WORKING PAPERS

N° 1374

December 2025

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Jean-Paul Décamps, Fabien Gensbittel and Thomas Mariotti

Toulouse School of Economics

The War of Attrition under Uncertainty: Theory and Robust Testable Implications*

Jean-Paul Décamps[†] Fabien Gensbittel[‡] Thomas Mariotti[§]

December 18, 2025

Abstract

We study a symmetric-information war of attrition in which the players' rewards depend on exogenous market conditions evolving as a homogeneous linear diffusion process. Using that a Markovian strategy can be represented by a stopping region along with an intensity measure of stopping, we fully characterize mixed-strategy Markov-perfect equilibria through a variational system for the players' value functions. When players are asymmetric, in any such equilibrium each player randomizes at a discrete set of thresholds for market conditions. As a result, players may alternatively find themselves in a position of strength or weakness on the equilibrium path. Delayed concessions occur because a player currently in a position of weakness can hope for market conditions to eventually turn in his favor. In the standard duopoly model of exit under uncertainty, the firms' stock prices and their return volatilities comove negatively over the attrition region and exhibit patterns documented by technical analysis.

Keywords: War of Attrition, Mixed-Strategy Equilibrium, Uncertainty.
JEL Classification: C61, D25, D83.

*This paper was first circulated by the same authors under the title "Mixed-Strategy Equilibria in the War of Attrition under Uncertainty" (arXiv:2210.08848). We thank René Aïd, Olivier Compte, Tiziano De Angelis, Erik Ekström, Bertrand Gobillard, Aaron Kolb, Pauli Murto, Frank Riedel, Jean-Charles Rochet, Julia Salmi, Sara Shahanaghi, Jan-Henrik Steg, Jean Tirole, Juuso Välimäki, Karine Van der Straeten, Nikhil Vellodi, and Stéphane Villeneuve for very valuable feedback. We also thank seminar audiences at Collegio Carlo Alberto, HEC Paris, Helsinki Graduate School of Economics, London Mathematical Finance Seminar, Paris School of Economics, Séminaire Bachelier, Toulouse Business School, Toulouse School of Economics, and University of South California, as well as conference participants at the 2021 Quimper Workshop on Dynamic Games, the 2024 Universität Bielefeld Conference on Strategic Timing Problems in Economics and Mathematical Finance, and the 2025 University of Leeds Conference on Stochastic Games and Randomised Strategies for many useful discussions. This research has benefited from financial support of the ANR (Programmes d'Investissements d'Avenir CHESS ANR-17-EURE-0010 and ANITI ANR-19-PI3A-0004) and the research foundation TSE-Partnership (Chaire Marchés des Risques et Création de Valeur, Fondation du Risque/SCOR).

[†]Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: `jean-paul.decamps@tse-fr.eu`.

[‡]Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: `fabien.gensbittel@tse-fr.eu`.

[§]Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France, CEPR, and CESifo. Email: `thomas.mariotti@tse-fr.eu`.

1 Introduction

The war of attrition (woa) is a workhorse model of situations in which each player has to decide when to concede and forfeit a prize to his opponent. Examples include animal conflict (Maynard Smith (1974)), public good provision (Bliss and Nalebuff (1984)), exit from a declining industry (Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986)), labor strikes (Kennan and Wilson (1989)), macroeconomic stabilizations (Alesina and Drazen (1991), Drazen and Grilli (1993)), competing standards (Bulow and Klemperer (1999)), bargaining (Abreu and Gul (2000)), investment under learning externalities (Décamps and Mariotti (2004)), and boycotts (Egorov and Harstad (2017)). A growing literature attempts to test the predictions of these models and to estimate the welfare cost of delayed concessions (Hendricks and Porter (1996), Ghemawat (1997), Padovano and Venturi (2001), Geraghty and Wiseman (2008), Wang (2009), Takahashi (2015)).

Because attrition generates costs for all players, a natural question is why it may occur in the first place.[1] Under complete information, if a player believes that his opponent is stubborn, so that she will never concede, then he is better off avoiding conflict altogether by conceding immediately. If his opponent correctly anticipates this, then she in turn has every reason to be stubborn. Hence attrition can occur in equilibrium only if players do not know each other's intentions, that is, if each player believes that his opponent concedes in a random manner. A common alternative explanation for attrition is asymmetric information, which arises when some player does not know how strong—how powerful, patient, enduring, or committed—his opponent is. Yet, this explanation is debatable, because conflicts can last for a considerable amount of time despite large observable differences in strength between the players.[2] Besides, asymmetric information is not a sufficient condition for attrition: immediate concession can occur in equilibrium if players have different distributions of waiting costs (Martinelli and Escorza (2007), Myatt (2025)).

This paper argues that delayed concessions in the woa may be caused by unpredictable changes in the environment rather than by players' asymmetric information about each others' characteristics. Our approach is motivated by the simple observation that uncertainty is a pervasive feature of the woas that are deemed to take place in practice. For instance, firms fighting to be the last to exit from a declining industry may still be uncertain about the future evolution of demand (Dixit and Pindyck (1994)). Similarly, political groups fighting about how to share the tax increase necessary to stabilize the economy may face

[1] Fearon (1995) raises this general point in the context of wars: given that wars are costly for all the belligerents, how can one rationalize the fact that they nonetheless recur and persist?

[2] Wars, labor strikes, and boycotts are obvious cases in point, as well as conflicts over macroeconomic stabilizations when opposed political groups represent electorates or constituencies of unequal sizes or means (Labán and Sturzenegger (1994)).

random fluctuations in interest rates as well as random shifts in the level of aid or foreign intervention (Alesina and Drazen (1991)). The new mechanism we emphasize is that, under these circumstances, a player currently finding himself in a position of weakness vis-à-vis his opponent may want to delay concession because he hopes for events to turn again in his favor while possibly remaining uncertain how his opponent will react to this, for her unfavorable, turn of events. The challenge is to show that this mechanism can cause delays over and above those due to the mere presence of uncertainty, even when changes in the environment are not assumed to a priori favor either player—so that being in a position of strength or weakness is not a built-in feature of players' preferences, but rather an endogenous feature of equilibrium play. In so doing, we identify a new class of mixed-strategy equilibria that have robust and novel testable implications.

To this end, we study a general two-player symmetric-information woa in which players' rewards depend on an exogenous state variable, hereafter generically referred to as *market conditions*. Both players continuously observe the evolution of market conditions, which follow a homogeneous linear diffusion. Based on this information, each player then decides whether to stay in the market or to irreversibly exit. There is a second-mover advantage in the sense that, if and when a player exits, his reward is lower than the reward he would have obtained had his opponent exited under the same market conditions. Players may be asymmetric, allowing us to capture observable differences in waiting costs or outside options. In the example of Section 6, two firms face uncertainty about future demand. A woa arises because each firm would like to liquidate its assets if demand were to deteriorate enough, but would meanwhile individually fare better as a monopolist than as a duopolist. Firms may be asymmetric; for instance, one firm's assets may be liquidated for a higher value than its opponent's, making it a priori more willing to exit the market.

Special cases of this example have been studied in the literature, with a natural focus on Markov-perfect equilibria (Mpes) in which firms' exit decisions at any point in time only depend on current market conditions (Maskin and Tirole (2001)). Murto (2004) characterizes pure-strategy Mpes in which each firm exits with probability 1 over some region of the state space. In the case of symmetric firms, Steg (2015) and Georgiadis, Kim, and Kwon (2022) construct a symmetric mixed-strategy Mpe in which both firms exit at a stochastic rate over an interval of market conditions. Attrition in this Mpe is maximal, in that each firm obtains the stand-alone value it would obtain if its opponent were stubborn. Importantly, Georgiadis, Kim, and Kwon (2022) show that this Mpe has no counterpart if there is even the slightest asymmetry between the firms.

As we argue in this paper, however, this negative result does not imply that attrition cannot occur in equilibrium when players are asymmetric. The reason is that the set of mixed Markovian strategies is not exhausted by strategies defined by stochastic exit rates. The latter, for instance, do not allow one to capture the behavior of a player who would wish to exit the market, with positive but finite intensity, each time the market conditions hit a single threshold value.[3] Such a Markovian strategy implies an infinite exit *rate* at this threshold, because the probability that the player exits within a short duration $\mathrm{d}t$ is of order $\sqrt{\mathrm{d}t}$ due to the infinite random fluctuations of Brownian motion. Yet, far from being artificial or exotic, this behavior naturally emerges as the limit of strategies with stochastic exit rates defined over a sequence of intervals degenerating to the randomization threshold, or, given appropriate normalizations, of mixed Markovian strategies defined over discretized state spaces and time grids with increasingly finer mesh.

In line with this observation, our first contribution is to provide a representation of mixed Markovian strategies, modeled as randomized stopping times. Our first main result, Theorem 1, states that a randomized stopping time is Markovian if and only if it can be represented by a pair (μ, S), where S is a subset of the state space over which the player exits with probability 1 and μ is a measure over the complement of S representing the player's intensity of exit at states at which he randomizes. Special cases include Markovian *stopping times*, in which the intensity measure μ is degenerate, *regular strategies* with a stochastic exit rate, in which μ is absolutely continuous with respect to Lebesgue measure, and *singular strategies* in which μ is discrete, prescribing to exit the market with finite intensity each time market conditions hit a countable set of states.[4]

Our second main result, Theorem 2, states that, if players are asymmetric, then all mixed-strategy Mpes are singular and involve strategies with discrete intensity measures whose supports form two intertwined sequences of randomization thresholds, one for each player. At any such threshold, the corresponding player is indifferent between staying in the market and exiting. This implies that the state space is endogenously partitioned into intervals that can all be visited on the equilibrium path and over which each player is alternatively in a position of strength or weakness depending on how far current market conditions are from one of his randomization thresholds. The reason why a player currently in a position of weakness has an incentive to wait and to randomize at specific thresholds for market conditions is that he can hope for a reversal of situation in his favor—that is, for market conditions to transit to a neighboring interval of the state space over which he will be again in a position of strength. As a result, the balance of power between the players

[3]Importantly, exiting with finite *intensity* at a point should be distinguished from exiting with positive *probability* each time the market conditions hit that point: given the infinite random fluctuations of Brownian motion, the latter is indistinguishable from exiting with probability 1 at that point.

[4]More generally, one can conceive mixed Markovian strategies in which the intensity measure μ has an arbitrary component that is singular with respect to Lebesgue measure—for instance, a component with support equal to a Cantor-like set—but these strategies play no role in our equilibrium analysis.

randomly fluctuates as market conditions vary over time. This new finding contrasts with the outcomes of pure-strategy Mpes, in which one of the players always remains in a position of strength until his opponent eventually exits. It also contrasts, when players are symmetric, with the outcome of the regular Mpe in which the players exit at a stochastic rate over an interval of the state space. This illustrates the general point that, because of their alternating structure, incentives in singular Mpes are nonlocal: that is, a player's equilibrium behavior at any state is disciplined by his uncertainty about his opponent's *future* intentions and behavior at other states.

Our third main result, Theorem 3, characterizes these singular Mpes by a variational system for the players' equilibrium value functions, which are linked by the intensity with which each player exits at each of his randomization thresholds. At any such point, the other player's value function reaches a peak and exhibits a kink whose size is proportional to the intensity with which the randomizing player exits the market; this kink reflects that exit by the latter is unpredictable given the current market conditions. Importantly, this characterization also applies to the case of symmetric players.

Finally, we use this variational characterization to prove in Theorem 4 an existence result for singular Mpes in the case of symmetric or nearly symmetric players, and, relatedly, to show in Theorem 5 that, contrary to the conclusions of Georgiadis, Kim, and Kwon (2022), the regular Mpe in the symmetric case is not isolated and can be approximated by singular Mpes of nearly symmetric woas featuring many randomization thresholds. Our existence proof is semi-constructive and thus could easily be converted into a numerical procedure for the construction of singular Mpes.

Admittedly, these results do not contribute to solving the multiplicity problem that plagues woa models (Riley (1980), Hendricks, Weiss, and Wilson (1988)): if anything, we exhibit new equilibria that have escaped attention in the literature, both in the cases of symmetric and asymmetric players. However, the key point is that the singular Mpes that survive when players are asymmetric share a common structure and lead to qualitatively similar and, hence, robust testable implications. The first, naturally, is that exit takes place at endogenously determined salient thresholds for market conditions, an intuitively appealing feature of singular Mpes that, notwithstanding their randomized nature, is reminiscent of pure-strategy Mpes. The second, which follows from our variational characterization, is that players' rents, defined as the difference between their equilibrium and stand-alone values, are non-monotonic with respect to market conditions and comove negatively over time across players as long as market conditions remain in the attrition region. Thus the antagonism between the players can be identified from the joint dynamics of their continuation values. This contrasts with the regular Mpe of a symmetric woa, in which these rents are identically zero over the entire attrition region and thus players' values are the same as those they would obtain if there were no strategic interaction between them.

Singular Mpes with many randomization thresholds have three shortcomings. First, they require a high degree of coordination between the players. Second, they only exist when players are symmetric or nearly so. Third, they lead to predictions that are qualitatively not very different from those of the regular Mpe of a symmetric woa. To address these issues, illustrate our findings, and further explore the testable implications of our model, we revisit the standard model of exit in duopoly in the light of our analysis.

We first provide sufficient conditions under which there exists a simple singular Mpe in which one firm randomizes at a single threshold while the other firm plays a pure strategy, and we use numerical simulations to show that the existence of such an Mpe is consistent with large asymmetries between the firms. In equilibrium, one firm, say, firm 1, randomizes between staying in the market and exiting at its stand-alone exit threshold, while firm 2 exits with probability 1 if market conditions fall below a lower threshold, the value of which is determined precisely so as to meet firm 1's indifference condition. The intensity with which firm 1 exits the market at its randomization threshold in turn makes it optimal for firm 2 to exit at its lower threshold.

In this simple singular Mpe, the total value of firm 1 goes down to its liquidation value at its randomization threshold, while, if firm 2's assets have a higher liquidation value than firm 1's, the total value of firm 2 simultaneously reaches a peak. Novel asset-pricing implications ensue when these firms are publicly traded on a frictionless financial market.

First, along any path of the diffusion process modeling the evolution of market conditions, the firms' stock prices and the volatilities of their returns fluctuate randomly over the attrition region, moving in opposite directions as long as no firm exits the market. These negative comovements of the firms' stock prices and of the volatilities of their returns stand in sharp contrast with the predictions of the regular Mpe that arises when firms have identical liquidation values, namely, that firms' stock prices stay constant and equal to their liquidation value over the attrition region—a very strong prediction that is likely to be rejected by the data.

Second, when the stock price of a firm reaches a peak of its total value function, two events may occur. Either its opponent does not exit the market, causing the firm's stock price to bounce downward. Or its opponent exits the market, causing the firm's stock price to jump upwards to its monopoly value. Because exit by the opponent is unpredictable, these downward bounces exactly compensate for this upward jump. As a result, rational investors have no means to arbitrage away the profits associated to these downward bounces by short-selling the firm's stock at its peak without incurring the risk of a sudden upward jump in its price. We argue that this pattern is consistent with what technical analysts describe as a resistance level in stock prices, for which our analysis provides an illustration in a setting where markets are efficient and stock prices are only driven by fundamentals. Needless to say, we do not claim that the patterns documented by technical analysis can only be rationalized by our model, as many other rational or behavioral factors may be at play. Still, our model seems to capture reasonably well the intuitive idea that resistance levels in stock prices can be discontinuously broken by unpredictable changes in the environment above investors' expectations—in this instance, the exit of a competitor.

Third, if firm 2's assets have a lower liquidation value than firm 1's, firm 2's total value need not reach a peak at firm 1's randomization threshold. In that case, it is no longer true that firm 2's stock price bounces back each time market condition hit firm 1's randomization threshold but firm 1 does not exit. However, because firm 2's value function still exhibits a kink at this point, a new phenomenon can arise, namely, a form of stickiness in firm 2's stock price at the corresponding level. To a technical analyst unaware of the strategic interaction between the firms and its impact on their valuations, it may seem from the resulting historical price movements that firm 2's stock price undergoes episodes of strong autocorrelation. For the same reasons as above, however, it does not follow that he would be able to predict future price movements.

Our findings pave the way to many other applications. For instance, woa models of macroeconomic stabilizations typically do not account for exogenous random changes in the environment. Our analysis suggests that such changes may considerably delay the needed reforms even if the parties involved—say, workers and capitalists—are well aware of each other's waiting costs. The singular Mpes we construct, in which concession by one player is stochastic and tied to the hitting of critical levels of the exogenous state variable, may also help explain why actual stabilizations need not follow significant observable changes in the macroeconomic environment (Alesina and Drazen (1991)).

Related Literature This paper belongs to the large literature on the continuous-time woa, starting with the seminal contribution of Maynard Smith (1974) on animal conflict. Ghemawat and Nalebuff (1985) study a woa between duopolists that must decide when to exit from a declining industry. Hendricks, Weiss, and Wilson (1988) offer an exhaustive characterization of pure- and mixed-strategy equilibria in the symmetric-information woa when players have potentially asymmetric rewards that are deterministic functions of time. Riley (1980), Bliss and Nalebuff (1984), and Fudenberg and Tirole (1986) extend the analysis to asymmetric-information settings where, for instance, a player is uncertain about his opponent's waiting cost, and Myatt (2025) studies the impact of players' perceived strengths on equilibrium concession times when there is an exogenous deadline. Décamps and Mariotti (2004) study an investment game that has the structure of a woa because a firm's investment generates a public signal about the return of a common-value project.

These papers except the last one focus on deterministic settings. By contrast, starting with Lambrecht (2001) and Murto (2004), a small literature examines the case where players in a woa have symmetric information but are uncertain about their future rewards, which are driven by a diffusion process. Lambrecht (2001) analyzes how the order in which firms go bankrupt in an industry is influenced by aggregate factors and firm-specific factors such as their financial structure. Murto (2004) studies a stochastic version of Ghemawat and Nalebuff (1985)'s exit model, and shows that the firm with the lowest liquidation value may end up exiting the market first in equilibrium, despite being a priori more enduring than its opponent. These papers allow for asymmetries between players but restrict attention to pure-strategy Mpes, as in Fine and Li (1989)'s discrete-time model of exit from a stochastically declining industry. By contrast, Steg (2015) characterizes the regular Mpe of the symmetric woa. Kwon and Palczewski (2025) extend this construction to a woa with asymmetric information and a continuum of types; they show that the symmetric-equilibrium pure strategy, seen as a randomized strategy using each player's type as a randomization device, has an absolutely continuous intensity that depends on the exogenous diffusion process and on an endogenous belief process.

Closest to the present paper in this literature is Georgiadis, Kim, and Kwon (2022). In a setting that extends Murto (2004), they show that, as soon as firms have different liquidation values, there exists no mixed-strategy Mpe in which firms exit the market at a stochastic rate, that is, according to a regular strategy. They argue that the regular Mpe characterized by Steg (2015) is not robust to even small asymmetries between firms. They conclude from this result that, when firms are asymmetric, only pure-strategy Mpes exist, and, therefore, that attrition cannot actually take place on the equilibrium path. Our analysis shows that these conclusions are unwarranted once the possibility for firms to exit the market according to singular strategies is accounted for.

Gieczewski (2025) studies a symmetric-information woa that has the structure of a supermodular game because changes in the underlying state variable affect the players' waiting costs in opposite ways. He shows that there exists a unique pure-strategy Mpe in which one player exits when the state hits a lower threshold from above, while his opponent exits when the state hits an upper threshold from below. In the intermediate region, each player is willing to wait because he hopes that the state will hit his opponent's exit threshold first. A similar equilibrium structure arises in Gul and Pesendorfer (2012)'s closely related war of information. Our model differs in that, although players may be asymmetric, changes in market conditions affect them in similar ways. As a result, market conditions do not per se determine whether a player is in a position of strength or weakness; instead, this is endogenously determined in a mixed-strategy Mpe.

We have borrowed from Touzi and Vieille (2002) our concept of a randomized stopping time, which they use to show that any continuous-time zero-sum Dynkin game admits a value. Singular mixed strategies have been emphasized in the recent literature on dynamic games of incomplete information where, in equilibrium, an informed player uses a randomized stopping time to control the belief process of an uninformed player using a pure strategy (Daley and Green (2012), Kolb (2019), De Angelis, Ekström, and Glover (2022)). The resulting singular cumulative distribution function is related to the local time of the uninformed player's belief process at some critical threshold and, at this threshold, can generate a kink in the value function of the uninformed player (Kolb (2019)).

In our woa model, the players have symmetric information, and randomized stopping times with singular cumulative distribution functions play a different role. Specifically, in equilibrium, each player randomizes at some critical threshold values for market conditions at which he is indifferent between exiting and staying in the market. The intensity with which he exits at each of these thresholds makes his opponent willing to stay in the market in intervals of states around these thresholds, and indifferent between exiting and staying in the market at the endpoints of such intervals. This gives rise to the alternating structure of singular Mpes. The resulting singular cumulative distribution function is related to the local times of the exogenous diffusion process of market conditions at these critical thresholds; such local times, in the exit-in-duopoly example, play a key role in the dynamics of firms' stock prices. Finally, the fact that the cumulative distribution function of any Markovian randomized stopping time can be represented via an integral of such local-time processes is precisely what enables us to obtain a full characterization of mixed-strategy Mpes. This representation result is of independent interest and may prove useful for the study of general stochastic timing games in which the state variable is driven by a Brownian motion. In this vein, Décamps, Gensbittel, and Mariotti (2025b) provide a general existence theorem for pure- or mixed-strategy Mpes for woas under weaker assumptions on reward functions than those in the present paper; however, the fixed-point approach they devise to this end falls short of delivering a characterization result.

Keller, Rady, and Cripps (2005) study a two-armed-bandit strategic-experimentation model in which the risky arm might distribute lump-sum rewards according to a Poisson process. They show that the unique symmetric Mpe, in which, over some range of beliefs, each player devotes an interior amount of resource to experimentation, is Pareto-dominated over this range by asymmetric Mpes in which players take turns in experimenting and playing the safe action. This alternating structure of asymmetric Mpes, as well as the idea that the latter may lead to Pareto improvements over the symmetric Mpe, is reminiscent of that which arises in the singular Mpes of our model. Keller and Rady (2010) generalize this insight to a situation in which a single success on the risky arm does not fully reveal its type. They show that this generates an encouragement effect and that players may alternatively find themselves in a position of strength or weakness. Despite these similarities with our results, the strategic interaction between the players is entirely different as the experimentation model does not involve irreversible decisions. Another difference is that our analysis is cast in a Brownian rather than in a Poisson setup.

The paper is organized as follows. Section 2 describes the model. Section 3 defines our strategy and equilibrium concepts. Section 4 provides preliminary properties of Mpes and discusses relevant benchmarks. Section 5 states our main characterization, existence, and convergence results for singular Mpes. Section 6 discuss the implications of our analysis for the exit-in-duopoly example. Section 7 concludes. The main Appendix provides the proofs of Theorems 1–3. The Online Supplement collects detailed proofs of technical lemmas and claims used in the derivation of these theorems, as well as the proofs of Theorems 4–5.

2 A Brownian Model of the War of Attrition

The Stochastic Environment Time is continuous, with an infinite horizon, and indexed by $t \geq 0$. Market conditions evolve as a one-dimensional time-homogeneous diffusion process $X \equiv (X_t)_{t\geq 0}$ defined over the canonical space $(\Omega, \mathcal{F}, \mathbf{P}_x)$ of continuous trajectories with $X_0 = x$ under $\mathbf{P}_x$, and solution in law to

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t \,+\, \sigma(X_t)\,\mathrm{d}W_t, \quad t \geq 0, \tag{1}$$

for some Brownian motion $W \equiv (W_t)_{t\geq 0}$. The state space for X is an interval $\mathcal{I} \equiv (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$, and the functions b and σ are continuous, with $\sigma > 0$ over $\mathcal{I}$. We assume that α and β are inaccessible (natural) boundaries for X.[5] Therefore, X is regular over $\mathcal{I}$ and (1) admits a weak solution that is unique in law.[6]

Players, Actions, and Payoffs Two players, 1 and 2, face uncertainty about future market conditions, which is gradually resolved over time. Upon observing the evolution of

[5] That is, X cannot reach the boundaries α and β in finite time.

[6] That X is regular means that X starting at x reaches y with positive $\mathbf{P}_x$-probability, for all x and y in $\mathcal{I}$. Our assumptions on b and σ imply that X satisfies non-degeneracy and local integrability properties that ensure the existence and uniqueness in law of a weak solution up to an explosion time at which X goes to $\pm\infty$ (Karatzas and Shreve (1991, Chapter 5, Theorem 5.15)). This already disposes of the case $-\infty < \alpha < \beta < \infty$. If $\alpha = -\infty$ or $\beta = \infty$, Feller's test for explosions (Karatzas and Shreve (1991, Chapter 5, Theorem 5.29)) ensures that the explosion time for X is $\mathbf{P}_x$-almost surely infinite.

market conditions, they must decide at each instant whether to hold fast or to concede to their opponent, where the latter decision terminates the woa. Letting τ^1 and τ^2 be the random times at which players 1 and 2 plan to concede, every player i's expected payoff is

$$J^i(x, \tau^1, \tau^2) = \mathbf{E}_x \Big[\mathbb{1}_{\{\tau^i \leq \tau^j\}} \, \mathrm{e}^{-r\tau^i} R^i(X_{\tau^i}) + \mathbb{1}_{\{\tau^i > \tau^j\}} \, \mathrm{e}^{-r\tau^j} G^i(X_{\tau^j}) \Big], \tag{2}$$

where $r > 0$ is the players' common discount rate.[7] There is a second-mover advantage, in the sense that the reward $G^i(x)$ player i obtains when player j concedes to him in any state x is at least as large as and sometimes strictly higher than the reward $R^i(x)$ player i would have obtained by conceding to player j in state x. Specifically,

A0 $G^i \geq R^i$ over $\mathcal{I}$ and, for some $\alpha^i \in [\alpha, \beta)$, $G^i(x) > R^i(x)$ if and only if $x > \alpha^i$.[8]

Information We assume that the functions b, σ, R^i, and G^i are common knowledge, so that our model of the woa is one of symmetric information.

Assumptions on the Reward Functions Let $\mathcal{L}$ be the infinitesimal generator of X, defined by $\mathcal{L}u \equiv bu' + \frac{1}{2}\sigma^2 u''$ for all $u \in \mathcal{C}^2(\mathcal{I})$.

For each i, R^i is $\mathcal{C}^2$ over $\mathcal{I}$ and satisfies

A1 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}|R^i(X_t)|] < \infty$.

A2 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt} R^i(X_t) = 0$ $\mathbf{P}_x$-a.s.

A3 For some $x_0^i \in \mathcal{I}$, $\mathcal{L}R^i - rR^i < 0$ over (α, x_0^i) and $\mathcal{L}R^i - rR^i > 0$ over (x_0^i, β).

A1–A2 are standard integrability and transversality conditions; A1 notably guarantees that the family $(\mathrm{e}^{-r\tau} R^i(X_\tau))_{\tau\in\mathcal{T}}$ is uniformly integrable. A3 is substantial and implies that, no matter how player j behaves, marginally delaying concession is beneficial for player i if the current state is in (x_0^i, β), that is, if market conditions are favorable enough.

For each i, G^i is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$ over $\mathcal{I}$ and satisfies

A4 For each $x \in \mathcal{I}$, $\mathbf{E}_x[\sup_{t\geq 0} \mathrm{e}^{-rt}|G^i(X_t)|] < \infty$.

A5 For each $x \in \mathcal{I}$, $\lim_{t\to\infty} \mathrm{e}^{-rt} G^i(X_t) = 0$ $\mathbf{P}_x$-a.s.

A6 $\mathcal{L}G^i - rG^i \leq 0$ everywhere $G^{i\prime\prime}$ is defined.

Again, A4–A5 are standard integrability and transversality conditions. A6 is substantial and implies that player i prefers to obtain the reward $G^i(X_t)$ sooner than later, as is the case when G^i is itself the value function of an ulterior optimal stopping problem.[9]

[7]Here and in what follows, i (he) refers to an arbitrary player and j (she) to his opponent. By convention, we let $f(X_\tau) \equiv 0$ over $\{\tau = \infty\}$ for any Borel function f and any random time τ.

[8]The case $\alpha^i > \alpha$ intuitively captures the idea that, if market conditions become adverse enough, then player i wants to exit the market no matter how player j behaves; see the example in Section 6.

[9]In this case, G^i is typically $\mathcal{C}^1$ but only piecewise $\mathcal{C}^2$ over $\mathcal{I}$; see the example in Section 6.

The Stand-Alone Problem If player j is *stubborn*, that is, plays $\tau^j = \infty$, then maximizing (2) is tantamount for player i to solving the *stand-alone problem*

$$V_{R^i}(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x [\mathrm{e}^{-r\tau} R^i(X_\tau)], \tag{3}$$

where $\mathcal{T}$ is the set of all stopping times of the usual augmentation $(\mathcal{F}_t)_{t \geq 0}$ of the natural filtration generated by X. It follows from A3 that the optimal stopping region $\{x \in \mathcal{I} : V_{R^i}(x) = R^i(x)\}$ for the stand-alone problem (3) is of the form $(\alpha, x_{R^i}]$ for some optimal stand-alone threshold $x_{R^i} < x_0^i$. Therefore,

$$V_{R^i}(x) = \begin{cases} R^i(x) & \text{if } x \leq x_{R^i} \\ \mathbf{E}_x [\mathrm{e}^{-r\tau_{x_{R^i}}}] R^i(x_{R^i}) & \text{if } x > x_{R^i} \end{cases}, \tag{4}$$

where, for each $y \in \mathcal{I}$, $\tau_y \equiv \inf\{t \geq 0 : X_t = y\}$ is the hitting time by X of y from $X_0 = x$. The smooth-fit property applies at x_{R^i}, that is, $V'_{R^i}(x_{R^i}) = R^{i\prime}(x_{R^i})$ (Dayanik and Karatzas (2003, Corollary 7.1)). From standard optimal stopping theory, $(\mathrm{e}^{-rt} V_{R^i}(X_t))_{t \geq 0}$ is a supermartingale and $\mathcal{L}V_{R^i} - rV_{R^i} \leq 0$ over $\mathcal{I} \setminus \{x_{R^i}\}$ (El Karoui (1981)). The following lemma is also standard (Décamps, Gensbittel, and Mariotti (2025a, Lemma 1)).

Lemma 1 *$V_{R^i} > 0$ over $\mathcal{I}$ and $R^i > 0$ over $(\alpha, x_{R^i}]$.*

From A0 and A6, it is easy to verify that $G^i \geq V_{R^i}$. From A0 and the fact that $V_{R^i} > R^i$ over (x_{R^i}, β), it then follows that $x_{R^i} \geq \alpha^i$. To rule out trivial cases from our analysis, such as the case where $G^i = V_{R^i}$ over $\mathcal{I}$, we make a final assumption.

A7 $x_{R^i} > \alpha^i$.

The resulting lemma summarizes the ordering of the reward and stand-alone value functions.

Lemma 2 *$G^i \geq V_{R^i} \geq R^i$ over $\mathcal{I}$ and $G^i(x) > V_{R^i}(x)$ if and only if $x > \alpha^i$.*

3 Markovian Strategies

We first recall the definition and basic properties of randomized stopping times, interpreted as mixed strategies for the players in the woa. Imposing a Markov restriction leads to our first main result, which is a representation theorem for Markovian strategies.

