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“Acting in the Darkness”

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Abstract. Environmental decisions involve both irreversibility and uncertainty. We study optimal behavior in a dynamic setting where actions yield a flow payoff but contribute to a stock that, once past an unknown tipping point, raises the probability of an environmental catastrophe. Optimal actions reflect an *Irreversibility Effect,* present even when the tipping point is known, a *Pseudo-Learning Effect*, arising because surviving to current acts conveys good news on the fact that the tipping point may lie ahead, and a *"End-of-Support" Effect* that arises once the decision-maker is almost sure that the tipping point has been passed. We also provide various comparative statics on value functions and feedback rules as the distribution of tipping points varies.

Keywords. Environmental Risk, Tipping Point, Uncertainty, Irreversibility.
JEL classification. D83, Q55.

1. Introduction

Many major environmental and health risks in today's *Risk Society* stem from our own production and consumption choices (Beck, 1992). Two features make standard cost-benefit

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analysis poorly suited to such problems. First, actions often involve irreversibility: greenhouse gas emissions, for example, accumulate in the atmosphere and raise temperatures, and current policies can only slow this trend, not reverse it. Second, decisions must be made under deep uncertainty. The environmental consequences of our actions-and the likelihood of harmful outcomes-are not fully known and may only become clearer over time.

This paper develops a simple dynamic model of decision-making under irreversibility and uncertainty. It analyzes how a decision-maker (thereafter DM) chooses actions when she is unaware whether current actions may have irreversible consequences.

MODEL. Our model features both irreversibility and uncertainty in a simple dynamic framework. At any point in time, by choosing an action related to consumption or production, DM obtains a flow payoff. If the future were irrelevant, DM would simply choose the myopically optimal action at each point in time. However, past actions accumulate into a stock that increases the arrival rate of an environmental catastrophe. For instance, consumption and production may generate pollution that raises the likelihood of a future disruptive event in which all opportunities for further payoff disappear.[1] When the stock passes an unknown tipping point (thereafter TP), the probability of catastrophe irreversibly jumps up.[2] DM knows only the distribution of the TP. Whether it has already been passed remains unknown until the trajectory reaches the upper bound of this distribution.[3]

OPTIMAL TRAJECTORY. Acting today affects how soon the TP may be reached but it also changes DM's belief on survival. The optimal policy *a priori* takes the form of a feedback rule that depends on two state variables: (i) the stock, which determines how close the system may be to have passed the TP, and (ii) the survival probability (i.e., the probability that the catastrophe has not not occurred yet) which captures DM's belief about whether the TP has already been crossed. Different action paths can lead to the same stock at a given date but still imply different beliefs and therefore different optimal continuations. To illustrate, consider two trajectories reaching the same stock today. In the first trajectory, actions were high early on and then remained small so that the stock was kept almost constant. Had the TP been crossed, it likely occurred long ago. Upon survival, DM significantly updates beliefs toward the TP being still ahead. In the second trajectory,

[1] See Cropper (1976), Gjerde et al. (1999), and Clarke and Reed (1994) for similar assumptions.

[2] Tipping-point dynamics are common in ecology and climate science (Lenton et al., 2008). The World Bank notes that exceeding 2^{o}C of warming may trigger nonlinear events such as ice sheet collapse or methane release from thawing permafrost.

[3] Roe and Baker (2007) argue that regime shifts may remain undetected for a long time.

actions were initially low and were raised only after a while so that the stock has sharply increased recently. Here, survival conveys almost no information and beliefs remain close to the prior.

Driving Forces. Several forces shape behavior in this context. First, there is an *Irreversibility Effect.* As long as DM believes that the TP is still ahead, she acts cautiously to slow down stock accumulation and refrain from irreversibly raising the probability of a catastrophe. This effect would already be present when the location of the tipping is known to be ahead for sure.

Second, a *Pseudo-Learning Effect* that is specific to the case of uncertainty now also matters. DM interprets survival as *"good news"* that the TP is still ahead. After having updated beliefs, acting even more cautiously becomes more attractive.

Third, while delaying action is valuable early on because future adjustments remain feasible and are only mildly discounted, when DM comes closer to know that the TP will have been passed for sure, the remaining time over which the control can be adjusted shrinks, so the marginal value of postponement becomes negligible. Actions are raised accordingly in a final phase of the trajectory: an *"End-of-Support" Effect.*

Comparative Statics. Dynamic programming techniques allow us to characterize the optimal policy. When the stock of past actions and DM's beliefs are treated as joint state variables, the associated value function satisfies a nonlinear partial differential equation that is generally intractable. Although closed-form solutions are rarely available, the structure of the problem yields informative comparative statics in a number of relevant cases.

We develop three such cases that may be useful for future applications. First, when the tipping point follows a binomial distribution with a strictly positive probability of having already been crossed at the initial date, the optimal action path admits a closed-form solution.[4] Relative to the benchmark in which the tipping point is known to lie ahead, optimal actions cannot be uniformly ranked. While earlier actions are shifted upward under uncertainty because DM believes that the TP may have already passed, survival suggests that the TP is still ahead and actions are reduced later on.

Second, when the tipping point is drawn from a continuous distribution, we show that the problem can be substantially simplified through an appropriate change of variables. By introducing a composite index that jointly summarizes the stock and the posterior beliefs,

[4] In Guillouet and Martimort (2026), we use this binomial model to investigate institutional design for environmental regulation.

the original value function can be transformed into a new function that solves a partial differential equation depending only on the hazard rate of the distribution. In the special case of an exponential distribution, the hazard rate is constant and quasi-explicit solutions become now available. Asymptotic analysis shows that optimal actions converge over time toward the myopic benchmark at an exponential rate.

Finally, this reduction yields an additional insight. It allows for a systematic comparison of value functions and optimal policies across tipping-point distributions when those distributions are ranked by means of the hazard-rate stochastic order. Consider a uniform increase in the hazard rate, corresponding to a shift in beliefs that places more weight on the tipping point having been passed earlier. The tipping point is thus thought to be *"just around the corner."* Under mild regularity conditions, such a shift of mass toward earlier thresholds strengthens irreversibility concerns and lowers the value function at all states, reflecting the reduced likelihood of survival. Moreover, optimal actions are shifted downward at the beginning of the trajectory, when irreversibility concerns are most acute.

STOCK-BASED FEEDBACK RULES. Even though the optimal feedback rule is *a priori* defined as a function of both the stock of past actions and DM's beliefs, these two state variables evolve deterministically along the optimal trajectory. As a result, the stock alone is sufficient to infer beliefs at any point in time, and the feedback rule can be expressed as a function of the stock only. By committing ex ante to follow this one-dimensional rule, DM implements the socially optimal policy.

This reduction delivers two main benefits. First, it clarifies how the *Irreversibility Effect*, the *Pseudo-Learning Effect* and the *"End-of-Support" Effect* interact along the optimal path. Second, it allows for a transparent comparison with benchmark environments in which the stock of past actions is the sole state variable. In this respect, a first benchmark is the case of complete information about the location of the tipping point. The problem then reduces to a standard irreversible control problem in which only the *Irreversibility Effect* is operative. A second benchmark arises when the exact location of the TP is immediately revealed when it is crossed. In that scenario, DM's posterior beliefs that the tipping point has been passed is still zero till it has been passed. This stands in sharp contrast with our main scenario, where DM is almost sure that the TP has been passed as the trajectory hits the upper bound of the TP distribution and, as a result, chooses action closer to the myopic optimum in that neighborhood. This *"End-of-Support" Effect* disappears when the exact location of the TP is immediately revealed. Also, the *Pseudo-Learning Effect* is now replaced by a genuine *Learning Effect*: higher current actions increase the probability of discovering that the tipping point has been reached. Because such information lowers continuation payoffs, DM again has an incentive to restrain current actions.

ORGANIZATION. Section 2 reviews the literature. Section 3 presents the model. Section 4 discusses how beliefs evolve other time (Section 4.1), and offers a recursive representation of the value function (Section 4.2). The optimal trajectory is characterized in Section 5 where, we also discuss various comparative statics. Section 6 analyzes the one-dimensional reduction of the problem in terms of stock-based feedback rules and value functions. Section 7 recaps our results and discusses possible extensions. Proofs are relegated into Appendices.

2. Literature Review

This paper contributes to several strands of the literature on optimal decision-making under uncertainty and irreversibility, particularly in environmental economics.

EVENT UNCERTAINTY IN ENVIRONMENTAL ECONOMICS. The environmental economics literature on catastrophic climate risk typically models uncertainty as the possibility of a sudden, irreversible event that causes a discrete drop in welfare. Early works link economic activity to catastrophe risk by making the probability of such events depend on the stock of pollution (Cropper, 1976; Heal, 1984; Clarke and Reed, 1994) and eventually on climate policy (Kelly and Kolstad, 1999). In these frameworks, the catastrophe itself is the event of concern, becoming more likely as the stock approaches a threshold. This literature generally assumes that DM observes the evolution of risk, including proximity to thresholds. Related work considers regime shifts caused by crossing thresholds, but still assumes that such shifts and their implications are known (Brock and Starrett, 2003; Barrett, 2013). In our model, the key stochastic event is not the catastrophe itself but the (unobserved) transition from a low- to a high-risk regime when the stock crosses a threshold. Because DM does not observe this crossing, she must form beliefs about whether the system has entered the high-risk regime. Thus, unlike prior work that focuses on uncertainty over the timing of catastrophic events, this paper studies uncertainty on both this timing and the probability of a catastrophe itself. This distinction matters because, in practice, optimal policy may first respond to changes in the perception of risk rather than to (not yet) realized damages.

IRREVERSIBILITY AND INFORMATION. Arrow and Fisher (1974), Henry (1974), and Freixas and Laffont (1984) show that under irreversible decisions with uncertain consequences, DMs should value waiting to preserve flexibility. Applied to climate policy, this logic implies stronger current abatement when future information is expected to improve (Chichilnisky and Heal, 1993; Kolstad, 1996). Relatedly, Gollier et al. (2000) argue that irreversible actions should be avoided more when the information system is worse.[5] Our *Irreversibility Effect* is in lines with this literature. However, information is treated there as exogenous, whereas in many environmental settings actions themselves shape the information structure. Hereafter, we instead emphasize how beliefs endogenously react to past actions. Laiho et al. (2025) also follow this route and study irreversible decisions when information is generated by the actions of multiple agents. Agents act too late because they fail to internalize an informational externality. Because in our framework there is a single DM, no such externality arises. Our *Pseudo-Learning Effect*, although it contributes to further reducing actions, is a feature of optimality under uncertainty.

Tipping Points. Tsur and Zemel (1995) study optimal resource management when a stock must be kept above an uncertain threshold to avoid irreversible pollution, showing that uncertainty leads to more conservative exploitation. Sims and Finoff (2016) analyze the interaction between irreversible environmental damage and sunk investment under TP uncertainty. Tsur and Zemel (1996, 2021; Nævdal, 2006) are closer to our setting in that crossing a threshold increases the arrival rate of catastrophes rather than causing them deterministically. While these and related models provide valuable insights into resource management under regime-shift uncertainty, they typically assume that crossing the TP is immediately observed by DM. We instead assume that this event is unobserved, introducing a belief state to capture DM's ignorance. In this respect, we follow Clarke and Reed (1994) and Tsur and Zemel (1995) in using the survivor probability as a state variable for our dynamic optimization problem. This approach is also related to Crépin and Nævdal (2020) who add a stress variable to govern regime shifts.

Lemoine and Traeger (2014, 2016) study learning about a TP through observed system responses and show that regime-switching risk substantially raises optimal carbon taxes, a result that is echoed by Cai and Lontzek (2019). Van der Ploeg (2014) analyzes how uncertainty on TPs may modify the design of an optimal dynamic path for carbon taxes. Liski and Salanié (2020) study a model in which the stock can induce catastrophes with a delay. While developed independently and sharing similar motivations, their framework differs in important ways: the stock directly enters the utility function and mitigation is allowed.

[5] It is nevertheless possible that anticipation of more precise future information increases current emissions in some contexts as in Karp and Zhang (2006).

By contrast, in our model the stock shapes risk through the threshold mechanism, which is particularly relevant for settings such as carbon accumulation with an unknown TP. Moreover, whereas Liski and Salanié (2020) rely on optimal control, our dynamic programming approach provides a feedback-oriented intuitive description of optimal trajectories. This framework makes transparent the link between beliefs and actions and allows a clear comparison between environments with and without uncertainty about the TP.

3. Model

Preferences. DM chooses actions over time. Time is continuous. Let $r>0$ be the discount rate. Let $\mathbf{x}=(x(\tau))_{\tau\geq 0}$ (resp. $\mathbf{x}_t=(x(\tau))_{\tau\geq t}$ and $\mathbf{x}^t=(x(\tau))_{\tau<t}$) denote an action plan (resp. the continuation of a plan from date t on and the history of past actions up to date t^-). Action $x(\tau)$ at date τ yields a flow payoff (net of the action cost) worth $u(x(\tau))=\zeta x(\tau)-\frac{x^2(\tau)}{2}$ where $\zeta x(\tau)$ is the benefit of choosing $x(\tau)$ (the consumption side) and $\frac{x^2(\tau)}{2}$ its cost (the production side).[6] The set of feasible actions $\mathcal{X}=[0,2\zeta]$ is such that flow payoff always remains non-negative.

Catastrophes. An environmental catastrophe may arise at a rate $\theta(t)$ that depends on the stock $X(t)=\int_0^t x(\tau)d\tau$ of actions taken before date t. More precisely, we postulate $\theta(t)=\Delta\mathbb{1}_{\{X(t)\geq\widetilde{X}\}}$ where $\widetilde{X}$ is the location of a TP. When the stock $X(t)$ passes the threshold $\widetilde{X}$, the rate jumps from 0 to $\Delta>0$.[7] Beyond that TP, actions no longer affect the arrival rate of a catastrophe.To capture its detrimental and irreversible impact, we assume that, if a catastrophe arises, the flow payoff is no longer realized from that date on.[8]

Benchmark. Suppose that DM has no control over the arrival rate of a catastrophe and assume that the TP is at 0, i.e., it is passed at the start. DM's expected payoff is $\int_0^{+\infty}e^{-\lambda\tau}u(x(\tau))d\tau$ where $\lambda=r+\Delta$ stands for the effective discount rate that applies with the possibility of a catastrophe. Since she cannot influence the arrival rate, DM maximizes

[6] Sharp analytical results are obtained for this quadratic specification. Our results could nevertheless be extended to more general concave utility functions but at the cost of a less clear exposition at places.

[7] This formulation, although it simplifies exposition, can be viewed as a reasonable approximation for a smoothly and quickly increasing hazard rate around a critical threshold; a feature that appears in much physical and climatological phenomena.

[8] A justification is that production is no longer possible afterwards. Notice that the payoff of the catastrophe is thus the same as when not acting. Adding of a positive harm D that would reduce flow payoffs following a catastrophe would be akin to assuming that DM enjoys a positive (opportunity) benefit D till the catastrophe arrives. Accordingly, DM would be even more reluctant to avoid a catastrophe than in our setup. Relatedly, we also assume that there is no flow damage $D(X(t))$ due to the stock but this assumption could also be easily relaxed.

her intertemporal payoff with the *myopic action* $x^m(\tau)=\zeta$ for all $\tau\geq 0$ and gets thereby the myopic payoff $\mathcal{V}_\infty=\frac{u(\zeta)}{\lambda}$. The same myopic action and payoff are obtained in all scenarios below once it is known that the TP has been passed for sure.

4. Beliefs, Value Function and Optimal Trajectory

4.1 *Beliefs*

Suppose now that DM does not know where the TP lies. Let F be the distribution of possible values for the TP. This distribution has, unless stated otherwise, no mass point, a (positive) continuous density function f and a bounded support $\left[0,\overline{X}\right]$ (i.e., $\overline{X}<+\infty$).[9]

Consider a history of past actions $\mathbf{x}^t$ with survival till date t. The stock trajectory has reached a level X at date t. The stock at any date $\tau\geq t$ solves the following ODE:

$$\dot{X}(\tau)=x(\tau),\quad \forall\tau\geq t \text{ with } X(t)=X. \tag{4.1}$$

Let $\hat{X}(\tau-t;t,X)=X+\int_0^{\tau-t}x(t+s)ds$ be this stock at any date $\tau\geq t$.[10]

To evaluate DM's continuation payoff following a history $\mathbf{x}^t$ that has reached stock $\hat{X}(t;0,0)=X$ at date t, we compute her posterior beliefs $f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}$ that the TP is in $\left[\widetilde{X},\widetilde{X}+d\widetilde{X}\right]$. The density function $f(\widetilde{X}|t,\mathbf{x}^t)$ should take into account that, if the TP lies at $\widetilde{X}\leq\hat{X}(t;0,0)$, the arrival rate has already jumped from 0 to Δ at an earlier date $T(\widetilde{X};0,0)\leq t$. If instead the TP is at $\widetilde{X}>\hat{X}(t;0,0)$, no such jump occurred. A key variable to describe how the posterior density evolves is thus the expected probability of survival up to date t given history $\mathbf{x}^t$, namely

$$\hat{Z}(t,\mathbf{x}^t)=\int_0^{\hat{X}(t;0,0)}f(\widetilde{X})e^{-\Delta(t-T(\widetilde{X};0,0))}d\widetilde{X}+1-F(\hat{X}(t;0,0)). \tag{4.2}$$

The first term in this expectation accounts for all possible scenarios where the TP may have already been passed at a previous date $T(\widetilde{X};0,0)\leq t$ and the rate of arrival has jumped to Δ from that date on. The second term takes into account that the TP might not have been passed yet. After manipulations, we rewrite:

$$\hat{Z}(t,\mathbf{x}^t)=1-\Delta e^{-\Delta t}\int_0^t F(\hat{X}(\tau;0,0))e^{\Delta\tau}d\tau.^{11} \tag{4.3}$$

[9]The fact that the distribution has no mass point implies that actions are continuous at $\overline{X}$ (see Appendix A). This continuity does not hold when the distribution has a mass point at $\overline{X}$ as Scenario 1 below illustrates or when the support is unbounded ($\overline{X}=+\infty$).

