games that might fail the assumption of Benoît and Krishna (1985), but satisfy Smith's (1995) assumption of recursively distinct Nash payoffs). This can be even further weakened by assuming that the convex hull of the set of sequential equilibrium payoffs (in private strategies) of the $n$-times repeated game has non-empty interior. (Of course, the resulting folk theorem would be in private strategies.)

Further, we introduce the following standard assumptions. Define the $|A_i| \times |Y|$-matrix

$$\Pi_i(\alpha) = (\pi(y|a_i, \alpha_{-i}))_{a_i,y}.$$

The profile $\alpha$ has pairwise full rank for $i$ and $j$ if the $(|A_i| + |A_j|) \times |Y|$-matrix

$$\Pi_{ij}(\alpha) = \begin{pmatrix} \Pi_i(\alpha) \\ \Pi_j(\alpha) \end{pmatrix}$$

has rank $|A_i| + |A_j| - 1$ (*i.e.*, maximal rank).

**Assumption 2 (A2, Pairwise Full Rank)** *For each pair of players, there exists a (possibly mixed) action profile $\alpha$ that satisfies pairwise full rank for this pair.*

As shown by Fudenberg, Levine and Maskin (1994, Lemma 6.2), this assumption implies that there exists an open and dense subset of action profiles that satisfy pairwise full rank for all pairs of players.

Player $i$'s minmax payoff is denoted $u^i$, and the corresponding (possibly mixed, but uncorrelated) action profile $m^i$. The following assumption is due to Kandori and Matsushima (1998).

**Assumption 3 (A3, Admissibility)** *For each $i$, there is a minmax profile $m^i$ such that, for all $j \neq i$, if there exists $\alpha_j \in \Delta(A_j)$ such that $p(\cdot \mid m^i_{-j}, \alpha_j) = p(\cdot \mid m^i)$, then $g_j(m^i) \geq g_j(m^i_{-j}, \alpha_j)$.*

Assumption A3 is satisfied, for instance, if the minmax profile is a Nash equilibrium of the stage game, as in the simple example of Section 2. Assumptions A1–A3 are sufficient for the folk theorem for infinitely repeated games to hold. (Specifically, see Fudenberg, Levine and Maskin, Theorem 6.2, with the modification in coordinate directions called for by Assumption A3, see Kandori and Matsushima.)

# 4 A Folk Theorem with a Randomization Device

In this section, we assume a public randomization device. This drastically simplifies arguments, as it guarantees that $E_n$ is convex, for all $n$.

**Theorem 4** *Assume* A1–A3. Then $E = V$, the set of feasible and individually rational payoffs.

**Proof.** Recall that $E$ is convex. Therefore, by a theorem due to Alexandrov, almost all its boundary points are normal, *i.e.*, the representing function is differentiable at these points. Therefore, if $E \neq V$, there exists a vector $\lambda \in S^1 := \{x \in \mathbb{R}^I : \|x\| = 1\}$ and a point $v \in bd(E)$ with (unique) normal vector $\lambda$, and such that $\max_{v' \in V} \lambda \cdot (v' - v) = 3\kappa > 0$.

Assume first that $\lambda$ is not a coordinate direction, and let $l$ be the line through $v$ with direction $\lambda$. Let $\alpha \in \Delta(A)$ refers to an action profile in the open dense subset of action profiles satisfying pairwise full rank guaranteed by Assumption A2) such that $\lambda \cdot g(\alpha) \geq \lambda \cdot v + 2\kappa$ (fix one of them if multiple exist). Let $\bar{v} \in \mathbb{R}^I$ denote the unique point of $l$ such that $\lambda \cdot \bar{v} = \lambda \cdot g(\alpha)$. Clearly, $\bar{v} \notin E$. For all $k \in \mathbb{R}$, let $H_\lambda(k) := \{x \in \mathbb{R}^I : \lambda \cdot x = k\}$, and $H_\lambda^+(k) := \{x \in \mathbb{R}^I : \lambda \cdot x \geq k\}$.