3.1 Randomized Stopping Times

A classical definition of a randomized stopping time consists, following Aumann (1964), in enlarging the probability space; this compensates for the absence of a natural measurable structure over the space of stopping times. For every player i, the enlarged probability space is $(\Omega^i, \mathcal{F}^i) \equiv (\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]))$, endowed with the product probability $\mathbf{P}_x^i \equiv \mathbf{P}_x \otimes Leb$, where $\mathcal{B}([0,1])$ is the Borel σ-field over the sampling space $[0,1]$ and Leb is Lebesgue measure. Hence the following definition (Touzi and Vieille (2002)).

Definition 1 *A randomized stopping time for player i is an $\mathcal{F} \otimes \mathcal{B}([0,1])$-measurable function $\gamma^i : \Omega^i \to \mathbb{R}_+$ such that, for Leb-a.e. $u^i \in [0,1]$, $\gamma^i(\cdot, u^i) \in \mathcal{T}$. The process $\Gamma^i \equiv (\Gamma^i_t)_{t \geq 0}$ defined by*

$$\Gamma^i_t(\omega) \equiv \int_{[0,1]} \mathbb{1}_{\{\gamma^i(\omega,u^i) \leq t\}} \, \mathrm{d}u^i, \quad (\omega, t) \in \Omega \times \mathbb{R}_+,$$

is the conditional cumulative distribution function (ccdf) of the randomized stopping time γ^i. The process $\Lambda^i \equiv (\Lambda^i_t)_{t \geq 0}$ defined by $\Lambda^i_t \equiv 1 - \Gamma^i_t$ is the conditional survival function (csf) of the randomized stopping time γ^i.

The terminology can be justified as follows. First, Γ^i takes its values in $[0,1]$ and has nondecreasing right-continuous trajectories that can be interpreted as cdfs.[10] Second, for all $x \in \mathcal{I}$ and $t \geq 0$, Γ^i_t is $\mathbf{P}_x$-a.s. equal to the probability that $\gamma^i \leq t$ conditionally on the information $\mathcal{F}_t$ generated by the process X until t, and thus Γ^i is $(\mathcal{F}_t)_{t \geq 0}$-adapted.[11]

If the players use randomized stopping times γ^1 and γ^2, then their expected payoffs are defined over the product probability space $\Omega \times [0,1] \times [0,1]$ with canonical element (ω, u^1, u^2), endowed with the product probability $\overline{\mathbf{P}}_x \equiv \mathbf{P}_x \otimes Leb \otimes Leb$. We have $\gamma^1 \equiv \gamma^1(\omega, u^1)$ and $\gamma^2 \equiv \gamma^2(\omega, u^2)$, reflecting that the players use independent randomization devices u^1 and u^2. In line with Touzi and Vieille (2002) and Riedel and Steg (2017), expected payoffs depend on (γ^1, γ^2) only through their ccdfs (Γ^1, Γ^2). Specifically,

$$\begin{aligned} J^i(x, \gamma^1, \gamma^2) &\equiv \overline{\mathbf{E}}_x \left[\mathbb{1}_{\{\gamma^i \leq \gamma^j\}} \, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i}) + \mathbb{1}_{\{\gamma^i > \gamma^j\}} \, \mathrm{e}^{-r\gamma^j} G^i(X_{\gamma^j}) \right] \\ &= \mathbf{E}_x \left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-} \, \mathrm{d}\Gamma^i_t + \int_{[0,\infty)} \mathrm{e}^{-rt} G^i(X_t) \Lambda^i_t \, \mathrm{d}\Gamma^j_t \right] \equiv J^i(x, \Gamma^1, \Gamma^2). \quad (5) \end{aligned}$$

The second half of (5) is nothing but a stochastic version of the formula for the payoffs from a mixed-strategy profile in the deterministic woa. Conversely, any nondecreasing, right-continuous, $(\mathcal{F}_t)_{t \geq 0}$-adapted, $[0,1]$-valued process Γ^i is the ccdf of the randomized stopping time $\hat{\gamma}^i$ defined by $\hat{\gamma}^i(u^i) \equiv \inf\{t \geq 0 : \Gamma^i_t > u^i\}$.[12] Thus we may interchangeably work with randomized stopping times or ccdfs.

[10] By convention, we let $\Gamma^i_{0-} \equiv 0$. This allows us in what follows to interpret integrals of the form $\int_{[0,\tau)} \cdot \, \mathrm{d}\Gamma^i_t$ in the Stieltjes sense for any ccdf Γ^i.

[11] This second statement is established in Lemma S.2.1 in Online Supplement S.2.

[12] This statement, as well as formula (5) for the players' expected payoffs, is established in Lemma S.2.2 in Online Supplement S.2.

3.2 A Representation Theorem for Markovian Strategies

Our goal in this paper is to characterize equilibria in which players concede according to mixed Markovian strategies, which only depend on current market conditions and thus have to be defined for any initial market conditions. To obtain a tractable definition of such strategies, it is useful to introduce, for each $t \geq 0$, the shift operator $\theta_t : \Omega \to \Omega$, whose effect on a trajectory ω of X is to forget the part of the trajectory prior to t and to shift back the remaining part by t units of time (Revuz and Yor (1999, Chapter I, §3)). We then have the following definition.

Definition 2 *A randomized stopping time for player i with csf $\Lambda^i : \Omega \times \mathbb{R}_+ \to [0,1]$ is Markovian if, for all $x \in \mathcal{I}$, $\tau \in \mathcal{T}$, and $s \geq 0$,*

$$\Lambda^i_{\tau+s} = \Lambda^i_\tau \cdot (\Lambda^i_s \circ \theta_\tau) \text{ over } \{\tau < \infty\} \ \mathbf{P}_x\text{-a.s.} \tag{6}$$

To motivate this definition, recall that $\Lambda^i_{\tau+s}$ is the probability that player i concedes after $\tau + s$ conditionally on $\mathcal{F}_{\tau+s}$. The Markov restriction states that, conditionally on player i not conceding by τ, the probability that he holds fast for at least s additional units of time does not depend on the trajectory of X prior to τ. This probability is thus given by $\Lambda^i_s \circ \theta_\tau$, that is, the probability induced by the randomized strategy applied to the shifted trajectory. Formula (6) then follows from the standard formula for conditional probabilities.

Combining a result by Sharpe (1971) with the standard representation result for additive functionals of regular diffusions (Borodin and Salminen (2002, Part I, Chapter II, Section 4, §23)), we obtain the following representation result.

Theorem 1 *$\Lambda^i : \Omega \times \mathbb{R}_+ \to [0,1]$ is the csf of a Markovian randomized stopping time for player i if and only if there exists a closed set $S^i \subset \mathcal{I}$ and a Radon measure μ^i over $\mathcal{I} \setminus S^i$ such that, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Lambda^i_t = \mathbb{1}_{\{t<\tau_{S^i}\}} \, \mathrm{e}^{-\int_{\mathcal{I}\setminus S^i} L^y_t \, \mu^i(\mathrm{d}y)} \ \mathbf{P}_x\text{-a.s.}, \tag{7}$$

where $L^y_t \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{(y-\varepsilon,y+\varepsilon)}(X_s)\sigma^2(X_s)\,\mathrm{d}s$ is the local time of X at y up to t and $\tau_{S^i} \equiv \inf\{t \geq 0 : X_t \in S^i\}$ is the hitting time by X of S^i.[13] *In particular, the mapping $t \mapsto \Lambda^i_t$ is continuous over $[0, \tau_{S^i})$ $\mathbf{P}_x$-a.s.*

Theorem 1 allows us to interchangeably refer to a Markovian strategy as a pair (μ^i, S^i), a ccdf Γ^i, or a csf Λ^i. Despite its technical appearance, formula (7) has a natural interpretation. The presence of the indicator $\mathbb{1}_{\{t<\tau_{S^i}\}}$ implies that Λ^i_t drops down to 0 as soon as X hits S^i.

[13] Recall that a Radon measure over an open set $U \subset \mathbb{R}$ is a nonnegative Borel measure that is locally finite in the sense that every point of U has a neighborhood having finite measure.

This reflects that player i concedes with probability 1, that is, with infinite intensity over the stopping region S^i. By contrast, the integral term $\int_{\mathcal{I}\setminus S^i} L_t^y \, \mu^i(\mathrm{d}y)$ reflects that player i concedes with finite intensity $\mu^i(\mathrm{d}y)$ each time the process hits $y \in \operatorname{supp} \mu^i \subset \mathcal{I} \setminus S^i$; this finite intensity is compounded with the amount of time that the process X spends in the vicinity of y, which is formally represented by the local time L_t^y. Finally, the exponential in (7) comes from the multiplicative form of (6).

4 Markov-Perfect Equilibria: General Properties and Classical Examples

After recalling the definition of Markov-perfect equilibrium, we provide some important properties of best replies and describe two types of such equilibria that have been emphasized in the literature.

4.1 Definition and Properties of Best Replies

The following standard result does not rely on the Markov assumption and reflects that a player, given the behavior of his opponent, cannot improve his payoff by merely randomizing over standard stopping times.

Lemma 3 *For each $x \in \mathcal{I}$ and for any pair of randomized stopping times with ccdfs (Γ^1, Γ^2), $J^i(x, \Gamma^i, \Gamma^j) \leq \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, \Gamma^j)$.*

This motivates the following definition.

Definition 3 *A Markov-perfect equilibrium (Mpe) of the woa is a profile $((\mu^1, S^1), (\mu^2, S^2))$ of Markovian strategies such that, for all i and $x \in \mathcal{I}$,*

$$J^i(x, (\mu^i, S^i), (\mu^j, S^j)) = \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j)).$$

That is, (μ^i, S^i) is a perfect best reply (pbr) to (μ^j, S^j) for every player i.[14]

For each $x \in \mathcal{I}$, we let $\bar{J}^i(x, (\mu^j, S^j)) \equiv \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j))$. That is, $\bar{J}^i(\cdot, (\mu^j, S^j))$ is player i's best-reply value function (brvf) to (μ^j, S^j). When no confusion can arise as to the strategy of player j, we write $\bar{J}^i(x)$ instead of $\bar{J}^i(x, (\mu^j, S^j))$. The next proposition provides useful general properties of pbrs and brvfs.

[14] Murto (2004) adds to this definition that $(\alpha, \alpha^i] \subset S^i$ for every player i. The rationale is that, because $G^i = V_{R^i} = R^i$ over $(\alpha, \alpha^i]$, holding fast over (α, α^i) is weakly dominated for player i by conceding with probability 1. For instance, being stubborn is a best reply for player i over (α, α^i) only if player j concedes with probability 1 over this interval, except perhaps over a set of *Leb*-measure 0. This behavior is not per se inconsistent with an Mpe, but, as pointed out by Ghemawat and Nalebuff (1985) in a deterministic context, it is not consistent with trembling-hand perfection in the spirit of Selten (1975). Although we do not systematically impose this refinement, we indicate which Mpes can be modified so as to satisfy it.

Proposition 1 *If* (μ^i, S^i) *is a pbr to* (μ^j, S^j) *with associated brvf* $\bar{J}^i$*, then* $V_{R^i} \leq \bar{J}^i \leq G^i$*. Furthermore,*

(i) $S^1 \cap S^2 \cap (\alpha^i, \beta) = \emptyset$;

(ii) $S^i \subset C^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = R^i(x)\}$;

(iii) $\operatorname{supp} \mu^i \setminus S^j \subset C^i$ *and* $\operatorname{supp} \mu^i \cap S^j \subset D^i \equiv \{x \in \mathcal{I} : \bar{J}^i(x) = G^i(x)\}$;

(iv) $S^i \cup (\operatorname{supp} \mu^i \setminus S^j) \subset (\alpha, x_{R^i}]$;

(v) $(0, S^i)$ *is also a pbr to* (μ^j, S^j)*; more generally,* $(\tilde{\mu}^i, S^i)$ *is a pbr to* (μ^j, S^j) *for any* $\tilde{\mu}^i$ *such that* $\operatorname{supp} \tilde{\mu}^i \subset C^i \cup S^j$.

Property (i) intuitively states that player i should never concede when market conditions are such that player j concedes with probability 1 and player i's payoff from conceding is strictly less than the payoff from letting player j concede, that is, $x \in S^j$ and $G^i(x) > V_{R^i}(x)$. Property (ii) simply expresses the fact that player i's brvf coincides with R^i over the portion S^i of the state space over which he concedes with probability 1. Property (iii) states that player i's payoff is R^i when he concedes with positive intensity outside of player j's stopping region S^j. Property (iv) reflects that player i should never concede when market conditions are above the optimal threshold x_{R^i} for his stand-alone problem; intuitively, this is because waiting for X to drop down to x_{R^i} before conceding is player i's optimal strategy even in the worst-case scenario in which player j is stubborn, that is, $(\mu^j, S^j) = (0, \emptyset)$. Finally, property (v) states that, when conceding with positive intensity outside of S^i, player i should be indifferent between holding fast and conceding.

The following global regularity result reflects the smoothing effect of Brownian motion.

Proposition 2 *In any Mpe, every player* i*'s brvf* $\bar{J}^i$ *is continuous over* $\mathcal{I}$*.*

4.2 Benchmarks

To describe our two benchmarks Mpes, the following terminology is useful. We will say that player 2 is *as least as enduring* as player 1 if $\alpha^2 \leq \alpha^1$ and $x_{R^2} \leq x_{R^1}$; intuitively, player 2 is at least as willing to hold fast as player 1. When these two inequalities are equalities, we say that the players are *as enduring* as each other.

A Pure-Strategy Mpe Consider first the case in which players do not mix in equilibrium. This corresponds to the special case of our representation (7) in which $\mu^i \equiv 0$, so that the Markovian strategy $(0, S^i)$ is just the stopping time τ_{S^i}. The following result is standard.

Proposition 3 (Murto (2004), Georgiadis, Kim, and Kwon (2022)) *If player 2 is at least as enduring as player 1, then* $((0,(\alpha, x_{R^1}]),(0,(\alpha,\alpha^2]))$ *is a pure-strategy Mpe.*

In this equilibrium, player 2 threatens to concede only if X goes down below α^2, at which point player 1 is indifferent between conceding or winning the woa because $\alpha^2 \leq \alpha^1$. As a result, player 1's unique pbr is to concede as soon as X goes down below x_{R^1}. Because $\alpha^2 < x_{R^2} \leq x_{R^1}$ and it is never a best response for player 2 to concede before X goes down below x_{R^2}, holding fast down to α^2 is a pbr for player 2. By construction, this Mpe satisfies Murto (2004)'s refinement.[15]

A Regular Mixed-Strategy Mpe in the Symmetric Case Consider next the case in which players mix in equilibrium with an absolutely continuous intensity $\mu^i \equiv g^i \cdot Leb$. Then, from the occupation-time formula (Revuz and Yor (1999, Chapter VI, §1, Corollary 1.6)),

$$\Lambda^i_t = \mathbb{1}_{\{t<\tau_{S^i}\}}\, \mathrm{e}^{-\int_{\mathcal{I}\setminus S^i} L^y_t g^i(y)\,\mathrm{d}y} = \mathbb{1}_{\{t<\tau_{S^i}\}}\, \mathrm{e}^{-\int_0^t g^i(X_s)\sigma^2(X_s)\,\mathrm{d}s}. \tag{8}$$

Thus, when $X_t \in \mathcal{I} \setminus S^i$, player i concedes within a short duration $\mathrm{d}t$ with probability $g^i(X_t)\sigma^2(X_t)\,\mathrm{d}t$. The following result is standard.

Proposition 4 (Steg (2015), Georgiadis, Kim, and Kwon (2022)) *If the players are as enduring as each other, then* $((g^1 \cdot Leb, (\alpha,\alpha^*]),(g^2 \cdot Leb,(\alpha,\alpha^*]))$ *with*

$$g^i(x) \equiv \frac{1}{\sigma^2(x)} \cdot \frac{rR^j(x) - \mathcal{L}R^j(x)}{G^j(x) - R^j(x)}\, \mathbb{1}_{\{\alpha^*<x\leq x^*\}} \tag{9}$$

for $\alpha^* \equiv \alpha^1 = \alpha^2$ *and* $x^* \equiv x_{R^1} = x_{R^2}$ *is a mixed-strategy Mpe.*

In this regular Mpe, each player concedes with an intensity measure over $(\alpha^*, x^*]$ that makes his opponent indifferent between holding fast and conceding over this whole interval. Each player's Mpe value function coincides with his stand-alone value function. Thus, in expectation, all rents are dissipated in equilibrium.

5 Characterization, Existence, and Convergence of Singular Markov-Perfect Equilibria

We first motivate a new class of mixed Markovian strategies whereby a player concedes according to a singular intensity measure. We next provide a necessary condition for

[15] When the asymmetry between the players is small, $((0,\emptyset),(0,(\alpha,x_{R^2}]))$ is also an Mpe in which the more enduring player 2 follows his stand-alone optimal strategy because the less enduring player 1 is stubborn (Georgiadis, Kim, and Kwon (2022)). This Mpe does not satisfy Murto (2004)'s refinement, because, for $x \in (\alpha^2, \alpha^1)$, player 2's strategy is no longer a best response when player 1 does not concede with probability 1 in a small neighborhood of x. Murto (2004) shows that there may exist an Mpe satisfying his refinement in which S^2 is disconnected and player 2 concedes when X hits x_{R^2} from above.

mixed-strategy Mpes, establishing that any such equilibrium is either singular and exhibits an alternating threshold structure or, and only if $x_{R^1} = x_{R^2}$, is regular and involves absolutely continuous intensity measures. We then characterize singular Mpes by a variational system satisfied by the players' value functions. We finally use this characterization to show that woas with nearly symmetric players always admit singular Mpes and that the regular Mpe of the limiting symmetric woa can be approximated by such singular Mpes.

5.1 Singular Markovian Strategies

An important result in the recent literature is that, as soon as $x_{R^1} \neq x_{R^2}$, there exists no mixed-strategy Mpe in which the players concede according to absolutely continuous intensity measures (Georgiadis, Kim, and Kwon (2022)). For all that, it would be incorrect to conclude that only pure-strategy Mpes exist in this case, because other types of mixed Markovian strategies can be conceived of.

For instance, consider an intensity measure $\mu^i \equiv a^i \delta_{x^i}$, where $a^i > 0$ and δ_{x^i} is the Dirac mass at $x^i \in \mathcal{I} \setminus S^i$. Then, according to (7), the corresponding csf writes as

$$\Lambda_t^i = \mathbb{1}_{\{t < \tau_{S^i}\}} \, \mathrm{e}^{-a^i L_t^{x^i}}. \tag{10}$$

In particular, the mapping $t \mapsto \Lambda_t^i(\omega)$ is singular over $[0, \tau_{S^i}(\omega))$ for $\mathbf{P}_x$-a.e. trajectory ω of X that crosses x^i; that is, its derivative is zero for *Leb*-a.e. $t \in [0, \tau_{S^i}(\omega))$, while $\Lambda_t^i(\omega)$ only decreases each time ω crosses x^i. Heuristically, (10) expresses that, when $X_t = x^i$, player i concedes with instantaneous probability $a^i \mathrm{d} L_t^{x^i}$. As a result, a^i truly represents an *intensity* of conceding; this should not be confused with a positive *probability* of conceding at x^i, which is indistinguishable from conceding with probability 1 at x^i as, starting from x^i, X crosses x^i $\mathbf{P}_{x^i}$-a.s. infinitely often within an arbitrarily short duration due to the infinite random fluctuations of Brownian motion.

Whereas strategies such as (10) have not been considered in the woa literature, they naturally emerge as limits of more familiar ones:

(i) Discretizing the state space and the time dimension, suppose that player i concedes with positive probability only when the current state is x^i. Then the limit of such Markovian strategies when the mesh of the discretization goes to 0 corresponds to a distribution with hazard rate proportional to the local time of the diffusion at x^i.[16]

(ii) Consider the Markovian strategy that, outside S^i, consists in conceding according to an intensity measure $g_\varepsilon^i \cdot \textit{Leb}$, where $g_\varepsilon^i \equiv \frac{a^i}{2\varepsilon} \mathbb{1}_{(x^i - \varepsilon, x^i + \varepsilon)}$ for $a^i > 0$ and some small $\varepsilon > 0$.

[16] The appropriate normalization consists, for a short time period of duration $\mathrm{d}t$, to concede at x^i with a probability of order $\sqrt{\mathrm{d}t}$ (Feller (1968, Chapter III, §5, Theorem 1)).

Then, by (8), the resulting csf writes as $\Lambda^i_{\varepsilon,t} = \mathbb{1}_{\{t<\tau_{S^i}\}}\, \mathrm{e}^{-\frac{a^i}{2\varepsilon}\int_0^t \mathbb{1}_{(x^i-\varepsilon,x^i+\varepsilon)}(X_s)\sigma^2(X_s)\,\mathrm{d}s}$, which, by definition of the local time, converges $\mathbf{P}_x$-a.s. to Λ^i_t in (10) as ε goes to 0.

The second example suggests that the space of Markovian strategies allowed for by Theorem 1 is, in a topological sense, a natural completion of the space of Markovian strategies with absolutely continuous intensity measures. This is indeed the case, provided an adequate weak topology is defined over the space of intensity measures, though a formal proof of this fact is beyond the scope of this paper.[17]

A key property of a Markovian strategy, such as (10), associated to a singular intensity measure with an atom at x^i, is that, starting from x^i, the probability of conceding within a short duration $\mathrm{d}t$ is itself of order $\sqrt{\mathrm{d}t}$,[18] whereas the same quantity is of order $\mathrm{d}t$ for a Markovian strategy, such as (8), associated to an absolutely continuous intensity measure. As we explain in Section 5.3, this generates a kink in player j's Mpe value function.

5.2 The Alternating Structure

The following regularity assumption is maintained throughout the remainder of the paper.

A8 The functions b, σ, and $R^{i\prime\prime}$ are locally Lipschitz.

By convention, we let $\max \emptyset \equiv \alpha$ and, for any Mpe $((\mu^1, S^1), (\mu^2, S^2))$, we let $s^i \equiv \max S^i$. The following result then holds.

Theorem 2 *For any mixed-strategy Mpe* $((\mu^1, S^1), (\mu^2, S^2))$,

(i) if $x_{R^1} \neq x_{R^2}$, *then the restrictions of the intensity measures* μ^1 *and* μ^2 *to* $(s^1 \vee s^2, \beta)$ *are purely atomic*;

(ii) if $x_{R^1} = x_{R^2}$, *either the restrictions of the intensity measures* μ^1 *and* μ^2 *to* $(s^1 \vee s^2, \beta)$ *are purely atomic*, *or they are absolutely continuous*, *with densities characterized by (9) with* α^* *replaced by* $\alpha^1 \vee \alpha^2$.

Theorem 2 confirms the basic insight of Georgiadis, Kim, and Kwon (2022), according to which there exists no mixed-strategy Mpe with absolutely continuous intensity measures when $x_{R^1} \neq x_{R^2}$. Thus, if a mixed-strategy Mpe exists in this case, it must feature intensity measures that are singular with respect to Lebesgue measure. The new information provided by Theorem 2 is that these measures charge a discrete set of randomization thresholds.

[17] Décamps, Gensbittel, and Mariotti (2025b, Theorem 2.9) build on Theorem 1 to show that, endowed with a topology that extends the vague topology over Radon measures, the space of Markovian strategies is metrizable and compact, and admits the space of Markovian strategies with absolutely continuous intensity measures as a dense subset.

[18] This follows from the properties of the local time, see Peskir (2019, Lemma 15).

The proof can be sketched as follows.

Let us consider a mixed-strategy Mpe $((\mu^1, S^1), (\mu^2, S^2))$, supposing one exists. First, Proposition 1(iv) implies $\max \operatorname{supp} \mu^i \cap (s^1 \vee s^2, \beta) \leq x_{R^i}$ for every player i; we show that this must in fact be an equality for the largest of the maxima of the supports. Next, Proposition 1(v) and standard dynamic-programming arguments imply that, for every player i, the Hamilton–Jacobi–Bellman (HJB) equation $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ holds over any interval (q, q') where player j does not concede; it also follows from Proposition 1 that $\bar{J}^i \geq V_{R^i}$ and that $\bar{J}^i(q^i) = R^i(q^i)$ for all $q^i \in \operatorname{supp} \mu^i$. Fixing such an interval (q, q'), and assuming that $q, q' \in \operatorname{supp} \mu^j$, we deduce from this that there must exist a single point $q^i \in (q, q') \cap \operatorname{supp} \mu^i$ at which player i is indifferent between conceding or holding fast. The reason why such a point q^i must exist is that, otherwise, player j would expect, starting from any initial market condition $x \in (q, q')$, to obtain either $R^j(q)$ or $R^j(q')$ when leaving this interval. However, because $\mathcal{L}R^j - rR^j < 0$ over (q, q') by A3 as $q < q' \leq x_{R^j} < x_0^j$, player j would be strictly better off conceding immediately and obtaining $R^j(x)$ at x, a contradiction. It follows that $\bar{J}^i$ coincides with the solution to $\mathcal{L}u - ru = 0$ that is tangent to R^i at q^i. This, together with $\mathcal{L}R^i - rR^i < 0 = \mathcal{L}\bar{J}^i - r\bar{J}^i$, implies that q^i is unique. As a result, the set of accumulation points of the supports of μ^1 and μ^2 in $(s^1 \vee s^2, \beta)$ must coincide.

Consider first the asymmetric case $x_{R^1} \neq x_{R^2}$, and, to fix ideas, assume that $q_1^1 = x_{R^1}$ and $q_1^1 \geq q_1^2$ for $q_1^i \equiv \max \operatorname{supp} \mu^i$. We verify that it is not optimal for player 2 to concede at q_1^1. Therefore, q_1^1 must be an isolated point of $\operatorname{supp} \mu^1$ and $q_1^1 > q_1^2$. Iterating this argument and using the preceding remarks, we show that, for every player i, and for any two consecutive points $q_n^i > q_{n+1}^i > s^1 \vee s^2$ in the support of μ^i, there must exist a single point $q^j \in (q_{n+1}^i, q_n^i)$ in the support of μ^j at which player j is indifferent between conceding or holding fast. We thus obtain two decreasing sequences of randomization thresholds $(q_n^1)_{n=1}^{N^1}$ and $(q_n^2)_{n=1}^{N^2}$, with either $N^1 = N^2 = \infty$ or $0 \leq N^1 - N^2 \leq 1$, which are intertwined in the sense that $q_1^1 > q_1^2 > q_2^1 > q_2^2 > \ldots$ as long as these thresholds are defined. We also show that, if $N^1 = N^2 = \infty$, any such intertwined sequences must converge to α. These two sequences fully characterize the restrictions of μ^1 and μ^2 to $(s^1 \vee s^2, \beta)$. It follows that, when $x_{R^1} \neq x_{R^2}$, any mixed-strategy Mpe must fall into one of three categories, which are delineated in Corollary 1 below.

In the symmetric case $x_{R^1} = x_{R^2}$, analogous arguments show that the common set of accumulation points of the supports of μ^1 and μ^2 is either empty or equal to $(s^1 \vee s^2, x_{R^1}]$. In the latter case, analytic arguments imply that the measures μ^i are absolutely continuous, with densities characterized by (9) for $\alpha^* \equiv s^1 \vee s^2$.

Corollary 1 *Let $((\mu^1, S^1), (\mu^2, S^2))$ be a singular Mpe. Then, for every player i,* $\operatorname{supp} \mu^i \cap$

$(s^1 \vee s^2, \beta) = \{q_n^i : n = 1, \ldots, N^i\}$ *for intertwined decreasing sequences of randomization thresholds* $(q_n^1)_{n=1}^{N^1}$ *and* $(q_n^2)_{n=1}^{N^2}$ *satisfying, with no loss of generality,* $q_1^1 > q_1^2$*. Moreover,* $q_1^1 = x_{R^1}$ *and one of the following three conditions holds*:

1. $N^1 = N^2 \equiv N \in \mathbb{N} \setminus \{0\}$ *and* $q_N^1 > q_N^2 > s^1 > s^2$;

2. $N^1 = N^2 + 1 \equiv N \in \mathbb{N} \setminus \{0\}$ *and* $q_{N-1}^2 > q_N^1 > s^2 > s^1$*, with* $q_0^2 \equiv \beta$ *by convention*;

3. $N^1 = N^2 = \infty$ *and* $\lim_{n\to\infty} q_n^1 = \lim_{n\to\infty} q_n^2 = s^1 = s^2 = \alpha$*, so that* $S^1 = S^2 = \emptyset$*.*

In an Mpe of type 1, player 1 concedes with probability 1 at s^1, and player 2 has the lowest randomization threshold. In an Mpe of type 2, player 1 has the lowest randomization threshold, and player 2 concedes with probability 1 at s^2; the example in Section 6.2 is a case in point, with $N^1 = 1$ and $N^2 = 0$. In an Mpe of type 3, neither player concedes with probability 1 at any point of the state space, and players keep randomizing all the way down to α. It should be noted that an Mpe of type 3 can exist only if $\alpha^1 = \alpha^2 = \alpha$; indeed, every player i such that $\alpha^i > \alpha$ would not be willing to delay concession over (α, α^i) if his opponent were to do the same.

Corollary 1 fully characterizes equilibrium outcomes for an Mpe of type 3, because any market condition in $\mathcal{I}$ can be reached with positive probability from any initial market conditions $x \in \mathcal{I}$. The same holds true for Mpes of types 1 and 2, provided $x > x_{R^1}$, with $q_1^1 > q_1^2$ by convention. Indeed, for any such Mpe $((\mu^1, S^1), (\mu^2, S^2))$ and for each $x > x_{R^1}$, there exists an outcome-equivalent Mpe $((\tilde{\mu}^1, \tilde{S}^1), (\tilde{\mu}^2, \tilde{S}^2))$ such that $\operatorname{supp} \tilde{\mu}^i = \operatorname{supp} \mu^i \cap (s^1 \vee s^2, \beta)$ for every player i and $\tilde{S}^1 = (\alpha, s^1)$ and $\tilde{S}^2 = \emptyset$ (for an Mpe of type 1), or $\tilde{S}^1 = \emptyset$ and $\tilde{S}^2 = (\alpha, s^2)$ (for an Mpe of type 2). By contrast, Corollary 1 does not pin down equilibrium outcomes of Mpes of types 1 and 2 for lower initial market conditions. Indeed, as in Murto (2004), one may construct Mpes with stopping regions S^1 and S^2 exhibiting gaps; moreover, these gaps may themselves include randomization thresholds.

The upshot from Theorem 2 and Corollary 1 is that, when players have different optimal stand-alone thresholds and are thus not as enduring as each other, alternation is a robust feature of mixed-strategy Mpes. In the attrition region, players randomize between conceding and holding fast at isolated thresholds. Thus players can alternatively find themselves in a relative position of strength or weakness on the equilibrium path. Consider, for instance, an Mpe with randomization thresholds $q_n^1 > q_n^2 > q_{n+1}^1$ for players 1 and 2; then player 1 and player 2 are respectively in relative positions of weakness and strength when market conditions are close to player 1's randomization thresholds q_n^1 and q_{n+1}^1, while these positions are reversed when market conditions are close to player 2's randomization threshold q_n^2. In an Mpe of type 1 and type 2, this alternation process may persist until one player eventually reaches his stopping region and concedes with probability 1. By contrast, in an Mpe of type 3, concession must take place at a randomization threshold.

Such alternation in the balance of power fundamentally reflects that a player currently in a position of weakness can hope for a reversal of future market conditions in his favor. This perspective is precisely what makes him indifferent between conceding and holding fast at his randomization thresholds. It should be noted that whether a player is currently in a position of strength or weakness is an endogenous feature of the equilibrium under consideration. Indeed, assumptions A3 and A6 of our model imply that the evolution of market conditions affects both players' rewards from conceding or winning the woa in similar ways, with symmetric rewards as a limiting case. In particular, high or low market conditions do not a priori favor one player to the detriment of the other, and attrition only arises as a consequence of players mixing on the equilibrium path, unlike in the opposite-interest models of Gul and Pesendorfer (2012) and Gieczewski (2025).