[10]Observe that $\hat{X}(\tau-t;t,\hat{X}(t;0,0))=\hat{X}(\tau;0,0)$.

A higher value of $\hat{Z}(t,\mathbf{x}^t)$ reflects a more likely survival. When the current stock $\hat{X}(t;0,0)$ is close to 0, the probability that the TP has already been crossed is low, so the survival probability is similar to the case in which the arrival rate is known to be zero. As $\hat{X}(t;0,0)$ increases toward $\overline{X}$, it becomes more likely that the TP has been passed, and $\hat{Z}(t,\mathbf{x}^t)$ declines. Hence, both the stock trajectory and the distribution F matter. If F places more mass near the origin, crossing the TP is more likely earlier, and survival probabilities fall more rapidly.

To further illustrate the role of $\hat{Z}(t,\mathbf{x}^t)$, especially in view of computing DM's value function, consider values $\widetilde{X}$ of the TP which are ahead of where DM currently stands, i.e., $\hat{X}(t;0,0)\leq\widetilde{X}$. For such values, the posterior belief that the TP lies in the interval $\left[\widetilde{X},\widetilde{X}+d\widetilde{X}\right]$ following a past history $\mathbf{x}^t$ with no catastrophe, writes as

$$f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}=\frac{f(\widetilde{X})}{\hat{Z}(t,\mathbf{x}^t)}d\widetilde{X}.$$

In other words, as $\hat{X}(t;0,0)$ comes closer to $\overline{X}$, DM believes that the distribution of TPs ahead has a higher density on $[\hat{X}(t;0,0),\overline{X}]$. Surviving is "good news" and DM's posterior beliefs puts increasingly higher mass on the remaining interval.

To better understand how the survival probability evolves over time, consider an arbitrary future date $t+\tau$ (with $\tau\geq 0$) and the corresponding trajectory $\mathbf{x}^{t+\tau}$. We may again use (4.3) and directly compute

$$\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})=1-\Delta e^{-\Delta(t+\tau)}\int_0^{t+\tau}F(\hat{X}(s;0,0))e^{\Delta s}ds. \tag{4.4}$$

This expression can in fact be decomposed in a more illuminative way as

$$\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})=\left(1-\Delta e^{-\Delta\tau}\int_0^{\tau}F(\hat{X}(t+s;t,X))e^{\Delta s}ds\right)-\left(1-\hat{Z}(t,\mathbf{x}^t)\right)e^{-\Delta\tau}. \tag{4.5}$$

The first bracketed term, stems for the survival probability if DM were to ignore past history and treat the current stock $\hat{X}(t;0,0)=X$ as a fresh starting point for a trajectory from date t on that would be $\hat{X}(t+s;t,X)$ for $s\geq 0$. The second term on the r.-h.s. of (4.5) shows that DM's pessimism (i.e., $\hat{Z}(t,\mathbf{x}^t)<1$) persists and is carried forward along the trajectory, even though it is at a declining rate.

Scenario 1. Binomial Distribution. Suppose that the TP is located at 0 or $\overline{X}$ with respective probabilities q and $1-q$. DM is uncertain whether the TP is passed at the

[11] See the Proof of Lemma A.1 in the Appendix.

start or whether it will be later found at $\overline{X}$. For any $t>0$ and history $\mathbf{x}^t$ that has not yet reached $\overline{X}$, the probability of survival is $\hat{Z}(t,\mathbf{x}^t)=1-q+qe^{-\Delta t}$. The posterior probability that the TP lies at 0 conditional on survival is thus $q(t)=\frac{qe^{-\Delta t}}{\hat{Z}(t,\mathbf{x}^t)}\leq q$. Survival is "good news." Upon survival, DM becomes more optimistic that TP lies ahead at $\overline{X}$. ■

4.2 *Value Function*

The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ is defined as DM's continuation payoff starting from date t given the past history $\mathbf{x}^t$ and survival till that date. This value function averages continuation payoffs using the simple expression for the posterior density $f(\widetilde{X}|t,\mathbf{x}^t)$ for realizations of the TP that lie ahead of the current stock $X=\hat{X}(t;0,0)$ reached at date t together with the surviving probability mass computed for realizations that have already been passed.

Lemma 1 offers a compact representation for this value function.

Lemma 1. *The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ satisfies*

$$\hat{Z}(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t)=\max_{\mathbf{x}_t}\int_0^{+\infty}e^{-r\tau}\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})u(x(t+\tau))d\tau. \tag{4.6}$$

Scenario 1 (Continued). Inserting the expression of $\hat{Z}(t,\mathbf{x}^t)$ above into (4.6) yields

$$\hat{\mathcal{V}}(t,\mathbf{x}^t)=\max_{\mathbf{x}_t}\int_0^{+\infty}e^{-r\tau}(1-q(t)+q(t)e^{-\Delta\tau})u(x(t+\tau))d\tau.$$

Because $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ is the value function conditional on survival, the posterior probability $q(t)$ is now used to compute intertemporal payoffs. Equation (4.6) shows how to generalize this construction beyond the binomial case. ■.

5. Optimal Trajectory

5.1 *Representation of the Value Function*

The representation (4.6) suggests that the state of the system might be best described by adding to the stock a second state variable, the survival probability. Two trajectories that have reached the same stock $X=\hat{X}(t;0,0)$ with the same beliefs $Z=\hat{Z}(t,\mathbf{x}^t)$ at date t have the same continuation. Instead, two trajectories that have reached the same stock but with different survival probabilities might be pursued differently. To complete the state of the system, we must thus add to the law of motion for the stock

$$\dot{X}(\tau) = x(\tau), \quad \forall \tau \geq 0 \text{ with } X(0) = X. \tag{5.1}$$

the law of motion for the survival probability. Adapting (4.5), we find that the survival probability $Z(\tau)$ starting from date 0 with an initial value $Z(0) = Z$ satisfies

$$Z(\tau) = 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X(s)) e^{\Delta s} ds - (1 - Z) e^{-\Delta\tau}, \tag{5.2}$$

a condition rewritten in differential form as

$$\dot{Z}(\tau) = \Delta(1 - F(X(\tau)) - Z(\tau)) \text{ with } Z(0) = Z. \tag{5.3}$$

Using (4.6) gives us a representation of the value function, now denoted as $\mathcal{V}^e(X, Z)$, in terms of the state variables (X, Z) for $X \geq 0$ and $Z \in (0, 1]$ as

$$Z\mathcal{V}^e(X,Z) = \max_{x(\cdot),X(\cdot),Z(\cdot),\overline{T}, \text{ s.t. (5.1)-(5.3)}, X(\overline{T})=\overline{X}} \int_0^{\overline{T}} e^{-r\tau} Z(\tau) u(x(\tau)) d\tau + e^{-r\overline{T}} Z(\overline{T}) \mathcal{V}_\infty. \tag{5.4}$$

Starting from any initial pair (X, Z), DM chooses an optimal path that eventually reaches $\overline{X}$ at a date $\overline{T}$. Once this level is attained, the TP is surely behind, and DM switches permanently to the myopic action with payoff $\mathcal{V}_\infty$.[12] Before reaching $\overline{X}$, the TP may already have been crossed, but DM does not know.

The expression (5.4) showcases that the effective discount rate, $\lambda^e(\tau) = r - \frac{\dot{Z}(\tau)}{Z(\tau)}$, is time-dependent. Using the survival probability as a state variable keeps track of this time-dependency. The action chosen at any given date has no direct impact on the evolution of this implicit discount rate since the law of motion (5.3) does not depend on current action. Because stock and beliefs evolve over time, this implicit discount rate nevertheless keeps on changing. Specifically, DM is using $\lambda^e(\tau) \approx r$ to discount future payoffs earlier on but, eventually, will switch to $\lambda^e(\tau) \approx \lambda = r + \Delta$.

SCENARIO 2: TP LOCATED AT $\overline{X}$. As a preamble, it is useful to investigate the complete information scenario where the TP is known to be located at $\overline{X} > 0$; a degenerate case of SCENARIO 1 for which $Z(\tau) = 1$ for all $\tau \in [0, \overline{T}]$. Proposition 1 characterizes the corresponding value function $\mathcal{V}^k(X)$ and feedback rule $\sigma^k(X)$.[13]

[12] We will demonstrate that, under all circumstances below, optimal actions are always bounded from below by a positive minimal action; which in turn implies that the finite upper bound on its distribution is always reached in finite time and thus $\overline{T} < +\infty$.

[13] The solution looks familiar since this optimization problem amounts to a "cake-eating" or resource extraction problem where the available stock of resource would be $\overline{X}$.

PROPOSITION 1. *The value function $\mathcal{V}^k(X)$ satisfies*

$$\dot{\mathcal{V}}^k(X) = -\zeta + \sqrt{2r\mathcal{V}^k(X)}, \quad \forall X \in [0, \overline{X}) \tag{5.5}$$

with the boundary condition $\mathcal{V}^k(\overline{X}) = \mathcal{V}_\infty$. The feedback rule $\sigma^k(X)$ is

$$\sigma^k(X) = \zeta + \dot{\mathcal{V}}^k(X) \quad \forall X \in [0, \overline{X}). \tag{5.6}$$

All actions which are taken before reaching $\overline{X}$ at a date $\overline{T}^k$ have a long-lasting impact. Reducing those actions decreases the probability that a catastrophe arises earlier, an *Irreversibility Effect*. The quantity $-\dot{\mathcal{V}}^k(X)$ on the r.-h.s. of (5.6) is actually the shadow cost of the irreversibility constraint. It thus depends on how much stock remains before reaching $\overline{X}$. As X increases towards $\overline{X}$, the irreversibility constraint is more demanding, and the shadow cost $-\dot{\mathcal{V}}^k(X)$ increases.[14]

While our analysis focuses on abstract actions and stocks, the mechanism has clear implications for the choice of environmental instruments. To the extent that $\overline{X}$ can be interpreted as the total amount of pollutants that can be dumped in the atmosphere before changing climate regime, the shadow cost $-\dot{\mathcal{V}}^k(X)$ stands as the social cost of carbon. The model shows that, when TP is known, the carbon price must raise as the trajectory gets closer to $\overline{X}$.

The intuition for decreasing actions is straightforward. All actions taken before reaching $\overline{X}$ contribute to harden the irreversibility constraint with the same intensity. Because of discounting, DM chooses higher actions earlier on and lower ones when approaching $\overline{X}$. The optimal action $x^k(\tau)$ thus decreases over time before $\overline{X}$ and then jumps to the myopic optimum afterwards:

$$x^k(\tau) = \begin{cases} \zeta\left(1 - \left(1 - \sqrt{\frac{r}{\lambda}}\right) e^{-r(\overline{T}^k - \tau)}\right) & \text{for } \tau \in [0, \overline{T}^k), \\ \zeta & \text{for } \tau \geq \overline{T}^k. \end{cases} \tag{5.7}$$

Actions remain positive since

$$\zeta\sqrt{\frac{r}{\lambda}} = x^k(\overline{T}^{k-}) = \sigma^k(\overline{X}^-) \leq \sigma^k(X^k(\tau)) = x^k(\tau) \leq \zeta \quad \forall \tau \in [0, \overline{T}^k). \tag{5.8}$$

[14] From (5.5), $\mathcal{V}^k$ is strictly concave for $X \in [0, \overline{X})$.

Because actions are now lower than the myopic optimum over the first phase but remain poitive, $\overline{X}$ is reached in finite time but later than when acting myopically:

$$\overline{T}^k = \overline{T}^m + \left(1 - \sqrt{\frac{r}{\lambda}}\right)\frac{1 - e^{-r\overline{T}^k}}{r} > \overline{T}^m = \frac{\overline{X}}{\zeta}. \tag{5.9}$$

■

5.2 *Optimal Trajectory*

We now come back to the setting where DM does not know where the TP lies.

PROPOSITION 2. *The value function* $\mathcal{V}^e(X,Z)$[15] *satisfies*

$$\frac{\partial \mathcal{V}^e}{\partial X}(X,Z) = -\zeta + \sqrt{2\left(r - \frac{\Delta(1-F(X)-Z)}{Z}\right)\mathcal{V}^e(X,Z) - 2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)} \tag{5.10}$$

with the boundary conditions $\mathcal{V}^e(\overline{X},Z) = \mathcal{V}_\infty \ \forall Z \in (0,1]$. *The feedback rule* $\sigma^e(X,Z)$ *is*

$$\sigma^e(X,Z) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X,Z). \tag{5.11}$$

The comparison of the feedback rule (5.11) with its complete information counterpart (5.6) shows that the term $\frac{\partial \mathcal{V}^e}{\partial X}(X,Z)$ can again be interpreted as the shadow cost of irreversibility. This shadow cost now depends on beliefs. Coming back to our previous interpretation of the shadow cost as the social cost of carbon, our findings suggest that the carbon price should be sensitive to beliefs on whether the TP has been passed or not. Yet, the analysis of specific scenarios below will show that this dependence on beliefs can be quite sensitive on whether one stands at the start of the trajectory or whether more time has elapsed.

To further stress the role of those beliefs, it is nevertheless useful to decompose the feedback rule into two components. Suppose first that beliefs do not change along the trajectory, which amounts to keeping Z constant. Then, using (5.10) and (5.11), the optimal action in that putative scenario would be reduced to

$$\sigma^e_*(X,Z) = \sqrt{2\left(r - \frac{\Delta(1-F(X)-Z)}{Z}\right)\mathcal{V}^e(X,Z)} < \sqrt{2\lambda\mathcal{V}^e(X,Z)}$$

[15] Thereafter, we restrict the analysis to differentiable value functions. All specific scenarios we investigate below satisfy this requirement. We nevertheless prove in Appendix A that, more generally, differentiability should hold more almost everywhere.

where the r.-h.s. inequality follows from observing that the effective discount rate $r - \frac{\Delta(1-F(X)-Z)}{Z}$ is bounded above by λ. This r.-h.s is similar to the feedback rule that would be obtained in Scenario 2, namely $\sigma^k(X) = \sqrt{2rV^k(X)}$, modulo of course the fact that value functions differ. Despite variations in the effective discount rate that now applies in comparison with Scenario 2, the feedback rule $\sigma^e_*(X,Z)$ would essentially capture the sole effect of the *Irreversibility Effect* in our context.

It is important to notice that the effective discount rate is equal to λ only at the end of the support and is lower before. As the effective discount rate increases, the action $\sigma^e_*(X,Z)$ is less distorted since the future matters less and DM behaves more myopically. In other words, the optimal trajectory will exhibit an *"End-of-Support" Effect* when X comes closer to $\overline{X}$. The intuition for this effect is the following. While delaying action is valuable early on because future adjustments remain feasible and are only mildly discounted, close to $\overline{X}$, the remaining time over which the control can be adjusted shrinks, so the marginal value of postponement becomes negligible. At the same time, the solution must hit the myopic outcome at $\overline{X}$ and remain there thereafter. As a result, the optimal policy calls for raising actions near the end of the support in order to satisfy this terminal requirement.

Under uncertainty, the value function depends also on beliefs and a new term, $-2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)$, appears on the r.-h.s. of (5.10) to capture this effect. For a fixed stock, the posterior probability that the TP lies ahead conditional on survival is $\frac{1-F(X)}{Z}$. A higher Z therefore corresponds to a higher posterior probability that the TP has already been passed. This shift of conditional probability mass toward the low-value regime is the natural reason why $\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z) < 0$.[16] Along the optimal trajectory starting at $(X=0, Z=1)$, $Z(\tau) \geq 1 - F(X(\tau))$ (in other words, the survival probability is always no less than the ex ante probability that the TP lies ahead $X(\tau)$).[17] Therefore, $-2\Delta(1-F(X)-Z)\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z)$ is non-positive; forcing DM to decreases current actions. Because DM now believes more strongly that the TP has not been passed yet, current acts are thought as having strong irreversible consequences. As a result, actions should be further reduced: a *Pseudo-Learning Effect* which is specific to the scenario of uncertainty.

[16]See the Appendix, Equation (A.18), for a proof of that sign.

[17]To prove this result, observe that a trajectory $Z(\tau)$ that crosses the curve $Z = 1 - F(X)$ at a date $\tau_0 \geq 0$ does so with zero derivative, $\dot{Z}(\tau_0) = 0$. Because $\frac{d}{d\tau}\left(Z(\tau) - 1 + F(X(\tau))\right)|_{\tau_0} = f(X(\tau_0))x(\tau_0) > 0$, crossing happens only once if it happens at all and $Z(\tau) > 1 - F(X(\tau))$ for $\tau > \tau_0$. Then, notice that, for the optimal trajectory $X(0) = 0$ and $Z(0) = 1$, so that $\tau_0 = 0$.

SCENARIO 1 (CONTINUED). Although $\mathcal{V}^e(X,Z)$ has no closed form in that case, the optimal actions $\mathbf{x}^e$ along the trajectory and the delay $\overline{T}^e$ till reaching $\overline{X}$, have explicit expressions.