The pairwise full rank assumption at the action profile $\alpha$ ensures that there exists $x : Y \to \mathbb{R}^I$ such that $\alpha$ is a Nash equilibrium of the game with payoff function

$$g(\cdot) + \sum \pi(y|\cdot)\, x(y),$$

and $\lambda \cdot x(y) = 0$ for all $y \in Y$ (See Mailath and Samuelson, 2006, Lemma 8.1.1(4) and 9.2.2). Without loss of generality, we may assume that the payoff from this equilibrium is equal to $\bar{v}$ (redefine each $x(y)$ as $x(y) + \bar{v} - g(\alpha) - \sum_{y \in Y} \pi(y|\alpha)x(y)$). Let $M = \max_y \|x(y)\|$, and set $\kappa_0 := \frac{\kappa}{2\sqrt{\kappa^2 + M^2}}$.

Given $z \in \mathbb{R}^I$, define the convex cone:

$$C_z := \{z' \in \mathbb{R}^I : \lambda \cdot (z - z') > \kappa_0 \|z - z'\|\}.$$

See Figure **??** (here, $\alpha = a$ is pure). Since $A1$ holds, $E$ is the closure of its interior. Because $E$ is smooth at $v$, there exists $k < \lambda \cdot v$ and a compact set $D \subset \mathbb{R}^I$ such that

$$H_\lambda(k) \cap C_v \subset D \subset int(E).$$

Because $(E_n)_n$ is a sequence of convex sets converging to $E$, we can assume that $D \subset E_n$ for $n$ large. Let $v_n = \arg\max_{x \in l \cap E_n} \lambda \cdot x$ be the highest point of $E_n$ on the line $l$, and set $k_n = \lambda \cdot v_n$. We restrict attention to $n \geq n_0$ such that $k_n - \kappa/n > k$. This implies that $C_{v_n} \cap H_\lambda^+(k) \subset E_n$, because $C_{v_n} \subset C_v$, $v_n$ and $D$ are in $E_n$, and $E_n$ is convex.

Given $v_n$, let

$$w_n(y) := v_n + \frac{1}{n}x(y) - \frac{\kappa}{n}\lambda,$$

for all $y \in Y$. Note that

$$\frac{\lambda \cdot (v_n - w_n(y))}{\|v_n - w_n(y)\|} \geq \frac{\kappa}{\sqrt{\kappa^2 + M^2}} > \kappa_0,$$

so that $w_n(y) \in C_{v_n}$ and hence $w_n(y) \in E_n$ (note that $\lambda \cdot w_n(y) = k_n - \kappa/n > k$).

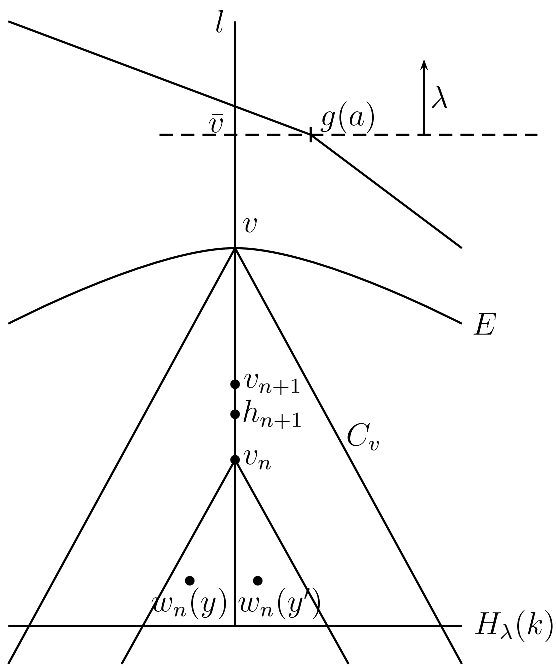


Figure 2: Proof of Theorem **??**

Note that the action profile $\alpha$ is a Nash equilibrium of the game with payoff vector

$$\begin{aligned} & \frac{1}{n+1} g(\cdot) + \frac{n}{n+1} \sum \pi(y|\cdot) w_n(y) \\ = \; & \frac{1}{n+1} \Big( g(\cdot) + \sum \pi(y|\cdot) x(y) \Big) + \frac{n}{n+1} v_n - \frac{\kappa \lambda}{n+1}. \end{aligned}$$