5.3 A Variational Characterization

Our second result establishes a variational characterization of Mpes of types 1 and 2 and, hence, provides a semi-explicit criterion for their existence. Analogous results hold for Mpes of types 3; their statements and proofs proceed along similar lines, and are omitted for the sake of brevity.

Theorem 3 *Let $N^1 \geq 1$ and $N^2 \geq 0$ be integers such that $0 \leq N^2 - N^1 \leq 1$ and let be given*

- *two finite sequences $(q^1_n)_{n=1}^{N^1}$ and $(q^2_n)_{n=0}^{N^2}$ of numbers in $\mathcal{I}$, with $q^2_0 \equiv \beta$ by convention, and a number $s^i \in \mathcal{I}$ such that $q^1_1 = x_{R^1} > q^2_1 > q^1_2 > \ldots > q^i_{N^i} > q^j_{N^j} > s^i$;*
- *two finite sequences $(a^1_n)_{n=1}^{N^1}$ and $(a^2_n)_{n=0}^{N^2}$ of positive real numbers.*

Then the strategy profile $((\mu^i, S^i), (\mu^j, S^j)) \equiv ((\sum_{n=1}^{N^i} a^i_n \delta_{q^i_n}, (\alpha, s^i]), (\sum_{n=1}^{N^j} a^j_n \delta_{q^j_n}, \emptyset))$, with $\sum_{n=1}^{0} \equiv 0$ by convention, is an Mpe of type $i = 1, 2$ if and only if $s^i > \alpha^i$ and there exist two functions $w^i \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q^j_n : 1 \leq n \leq N^j\} \cup \{s^i\}))$ and $w^j \in \mathcal{C}^0(\mathcal{I}) \cap \mathcal{C}^2(\mathcal{I} \setminus (\{q^i_n : 1 \leq n \leq N^i\} \cup \{s^i\}))$ that satisfy the variational system

$$w^i \geq R^i \text{ over } \mathcal{I}, \tag{11}$$

$$w^i = R^i \text{ over } (\alpha, s^i], \tag{12}$$

$$w^{i\prime}(s^i) = R^{i\prime}(s^i), \tag{13}$$

$$w^i(\beta-) = 0, \tag{14}$$

$$w^i(q^i_n) = R^i(q^i_n), \ 1 \leq n \leq N^i, \tag{15}$$

$$\mathcal{L}w^i - rw^i = 0 \text{ over } (s^i, \beta) \setminus \{q^j_n : 1 \leq n \leq N^j\}, \tag{16}$$

$$a^j_n[G^i(q^j_n) - w^i(q^j_n)] + \frac{1}{2}\,\Delta w^{i\prime}(q^j_n) = 0,\ 1 \leq n \leq N^j, \tag{17}$$

$$w^j \geq R^j \ \textit{over}\ \mathcal{I}, \tag{18}$$

$$w^j = G^j \ \textit{over}\ (\alpha, s^i], \tag{19}$$

$$w^j(\beta-) = 0, \tag{20}$$

$$w^j(q^j_n) = R^j(q^j_n),\ 1 \leq n \leq N^j, \tag{21}$$

$$\mathcal{L}w^j - rw^j = 0 \ \textit{over}\ (s^i, \beta) \setminus \{q^i_n : 1 \leq n \leq N^i\}, \tag{22}$$

$$a^i_n[G^j(q^i_n) - w^j(q^i_n)] + \frac{1}{2}\,\Delta w^{j\prime}(q^i_n) = 0,\ 1 \leq n \leq N^i, \tag{23}$$

Moreover, whenever $\alpha^j \leq \alpha^i$, $((\sum_{n=1}^{N^i} a^i_n \delta_{q^i_n}, (\alpha, s^i]), (\sum_{n=1}^{N^j} a^j_n \delta_{q^j_n}, (\alpha, \alpha^j]))$ *is an outcome-equivalent Mpe of type* i *that satisfies Murto (2004)'s refinement.*

The proof of Theorem 3 is based on the properties obtained in the proof of Theorem 2, together with classical methods employed in verification theorems for optimal-stopping and stopping-game theory.

The system (11)–(23) intuitively characterizes the Mpe value functions $\bar{J}^1$ and $\bar{J}^2$ of players 1 and 2. This system can be interpreted as follows. First, (11) and (18) express that every player can guarantee himself his stand-alone reward at x by conceding immediately. Second, (12) and (19) express that player i concedes with probability 1 over $(\alpha, s^i]$, and (13) states that the smooth-fit property must apply at s^i for player i. Third, (14) and (20) are boundary conditions that reflect that, as x gets large, each player's continuation value becomes arbitrarily small due to discounting. Fourth, (15) and (21) state that each player must be indifferent between conceding and holding fast at any of his randomization thresholds. Fifth, (16) and (22) state that the players' Mpe value functions must satisfy the HJB equation over (s^i, β), except at the randomization thresholds of their opponent. Sixth, and finally, (17) and (23) characterize how each player's Mpe value function behaves at the randomization thresholds of his opponent.

Consider, for instance, what happens at $q^1_1 = x_{R^1}$, the highest randomization threshold of player 1. Because player 1 randomizes between holding fast and conceding at x_{R^1} with positive but finite intensity a^1_1, we expect that $G^2(x_{R^1}) > \bar{J}^2(x_{R^1}) > R^2(x_{R^1})$. As we shall now see, this implies that $\bar{J}^2$ is not differentiable at x_{R^1}. Indeed, by the properties of the local time, starting from x_{R^1}, player 1 concedes within a short duration $\mathrm{d}t$ with probability $\mathbf{E}_{x_{R^1}}[\Gamma_{\mathrm{d}t}] = a^1_1 c\sqrt{\mathrm{d}t} + o(\sqrt{\mathrm{d}t})$, where $\Gamma_{\mathrm{d}t} = 1 - \mathrm{e}^{-a^1_1 L^{x_{R^1}}_{\mathrm{d}t}}$ and c is a positive constant.[19] If player 1 concedes, then player 2 wins the woa and obtains $G^2(x_{R^1})$, whereas if player 1 holds

[19] Strictly speaking, we should restrict attention to trajectories of X that do not hit player 2's highest randomization threshold q^1_2 until $\mathrm{d}t$ units of time have elapsed; that is, we should consider a short duration $\mathrm{d}t \wedge \tau_{q^1_2}$. This, however, would not affect the derivations below in any significant way.

fast, then player 2 achieves the value $\bar{J}^2(X_{\mathrm{dt}})$. Thus

$$\bar{J}^2(x_{R^1}) = a_1^1 c\sqrt{\mathrm{dt}}\, G^2(x_{R^1}) + (1 - a_1^1 c\sqrt{\mathrm{dt}})\, \mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}}\bar{J}^2(X_{\mathrm{dt}})] + o(\sqrt{\mathrm{dt}}). \tag{24}$$

Now, suppose, by way of contradiction, that $\bar{J}^2$ is $\mathcal{C}^2$ in a neighborhood of x_{R^1}. Then, from Itô's formula, we have

$$\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}}\bar{J}^2(X_{\mathrm{dt}})] = \bar{J}^2(x_{R^1}) + (\mathcal{L}\bar{J}^2 - r\bar{J}^2)(x_{R^1})\,\mathrm{dt} + o(\mathrm{dt}). \tag{25}$$

Plugging (25) into (24) yields $a_1^1 c[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})]\sqrt{dt} + o(\sqrt{\mathrm{dt}}) = 0$, a contradiction as $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$ and a_1^1 and c are positive constants. This is an indication that $\bar{J}^2$ is not differentiable at x_{R^1}, so let us denote by $\Delta\bar{J}^{2\prime}(x_{R^1}) \equiv \bar{J}^{2\prime+}(x_{R^1}) - \bar{J}^{2\prime-}(x_{R^1})$ the corresponding derivative jump. From the Itô–Tanaka–Meyer formula (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.5)), we have

$$\begin{aligned}
\mathbf{E}_{x_{R^1}}[\mathrm{e}^{-r\mathrm{dt}}\bar{J}^2(X_{\mathrm{dt}})] &= \bar{J}^2(x_{R^1}) + \mathbf{E}_{x_{R^1}}\Bigg[\int_0^{\mathrm{dt}} \mathrm{e}^{-rs}(\mathcal{L}\bar{J}^2 - r\bar{J}^2)(X_s)\,\mathrm{d}s \\
&\qquad + \int_0^{\mathrm{dt}} \mathrm{e}^{-rs}\bar{J}^{2\prime-}(X_s)\sigma(X_s)\,\mathrm{d}W_s + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1})L_{\mathrm{dt}}^{x_{R^1}}\Bigg] \\
&= \bar{J}^2(x_{R^1}) + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1})c\sqrt{dt} + o(\sqrt{\mathrm{dt}}),
\end{aligned} \tag{26}$$

where the second equality follows from the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over $(q_2^1, \beta) \setminus \{x_{R^1}\}$ and from the properties of local time. Plugging (26) into (24) yields

$$a_1^1[G^2(x_{R^1}) - \bar{J}^2(x_{R^1})] + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(x_{R^1}) = 0, \tag{27}$$

which corresponds to (17) at $q_1^1 = x_{R^1}$. From (27) and $G^2(x_{R^1}) > \bar{J}^2(x_{R^1})$, we obtain that the derivative jump $\Delta\bar{J}^{2\prime}(x_{R^1})$ of $\bar{J}^2$ at x_{R^1} is negative: intuitively, this reflects that player 2 gets increasingly optimistic as X approaches x_{R^1}, but is disappointed if X crosses x_{R^1} but player 1 holds fast at x_{R^1}. Notice that the magnitude of this disappointment is proportional to the intensity a_1^1 with which player 1 concedes at x_{R^1} and to the foregone payoff $G^2(x_{R^1}) - \bar{J}^2(x_{R^1})$. Similar reasonings can be applied to every player and any randomization thresholds of his opponent to obtain (17) and (23).

A key insight from this variational characterization is that incentives in singular Mpes are nonlocal. To see this, suppose for instance that $N^1 \geq 2$, and let $1 \leq n \leq N^1 - 1$. Then the intensities a_n^1 and a_{n+1}^1 with which player 1 randomizes at the thresholds q_n^1 and q_{n+1}^1 just make player 2 indifferent between conceding and holding fast at his randomizing threshold $q_n^2 \in (q_{n+1}^1, q_n^1)$, and vice versa. Thus what makes a player willing to randomize at some point of the state space is his uncertainty about his opponent's *future* intentions and behavior at other points of the state space. This contrasts, when players are as enduring as each other, with the regular Mpe characterized in Proposition 4, in which, at any point of the attrition region, what makes a player willing to randomize is his uncertainty about his opponent's *present* intentions and behavior at the very same point.

Importantly, the variational characterization in Theorem 3 and the analogous result for Mpes of types 3 hold for both symmetric and asymmetric players. Thus Theorem 3 provides a characterization of a class of mixed-strategy Mpe outcomes in the woa that are robust to even the slightest degree of heterogeneity between players.

5.4 Existence and Convergence

We now show that, provided the asymmetry between the players is not too large, the variational system (11)–(23) admits solutions, which by Theorem 3 implies that singular Mpes exist. To parameterize asymmetry in a flexible way, we consider a family of woas with reward functions $(R^i(p,\cdot), G^i(p,\cdot))_{p\in P}$ indexed by a parameter p in a compact metric space (P,d), with a reference value $p_\infty \in P$ corresponding to the symmetric case; that is, $R^1(p_\infty,\cdot) = R^2(p_\infty,\cdot)$ and $G^1(p_\infty,\cdot) = G^2(p_\infty,\cdot)$. For each $p \in P$, the functions $R^i(p,\cdot)$ and $G^i(p,\cdot)$ as well as the associated thresholds $\alpha^i(p)$ and $x^i_0(p)$ satisfy A1–A8 and

A9 $R^i(p,\alpha+) > 0$.

A10 $\alpha^i(p) > \alpha$.

To simplify notation, we denote by $x_{R^i}(p) \equiv x_{R^i(p,\cdot)}$ the optimal stand-alone threshold for player i. By definition of p_∞, we have $\alpha^1(p_\infty) = \alpha^2(p_\infty)$, $x^1_0(p_\infty) = x^2_0(p_\infty)$, and $x_{R^1}(p_\infty) = x_{R^2}(p_\infty)$. We finally make the following regularity assumptions.

A11 (i) R^i, $R^{i\prime}$, $R^{i\prime\prime}$, and G^i are continuous over $P \times \mathcal{I}$;

(ii) R^i, $R^{i\prime}$, $R^{i\prime\prime}$ are Lipschitz in x over any compact subset of $P \times \mathcal{I}$;

(iii) the mapping $p \mapsto \alpha^i(p)$ is continuous over P.

Our existence result can now be stated as follows.

Theorem 4 *For all $\bar{N} \in \mathbb{N}$, $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, and $\kappa > 0$, there exists $\eta_0 > 0$ such that, using the notation of Corollary 1, for each p satisfying $d(p,p_\infty) \le \eta_0$, there exists an Mpe of type 1 or 2 associated with intertwined sequences of randomization thresholds*

$$q^1_1 > q^2_1 > \ldots > q^i_{N^i} > q^j_{N^j} > s^i$$

(with $i = 1$ for an Mpe of type 1 and $i = 2$ for an Mpe of type 2) that satisfy

$$N^j \ge \bar{N},\ s^i \le M, \text{ and } (q^1_n - q^2_n) \vee (q^2_n - q^1_{n+1}) \le \kappa \text{ for all } n \ge 1 \text{ such that } q^1_{n+1} \ge M.$$

A sketch of the proof is provided below for a special case of the model. We refer to Online Supplement S.6 for complete details.

The main insight of Theorem 4 is that singular Mpes with arbitrarily many randomization thresholds exist in the vicinity of any symmetric woa. Specifically, the closest p is to the symmetric case p_∞, the more condensed the set of equilibrium randomization thresholds can be chosen in $(\alpha^1(p_\infty), x_{R^1}(p_\infty))$.

To illustrate the main ideas of the proof, we focus on the limiting case $p = p_\infty$ and on the completely symmetric model in which $R^1 = R^2 \equiv R$ and $G^1 = G^2 \equiv G$, leaving the general case to the Appendix. We let $x^* \equiv x_{R^1} = x_{R^2}$ and $\alpha^* \equiv \alpha^1 = \alpha^2$. By A9–A10, $R(\alpha+) > 0$ and $\alpha^* > \alpha$. We also denote by T_q the unique solution to $\mathcal{L}u - ru = 0$ over $\mathcal{I}$ that is tangent to R at q. In this simplified setting, the proof can be sketched as follows.

First, we show that, given $q_1^1 = x^*$, all the players' randomization thresholds and the largest of the maxima of their stopping regions are fully determined by the highest randomization threshold q_1^2 of player 2. To see this, consider an Mpe of type $i = 1, 2$.[20] We know from Theorem 3 that the value functions (w^1, w^2) are solutions to the variational system (11)–(23). In particular, using that $w^2 \geq R^2$, with equality at q_n^2, and that w^2 is continuous and satisfies $\mathcal{L}w^2 - rw^2 = 0$ over (q_{n+1}^1, q_n^1), we obtain that $w^2 = T_{q_n^2}$ over $[q_{n+1}^1, q_n^1]$ for all $1 \leq n \leq N^2 - 1$. Because w^2 is continuous at q_{n+1}^1, it follows that

$$T_{q_{n+1}^2}(q_{n+1}^1) = T_{q_n^2}(q_{n+1}^1), \quad 1 \leq n \leq N^2 - 1. \tag{28}$$

It is not difficult to show that the mapping $x \mapsto T_x(q_{n+1}^1)$ is continuously decreasing from ∞ to $R(q_{n+1}^1)$ over (α, q_{n+1}^1). Because the right-hand side of (28) satisfies $T_{q_n^2}(q_{n+1}^1) > T_{q_{n+1}^1}(q_{n+1}^1)$, the equation (28) characterizes q_{n+1}^2 as a function of q_n^2 and q_{n+1}^1. A similar analysis for w^1 leads to

$$T_{q_{n+1}^1}(q_n^2) = T_{q_n^1}(q_n^2), \quad 1 \leq n \leq N^1 - 1. \tag{29}$$

Likewise,

$$T_{s^i}(q_{N^j}^j) = T_{q_{N^i}^i}(q_{N^j}^j) \tag{30}$$

characterizes s^i as a function of $q_{N^i}^i$ and $q_{N^j}^j$, reflecting the value-matching and smooth-fit conditions (12)–(13) for player i at s^i. Figure 1 illustrates this recursive structure. Putting together equations (28)–(30), we obtain that the whole sequence $q_1^1 > q_1^2 > \ldots > q_{N^i}^i > q_{N^j}^j > s^i$ is pinned down by $q_1^1 = x^*$ and the choice of $q_1^2 \in (\alpha^*, x^*)$, as claimed.[21]

[20] Mpes of type 3 are ruled out by the assumption $\alpha^* > \alpha$.

[21] Whereas equations (28)–(29) cannot be solved explicitly in general, it is easy to approximate their solutions numerically. Hence our shooting method is semi-constructive.

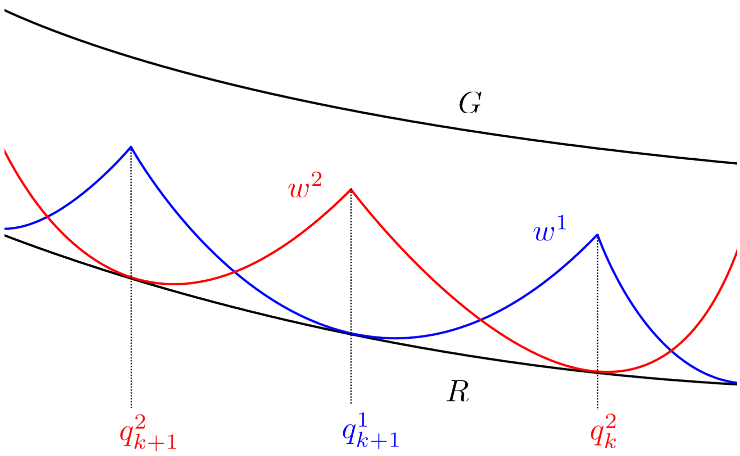

Figure 1: The recursive structure of randomization thresholds.

Motivated by this analysis, the construction of an Mpe of type i involves selecting an appropriate value for $q_1^2 \in (\alpha^*, x^*)$. Given the outlined recursive structure, any such q_1^2 determines two intertwined decreasing sequences $q_1^1 > q_1^2 > \ldots > q_n^1 > q_n^2 > \ldots$ that can be shown to converge to α. Now, suppose that we can find some value of $q_1^2 \in (\alpha^*, x^*)$ and some integers $N^1 \geq 1$ and $N^2 \geq 0$ such that $0 \leq N^1 - N^2 \leq 1$ and

$$T_{q^j_{N^j}}(q^i_{N^i+1}) = G(q^i_{N^i+1}), \tag{31}$$

$$T_{q^j_n}(q^i_n) < G(q^i_n), \quad 1 \leq n \leq N^i, \tag{32}$$

$$T_{q^i_n}(q^j_n) < G(q^j_n), \quad 1 \leq n \leq N^j, \tag{33}$$

where (31) reflects that player i concedes with probability 1 at $q^i_{N^i+1}$, and (32)–(33) reflect that both players concede with finite intensity at their randomization thresholds. We can then let $s^i \equiv q^i_{N^i+1}$ and construct candidate Mpe value functions as in Figure 1. For instance, in an Mpe of type 1, we have

$$w^1 \equiv \mathbb{1}_{(\alpha,s^1]}\, R + \mathbb{1}_{(s^1,q^2_N]}\, T_{s^1} + \sum_{n=1}^{N-1} \mathbb{1}_{(q^2_{n+1},q^2_n]}\, T_{q^1_{n+1}} + \mathbb{1}_{(q^2_1,\beta)}\, T_{q^1_1}, \tag{34}$$

$$w^2 \equiv \mathbb{1}_{(\alpha,s^1]}\, G + \sum_{n=1}^{N} \mathbb{1}_{(q^1_{n+1},q^1_n]}\, T_{q^2_n} + \mathbb{1}_{(q^1_1,\beta)}\, u^2 \tag{35}$$

where $N \equiv N^1 = N^2$ and u^2 is the unique solution to $\mathcal{L}u - ru = 0$ satisfying $u(\beta-) = 0$ and $u(q_1^1) = T_{q_1^2}(q_1^1)$. Using that $\mathcal{L}R - rR < 0$ over (α, x^*), one can verify that each player's candidate Mpe value function features a negative derivative jump at all randomization thresholds of his opponent. This, together with (32)–(33), pins down concession intensities $(a^j_n)_{n=1}^{N^j}$ and $(a^i_n)_{n=1}^{N^i}$ according to (17) and (23). The upshot of this analysis is that, if q_1^2 is chosen so that (31)–(33) are satisfied, then (w^1, w^2) is a solution to the variational system

(18)–(17) given the constructed randomization thresholds and concession intensities; hence the latter determine an Mpe of type 1 by Theorem 3.

In the end, we only need to show that (31)–(33) are satisfied for some i and some well-chosen q_1^2. To do so, we use a shooting method that relies on two regularity properties of the sequences $(q_n^1)_{n\geq 1}$ and $(q_n^2)_{n\geq 1}$, and of the pseudo-value functions

$$w_\infty^1 \equiv \sum_{n=1}^{\infty} \mathbb{1}_{(q_{n+1}^2, q_n^2]}\, T_{q_{n+1}^1} + \mathbb{1}_{(q_1^2,\beta)}\, T_{q_1^1} \ \text{ and } \ w_\infty^2 \equiv \sum_{n=1}^{\infty} \mathbb{1}_{(q_{n+1}^1, q_n^1]}\, T_{q_n^2} + \mathbb{1}_{(q_1^1,\beta)}\, u^2$$

associated to them. Specifically, for each i, it holds that

(i) For each $n > 1$, q_n^i varies continuously with q_1^2 and goes to x^* as q_1^2 goes to x^*.

(ii) For each $M \in (\alpha^*, x^*)$, w_∞^i converges uniformly to R over $[M, x^*]$ as q_1^2 goes to x^*.

Given $\bar{N} \in \mathbb{N}$ and $M \in (\alpha^*, x^*)$, it follows from (i)–(ii) that, for each q_1^2 close to x^*, we have $q_{\bar{N}}^2 \geq M$ and $w_\infty^i < G$ over $[M, q_1^1]$ for every player i. Fixing such a $q_{1,\varepsilon}^2 \equiv x^* - \varepsilon$, the corresponding sequence $(q_{n,\varepsilon}^1)_{n\geq 1}$, and the corresponding pseudo-value function $w_{\infty,\varepsilon}^2$ for player 2, we can use that $G = R$ over $(\alpha, \alpha^*]$, that $(q_{n,\varepsilon}^1)_{n\geq 1}$ converges to α, and that $w_{\infty,\varepsilon}^2(q_{n,\varepsilon}^1) > R(q_{n,\varepsilon}^1)$ for all $n \geq 1$ to obtain that there exists an integer N_ε such that $w_{\infty,\varepsilon}^2(q_{N_\varepsilon,\varepsilon}^1) \geq G(q_{N_\varepsilon,\varepsilon}^1)$. Up to replacing the point $q_{N_\varepsilon,\varepsilon}^1$ by a larger one in the sequence, we may assume with no loss of generality that $w_{\infty,\varepsilon}^2(q_{n,\varepsilon}^1) < G(q_{n,\varepsilon}^1)$ for $1 \leq n < N_\varepsilon$. This is illustrated in Figure 2.

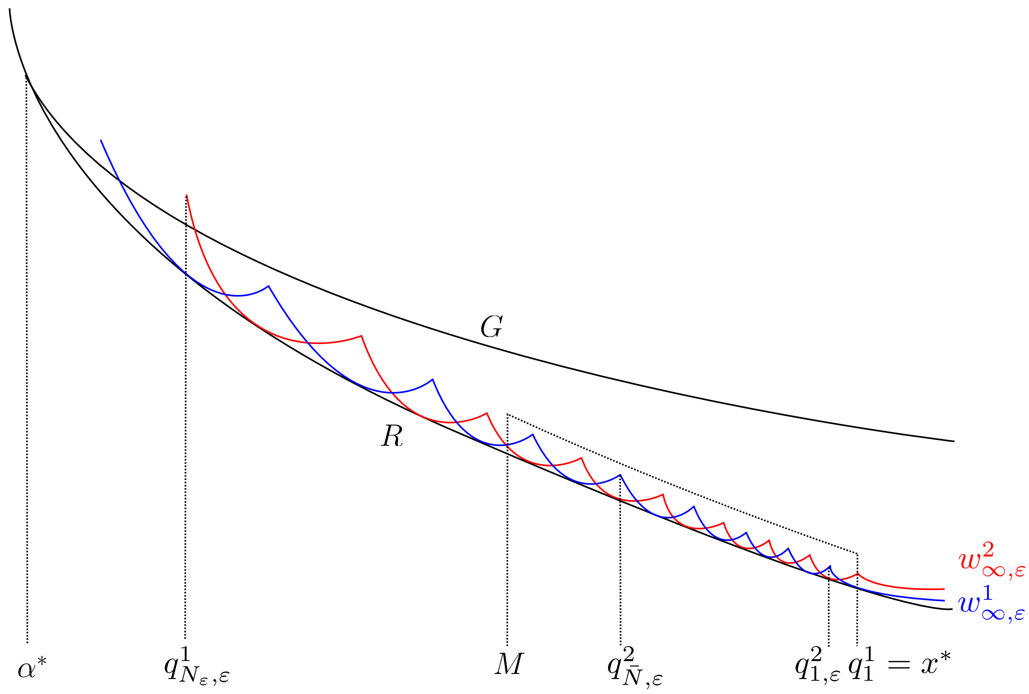

Figure 2: The pseudo-value functions and the condensation of randomization thresholds.

Letting q_1^2 increase to x^* while keeping N_ε fixed, the point $q_{N_\varepsilon}^1$ eventually reaches $[M, x^*]$, where it must be that $w_\infty^2 < G$. Thus $w_\infty^2(q_{N_\varepsilon}^1) = G(q_{N_\varepsilon}^1)$ for some $q_1^2 \in (x^* - \varepsilon, x^*)$ by continuity. The set of values of $q_1^2 \in [x^* - \varepsilon, x^*)$ for which there exist i and N such that

$$w_\infty^j(q_{N+1}^i) = G(q_{N+1}^i) \tag{36}$$

is therefore nonempty and one easily checks that it admits a minimum. We choose q_1^2 equal to this minimum and then pick the smallest N satisfying (36), which generically pins down i and j.[22] The values of q_1^2, $N^i = N$, i, and j determined by this procedure satisfy (31)–(33) and thus define an Mpe of type i with at least $2\bar{N}$ randomization thresholds and in which none of the players stops with probability 1 before market conditions drop below M. Moreover, ε can be chosen so that the distance between two consecutive randomization thresholds does not exceed a given constant κ as long these thresholds stay above M.

The condensation of randomization thresholds over any interval $[M, x^*]$ highlighted in the above reasoning suggests that, when q_1^2 increases to q_1^1, the corresponding sequence of Mpes constructed through this procedure should, in some sense, converge to the regular Mpe described in Proposition 4. That this is indeed the case follows from our last result.

Theorem 5 *Let $((\mu_n^1, S_n^1), (\mu_n^2, S_n^2))_{n\geq 1}$ be a sequence of Mpes of type 1 or 2 in the sequence of woas with parameters $p_n \in P$ converging to p_∞ such that $q_{1,n}^2 - q_{1,n}^1$ converges to 0. Then, for each i, the intensity measures μ_n^i converge weakly to $\mu_\infty^i = g_\infty^i \cdot Leb$ over any compact subset of $(\alpha^j(p_\infty), \beta)$, with*

$$g_\infty^i(x) \equiv \frac{1}{\sigma^2(x)} \cdot \frac{rR^j(p_\infty, x) - \mathcal{L}R^j(p_\infty, x)}{G^j(p_\infty, x) - R^j(p_\infty, x)} \mathbb{1}_{\{\alpha^j(p_\infty) < x \leq x_{R^j}(p_\infty)\}}, \tag{37}$$

and the maxima $\max S_n^1 \cup S_n^2$ of the stopping regions converge to $\alpha^1(p_\infty) = \alpha^2(p_\infty)$. Moreover, for any initial market conditions, the players' equilibrium payoffs along this sequence of singular Mpes converge to those in the regular Mpe at p_∞.

The proof is technical, and relegated to Online Supplement S.7.[23]

The main insight of Theorem 5 is that, contrary to the argument developed by Georgiadis, Kim, and Kwon (2022), the regular Mpe in the symmetric case $p = p_\infty$ is not isolated, and can be approached by singular Mpes of (nearly) symmetric woas. There are two ways of interpreting this result. On a negative note, one may first argue that the regular Mpe in the symmetric case is somewhat degenerate, and to a certain extent hides the singularities that necessarily arise in approximating mixed-strategy Mpes of nearby asymmetric woas. On a

[22] In the non-generic case where, for this q_1^2 and this N, (36) holds for both players, we set $i \equiv 2$.

[23] The convergence of singular Mpes to the regular Mpe in Theorem 5 is consistent with the compact metrizable topology over Markovian strategies introduced by Décamps, Gensbittel, and Mariotti (2025b).

more positive note, one may on the contrary argue that, if asymmetries between the players are so small as to be considered negligible, the regular Mpe of the symmetric woa provides a sensible prediction about the outcome of the woa. This second line of argumentation, however, turns out to be fragile. The reason, as we show in Section 6, is that even (nearly) symmetric woas admit singular Mpes that feature only a few randomization thresholds, and thus lead to qualitatively very different predictions.

5.5 Taking Stock

Before we move to the discussion of the exit-in-duopoly example in Section 6, it is useful at this stage to step back and draw some general lessons about singular Mpes, both in themselves and in relation to the benchmark Mpes discussed in Section 4.2.

Randomization in Singular Mpes The hallmark of singular Mpes is that concession is random yet tied to certain endogenously determined salient values of market conditions. These two features distinguish them both from the pure-strategy Mpes of threshold type and from the regular Mpe that have been emphasized in the literature, though singular Mpes somehow retain and combine key aspects of such Mpes. A direct implication of this is that a player may hold fast and later concede despite no observable differences in current market conditions. This may for instance help address some of the puzzles noted by Alesina and Drazen (1991) in their discussion of delayed macroeconomic stabilizations, namely, that countries do not seem to stabilize as soon as unfavorable shocks occur, and that stabilization when it is undertaken need not be triggered by significant observable changes in external circumstances. Singular Mpes provide a flexible way to reconcile the authors' view that stabilizations are caused by a woa between opposed political groups with more traditional explanations that emphasize the role of exogenous shocks.

Singular versus Pure-Strategy Mpes When initial market conditions are favorable enough, any singular Mpe clearly involves longer concession delays than the pure-strategy Mpe constructed in Proposition 3, in which the less enduring player concedes with probability 1 at his stand-alone threshold. More generally, and apart from exceptional cases involving indifference in pure-strategy Mpes, it holds that $s^1 \vee s^2 < x_{R^1} \wedge x_{R^2}$ in any singular Mpe. Hence, generically, mixing by any player delays the time at which some player must exit the market with probability 1 compared to any pure-strategy Mpe. Regarding welfare comparisons, for favorable enough initial market conditions, any singular Mpe in which, say, player 1 is the first player to randomize, at $q_1^1 = x_{R^1}$, is Pareto-dominated by any pure-strategy Mpe in which he concedes with probability 1 at x_{R^1}; this is because player 1 in either case obtains his stand-alone value, as he must be indifferent between conceding or holding fast at x_{R^1}, while player 2 is worse off, as player 1 only concedes with finite intensity at x_{R^1}. For less favorable initial market conditions, player 2 is a fortiori worse off in any singular Mpe; however, player 1 may now be better off, as he may now end up winning the woa. Hence no Pareto-dominance arises in this case.