PROPOSITION 3. *The optimal action $x^e(\tau)$ in* SCENARIO 1 *decreases over $[0,\overline{T}^e)$ and is the myopic optimum thereafter:*

$$x^e(\tau) = \begin{cases} \zeta\left(1 - e^{-r(\overline{T}^e-\tau)}\frac{Z(\overline{T}^e)}{Z(\tau)}\left(1-\sqrt{\frac{r-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda}}\right)\right) < \zeta & \text{for } \tau \in [0,\overline{T}^e), \\ \zeta & \text{for } \tau \geq \overline{T}^e \end{cases} \tag{5.12}$$

where the survival probability is $Z(\tau) = 1-q+qe^{-\Delta\tau}$ for all $\tau \in [0,\overline{T}^e]$ and $\overline{T}^e$, the date at which $\overline{X}$ is reached, satisfies

$$\overline{T}^e = \overline{T}^m + \left(1-\sqrt{\frac{r-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda}}\right)\int_0^{\overline{T}^e}\frac{Z(\overline{T}^e)}{Z(\tau)}e^{-r(\overline{T}^e-\tau)}d\tau > \overline{T}^m. \tag{5.13}$$

Because the TP is ahead with some probability, the *Irreversibility Effect* is still at play although with a lesser magnitude than for SCENARIO 2 (that corresponds to $q=0$). Actions thus remain below the myopic optimum as long as $\overline{X}$ has not been reached.

Comparative statics w.r.t. q are complex. To illustrate, Figure 1 displays the paths $x^e(\tau)$ (blue) and $x^k(\tau)$ (orange) for some reasonable parameters values. Those actions decrease over time until $\overline{X}$ is reached after which they are equal to the myopic outcome. Yet, the ordering of $x^e(\tau)$ and $x^k(\tau)$ is not uniform over time.

Because the TP may already have been passed at the start, the *Irreversibility Effect* is of lower magnitude in SCENARIO 1 and, as a result, $x^e(\tau)$ significantly dominates $x^k(\tau)$ for τ small enough. Indeed, we find $x^e(0) \approx 1.85 > x^k(0) \approx 0.59$ for the chosen parameter values. In the middle of the trajectory, the *Pseudo-Learning Effect* bites under SCENARIO 1 and $x^e(\tau)$ decreases faster than $x^k(\tau)$. Over this range, survival is interpreted as "good news" which triggers reinforced precaution and low actions.

As a result of the lower magnitude of the *Irreversibility Effect*, $\overline{X}$ is reached earlier in SCENARIO 1; $\overline{T}^e \approx 18.31 < \overline{T}^k \approx 24.73$. Numerical evaluations then show that the difference $x^e(\tau) - x^k(\tau)$ changes sign once on the interval $[0,\overline{T}^e)$, at approximately $\tau^\star \approx 4.91$ while for $\tau^\star < \tau < \overline{T}^e$, we have $x^e(\tau) < x^k(\tau)$ and, finally, for $\overline{T}^e \leq \tau < \overline{T}^k$, $x^e(\tau) = \zeta > x^k(\tau)$.

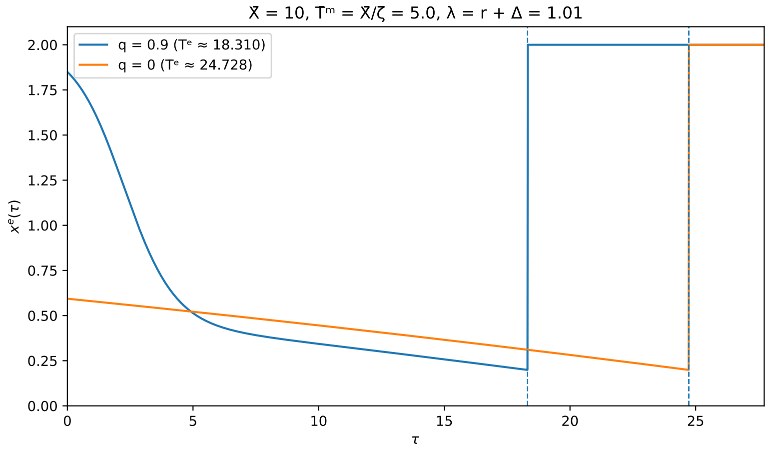

FIGURE 1. Optimal actions $x^e(\tau)$ $(q=0.9)$ and $x^k(\tau)$ $(q=0)$ for $\overline{X}=10$. Parameters are $\zeta=2$, $r=0.01$, $\Delta=1$, $\lambda=r+\Delta=1.01$. Vertical dashed lines indicate dates $\overline{T}^e$ and $\overline{T}^k$.

When the trajectory comes closer to the threshold $\overline{X}$, the *"End-of-Support" Effect* pushes actions up in SCENARIO 1 relative to SCENARIO 2. Indeed, the last action before reaching $\overline{X}$ (and thus, by continuity, actions in a left-neighborhood of $\overline{X}$) is raised towards the myopic solution when $q>0$:

$$x^e(\overline{T}^{e-})=\zeta\sqrt{\frac{r-\dfrac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda}}>\zeta\sqrt{\frac{r}{\lambda}}=x^k(\overline{T}^{k-}). \tag{5.14}$$

Because the survival probability $Z(\tau)$ decreases quickly, a higher q leads only to a slightly larger left-limit value of the action at the switching threshold ($\dot{Z}(\overline{T}^e)$ is small). Under the chosen parameter values, $x^k(\overline{T}^{k-})=2\sqrt{\frac{0.01}{1.01}}\approx 0.1990074380$ while $x^e(\overline{T}^{e-})=2\sqrt{\frac{0.0100001005}{1.01}}\approx 0.1990084381$ is greater as predicted but only by a tiny difference of order 10^{-6}. This result, which holds more broadly across parameter values, means that the *"End-of-Support" Effect* might remain quite small. ■

TRANSFORMED VALUE FUNCTION. We look for a solution to (5.10) of the form $\mathcal{V}^e(X,Z)=\widetilde{\mathcal{V}}(X,Y)$, where we define the index $Y=\frac{Z}{1-F(X)}$ and observe that we may restrict to $Y\geq 1$ since, as shown above, optimal trajectories satisfy $Z(\tau)\geq 1-F(X(\tau))$. This transformed value function $\widetilde{\mathcal{V}}(X,Y)$ satisfies

$$\frac{\partial \widetilde{\mathcal{V}}}{\partial X}(X,Y)+\frac{f(X)}{1-F(X)}Y\frac{\partial \widetilde{\mathcal{V}}}{\partial Y}(X,Y)=-\zeta+\sqrt{2\left(r+\frac{\Delta(Y-1)}{Y}\right)\widetilde{\mathcal{V}}(X,Y)+2\Delta(Y-1)\frac{\partial \widetilde{\mathcal{V}}}{\partial Y}(X,Y)}. \tag{5.15}$$

When $\widetilde{\mathcal{V}}$ is continuously extended to $X=\bar{X}$ so that the original boundary $\left\{(\bar{X},Z):Z\in(0,1]\right\}$ corresponds to the limit $Y\to\infty$ in the (X,Y) coordinates, the boundary condition becomes $\lim_{Y\to+\infty}\widetilde{\mathcal{V}}(\overline{X},Y)=\mathcal{V}_\infty$. The feedback rule can also be expressed in terms of (X,Y) as

$$\widetilde{\sigma}(X,Y)=\sqrt{2\left(r+\frac{\Delta(Y-1)}{Y}\right)\widetilde{\mathcal{V}}(X,Y)+2\Delta(Y-1)\frac{\partial \widetilde{\mathcal{V}}}{\partial Y}(X,Y)}. \tag{5.16}$$

For future reference, observe that $\widetilde{\sigma}(0,1)$ is the optimal action at the start of the optimal trajectory (since $X=0$, $Z=1$, and thus $Y=1$ for this trajectory).

The solution to (5.15) is entirely determined by the hazard rate $\frac{f(X)}{1-F(X)}$. This reduction suggests that further properties of the value functions and feedback rules can be simply obtained for exponential distributions and comparative statics are easier to get when distributions are ranked in terms of their hazard rates. Next paragraphs take those routes.

Scenario 3: Exponential Distributions. Suppose that the TP is exponentially distributed over $\mathbb{R}_+$ with mean $\mu>0$, i.e., $F(X)=1-e^{-\frac{X}{\mu}}$. The hazard rate is now constant, $\frac{f(X)}{1-F(X)}=\frac{1}{\mu}$. To simplify the analysis, we also assume $\zeta\in[\mu\Delta,\mu\lambda]$.[18] We look for a solution to (5.15) that only depends on $Y=Ze^{\frac{X}{\mu}}$, namely $\widetilde{\mathcal{V}}(X,Y)=v_\mu(Y)$ (where v_μ is continuously differentiable), and which also induces a feedback rule of the form $\sigma^e(X,Z)=\widetilde{\sigma}_\mu(Ze^{\frac{X}{\mu}})$.

Proposition 4. *The value function $\mathcal{V}^e(X,Z)$ satisfies $\mathcal{V}^e(X,Z)=v_\mu(Ze^{\frac{X}{\mu}})$ where v_μ is the unique solution to*

$$\frac{Yv_\mu'(Y)}{\mu}=-\zeta+\sqrt{2\left(r+\frac{\Delta(Y-1)}{Y}\right)v_\mu(Y)+2\Delta(Y-1)v_\mu'(Y)} \tag{5.17}$$

with the boundary condition $\lim_{Y\to+\infty}v_\mu(Y)=\mathcal{V}_\infty$. Moreover, v_μ is everywhere decreasing and admits the following asymptotic development for Y large

$$v_\mu(Y)=\mathcal{V}_\infty+\frac{\Delta\mathcal{V}_\infty}{r+\frac{\zeta}{\mu}}\frac{1}{Y}+o\left(\frac{1}{Y}\right). \tag{5.18}$$

[18] The scenarios $\zeta\in[0,\mu\Delta)$ and $\zeta\geq\mu\lambda$ require a slightly different analysis and are thus omitted because of space constraints.

The feedback rule satisfies

$$\widetilde{\sigma}_\mu(Y) = \zeta + \frac{Y v'_\mu(Y)}{\mu} < \zeta, \quad \forall Y \geq 1. \tag{5.19}$$

The optimal action $x^e(\tau) = \sigma_\mu(Y(\tau))$ *converges towards the myopic outcome at an exponential rate:*

$$\frac{\dot{x}^e(\tau)}{\zeta - x^e(\tau)} \sim_{\tau \to +\infty} \frac{\zeta}{\mu} - \Delta. \tag{5.20}$$

Because of the compounding impacts of both the *Irreversibility* and *Pseudo-Learning Effect*, actions always remain here also below the myopic optimum. Observe that actions are kept constant on any iso-Y locus. As the stock increases, the survival probability must thus decrease to keep actions constant. Along the optimal trajectory starting from $X(0) = 0$ and $Z(0) = 1$ and satisfying (5.1)-(5.3), the index $Y(\tau)$ is increasing and goes to $+\infty$. In other words, the stock increases over time much faster than the survival probability decreases. Although the feedback rule is not necessarily monotonic in Y, actions certainly converge over time towards the myopic outcome without reaching it because the support of the distribution is now unbounded and there is always a chance that the TP remains ahead. The rate of convergence is almost constant in the long-run and convergence is faster as it becomes more likely that the TP is passed earlier on, i.e., when the distribution has a lower mean μ. ■

Hazard Rate Stochastic Dominance. Consider the distribution F_ε constructed as

$$F_\varepsilon(X) = 1 - e^{-\epsilon X} + e^{-\epsilon X} F(X), \quad \forall X \in \left[0, \overline{X}\right].$$

For any $\epsilon \geq 0$, we have $F_\varepsilon(X) \geq F(X)$ for all X, and thus F dominates F_ε in the sense of first-order stochastic dominance. More interestingly, F dominates F_ε in the sense of hazard rate stochastic order[19] since

$$\frac{f_\varepsilon(X)}{1 - F_\varepsilon(X)} = \frac{f(X)}{1 - F(X)} + \epsilon, \quad \forall X \in \left[0, \overline{X}\right].$$

We are now interested in how solutions are modified when replacing F with F_ε. To this end, let denote by $\widetilde{\mathcal{V}_\varepsilon}$ and $\widetilde{\sigma_\varepsilon}$ (resp. $\widetilde{\mathcal{V}}$ and $\widetilde{\sigma}$) the transformed value function solution to (5.15) and the associated feedback rule when the distribution is F_ε (resp. F). To perform Taylor

[19] Shaked and Shanthikumar (1994).

expansions for ε small enough, we assume that $\widetilde{\mathcal{V}_\varepsilon}$ and $\widetilde{\sigma_\varepsilon}$ are continuously differentiable in ε around zero. We also require that $\widetilde{\mathcal{V}}$ first satisfies

$$\frac{\partial \widetilde{\mathcal{V}}}{\partial Y}(X,Y) < 0 \quad \forall (X,Y). \tag{5.21}$$

Consider now a stock-beliefs trajectory $(X(\tau, X_0), Z(\tau, X_0))$ with initial conditions $X(0) = X_0 \in [0, \overline{X}]$ and $Z(0) = 1$. Let also denote the corresponding index $Y(\tau, X_0) = \frac{Z(\tau,X_0)}{1-F(X(\tau,X_0))}$. Along such a trajectory, we assume that

$$\begin{cases} \hat{w}(\tau, X_0) = \frac{Y(\tau,X_0)\widetilde{\sigma}(X(\tau,X_0),Y(\tau,X_0))\frac{\partial \widetilde{\mathcal{V}}}{\partial Y}(X(\tau,X_0),Y(\tau,X_0))}{\lambda - \frac{\Delta}{Y(\tau,X_0)}} & \text{increasing,} \\ \lim_{\tau\to+\infty} \hat{w}(\tau, X_0) = 0. \end{cases} \tag{5.22}$$

Conditions (5.21) holds in the exponential Scenario 3 above but is likely to be satisfied more broadly for “nearby" distributions. Condition (5.22) is more technical but also holds in that case, at least for τ large enough.[20]

Proposition 5. *Suppose that (5.21) and (5.22) both hold. For ε small enough but positive,*

$$\widetilde{\mathcal{V}_\varepsilon}(X,Y) < \widetilde{\mathcal{V}}(X,Y) \quad \forall (X,Y) \in \mathbb{R}_+ \times [1, +\infty), \tag{5.23}$$

$$\widetilde{\sigma_\varepsilon}(0,1) < \widetilde{\sigma}(0,1). \tag{5.24}$$

Moving from F to F_ε, it becomes more likely that the TP is passed earlier on. The value function decreases accordingly. From (5.16), optimal actions at the start of the trajectory $(X = 0,\, Y = 1)$, are entirely determined by the value function at that point. A lower value function suggests that the *Irreversibility* and *Pseudo-Learning Effects* are of greater magnitude; which pushes DM to choose lower actions earlier on. ∎

6. Stock-Based Feedback Rules

The value function $\mathcal{V}^e(X,Z)$ is a technical device to use dynamic programming techniques and compute a feedback rule $\sigma^e(X,Z)$ that guides behavior along the optimal trajectory. Thanks to its generality, this approach could also be attractive to understand how the trajectory is modified if additional shocks might impact the stock trajectory and modify

[20] Unfortunately, we have been unable to find general conditions on primitives that ensures those conditions hold.

beliefs. In our context, stock and beliefs evolve deterministically and there is actually a one-to-one relationship, say $Z^o(X)$, between stock and beliefs along this trajectory. Knowing how the stock evolves over time, DM can thus always reconstruct beliefs from the whole past history of actions and correctly infers how to discount future payoffs along the optimal path. Along this path, $\sigma^e(X,Z)$ induces a simpler feedback rule

$$\sigma^o(X) = \sigma^e(X, Z^o(X)) \quad \forall X \in \left[0, \overline{X}\right]. \tag{6.1}$$

Deviations from the optimal path would presumably decouple stocks and beliefs. Consequently, arriving at an off-path stock following a deviation, DM would presumably hold beliefs different from those assumed by abiding to $\sigma^o(X)$ and typically the survival probability would be different from $Z^o(X)$. This suggests that implementing the optimal path by following $\sigma^o(X)$ effectively requires commitment to this stock-based feedback rule. Nevertheless, viewing action profile and stock evolution through this lens makes the comparison with SCENARIO 2 (the case where the TP is known to lie ahead for sure) and SCENARIO 4 below (the case where DM learns TP when it is passed) more compelling. Along this path, the stock $X^o(\tau, X)$ evolves as

$$\frac{\partial X^o}{\partial \tau}(\tau, X) = \sigma^o(X^o(\tau, X)) \text{ with } X^o(0; X) = X. \tag{6.2}$$

From (5.3), the survival probability that takes value $Z^o(X)$ at date 0 is

$$Z^o(X^o(\tau, X)) = 1 - \Delta e^{-\Delta\tau} \int_0^\tau F(X^o(s, X)) e^{\Delta s} ds - (1 - Z^o(X)) e^{-\Delta\tau}. \tag{6.3}$$

Differentiating w.r.t. τ unveils how $Z^o(X)$ evolves with the stock as

$$\sigma^o(X)\dot{Z}^o(X) = \Delta(1 - F(X) - Z^o(X)) \quad \forall X \geq 0 \text{ with } Z^o(0) = 1. \tag{6.4}$$

The value function can also be expressed in terms of X only as

$$\mathcal{V}^o(X) = \mathcal{V}^e(X, Z^o(X)) = \int_0^{+\infty} e^{-r\tau} \frac{Z^o(X^o(\tau, X))}{Z^o(X)} u(\sigma^o(X^o(\tau, X))) d\tau. \tag{6.5}$$

We also define DM's intertemporal payoff once the TP has been passed but, being not aware of that event, DM still follows the feedback rule σ^o, as

$$\varphi^o(X) = \int_0^{+\infty} e^{-\lambda\tau} u(\sigma^o(X^o(\tau, X))) d\tau. \tag{6.6}$$

Observe that, $F(X^o(s, X)) \leq 1$ implies that $Z^o(X^o(\tau, X)) \geq Z^o(X)e^{-\Delta\tau}$, and thus $\mathcal{V}^o(X) \geq \varphi^o(X)$. The payoff difference $\mathcal{V}^o(X) - \varphi^o(X)$ stands for the *value of optimism*.