Because $w_n \in E_n$, this implies that the resulting equilibrium payoff vector $h_{n+1}$ is in $E_{n+1}$. Observe that

$$\lambda \cdot (h_{n+1} - v_n) = \frac{1}{n+1} \lambda \cdot (g(\alpha) - v_n) - \frac{\kappa}{n+1} \geq \frac{\kappa}{n+1}.$$

Furthermore, observe that, by construction, $h_{n+1}$ is on the line $l$ (recall that $g(\alpha) + \sum \pi(y|\alpha) x(y) =$

$\bar{v} \in l$). Hence,

$$k_{n+1} \geq \lambda \cdot h_{n+1} \geq \lambda \cdot v_n + \frac{\kappa}{n+1} = k_n + \frac{\kappa}{n+1}.$$

This is not possible, as it implies that $k_n \to \infty$.

Consider the case in which $\lambda$ is a coordinate direction, *i.e.*, $\lambda = \pm e^i$, for some basis vector $e^i$. Then it is also the case, by Assumption A2 and A3 (see Mailath and Samuelson, 2006, Lemma 9.2.1(1) and (3)) that there exists $x(y)$ such that $\lambda \cdot x(y) = 0$ for all $y$, and such that the action profile $a$ (resp., $m^i$) that maximizes (or is arbitrarily close to maximize, in case $a$ fails pairwise full rank) $g_i(a)$ (resp., minmaxes player $i$) is an equilibrium of the game with payoffs $g(\cdot) + \sum_{y \in Y} \pi(y|\cdot)x(y)$. Whether $\lambda = e^i$ or $\lambda = -e^i$, the remainder of the proof is identical to the previous one. ■

# 5 Discussion

The example in Section 2 satisfies A2 and A3, but fails A1, establishing its necessity for the folk theorem. While A2 and A3 are more standard, there is no simple example we are aware of that would establish their necessity, even in the context of infinitely repeated games (and even without assuming A1). The next example, Example 2, has some similarities with Example 3.1 in Renault and Tomala (2004). Here A1 and A3 are satisfied, but A2 is not and the folk theorem fails.

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(0,0,1)$ | $(0,3,0)$ |
| $B$ | $(3,0,0)$ | $(2,2,0)$ |

$W$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(0,4,0)$ | $(0,4,0)$ |
| $B$ | $(0,4,0)$ | $(0,4,0)$ |

$M_1$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(4,0,0)$ | $(4,0,0)$ |
| $B$ | $(4,0,0)$ | $(4,0,0)$ |

$M_2$

| | $L$ | $R$ |
|---|---|---|
| $T$ | $(1,1,2)$ | $(1,1,0)$ |
| $B$ | $(1,1,0)$ | $(1,1,0)$ |

$E$

Figure 3: Example 2, that satisfies A1 and A3, but not A2.

This is a 3-player game where $A_1 = \{T, B\}$, $A_2 = \{L, R\}$ and $A_3 = \{W, M_1, M_2, E\}$. The public signal consists of the action and payoff of player 3. $E_1$ contains $(0,4,0)$, $(4,0,0)$, $(1,1,0)$ and $(1,1,2)$, hence the full dimensionality assumption A1 is satisfied. All minmax payoffs are 0, and can be obtained with Nash equilibria of the stage game, so admissibility (A3) also holds. The profile $(T, L, W)$ has individual full rank, but there is no action profile in $\Delta(A)$ having pairwise full rank for players 1 and 2, so A2 is not satisfied here. We claim that the Folk theorem does not hold.