Singular versus Regular Mpes When players are as enduring as each other, one can also compare singular Mpes with the regular Mpe of Proposition 4. We have already noted that incentives are nonlocal in the former and local in the latter. Another difference is that, in the regular Mpe, the probability that any player concedes within a short duration $\mathrm{d}t$ is also of order $\mathrm{d}t$ at any point of the attrition region $(\alpha^*, x^*]$. As a result, both players overall obtain their stand-alone values and rents are fully dissipated, which unambiguously makes the regular Mpe the worst Mpe in the Pareto sense for any initial market conditions $x \in (\alpha^*, \beta)$. By contrast, in all singular Mpes, the probability that player i concedes within a short duration $\mathrm{d}t$ starting from any of his randomization thresholds is of order $\sqrt{\mathrm{d}t}$, so that his concession *rate* at these thresholds becomes infinite. The resulting kinks in player j's Mpe value function reflect that she now earns a rent, and contrast with the differentiability of the regular Mpe value functions. In practice, such kinks can be non-parametrically identified if detailed enough data on market conditions and, crucially, players' continuation values are available. When players are firms, as in the example of Section 6, a proxy for continuation values is stock prices, provided that financial markets are efficient.

Negative Comovements of Rents Our analysis has important consequences for the players' equilibrium rent schedules, defined for every player i as the difference between his Mpe value function $\bar{J}^i$ and his stand-alone value function V_{R^i}. We have already noted that, when players are as enduring as each other, players' rents in the regular Mpe are identically zero. Consider now a singular Mpe satisfying, with no loss of generality, $q^1_1 = x_{R^1}$. An important byproduct of the proof of Theorem 4 is that, for $1 \leq n \leq N^1 - 1$, the rent of player 2 is first decreasing and then increasing over $[q^1_{n+1}, q^1_n]$, and reaches its minimum value of 0 at q^2_n; similarly, for $1 \leq n \leq N^2 - 1$, the rent of player 1 is first decreasing and then increasing over $[q^2_{n+1}, q^2_n]$, and reaches its minimum value of 0 at q^1_{n+1}. Thus a robust implication of singular Mpes is that players' rents comove negatively as long as market conditions remain in the interval $(s^1 \vee s^2, x_{R^1})$, as illustrated on Figures 1–2. Admittedly, negative comovements of rents also arise in the pure-strategy Mpes of the opposite-interest woas of Gul and Pesendorfer (2012) and Gieczewski (2025).[24] Yet in the (typically unique) Mpe of an opposite-interest woa, one player's rent schedule is globally increasing while his

[24]It should be noted that this is also the case in the gaps of the disconnected stopping regions in the pure-strategy Mpes constructed by Murto (2004), where players' interests are clearly opposed. However, such gaps can never be reached on the equilibrium path if initial market conditions are favorable enough.

opponent's is globally decreasing. By contrast, a distinctive feature of our singular Mpes is the non-monotonicity of each player's rent schedule, which reflects that his opponent concedes according to a mixed Markovian strategy with a discrete intensity measure. These predictions also contrast, when initial market conditions are favorable enough, with the positive comovements of players' values in pure-strategy Mpes, reflecting that no attrition *region* is reached on the equilibrium path.

6 Exit in Duopoly

We have established in Theorem 4 that singular Mpes with arbitrarily many randomization thresholds exist in the vicinity of any symmetric woa. However, the topological method we used to this end does not allow us to construct simple singular Mpes with few randomization thresholds—which is arguably desirable as such Mpes would (1) require less coordination on the part of the players and (2) lead to outcomes that would qualitatively differ the most from the regular Mpe when the latter exists. Furthermore, the approach followed so far does not ensure that singular Mpes exist away from the case of nearly symmetric players. To address these questions, and to further explore the testable implications of our analysis, we revisit the standard model of exit in duopoly (Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986), Fine and Li (1989), Murto (2004), Georgiadis, Kim, and Kwon (2022)).

6.1 The Model

Two firms are initially present in a market. As long as they both stay in the market, each earns a flow duopoly profit X_t, where X follows a geometric Brownian motion with drift $b < r$ and volatility σ over $\mathcal{I} \equiv (0, \infty)$,

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad t \geq 0.$$

When firm i exits the market, its assets are liquidated for a value $l^i > 0$. If a firm exits before its rival, then the latter thereafter earns a flow monopoly profit mX_t, $m > 1$, as long as it stays in the market. The total value of every firm i is thus

$$F^i(x, \tau^1, \tau^2) \equiv \mathbf{E}_x\left[\int_0^{\tau_1 \wedge \tau_2} \mathrm{e}^{-rt} X_t\,\mathrm{d}t + \mathbb{1}_{\{\tau^i \leq \tau^j\}}\,\mathrm{e}^{-r\tau^i} l^i + \mathbb{1}_{\{\tau^i > \tau^j\}}\,\mathrm{e}^{-r\tau_j} V_m^i(X_{\tau^j})\right],$$

where (τ^1, τ^2) are the firms' planned exit times and

$$V_m^i(x) \equiv \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\left[\int_0^{\tau} \mathrm{e}^{-rt} m X_t\,\mathrm{d}t + \mathrm{e}^{-r\tau} l^i\right]$$

is firm i's monopoly value. Letting $E(x) \equiv \frac{x}{r-b}$, $R^i \equiv l^i - E$, and $G^i \equiv V_m^i - E$ yields that $J^i(\cdot, \tau^1, \tau^2) \equiv F^i(\cdot, \tau^1, \tau^2) - E$ satisfies (2). It follows from standard computations (Dixit and Pindyck (1994)) that $x_{R^i} = \frac{\rho^-}{\rho^- - 1}(r-b)l^i$, where $\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}$, that $\alpha^i = \frac{x_{R^i}}{m}$, and that $G^i = R^i$ over $(0, \alpha^i]$. Notice that the most enduring firm is the one with the lowest liquidation value because it has less to gain from exiting the market. A1–A8 can be checked along the lines of Décamps, Gensbittel, and Mariotti (2025a), and A9–A11 are satisfied for $p \equiv (l^1, l^2)$ in a compact subset P of $(0,\infty) \times (0,\infty)$ that intersects the diagonal at some point $p_\infty \equiv (l_\infty, l_\infty)$.[25]

6.2 A Simple Singular Mpe

We construct a simple singular Mpe of type 2 in which only firm 1 randomizes, at x_{R^1}, and firm 2 exits with probability 1 at some $s^2 < x_{R^1}$. For firm 1 to be indifferent between exiting at x_{R^1} and obtaining $R^1(x_{R^1})$ immediately and waiting for firm 2 to exit at s^2 and obtaining $G^1(s^2)$ only then, it must be that

$$R^1(x_{R^1}) = \mathbf{E}_x[\mathrm{e}^{-r\tau_{s^2}}]G^1(s^2). \tag{38}$$

Lemma S.8.1 in Online Supplement S.8 shows that (38) always admits a unique solution $s^2 \in (\alpha^1, x_{R^1})$. In the exit-in-duopoly example, it follows from standard computations that $s^2 = \left[\frac{1-m^{\rho^-}}{\rho^-(1-m)}\right]^{\frac{1}{1-\rho^-}} x_{R^1}$ and that

$$\bar{J}^1(x) \equiv \begin{cases} \left(\frac{x}{x_{R^1}}\right)^{\rho^-} R^1(x_{R^1}) & \text{if } x > s^2 \\ G^1(x) & \text{if } x \leq s^2 \end{cases} \tag{39}$$

satisfies (18)–(22).[26] By construction, $\bar{J}^1 = V_{R^1}$ over $[x_{R^1}, \infty)$, reflecting that firm 1 cannot benefit from the presence of firm 2 over $[x_{R^1}, \infty)$. Compared to a pure-strategy Mpe in which a less enduring firm 2 exits first, at $x_{R^2} \geq x_{R^1}$, this reflects that firm 2 adopts a tougher stance by threatening to exit only at $s^2 < x_{R^1}$, which makes firm 1 indifferent between staying in and exiting from the market at x_{R^1}. By contrast, $\bar{J}^1 > V_{R^1}$ over $[s^2, x_{R^1})$ because firm 1 can hope for firm 2 to exit at s^2 before it itself exits at x_{R^1}. Compared to a pure-strategy Mpe in which a less enduring firm 1 exits first, at $x_{R^1} \geq x_{R^2}$, this reflects that firm 1 now only exits probabilistically at x_{R^1}.

As for firm 2, let T^2_q be the unique solution to $\mathcal{L}u - ru = 0$ over $\mathcal{I}$ that is tangent to R^2 at q. It then follows from Theorem 3 that $\bar{J}^2$ must coincide, over $[s^2, x_{R^1})$, with $T^2_{s^2}$, and, over $[x_{R^1}, \beta)$, with the unique solution u^2 to $\mathcal{L}u - ru = 0$ satisfying $u(\beta-) = 0$ and $u(x_{R^1}) = T^2_{s^2}(x_{R^1})$. In fact, independently of the ordering of the firms' stand-alone thresholds

[25] The assumption that firms can only differ in the liquidation values of their assets is not essential. Indeed, because our analysis is entirely conducted in terms of thresholds for the state variable, similar arguments would apply if firms had different profit or discount rates.

[26] Notice that $N^2 = 0$ because firm 2 plays a pure strategy, so that (23) is emptily satisfied.

x_{R^1} and x_{R^2}, a general necessary and sufficient condition for the existence of a simple singular Mpe of type 2 turns out to be

$$G^2(x_{R^1}) > T^2_{s^2}(x_{R^1}) > T^2_{x_{R^2}}(x_{R^1}). \tag{40}$$

The first inequality in (40) ensures by (27) that the intensity a^1_1 with which firm 1 exits at x_{R^1} is positive and finite—reflecting firm 2's incentive-compatibility constraint—and the second inequality ensures that $s^2 < x_{R^2}$ and $\bar{J}^2 \geq R^2$—reflecting firm 2's individual-rationality constraint. The following result delineates a few circumstances in which (40) holds and thus a simple singular Mpe exists.

Proposition 5 *In the exit-in-duopoly example, when the firms' liquidation values l^1 and l^2 are close to each other in relative terms, the following implications hold:*

(i) if $b > 0$ and m is sufficiently large, then a simple singular Mpe exists;

(ii) if $b < 0$ and m is sufficiently large, then a simple singular Mpe does not exist;

(iii) if m is close to 1, then a simple singular Mpe exists.

Moreover, the Mpes in (i) and (iii) can be taken to satisfy Murto (2004)'s refinement.

It should be noted that, whereas the regular Mpe of Proposition 4 has no counterpart when there is even the slightest asymmetry in the firms' liquidation values, the simple singular Mpe of Proposition 5 also exists in the case of symmetric liquidation values and is robust to asymmetry.

Proposition 5 still requires that the firms' liquidation values be close to each other—albeit in a relative sense, so that the difference $|l^2 - l^1|$ can be chosen as high as we wish as long as the ratio $\frac{l^2}{l^1}$ does not differ too much from 1. Due to the high nonlinearity of the inequalities (40), it is however difficult to obtain much more general analytical results.

To get a clearer picture of the range of parameters over which (40) holds, and of whether these are consistent with large relative asymmetries between firms, we thus turn to numerical simulations. To this end, it is useful to think in terms of $\xi \equiv \left[\frac{1-m\rho^-}{\rho^-(1-m)}\right]^{\frac{1}{1-\rho^-}} \in (0,1)$ and $\pi \equiv \frac{l^2}{l^1} \in (0,\infty)$. It is readily verified that a necessary condition for the second inequality in (40) is $\pi > \xi$. From this, it can be shown that this inequality is satisfied as long as $1 \geq \pi > \xi$, that is, firm 2 is at least as enduring as firm 1; when $\pi > 1$, the condition becomes that π is below a threshold $\bar{\pi}(\xi)$ that depends in a nontrivial manner on ξ.

To move forward, we consider in our numerical illustration the special case of our model where ρ^- is fixed equal to -1, that is, $b + r = \sigma^2$, while $\rho^+ \equiv \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} >$

1 is allowed to vary.[27] As shown in Figure 3, the range of parameters under which a simple singular Mpe exists depends crucially on whether b is positive or negative—so that market conditions drift upward or downward on average—and thus, concomitantly, given our parameter restriction $b + r = \sigma^2$, on whether r is low or high—so that the players are relatively patient or impatient.

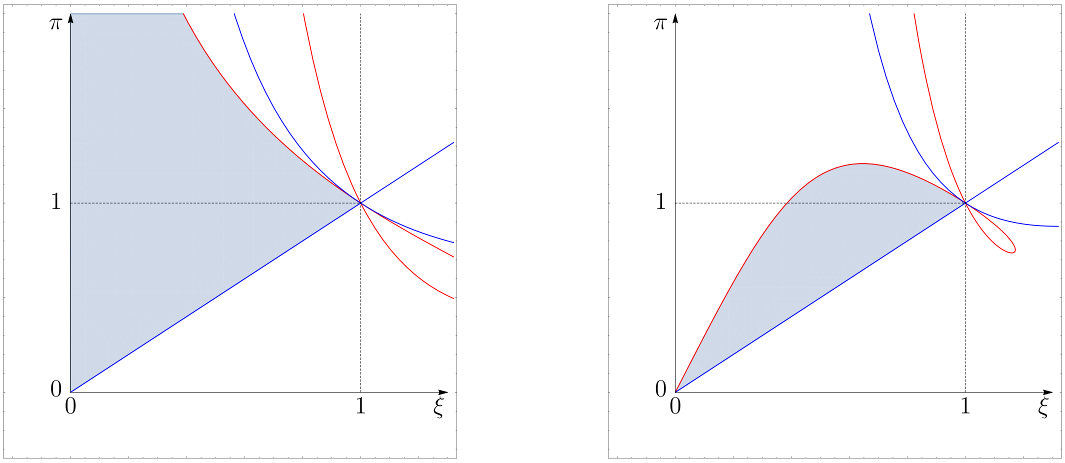

Figure 3: The shaded regions show the parameter values in (ξ, π)-space under which a simple singular Mpe exists when $b > 0$ (left, $\rho^+ = 1.5$) and $b < 0$ (right, $\rho^+ = 3$). The red and blue lines correspond to (ξ, π)-configurations such that the left and right inequalities in (40) bind, respectively.

(a) In case $b > 0$, the existence of a simple singular Mpe is consistent with large asymmetries between firms. Indeed, if ξ is low enough, then, even in case $\pi \gg 1$, the much less enduring firm 2 can be incentivized to wait until $s^2 = \xi x_{R^1}$ to exit the market; this is because firm 2 is patient and market conditions drift upward on average, so that firm 2 can hope for market conditions to hit x_{R^1}—at which point firm 1 may exit—before they hit s^2—at which point firm 2 exits with probability 1. This is a fortiori true if firm 2 is more enduring than firm 1, that is, $\pi \leq 1$. The interval $(\xi, \bar{\pi}(\xi))$ of values for π such that a simple singular Mpe exists decreases in ξ and thus increases in $m = \frac{1}{\xi^2}$. This reflects that, when the market drifts upward on average, the larger is the monopoly payoff rate, the easier it is to incentivize firm 2 to wait and bet on the fact that firm 1 will exit at x_{R^1}.

(b) In case $b < 0$, the existence of a simple singular Mpe need not be consistent with large asymmetries between firms. If ξ is low enough and $\pi \geq 1$, the above argument can be reversed: the less enduring firm 2 can no longer be incentivized to wait until $s^2 = \xi x_{R^1}$ to exit the market, because it is impatient and market conditions drift downward on average. This is true even if firm 1 exits with very high intensity at x_{R^1}: the only way to induce firm

[27] Notice that $\rho^- + \rho^+ \lesseqgtr 1$ according to $b \gtreqless 0$ and that $\xi = \frac{1}{\sqrt{m}}$ if $b + r = \sigma^2$. In their baseline specification, Dixit and Pindyck (1994, Chapter 5, §4) take $b = 0$, $\sigma = 0.2$, and $r = 0.04$ (at annual rates).

2 to wait for so long would be to promise it a payoff larger than $G^2(x_{R^1})$ at x_{R^1}, which is of course impossible. The novel insight is that this situation may arise even in case firm 2 is much more enduring than firm 1, that is, $\pi \ll 1$, and that the upper bound of the interval $(\xi, \bar{\pi}(\xi))$ of values for π such that a simple singular Mpe exists is no longer monotone in ξ. Specifically, holding π constant, a decrease in ξ can make a simple singular Mpe cease to exist. The reason is that a decrease in ξ leads to a decrease in s^2, which is costly for firm 2 as the market drifts downward on average. This, therefore, has to be matched by a higher intensity of exit for firm 1 at x_{R^1}; eventually, this intensity becomes infinite, and a simple singular Mpe no longer exists.

6.3 Asset-Pricing Implications

We now draw the asset-pricing implications of our simple singular Mpe.

Assets and Investors Suppose that both firms are all-equity firms whose stocks are traded on a frictionless financial market. At any time t, each firm's stock distributes a payout flow X_t if neither firm has conceded, and a 0 or mX_t payout flow otherwise, depending on whether or not the firm has exited the market. Shareholders are risk-neutral and observe market conditions and firms' exit decisions. Thus their information set at any time t is

$$\hat{\mathcal{F}}_t \equiv \mathcal{F}_t \vee \sigma(\mathbb{1}_{\{\gamma^1 \leq s\}}, 0 \leq s \leq t). \tag{41}$$

Notice that (41) reflects that, as the exit time τ_{s^2} of firm 2 is $(\mathcal{F}_t)_{t\geq 0}$-adapted, the event that firm 2 has exited by time t is included in $\hat{\mathcal{F}}_t$. By contrast, because firm 1's plays a mixed strategy in equilibrium, the event that it has exited by time t does not belong to $\mathcal{F}_t$. Shareholders are assumed to have rational expectations about the Mpe played by the firms.

Stock Prices Because the market is efficient and shareholders are risk-neutral, the firms' stock prices up to the first time $\tau^e \equiv \gamma^1 \wedge \tau_{s^2}$ at which one of them exits are given by

$$\begin{aligned} V_t^{1,\tau^e} &\equiv F^1(X_{t\wedge\tau^e}), \\ V_t^{2,\tau^e} &\equiv F^2(X_{t\wedge\tau^e}) + [V_m^2(x_{R^1}) - F^2(x_{R^1})]\mathbb{1}_{\{t\wedge\tau_{s^2} \geq \gamma^1\}}, \end{aligned}$$

where the second term in the definition of V_t^{2,τ^e} reflects that firm 1 concedes over $\{\tau_{s^2} > \gamma^1\}$, so that firm 2's market value jumps upwards to its monopoly value at γ^1; there is no analogous term in the definition of V_t^{1,τ^e} as $F^1(s^2) = V_m^1(s^2)$. Applying Itô's formula to F^1 and the Itô–Tanaka–Meyer formula to F^2 yields

$$V_t^{1,\tau^e} = F_1(x) + \int_0^{t\wedge\tau^c} [rF^1(X_s) - X_s]\,\mathrm{d}s + \int_0^{t\wedge\tau^e} \sigma X_s F^{1\prime}(X_s)\,\mathrm{d}W_s, \tag{42}$$

$$V_t^{2,\tau^e} = F_2(x) + \int_0^{t\wedge\tau^c} [rF^2(X_s) - X_s]\,\mathrm{d}s + \int_0^{t\wedge\tau^e} \sigma X_s F^{2\prime-}(X_s)\,\mathrm{d}W_s$$

$$+ [V_m^2(x_{R^1}) - F^2(x_{R^1})](\mathbb{1}_{\{t\wedge\tau_{s^2}\geq\gamma^1\}} - a_1^1 L_{t\wedge\tau^e}^{x_{R^1}}). \quad (43)$$

The Martingale Property The absence of arbitrage opportunities requires that each firm i's discounted cum-dividend stock-price process $(\mathrm{e}^{-rt\wedge\tau^e} V_t^{i,\tau^e} + \int_0^{t\wedge\tau^e} \mathrm{e}^{-rs} X_s \,\mathrm{d}s)_{t\geq 0}$ be a martingale with respect to the shareholders' filtration $(\hat{\mathcal{F}}_t)_{t\geq 0}$. For firm 1, this readily follows from (42). The analysis of firm 1's stock price is then the same as in the corporate-finance models of Merton (1974), Leland (1994), and Goldstein, Ju, and Leland (2001), except that it is not stopped with probability 1 at $\tau_{x_{R^1}}$. As a result, firm 1's stock price is not a monotonic function of its current payout level, as illustrated on Figures 4 and 6 below. For firm 2, the martingale property is more subtle. At first sight, the presence of the local-time term $L_{t\wedge\tau^e}^{x_{R^1}}$ in (43) seems to create an arbitrage opportunity that consists to sell firm 2's stock each time X hits x_{R^1} at price $F^2(x_{R^1})$ and then to repurchase firm 2's stock at price $F^2(X_{t+\mathrm{d}t})$ at $t+\mathrm{d}t$; to a naive investor, this strategy seems to yield a gain of order $\mathrm{d}L_t^{x_{R^1}}$ each time X hits x_{R^1}.[28] However, this does not account for the possibility that firm 1 may exit at x_{R^1}, causing firm 2's stock price to jump upwards to $V_m^2(x_{R^1})$. Once this risk is taken into account, the expected gain of this strategy is exactly zero, reflecting that the term $\mathbb{1}_{\{t\wedge\tau_{\underline{x}^2}\geq\gamma^1\}} - a_1^1 L_{t\wedge\tau^c}^{x_{R^1}}$ in (43) is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingale.

Comovements of Stock Prices and their Volatilities We know from Section 5.5 that a robust implication of any singular Mpe is that players' rent schedules $\bar{J}^1 - R^1$ and $\bar{J}^2 - R^2$ comove negatively as long as market conditions remain in $(s^1 \vee s^2, x_{R^1})$. As $\bar{J}^i - R^i = F^i - l^i$ in the exit-in-duopoly example, this translates, as long as no firm exits the market, into negative comovements of firms' stock prices in bad times, that is, when $X_t \in (s^1 \vee s^2, x_{R^1})$. It should be noted that this property—which, importantly, does not rely on any specific assumption on the dynamics of X, nor on which singular Mpe is considered—holds in spite of the fact that the firms' cash-flows are perfectly correlated as long as neither exits the market. This illustrates once again the general point that the impact of market conditions on players' values is endogenous to the Mpe under consideration and, in particular, to its sequence of randomization thresholds. Our simple singular Mpe corresponds to the case in which firm 1's and firm 2's stock prices are decreasing and increasing functions of market conditions over $[s^2, x_{R^1}]$, respectively. Moreover, as Lemma S.8.3 in Online Supplement S.8 shows, (43) predicts that the volatility $\left|\frac{\sigma X_t F^{2\prime -}(X_t)}{F^2(X_t)}\right|$ of firm 2's stock returns peaks when firm 2's stock price approaches $F^2(x_{R^1})$, and drops to zero when firm 2's stock price approaches

[28] This strategy is in the spirit of Karatzas and Shreve (1998, Appendix B) and Jarrow and Protter (2005, Theorem 4.3), who show that a singular term in the dynamics of a cum-dividend stock prices leads to arbitrage opportunities. Of course, this is not the case for an ex-dividend stock-price process, as in dynamic security-design models (DeMarzo and Sannikov (2006), Biais, Mariotti, Plantin, and Rochet (2007)) or cash-management models (Bolton, Chen, and Wang (2011), Décamps, Mariotti, Rochet, and Villeneuve (2011)), where the dividend process is of the local-time form.

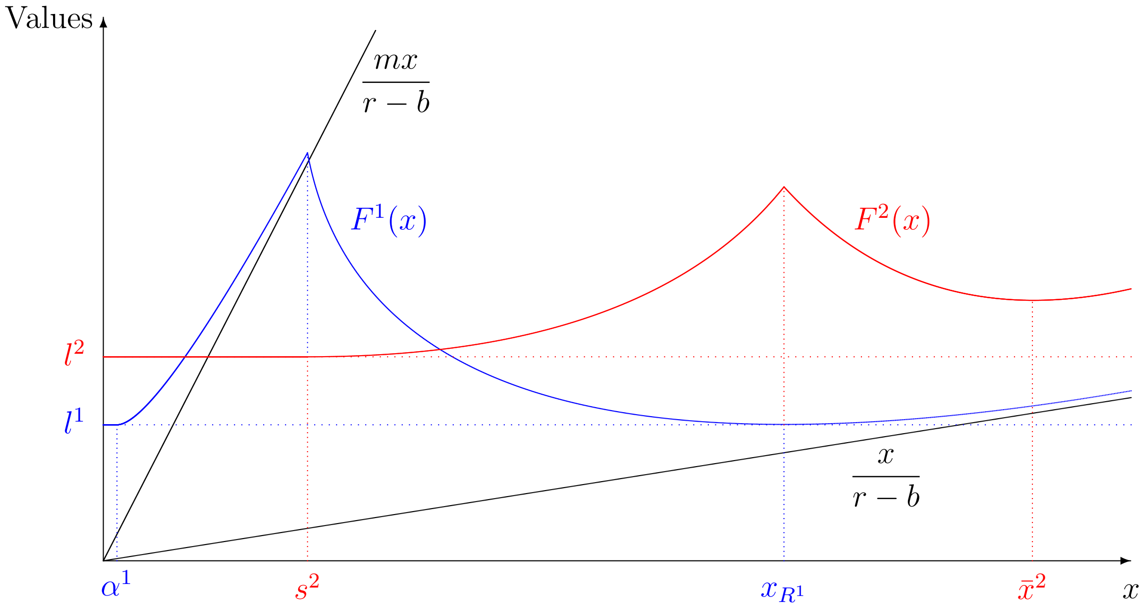

Figure 4: The total values of never exiting the market in a duopoly and in a monopoly (in black), firm 1's total value (in blue), and firm 2's total value (in red) in the simple singular Mpe when $F^{2\prime+}(x_{R^1}) < 0$.

its liquidation value l^2. Similarly, (42) predicts that the volatility $\left|\frac{\sigma X_t F^{1\prime}(X_t)}{F^1(X_t)}\right|$ of firm 1's stock returns peaks when firm 1's stock price approaches its monopoly value $V_m^1(s^2)$ at s^2, and drops to zero when firm 1's stock price approaches its liquidation value l^1. Thus the volatilities of firms' stock returns comove negatively as long as no firm exits the market and the market conditions remain in (s^2, x_{R^1}). Again, this is a robust feature of singular Mpes in the exit-in-duopoly example.

Sample-Path Properties We now turn to the sample-path properties of firms' stock prices. It is useful to distinguish two cases.

Case 1: $F^{2\prime+}(x_{R^1}) < 0$ As shown in Online Supplement S.8, this situation arises whenever $\bar{J}^2(x_{R^1}) > R^1(x_{R^1})$, for which a sufficient condition is $l^2 \geq l^1$. Figure 4 illustrates the firms' total values in that case. The key observation is that the firms' value functions are non-monotonic with respect to the payout rate. For firm 1, this follows readily from (39) along with the smooth-pasting condition for firm 1 at x_{R^1}. For firm 2, this follows from the fact that the kink in its value function $\bar{J}^2$ at x_{R^1} is very pronounced, reflecting a large disappointment in case firm 1 fails to exit at x_{R^1}. A takeaway from Figure 4 is that the firms's stock prices comove negatively in good times, when $X_t < \bar{x}^2$, while they comove positively in good times, when $X_t > \bar{x}^2$. Figure 5 illustrates sample paths of stock prices before any firm exits the market. In line with Figure 4, each time X hits x_{R^1} without firm 1 exiting the market, firm 2's stock price is continuously reflected downward.

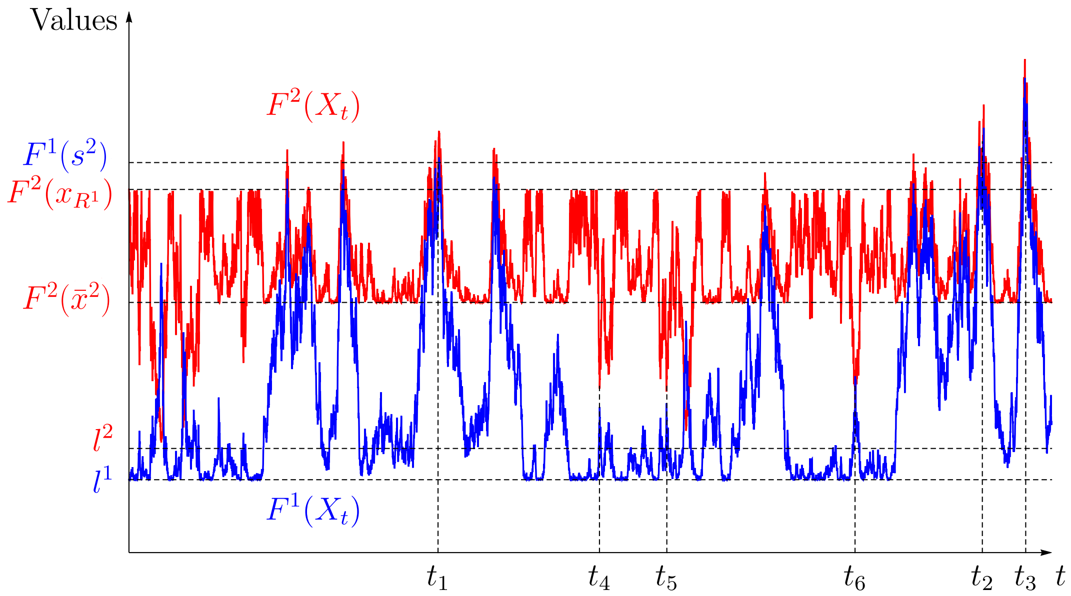

Figure 5: Sample paths of firm 1's stock price (in blue) and firm 2's stock price (in red) before any firm exits the market. The selected parameter values are $l^1 = 0.97$, $l^2 = 1$, $m = 2$, $b = 0.15$, $r = 0.8$, and $\sigma = 0.3$.

These results can shed some light on patterns documented by technical analysis of market data. Technical analysts claim that they can predict financial price movements using limited information sets, including past prices (Edwards, Magee, and Bassetti (2013)). Faced with a chart such as Figure 5, a technical analyst unaware of the strategic interaction between the firms and its impact on their valuations, and, hence, of the fundamental relationship between market conditions and stock prices, may interpret $F^2(x_{R^1})$ as a predictable *resistance level* for firm 2's stock price, at which upward trends tend to be reversed. Similarly, he may interpret l^1 and $F^2(\bar{x}^2)$ as predictable *support levels* for firm 1's and firm 2's stock prices.[29] Our analysis is consistent with these stylized facts while maintaining the assumption that markets are efficient and stock prices are only driven by fundamentals.