Thinking that the TP may still be ahead with some probability, DM's payoff $\mathcal{V}^o(X)$ is computed with more optimistic beliefs and less discounting than $\varphi^o(X)$.

Next proposition characterizes how $\mathcal{V}^o(X)$ evolves and defines the stock-based feedback rule $\sigma^o(X)$ along the optimal trajectory.

PROPOSITION 6. *The value function* $\mathcal{V}^o(X)$ *satisfies*

$$\dot{\mathcal{V}}^o(X) = -\zeta - \frac{\dot{Z}^o(X)}{Z^o(X)}\mathcal{V}^o(X) + \sqrt{2r\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2} \quad \forall X \in [0, \overline{X}) \tag{6.7}$$

with the boundary condition $\mathcal{V}^o(\overline{X}) = \mathcal{V}_\infty$.[21] *The feedback rule* $\sigma^o(X)$ *satisfies*

$$\sigma^o(X) = \zeta + \dot{\mathcal{V}}^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}(\mathcal{V}^o(X) - \varphi^o(X)) \quad \forall X \in [0, \overline{X}). \tag{6.8}$$

The feedback rule (6.8) looks like its counterpart (5.6) found in SCENARIO 2, where, by definition, the sole state variable is the level of stock. To understand the changes, it is useful to come back on the expression of the value function (6.5). Starting from a current stock X and on-path beliefs $Z^o(X)$, consider an impulse deviation that would consist in adopting action x over an interval of infinitesimal length ε.[22] Accordingly, the stock has jumped by $x\varepsilon$ at the end of the deviation and stock and belief move to new trajectories $\widetilde{X}(\varepsilon, x, \tau, X)$ and $\widetilde{Z}(\varepsilon, x, \tau, X)$ afterwards, with the payoff from the deviation being

$$\widetilde{\mathcal{V}}(\varepsilon, x, X) = \int_0^{+\infty} e^{-r\tau} \frac{\widetilde{Z}(\varepsilon, x, \tau, X)}{Z^o(X)} u(\sigma^o(\widetilde{X}(\varepsilon, x, \tau, X)))d\tau.$$

First-order Taylor expansions w.r.t. ε show that the stock trajectory is modified as

$$\widetilde{X}(\varepsilon, x, \tau, X) - X^o(\tau, X) = \varepsilon\,(x - \sigma^o(X))\,\frac{\partial X^o}{\partial X}(\tau, X) + o(\varepsilon). \tag{6.9}$$

The impulse deviation has a long-lasting impact on stock because, just after the deviation, actions change and so do future levels of stock and subsequent actions.

[21] The functional equation (6.7) that describes how the value function evolves is non-local. Indeed, $\dot{\mathcal{V}}^o(X)$ depends on $\varphi^o(X)$ which itself depends on actions that will be taken along the whole future trajectory. In turn, those actions are determined by the feedback rule (6.8) and depend on the value function at all future dates. In other words, (6.7) is not a simple differential equation which would determine how the value function varies locally but a functional equation which is the projection of the partial differential equation (5.10) along the optimal trajectory when $Z = Z^o(X)$.

[22] This approach of looking at the benefit of any impulse deviation is similar to that that was developed in Karp and Lee (2003), Karp (2005, 2007), Ekeland, Karp and Sumaila (2015), Ekeland and Lazrak (2010) and Auster, Che and Mierendorff (2024) in other contexts.

An impulse deviation has also an impact on beliefs as

$$\widetilde{Z}(\varepsilon,x,\tau,X)-Z^o(X^o(\tau,X))=\varepsilon\,(x-\sigma^o(X))\left(\frac{d}{dX}(Z^o(X^o(\tau,X)))-\dot{Z}^o(X)e^{-\Delta\tau}\right)+o(\varepsilon). \tag{6.10}$$

To understand the nature of the beliefs variation, consider a hypothetical scenario where beliefs evolve with the stock according to a fixed rule of thumb, say $Z^*(X)$, and suppose that, as the stock increases, those beliefs become more pessimistic on survival, i.e., $\dot{Z}^*(X)<0$. The sole effect of a deviation would thus be to make DM think that survival is less likely. The beliefs variation would be reduced to the first term on the r.-h. s. of (6.10)

$$\widetilde{Z}^*(\varepsilon,x,\tau,X)-Z^*(X^*(\tau,X))=\varepsilon\,(x-\sigma^*(X))\,\frac{d}{dX}(Z^*(X^*(\tau,X)))+o(\varepsilon).$$

When beliefs evolve endogenously, an impulse deviation has another effect. That no catastrophe will arise in the immediate future will bring "good news"-the TP might not have been passed after all-and that view is carried over in the future as captured by the second term of (6.10). Of course, the impact of this term decreases over time because information that has been learned from survival at a given date by an impulse deviation at that time only becomes less and less relevant in the future as we already know from the discussion surrounding (4.5).

Finally, we may get a first-order Taylor expansion for DM's payoff as

$$\widetilde{\mathcal{V}}(\varepsilon,x,X)-\mathcal{V}^o(X)=\varepsilon\left(u(x)-u(\sigma^o(X))+(x-\sigma^o(X))\,\frac{\partial\mathcal{V}^e}{\partial X}(X,Z^o(X))\right)+o(\varepsilon). \tag{6.11}$$

The first term on the r.-h. s. of (6.11) represents the marginal benefit of the deviation on flow payoff. Because $\sigma^o(X)<\zeta$, an upward impulse deviation is desirable. The second term quantifies the cost of this deviation in terms of continuation values, taking the belief dynamics as given; a standard marginal term from dynamic programming. This marginal cost can itself be split into two components as[23]

$$\frac{\partial\mathcal{V}^e}{\partial X}(X,Z^o(X))=\dot{\mathcal{V}}^o(X)+\frac{\dot{Z}^o(X)}{Z^o(X)}(\mathcal{V}^o(X)-\varphi^o(X)). \tag{6.12}$$

The first term on the r.-h. s. captures the *Irreversibility Effect* in a context of uncertainty on where the TP is located. It was already present in the complete information SCENARIO 2. The second term captures the effect of the deviation on beliefs. Following an upward deviation, and conditional on survival, DM becomes more optimistic that the TP lies ahead.

[23] See Appendix C.

The *Pseudo-Learning Effect* pushes toward more precaution and lower actions. Along an optimal path, no impulse deviation should be profitable. Accordingly, the r.-h. s. of (6.11) is maximized at $\sigma^o(X)$ as defined in (6.8).

Viewed in terms of shadow cost, the comparison of (6.8) and (5.6) highlights that, under uncertainty, the shadow cost of carbon should entail an *"uncertainty premium"* worth $-\frac{\dot{Z}^o(X)}{Z^o(X)}(\mathcal{V}^o(X)-\varphi^o(X))$. That premium vanishes at $\overline{X}$, reflecting the *"End-of-Support" Effect*.

SCENARIO 4: LEARNING BAD NEWS. Consider a scenario where DM still does not know where the TP lies but immediately learns this TP upon passing it as in Tsur and Zemel (1996, 2021).[24] DM thus knows that her payoffs should be discounted at rate r as long as she has not yet learned having passed the TP and at higher rate λ afterwards. The stock X is now the sole state variable. Observe also that the probability of not having yet switched regime is then $1-F(X)$ and that, once the TP has been passed, the myopic action is again chosen which yields payoff $\mathcal{V}_\infty$. We define $\mathcal{V}^u(X)$, the value function conditionally on not having yet learned that the TP has been passed, as

$$(1-F(X))\mathcal{V}^u(X) = \max_{x(\cdot),X(\cdot)\text{ s.t. }(5.1)} \int_0^{+\infty} e^{-r\tau}\left((1-F(X(\tau)))u(x(\tau)) + f(X(\tau))x(\tau)\mathcal{V}_\infty\right)d\tau \tag{6.13}$$

where $f(X(\tau))x(\tau)d\tau$ is the probability of passing the TP within the interval $[\tau,\tau+d\tau]$.

PROPOSITION 7. *The value function $\mathcal{V}^u(X)$ satisfies*

$$\dot{\mathcal{V}}^u(X) = -\zeta + \frac{f(X)}{1-F(X)}(\mathcal{V}^u(X)-\mathcal{V}_\infty) + \sqrt{2r\mathcal{V}^u(X)} \quad \forall X \in [0,\overline{X}) \tag{6.14}$$

with the boundary condition $\mathcal{V}^u(\overline{X}) = \mathcal{V}_\infty$. The feedback rule $\sigma^u(X)$ is

$$\sigma^u(X) = \zeta + \dot{\mathcal{V}}^u(X) - \frac{f(X)}{1-F(X)}(\mathcal{V}^u(X)-\mathcal{V}_\infty) \quad \forall X \in [0,\overline{X}). \tag{6.15}$$

The comparison with (6.8) is straightforward. When aware of having passed the TP within the interval $[\tau,\tau+d\tau]$ following an impulse deviation with action x, DM knows for sure that her continuation payoff will drop from $\mathcal{V}^u(X^u(\tau))$ to $\mathcal{V}_\infty$; a true *Learning Effect*. This drop in payoff arises at rate $\frac{f(X^u(\tau))}{1-F(X^u(\tau))}x$. The corresponding loss in terms of expected payoff within that small interval is thus

$$\frac{f(X^u(\tau))}{1-F(X^u(\tau))}(\mathcal{V}^u(X^u(\tau))-\mathcal{V}_\infty)x d\tau.$$

[24] This scenario is also similar to Loury (1978)'s analysis of how to exploit an unknown resource.

When unaware whether the TP has been passed, DM's payoff drops by the value of optimism $\mathcal{V}^o(X^o(\tau)) - \varphi^o(X)$ as survival probability declines. Given that this probability declines proportionally at rate $-\frac{\dot{Z}^o(X^o(\tau))}{Z^o(X^o(\tau))}x$, the corresponding loss within a small interval of length $d\tau$ is thus

$$-\frac{\dot{Z}^o(X^o(\tau))}{Z^o(X^o(\tau))}(\mathcal{V}^o(X^o(\tau)) - \varphi^o(X))xd\tau.$$

Comparative Statics. Proposition 6 expresses the optimal feedback rule and the value function exclusively in terms of the stock. This feature allows a direct comparison with the feedback rule and value function that apply in Scenarios 2 and 4.

Proposition 8. *For X in a left-neighborhood of $\overline{X}$,*

$$\mathcal{V}^k(X) \geq \mathcal{V}^u(X) \geq \mathcal{V}^o(X) \text{ (with } \mathcal{V}^o(\overline{X}) = \mathcal{V}^u(\overline{X}) = \mathcal{V}^k(\overline{X}) = \mathcal{V}_\infty\text{)}, \tag{6.16}$$

$$\sigma^o(X) > \sigma^k(X) \geq \sigma^u(X) \geq \zeta\sqrt{\frac{r}{\lambda}} = \sigma^k(\overline{X}^-) = \sigma^u(\overline{X}^-) \text{ and } \sigma^o(\overline{X}^-) = \zeta. \tag{6.17}$$

When the upper bound of the support $\overline{X}$ is reached, DM knows for sure that the TP has been passed and all scenarios have the same continuation. Right before $\overline{X}$, DM's payoff is higher when she is better informed as posited in (6.16). Turning now to actions, we observe that, in the left-neighborhood of $\overline{X}$, there is little difference between Scenarios 2 and 4. In both cases, DM knows that either the TP is at $\overline{X}$ for sure (Scenarios 2) or it very likely to be around that value if nothing has been learned yet (Scenarios 4). Actions are thus almost the same under both scenarios. The *Irreversibility Effect* dominates and strong distortions away from the myopic optimum are needed just before $\overline{X}$ before jumping to the myopic outcome. Instead, when DM does not know whether the TP has been passed, the *"End-of-Support" Effect* matters in a left-neighborhood of $\overline{X}$. Actions are thus already close to the myopic optimum in a left-neighborhood of $\overline{X}$.

7. Concluding Remarks

We have studied a dynamic decision-making problem under irreversibility and uncertainty. Acting generates an immediate surplus, but higher current actions increase the likelihood that the stock will cross a TP, triggering an irreversible rise in the probability of an environmental catastrophe. The location of this TP is unknown. We have characterized DM's value function and optimal feedback rule, and provided comparative statics and detailed features of the trajectories under a number of important scenarios. Optimal actions at any moment reflect beliefs about whether the TP has already been crossed. The optimal action results from compounding several effects. First, acting today entails some irreversibility; an *Irreversibility Effect* that would be also present if the TP was known for sure to be ahead and that calls for reducing current actions. Second, acting today and surviving makes DM more optimistic on the fact that that the TP is still ahead; a *Pseudo-Learning Effect* which reinforces her incentives to reduce actions. Third, near the end of the support of the distribution of TPs, an *"End-of-Support" Effect* pushes actions up as DM believes that the TP has certainly been passed anyway.

While our model captures key intuitions about decision-making in environmental contexts, it could be extended in several ways that would strengthen the model's applicability to physical and climatological systems. The arrival rate of a catastrophe could depend nonlinearly on the stock rather than moving through a single discrete jump, or could involve multiple jumps, without altering the qualitative nature of the optimal solution.

A second extension would allow the decision-maker to learn about the TP through external signals, for example via scientific progress. As the trajectory approaches the TP, accumulating evidence would make past actions insufficient to determine beliefs. We conjecture, however, that the optimal feedback rule would call for higher actions as more precise information confirms that the TP has been crossed.

Another extension would allow active mitigation: the decision-maker could incur a cost to reduce the stock. This fits naturally into our framework, though at the expense of more complex dynamics. For example, with a known TP and limited depreciation, optimal actions may initially be high-because the stock moves slowly-followed by lower actions just after the TP as the decision-maker attempts to bring the stock below the threshold again. Depreciation would therefore smooth the variation in actions. How much smoothing persists when the TP is uncertain remains an open question.

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Appendix A: Value Function and Feedback Rule

Beliefs

We first present the evolution of the posterior density function $f(\widetilde{X}|t,\mathbf{x}^t)$. Notice that, as times passes, a stock process $\hat{X}(t;0,0)$ that solves (5.1) with $\hat{X}(0;0,0)=0$ goes through various possible values $\widetilde{X}$ of the TP. We may thus also describe this process by the time $T(\widetilde{X};0,0)$ at which this stock reaches $\widetilde{X}$. If $\hat{X}(t;0,0)$ is smooth, increasing and differentiable in t, with no flat part, $T(\widetilde{X};0,0)$ is itself increasing and smooth and differentiable with finite derivative.