**Lemma.** $E \subset \{x = (x_i)_i \in \mathbb{R}^3, x_1 + x_2 \geq 2\}$.

**Proof.** Let $\sigma = (\sigma_i)_i$ be an equilibrium of the $n$-stage game, with payoff $(\gamma_i(\sigma))_i$ in $\mathbb{R}^3$. Consider the deviation $\tau_1$, resp. $\tau_2$, of player 1, resp. 2, which plays $B$, resp. $R$ at every stage independently of the past. The point is that both profiles $(\tau_1, \sigma_{-1})$ and $(\tau_2, \sigma_{-2})$ induce the same distribution over sequences of player 3's actions. Let us denote by $\alpha_0$ (resp. $\alpha_1$, resp. $\alpha_2$, resp. $\alpha_3$), the expected frequencies across stages of player 3 playing $W$ (resp. $M_1$, resp. $M_2$, resp. $E_3$) under this probability distribution. Since $\sigma_1$ is a best reply to $\sigma_{-1}$, we have $\gamma_1(\sigma) \geq 2\alpha_0 + 4\alpha_2 + \alpha_3$. Similarly $\sigma_2$ is a best reply to $\sigma_{-2}$, we have $\gamma_2(\sigma) \geq 2\alpha_0 + 4\alpha_1 + \alpha_3$. So $\gamma_1(\sigma) + \gamma_2(\sigma) \geq 4(1 - \alpha_3) + 2\alpha_3 \geq 2$.

Our last example (shown in Figure 4) satisfies Assumptions A1 (the Nash equilibria payoffs of the one-shot game include the vectors $(2, 0)$, $(2, 1)$ and $(3, 1)$) as well Assumption A2 (clearly, $u$ uniquely discriminates among player 1's two actions, while $v, w$ and $x$ statistically distinguish player 2's actions, for any action profile such that player 1 does not assign probability 1 to $T$), yet fails Assumption A3. (Player 2's action is not statistically identifiable, when player 1 plays the action $T$, which is part of the unique action profile that minmaxes player 1.)

| | $L$ | $M$ | $R$ |
|---|---|---|---|
| $T$ | $(0,0)_{u/4+v/4+w/4+x/4}$ | $(1,1)_{u/4+v/4+w/4+x/4}$ | $(3,1)_{u/4+v/4+w/4+x/4}$ |
| $B$ | $(-1,-1)_{u/2+v/4+w/8+x/8}$ | $(2,0)_{u/2+v/8+w/4+x/8}$ | $(1,0)_{u/2+v/8+w/8+x/4}$ |

Figure 4: Example 3, that satisfies A1 and A2, but not A3.

The minmax payoff of player 1 is 0 (achieved by the action profile $(T, L)$), yet the score $k(-e^1)$ of Fudenberg and Levine's (1994) algorithm is readily seen to be $1/2$, which is therefore a lower bound on the equilibrium payoff of player 1. The difficulty is that player 2's minmaxing action $L$ is strictly dominated, yet deviations from $L$ cannot be detected if player 1 plays $T$, as he should to minmax player 1. Hence, player 1 must assign positive probability to $B$, but given that it is costly to do so, continuation payoffs must be such that he gets rewarded for it. Yet, imperfect (full-support) monitoring ensures that this reward comes "too often," and so at an efficiency cost –the $1/2$ that separate the minmax payoff of player 1 from the score in direction $-e^1$.

To conclude, we discuss the limitations of our result. The main limitation is already in the title of Section 4. Theorem 4 assumes a public randomization device. Dispensing with it raises significant issues that are already present in the case of perfect monitoring. Because the set of continuation payoff vectors need not be convex, it is not obvious that mixed action profiles can be enforced, as it requires continuation payoffs to be fined-tuned to the flow payoffs. Gossner's (1995) remarkable result for perfect monitoring does not readily adapt to the case of imperfect public monitoring, as approachability works best when signals (in his case, actions) can be attributed to specific players, as under perfect monitoring, or imperfect public but product monitoring. Whether his or other techniques can be extended to public monitoring with a finite horizon to dispense with public randomization (assuming it is possible at all) is an important open question.

We have chosen not to discount payoffs in the finitely repeated game, in line with the tradition of the literature. The result remains unchanged for vanishing discount factors. To the extent that the horizon length ($T$) can be viewed as the first occurrence of a discount factor equal to zero in some infinite sequence of discount factors, it is natural to wonder about a more general payoff criterion (*i.e.*, a characterization of discount factor sequences) for which the folk theorem would hold under Assumptions A1–A3. This is left for future research.

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