In the terminology of technical analysis, a *breakup* of the resistance level $F^2(x_{R^1})$ for firm 2 can occur in two types of situations. (1) In good times, when a large improvement in market conditions above $\bar{x}^2$ leads firm 2's stock price to break, in a continuous way, the resistance level $F^2(x_{R^1})$. This happens at times t_1, t_2, and t_3 in Figure 5. These events are preceded and followed by episodes of positive comovements of stock prices, reflecting that market conditions have left the attrition region $(s^2, \bar{x}^2)$. (2) In bad times, if firm 1 concedes at x_{R^1},

[29]The interpretation of l^1 and $F^2(\bar{x}^2)$ as support levels for firms 1 and 2, respectively, is a little less clear-cut than that of $F^2(x_{R^1})$ as a resistance level for firm 2. Indeed, because $F^{1\prime}(x_{R^1}) = F^{2\prime}(\bar{x}^2) = 0$, the volatilities of firm 1's and firm 2's stock prices drop to zero at x_{R^1} and $\bar{x}^2$, respectively, making it less likely to detect a trend reversal at l^1 and $F^2(\bar{x}^2)$ than at $F^2(x_{R^1})$, where the volatility of firm 2's stock price reaches a peak, as illustrated in Figure 5. Still, the fact remains that, as long as firm 1 does not exit at x_{R^1}, its stock price is not absorbed at l^1, although its fluctuations are damped down at this price level.

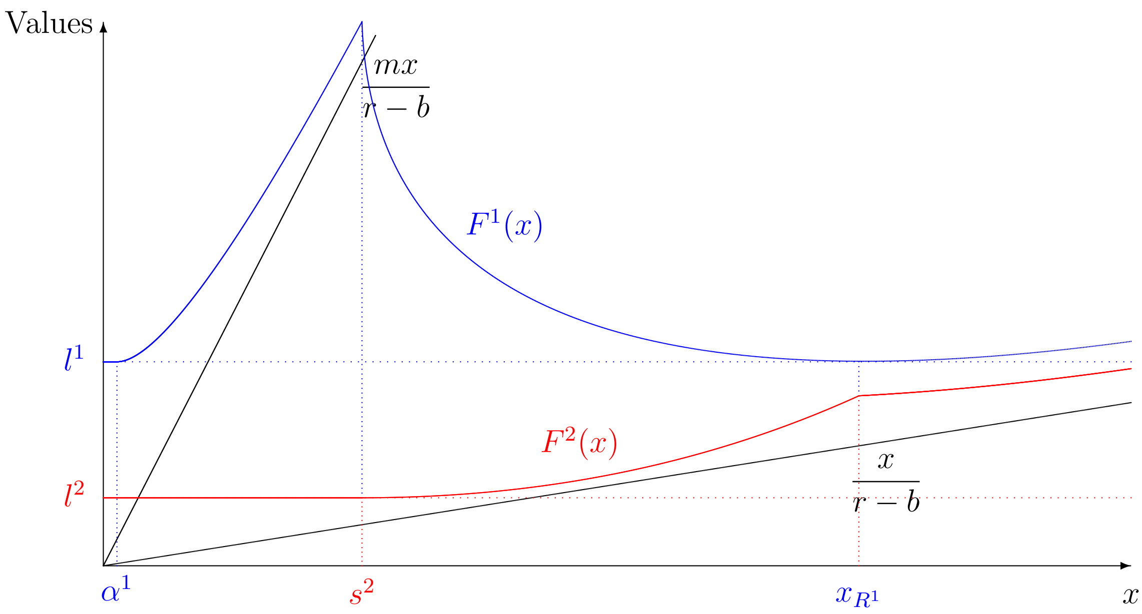

Figure 6: The total values of never exiting the market in a duopoly and in a monopoly (in black), firm 1's total value (in blue), and firm 2's total value (in red) in the simple singular Mpe when $F^{2\prime+}(x_{R^1}) \geq 0$.

causing an upward jump in firm 2's stock price from $F^2(x_{R^1})$ to its monopoly value $V_m^2(x_{R^1})$. This second case is in line with the observation often made in technical analysis that, when prices rise above their resistance levels, they tend to do so decisively. In contrast with continuous breakups, a decisive breakup is preceded by an episode of negative comovements of stock prices. A *breakdown* of the support level $F^2(\bar{x}^2)$ can only happen in a continuous way, and only after firm 2's stock price has reached its resistance level $F^2(x_{R^1})$. This happens at times t_4, t_5, and t_6 in Figure 5.

Technical analysts often explain decisive breakups of resistance levels by unpredictable changes in earnings, management, or competition above investors' expectations. This is exactly what happens in our simple singular Mpe when firm 1 exits at x_{R^1}. Where we fundamentally differ from technical analysis is that the downward bounces in firm 2's stock price at the resistance level $F^2(x_{R^1})$ are no more predictable from past prices than the upward jump in firm 2's stock price that occurs when firm 1 exits the market at x_{R^1}, and thus cannot be arbitraged away by rational investors.[30]

Case 2: $F^{2\prime+}(x_{R^1}) \geq 0$ As shown in Online Supplement S.8, this situation arises whenever $\bar{J}^2(x_{R^1}) \leq R^1(x_{R^1})$, for which a sufficient condition is that π is close to ξ, consistent with

[30]Brown and Jennings (1989), Grundy and McNichols (1989), and Blume, Easley, and O'Hara (1994) construct rational-expectations models in which information about past prices or volume can be valuable for investors in equilibrium. By contrast, in our model, using technical analysis to predict future prices would be fundamentally misguided. What we show is that the price processes generated in equilibrium exhibit patterns that *apparently* lend support to technical analysis.

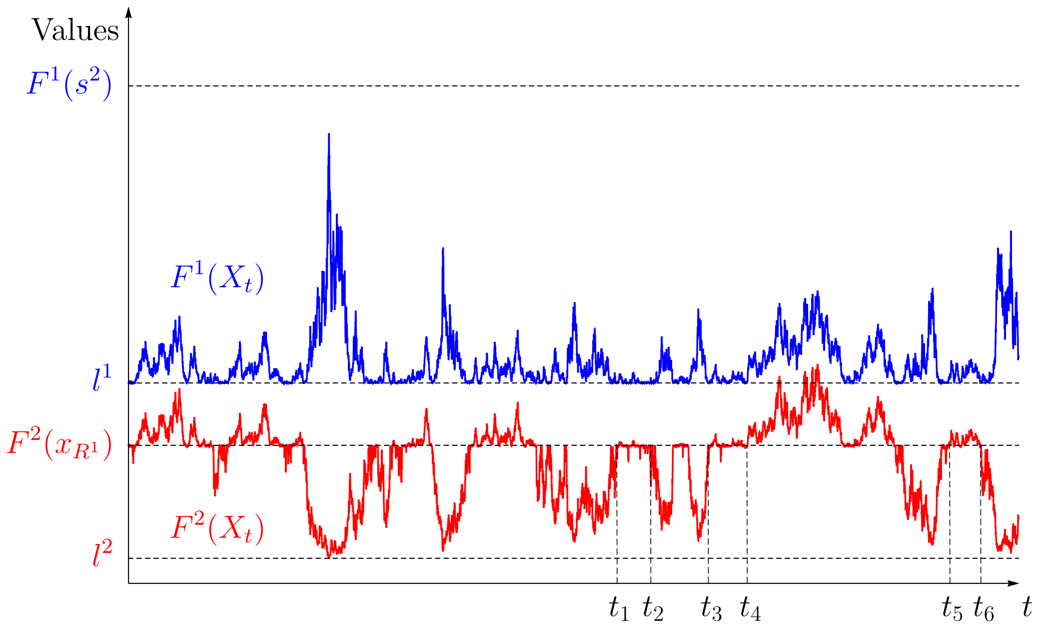

Figure 7: Sample paths of firm 1's stock price (in blue) and firm 2's stock price (in red) before any firm exits the market. The selected parameter values are $l^1 = 1$, $l^2 = 0.6$, $m = 5$, $b = 0.01$, $r = 2$, and $\sigma = 0.5$; $F^2(X_t)$ and l^2 have been translated downward for distinctness.

firm 2 being significantly more enduring than firm 1. Figure 6 illustrates the firms' total values in that case, and Figure 7 illustrates sample paths of stock prices before any firm exits the market. Compared to Figures 4–5, the novelty is that firm 2's value function no longer exhibits a peak at x_{R^1}, so that its stock price is no longer reflected downwards at the level $F^2(x_{R^1})$. Nevertheless, because firm 2's value function still exhibits a kink at x_{R^1}, though a less pronounced one than in Case 1, a new phenomenon arises, namely, a form of upward stickiness of firm 2's stock price at the level $F^2(x_{R^1})$. This happens over the time intervals $[t_1, t_2]$, $[t_3, t_4]$, and $[t_5, t_6]$ in Figure 7. Faced with a chart such as Figure 7, a technical analyst unaware of the strategic interaction between the firms and its impact on their valuations may conclude that firm 2's stock price undergoes episodes of strong autocorrelation. For the same reasons as in Case 1, his ignorance of the fundamental relationship between market conditions and stock prices would then lead him to believe, incorrectly, that past prices have predictive power for future price movements.

For simplicity, we have focused in this section on simple singular Mpes with a single randomization threshold, but our analysis—in particular, that the firms' stock prices and the volatilities of their returns comove negatively in the attrition region—straightforwardly extends to singular Mpes with multiple randomization thresholds. The difference is that firms' stock prices will now exhibit several resistance levels, $F^1(q^2_n)$, $1 \leq n \leq N^2$, for firm 1 and $F^2(q^1_n)$, $1 \leq n \leq N^1$, for firm 2, resulting in richer price dynamics.

We do not claim that exit by a competitor is the sole, or even the main driver of resistance levels in stock prices. Our goal in this section has rather been to show that such resistance levels are consistent with a rational model and efficient financial markets. One can speculate that singular Markovian strategies governing other firms' decisions such as product launch and changes in management or governance can generate similar patterns in stock prices.

7 Concluding Remarks

This paper has offered a detailed study of mixed-strategy Mpe outcomes in the symmetric-information woa when the players' rewards are driven by a homogenous linear diffusion. Our contribution is fourfold.

First, we have provided a characterization result for Markovian strategies in terms of an intensity measure over the state space together with a subset of the state space over which the player concedes with probability 1. This covers the standard cases of pure strategies and of mixed strategies with absolutely continuous intensity measures. In addition, this representation allows for mixed Markovian strategies with singular intensity measures, a possibility that has been disregarded in the literature.

Second, we have argued that, far from being artificial or exotic, such singular Markovian strategies are key to the identification of robust mixed-strategy Mpe outcomes, both in the cases of symmetric and asymmetric players. We have provided a variational characterization of singular Mpes and we have shown that they are characterized by intertwined sequences of randomization thresholds for the players. As a result, the players on the equilibrium path can alternately find themselves in a position of strength or weakness, and players' equilibrium rents comove negatively and are typically non-monotonic over the attrition region, which are novel insights in the literature.

Third, we have shown that such singular Mpes exist in the vicinity of any symmetric woa, and that the standard regular Mpe in the latter case is not isolated, as it can be approximated by singular Mpes of nearly symmetric woas with many randomization thresholds. Our favored interpretation of this result is that this standard regular Mpe is a limiting case, hiding the singularities arising in any neighboring Mpe of the symmetric woa, or of any nearby asymmetric woa. This reinforces our emphasis on the class of singular Mpes as providing robust implications of the woa under uncertainty.

Fourth, we have seen that, in the standard model of exit in a duopoly, simple singular Mpes with few randomization thresholds can be constructed that are consistent with large asymmetries between the firms. In this context, our characterization results lead to new testable asset-pricing implications when firms are publicly traded. Namely, the firms' stock prices and the volatilities of their returns comove negatively over the attrition region and exhibit patterns similar to some documented by technical analysis. This contrasts with the predictions of the standard regular Mpe in the case of symmetric firms, in which firms' stock prices are perfectly aligned and are constant and equal to their common liquidation value over the attrition region.

Taken together, our results show that mixed-strategy Mpes that are robust to even slight asymmetries between players' rewards share a common structure, and lead to qualitatively similar empirical implications. This yields rich and robust predictions for the woa under uncertainty—something that is precluded by focusing on pure-strategy Mpes, or regular Mpes of symmetric woas, whose implications are too stark to fruitfully lend themselves to applied analysis. Our hope is that these insights may pave new avenues for empirical work.

Appendix

Notation A property is satisfied a.s. if, for each $x \in \mathcal{I}$, it is satisfied for $\mathbf{P}_x$-a.e. $\omega \in \Omega$.

Discount Factors Because $\sigma > 0$ over $\mathcal{I}$, the ode $\mathcal{L}u - ru = 0$ admits a two-dimensional vector space of solutions in $\mathcal{C}^2(\mathcal{I})$, spanned by two positive fundamental solutions ψ and ϕ, respectively increasing and decreasing, and uniquely defined up to a linear transformation. Moreover, because the boundaries α and β of $\mathcal{I}$ are natural,

$$\lim_{x \to \alpha+} \psi(x) = 0, \quad \lim_{x \to \beta-} \psi(x) = \infty, \quad \lim_{x \to \alpha+} \phi(x) = \infty, \quad \lim_{x \to \beta-} \phi(x) = 0. \tag{A.1}$$

(Borodin and Salminen (2002, Part I, Chapter II, Section 1, §10)). Letting $\tau_y \equiv \inf\{t \geq 0 : X_t = y\}$ be the hitting time by X of $y \in \mathcal{I}$ from $X_0 = x$, we then obtain the following formula for the expected discount factor associated to x and τ_y:

$$\mathbf{E}_x[\mathrm{e}^{-r\tau_y}] = \begin{cases} \frac{\psi(x)}{\psi(y)} & \text{if } x \leq y \\ \frac{\phi(x)}{\phi(y)} & \text{if } x > y \end{cases}. \tag{A.2}$$

A1–A2 imply

$$\lim_{x \to \alpha+} \frac{R^i(x)}{\phi(x)} = \lim_{x \to \beta-} \frac{R^i(x)}{\psi(x)} = 0. \tag{A.3}$$

By Lemmas 1–2, $G^i > 0$ over $\mathcal{I}$; hence A4 guarantees that the family $(\mathrm{e}^{-r\tau} G^i(X_\tau))_{\tau \in \mathcal{T}}$ is uniformly integrable. A4–A5 imply

$$\lim_{x \to \alpha+} \frac{G^i(x)}{\phi(x)} = \lim_{x \to \beta-} \frac{G^i(x)}{\psi(x)} = 0. \tag{A.4}$$

Proof of Theorem 1: (Necessity) We hereafter omit the index i for the sake of clarity. If Λ is the csf of a Markovian randomized stopping time, then, for all $t, s \geq 0$,

$\Lambda_{t+s} = \Lambda_t(\Lambda_s \circ \theta_t)$ a.s. In particular, applying this property at $t = s = 0$ yields $\Lambda_0 = (\Lambda_0)^2$ and, hence, $\Lambda_0 \in \{0, 1\}$ a.s. In the terminology of Blumenthal and Getoor (1968, Definition III.1.1), Λ is a right-continuous multiplicative functional of X adapted to $(\mathcal{F}_t)_{t\geq 0}$. The set $E_\Lambda \equiv \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 1] = 1\}$ is called the set of permanent points for Λ. Using Blumenthal's 0–1 law (Blumenthal and Getoor (1968, Proposition I.5.17)) and the fact that $\Lambda_0 \in \{0, 1\}$ a.s., we have $\mathcal{I} \setminus E_\Lambda = \{x \in \mathcal{I} : \mathbf{P}_x[\Lambda_0 = 0] = 1\}$. The stopping time $\tau \equiv \inf \{t > 0 : \Lambda_t = 0\} \in \mathcal{T}$ is called the lifetime of Λ. The proof consists of three steps.

Step 1 In order to apply the main result of Sharpe (1971), we need to check that Λ is an exact multiplicative functional in the sense of Blumenthal and Getoor (1968, Definition III.4.13). According to Blumenthal and Getoor (1968, Proposition III.5.9) it is sufficient to prove that, for all $x \in \mathcal{I} \setminus E_\Lambda$ and $t > 0$,

$$\lim_{u\downarrow 0} \mathbf{E}_x [\Lambda_{t-u} \circ \theta_u] = 0. \tag{A.5}$$

To this end, we first claim that, for any such x and t, and for each $u \in (0, t)$, we have $\mathbb{1}_{\{t-u\geq\tau_x\circ\theta_u\}}(\Lambda_{t-u} \circ \theta_u) = 0$ $\mathbf{P}_x$-a.s. Indeed, if $t - u \geq \tau_x \circ \theta_u(\omega)$ for some $\omega \in \Omega$, then the trajectory $\theta_u(\omega)$ crosses x over the interval $[0, t - u]$. Because, by (6), $\Lambda_{\tau_x\circ\theta_u(\omega)}(\theta_u(\omega)) = \Lambda_{\tau_x\circ\theta_u(\omega)}(\theta_u(\omega))\Lambda_0(\theta_{\tau_x\circ\theta_u(\omega)}(\theta_u(\omega))) = 0$ $\mathbf{P}_x$-a.s. as $x \in \mathcal{I}\setminus E_\Lambda$, this implies that $\Lambda_{t-u}(\theta_u(\omega)) = 0$ as the mapping $s \mapsto \Lambda_s(\theta_u(\omega))$ is nonincreasing and nonnegative. The claim follows. This implies in particular that, for $u < \frac{t}{2}$,

$$\mathbf{E}_x [\Lambda_{t-u} \circ \theta_u] \leq \mathbf{P}_x [t - u < \tau_x \circ \theta_u] = \mathbf{E}_x [\mathbf{P}_{X_u} [t - u < \tau_x]] \leq \mathbf{E}_x \left[\mathbf{P}_{X_u}\left[\frac{t}{2} < \tau_x\right]\right].$$

The mapping $y \mapsto \mathbf{P}_y[\frac{t}{2} < \tau_x]$ is bounded and $\lim_{y\to x} \mathbf{P}_y[\frac{t}{2} < \tau_x] = 0$ as X is a regular diffusion. Hence (A.5) follows by bounded convergence along with the fact that $\lim_{u\downarrow 0} X_u = x$ $\mathbf{P}_x$-a.s. Exactness of Λ implies that E_Λ is open and thus that $\mathcal{I}\setminus E_\Lambda$ is closed, see Blumenthal and Getoor (1968, page 126, last paragraph) together with the fact that the fine topology over $\mathcal{I}$ associated to X coincides with the usual topology, see Blumenthal and Getoor (1968, Definition II.4.1 and Exercise II.4.16).

Step 2 We are now in a position to apply Sharpe (1971, Theorem 7.1, Formula (7.1)), which expresses Λ_t as the product of three factors.

1. The first factor is equal to 1 because X has continuous trajectories, so that the terms $F(X_{s-}, X_s)$ vanish as $F = 0$ over the diagonal of $\mathcal{I}$, see Sharpe (1971, Theorem 5.1 and proof of Theorem 7.1).

2. The second factor can be written as $\mathbb{1}_{\{t<\tau_B\}}$, where τ_B is the hitting time by X of a Borel subset B of $\mathcal{I}$; this is because the lifetime of X is infinite and X has continuous

trajectories. In turn, because X is a diffusion process and $\sigma > 0$ over $\mathcal{I}$, this term is a.s. equal to $\mathbb{1}_{\{t<\tau_S\}}$, where S is the closure of B.

3. The third factor is of the form $\mathrm{e}^{-\int_0^t f(X_s)\,\mathrm{d}A_s}$, where $f : \mathcal{I} \to \mathbb{R}_+$ is Borel-measurable and A is a continuous additive functional of X (Revuz and Yor (1999, Chapter X, §1, Definition 1.1)) such that the mapping $x \mapsto \mathbf{E}_x[\int_0^\infty \mathrm{e}^{-t}\,\mathrm{d}A_t]$ is bounded.

Thus, for each $t \geq 0$, we have the representation

$$\Lambda_t = \mathbb{1}_{\{t<\tau_S\}}\,\mathrm{e}^{-\int_0^t f(X_s)\,\mathrm{d}A_s} \text{ a.s.} \tag{A.6}$$

Moreover, the integral $\int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-a.s. finite for all $t < \tau_S$ except maybe for x in an M-polar set, where M is the multiplicative functional defined by $M_t \equiv \mathbb{1}_{\{t<\tau_S\}}$ for all $t \geq 0$ (Blumenthal and Getoor (1968, II.2.18 and III.1.4)). According to Sharpe (1971, Definition, page 29), $B \subset \mathcal{I}$ is an M-polar set if there exists a nearly Borel subset (Blumenthal and Getoor (1968, Definition I.10.21)) $C \supset B$ of $\mathcal{I}$ such that the hitting time by X of C is a.s. greater than or equal to the lifetime of M, that is, τ_S. Hence, because the trajectories of X are continuous and S is closed, an M-polar set must be a subset of S, and it follows that $\int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-a.s. finite for all $t < \tau_S$ and $x \in \mathcal{I} \setminus S$. Finally, observe that we can with no loss of generality assume that $f = 0$ over S, as replacing f by $f\mathbb{1}_{\mathcal{I}\setminus S}$ does not alter the right-hand side of (A.6).

Step 3 Using the classical representation result for additive functionals of X (Borodin and Salminen (2002, Part I, Chapter I, Section 4, §23)), there exists a Radon measure ν over $\mathcal{I} \setminus S$ such that $A_t = \int_{\mathcal{I}\setminus S} L_t^y\,\nu(\mathrm{d}y)$ a.s. Therefore, for each $t < \tau_S$,

$$\tilde{A}_t \equiv \int_0^t f(X_s)\,\mathrm{d}A_s = \int_0^t \int_{\mathcal{I}\setminus S} f(X_s)\,\mathrm{d}L_s^y\,\nu(\mathrm{d}y) = \int_{\mathcal{I}\setminus S} L_t^y f(y)\,\nu(\mathrm{d}y) \text{ a.s.}$$

We claim that $\mu \equiv f \cdot \nu$ is a Radon measure, which concludes the first part of the proof. To this end, we only need to prove that μ is locally finite. Indeed, if it were not so, then there would exist $x \in \mathcal{I} \setminus S$ such that $\int_{[x-\varepsilon,x+\varepsilon]} f(y)\,\nu(\mathrm{d}y) = \infty$ for all $\varepsilon > 0$ such that $[x-\varepsilon, x+\varepsilon] \subset \mathcal{I} \setminus S$. For each $t > 0$, $L_t^x(\omega) > 0$ for all ω in a set of $\mathbf{P}_x$-probability 1. Therefore, as the local time of X is a.s. jointly continuous (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), we have that, for any such ω, there exists $\varepsilon(\omega) > 0$ such that $[x-\varepsilon(\omega), x+\varepsilon(\omega)] \subset \mathcal{I} \setminus S$ and $L_t^y(\omega) > 0$ for all $y \in [x-\varepsilon(\omega), x+\varepsilon(\omega)]$. This implies that, if $0 < t < \tau_S(\omega)$, then

$$\tilde{A}_t(\omega) = \int_{\mathcal{I}\setminus S} L_t^y(\omega) f(y)\,\nu(\mathrm{d}y) \geq \min_{y\in[x-\varepsilon(\omega),x+\varepsilon(\omega)]} L_t^y(\omega) \int_{[x-\varepsilon(\omega),x+\varepsilon(\omega)]} f(y)\,\nu(\mathrm{d}x),$$

which is infinite by assumption. Because $\mathbf{P}_x[\tau_S > 0] = 1$ as $x \in \mathcal{I} \setminus S$, this contradicts the fact that, for each $t < \tau_S$, $\tilde{A}_t = \int_0^t f(X_s)\,\mathrm{d}A_s$ is $\mathbf{P}_x$-a.s. finite. The claim follows.

(Sufficiency) Reciprocally, if S is a closed subset of $\mathcal{I}$ and μ is a Radon measure over $\mathcal{I} \setminus S$, then the process defined by $\Lambda_t = \mathbb{1}_{\{t<\tau_S\}} \, \mathrm{e}^{-\int_{\mathcal{I}\setminus S} L_t^y \, \mu(\mathrm{d}y)}$ is well-defined and, as the local time of X is a strong additive functional of X (Revuz and Yor (1999, Chapter X, §1, Proposition 1.2)), is a right-continuous multiplicative functional that satisfies (6). In particular, $\Gamma \equiv 1 - \Lambda$ satisfies the assumptions of Lemma S.2.2 and thus is the ccdf of a randomized stopping time. Hence the result. ■

Proof of Theorem 2: Let $s \equiv s^1 \vee s^2$ and $E^i \equiv \operatorname{supp} \mu^i \cap (s, \beta)$ for $i = 1, 2$. E^i is a relatively closed subset of (s, β) that can be written as a disjoint union $E^i = A^i \cup K^i$, where A^i is the set of accumulation points of E^i in (s, β), which is relatively closed in (s, β), and K^i is the (countable) set of isolated points of E^i. Observe that $E^i \subset (s, x_{R^i}]$ by Proposition 1(iv) as $E^i \cap S^j = \emptyset$. If $E^1 = E^2 = \emptyset$, there is nothing to prove and the Mpe under consideration is outcome-equivalent to a pure-strategy Mpe. Let us otherwise denote by $\bar{J}^i$ player i's equilibrium value function. The proof then consists of four steps and repeatedly uses assertions (i)–(iii) of Lemma A.1 below.

Lemma A.1 *Let u be a $\mathcal{C}^2$ function defined over an open interval $(a, b) \subset \mathcal{I}$ and such that $\mathcal{L}u - ru = 0$ over (a, b). Then, the following holds:*

(i) if $b = \beta$, $u(\beta-) = 0$, $u(a+) = R^i(a)$, and $u \geq V_{R^i}$ over (a, β), then $a = x_{R^i}$;

(ii) if $u \geq V_{R^i}$ over (a, b), then $\{x \in (a, b) : u(x) = R^i(x)\}$ contains at most one point;

(iii) if $b \leq x_{R^i}$, $u(b-) = R^i(b)$, and either $a > \alpha$ and $u(a+) = R^i(a)$ or $a = \alpha$ and $u(a+) = 0$, then $u < R^i$ over (a, b);

(iv) if $\alpha < a \leq x_{R^i}$, $u \geq R^i$ over (a, b), $u(a) = R^i(a)$, and $u'(a+) > R^{i\prime}(a)$, then, for every sufficiently small $\varepsilon > 0$, the function f_ε solution to $\mathcal{L}f - rf = 0$ over $(a - \varepsilon, a + \varepsilon)$ with $f_\varepsilon(a - \varepsilon) = R^i(a - \varepsilon)$ and $f_\varepsilon(a + \varepsilon) = u(a + \varepsilon)$ satisfies $f_\varepsilon(a) > u(a)$.

Step 1 We first claim that every connected component (a, b) of $(s, \beta) \setminus E^i$ such that (a) $a > s$ or $a = s = s^i$ or $a = s = \alpha$, and (b) $b \leq x_{R^i}$, contains exactly one point of E^j.

Suppose first, by way of contradiction, that $E^j \cap (a, b) = \emptyset$. By Proposition 1(v), the strategy $(0, S^i)$ is a pbr to the strategy (μ^j, S^j). Therefore, τ_{S^i} is a solution to the optimal-stopping problem $\bar{J}^i(x) = \sup_{\tau^i \in \mathcal{T}} J^i(x, \tau^i, (\mu^j, S^j))$. Letting τ be the first exit time of X from (a, b), we have $\tau_{S^i} \geq \tau$ $\mathbf{P}_x$-a.s. for all $x \in (a, b)$. We deduce from this that the brvf $\bar{J}^i$ satisfies, for each $x \in (a, b)$,

$$\bar{J}^i(x) = J^i(x, (0, S^i), (\mu^j, S^j)) = \mathbf{E}_x[\mathrm{e}^{-r\tau} \bar{J}^i(X_\tau)], \tag{A.7}$$

where the last inequality follows from the strong Markov property (S.9). As $E^j \cap (a, b) = \emptyset$, it then follows from standard arguments that $\bar{J}^i$ is $\mathcal{C}^2$ and $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (a, b). Now, consider the conditions in the claim. First, if $a > s$, then $a \in \operatorname{supp} \mu^i$ by definition of a connected component of $(s, \beta) \setminus E^i$, and thus $\bar{J}^i(a) = R^i(a)$ by Proposition 1(iii); the same reasoning shows that $\bar{J}^i(b) = R^i(b)$. Next, if $a = s = s^i$, then $\bar{J}^i(a) = R^i(a)$ by Proposition 1(iii). Finally, if $a = s = \alpha$, then τ coincides with the hitting time of b, and thus (A.7) and (A.1)–(A.2) together imply, letting x go to $\alpha+$, that $\bar{J}^i(a+) = 0$. Thanks to $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (a, b) and $b \leq x_{R^i}$, we are thus in a position to apply Lemma A.1(iii); we obtain $\bar{J}^i < R^i$ over (a, b), a contradiction as $\bar{J}^i \geq V_{R^i}$ over $\mathcal{I}$. Therefore, $E^j \cap (a, b) \neq \emptyset$. Finally, using again standard arguments, it must be that $\mathcal{L}\bar{J}^j - r\bar{J}^j = 0$ over (a, b). Because $\bar{J}^j \geq V_{R^j}$, Lemma A.1(ii) implies that $E^j \cap (a, b)$ contains exactly one point. The claim follows. It should be noted that the same arguments show that every interval $(a, b) \subset (s, \beta)$ such that (a) $a > s$ or $a = s = s^i$ or $a = s = \alpha$, (b) $b \leq x_{R^i}$, and (c) $\bar{J}^i(a+) = R^i(a+)$ and $\bar{J}^i(b) = R^i(b)$ contains at least one point of E^j.

Step 2 We next claim that $A^1 = A^2$.

Let $x \in A^i$. Suppose first, by way of contradiction, that $x \notin E^j$. Then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap E^j = \emptyset$, where ε can be chosen sufficiently small so that $x - \varepsilon > s$. As x is an accumulation point of E^i and E^i is relatively closed in (s, β), one of the two following conditions must hold:

(i) $(x - \varepsilon, x + \varepsilon)$ includes a connected component (a, b) of $(s, \beta) \setminus E^i$ such that $a > s$ and $b \leq x_{R^i}$;

(ii) E^i includes a nondegenerate interval $\mathcal{I}_0 \subset (x - \varepsilon, x + \varepsilon)$ that contains x.

In case (i), the connected component (a, b) must contain one point of E^j by Step 1, a contradiction. In case (ii), notice that $\mathcal{I}_0 \cap S^j = \emptyset$ by definition of s, E^i, and E^j. Thus, by Proposition 1(ii), it must be that $\bar{J}^i = R^i$ over $\mathcal{I}_0$. On the other hand, because $(0, S^i)$ is also a pbr to (μ^j, S^j) and $E^j \cap \mathcal{I}_0 = \emptyset$, we obtain as in Step 1 that $\bar{J}^i$ must be $\mathcal{C}^2$ and satisfy $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over the interior of $\mathcal{I}_0$. But then $\mathcal{L}R^i - rR^i = 0$ over a nondegenerate interval, a contradiction by A3. We conclude that $x \in E^j$ and in turn that $A^i \subset E^j$. Let us now prove that $A^i \subset A^j$. If $x \in A^i$ belongs to the relative closure of $A^i \setminus \{x\}$ in (s, β), then $x \in A^j$ as $A^i \setminus \{x\} \subset E^j$. If not, then x must be the limit of a sequence of points $(x_n)_{n \geq 1}$ in K^i, which we can assume to be strictly monotone. By Step 1, for every sufficiently large n, the interval formed by two consecutive elements x_n and x_{n+1} of this sequence contains exactly one point y_n of E^j, and thus $x = \lim_{n \to \infty} y_n \in A^j$ as it is an accumulation point of E^j. We conclude that $A^i \subset A^j$ and in turn that $A^i = A^j$ by exchanging the role of the players. The claim follows.

Step 3 We then claim that, if $A^1 = A^2 = \emptyset$ and $K^1 \cup K^2 \neq \emptyset$, then the measures μ^1 and μ^2 are discrete or degenerate, with at least a nondegenerate one, and their supports are described by one of the cases in Corollary 1.