Lemma A.1. *The posterior density function $f(\widetilde{X}|t,\mathbf{x}^t)$ conditional on survival up to date t and history $\mathbf{x}^t$ satisfies:*

$$f(\widetilde{X}|t,\mathbf{x}^t)=\begin{cases}\frac{f(\widetilde{X})}{\hat{Z}(t,\mathbf{x}^t)} & \text{if } \hat{X}(t;0,0)\leq\widetilde{X}\\ \frac{e^{-\Delta(t-T(\widetilde{X};0,0))}}{\hat{Z}(t,\mathbf{x}^t)}f(\widetilde{X}) & \text{otherwise.}\end{cases}\tag{A.1}$$

Proof of Lemma A.1. We first compute the survival probability $\hat{Z}(t,\mathbf{x}^t)$ as (4.2). The first term on the r.-h.s. of (4.2) stems for the probability that the TP is below $\hat{X}(t;0,0)$, and the rate of survival then jumps up to Δ at a date $T(\widetilde{X};0,0)\leq t$. The second term is the probability that the TP is above $\hat{X}(t;0,0)$ and the rate of arrival of a catastrophe is still zero. Denote these terms respectively by P_{1t} and P_{2t}. We immediately compute

$$P_{2t}=1-F(\hat{X}(t;0,0)). \tag{A.2}$$

Changing variables and letting $\hat{X}(\tau;0,0)=\widetilde{X}$ with $\frac{\partial\hat{X}}{\partial\tau}(\tau;0,0)d\tau=d\widetilde{X}$, we rewrite

$$P_{1t}=\int_0^{\hat{X}(t;0,0)} f(\widetilde{X})e^{-\Delta(t-T(\widetilde{X};0,0))}d\widetilde{X}=\int_0^t f(\hat{X}(\tau;0,0))\frac{\partial\hat{X}}{\partial\tau}(\tau,0)e^{-\Delta(t-\tau)}d\tau.$$

Integrating by parts yields

$$P_{1t}=F(\hat{X}(t;0,0))-\Delta\int_0^t F(\hat{X}(\tau;0,0))e^{\Delta(\tau-t)}d\tau. \tag{A.3}$$

Inserting (A.2) and (A.3) into (4.2) yields (4.3) and thus (A.1). □

Value Function

Proof of Lemma 1. Following history $\mathbf{x}^t$, the stock at date $\tau\geq t$ is $\hat{X}(\tau;t,X)=X+\int_t^\tau x(s)ds$ with a stream of future actions $\mathbf{x}_t=(x(\tau))_{\tau\geq t}$. Let $T(\widetilde{X};t,X)$ accordingly denote the inverse function defined for $\widetilde{X}\geq X$. The value function $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ can be written as

$$\hat{\mathcal{V}}(t,\mathbf{x}^t)=\max_{\mathbf{x}_t,\hat{X}(\cdot;t,X)}\int_0^X\left(\int_t^{+\infty}e^{-r(\tau-t)}e^{-\Delta(\tau-t)}u(x(\tau))d\tau\right)f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}$$

$$+\int_X^{+\infty}\left(\int_t^{T(\widetilde{X};t,X)}e^{-r(\tau-t)}u(x(\tau))d\tau+\int_{T(\widetilde{X};t,X)}^{+\infty}e^{-r(\tau-t)}e^{-\Delta(\tau-T(\widetilde{X};t,X))}u(x(\tau))d\tau\right)f(\widetilde{X}|t,\mathbf{x}^t)d\widetilde{X}.$$

Taking into account (A.1), we rewrite $\hat{\mathcal{V}}(t,\mathbf{x}^t)$ as

$$\hat{Z}(t,\mathbf{x}^t)\hat{\mathcal{V}}(t,\mathbf{x}^t)=\max_{\mathbf{x}_t,\hat{X}(\cdot,X)}\int_0^X\left(\int_t^{+\infty}e^{-r(\tau-t)}e^{-\Delta(\tau-t)}u(x(\tau))d\tau\right)e^{-\Delta(t-T(\widetilde{X};0,0))}f(\widetilde{X})d\widetilde{X} \tag{A.4}$$

$$+\int_X^{+\infty}\left(\int_t^{T(\widetilde{X};t,X)}e^{-r(\tau-t)}u(x(\tau))d\tau+\int_{T(\widetilde{X};t,X)}^{+\infty}e^{-r(\tau-t)}e^{-\Delta(\tau-T(\widetilde{X};t,X))}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X}.$$

Let

$$\mathcal{I}_1=\int_0^X\left(\int_t^{+\infty}e^{-r(\tau-t)}e^{-\Delta(\tau-t)}u(x(\tau))d\tau\right)e^{-\Delta(t-T(\widetilde{X};0,0))}f(\widetilde{X})d\widetilde{X}$$

which rewrites as

$$\mathcal{I}_1=\left(\int_t^{+\infty}e^{-\lambda(\tau-t)}u(x(\tau))d\tau\right)\left(\int_0^X e^{-\Delta(t-T(\widetilde{X};0,0))}f(\widetilde{X})d\widetilde{X}\right). \tag{A.5}$$

Changing variables and letting $\hat{X}(\tau;0,0)=\widetilde{X}$ for $\tau\le t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau;0,0)d\tau=d\widetilde{X}$, we also rewrite

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0,0))}f(\widetilde{X})d\widetilde{X}=\int_0^t e^{-\Delta(t-\tau)}f(\hat{X}(\tau;0,0))\frac{\partial \hat{X}}{\partial \tau}(\tau,0)d\tau.$$

Integrating by parts thus yields

$$\int_0^X e^{-\Delta(t-T(\widetilde{X};0,0))}f(\widetilde{X})d\widetilde{X}=F(X)-\Delta e^{-\Delta t}\int_0^t F(\hat{X}(\tau;0,0))e^{\Delta\tau}d\tau$$

where the last equality follows from $\hat{X}(t;0,0)=X$. Inserting into (A.5) yields

$$\mathcal{I}_1=\left(\int_t^{+\infty} e^{-\lambda(\tau-t)}u(x(\tau))d\tau\right)\left(F(X)-\Delta e^{-\Delta t}\int_0^t F(\hat{X}(s;0,0))e^{\Delta s}ds\right). \tag{A.6}$$

We now compute

$$\mathcal{I}_2=\int_X^{+\infty}\left(\int_t^{T(\widetilde{X};t,X)} e^{-r(\tau-t)}u(x(\tau))d\tau+\int_{T(\widetilde{X};t,X)}^{+\infty} e^{-r(\tau-t)}e^{-\Delta(\tau-T(\widetilde{X};t,X))}u(x(\tau))d\tau\right)f(\widetilde{X})d\widetilde{X}.$$

Changing variables and letting $\hat{X}(\tau-t;t,X)=\widetilde{X}$ for $\tau\ge t$ with $\frac{\partial \hat{X}}{\partial \tau}(\tau-t;t,X)d\tau=d\widetilde{X}$ and $\hat{X}(0;t,X)=X$, we also rewrite

$$\mathcal{I}_2=\int_t^{+\infty}\left(\int_t^{\tau} e^{-r(s-t)}u(x(s))ds+e^{\Delta(\tau-t)}\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)f(\hat{X}(\tau-t;t,X))\frac{\partial \hat{X}}{\partial \tau}(\tau-t;t,X)d\tau.$$

Integrating by parts yields

$$\mathcal{I}_2=\left[F(\hat{X}(\tau-t;t,X))\left(\int_t^{\tau} e^{-r(s-t)}u(x(s))ds+e^{\Delta(\tau-t)}\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)\right]_t^{+\infty}$$

$$-\Delta\int_t^{+\infty}F(\hat{X}(\tau-t;t,X))e^{\Delta(\tau-t)}\left(\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)d\tau.$$

Using that $\lim_{\tau\to+\infty}F(\hat{X}(\tau-t;t,X))=1$ if the stock passes $\overline{X}$ (which holds when the minimal action is positive at any point of time as we will see below), we get

$$\mathcal{I}_2=\int_t^{+\infty} e^{-r(s-t)}u(x(s))ds-F(X)\int_t^{+\infty} e^{-\lambda(s-t)}u(x(s))ds \tag{A.7}$$

$$-\Delta\int_t^{+\infty}F(\hat{X}(\tau-t;t,X))e^{\Delta(\tau-t)}\left(\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)d\tau.$$

Integrating by parts, we obtain

$$\int_t^{+\infty}F(\hat{X}(\tau-t;t,X))e^{\Delta\tau}\left(\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)d\tau$$

$$=\left[\left(\int_t^{\tau}F(\hat{X}(s-t;t,X))e^{\Delta s}ds\right)\left(\int_\tau^{+\infty} e^{-\lambda(s-t)}u(x(s))ds\right)\right]_t^{+\infty}$$

$$+\int_t^{+\infty} e^{-\lambda(\tau-t)}\left(\int_t^{\tau} F(\hat{X}(s-t;t,X))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

$$=\int_t^{+\infty} e^{-\lambda(\tau-t)}\left(\int_t^{\tau} F(\hat{X}(s-t;t,X))e^{\Delta s}ds\right)u(x(\tau))d\tau.$$

Inserting into (A.7), we thus obtain

$$\mathcal{I}_2=\int_t^{+\infty} e^{-r(s-t)}u(x(s))ds-F(X)\int_t^{+\infty} e^{-\lambda(s-t)}u(x(s))ds \tag{A.8}$$

$$-\Delta e^{-\Delta t}\int_t^{+\infty} e^{-\lambda(\tau-t)}\left(\int_t^{\tau} F(\hat{X}(s-t;t,X))e^{\Delta s}ds\right)u(x(\tau))d\tau.$$

Summing up (A.6) and (A.8) and taking into account that $\hat{X}(s-t;t,X)=\hat{X}(s-t;t,\hat{X}(t;0,0))=\hat{X}(s;0,0)$ for $s\geq t$, yields

$$\mathcal{I}=\int_t^{+\infty} e^{-r(\tau-t)}u(x(\tau))d\tau-\Delta e^{-\Delta t}\int_t^{+\infty} e^{-\lambda(\tau-t)}\left(\int_0^{\tau} F(\hat{X}(s;0,0))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

and thus

$$\mathcal{I}=\int_t^{+\infty} e^{-r(\tau-t)}\left(1-\Delta e^{-\Delta\tau}\int_0^{\tau} F(\hat{X}(s;0,0))e^{\Delta s}ds\right)u(x(\tau))d\tau.$$

Changing variables and letting $s=\tau-t$ yields

$$\mathcal{I}=\int_0^{+\infty} e^{-rs}\left(1-\Delta e^{-\Delta(s+t)}\int_0^{s+t} F(\hat{X}(s;0,0))e^{\Delta s}ds\right)u(x(s+t))ds. \tag{A.9}$$

Using the definition of $\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})$ given in (4.4), inserting into (A.9) and changing the name of dummy variables yields

$$\mathcal{I}=\int_0^{+\infty} e^{-r\tau}\hat{Z}(t+\tau,\mathbf{x}^{t+\tau})u(x(\tau+t))d\tau.$$

Inserting into (A.4) yields (4.6). □

Next proposition provides properties of $\mathcal{V}^e(X,Z)$. As X increases, $\mathcal{V}^e(X,Z)$ necessarily decreases since the irreversibility constraint is hardened as X comes closer to $\overline{X}$.

Proposition A.1. *There exists a solution to the optimization problem (5.4). $Z\mathcal{V}^e(X,Z)$ is non-increasing in X, convex in Z, Lipschitz-continuous and thus a.e. differentiable.*

Proof of Proposition A.1. We first define $\mathcal{W}^e(X,Z)=Z\mathcal{V}^e(X,Z)$. Inserting (5.2) into the r.-h.s. of (5.4), we thus rewrite

$$\mathcal{W}^e(X,Z)=\max_{x(\cdot),X(\cdot),Z(\cdot),\overline{T},\text{ s.t. (5.1)-(5.3)},X(\overline{T})=\overline{X}}(Z-1)\left(\int_0^{\overline{T}} e^{-\lambda\tau}u(x(\tau))d\tau\right. \tag{A.10}$$

$$+\mathcal{V}_\infty e^{-\lambda\overline{T}}\Bigg) + \int_0^{\overline{T}} e^{-r\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds\right)u(x(\tau))d\tau$$

$$+\int_{\overline{T}}^{+\infty} e^{-r\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F(X(s))e^{\Delta s}ds\right)\lambda\mathcal{V}_\infty d\tau.$$

Existence. Existence of a solution to the optimization problem (A.10) follows from applying Filipov-Cesari Theorem with free final time (see Seierstad and Sydsaeter, 1987, Theorem 12, p. 145). To check that all conditions for this theorem are satisfied, first observe that $\mathcal{X}$ is closed and bounded, while X is bounded above by $\overline{X}$ and Z is also bounded ($Z\in[0,1]$). Denote

$$N(X,Z,\mathcal{X},\tau)=\{\,(\gamma^1,\gamma^2,\gamma^3,x)\mid e^{-r\tau}Zu(x)+\gamma^1\geq 0,\gamma^2=x,\gamma^3=\Delta(1-F(X)-Z),x\in\mathcal{X}\}.$$

We check that $N(X,Z,\mathcal{X},\tau)$ is convex for each (X,Z,τ). Take a pair $(\gamma_i^1,\gamma_i^2,\gamma_i^3,x_i)\in N(X,Z,\mathcal{X},\tau)$ for $i=1,2$ Observe that $,\gamma_1^3=\gamma_2^3=\gamma^3$ since X and Z are fixed. Consider now $\lambda(\gamma_1^1,\gamma_1^2,\gamma^3,x_1)+(1-\lambda)(\gamma_2^1,\gamma_2^2,\gamma^3,x_2)$ for any $\lambda\in[0,1]$. By convexity of $\mathcal{X}$, $\lambda x_1+(1-\lambda)x_2\in\mathcal{X}$. Observe also that u concave implies

$$e^{-r\tau}Zu(\lambda x_1+(1-\lambda)x_2)+\lambda\gamma_1^1+(1-\lambda)\gamma_2^1\geq\lambda(e^{-r\tau}Zu(x_1)+\gamma_1^1)+(1-\lambda)(e^{-r\tau}Zu(x_2)+\gamma_2^1)\geq 0.$$

Hence, $N(X,Z,\mathcal{X},\tau)$ is convex as requested. From Filipov-Cesari Theorem, an optimal arc $(X^e(\tau,X,Z),Z^e(\tau,X,Z),x^e(\tau,X,Z),\overline{T}^e(\tau,X,Z))$ exists.

Properties. Fixing an action path $\mathbf{x}$ and taking $X'\geq X$, the corresponding stock processes satisfy $X(s;X)\leq X(s;X')$. The r.-h.s. of (A.10) is thus lower at X' for any action path. Taking the max-operator proves that $\mathcal{W}^e(X,Z)$ is non-increasing in X. From (A.10), it also follows that $\mathcal{W}^e(X,Z)$ is convex as a maximum of linear functions of Z.

Consider an alternative pair (X',Z'). Because an optimal arc for (X',Z'), is suboptimal for (X,Z), the following inequality holds:

$$\mathcal{W}^e(X,Z)\geq(Z-1)\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda\tau}u(x^e(\tau;X',Z')d\tau+\lambda\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^\infty e^{-\lambda\tau}d\tau\right)$$

$$+\int_0^{\overline{T}^e(X',Z')} e^{-r\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)u(x^e(\tau;X',Z'))d\tau$$

$$+\int_{\overline{T}^e(X',Z')}^{+\infty} e^{-r\tau}\left(1-\Delta e^{-\Delta\tau}\int_0^\tau F\left(X+\int_0^s x^e(s';X',Z')ds'\right)e^{\Delta s}ds\right)\lambda\mathcal{V}_\infty d\tau.$$

We express the r.-h.s. in terms of $\mathcal{W}^e(X',Z')$ to get:

$$\mathcal{W}^e(X,Z)-\mathcal{W}^e(X',Z')\geq(Z-Z')\left(\int_0^{\overline{T}^e(X',Z')} e^{-\lambda\tau}u(x^e(\tau;X',Z'))d\tau+\lambda\mathcal{V}_\infty\int_{\overline{T}^e(X',Z')}^\infty e^{-\lambda\tau}d\tau\right)+$$

$$\Delta\left(\int_0^{\overline{T}^e(X',Z')} e^{-r\tau}\left(\int_0^\tau\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)u(x^e(\tau;$$

$$+\Delta\left(\int_{\overline{T}^e(X',Z')}^{\infty} e^{-r\tau}\left(\int_0^{\tau}\left(F\left(X'+\int_0^s x^e(s';X',Z')ds'\right)-F\left(X+\int_0^s x^e(s';X',Z')ds'\right)\right)e^{\Delta s}ds\right)\lambda\mathcal{V}\right.$$

Permuting the roles of (X,Z) and (X',Z'), we deduce a similar inequality. Putting together those conditions implies

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X',Z')|\leq \mathcal{V}_\infty(\|f\|_\infty|X'-X|+|Z'-Z|).$$

From which, we deduce that there exists a positive constant $k=2\mathcal{V}_\infty\max\{\|f\|_\infty,1\}$ such that

$$|\mathcal{W}^e(X,Z)-\mathcal{W}^e(X',Z')|\leq k||(X',Z')-(X,Z)||$$

where $||\cdot||$ is the Euclidian norm. $\mathcal{W}^e(X,Z)$ is Lipschitz continuous and thus a.e. differentiable. In the sequel, we will consider differentiable value functions; a condition satisfied by the examples found in the various scenarios. □

Proof of Proposition 2. Characterization. Looking for continuously differentiable solution, $\mathcal{W}^e(X,Z)$ solves the following Hamilton-Bellman-Jacobi equation:

$$r\mathcal{W}^e(X,Z)=\max_{x\in\mathcal{X}}\left\{Zu(x)+x\frac{\partial\mathcal{W}^e}{\partial X}(X,Z)+\Delta(1-F(X)-Z)\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)\right\}. \tag{A.11}$$

Feedback Rule. The maximand on the r.-h.s. of (A.11) is strictly concave. It immediately follows that an interior feedback rule $\sigma^e(X,Z)$ is given by the first-order condition

$$\sigma^e(X,Z)=\zeta+\frac{1}{Z}\frac{\partial\mathcal{W}^e}{\partial X}(X,Z). \tag{A.12}$$

Rewriting this condition in terms of $\mathcal{V}^e(X,Z)$ finally yields (5.11).
Partial Differential Equation. Simplifying (A.11) by using the feedback rule (A.12) and rewriting this condition in terms of $\mathcal{V}^e(X,Z)$, we obtain

$$r\mathcal{V}^e(X,Z)=\frac{1}{2}\left(\zeta+\frac{\partial\mathcal{V}^e}{\partial X}(X,Z)\right)^2+\frac{\Delta(1-F(X)-Z)}{Z}\left(\mathcal{V}^e(X,Z)+Z\frac{\partial\mathcal{V}^e}{\partial Z}(X,Z)\right). \tag{A.13}$$

Keeping the positive root gives us (5.10).

Denoting the optimal solution to (A.10) by $(x^e(\tau;X,Z),X^e(\tau;X,Z),Z^e(\tau;X,Z),\overline{T}^e(X,Z))$ with $x^e(\tau;X,Z)=\sigma^e(X^e(\tau;X,Z),Z^e(\tau;X,Z))$, we write

$$\mathcal{W}^e(X,Z)=\int_0^{\overline{T}^e(X,Z)} e^{-r\tau}Z^e(\tau;X,Z)u(x^e(\tau;X,Z))d\tau+Z^e(\overline{T}^e(X,Z);X,Z)e^{-r\overline{T}^e(X,Z)}\mathcal{V}_\infty. \tag{A.14}$$

Integrating (5.3), we obtain

$$Z^e(\tau,X,Z)=(Z-1)e^{-\Delta\tau}+1-\Delta e^{-\Delta\tau}\int_0^{\tau}F(X^e(s;X,Z))e^{\Delta s}ds\quad \forall\tau\geq 0,X,Z. \tag{A.15}$$

Applying the Envelope Theorem to (A.10) thus yields

$$\frac{\partial\mathcal{W}^e}{\partial Z}(X,Z)=Z\frac{\partial\mathcal{V}^e}{\partial Z}(X,Z)+\mathcal{V}^e(X,Z)=\varphi^e(X,Z) \tag{A.16}$$

where

$$\varphi^e(X,Z)=\int_0^{\overline{T}^e(X,Z)} e^{-\lambda\tau}u(\sigma^e(X^e(\tau,X,Z),Z^e(\tau,X,Z)))d\tau+e^{-\lambda\overline{T}^e(X,Z)}\mathcal{V}_\infty. \tag{A.17}$$

Observe that $\mathcal{V}^e(X,Z) \geq \varphi^e(X,Z)$ because payoffs are more heavily discounted when computing $\varphi^e(X,Z)$ than when computing $\mathcal{V}^e(X,Z)$ and thus

$$\frac{\partial \mathcal{V}^e}{\partial Z}(X,Z) \leq 0. \tag{A.18}$$

Boundary Condition. Using (5.4) for $X \geq \overline{X}$, we notice that $\overline{T}=0$. Thus, $Z\mathcal{V}^e(X,Z) = Z\int_0^{+\infty} e^{-\lambda\tau}u(\zeta)d\tau = Z\mathcal{V}_\infty$ which gives the condition in the text. □

Optimal Path

The intertemporal date 0-payoff $\mathcal{V}^e(0,1)$ is achieved by adopting the action profile $\sigma^e(X^e(\tau;0,1),Z^e(\tau;0,1))$ for all $\tau \geq 0$ starting from the initial conditions $X=0$ and $Z=1$. Next Proposition provides necessary conditions for an optimal arc.

Proposition A.2. *An optimal action path $x^e(\tau)$ satisfies the following necessary condition:*

$$x^e(\tau) = \zeta - \frac{\Delta e^{r\tau}}{Z^e(\tau)} \int_\tau^{\overline{T}^e} f(X^e(s))e^{\Delta s}\left(e^{-\lambda \overline{T}^e}\mathcal{V}_\infty + \int_s^{\overline{T}^e} e^{-\lambda s'}u(x^e(s'))ds'\right)ds \tag{A.19}$$

where the survival probability writes as

$$Z^e(\tau) = 1 - \Delta e^{-\Delta\tau}\int_0^\tau F(X^e(s))e^{\Delta s}ds.$$

$\overline{X}$ is reached at a date $\overline{T}^e > \overline{T}^m$ with

$$\overline{X} = \zeta\overline{T}^e - \int_0^{\overline{T}^e}\left(\frac{\Delta e^{r\tau}}{Z^e(\tau)}\int_\tau^{\overline{T}^e} f(X^e(s))e^{\Delta s}\left(e^{-\lambda\overline{T}^e}\mathcal{V}_\infty + \int_s^{\overline{T}^e} e^{-\lambda s'}u(x^e(s'))ds'\right)ds\right)d\tau \tag{A.20}$$

Proof of Proposition A.2. From (4.6), DM's intertemporal payoff writes as

$$\mathcal{V}^e(0,1) = \max_{x(\cdot),X(\cdot),Z(\cdot),\overline{T},\text{ s.t. (5.1)-(5.3)},X(\overline{T})=\overline{X}} \int_0^{\overline{T}} e^{-r\tau}Z(\tau)u(x(\tau))d\tau + Z(\overline{T})e^{-r\overline{T}}\mathcal{V}_\infty. \tag{A.21}$$

Existence. A solution to (A.21) exists. (Same argument as in the Proof of Proposition 2.)
Maximum Principle. We now define the Hamiltonian for (A.21) as

$$\mathcal{H}^e(X,Z,x,\tau,\mu,\nu) = e^{-r\tau}Zu(x) + \mu x + \nu\Delta(1-F(X)-Z) \tag{A.22}$$

where μ and ν are respectively the costate variables for (4.1) and (5.3). Observe that $\mathcal{H}^e$ is differentiable in X (since, when F has no mass point and admits a positive density on $[0,\overline{X}]$, it has a derivative $F'=f$) and Z. The *Maximum Principle* with free final time $\overline{T}$ and scrap value $Z(\overline{T})e^{-r\overline{T}}\mathcal{V}_\infty$ gives us the necessary conditions for an optimal arc $(X^e(\tau),Z^e(\tau),x^e(\tau),\overline{T}^e)$. (See Seierstad and Sydsaeter, 1987, Theorem 3, p. 182 and Note 2, p. 183).)
Costate variables. $\mu(\tau)$ and $\nu(\tau)$ are both piecewise continuously differentiable with

$$\dot{\mu}(\tau) = -\frac{\partial\mathcal{H}^e}{\partial X}(X^e(\tau),Z^e(\tau),x^e(\tau),\tau,\mu(\tau),\nu(\tau)) = \Delta f(X^e(\tau))\nu(\tau) \quad \forall\tau\in\left[0,\overline{T}^e\right], \tag{A.23}$$

$$\dot{\nu}(\tau) = -\frac{\partial \mathcal{H}^e}{\partial Z}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau)) = -e^{-r\tau}u(x^e(\tau)) + \Delta\nu(\tau) \quad \forall \tau \in \left[0, \overline{T}^e\right]. \tag{A.24}$$

Transversality conditions. The boundary conditions $X^e(0) = 0$, $X^e(\overline{T}^e) = \overline{X}$ and $Z^e(0) = 1$ imply that $\mu(0)$, $\mu(\overline{T}^e)$ and $\nu(0)$ are free. Applying Seierstad and Sydsaeter (1987, Theorem 3), we get the last transversality condition

$$\nu(\overline{T}^e) = \frac{\partial (Ze^{-r\overline{T}}\mathcal{V}_\infty)}{\partial Z}|_{Z=Z^e(\overline{T}^e), \overline{T}=\overline{T}^e} = e^{-r\overline{T}^e}\mathcal{V}_\infty. \tag{A.25}$$

Free-end point condition. Applying Seierstad and Sydsaeter (1987, Note 2, p. 183), the necessary condition w.r.t. $\overline{T}$ is

$$\mathcal{H}^e(X^e(\overline{T}^e), Z^e(\overline{T}^e), x^e(\overline{T}^e), \overline{T}^e, \mu(\overline{T}^e), \nu(\overline{T}^e)) + \frac{\partial (Ze^{-r\overline{T}}\mathcal{V}_\infty)}{\partial \overline{T}}|_{Z=Z^e(\overline{T}^e), \overline{T}=\overline{T}^e} = 0. \tag{A.26}$$

Using the definition (A.22) and (A.25), we write (A.26) as

$$e^{-r\overline{T}^e} Z^e(\overline{T}^e)\left(u(x^e(\overline{T}^{e-})) - \lambda\mathcal{V}_\infty\right) + \mu(\overline{T}^e)x^e(\overline{T}^{e-}) = 0$$

or

$$-\frac{1}{2}e^{-r\overline{T}^e} Z^e(\overline{T}^e)(x^e(\overline{T}^{e-}) - \zeta)^2 + \mu(\overline{T}^e)x^e(\overline{T}^{e-}) = 0 \tag{A.27}$$

where $x^e(\overline{T}^{e-})$ denotes the l.-h. s. limit of $x^e(\tau)$ as $\tau \to \overline{T}^{e-}$.
Control variable. We should have $x^e(\tau) \in \arg\max_{x\geq 0} \mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \mu(\tau), \nu(\tau))$. Because $\mathcal{H}^e(X^e(\tau), Z^e(\tau), x, \tau, \mu(\tau), \nu(\tau))$ is strictly concave in x, an interior solution satisfies

$$\frac{\partial \mathcal{H}^e}{\partial x}(X^e(\tau), Z^e(\tau), x^e(\tau), \tau, \mu(\tau), \nu(\tau)) = 0 \Rightarrow x^e(\tau) = \zeta + e^{r\tau}\frac{\mu(\tau)}{Z^e(\tau)}. \tag{A.28}$$

Characterization. Inserting (A.28) taken for $\overline{T}^e$ into (A.27) yields

$$\frac{e^{r\overline{T}^e}(\mu(\overline{T}^e))^2}{2Z^e(\overline{T}^e)} + \mu(\overline{T}^e)\zeta = 0.$$

The only solution consistent with a non-negative action at date $\overline{T}^e$ is thus

$$\mu(\overline{T}^e) = 0. \tag{A.29}$$

It follows that the optimal action is continuous at $\overline{T}^e$, $x^e(\overline{T}^{e-}) = x^e(\overline{T}^{e+}) = \zeta$. The solution for (A.24) that satisfies the transversality condition (A.25) is

$$\nu(\tau) = e^{\Delta\tau}\left(e^{-\lambda\overline{T}^e}\mathcal{V}_\infty + \int_\tau^{\overline{T}^e} e^{-\lambda s}u(x^e(s))ds\right). \tag{A.30}$$

Inserting into (A.23), and integrating and using (A.29) yields

$$\mu(\tau) = -\int_\tau^{\overline{T}^e} \Delta f(X^e(s))e^{\Delta s}\left(e^{-\lambda\overline{T}^e}\mathcal{V}_\infty + \int_s^{\overline{T}^e} e^{-\lambda s'}u(x^e(s'))ds'\right)ds. \tag{A.31}$$

Inserting into (A.28), we obtain (A.19). Finally, the value of $\overline{T}^e$ is obtained when $\int_0^{\overline{T}^e} x^e(\tau)d\tau = \overline{X}$. Using the expression of $x^e(\tau)$ in (A.19) yields (A.20). That $\overline{T}^e > \overline{T}^m$ is then immediate. □

Appendix B: Scenarios 1-3

Proof of Proposition 3. Observe that (5.2) rewrites now as $Z(\tau)=1-q+(Z+q-1)e^{-\Delta\tau}$. This expression of $Z(\tau)$ allows us to rewrite the definition (5.4) for $\mathcal{V}^e(X,Z)$ as

$$Z\mathcal{V}^e(X,Z)=\max_{x(\cdot),X(\tau)=X+\int_0^\tau x(s)ds,Z(\cdot)\text{ s.t. }(5.3),\overline{T},X(\overline{T})=\overline{X}}\int_0^{\overline{T}} e^{-r\tau}\left(1-q+(Z+q-1)e^{-\Delta\tau}\right)u(x(\tau))d\tau$$

$$+e^{-r\overline{T}}\left(1-q+(Z+q-1)e^{-\Delta\overline{T}}\right)\mathcal{V}_\infty \text{ s.t. } \int_0^{\overline{T}} x(\tau)d\tau=\overline{X}-X.$$

Let denote by μ the multiplier for this constraint. We form the corresponding Lagrangean as

$$\int_0^{\overline{T}} e^{-r\tau}\left(1-q+(Z+q-1)e^{-\Delta\tau}\right)u(x(\tau))d\tau$$

$$+e^{-r\overline{T}}\left(1-q+(Z+q-1)e^{-\Delta\overline{T}}\right)\mathcal{V}_\infty+\mu\left(\overline{X}-X-\int_0^{\overline{T}} x(\tau)d\tau\right).$$

Pointwise optimization for this strictly concave objective w.r.t. x yields

$$\zeta-x^e(\tau)=\frac{\mu e^{r\tau}}{Z(\tau)} \tag{B.1}$$

where, for simplicity, we omit dependence on (X,Z). Integrating over $\left[0,\overline{T}^e\right]$ yields

$$\zeta\overline{T}^e-(\overline{X}-X)=\mu\int_0^{\overline{T}^e}\frac{e^{r\tau}}{Z(\tau)}d\tau. \tag{B.2}$$

Optimizing now w.r.t. $\overline{T}$ yields the following necessary first-order condition

$$e^{-r\overline{T}^e}Z(\overline{T}^e)u(x^e(\overline{T}^{e-}))+\mathcal{V}_\infty e^{-r\overline{T}^e}\left(-rZ(\overline{T}^e)+\dot{Z}(\overline{T}^e)\right)=\mu x^e(\overline{T}^{e-})$$

where $x^e(\overline{T}^{e-})$ denotes the l.h.-s. limit of $x^e(\tau)$ at $\overline{T}^e$. Simplifying, we get

$$\zeta x^e(\overline{T}^{e-})-\frac{(x^e(\overline{T}^{e-}))^2}{2}+\mathcal{V}_\infty\left(-r+\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right)=\mu\frac{e^{r\overline{T}^e}}{Z(\overline{T}^e)}x^e(\overline{T}^{e-})$$

Using (B.1) taken at $\tau=\overline{T}^e$, we rewrite the r.-h.s. and get

$$\zeta x^e(\overline{T}^{e-})-\frac{(x^e(\overline{T}^{e-}))^2}{2}+\mathcal{V}_\infty\left(-r+\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}\right)=x^e(\overline{T}^{e-})(\zeta-x^e(\overline{T}^{e-}))$$

Simplifying further yields

$$x^e(\overline{T}^{e-})=\zeta\sqrt{\frac{r-\frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda}}.$$

From (B.1) taken at $\tau = \overline{T}^e$, we then get

$$\mu \frac{e^{r\overline{T}^e}}{Z(\overline{T}^e)} = \zeta \left(1 - \sqrt{\frac{r - \frac{\dot{Z}(\overline{T}^e)}{Z(\overline{T}^e)}}{\lambda}} \right). \tag{B.3}$$

Specializing to the case $X = 0$ and $Z = 1$, and inserting the value of μ obtained from (B.3) into (B.2) yields the expression of $\overline{T}^e$ in (5.13). Inserting into (B.1) yields (5.12). Because $\frac{e^{r\tau}}{Z(\tau)} = \frac{e^{r\tau}}{1-q+qe^{-\Delta\tau}}$ is increasing, $x^e(\tau)$ is decreasing over $[0, \overline{T}^e)$. Specializing further to the case $q = 0$ yields the optimal trajectory when the TP is known being at $\overline{X}$ for sure. In this case, the optimal action is given by (5.7) and the the time to reach $\overline{X}$ is given by (5.9). □

Proof of Proposition 4. Solving for $v'_\mu(Y)$ in (5.17) a second-degree polynomial in $v'_\mu(Y)$ and taking roots, we obtain

$$\frac{Yv'_\mu(Y)}{\mu} + \zeta + \frac{\mu\Delta(1-Y)}{Y} = \pm\sqrt{2\left(\left(r + \frac{\Delta(Y-1)}{Y}\right)v_\mu(Y) - \lambda\mathcal{V}_\infty\right) + \left(\zeta + \frac{\mu\Delta(1-Y)}{Y}\right)^2}. \tag{B.4}$$

Denote $\hat{v}(Y) = \frac{\lambda\mathcal{V}_\infty}{\lambda - \frac{\Delta}{Y}}$ and observe that $\hat{v}$ is decreasing on $[1, +\infty)$ with $\lim_{Y\to+\infty} \hat{v}(Y) = \mathcal{V}_\infty$ and $\hat{v}(1) = \frac{\lambda\mathcal{V}_\infty}{r}$.

When $\zeta \geq \mu\Delta$, we have $\zeta + \frac{\mu\Delta(1-Y)}{Y} > 0$ for all $Y \geq 1$. Considering solutions to (**??**) obtained with + on the r.-h.-s., we observe that those trajectories have positive (resp. negative, zero) derivative if they are above (resp. below, cross) $\hat{v}$. Solutions that cross $\hat{v}$ cross it only once as it can be easily seen. The reciprocal image of the curve $v = v_\mu(Y)$ for $Y \geq 1$ by the continuous flow associated to the differential equation (**??**) is thus of the form $(w_1, \hat{v}(1)] \subset [V_\infty, \hat{v}(1)]$. Because $\hat{v}(Y) > \mathcal{V}_\infty$ for all $Y \geq 1$, any solution to (**??**) that crosses the horizontal line $v = \mathcal{V}_\infty$, crosses it with a negative slope. It follows that the reciprocal image of the curve $v = \mathcal{V}_\infty$ for $Y \geq 1$ by the continuous flow is of the form $[\mathcal{V}_\infty, w_2) \subset [\mathcal{V}_\infty, \hat{v}(1)]$. Of course, $w_1 \geq w_2$. Hence, any solution to (**??**) with $v_\mu(1) \in [w_2, w_1]$ satisfies $\lim_{Y\to+\infty} v_\mu(Y) = \mathcal{V}_\infty$; which proves existence of a solution. To prove uniqueness, suppose that there exist two such solutions v_1 and v_2 with $v_2(1) = w_2 < w_1 = v_1(1)$ and denote $v_3 = v_1 - v_2$. Because $\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$, v_3 satisfies

$$\frac{Yv'_3(Y)}{\mu} = \frac{2\left(r + \frac{\Delta(Y-1)}{Y}\right)v_3(Y)}{\sum_{i=1}^{2}\sqrt{2\left(\left(r + \frac{\Delta(Y-1)}{Y}\right)v_i(Y) - \lambda\mathcal{V}_\infty\right) + \left(\zeta + \frac{\mu\Delta(1-Y)}{Y}\right)^2}}.$$

Now observe that

$$\sqrt{2\left(\left(r + \frac{\Delta(Y-1)}{Y}\right)v_i(Y) - \lambda\mathcal{V}_\infty\right) + \left(\zeta + \frac{\mu\Delta(1-Y)}{Y}\right)^2} \leq |\zeta + \frac{\mu\Delta(1-Y)}{Y}| \leq \zeta + \mu\Delta$$

where the first inequality follows from $v_i(Y) \leq \hat{v}(Y)$ for all $Y \geq 1$ and the second from the triangular inequality and $Y \geq 1$. The following lower bound thus holds

$$v'_3(Y) \geq \frac{\mu}{(\zeta + \Delta\mu)Y}\left(r + \frac{\Delta(Y-1)}{Y}\right)v_3(Y).$$

Denote $g(Y)=\frac{\mu}{(\zeta+\Delta\mu)}\left(\frac{\lambda}{Y}-\frac{\Delta}{Y^2}\right)$ and observe that $\lim_{Y\to+\infty}\int_1^Y g(\widetilde{Y})d\widetilde{Y}=+\infty$. Gronwall's Inequality then implies

$$v_3(Y)\geq v_3(1)e^{\int_1^Y g(\widetilde{Y})d\widetilde{Y}}$$

and, if $v_3(1)>0$, $\lim_{Y\to+\infty}v_3(Y)=+\infty$; a contradiction with the fact that $\lim_{Y\to+\infty}v_1(Y)=\lim_{Y\to+\infty}v_2(Y)=\mathcal{V}_\infty$. Hence, necessarily $v_3(1)=0$ and there exists a unique solution v_μ to (**??**) satisfying the boundary condition. This solution lies below $\hat{v}$ and thus is everywhere decreasing. The r.-h.s. of (5.19) follows. Figure **??** represents the solution.