By assumption, μ^1 and μ^2 are discrete measures and their supports have no accumulation points in $(s, x_{R^i}]$. Therefore, either their supports are finite, or they are infinite, with s as a unique accumulation point. In both cases, for each $i = 1, 2$, $E^i = K^i \equiv \{q^i_n : 1 \leq n \leq N^i\}$ for some decreasing sequence $(q^i_n)_{n=1}^{N^i}$ in $(s, x_{R^i}]$, with N^i finite or infinite, and possibly equal to 0 for some i, in which case μ^i is degenerate. We now establish three key properties of the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, which together imply the claim.

First, it must be that $q^i_1 = x_{R^i}$ for some i. Indeed, suppose that $K^i \neq \emptyset$ and $\max E^j \leq q^i_1$, where $\max \emptyset = \alpha$. We first have $\bar{J}^i(q^i_1) = R^i(q^i_1)$ by Proposition 1(ii)–(iii) as $q^i_1 > s \geq s^j$. Next, because $E^j \cap (q^i_1, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that $\bar{J}^i$ is $\mathcal{C}^2$ and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (q^i_1, β). As a result, $\bar{J}^i = A\phi + B\psi$ over this interval for some constants A and B. From this, it follows in turn that $\bar{J}^i(\beta^-) = 0$. Indeed, by Lemma 1 and Proposition 1, we have $0 \leq \bar{J}^i \leq G^i$, which, together with (A.4), implies $B = 0$. That $\bar{J}^i(\beta-) = 0$ follows then from (A.1). Finally, $J^i \geq V_{R^i}$ by Proposition 1. Thus $\bar{J}^i$ satisfies all the conditions of Lemma A.1(i), from which we conclude that $q^i_1 = x_{R^i}$.

Next, it must be that the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, are intertwined. Indeed, Step 1 implies that, if at least one of these sequence has at least two elements, then, between two consecutive elements of each sequence, there must be exactly one element of the other sequence. Similarly, if $1 \leq N^i < \infty$ and $s = s^i$ or $s = \alpha$, then $s < q^i_{N^i}$ and there must be one element of K^j in $(s, q^i_{N^i})$. These properties have two main implications. (a) First, the sequences $(q^i_n)_{n=1}^{N^i}$, $i = 1, 2$, have no common element. Indeed, suppose, by way of contradiction, that $q^1 = q^2 = q$ for two components of these two sequences. We distinguish two cases. If at least one of the sets K^1 and K^2 is not a singleton, then, because K^1 and K^2 have s as their only possible accumulation point, there exists some $i = 1, 2$ for which the distance $\inf_{q' \in K^i \setminus \{q\}} |q' - q| > 0$ is minimized, with $\inf_{q' \in \emptyset} |q' - q| \equiv \infty$ for all $q \in \mathcal{I}$ by convention. Let this minimal distance be reached at q'. But then, as argued above, there must exist $q'' \in K^j$ in between q and q', so that $|q'' - q| < |q' - q|$, in contradiction with the definition of q'. If both K^1 and K^2 are singletons, then it must be that $K^1 = K^2 = \{x_{R^1}\} = \{x_{R^2}\}$ by the first property above. Applying Step 1 to the connected component (s^i, x_{R^i}) of $(s, \beta) \setminus E^i$ for a player i such that $s = s^i$, we obtain that (s^i, x_{R^i}) contains exactly one point of E^j, a contradiction. (b) Second, and as a result, if $\max S^j \cup E^j < q^i_1$, then either N^i is finite and $N^j \in \{N^i - 1, N^i\}$, or $N^i = N^j = \infty$.

Finally, if $N^1 = N^2 = \infty$, the sequences $(q^i_n)_{n \geq 1}$, $i = 1, 2$, must converge to α, so that $s = \alpha$ and $S^1 = S^2 = \emptyset$. This is a consequence of the following general lemma.

Lemma A.2 *Let* $((\mu^1, S^1), (\mu^2, S^2))$ *be a mixed-strategy Mpe for which there exists two intertwined decreasing sequences* $(\chi_n^1)_{n\geq 1}$ *and* $(\chi_n^2)_{n\geq 1}$ *in* $\operatorname{supp}\mu^1 \cap (s,\beta)$ *and* $\operatorname{supp}\mu^2 \cap (s,\beta)$*, respectively, such that, for each* $i = 1, 2$*,* $\operatorname{supp}\mu^i \cap (\inf_{n\geq 1}\chi_n^i, \chi_1^i] = \{\chi_n^i : n \geq 1\}$*. Then these two sequences converge to* α*. Similarly, there are no intertwined increasing sequences* $(\chi_n^1)_{n\geq 1}$ *and* $(\chi_n^2)_{n\geq 1}$ *in* $\operatorname{supp}\mu^1 \cap (s,\beta)$ *and* $\operatorname{supp}\mu^2 \cap (s,\beta)$*, respectively, such that, for each* $i = 1, 2$*,* $\operatorname{supp}\mu^i \cap [\chi_0^i, \sup_{n\geq 1}\chi_n^i) = \{\chi_n^i : n \geq 1\}$*.*

The claim follows.

Step 4 We finally claim that, if $A \equiv A^1 = A^2 \neq \emptyset$, then $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, $s = \alpha^1 \vee \alpha^2$, and, for each $i = 1, 2$, the restriction of μ^i to $(s, x_R]$ is absolutely continuous with density $\sigma^{-2}\lambda^i$, where λ^i is given by (9) with s instead of α^*.

We first show that $x_{R^1} = x_{R^2} \equiv x_R$, $A = (s, x_R]$, and $s = \alpha^1 \vee \alpha^2$. The argument is fourfold.

We first claim that $A \subset (s, x_{R^1} \wedge x_{R^2}]$ is an interval. Indeed, suppose, by way of contradiction, that this is not so. Then there exists an interval $(a, b) \subset (s,\beta) \setminus A$ such that $a > s$ and $a, b \in A$. Because (a, b) cannot be a connected component of both $(s,\beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (a, b) \neq \emptyset$ for some i. Fix some $\chi_1^i \in K^i \cap (a, b)$. Then $\bar{J}^i(a) = R^i(a)$ and $\bar{J}^i(\chi_1^i) = R^i(\chi_1^i)$ by Proposition 1(ii)–(iii) as $a > s \geq s^j$, so that $K^j \cap (a, \chi_1^i) \neq \emptyset$ by the final remark of Step 1. Because $\chi_1^i \in (a, b)$ is not an accumulation point of E^j, we have $\chi_1^i > \chi_1^j \equiv \sup K^j \cap (a, \chi_1^i) \in K^j$. Applying this argument recursively, we obtain two infinite intertwined decreasing sequences $(\chi_n^1)_{n\geq 1}$ and $(\chi_n^2)_{n\geq 1}$ in $K^1 \cap (a, b)$ and $K^2 \cap (a, b)$, respectively. Because these sequences are bounded below by $a > s$ and (a, b) is a connected component of $(s,\beta) \setminus A$, they both converge to a. Moreover, arguing as in Step 3, it is easy to check that $\operatorname{supp}\mu^1 \cap (\inf_{n\geq 1}\chi_n^1, \chi_1^1] = \{\chi_n^1 : n \geq 1\}$, and similarly for player 2. Thus, by Lemma A.2, it must be the case that $a = \alpha$, a contradiction as $a > s$. The claim follows. As A is relatively closed in (s,β), $\sup A = \max A \in A$.

We next claim that $\max A = x_{R^1} = x_{R^2}$. Indeed, suppose first, by way of contradiction, that $\max A < x_{R^1} \wedge x_{R^2}$. Arguing as in Step 3, we obtain that $x_{R^i} \in K^i$ for some $i = 1, 2$. Hence $\bar{J}^i(\max A) = R^i(\max A)$ and $\bar{J}^i(x_{R^i}) = R^i(x_{R^i})$ by Proposition 1(ii)–(iii) as $\max A > s \geq s^j$, so that $K^j \cap (\max A, x_{R^1} \wedge x_{R^2}] \neq \emptyset$ by the final remark of Step 1 along with the fact that $K^j \subset (s, x_{R^j}]$. We can then repeat the above argument, leading again to a contradiction. We conclude that $\max A = x_{R^1} \wedge x_{R^2} = x_{R^j}$ for some $j = 1, 2$, so that $\max E^i \geq x_{R^j}$. Now, $\bar{J}^i(x_{R^j}) = R^i(x_{R^j})$ by Proposition 1(ii)–(iii) as $x_{R^j} > s \geq s^j$. Because $E^j \cap (x_{R^j}, \beta) = \emptyset$, we can use similar arguments as in Step 1 to show that $\bar{J}^i$ is $\mathcal{C}^2$ and satisfies $\mathcal{L}\bar{J}^i - r\bar{J}^i = 0$ over (x_{R^j}, β). Finally, we can use similar arguments as in Step 3 to show that $\bar{J}^i(\beta-) = 0$. As $\bar{J}^i \geq V_{R^i}$ by Proposition 1, $\bar{J}^i$ satisfies all the conditions of

Lemma A.1(i), from which we conclude that $x_{R^j} = x_{R^i} \equiv x_R$. The claim follows.

We then claim that $\inf A = s$. Indeed, suppose, by way of contradiction, that $\inf A > s$. Because $(s, \inf A)$ cannot be a connected component of both $(s, \beta) \setminus E^i$, $i = 1, 2$, by Step 1, it must be that $K^i \cap (s, \inf A) \neq \emptyset$ for some i. Fixing some $\chi_1^i \in K^i \cap (s, \inf A)$, we can then mirror the above argument to obtain two infinite intertwined *increasing* sequences $(\chi_n^1)_{n\geq 1}$ and $(\chi_n^2)_{n\geq 1}$ in K^1 and K^2, respectively, converging to $\inf A$, and such that for $i = 1, 2$, $\operatorname{supp} \mu^i \cap [\chi_1^i, \sup_{n\geq 1} \chi_n^i) = \{\chi_n^i : n \geq 1\}$, a contradiction by Lemma A.2. We conclude that $\inf A = s$ and thus that $A = (s, x_R]$. The claim follows.

We finally claim that $s = \alpha^1 \vee \alpha^2$. Notice first that $s \geq \alpha^1 \vee \alpha^2$ by Lemma S.3.3(ii) in Online Supplement S.3. Now, suppose, by way of contradiction that $s > \alpha^1 \vee \alpha^2$ and $s \in S^i$. Then, by Proposition 1(i), $s \notin S^j$, so that $\bar{J}^j(s) = G^j(s)$. But $\bar{J}^j(s+) = R^j(s) < G^j(s)$ as $(s, x_{R^j}] \subset \operatorname{supp} \mu^j$ and $s > \alpha^j$, a contradiction as $\bar{J}^j$ is continuous by Proposition 2. The claim follows.

We have thus shown that, if $A \neq \emptyset$, then $A = (s, x_R]$, with $s = \alpha^1 \vee \alpha^2$ and $x_R = x_{R^1} = x_{R^2}$. By Proposition 1(iii), it follows that, for each $i = 1, 2$, $\bar{J}^i = R^i$ over $(s, x_R]$. Therefore, by Lemma S.2.2, Proposition 1(v), and the strong Markov property, we have

$$\bar{J}^i(x) = f^i(x, \mu^j) \equiv \mathbf{E}_x \left[\int_{[0,\tau_s)} \mathrm{e}^{-rt} G^i(X_t) \, \mathrm{d}\Gamma_t^j + \mathrm{e}^{-r\tau_s} R^i(s)(1 - \Gamma_{\tau_s}^j) \right] = R^i(x) \tag{A.8}$$

for all $x \in (s, x_R]$, where $\Gamma_t^j \equiv 1 - \mathrm{e}^{-\int_{(s,x_R]} L_t^y \, \mu^j(\mathrm{d}y)}$. Notice that the right-hand side of (A.8) does not depend on μ^j, so that neither does $f^i(x, \mu^j)$ in equilibrium for all $x \in (s, x_R]$. Defining the measure $\bar{\mu}^j \equiv \sigma^{-2} \lambda^j \cdot Leb$ over (α^i, β), where $\lambda^j(x) \equiv \frac{rR^i(x) - \mathcal{L}R^i(x)}{G^i(x) - R^i(x)} \mathbb{1}_{\{\alpha^i < x \leq x_R\}}$, it can be verified as in Steg (2015) and Georgiadis, Kim, and Kwon (2022) that, as in Proposition 4, the pair $(((\alpha, \alpha^2], \bar{\mu}^1), ((\alpha, \alpha^1], \bar{\mu}^2))$ is an Mpe with equilibrium value functions (V_{R^1}, V_{R^2}). In particular, Proposition 1 implies that player j's strategy makes player i indifferent between holding fast and conceding over $(\alpha^i, x_{R^i}]$, which, together with the Markov property, implies that $\bar{\mu}^j$ is solution to (A.8).

To conclude, we show that $\mu^j = \bar{\mu}^j_{|\mathcal{B}((s,\beta))}$, that is, (A.8) has a unique solution over the Borel σ-field $\mathcal{B}((s, \beta))$. The strong Markov property implies that, for each $x \in (s, x_R]$ and for every stopping time $\tau < \tau_s$,

$$\begin{aligned} f^i(x, \mu^j) &= \mathbf{E}_x \left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t) \, \mathrm{d}\Gamma_t^j + \mathrm{e}^{-r\tau} f^i(X_\tau, \mu^j) \right] \\ &= \mathbf{E}_x \left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t) \, \mathrm{d}\Gamma_t^j + \mathrm{e}^{-r\tau} R^i(X_\tau) \right]. \end{aligned}$$

Because $f^i(x, \bar{\mu}^j) = f^i(x, \mu^j) = R^i(x)$ and similarly for $f^i(x, \bar{\mu}^j)$, it follows that

$$\mathbf{E}_x \left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^i(X_t) \, \mathrm{d}(\Gamma_t^j - \bar{\Gamma}_t^j) \right] = 0,$$

where $\bar{\Gamma}^j_t \equiv 1 - \mathrm{e}^{-\int_{(s,x_R]} L^y_t \bar{\mu}^j(\mathrm{d}y)}$. Because this equality holds for any stopping time $\tau < \tau_s$, the process $u \mapsto M_u \equiv \int_{[0,u]} \mathrm{e}^{-rt} G^i(X_t)\,\mathrm{d}(\Gamma^j_t - \bar{\Gamma}^j_t)$ is a martingale over $[0, \tau_s)$ (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Therefore, being a continuous process of bounded variation, it is indistinguishable from 0 over $[0, \tau_s)$. As $G^i > V_{R^i} > 0$ by Lemma 1, it follows that the process $u \mapsto \Gamma^j_u - \bar{\Gamma}^j_u = \int_{[0,u]} \frac{\mathrm{e}^{rt}}{G^i(X_t)}\,\mathrm{d}M_t$ is indistinguishable from 0 over $[0, \tau_s)$, so that the processes $u \mapsto \int_{(s,x_{R^i}]} L^y_u\,\mu^j(\mathrm{d}y)$ and $u \mapsto \int_{(s,x_{R^i}]} L^y_u\,\bar{\mu}^j(\mathrm{d}y)$ are indistinguishable from each other over $[0, \tau_s)$. In turn, these two processes can be seen as additive functionals of the diffusion X over (s, β), where s is modified into a killing boundary. This implies that $\mu^j = \bar{\mu}^j$, because both the measure associated to an additive functional of a diffusion and the killing measure of a diffusion are unique (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §4, and Section 4, §23)). Hence the result. ■

Proof of Theorem 3: Hereafter, we suppose with no loss of generality that $i = 2$.

(Necessity) Let $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^{N} a^1_n \delta_{q^1_n}, \emptyset), (\sum_{n=1}^{N-1} a^2_n \delta_{q^2_n}, (\alpha, s^2]))$ be an Mpe of type 2, and consider the brvf $\bar{J}^2$ to (μ^1, S^1). Our goal is to show that $\bar{J}^2$ satisfies the variational system (11)–(17).

We start with some simple observations. First, $\bar{J}^2 \in \mathcal{C}^0(\mathcal{I})$ by Proposition 2, as requested. Second, we know from Proposition 1 that $\bar{J}^2 \geq V_{R^2}$ over $\mathcal{I}$ and from (3) that $V_{R^2} \geq R^2$ over $\mathcal{I}$. Hence $\bar{J}^2$ satisfies (11). Third, $\bar{J}^2 = R^2$ over $S^2 = (\alpha, s^2]$ by Proposition 1(ii). Hence $\bar{J}^2$ satisfies (12). Fourth, $\operatorname{supp} \mu^2 = \{q^2_n : 1 \leq n \leq N-1\} \subset \{x \in \mathcal{I} : \bar{J}^2(x) = R^2(x)\}$ by Proposition 1(iii). Hence $\bar{J}^2$ satisfies (15). Fifth, as in Steps 1 and 3 of the proof of Theorem 2, it can be verified that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q^1_1, β) and (s^2, q^1_N), and that $\bar{J}^2 = T^2_{q^2_n}$ over (q^1_{n+1}, q^1_n) for $1 \leq n \leq N-1$, where T^2_q is the solution to $\mathcal{L}u - ru = 0$ that is tangent to R^2 at q. Hence $\bar{J}^2$ satisfies (16). Sixth, as in Step 3 of the proof of Theorem 2, the fact that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (q^1_1, β) implies that $\bar{J}^2 = A\phi + B\psi$ over this interval for some constants A, B, and the fact that $0 \leq \bar{J}^2 \leq G^2$ together with (A.4) implies that $B = 0$ and thus $\bar{J}^2(\beta-) = 0$. Hence $\bar{J}^2$ satisfies (14). Seventh, and as a result, $\bar{J}^2 \in \mathcal{C}^2(\mathcal{I} \setminus (\{q^1_n : 1 \leq n \leq N\} \cup \{s_2\}))$ and $|\bar{J}^{2\prime -}(x)| \vee |\bar{J}^{2\prime +}(x)| < \infty$ for all $x \in \{q^1_n : 1 \leq n \leq N\} \cup \{s_2\}$, as requested.

Let us now check that $\bar{J}^2$ satisfies (13). Because $\bar{J}^2 \geq R^2$, with equality at s^2, it must be that $\bar{J}^{2\prime +}(s^2) \geq R^{2\prime}(s^2)$. Suppose, by way of contradiction, that this inequality is strict. Consider the stopping time $\tau_\varepsilon \equiv \inf\{t \geq 0 : X_t \notin (s^2 - \varepsilon, s^2 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $\alpha < s^2 - \varepsilon < s^2 + \varepsilon < q^1_N$. Define $f_\varepsilon(x) \equiv \mathbf{E}_x[\mathrm{e}^{-r\tau_\varepsilon} \bar{J}^2(X_{\tau_\varepsilon})]$ for $x \in (s^2 - \varepsilon, s^2 + \varepsilon)$. Recalling that τ_{S^2} is a best reply to (μ^1, S^1) by Proposition 1(v) and invoking the strong Markov property, we obtain that $f_\varepsilon(x)$ is the payoff of player 2 against (μ^1, S^1) when using the non-Markovian stopping time $\tau_\varepsilon + \tau_{S^2} \circ \theta_{\tau_\varepsilon}$ that consists in holding fast up to τ_ε and then conceding the first time X hits S^2 in the continuation game. By construction, $\mathcal{L}f_\varepsilon - rf_\varepsilon = 0$ over $(s^2-\varepsilon, s^2+\varepsilon)$. Applying Lemma A.1(iv) with $i=2$, $a=s^2$, $b=q_N^1$, and $u=\bar{J}^2$, we deduce that $f_\varepsilon(s^2) > \bar{J}^2(s^2)$ for ε sufficiently small, a contradiction as (μ^2, S^2) is a pbr to (μ^1, S^1). Hence $\bar{J}^2$ satisfies (13).

Let us finally check that $\bar{J}^2$ satisfies (17). The following lemma provides two expressions for $\bar{J}^2$ that result from the Markov property and the Itô–Tanaka–Meyer formula.

Lemma A.3 *Let* $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n^1 \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} a_n^2 \delta_{q_n^2}, (\alpha, s^2]))$ *be an Mpe of type 2. Then, for all* $x \in \mathcal{I}$ *and* $\tau \in \mathcal{T}$,

$$\begin{aligned} \bar{J}^2(x) &= \mathbf{E}_x\Bigg[\sum_{n=1}^N \int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt} G^2(q_n^1)\Lambda_t^1 a_n^1 \,\mathrm{d}L_t^{q_n^1} \\ &\qquad + \mathbb{1}_{\{\tau_{S^2}<\tau\}}\, \mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}} + \mathbb{1}_{\{\tau_{S^2}\geq\tau\}}\, \mathrm{e}^{-r\tau} \bar{J}^2(X_\tau)\Lambda^1_\tau\Bigg] \qquad (A.9) \\ &= \mathbf{E}_x\Bigg[\sum_{n=1}^N \int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n^1 - \frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1 \,\mathrm{d}L_t^{q_n^1} \\ &\qquad + \mathbb{1}_{\{\tau_{S^2}<\tau\}}\, \mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}} + \mathbb{1}_{\{\tau_{S^2}\geq\tau\}}\, \mathrm{e}^{-r\tau} \bar{J}^2(X_\tau)\Lambda^1_\tau\Bigg]. \qquad (A.10) \end{aligned}$$

An immediate implication of (A.9)–(A.10) is that, for each $\tau \in \mathcal{T}$,

$$\begin{aligned} &\mathbf{E}_x\Bigg[\sum_{n=1}^N \int_{[0,\tau)} \mathbb{1}_{\{\tau_{S^2}>t\}}\, \mathrm{e}^{-rt} G^2(q_n^1)\Lambda_t^1 a_n^1 \,\mathrm{d}L_t^{q_n^1}\Bigg] \\ &\qquad = \mathbf{E}_x\Bigg[\sum_{n=1}^N \int_{[0,\tau)} \mathbb{1}_{\{\tau_{S^2}>t\}}\, \mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n^1 - \frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1 \,\mathrm{d}L_t^{q_n^1}\Bigg]. \end{aligned}$$

Equivalently, for each $\tau \in \mathcal{T}$, $\mathbf{E}_x[M_\tau] = \mathbf{E}_x[M_0] = 0$, where

$$M_t \equiv \sum_{n=1}^N \int_{[0,t)} \mathbb{1}_{\{\tau_{S^2}>s\}}\, \mathrm{e}^{-rs}\left\{a_n^1[G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1)\right\}\Lambda_s^1 \,\mathrm{d}L_s^{q_n^1} \qquad (A.11)$$

for all $t \geq 0$. It follows that the process $(M_t)_{t\geq 0}$ is a martingale (Revuz and Yor (1999, Chapter II, §3, Proposition 3.5)). Because it is a continuous process of bounded variation, it must then be that, for each $\tau \in \mathcal{T}$, $M_\tau = M_0 = 0$ $\mathbf{P}_x$-a.s. Now, suppose, by way of contradiction, that $a_n^1[G^2(q_n^1) - \bar{J}^2(q_n^1)] + \frac{1}{2}\,\Delta\bar{J}^{2\prime}(q_n^1) \neq 0$ for some n such that $1 \leq n \leq N$. Let $x \equiv q_n^1$ and $\tau_\varepsilon \equiv \inf\{t \geq 0 : X_t \notin (q_n^1 - \varepsilon, q_n^1 + \varepsilon)\}$, where $\varepsilon > 0$ is such that $q_{n+1}^1 < q_n^1 - \varepsilon < q_n^1 + \varepsilon < q_{n-1}^1$, with $q_0^1 \equiv \beta$ and $q_{N+1}^1 \equiv s^2$ by convention. From the properties of local time, we have that, for each $t > 0$, $L_t^{q_n^1} > 0$ $\mathbf{P}_{q_n^1}$-a.s. (see, for instance, Revuz and Yor (1999, Chapter VI, §2, Proof of Proposition 2.5)). It then follows from (A.11) that $M_{\tau_\varepsilon} \neq 0$ $\mathbf{P}_{q_n^1}$-a.s., a contradiction. Hence $\bar{J}^2$ satisfies (17). This completes the proof that $\bar{J}^2$ satisfies the variational system (11)–(17). The proof that $\bar{J}^1$ satisfies the variational system (18)–(23) is similar, and is omitted for the sake of brevity.

(Sufficiency) That the variational system (11)–(23) characterizes players' value functions in Mpes of type 2 is an immediate consequence of the following verification lemma.

Lemma A.4 *Let w^1 and w^2 be solutions to the variational systems (18)–(23) and (11)–(17), respectively, for some $N \in \mathbb{N} \setminus \{0\}$, four sequences $(q^1_n)^N_{n=1}$, $(q^2_n)^{N-1}_{n=0}$, $(a^1_n)^N_{n=1}$, $(a^2_n)^{N-1}_{n=0}$, and a number s^2 as in the statement of Theorem 3. Then, for each $i = 1, 2$,*

$$w^i(x) \geq \sup_{\tau \in \mathcal{T}} J^i(x, \tau, (\mu^j, S^j)), \tag{A.12}$$

$$w^i(x) = J^i(x, (\mu^1, S^1), (\mu^2, S^2)), \tag{A.13}$$

where $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^{N} a^1_n \delta_{q^1_n}, \emptyset), (\sum_{n=1}^{N-1} a^2_n \delta_{q^2_n}, (\alpha, s^2]))$.

(Refinement) On the one hand, we have $\alpha^1 \leq \alpha^2 \leq s^2$, where the second inequality follows from Lemma S.3.3(ii) in Online Supplement S.3, and thus it can be easily checked that, for each $x \in \mathcal{I}$, $J^1(x, (\mu^1, (\alpha, \alpha^1]), (\mu^2, S^2)) = J^1(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^1(x)$, which implies that $(\mu^1, (\alpha, \alpha^1])$ is a pbr to (μ^2, S^2). On the other hand, using (5) along with the fact that $G^2 = R^2$ over $(\alpha, \alpha^1]$ as $\alpha^1 \leq \alpha^2$, it is easily checked that, for all $x \in \mathcal{I}$ and $\tau^2 \in \mathcal{T}$, $J^2(x, (\mu^1, (\alpha, \alpha^1]), \tau^2) = J^2(x, (\mu^1, \emptyset), \tau^2 \wedge \tau_{(\alpha, \alpha^1]}) \leq \bar{J}^2(x)$ and $J^2(x, (\mu^1, (\alpha, \alpha^1]), (\mu^2, S^2)) = J^2(x, (\mu^1, \emptyset), (\mu^2, S^2)) = \bar{J}^2(x)$, which implies that (μ^2, S^2) is a pbr to $(\mu^1, (\alpha, \alpha^1])$. Hence the result. ■

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Supplement to "The War of Attrition under Uncertainty: Theory and Robust Testable Implications": Additional Proofs

Jean-Paul Décamps* Fabien Gensbittel† Thomas Mariotti‡

December 18, 2025

Abstract

Section S.1 provides useful preliminary results. Section S.2 gathers proofs of lemmas that appear elsewhere in the literature or follow directly from existing results. Section S.3 provides the proofs of Propositions 1 and 2. Section S.4 proves two key lemmas for Theorem 2. Section S.5 proves two key lemmas for Theorem 3. Section S.6 provides the proof of Theorem 4. Section S.7 provides the proof of Theorem 5. Section S.8 gathers proofs of results in Section 6.

*Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: `jean-paul.decamps@tse-fr.eu`.

†Toulouse School of Economics, University of Toulouse Capitole, Toulouse, France. E-mail: `fabien.gensbittel@tse-fr.eu`.

‡Toulouse School of Economics, CNRS, University of Toulouse Capitole, Toulouse, France, CEPR, and CESifo. Email: `thomas.mariotti@tse-fr.eu`.

S.1 Preliminaries

S.1.1 The Fundamental Filtration

We start with some definitions (Revuz and Yor (1999, Chapter I, §4)). The process X is defined over the canonical space $(\Omega, \mathcal{F})$ of continuous trajectories, and $\mathbf{P}_\mu$ denotes the law of the process X given an initial distribution $\mu \in \Delta(\mathcal{I})$, where $\Delta(\mathcal{I})$ is the space of Borel probability measures over $\mathcal{I}$. We denote by $(\mathcal{F}_t^0)_{t\geq 0}$ the natural filtration $(\sigma(X_s; s \leq t))_{t\geq 0}$ generated by X, and we let $\mathcal{F}_\infty^0 \equiv \sigma(\bigcup_{t\geq 0} \mathcal{F}_t^0)$. For each $\mu \in \Delta(\mathcal{I})$, we denote by $\mathcal{F}_\infty^\mu$ the completion of $\mathcal{F}_\infty^0$ with respect to $\mathbf{P}_\mu$, and, for each $t \geq 0$, we let $\mathcal{F}_t^\mu$ be the augmentation of $\mathcal{F}_t^0$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}_\infty^\mu$-measurable sets. The usual augmented filtration $(\mathcal{F}_t)_{t\geq 0}$ is then defined by $\mathcal{F}_t \equiv \bigcap_{\mu\in\Delta(\mathcal{I})} \mathcal{F}_t^\mu$ for all $t \geq 0$. Because the process X is a Feller process in the sense of Revuz and Yor (1999, Chapter III, §2, Definition 2.5) and a standard process in the sense of Blumenthal and Getoor (1968, Chapter I, Definition 9.2), the filtration $(\mathcal{F}_t)_{t\geq 0}$ is actually right-continuous. As usual in this literature, we say that a property of the trajectories $\omega \in \Omega$ is satisfied *almost surely* if it is satisfied $\mathbf{P}_\mu$-almost surely for all $\mu \in \mathcal{I}$ or, equivalently, $\mathbf{P}_x$-almost surely for all $x \in \mathcal{I}$.