Asymptotic Behavior. For $Y\to+\infty$, we look for an asymptotic development of the form

$$v_\mu(Y)=\mathcal{V}_\infty+\frac{a}{Y}+o\left(\frac{1}{Y}\right).$$

Inserting into (**??**) and developing the r.-h.s. to the first order yields

$$-\frac{a}{\mu Y}+\zeta-\mu\Delta+\frac{\mu\Delta}{Y}=\pm|\zeta-\mu\Delta|\left(1+\frac{\mu\Delta}{Y(\zeta-\mu\Delta)}+\frac{1}{(\zeta-\mu\Delta)^2}\left(\frac{\lambda a}{Y}-\frac{\Delta\mathcal{V}_\infty}{Y}\right)\right)+o\left(\frac{1}{Y}\right).$$

Simplifying yields

$$a=\frac{\Delta\mathcal{V}_\infty}{r+\dfrac{\zeta}{\mu}}.$$

Trajectories for $Y(t)$. (**??**) implies

$$\widetilde{\sigma}_\mu(Y)+\frac{\mu\Delta(1-Y)}{Y}=\sqrt{R(Y)}\geq 0 \tag{B.5}$$

where $R(Y)=2\left(\left(r+\frac{\Delta(Y-1)}{Y}\right)v_\mu(Y)-\lambda\mathcal{V}_\infty\right)+(B(Y))^2$ and $B(Y)=\zeta+\frac{\mu\Delta(1-Y)}{Y}$. Suppose that there exists $Y^*<+\infty$ such that $R(Y^*)=0$. At that point, we also have

$$R'(Y^*)=2\left(\left(\lambda-\frac{\Delta}{Y^*}\right)v'_\mu(Y^*)+\frac{\Delta}{(Y^*)^2}v_\mu(Y^*)-\frac{\mu\Delta B(Y^*)}{(Y^*)^2}\right).$$

At such Y^*, the l.-h.-s. of (**??**) is also zero and thus $v'_\mu(Y^*)=-\frac{\mu B(Y^*)}{Y^*}$.Inserting above yields

$$R'(Y^*)=\frac{2}{Y^*}\left(\frac{\Delta}{Y^*}v_\mu(Y^*)-\mu\lambda B(Y^*))\right).$$

Because $R(Y^*)=0$, $v_\mu(Y^*)=\frac{\lambda\mathcal{V}_\infty-\frac{1}{2}(B(Y^*))^2}{\lambda-\frac{\Delta}{Y^*}}$. Inserting above yields

$$R'(Y^*)=\frac{2}{Y^*}\left(\frac{\Delta}{\lambda Y^*-\Delta}\left(\lambda\mathcal{V}_\infty-\frac{1}{2}\left(B(Y^*)\right)^2\right)-\mu\lambda B(Y^*)\right). \tag{B.6}$$

Introduce $d=\zeta-B(Y^*)=\mu\Delta\left(1-\frac{1}{Y^*}\right)\in[0,\mu\Delta)$ and observe that $Y^*=\frac{\mu\Delta}{\mu\Delta-d}$, $\lambda Y^*-\Delta=\Delta\frac{\mu r+d}{\mu\Delta-d}$, and $2\lambda\mathcal{V}_\infty-(B(Y^*))^2=d(2\zeta-d)$. Using these identities, and inserting into (**??**), we obtain $(\mu r+d)Y^*R'(Y^*)=T(d)$, where $T(d)=(\mu\Delta-d)Q(d)-2\mu^2\lambda^2(\zeta-\mu\Delta)$ and $Q(d)=-d^2+2\big(\zeta-\mu\lambda\big)(d+\mu\lambda)$. When $\zeta\leq\mu\lambda$, we have for all $d\geq 0$, $Q(d)\leq 0$. Since, $d\in[0,\mu\Delta)$ and $\zeta\geq\mu\Delta$, we thus have $T(d)<0$ and thus $R'(Y^*)<0$. Now observe that $\lim_{Y\to+\infty}R(Y)=(\zeta-\mu\Delta)^2>0$ when $\zeta>\mu\Delta$ and thus $R(Y)>0$ for Y large enough, a contradiction. For $\zeta=\mu\Delta$, the asymptotic behavior of v_μ in (5.18) implies that $R(Y)=\left(\frac{\mu\Delta}{Y}\left(1-\frac{\Delta^2}{2\lambda^2}\right)\right)^2+o(\frac{1}{Y^2})$. Hence, $\lambda>\Delta$ again implies $R(Y) > 0$ for Y large enough; still a contradiction. Hence, no such Y^* exists, so $R(Y) > 0$ for all $Y \geq 1$.

Along the optimal trajectory starting from $X(0) = 0$ and $Z(0) = 1$ and satisfying (4.1) and (5.3), we have $Y(\tau) = Z(\tau)e^{\frac{X(\tau)}{\mu}}$. Differentiating w.r.t. t yields

$$\dot{Y}(\tau) = \Delta(1 - Y(\tau)) + \frac{\widetilde{\sigma}_\mu(Y(\tau))}{\mu} Y(\tau) \text{ with } Y(0) = 1. \tag{B.7}$$

That $R(Y) > 0$ for all $Y \geq 1$ then implies that $Y(t)$ is increasing along the whole trajectory with $\lim_{\tau \to +\infty} Y(\tau) = +\infty$. (Suppose that it is bounded above, then it converges towards some $Y^* \geq 1$ such that $R(Y^*) = 0$ and we have proved above that such Y^* do not exist.) Hence, using (**??**) and (5.18), $x^e(\tau) = \widetilde{\sigma}(Y(\tau))$ satisfies

$$\frac{\dot{x}^e(\tau)}{\zeta - x^e(\tau)} \sim_{t \to +\infty} \frac{\dot{Y}(\tau)}{Y(\tau)} \sim_{t \to +\infty} \frac{\zeta}{\mu} - \Delta.$$

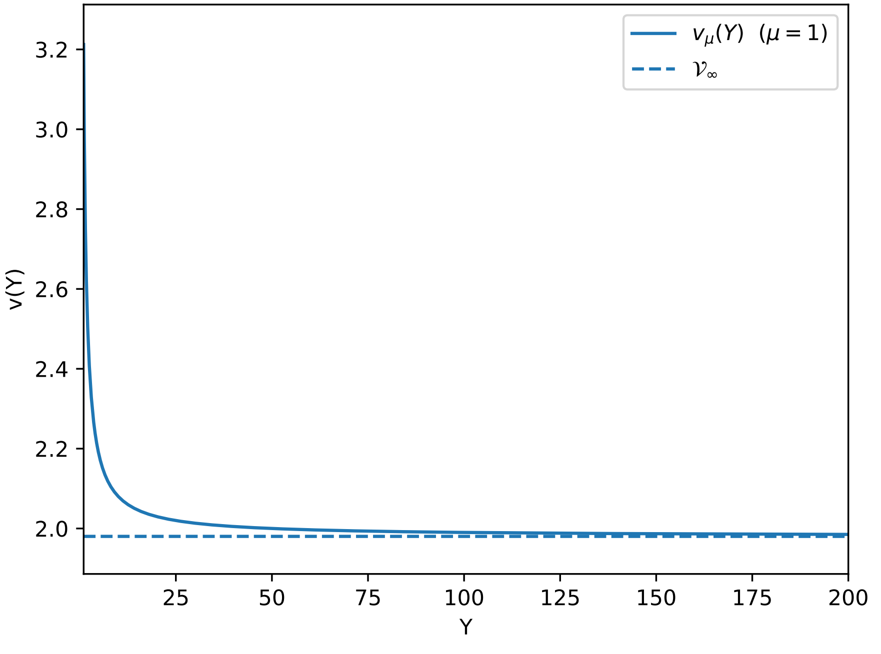

Figure 2. Solution of equation (5.17) for $\mu = 1$ and $Y \geq 1$. Parameters are $\zeta = 2$, $\Delta = 1$, and $r = 0.01$ (so $\lambda = r + \Delta$). The solid line plots $v_\mu(Y)$, while the dashed line represents the asymptotic value $\mathcal{V}_\infty = \zeta^2/(2\lambda)$.

□

Proof of Proposition 5. The value function $\widetilde{\mathcal{V}_\varepsilon}(X,Y)$ satisfies

$$\frac{\partial \widetilde{\mathcal{V}_\varepsilon}}{\partial X}(X,Y) + \frac{f_\varepsilon(X)}{1 - F_\varepsilon(X)} Y \frac{\partial \widetilde{\mathcal{V}_\varepsilon}}{\partial Y}(X,Y) = -\zeta + \sqrt{2\left(r + \frac{\Delta(Y-1)}{Y}\right)\widetilde{\mathcal{V}_\varepsilon}(X,Y) + 2\Delta(Y-1)\frac{\partial \widetilde{\mathcal{V}_\varepsilon}}{\partial Y}(X,Y)} \tag{B.8}$$

with the boundary condition being now $\lim_{Y\to+\infty}\widetilde{\mathcal{V}_\varepsilon}(\overline{X},Y)=\mathcal{V}_\infty$.

We look for first-order Taylor expansions of the form $\widetilde{\mathcal{V}_\varepsilon}=\widetilde{\mathcal{V}}+\epsilon v+o(\epsilon)$ and $\widetilde{\sigma_\varepsilon}=\widetilde{\sigma}+\epsilon\sigma+o(\epsilon)$. We linearize (**??**) in the neighborhood of $\varepsilon=0$ to get the following condition

$$\widetilde{\sigma}(X,Y)\left(\frac{\partial v}{\partial X}(X,Y)+\frac{f(X)}{1-F(X)}Y\frac{\partial v}{\partial Y}(X,Y)+Y\frac{\partial\widetilde{\mathcal{V}}}{\partial Y}(X,Y)\right)=\left(\lambda-\frac{\Delta}{Y}\right)v(X,Y)+\Delta(Y-1)\frac{\partial v}{\partial Y}(X,Y) \tag{B.9}$$

with the new boundary condition $\lim_{Y\to+\infty}v(\overline{X},Y)=0$. To solve the quasi-linear PDE (**??**) together with the boundary condition, we use the method of characteristics.[25] Consider the system of ODEs defined for $\tau\geq 0$ as

$$\dot{X}(\tau)=\widetilde{\sigma}(X(\tau),Y(\tau)) \text{ and } \dot{Y}(\tau)=\widetilde{\sigma}(X(\tau),Y(\tau))\frac{f(X(\tau))}{1-F(X(\tau))}Y(\tau)-\Delta(Y(\tau)-1) \tag{B.10}$$

together with the initial conditions

$$X(0)=X_0 \text{ and } Y(0)=\frac{1}{1-F(X_0)}. \tag{B.11}$$

By Cauchy Uniqueness Theorem, there exists a unique solution $(X(\tau,X_0),Y(\tau,X_0))$ to (**??**)-(**??**) where we make explicit the dependence on the initial condition X_0.

Let define $w(\tau,X_0)=v(X(\tau,X_0),Y(\tau,X_0))$. Using (**??**), w solves the following first-order quasi-linear ODE

$$\frac{\partial w}{\partial\tau}(\tau,X_0)=\left(\lambda-\frac{\Delta}{Y(\tau,X_0)}\right)w(\tau,X_0)-Y(\tau,X_0)\widetilde{\sigma}(X(\tau,X_0),Y(\tau,X_0))\frac{\partial\widetilde{\mathcal{V}}}{\partial Y}(X(\tau,X_0),Y(\tau,X_0)) \tag{B.12}$$

together with the boundary condition $\lim_{\tau\to+\infty}w(\tau,X_0)=0$. We may look for a general solution of the form $w(\tau,X_0)=w_0(\tau,X_0)e^{\int_0^\tau\left(\lambda-\frac{\Delta}{Y(s,X_0)}\right)ds}$ where

$$\frac{\partial w_0}{\partial\tau}(\tau,X_0)=-Y(\tau,X_0)\widetilde{\sigma}(X(\tau,X_0),Y(\tau,X_0))\frac{\partial\widetilde{\mathcal{V}}}{\partial Y}(X(\tau,X_0),Y(\tau,X_0))e^{-\int_0^\tau\left(\lambda-\frac{\Delta}{Y(s,X_0)}\right)ds}$$

and thus, for some constant $w_{00}(X_0)$

$$w_0(\tau,X_0)=w_{00}(X_0)-\int_0^\tau Y(s,X_0)\widetilde{\sigma}(X(s,X_0),Y(s,X_0))\frac{\partial\widetilde{\mathcal{V}}}{\partial Y}(X(s,X_0),Y(s,X_0))e^{-\int_0^s\left(\lambda-\frac{\Delta}{Y(u,X_0)}\right)du}.$$

Observe that (**??**) together with (**??**) imply that

$$Y(\tau,X_0)=\frac{e^{-\Delta\tau}}{1-F(X(\tau,X_0))}\left(1+\Delta\int_0^\tau(1-F(X(s,X_0)))e^{\Delta s}ds\right). \tag{B.13}$$

Thus, $Y(\tau,X_0)=\frac{Z(\tau,X_0)}{1-F(X(\tau,X_0))}$ where $X(\tau,X_0)$ and $Z(\tau,X_0)$ are the trajectories starting from $X(0)=X_0$ and $Z(0)=1$, and satisfying (5.1) and (5.3). From this, it follows that

$$\lambda-\frac{\Delta}{Y(\tau,X_0)}=\lambda-\frac{\Delta e^{\Delta\tau}(1-F(X(\tau,X_0))}{1+\Delta\displaystyle\int_0^\tau(1-F(X(s,X_0)))e^{\Delta s}ds}.$$

[25] John (1982, Chapter 1).

Because $\Delta\int_0^\tau (1-F(X(s,X_0)))e^{\Delta s} \geq (1-F(X(\tau,X_0)))(e^{\Delta\tau}-1)$, we have $\frac{e^{\Delta\tau}(1-F(X(\tau,X_0))}{1+\Delta\int_0^\tau(1-F(X(s,X_0)))e^{\Delta s}ds} \leq \frac{(1-F(X(\tau,X_0))e^{\Delta\tau}}{F(X(\tau,X_0))+(1-F(X(\tau,X_0)))e^{\Delta\tau}} \leq 1$. Hence $\lambda - \frac{\Delta}{Y(\tau,X_0)} \geq \lambda-\Delta = r$ and thus $\int_0^\tau \left(\lambda - \frac{\Delta}{Y(s,X_0)}\right) ds \geq r\tau$. The unique bounded solution to (**??**) is thus

$$w(\tau,X_0) = e^{\int_0^\tau\left(\lambda-\frac{\Delta}{Y(s,X_0)}\right)ds} \tag{B.14}$$

$$\times\left(\int_\tau^\infty Y(s,X_0)\widetilde{\sigma}(X(s,X_0),Y(s,X_0))\frac{\partial\widetilde{\mathcal{V}}}{\partial Y}(X(s,X_0),Y(s,X_0))e^{-\int_0^s\left(\lambda-\frac{\Delta}{Y(u,X_0)}\right)du}ds\right).$$

Integrating by parts the bracketed integral and simplifying, we obtain

$$w(\tau,X_0) = \hat{w}(\tau,X_0) + e^{\int_0^\tau\left(\lambda-\frac{\Delta}{Y(s,X_0)}\right)ds}\left(\int_\tau^\infty \frac{\partial\hat{w}}{\partial s}(s,X_0)e^{-\int_0^s\left(\lambda-\frac{\Delta}{Y(u,X_0)}\right)du}ds\right) \tag{B.15}$$

where $\hat{w}(\tau,X_0)$ is defined in (5.22). From (5.21), (5.22), (**??**) and (**??**), it follows that $0 \geq w(\tau,X_0) \geq \hat{w}(\tau,X_0), \quad \forall\tau\geq 0$. Again using (5.22), the boundary condition holds

$$\lim_{\tau\to+\infty} w(\tau,X_0) = 0. \tag{B.16}$$

Fix a pair (X,Y) reached by a characteristic curve. Then, let $T(X,Y)$ and $X_0(X,Y)$ be such that $X = X(T,X_0)$ and $Y = Y(T,X_0)$. Equipped with those notations, we obtain the solution to the quasi-linear PDE (**??**) together with the boundary condition for any pair (X,Y) as

$$v(X,Y) = w(T(X,Y),X_0(X,Y)). \tag{B.17}$$

Because of (**??**), (5.21) implies $v(X,Y) < 0$ for all (X,Y) and thus (5.23) holds.

Proceeding as in (5.16), we get

$$\widetilde{\sigma}_\varepsilon(X,Y) = \sqrt{2\left(r-\frac{\Delta(1-Y)}{Y}\right)\widetilde{\mathcal{V}_\varepsilon}(X,Y) - 2\Delta(1-Y)\frac{\partial\widetilde{\mathcal{V}_\varepsilon}}{\partial Y}(X,Y)}. \tag{B.18}$$

Using again (5.16) and (**??**), we get the following Taylor expansion for ε small enough:

$$\widetilde{\sigma}(X,Y)\sigma(X,Y) = \left(r-\frac{\Delta(1-Y)}{Y}\right)v(X,Y) - \Delta(1-Y)\frac{\partial v}{\partial Y}(X,Y) + o(1). \tag{B.19}$$

From this, we immediately get that $\sigma(X,1)$ has the sign of $v(X,1)$ and is thus negative for all X, in particular at $X=0$ and thus (5.23) holds. □

Appendix C: Stock-Based Value Functions, Feedback Rules. Scenarios 2-4.

Proof of Proposition 1. The value function $\mathcal{V}^k(X)$ satisfies

$$\mathcal{V}^k(X) = \max_{x(\cdot),X(\tau)=X+\int_0^\tau x(s)ds,\overline{T},X(\overline{T})=\overline{X}} \int_0^{\overline{T}} e^{-r\tau}u(x(\tau))d\tau + e^{-r\overline{T}}\mathcal{V}_\infty. \tag{C.1}$$

From which, we derive the Hamilton-Bellman-Jacobi equation

$$r\mathcal{V}^k(X) = \max_{x\in\mathcal{X}} u(x) + x\dot{\mathcal{V}}^k(X). \tag{C.2}$$

The optimal feedback rule $\sigma^k(X)$ is thus given by (5.6). Inserting into (**??**) yields (5.5) with the boundary condition $\mathcal{V}^k(\overline{X}) = \mathcal{V}_\infty$. □

Proof of Proposition 6. Using (A.16) for $Z = Z^o(X)$ gives us

$$Z^o(X)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z^o(X)) + \mathcal{V}^o(X) = \varphi^o(X). \tag{C.3}$$

Computing the total derivative of $\mathcal{V}^0(X) = \mathcal{V}^e(X, Z^o(X))$ yields

$$\dot{\mathcal{V}}^o(X) = \dot{Z}^o(X)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z^o(X)) + \frac{\partial \mathcal{V}^e}{\partial X}(X, Z^o(X)).$$

Inserting into (5.11) still for $Z = Z^o(X)$ yields (6.12). From there, we obtain the expression of the feedback rule (6.8).

Using (A.13) for $Z = Z^o(X)$, we obtain

$$r\mathcal{V}^o(X) = \frac{1}{2}\left(\zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X, Z^o(X))\right)^2 + \frac{\Delta(1 - F(X) - Z^o(X))}{Z^o(X)}\left(\mathcal{V}^o(X) + Z^o(X)\frac{\partial \mathcal{V}^e}{\partial Z}(X, Z^o(X))\right). \tag{C.4}$$

Using (5.11) for $Z = Z^o(X)$ yields

$$\sigma^o(X) = \zeta + \frac{\partial \mathcal{V}^e}{\partial X}(X, Z^o(X)). \tag{C.5}$$

Inserting (**??**) into (**??**) and using (**??**) gives

$$r\mathcal{V}^o(X) = \frac{1}{2}\left(\sigma^o(X)\right)^2 + \sigma^o(X)\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)$$

or

$$2r\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2 = \left(\sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2.$$

Taking then the positive root, we obtain

$$\sigma^o(X) + \frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X) = \sqrt{2r\mathcal{V}^o(X) + \left(\frac{\dot{Z}^o(X)}{Z^o(X)}\varphi^o(X)\right)^2}. \tag{C.6}$$

Inserting (6.8) into (**??**) and simplifying yields (6.7), to which we append the boundary condition. □

Conditions (6.9) and (6.10). Observe that $\widetilde{X}(0, x, \tau, X) = X^o(\tau, X)$, $\widetilde{Z}(0, x, \tau, X) = Z^o(X)$ and thus $\widetilde{\mathcal{V}}(0, x, X) = \mathcal{V}^o(X)$. Following an impulse deviation (x, ε), the stock becomes

$$\widetilde{X}(\varepsilon, x, \tau, X) = \begin{cases} X + x\tau & \text{if } \tau \in [0, \varepsilon], \\ X + x\varepsilon + \int_\varepsilon^\tau \sigma^o(X^0(s - \varepsilon, X + x\varepsilon))ds & \text{if } \tau \geq \varepsilon. \end{cases}$$

A first-order Taylor approximation in ε gives us

$$\widetilde{X}(\varepsilon, x, \tau, X) - X^o(\tau, X) = \varepsilon\left((x - \sigma^o(X) + \int_0^\tau \dot{\sigma}^o(X^0(s, X))\left(-\frac{\partial X^o}{\partial s}(s, X) + x\frac{\partial X^o}{\partial X}(s, X)\right)ds\right) + o(\varepsilon). \tag{C.7}$$

Let us remind that

$$\frac{\partial X^o}{\partial X}(\tau, X) = \frac{1}{\sigma^o(X)}\frac{\partial X^o}{\partial \tau}(\tau, X) = \frac{\sigma^o(X^o(\tau, X))}{\sigma^o(X)}. \tag{C.8}$$

Inserting into (**??**), we obtain

$$\widetilde{X}(\varepsilon,x,\tau,X)-X^o(\tau,X)=\varepsilon\left(x-\sigma^o(X)+\left(\frac{x}{\sigma^o(X)}-1\right)\int_0^\tau \dot{\sigma}^o(X^0(s,X))\frac{\partial X^o}{\partial s}(s,X)ds\right)+o(\varepsilon). \tag{C.9}$$

Integrating yields

$$\int_0^\tau \dot{\sigma}^o(X^0(s,X))\frac{\partial X^o}{\partial s}(s,X)ds=\sigma^o(X^0(\tau,X))-\sigma^o(X))=\sigma^o(X)\left(\frac{\partial X^o}{\partial X}(\tau,X)-1\right).$$

Inserting above and simplifying thus yields (6.9).

The survival probability also changes with an impulse deviation as

$$\widetilde{Z}(\varepsilon,x,\tau,X)=1-\Delta e^{-\Delta\tau}\int_0^\tau F(\widetilde{X}(\varepsilon,x,s,X))e^{\Delta s}ds-(1-Z^o(X))e^{-\Delta\tau}.$$

Using again a first-order Taylor approximation in ε, the impact of an impulse deviation on beliefs is approximatively worth

$$\widetilde{Z}(\varepsilon,x,\tau,X)-Z^o(\tau,X)=-\left(x-\sigma^o(X)\right)\varepsilon\Delta e^{-\Delta\tau}\int_0^\tau f(X^o(s,X))\frac{\partial X^o}{\partial X}(s,X)e^{\Delta s}ds+o(\varepsilon)$$

or

$$\widetilde{Z}(\varepsilon,x,\tau,X)-Z^o(\tau,X)=-\left(x-\sigma^o(X)\right)\varepsilon\Delta e^{-\Delta\tau}\int_0^\tau f(X^o(s,X))\frac{\sigma^o(X^o(s,X))}{\sigma^o(X)}e^{\Delta s}ds+o(\varepsilon)$$

that can also be written as

$$\widetilde{Z}(\varepsilon,x,\tau,X)-Z^o(\tau,X)=-\left(\frac{x}{\sigma^o(X)}-1\right)\varepsilon\Delta e^{-\Delta\tau}\int_0^\tau f(X^o(s,X))\frac{\partial X^o}{\partial s}(s,X)e^{\Delta s}ds+o(\varepsilon). \tag{C.10}$$

Integrating by parts the integral, the term in ε in the r.-h. s. above becomes

$$-\left(\frac{x}{\sigma^o(X)}-1\right)\varepsilon\Delta e^{-\Delta\tau}\left(F(X^o(\tau,X))e^{\Delta\tau}-F(X)-\Delta\int_0^\tau F(X^o(s,X))e^{\Delta s}ds\right).$$

Using (6.3), this expression can be rewritten as

$$\left(\frac{x}{\sigma^o(X)}-1\right)\varepsilon\Delta\left(1-F(X^o(\tau,X))-Z^o(X^o(\tau,X))-(1-F(X)-Z^o(X))e^{-\Delta\tau}\right).$$

Using (6.4) at $X^o(\tau,X)$ and X, we rewrite this expression as

$$\left(\frac{x}{\sigma^o(X)}-1\right)\varepsilon\left(\sigma^o(X^o(\tau,X))\dot{Z}^o(X^o(\tau,X))-\sigma^o(X)\dot{Z}^o(X)e^{-\Delta\tau}\right)$$

$$=\left(x-\sigma^o(X)\right)\varepsilon\left(\frac{\partial X^o}{\partial X}(\tau,X)\dot{Z}^o(X^o(\tau,X))-\dot{Z}^o(X)e^{-\Delta\tau}\right)$$

where the equality follows from (**??**). Inserting into (**??**), we finally obtain (6.10).

Now, we compute $\widetilde{\mathcal{V}}(\varepsilon,x,X)$ as

$$\widetilde{\mathcal{V}}(\varepsilon,x,X)=\int_0^\varepsilon e^{-r\tau}\frac{\widetilde{Z}(\varepsilon,x,\tau,X)}{Z^o(X)}u(x)d\tau+\int_\varepsilon^{+\infty}e^{-r\tau}\frac{\widetilde{Z}(\varepsilon,x,\tau,X)}{Z^o(X)}u(\sigma^o(X^o(\tau-\varepsilon,X+x\varepsilon)))d\tau.$$

Taking a first-order Taylor approximation in ε, we find

$$Z^o(X)\left(\widetilde{\mathcal{V}}(\varepsilon,x,X)-\mathcal{V}^o(X)\right)=\varepsilon Z^o(X)(u(x)-u(\sigma^o(X)))$$

$$+\varepsilon\int_0^{+\infty}e^{-r\tau}\left(\frac{\partial\widetilde{Z}}{\partial\varepsilon}(0,x,\tau,X)u(\sigma^o(X^o(\tau,X)))+Z^o(X^o(\tau,X))\frac{\partial}{\partial\varepsilon}(u(\sigma^o(\widetilde{X}(\varepsilon,x,\tau,X)))_{\varepsilon=0})\right)d\tau+o(\varepsilon). \tag{C.11}$$

Using (6.9), we get

$$\frac{\partial}{\partial\varepsilon}(u(\sigma^o(\widetilde{X}(\varepsilon,x,\tau,X)))_{\varepsilon=0}=\left(x-\sigma^o(X)\right)u'(\sigma^o(X(\tau,X)))\dot{\sigma}^o(X(\tau,X))\frac{\partial X^o}{\partial X}(\tau,X). \tag{C.12}$$

Using now (6.10), (**??**) and inserting into (**??**), we find

$$Z^o(X)\left(\widetilde{\mathcal{V}}(\varepsilon,x,X)-\mathcal{V}^o(X)\right)=\varepsilon Z^o(X)(u(x)-u(\sigma^o(X))) \tag{C.13}$$

$$+\varepsilon\left(x-\sigma^o(X)\right)\int_0^{+\infty}e^{-r\tau}\left(\left(\frac{d}{dX}(Z^o(X^o(\tau,X)))-\dot{Z}^o(X)e^{-\Delta\tau}\right)u(\sigma^o(X^o(\tau,X)))\right.$$

$$\left.+Z^o(X^o(\tau,X))\frac{d}{dX}(u(\sigma^o(X^o(\tau,X))))\right)d\tau+o(\varepsilon).$$

Now observe that

$$\frac{d}{dX}(Z^o(X)\mathcal{V}^o(X))=\int_0^{+\infty}e^{-r\tau}\left(\frac{d}{dX}(Z^o(X^o(\tau,X)))u(\sigma^o(X^o(\tau,X))\right.$$

$$\left.+Z^o(X^o(\tau,X))\frac{d}{dX}(u(\sigma^o(X^o(\tau,X))))\right)d\tau$$

and

$$\dot{Z}^o(X)\varphi^o(X)=\int_0^{+\infty}e^{-\lambda\tau}\dot{Z}^o(X)u(\sigma^o(X^o(\tau,X)))d\tau.$$

Inserting those two expressions into (**??**) yields (6.11). □

Proof of Proposition 7. Let $\mathcal{W}^u(X)=(1-F(X))\mathcal{V}^u(X)$. Consider an impulse deviation consisting in choosing action x over an interval of length ε. By the *Principle of Dynamic Programming,* it must be that

$$\mathcal{W}^u(X)=\max_{x\in\mathcal{X}}(1-F(X))u(x)\varepsilon+(1-r\varepsilon)\mathcal{W}^u(X+x\varepsilon)+x\varepsilon f(X)\mathcal{V}_\infty.$$

Taking first-order Taylor expansions and dividing by ε, we rewrite this condition as the Hamilton-Bellman-Jacobi equation

$$r\mathcal{W}^u(X)=\max_{x\in\mathcal{X}}(1-F(X))u(x)+x\dot{\mathcal{W}}^u(X)+xf(X)\mathcal{V}_\infty. \tag{C.14}$$

The maximand is strictly concave in x and optimizing yields the expression of the feedback rule in (6.15). Inserting the value of the maximand into (**??**) gives us

$$r\mathcal{V}^u(X)=\frac{1}{2}\left(\zeta+\dot{\mathcal{V}}^u(X)-\frac{f(X)}{1-F(X)}(\mathcal{V}^u(X)-\mathcal{V}_\infty)\right)^2.$$

Taking the positive root yields (6.14), to which we add the boundary condition $\mathcal{V}^u(\overline{X})=\mathcal{V}_\infty$. □

PROOF OF PROPOSITION 8. First, observe that $\mathcal{V}^o(\overline{X}) = \mathcal{V}^u(\overline{X}) = \mathcal{V}^k(\overline{X}) = \mathcal{V}_\infty$. Second, observe that when F has no mass point at $\overline{X}$, $\frac{\dot{Z}^o(\overline{X}^-)}{Z^o(\overline{X}^-)} = -\frac{\Delta}{\sigma^o(\overline{X}^-)}$. Inserting into (6.7) and (6.8), $\sigma^o(\overline{X}^-)$ solves $\sigma^o(\overline{X}^-) = \frac{\Delta}{\sigma^o(\overline{X}^-)}\mathcal{V}_\infty + \sqrt{2r\mathcal{V}_\infty + \left(\frac{\Delta}{\sigma^o(\overline{X}^-)}\mathcal{V}_\infty\right)^2}$ whose unique solution is $\sigma^o(\overline{X}^-) = \zeta$. From (6.7) and (6.8), it then follows that $\dot{\mathcal{V}}^o(\overline{X}^-) = 0$. Using (5.5), (5.6) and $\mathcal{V}^k(\overline{X}) = \mathcal{V}_\infty$, we get $\sigma^k(\overline{X}^-) = \zeta\sqrt{\frac{r}{\lambda}} < \sigma^o(\overline{X}^-)$. It then follows from (5.5) that $\dot{\mathcal{V}}^k(\overline{X}^-) = \zeta\left(\sqrt{\frac{r}{\lambda}} - 1\right) < 0$. Using (6.14) and L'Hôspital rule at $\overline{X}$, we obtain $2\dot{\mathcal{V}}^u(\overline{X}^-) = \zeta\left(\sqrt{\frac{r}{\lambda}} - 1\right)$. Hence,

$$\dot{\mathcal{V}}^k(\overline{X}^-) < \dot{\mathcal{V}}^u(\overline{X}^-) < \dot{\mathcal{V}}^0(\overline{X}^-) \tag{C.15}$$

and thus (6.16) holds. Using (6.15) yields

$$\sigma^u(\overline{X}^-) = \sigma^k(\overline{X}^-) = \zeta\sqrt{\frac{r}{\lambda}} < \zeta = \sigma^o(\overline{X}^-). \tag{C.16}$$

Differentiating now (5.5) w.r.t. X, we obtain

$$r\dot{\mathcal{V}}^k(\overline{X}^-) = \dot{\sigma}^k(\overline{X}^-)\sigma^k(\overline{X}^-). \tag{C.17}$$

Using (6.14) and (6.15) and yields

$$\sigma^u(X) = \sqrt{2r\mathcal{V}^u(X)} \quad \forall X \in [0, \overline{X}).$$

Differentiating this condition w.r.t. X, we obtain

$$r\dot{\mathcal{V}}^u(\overline{X}^-) = \dot{\sigma}^u(\overline{X}^-)\sigma^u(\overline{X}^-). \tag{C.18}$$

Using (**??**) and (**??**) and simplifying with (**??**) yields $\dot{\sigma}^u(\overline{X}^-) > \dot{\sigma}^k(\overline{X}^-)$ and thus $\sigma^u(X) \le \sigma^k(X)$ for X in a left-neighborhood of $\overline{X}$. which gives us the ranking of feedback rules as in (6.17). □