S.1.2 A Useful Change of Variables

Dayanik and Karatzas (2003) introduced an elegant change of variables that we use in several proofs. Specifically, for each $x \in \mathcal{I}$, define $\zeta(x) \equiv \frac{\phi(x)}{\psi(x)}$, which is strictly decreasing in x and maps $\mathcal{I}$ onto $(0, \infty)$. Then, for any function $g : \mathcal{I} \to \mathbb{R}$, define the function $\hat{g}$ by

$$\hat{g}(y) \equiv \frac{g}{\psi} \circ \zeta^{-1}(y), \quad y \in (0, \infty). \tag{S.1}$$

Observe that $\hat{\phi}(y) = y$ and $\hat{\psi}(y) = 1$ for all $y \in (0, \infty)$. A direct computation shows that, if $g \in \mathcal{C}^2(\mathcal{I})$, then

$$\hat{g}''(\zeta(x)) = \frac{2\psi(x)^3}{[\varrho\sigma(x)p'(x)]^2} (\mathcal{L}g - rg)(x), \quad x \in \mathcal{I}, \tag{S.2}$$

where p is the scale function of the diffusion X, which is uniquely defined up to an affine transformation by

$$p(x) \equiv \int_c^x \exp\left(-\int_c^y \frac{2\mu(z)}{\sigma^2(z)}\, \mathrm{d}z\right) \mathrm{d}y, \quad x \in \mathcal{I}, \tag{S.3}$$

for some fixed $c \in \mathcal{I}$ (Karatzas and Shreve (1991, Chapter 5, Section 5, §B)), and

$$\varrho \equiv \frac{\psi'(x)\phi(x) - \psi(x)\phi'(x)}{p'(x)} > 0, \tag{S.4}$$

the ratio of the Wronskian of ψ and ϕ and of the derivative of the scale function, is a constant independent of x by Abel's theorem. From A3 and (S.2), we deduce that $\hat{R}^{i\prime\prime}(\zeta(x)) < 0$ for all $x \in (\alpha, x_0^i)$ or, equivalently, that $\hat{R}^{i\prime\prime}(y) < 0$ for all $y \in (\zeta(x_0^i), \infty)$ and thus, in particular, for all $y \in (\zeta(x_{R^i}), \infty)$ as $x_{R^i} < x_0^i$. From A7 and (S.2), we deduce that $\hat{G}^{i\prime\prime} \leq 0$ everywhere $\hat{G}^{i\prime\prime}$ is defined. Another useful remark is that, from Lemma 1 and A6, we have $G^i > 0$ over $\mathcal{I}$. Thus, $\hat{G}^i > 0$ over $(0, \infty)$, and (A.4) implies

$$\lim_{y \to 0} \hat{G}^i(y) = \lim_{y \to \infty} \frac{\hat{G}^i(y)}{y} = 0. \tag{S.5}$$

S.2 Basic Lemmas

Proof of Lemma 1: The proof proceeds along the same lines as in Décamps, Gensbittel, and Mariotti (2025, Lemma 1). The result follows. ■

Proof of Lemma 2: By A0 and (4), $G^i = V_{R^i} = R^i$ over $(\beta, \alpha^i]$, and by A7 $G^i > R^i = V_{R^i}$ over $(\alpha^i, x_{R^i}]$. For each $x \in (x_{R^i}, \beta)$, we have

$$G^i(x) \geq \mathbf{E}_x[\mathrm{e}^{-r\tau_{x_{R^i}}} G^i(x_{R^i})] > \mathbf{E}_x[\mathrm{e}^{-r\tau_{x_{R^i}}} R^i(x_{R^i})] = V_{R^i}(x),$$

where the first inequality follows from the fact that $(\mathrm{e}^{-rt} G^i(X_t))_{t \geq 0}$ is a supermartingale by A6, the second inequality follows from A0 and A7, and the equality follows from (4). Thus $G^i > V_{R^i} > R^i$ over (x_{R^i}, β). The result follows. ■

Lemma S.2.1 *The ccdf process Γ^i is $(\mathcal{F}_t)_{t \geq 0}$-adapted and, for all $x \in \mathcal{I}$ and $t \geq 0$,*

$$\Gamma_t^i = \mathbf{P}_x^i[\gamma^i \leq t \,|\, \mathcal{F}_t] \;\; \mathbf{P}_x\text{-}a.s. \tag{S.6}$$

Proof: For each $\mu \in \Delta(\mathcal{I})$, ω and u^i are independent under $\mathbf{P}_\mu^i \equiv \mathbf{P}_\mu \otimes Leb$, and hence, for each $t \geq 0$,

$$\Gamma_t^i(\omega) = \mathbf{P}_\mu^i[\gamma^i \leq t \,|\, \mathcal{F}](\omega)$$

for $\mathbf{P}_\mu$-almost every $\omega \in \Omega$. We may assume that $\gamma(\cdot, u^i) \in \mathcal{T}$ for all u^i, as we can replace γ^i by the constant stopping time 0 for all u^i in a Borel set of Lebesgue measure zero without modifying the process Γ^i. Therefore, for all $u^i \in [0,1]$ and $t \geq 0$, we have $\{\omega \in \Omega : \gamma^i(\omega, u^i) \leq t\} \in \mathcal{F}_t$ as $\gamma(\cdot, u^i) \in \mathcal{T}$. It follows from Solan, Tsirelson, and Vieille (2012, Corollary 2) that this implies that Γ_t^i is measurable with respect to the augmentation of $\mathcal{F}_t$ by the $\mathbf{P}_\mu$-null, $\mathcal{F}_\infty^\mu$-measurable sets, which coincides with $\mathcal{F}_t^\mu$. As this is true for all $\mu \in \Delta(\mathcal{I})$, we deduce that Γ^i is adapted with respect to $\mathcal{F}_t$. In particular, letting $\mu \equiv \delta_x$ yields

$$\Gamma_t^i(\omega) = \mathbf{P}_x^i[\gamma^i \leq t \,|\, \mathcal{F}_t](\omega)$$

for $\mathbf{P}_x$-almost every $\omega \in \Omega$ by the law of iterated expectations. The result follows. ■

Lemma S.2.2 *If the players use randomized stopping times with ccdfs* Γ^1 *and* Γ^2*, then their expected payoffs write as* (5)*. Moreover, any nondecreasing, right-continuous,* $\mathcal{F}_t$*-adapted,* $[0,1]$*-valued process* Γ^i *is the ccdf of the randomized stopping time* $\hat{\gamma}^i$ *defined by*

$$\hat{\gamma}^i(u^i) \equiv \inf\{t \geq 0 : \Gamma^i_t > u^i\}. \tag{S.7}$$

PROOF: Suppose that, for each $i = 1, 2$, γ^i is a randomized stopping time with ccdf Γ^i. We have

$$\begin{aligned}
\overline{\mathbf{E}}_x\left[\mathbb{1}_{\{\gamma^i \leq \gamma^j\}}\, \mathrm{e}^{-r\gamma^i} R^i(X_{\gamma^i})\right] &= \int_0^1 \int_0^1 \mathbf{E}_x\left[\mathbb{1}_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}}\, \mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)})\right] \mathrm{d}u^j\, \mathrm{d}u^i \\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \int_0^1 \mathbb{1}_{\{\gamma^i(u^i) \leq \gamma^j(u^j)\}}\, \mathrm{d}u^j\right] \mathrm{d}u^i \\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda^j_{\gamma^i(u^i)-}\right] \mathrm{d}u^i \\
&= \mathbf{E}_x\left[\int_0^1 \mathrm{e}^{-r\gamma^i(u^i)} R^i(X_{\gamma^i(u^i)}) \Lambda^j_{\gamma^i(u^i)-}\, \mathrm{d}u^i\right] \\
&= \mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt} R^i(X_t) \Lambda^j_{t-}\, \mathrm{d}\Gamma^i_t\right],
\end{aligned}$$

where the second and fourth equalities follow from Fubini's theorem, and the third equality follows from the definition of Λ^j. The last equality follows from observing that, for $\mathbf{P}_x$-almost every $\omega \in \Omega$, $t \mapsto \Gamma^i_t(\omega)$ is the cdf of the random variable $\gamma^i(\omega, \cdot)$ defined on the probability space $([0,1], \mathcal{B}([0,1]), \mathit{Leb})$ and taking values in $[0, \infty]$, where $\Gamma^i_\infty(\omega) \equiv 1$ by convention; Fubini's theorem then implies that the random variable $u^i \mapsto \mathrm{e}^{-r\gamma^i(\omega,u^i)} R^i(X_{\gamma^i(\omega,u^i)}) \Lambda^j_{\gamma^i(\omega,u^i)-}$ is Lebesgue integrable over $[0,1]$ for $\mathbf{P}_x$-almost every $\omega \in \Omega$,[1] and we can thus apply the usual formula for the expectation. The proof for the terms in G^i appearing in (5) is similar and thus omitted.

Let us then verify that (S.7) defines a randomized stopping time in the sense of Definition 1. That $\hat{\gamma}^i(u^i) \in \mathcal{T}$ for *Leb*-almost every $u^i \in [0,1]$ is standard (Jacod and Shiryaev (1994, Proposition I.1.28)). The random variable $(\omega, u^i) \mapsto \hat{\gamma}^i(u^i)(\omega)$ is $\mathcal{F}_\infty \otimes \mathcal{B}([0,1])$-measurable as it is nondecreasing and right-continuous with respect to u^i. That the ccdf associated to $\hat{\gamma}^i$ is Γ^i is proven in De Angelis, Ferrari, and Moriarty (2018, Lemma 4.1), who use this representation as the definition of a randomized stopping time. The result follows. ■

PROOF OF LEMMA 3: We focus on player 1, the proof for player 2 being symmetrical. Observe from (5) that, for each $\tau^1 \in \mathcal{T}$, player 1's payoff from playing τ^1 against Γ^2 is

$$J^1(x, \tau^1, \Gamma^2) = \mathbf{E}_x\left[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) \Lambda^2_{\tau^1-} + \int_{[0,\tau^1)} \mathrm{e}^{-rt} G^1(X_t)\, \mathrm{d}\Gamma^2_t\right]. \tag{S.8}$$

[1]Recall that, by convention, this random variable is equal to 0 if $\gamma^i(\omega, u^i) = \infty$.

Letting $\hat{\gamma}^1$ be the randomized stopping time associated to the ccdf Γ^1 by (S.7), we have

$$\begin{aligned} J^1(x,\Gamma^1,\Gamma^2) &= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r\hat{\gamma}^1(u^1)}R^1(X_{\hat{\gamma}^1(u^1)})\Lambda^2_{\hat{\gamma}^1(u^1)-} + \int_{[0,\hat{\gamma}^1(u^1))} \mathrm{e}^{-rt}G^1(X_t)\,\mathrm{d}\Gamma^2_t\right]\mathrm{d}u^1 \\ &= \int_0^1 J^1(x,\hat{\gamma}^1(u^1),\Gamma^2)\,\mathrm{d}u^1 \\ &\leq \sup_{u^1\in[0,1]} J^1(x,\hat{\gamma}^1(u^1),\Gamma^2) \\ &\leq \sup_{\tau^1\in\mathcal{T}} J^1(x,\tau^1,\Gamma^2). \end{aligned}$$

where the first equality follows along the same steps as in the proof of Lemma S.2.2, and the second equality follows from (S.8). The result follows. ■

The following consequence of the strong Markov property will be used several times throughout this Online Supplement.

Lemma S.2.3 *If the players use Markovian randomized stopping times with ccdfs* (Γ^1,Γ^2), *then, for all* $x\in\mathcal{I}$ *and* $\tau\in\mathcal{T}$, *their expected payoffs write as*

$$\begin{aligned} J^i(x,\Gamma^1,\Gamma^2) = \mathbf{E}_x\bigg[\int_{[0,\tau)} \mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\tau)} \mathrm{e}^{-rt}G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t \\ + \mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-}\bigg]. \end{aligned} \tag{S.9}$$

Proof: It follows from Lemma S.2.2 that

$$\begin{aligned} J^i(x,\Gamma^1,\Gamma^2) = \mathbf{E}_x\bigg[&\int_{[0,\tau)} \mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{[0,\tau)} \mathrm{e}^{-rt}G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t \\ &+ \mathrm{e}^{-r\tau}R^i(X_\tau)\Lambda^j_{\tau-}(\Gamma^i_\tau-\Gamma^i_{\tau-}) + \mathrm{e}^{-r\tau}G^i(X_\tau)\Lambda^i_\tau(\Gamma^j_\tau-\Gamma^j_{\tau-}) \\ &+ \int_{(\tau,\infty)} \mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{(\tau,\infty)} \mathrm{e}^{-rt}G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\bigg]. \end{aligned} \tag{S.10}$$

Notice from (7) that the only jump of Λ^i occurs at τ_{S^i}, at which time Λ^i jumps down to 0 and remains there forever after, and similarly for Λ_j. Hence

$$\begin{aligned} &\mathrm{e}^{-r\tau}R^i(X_\tau)\Lambda^j_{\tau-}(\Gamma^i_\tau-\Gamma^i_{\tau-}) + \mathrm{e}^{-r\tau}G^i(X_\tau)\Lambda^i_\tau(\Gamma^j_\tau-\Gamma^j_{\tau-}) \\ &= \mathbb{1}_{\{\tau_{S^j}\geq\tau=\tau_{S^i}\}}\,\mathrm{e}^{-r\tau}R^i(X_\tau)\Lambda^j_{\tau-}\Lambda^i_{\tau-} + \mathbb{1}_{\{\tau_{S^i}>\tau=\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}G^i(X_\tau)\Lambda^i_{\tau-}\Lambda^j_{\tau-} \\ &= \mathbb{1}_{\{\tau_{S^j}\geq\tau=\tau_{S^i}\}}\,\mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-} + \mathbb{1}_{\{\tau_{S^i}>\tau=\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^i_{\tau-}\Lambda^j_{\tau-} \\ &= \mathbb{1}_{\{\tau\geq\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-}, \end{aligned} \tag{S.11}$$

where the last equality follows from the fact that $\mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^i_{\tau-}\Lambda^j_{\tau-}$ vanishes over $\{\tau>\tau_{S^i}\wedge\tau_{S^j}\}$. On the other hand, we have

$$\int_{(\tau,\infty)} \mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t + \int_{(\tau,\infty)} \mathrm{e}^{-rt}G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t$$

$$= \mathbb{1}_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\bigg[\int_{(0,\infty)}\mathrm{e}^{-rt}R^i(X_{\tau+t})\Lambda^j_{(\tau+t)-}\,\mathrm{d}\Gamma^i_{\tau+t}+\int_{(0,\infty)}\mathrm{e}^{-rt}G^i(X_{\tau+t})\Lambda^i_{\tau+t}\,\mathrm{d}\Gamma^j_{\tau+t}\bigg]$$

$$= \mathbb{1}_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\Lambda^j_\tau\Lambda^i_\tau\bigg[\int_{(0,\infty)}\mathrm{e}^{-rt}R^i(X_t\circ\theta_\tau)(\Lambda^j_{t-}\circ\theta_\tau)\,\mathrm{d}(\Gamma^i_t\circ\theta_\tau)$$

$$+\int_{(0,\infty)}\mathrm{e}^{-rt}G^i(X_t\circ\theta_\tau)(\Lambda^i_t\circ\theta_\tau)\,\mathrm{d}(\Gamma^2_t\circ\theta_\tau)\bigg]$$

$$= \mathbb{1}_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}\Lambda^j_{\tau-}\Lambda^i_{\tau-}\bigg[\int_{[0,\infty)}\mathrm{e}^{-rt}R^i(X_t\circ\theta_\tau)(\Lambda^j_{t-}\circ\theta_\tau)\,\mathrm{d}(\Gamma^i_t\circ\theta_\tau)$$

$$+\int_{[0,\infty)}\mathrm{e}^{-rt}G^i(X_t\circ\theta_\tau)(\Lambda^i_t\circ\theta_\tau)\,\mathrm{d}(\Gamma^j_t\circ\theta_\tau)\bigg],$$

where the second equality follows from (6). Taking expectations and applying the strong Markov property at τ yields

$$\mathbf{E}_x\bigg[\int_{(\tau,\infty)}\mathrm{e}^{-rt}R^i(X_t)\Lambda^j_{t-}\,\mathrm{d}\Gamma^i_t+\int_{(\tau,\infty)}\mathrm{e}^{-rt}G^i(X_t)\Lambda^i_t\,\mathrm{d}\Gamma^j_t\bigg]$$

$$= \mathbf{E}_x\big[\mathbb{1}_{\{\tau<\tau_{S^i}\wedge\tau_{S^j}\}}\,\mathrm{e}^{-r\tau}J^i(X_\tau,\Gamma^1,\Gamma^2)\Lambda^j_{\tau-}\Lambda^i_{\tau-}\big]. \qquad \text{(S.12)}$$

Inserting (S.11) and (S.12) into (S.10) yields (S.9). The result follows. ■

S.3 Proofs of Propositions 1 and 2

Proof of Proposition 1: Suppose, with no loss of generality, that $i=1$. We first prove that $V_{R^1}\le\bar{J}^1\le G^1$. For the first inequality, let $\tau^1\equiv\tau_{(\alpha,x_{R^1}]}$, the hitting time by X of $(\alpha,x_{R^1}]$, and let $\hat{\gamma}^2(u)$ be defined by (S.7). Using Lemma S.2.2 and $G^1\ge V_{R^1}$ by A6, we obtain

$$\begin{aligned}\bar{J}^1(x)&\ge J^1(x,\tau^1,\Gamma^2)\\&=\int_0^1 J^1(x,\tau^1,\hat{\gamma}^2(u))\,\mathrm{d}u\\&\ge\int_0^1\mathbf{E}_x\Big[\mathbb{1}_{\{\tau^1\le\hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1}R^1(X_{\tau^1})+\mathbb{1}_{\{\tau^1>\hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)}V_{R^1}(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u\end{aligned}$$

for all $x\in\mathcal{I}$. For each $u\in[0,1]$, we have

$$\mathrm{e}^{-r\hat{\gamma}^2(u)}V_{R^1}(X_{\hat{\gamma}^2(u)})=\mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1}R^1(X_{\tau^1})\,|\,\mathcal{F}_{\hat{\gamma}^2(u)}\Big]$$

$\mathbf{P}_x$-almost surely over $\{\tau^1>\hat{\gamma}^2(u)\}$. Thus, by the tower property of conditional expectation,

$$\mathbf{E}_x\Big[\mathbb{1}_{\{\tau^1\le\hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1}R^1(X_{\tau^1})+\mathbb{1}_{\{\tau^1>\hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)}V_{R^1}(X_{\hat{\gamma}^2(u)})\Big]=\mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1}R^1(X_{\tau^1})\Big]=V_{R^1}(x),$$

and we conclude that, for each $x\in\mathcal{I}$,

$$\bar{J}^1(x)\ge\int_0^1 V_{R^1}(x)\,\mathrm{d}u=V_{R^1}(x).$$

For the second inequality, we have $R^1 \leq V_{R^1} \leq G^1$ by A6. Hence, for each $\tau^1 \in \mathcal{T}$,

$$\begin{aligned}
J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))\,\mathrm{d}u \\
&\leq \int_0^1 \mathbf{E}_x\Big[\mathbb{1}_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} G^1(X_{\tau^1}) + \mathbb{1}_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u \\
&= \int_0^1 \mathbf{E}_x\Big[\mathrm{e}^{-r(\tau^1 \wedge \hat{\gamma}^2(u))} G^1(X_{\tau^1 \wedge \hat{\gamma}^2(u)})\Big]\,\mathrm{d}u \\
&\leq \int_0^1 G^1(x)\,\mathrm{d}u \\
&= G^1(x)
\end{aligned}$$

for all $x \in \mathcal{I}$, where the second inequality follows from the fact that $(\mathrm{e}^{-rt} G^1(X_t))_{t \geq 0}$ is a supermartingale by A7. We now prove properties (i)–(v) in turn.

(i) It is not optimal for player 1 to concede at $x \in S^2$ if $R^1(x) < G^1(x)$, that is, if $x > \alpha^1$. Therefore, if (μ^1, S^1) is a pbr to (μ^2, S^2), then $S^1 \cap S^2 \cap (\alpha^1, \beta) = \emptyset$.

(ii) This directly follows from the definition (5) of players' payoffs.

(iii) By Lemma S.2.2, for each $x \in \operatorname{supp} \mu^1$, we have

$$\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2)\,\mathrm{d}u,$$

where $\hat{\gamma}^1(u) = \inf\{t \geq 0 : \Gamma^1_t > u\}$. Thus the inequality $J^1(x, \hat{\gamma}^1(u), \Gamma^2) \leq \bar{J}^1(x)$, which holds for all $u \in [0,1]$, must be an equality for all u in a set U of Lebesgue measure 1. By definition of Γ^1, $\hat{\gamma}^1(u) = \inf\{t \geq 0 : 1 - \mathrm{e}^{-\int_{\mathcal{I}\setminus S^1} L^y_t\,\mu^1(\mathrm{d}y)} > u\} \wedge \tau_{S^1}$ for all $u \in [0,1)$. Notice that $\hat{\gamma}^1(u) > 0$ $\mathbf{P}_x$-almost surely for all $u \in (0,1)$ as the mapping $t \mapsto \mathbb{1}_{\{t < \tau_{S^1}\}}\,\mathrm{e}^{-\int_{\mathcal{I}\setminus S^1} L^y_t\,\mu^1(\mathrm{d}y)}$ is continuous over $[0, \tau_{S^1})$ by Theorem 1. We claim that, because $x \in \operatorname{supp} \mu^1$, we also have $\lim_{u \to 0} \hat{\gamma}^1(u) = 0$ $\mathbf{P}_x$-almost surely. Indeed, $\hat{\gamma}^1(u, \omega)$ is nondecreasing in u for all ω and converges to $\hat{\gamma}^1(0, \omega) = \inf\{t \geq 0 : \int_{\mathcal{I}\setminus S^1} L^y_t(\omega)\,\mu^1(\mathrm{d}y) > 0\} \wedge \tau_{S^1}(\omega)$. Let us fix a continuous version $(t, y) \mapsto L^y_t$ of the local time of X (Revuz and Yor (1999, Chapter VI, §1, Theorem 1.7)), and observe that $L^x_t > 0$ $\mathbf{P}_x$-almost surely for all $t > 0$. Thus there exist a sequence $(t_n)_{n \geq 1}$ converging to 0 and, for each $n \geq 1$, a set $\Omega_{t_n} \in \mathcal{F}$ of $\mathbf{P}_x$-probability 1 such that $L^x_{t_n}(\omega) > 0$ and $y \mapsto L^y_{t_n}(\omega)$ is continuous at x for all $\omega \in \Omega_{t_n}$. Now, $x \in \operatorname{supp} \mu^1$ and $\operatorname{supp} \mu^1$ being closed jointly imply that any open interval of $\mathcal{I}$ containing x has positive μ^1-measure. From these observations, it follows that, for each $n \geq 1$, $\int_{\mathcal{I}\setminus S^1} L^y_{t_n}(\omega)\,\mu^1(\mathrm{d}y) > 0$ for all $\omega \in \Omega_{t_n}$, so that $\hat{\gamma}^1(0, \omega) = 0$ for all $\omega \in \bigcap_{n \geq 1} \Omega_{t_n}$ and thus $\mathbf{P}_x$-almost surely, as claimed. Finally, for each $u \in U$,

$$\bar{J}^1(x) = J^1(x, \hat{\gamma}^1(u), \Gamma^2) = \mathbf{E}_x\left[\int_{[0, \hat{\gamma}^1(u))} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\hat{\gamma}^1(u)} R^1(X_{\hat{\gamma}^1(u)}) \Lambda^2_{\hat{\gamma}^1(u)-}\right]. \quad \text{(S.13)}$$

Using bounded convergence to take the limit as $u \in U$ goes to 0, two cases must be distinguished. If $x \notin S^2$, then Γ^2_t is continuous at $t = 0$, from which it follows that

$\bar{J}^1(x) = R^1(x)$. If $x \in S^2$, then $\Gamma^2_{0-} = 0$, $\Gamma^2_0 = 1$, and $\Lambda^2_{\hat{\gamma}^1(u)-} = 0$ for all $u \in (0,1)$, from which it follows that $\bar{J}^1(x) = G^1(x)$.

(iv) We claim that, for each $x \in \mathcal{I}$,

$$\bar{J}^1(x) \geq J^1(x, (0, (\alpha, x_{R^1}]), (\mu^2, S^2)) \geq J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)). \tag{S.14}$$

The first inequality in (S.14) directly follows from the fact that (μ^1, S^1) is a pbr to (μ^2, S^2). For the second one, recall that, by A6,

$$G^1(x) \geq V_{R^1}(x) = \sup_{\tau \in \mathcal{T}} \mathbf{E}_x\big[\mathrm{e}^{-r\tau} R^1(X_\tau)\big] = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1})\Big],$$

where $\tau^1 \equiv \tau_{(\alpha, x_{R^1}]}$. We have

$$\begin{aligned} J^1(x, \tau^1, \Gamma^2) &= \int_0^1 J^1(x, \tau^1, \hat{\gamma}^2(u))\,\mathrm{d}u \\ &= \int_0^1 \mathbf{E}_x\Big[\mathbb{1}_{\{\tau^1 \leq \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) + \mathbb{1}_{\{\tau^1 > \hat{\gamma}^2(u)\}}\,\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)})\Big]\,\mathrm{d}u. \end{aligned}$$

Over $\{\tau^1 > \hat{\gamma}^2(u)\}$, we have

$$\mathrm{e}^{-r\hat{\gamma}^2(u)} G^1(X_{\hat{\gamma}^2(u)}) \geq \mathrm{e}^{-r\hat{\gamma}^2(u)} V_{R^1}(X_{\hat{\gamma}^2(u)}) = \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1}) \,|\, \mathcal{F}_{\hat{\gamma}^2(u)}\Big].$$

$\mathbf{P}_x$-almost surely by A6. Therefore, using the tower property of conditional expectation,

$$J^1(x, \tau^1, \hat{\gamma}^2(u)) \geq \mathbf{E}_x\Big[\mathrm{e}^{-r\tau^1} R^1(X_{\tau^1})\Big],$$

which implies the second inequality of (S.14) upon integrating with respect to u. The conclusion follows from noticing that $J^1(x, (0, (\alpha, x_{R^1}]), (0, \emptyset)) = V_{R^1}(x) > R^1(x)$ for all $x > x_{R^1}$ and applying (ii) and the first assertion in (iii).

(v) Arguing as in (iii) yields that $\bar{J}^1(x) = \int_0^1 J^1(x, \hat{\gamma}^1(u), \Gamma^2)\,\mathrm{d}u$ for all u in a set U of Lebesgue measure 1. Moreover, using the explicit expression for $\hat{\gamma}^1(u)$ given in (iii), it is easy to check that $\lim_{u \to 1} \hat{\gamma}^1(u) = \tau_{S^1}$. Therefore, taking the limit in (S.13) as $u \in U$ goes to 1, we deduce that

$$\bar{J}^1(x) = \mathbf{E}_x\left[\int_{[0, \tau_{S^1})} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_{S^1}} R^1(X_{\tau_{S^1}}) \Lambda^2_{\tau_{S^1}-}\right] = J^1(x, \tau_{S^1}, \Gamma^2)$$

by bounded convergence, from which the first assertion follows. For the second assertion, let $\tilde{\Gamma}^1$ be the ccdf associated to $(\tilde{\mu}^1, S^1)$ and $\tilde{\gamma}^1(u) = \inf\{t \geq 0 : \tilde{\Gamma}^1_t > u\}$. By assumption,

$$\bar{J}^1(X_{\tilde{\gamma}^1(u)}) = R^1(X_{\tilde{\gamma}^1(u)}). \tag{S.15}$$

for all $u \in [0, 1]$. On the one hand,

$$J^1(x, \tilde{\Gamma}^1, \Gamma^2) = \int_0^1 \mathbf{E}_x\left[\int_{[0, \tilde{\gamma}^1(u))} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\gamma}^1(u)} R^1(X_{\tilde{\gamma}^1(u)}) \Lambda^2_{\tilde{\gamma}^1(u)-}\right] \mathrm{d}u. \tag{S.16}$$

On the other hand, using that $\bar{J}^1 = J^1(\cdot, \tau_{S^1}, \Gamma^2)$ and applying the strong Markov property at $\tilde{\gamma}^1(u)$ in (5) yields

$$\bar{J}^1(x) = \mathbf{E}_x\left[\int_{[0,\tilde{\gamma}^1(u))} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tilde{\gamma}^1(u)} \bar{J}^1(X_{\tilde{\gamma}^1(u)})\Lambda^2_{\tilde{\gamma}^1(u)-}\right]$$
$$= \mathbf{E}_x\left[\int_{[0,\tilde{\gamma}^1(u))} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tilde{\gamma}^1(u)} R^1(X_{\tilde{\gamma}^1(u)})\Lambda^2_{\tilde{\gamma}^1(u)-}\right], \quad \text{(S.17)}$$

where the second equality follows from (S.15). Integrating (S.17) with respect to u yields (S.16), from which the second assertion follows. Hence the result. ■

Proof of Proposition 2: Our argument requires some technical results on processes $A \equiv (A_t)_{t\geq 0}$ of the form $A_t \equiv \int_{\mathcal{I}\setminus S} L_t^x\,\mu(\mathrm{d}x)$, where $S \subset \mathcal{I}$ is a closed set and μ is a Radon measure over $\mathcal{I} \setminus S$. Precisely, if τ is the first exit time of $(a, b) \subset \mathcal{I} \setminus S$, with $[a, b] \subset \mathcal{I}$, then

$$\mathbf{E}_x[A_\tau] = \int_{\mathcal{I}\setminus S} \mathbf{E}_x[L_\tau^y]\,\mu(\mathrm{d}y) = \int_{(a,b)} \mathbf{E}_x[L_\tau^y]\,\mu(\mathrm{d}y) = \int_{(a,b)} 2[p'(y)]^{-1}\Phi_{a,b}(x, y)\,\mu(\mathrm{d}y), \quad \text{(S.18)}$$

where p' is the derivative of the scale function (S.3) of the diffusion X, and

$$\Phi_{a,b}(x, y) \equiv \frac{[p(x \wedge y) - p(a)][p(b) - p(x \vee y)]}{p(b) - p(a)}$$

is the Green function of the diffusion X killed at the boundaries a and b (Borodin and Salminen (2002, Part I, Chapter II, Section 1, §11, and Section 2, §13)). It is easy to check that $\mathbf{E}_x[A_\tau]$ is finite if and only if, for some $x \in (a, b)$,

$$\int_a^x [p(y) - p(a)]\,\mu(\mathrm{d}y) < \infty \quad \text{and} \quad \int_x^b [p(b) - p(y)]\,\mu(\mathrm{d}y) < \infty.$$

A more precise result can be stated as follows (Çetin (2018, Theorem 2.1)):

$$A_{\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}} = \infty \text{ a.s. if } \int_a^x [p(y) - p(a)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a, b), \quad \text{(S.19)}$$
$$A_{\tau_a}\mathbb{1}_{\{\tau_a<\tau_b\}} < \infty \text{ a.s. otherwise.} \quad \text{(S.20)}$$

A symmetric result holds for b. The following lemma is key to our continuity result.

Lemma S.3.1 *For each $t \geq 0$, let $A_t \equiv \int_{(a,b)} L_t^y\,\mu(\mathrm{d}y)$ for some Radon measure μ over $(a, b) \subset \mathcal{I}$. Then the function h defined, for nonnegative constants C_a and C_b, by*

$$h(x) = \mathbf{E}_x\left[C_a\mathbb{1}_{\{\tau_a<\tau_b\}}\,\mathrm{e}^{-A_{\tau_a}} + C_b\mathbb{1}_{\{\tau_b<\tau_a\}}\mathrm{e}^{-A_{\tau_b}}\right], \quad x \in (a, b),$$

is nonnegative, p-convex,[2] *and continuous over (a, b). Moreover, the limits $h(a+)$ and $h(b-)$*

[2]That is,

$$h(\lambda x_1 + (1-\lambda)x_2) \leq h(x_1)\,\frac{p(x_2) - p(\lambda x_1 + (1-\lambda)x_2)}{p(x_2) - p(x_1)} + h(x_2)\,\frac{p(\lambda x_1 + (1-\lambda)x_2) - p(x_1)}{p(x_2) - p(x_1)}$$

for all $x_1, x_2 \in (a, b)$ and $\lambda \in [0, 1]$.

exist and are given by

$$h(a+) = \begin{cases} 0 & \text{if } \int_a^x [p(y) - p(a)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a,b) \\ C_a & \text{otherwise} \end{cases}, \tag{S.21}$$

$$h(b-) = \begin{cases} 0 & \text{if } \int_x^b [p(b) - p(y)]\,\mu(\mathrm{d}y) = \infty \text{ for some } x \in (a,b) \\ C_b & \text{otherwise} \end{cases}. \tag{S.22}$$

PROOF: First, h is clearly nonnegative. Next, applying the strong Markov property to $h(\lambda x_1 + (1-\lambda)x_2)$ at $\tau_{x_1} \wedge \tau_{x_2}$ yields

$$h(\lambda x_1 + (1-\lambda)x_2) = \mathbf{E}_{\lambda x_1 + (1-\lambda)x_2}\Big[h(x_1)\mathbb{1}_{\{\tau_{x_1} < \tau_{x_2}\}}\,\mathrm{e}^{-A_{\tau_{x_1}}} + h(x_2)\mathbb{1}_{\{\tau_{x_2} < \tau_{x_1}\}}\,\mathrm{e}^{-A_{\tau_{x_2}}}\Big].$$

Using that $\mathrm{e}^{-A_t} \leq 1$, we then obtain from standard computations (Karatzas and Shreve (1991, Chapter 5, Section 5, §C)) that h is p-convex. Finally, that h is continuous follows from its being p-convex (Revuz and Yor (1999, Appendix, §3)).

Consider now (S.21). If $\int_a^x [p(y) - p(a)]\,\mu(\mathrm{d}y) = \infty$ for some $x \in (a,b)$, then by (S.19) $h(x) = \mathbf{E}_x\big[C_b \mathbb{1}_{\{\tau_b < \tau_a\}}\,\mathrm{e}^{-A_{\tau_b}}\big]$ and thus $0 \leq h(x) \leq C_b \mathbf{P}_x[\tau_b < \tau_a]$, which goes to 0 as x goes to a. Hence $h(a+) = 0$. If $\int_a^x [p(y) - p(a)]\,\mu(\mathrm{d}y) < \infty$ for some $x \in (a,b)$, then by (S.20) $\mathrm{e}^{-A_{\tau_a}} > 0$ $\mathbf{P}_x$-almost surely. If $(a_n)_{n \geq 1}$ is a decreasing sequence converging to a and strictly bounded above by x, then, applying the strong Markov property to $h(x)$ at τ_{a_n}, we have

$$h(x) = \mathbf{E}_x\Big[h(a_n)\mathbb{1}_{\{\tau_{a_n} < \tau_b\}}\,\mathrm{e}^{-A_{\tau_{a_n}}} + C_b \mathbb{1}_{\{\tau_b < \tau_{a_n}\}}\,\mathrm{e}^{-A_{\tau_b}}\Big].$$

Using bounded convergence to take the limit along any subsequence $(h(a_{n_k}))_{k \geq 1}$ converging to some $z < \infty$, we obtain that

$$h(x) = \mathbf{E}_x\Big[z \mathbb{1}_{\{\tau_a < \tau_b\}}\,\mathrm{e}^{-A_{\tau_a}} + C_b \mathbb{1}_{\{\tau_b < \tau_{a_n}\}}\,\mathrm{e}^{-A_{\tau_b}}\Big],$$

and thus $z = C_a$ as $\mathbf{E}_x\big[\mathbb{1}_{\{\tau_a < \tau_b\}}\,\mathrm{e}^{-A_{\tau_a}}\big] > 0$. It follows that $\lim_{n \to \infty} h(a_n) = C_a$. Because this is true for any decreasing sequence $(a_n)_{n \geq 0}$ converging to a, this implies that $h(a+)$ exists and is equal to C_a. This concludes the proof of (S.21). The argument for (S.22) proceeds along similar lines, using (S.20). The result follows. ∎

The proof of Proposition 2 relies on two preliminary lemmas.

Lemma S.3.2 *If (μ^i, S^i) is a pbr to (μ^j, S^j) with associated brvf $\bar{J}^i$, then the restriction of $\bar{J}^i$ to $[a,b]$ is continuous for any interval $[a,b]$ such that $(a,b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$.*

PROOF: Suppose, with no loss of generality, that $i = 1$. Given $x \notin S^1 \cup S^2$, and for each integer $n \geq 1$, let $\tilde{\tau}_n \equiv \tau_{x-\eta} \wedge \tau_{x+\varepsilon_n}$, where $\eta > 0$, $(\varepsilon_n)_{n \geq 1}$ is a decreasing sequence converging to 0, and $[x-\eta, x+\varepsilon_n] \subset \mathcal{I} \setminus (S^1 \cup S^2)$. Applying Lemma S.2.3 with $\tau \equiv \tilde{\tau}_n$ yields

$\bar{J}^1(x)$

$$= \mathbf{E}_x\Bigg[\int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} R^1(X_t)\Lambda^2_{t-}\,\mathrm{d}\Gamma^1_t + \int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} G^1(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\tau}_n}\bar{J}^1(X_{\tilde{\tau}_n})\Lambda^2_{\tilde{\tau}_n-}\Lambda^1_{\tilde{\tau}_n-}\Bigg]$$

$$= \mathbf{E}_x\Bigg[\int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} R^1(X_t)\Lambda^2_{t}\,\mathrm{d}\Gamma^1_t + \int_{[0,\tilde{\tau}_n)} \mathrm{e}^{-rt} G^1(X_t)\Lambda^1_t\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tilde{\tau}_n}\bar{J}^1(X_{\tilde{\tau}_n})\Lambda^2_{\tilde{\tau}_n}\Lambda^1_{\tilde{\tau}_n}\Bigg], \quad \text{(S.23)}$$

where the second equality follows from the fact that $\Lambda^i_{t-} = \Lambda^i_t$ over $\{t \leq \tilde{\tau}_n\}$. Consider a subsequence $(\bar{J}^1(x + \varepsilon_{n_k}))_{k\geq 1}$ converging to some z. Because η is fixed, $\tilde{\tau}_{n_k}$ goes to 0 $\mathbf{P}_x$-almost surely as k goes to ∞, and $\mathbf{P}_x[\tilde{\tau}_{n_k} = \tau_{x+\varepsilon_{n_k}}]$ goes to 1. The equality $X_{\tilde{\tau}_{n_k}} = \mathbb{1}_{\{\tilde{\tau}_{n_k}=\tau_{x-\eta}\}}(x-\eta) + \mathbb{1}_{\{\tilde{\tau}_{n_k}=\tau_{x+\varepsilon_{n_k}}\}}(x+\varepsilon_{n_k})$ then implies that $\bar{J}^1(X_{\tilde{\tau}_{n_k}})$ goes to z $\mathbf{P}_x$-almost surely as k goes to infinity. Using bounded convergence to take the limit in (S.23), and taking advantage of the fact that both Γ^1_t and Γ^2_t are continuous at $t = 0$ as $x \notin S^1 \cup S^2$, we obtain that $\bar{J}^1(x) = z$, from which it follows as in the proof of Lemma S.3.1 that $\bar{J}^1$ is right-continuous at x. The proof that $\bar{J}^1$ is left-continuous at x is similar.

Now, let us consider an interval $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$. That $\bar{J}^1$ is continuous over (a, b) follows from the preceding argument; but we need to check that $\bar{J}^1$ is right-continuous at a and left-continuous at b. We focus on a, the arguments for b being symmetrical. Because $S^1 \cup S^2$ is closed, the only difficulty arises when $a \in S^1 \cup S^2$. We distinguish two cases.

Case 1 Suppose first that $a \in S^1$, so that $\bar{J}^1(a) = R^1(a)$ by Proposition 1(ii). By Proposition 1(v), $\bar{J}^1 = J^1(\cdot, (0, S^1), (\mu^2, S^2))$. Applying Lemma S.2.3 with $\tau \equiv \tau_a \wedge \tau_b$ yields

$$\bar{J}^1(x) = \mathbf{E}_x\Bigg[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\Bigg]$$

for all $x \in (a, b)$. Moreover,

$$0 \leq \mathbf{E}_x\Bigg[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t\Bigg] \leq C\,\mathbf{E}_x\big[1 - \mathrm{e}^{-A^2_\tau}\big],$$

where C is an upper bound for G^1 over $[a, b]$. Because $a \notin S^2$, μ^2 is locally finite at a. Applying Lemma S.3.1 with $C_a = C_b \equiv 1$ and $\mu \equiv \mu^2$ then yields that $\mathbf{E}_x\big[1 - \mathrm{e}^{-A^2_\tau}\big]$ goes to 0 as $x > a$ goes to a. Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.3.1 also yields that $\mathbf{E}_x\big[\mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\big]$ goes to $\bar{J}^1(a) = R^1(a)$ as $x > a$ goes to a. Thus $\bar{J}^1$ is right-continuous at a.

Case 2 Suppose next that $a \in S^2$, so that $\bar{J}^1(a) = G^1(a)$ by Proposition 1(i) and (iii). Fix some $\varepsilon \in (0, b - a)$. As in Case 1 with $\tau \equiv \tau_a \wedge \tau_{a+\varepsilon}$, we have

$$\bar{J}^1(x) = \mathbf{E}_x\Bigg[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\Bigg]$$

for all $x \in (a, a+\varepsilon)$. If $\int_a^x [p(y) - p(a)]\,\mu^2(\mathrm{d}y) < \infty$, the proof proceeds along the same lines as in Case 1. Thus let us assume that $\int_a^x [p(y) - p(a)]\,\mu^2(\mathrm{d}y) = \infty$. Letting $\mu \equiv \mu^2 + rLeb$, Lemma S.3.1 yields that $\mathbf{E}_x\big[\mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\bar{J}^1(X_\tau)\big]$ goes to 0 as $x > a$ goes to a. Moreover,

$$\mathbf{E}_x\Bigg[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma^2_t\Bigg] \geq \min_{y\in[a,a+\varepsilon]} G^1(y)\,\mathbf{E}_x\big[\mathrm{e}^{-r\tau} - \mathrm{e}^{-r\tau}\mathrm{e}^{-A^2_\tau}\big].$$

By (A.2) and Lemma S.3.1, the last expectation goes to 1 as $x > 0$ goes to a. We deduce that $\liminf_{x\to a+} \bar{J}^1(x) \geq \min_{y\in[a,a+\varepsilon]} G^1(y)$ and thus that $\liminf_{x\to a+} \bar{J}^1(x) \geq G^1(a)$ by letting ε go to zero. Finally, we also have $\limsup_{x\to a+} \bar{J}^1(x) \leq G^1(a)$ as $\bar{J}^1 \leq G^1$ by Proposition 1, and this concludes the proof that $\bar{J}^1$ is right-continuous at a. The result follows. ■

Lemma S.3.3 *The following holds:*

(i) If (μ^i, S^i) is a pbr to (μ^j, S^j), then $(\alpha, \alpha^i] \subset S^1 \cup S^2$;

(ii) If $\alpha^1 < \alpha^2$ and $((\mu^1, S^1), (\mu^2, S^2))$ is an Mpe, then S^1 and S^2 cannot both intersect $(\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2]$, so that either $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^1$ or $[\alpha^1 \wedge \alpha^2, \alpha^1 \vee \alpha^2] \subset S^2$.

Proof: (i) Suppose, with no loss of generality, that $i = 1$, and recall that $\bar{J}^1 = R^1 = V_{R^1} = G^1$ over $(\alpha, \alpha^1]$. Suppose, by way of contradiction, that $x \in (\alpha, \alpha^1) \setminus (S^1 \cup S^2)$. Let $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$, with $b < \alpha^1$ and $x \in (a, b)$. Because $(0, S^1)$ is also a pbr to (μ^2, S^2) by Proposition 1(v), applying Lemma S.2.3 with $\tau \equiv \tau_{\mathcal{I}\setminus(a,b)}$ yields

$$\begin{aligned}
\bar{J}^1(x) &= \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\, \mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tau} \bar{J}^1(X_\tau)\Lambda_{\tau-}^2\right] \\
&= \mathbf{E}_x\left[\int_{[0,\tau)} \mathrm{e}^{-rt} R^1(X_t)\, \mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda_{\tau-}^2\right] \\
&= \int_0^1 \mathbf{E}_x\left[\mathbb{1}_{\{\hat{\gamma}^2(u,\cdot)<\tau\}}\, \mathrm{e}^{-r\hat{\gamma}^2(u,\cdot)} R^1(X_{\hat{\gamma}^2(u,\cdot)}) + \mathbb{1}_{\{\hat{\gamma}^2(u,\cdot)\geq\tau\}}\, \mathrm{e}^{-r\tau} R^1(X_\tau)\right] \mathrm{d}u \\
&= \int_0^1 \mathbf{E}_x\left[\mathrm{e}^{-r(\hat{\gamma}^2(u,\cdot)\wedge\tau)} R^1(X_{\hat{\gamma}^2(u,\cdot)\wedge\tau})\right] \mathrm{d}u \\
&< R^1(x),
\end{aligned} \tag{S.24}$$

where the third equality follows along the same lines as in Lemma S.2.2, and the inequality follows from A3 together with the fact that, for each $u > 0$, $\tau \wedge \hat{\gamma}^2(u, \cdot) > 0$ $\mathbf{P}_x$-almost surely as Γ^2 is continuous over $[0, \tau_{S^2})$ and $\tau_{S^2} > 0$ $\mathbf{P}_x$-almost surely. By (S.24), $J(x) < R^1(x)$, in contradiction with Proposition 1. Therefore, $(\alpha, \alpha^1) \subset S^1 \cup S^2$, from which (i) follows as $S^1 \cup S^2$ is closed.

(ii) Suppose, with no loss of generality, that $\alpha^1 < \alpha^2$. By Proposition 1(i), $S^1 \cap S^2 \cap (\alpha^1, \alpha^2] = \emptyset$, and, as shown in (i), $(\alpha^1, \alpha^2] \subset S^1 \cup S^2$. It follows that $S^1 \cap (\alpha^1, \alpha^2]$ and $S^2 \cap (\alpha^1, \alpha^2]$, which are both relatively closed sets in $(\alpha^1, \alpha^2]$, are complementary sets in $(\alpha^1, \alpha^2]$, and thus are both relatively open in $(\alpha^1, \alpha^2]$. As their union $(\alpha^1, \alpha^2]$ is a connected set, either one or the other must be empty. Thus either $(\alpha^1, \alpha^2] \subset S^1$ or $(\alpha^1, \alpha^2] \subset S^2$, from which (ii) follows as both S^1 and S^2 are closed sets. The result follows. ■

We are now ready to complete the proof of Proposition 2. We focus on the right-continuity of the functions $\bar{J}^i$, $i = 1, 2$, the arguments for their left-continuity being symmetrical. For any function $J : \mathcal{I} \to \mathbb{R}$ and for each $S \subset \mathcal{I}$, we denote by $J_{|S}$ the restriction of J to S. Suppose, with no loss of generality, that $\alpha^1 \leq \alpha^2$. For each $i = 1, 2$, $R^i = G^i$ over $(\alpha, \alpha^i]$ and $R^i \leq \bar{J}^i \leq G^i$ by Proposition 1. Thus $\bar{J}^i$ is continuous over $(\alpha, \alpha^i]$ and, in particular, over $(\alpha, \alpha^1]$. Moreover, by Lemma S.3.3(ii), J^1 coincides with R^1 or G^1 over $(\alpha^1, \alpha^2]$. We conclude that, for each $i = 1, 2$, $\bar{J}^i_{|(\alpha,\alpha^2]}$ is continuous. Notice that $\bar{J}^2$ is right-continuous at α^2 and that the same is true for $\bar{J}^1$ if $\alpha^1 = \alpha^2$. By Lemma S.3.2, for each $i = 1, 2$, $\bar{J}^i_{|[a,b]}$ is continuous for any interval $[a, b]$ such that $(a, b) \subset \mathcal{I} \setminus (S^1 \cup S^2)$; moreover, $\bar{J}^i_{|S^i} = R^i_{|S^i}$ and $\bar{J}^i_{|S^j} = G^i_{|S^j}$ are also continuous. Therefore, if $\bar{J}^1$ or $\bar{J}^2$ is not right-continuous at x, it must be that $x \geq \alpha^2$, that $x \in S^1 \cup S^2$, and that, for each $\varepsilon > 0$, $[x, x + \varepsilon)$ intersects both $S^1 \cup S^2$ and $\mathcal{I} \setminus (S^1 \cup S^2)$; we refer to this last property as Property P. We distinguish two cases.

Case 1 Let us first consider the case where $x \in S^2$ and $x > \alpha^2$, and suppose, by way of contradiction, that $\bar{J}^1$ or $\bar{J}^2$ is not right-continuous at x, so that Property P is satisfied. As $(\alpha^2, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^1$. Hence, because S^1 is closed, there exists $\varepsilon > 0$ such that $[x, x + \varepsilon) \cap S^1 = \emptyset$. If (a, b) is a connected component of the open set $[x, x+\varepsilon) \setminus S^2$, so that $a, b \in S^2$, then it must be that $\mu^1[(a, b)] > 0$. Indeed, suppose, by way of contradiction, that this is not the case. Then, for each $y \in (a, b)$, we have

$$\bar{J}^2(y) = J^2(y, (0, S^1), (\mu^2, S^2)) = \mathbf{E}_y\left[\mathrm{e}^{-r\tau_{S^2}} R^2(X_{\tau_{S^2}})\right] < R^2(y)$$

by A3 as $b \leq x_{R^2}$ by Proposition 1(iv), in contradiction with Proposition 1. Thus $\mu^1[(a, b)] > 0$ and, by Proposition 1(iii), there exists some $y \in (a, b)$ such that $\bar{J}^1(y) = R^1(y)$. As this is true for every connected component of $[x, x + \varepsilon) \setminus S^2$, Property P implies that there exists a decreasing sequence $(y_n)_{n\geq 1}$ converging to x such that $\bar{J}^1(y_n) = R^1(y_n)$, as well as a sequence of connected components $((a_n, b_n))_{n\geq 1}$ of $[x, x + \varepsilon) \setminus S^2$ such that $y_n \in (a_n, b_n)$ for all $n \geq 1$ and whose length goes to zero as n goes to ∞. By Proposition 1, $\bar{J}^1(a_n) = G^1(a_n)$ and $\bar{J}^1(b_n) = G^1(b_n)$. Because $x > \alpha^1$, $G^1(x) > R^1(x)$. For each $n \geq 1$, because $(0, S^2)$ is a best reply to (μ^1, S^1) by Proposition 1(v), applying Lemma S.2.3 to $\tau_n \equiv \tau_{a_n} \wedge \tau_{b_n}$ yields

$$\begin{aligned}\bar{J}^1(y_n) &= \mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt} G^1(X_t)\, \mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_n} \bar{J}^1(X_{\tau_n})\Lambda^2_{\tau_n-}\right] \\ &= \mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt} G^1(X_t)\, \mathrm{d}\Gamma^2_t + \mathrm{e}^{-r\tau_n} G^1(X_{\tau_n})\Lambda^2_{\tau_n-}\right].\end{aligned}$$

G^1 and R^1 being locally Lipschitz, there exists $\varepsilon > 0$ such that, for any sufficiently large n,

$$G^1(y) > R^1(y_n) + \varepsilon, \quad y \in (a_n, b_n).$$

Hence, for any such n,

$$\bar{J}^1(y_n) \geq [R^1(y_n) + \varepsilon]\, \mathbf{E}_{y_n}\left[\int_{[0,\tau_n)} \mathrm{e}^{-rt}\, \mathrm{d}\Gamma^2_s + \mathrm{e}^{-r\tau_n}\Lambda^2_{\tau_n-}\right] \geq [R^1(y_n) + \varepsilon]\, \mathbf{E}_{y_n}[\mathrm{e}^{-r\tau_n}].$$

We have $\mathbf{E}_{y_n}[e^{-r\tau_n}] = A_n\phi(y_n) + B_n\psi(y_n)$, where the coefficients A_n and B_n are such that

$$A_n\phi(a_n) + B_n\psi(a_n) = A_n\phi(b_n) + B_n\psi(b_n) = 1.$$

If follows that these coefficients are bounded, and, therefore, as ϕ and ψ are locally Lipschitz, that $\mathbf{E}_{y_n}[\mathrm{e}^{-r\tau_n}]$ goes to 1 as n goes ∞. This, for n sufficiently large, contradicts the fact that $\bar{J}^1(y_n) = R^1(y_n)$. Thus $\bar{J}^1$ and $\bar{J}^2$ are right-continuous at x. The right-continuity of $\bar{J}^1$ and $\bar{J}^2$ at x in case $x \in S^1$ and $x > \alpha^2$ and the right-continuity of $\bar{J}^1$ at x in case $x \in S^2$ and $x = \alpha^2 > \alpha^1$ can be proven in a similar way.

Case 2 It remains only to prove that $\bar{J}^1$ is right-continuous at x in case $x \in S^1$ and $x = \alpha^2 > \alpha^1$. Suppose that Property P is satisfied so that $\bar{J}^1$ may not be right-continuous at x. As $(\alpha^1, \beta) \cap S^1 \cap S^2 = \emptyset$ by Proposition 1(i), $x \notin S^2$. Hence, because S^2 is closed, there exists $\varepsilon > 0$ such that $[x, x+\varepsilon) \cap S^2 = \emptyset$. Notice that $\mu^2([x, x+\varepsilon)) < \infty$ as μ^2 is locally finite on $\mathcal{I} \setminus S^2$. If (a, b) is a connected component of the open set $(x, x+\varepsilon) \setminus S^1$, so that $a, b \in S^1$, then, for $y \in (a, b)$ and $\tau \equiv \tau_a \wedge \tau_b$, we have

$$\begin{aligned}\bar{J}^1(y) - R^1(y) &= J^1(y, (0, S^1), (\mu^2, S^2)) - R^1(y) \\ &= \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda_{\tau-}^2\right] - R^1(y) \\ &\geq 0, \end{aligned} \tag{S.25}$$

where the first equality follows from Proposition 1(v). We also have

$$\begin{aligned}&\bar{J}^1(y) - R^1(y) \\ &\quad = \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} G^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda_{\tau-}^2\right] - R^1(y) \\ &\quad = \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt} R^1(X_t)\,\mathrm{d}\Gamma_t^2 + \mathrm{e}^{-r\tau} R^1(X_\tau)\Lambda_{\tau-}^2 - R^1(y) + \int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma_t^2\right] \\ &\quad = \int_0^1 \mathbf{E}_y\left[\mathrm{e}^{-r[\tau\wedge\hat{\gamma}^2(u,\cdot)]} R^1(X_{\tau\wedge\hat{\gamma}^2(u,\cdot)}) - R^1(y)\right]\mathrm{d}u + \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}(G^1(X_t) - R^1(X_t))\,\mathrm{d}\Gamma_t^2\right] \\ &\quad = \int_0^1 \mathbf{E}_y\left[\int_0^{\tau\wedge\hat{\gamma}^2(u,\cdot)} (\mathcal{L}R^1 - rR^1)(X_t)\,\mathrm{d}t\right]\mathrm{d}u + \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma_t^2\right] \\ &\quad \leq \mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma_t^2\right], \end{aligned} \tag{S.26}$$

where the third equality follows along the same lines as in the proof of Lemma S.2.2, the fourth equality follows from Itô's formula, and the inequality follows from A3 and Proposition 1(iv). Letting $C > 0$ be an upper bound for $G^1 - R^1$ over $[x, x+\varepsilon)$, we then have

$$\begin{aligned}\mathbf{E}_y\left[\int_{[0,\tau)} \mathrm{e}^{-rt}[G^1(X_t) - R^1(X_t)]\,\mathrm{d}\Gamma_t^2\right] &\leq C\,\mathbf{E}_y[\Gamma_\tau^2] \\ &= C\,\mathbf{E}_y[1 - \Lambda_\tau^2]\end{aligned}$$

$$
\begin{aligned}
&= C\,\mathbf{E}_y[1-\mathrm{e}^{-A_\tau^2}] \\
&\leq C\,\mathbf{E}_y[A_\tau^2].
\end{aligned}
$$

From (S.18), we have, for some positive constant C',

$$
\mathbf{E}_y[A_\tau^2] = \int_a^b 2[p'(z)]^{-1}\Phi_{a,b}(y,z)\,\mu^2(\mathrm{d}z) \leq C'\mu^2[(a,b)],
$$

as the mapping $z \mapsto 2[p'(z)]^{-1}\Phi_{a,b}(y,z)$ is uniformly bounded over $[x, x+\varepsilon)$. Property P implies that there exists a sequence $((a_n,b_n))_{n\in\mathbb{N}}$ of connected components of $[x,x+\varepsilon)\setminus S^1$ whose length goes to zero as n goes to ∞. Because μ^2 is locally bounded at x, it must be that $\mu^2[(a_n,b_n)]$ goes to 0 as n goes to ∞, and the inequalities $0 \leq \bar{J}^1(y) - R^1(y) \leq C\,C'\,\mu^2[(a_n,b_n)]$ along with the fact that the constants C and C' are independent of n imply that $\bar{J}^1$ is right-continuous at x. Hence the result. ■

S.4 Proofs of Lemmas for Theorem 2

Proof of Lemma A.1: Recall that any solution $u \in \mathcal{C}^2((a,b))$ to the ODE $\mathcal{L}u - ru = 0$ is of the form $u = A\phi + B\psi$ for some constants A and B. Whenever needed, we use the change of variables (S.1) to reexpress the assumptions and the conclusions of (i)–(iv). For instance, $u(x) \geq V_{R^i}(x)$ for all $x \in (a,b)$ if and only if $\hat{u}(z) = Az + B \geq \hat{V}_{R^i}(z)$ for all $z \in (\zeta(b),\zeta(a))$. Recall also that $\hat{V}_{R^i} \in \mathcal{C}^1((0,\infty))$, that, for some $C^i > 0$, $\hat{V}_{R^i}(z) = C^i z > \hat{R}^i(z)$ for all $z \in (0,\zeta(x_{R^i}))$, and that $\hat{V}_{R^i} = \hat{R}^i$ is $\mathcal{C}^2$ and strictly concave over $[\zeta(x_{R^i}),\infty)$.

(i) The assumption $u(\beta-) = 0$ implies $B = 0$, and thus $\hat{u}(0+) = 0$. The assumption that $\hat{u} \geq \hat{V}_{R^i}$ over $(0,\zeta(a))$ implies $A \geq C^i$. If this inequality were strict, then we would have $Az > C^i z \geq \hat{R}^i(z)$ for all $z > 0$ as $\hat{V}_{R^i}$ is concave, in contradiction to the assumption $\hat{u}(\zeta(a)-) = A\zeta(a) = \hat{R}^i(\zeta(a))$. We conclude that $A = C^i$ and, from the properties of $\hat{V}_{R^i}$, that the unique solution to $Az = \hat{R}^i(z)$ is $\zeta(x_{R^i})$.

(ii) Notice that $V_{R^i} > R^i$ over (x_{R^i},β), so that $\hat{V}_{R^i} > \hat{R}^i$ over $(0,\zeta(x_{R^i}))$. Hence, if there exists $z_0 \in (\zeta(b),\zeta(a))$ such that $\hat{u}(z_0) = \hat{R}^i(z_0)$, then it must be that $z_0 \geq \zeta(x_{R^i})$. In this case, $\hat{u}$ is tangent to the concave $\mathcal{C}^1$ function $\hat{V}_{R^i}$ at z_0. Over $[\zeta(x_{R^i}),\infty)$, $\hat{V}_{R^i} = \hat{R}^i$ is strictly concave. As a result, $\hat{u}(z) > \hat{R}^i(z)$ for all $z \neq z_0$ in $[\zeta(x_{R^i}),\infty) \cap (\zeta(b),\zeta(a))$, and thus for all $z \neq z_0$ in $(\zeta(b),\zeta(a))$ by the preceding remark.

(iii) If $a > \alpha$, then $\hat{u}$ is an affine function over $(\zeta(b),\zeta(a))$ that coincides with $\hat{R}^i$ at both boundaries of this interval. The fact that $\hat{R}^i$ is strictly concave over $[\zeta(x_{R^i}),\infty)$ together with $\zeta(b) \geq \zeta(x_{R^i})$ then implies that $\hat{u} < \hat{R}^i$ over $(\zeta(b),\zeta(a))$. If $a = \alpha$, then $u(a+) = 0$ implies that $u = B\psi$ for some constant B by (A.1), and thus that $\hat{R}^i(\zeta(b)) = \hat{u}(\zeta(b)) = B$. The function $\hat{R}^i$ is strictly concave and, by Lemma 1, positive over $[\zeta(x_{R^i}),\infty)$. It is thus increasing over this interval, which implies that $\hat{u} = B < \hat{R}^i$ over $(\zeta(b),\infty)$.

(iv) The function $\hat{u}$ satisfies $\hat{u}(z) = Az + B$ for all $z \in (\zeta(b), \zeta(a))$ for some constants A and B. A direct computation yields

$$A = \hat{u}'(\zeta(a)-) = \frac{\psi(a)u'(a+) - \psi'(a)u(a)}{\psi(a)^2\zeta'(a)} \text{ and } \hat{R}^{i\prime}(\zeta(a)) = \frac{\psi(a)R^{i\prime}(a) - \psi'(a)R^i(a)}{\psi(a)^2\zeta'(a)},$$

so that, as $u(a) = R^i(a)$, $u'(a+) > R^{i\prime}(a)$, and $\zeta'(a) < 0$,

$$\hat{R}^{i\prime}(\zeta(a)) - A = \frac{R^{i\prime}(a) - u'(a+)}{\psi(a)\zeta'(a)} > 0.$$

Hence $\hat{R}^i(\zeta(a - \varepsilon)) > A\zeta(a - \varepsilon) + B$ for $\varepsilon > 0$ small enough. Similarly, the function $\hat{f}_\varepsilon$ satisfies $\hat{f}_\varepsilon(z) = A'z + B'$ for all $z \in (0, \infty)$ for some constants A' and B'. Moreover, $\hat{f}_\varepsilon(\zeta(a - \varepsilon)) = \hat{R}^i(\zeta(a - \varepsilon))$ and $\hat{f}_\varepsilon(\zeta(a + \varepsilon)) = \hat{u}(\zeta(a + \varepsilon))$. Hence

$$A'\zeta(a + \varepsilon) + B' = A\zeta(a + \varepsilon) + B \text{ and } A'\zeta(a - \varepsilon) + B' > A\zeta(a - \varepsilon) + B,$$

so that $A'\zeta(a) + B' > A\zeta(a) + B$ as $\zeta(a) \in (\zeta(a + \varepsilon), \zeta(a - \varepsilon))$. The result follows. ■

Proof of Lemma A.2: Let T^i_x denote, for each $x \in (\alpha, x_{R^i})$, the unique solution to $\mathcal{L}u - ru = 0$ that is tangent to R^i at x. Then $T^i_x \geq R^i$ over (x_{R^i}, β) and $T^i_x \equiv A_x\phi + B_x\psi$ for some coefficients A_x and B_x. For each $z \geq \zeta(x_{R^i})$, let $\hat{T}^i_z \equiv \widehat{T^i_{\zeta^{-1}(z)}}$ be the affine function tangent to $\hat{R}^i$ at z, which is given by

$$\hat{T}^i_z(y) = A_{\zeta^{-1}(z)}y + B_{\zeta^{-1}(z)} = \hat{R}^i(z) + \hat{R}^{i\prime}(z)(y - z), \quad y \in (0, \infty). \tag{S.27}$$

Recall that $\hat{V}_{R^i} = \hat{R}^i$ over $[\zeta(x_{R^i}), \infty)$ and that $\hat{V}_{R^i}$ is concave over $(0, \infty)$. Because $\hat{T}^i_z$ is tangent to $\hat{V}_{R^i}$ at z, we have $\hat{T}^i_z \geq \hat{V}_{R^i} \geq \hat{R}^i$ over $(0, \infty)$, so that $T^i_x \geq R^i$ over $\mathcal{I}$ for $x = \zeta^{-1}(z)$. Moreover, because, by Lemma 1, $\hat{R}^i$ is strictly concave and positive over $[\zeta(x_{R^i}), \infty)$, we have $A_x = \hat{R}^{i\prime}(z) > 0$. Taking the limit as y goes to 0 in (S.27), we obtain

$$B_x = \hat{R}^i(z) - \hat{R}^{i\prime}(z)z.$$

The right-hand side of this expression is equal to 0 for $z = \zeta(x_{R^i})$, and its derivative $-\hat{R}^{i\prime\prime}(z)$ is positive for $z \in [\zeta(x_{R^i}), \infty)$. Therefore, $B_x > 0$ for $x \in (\alpha, x_{R^i})$.

Now, suppose, by way of contradiction, that $\chi_\infty \equiv \lim_{n\to\infty} \chi^1_n = \lim_{n\to\infty} \chi^2_n > \alpha$. Also suppose, with no loss of generality, that $\chi^1_1 > \chi^2_1$, and let $y_{2n-1} \equiv \zeta(\chi^1_n)$ and $y_{2n} \equiv \zeta(\chi^2_n)$ for all $n \geq 1$. Because $(\chi^i_n)_{n\geq 1}$ is a sequence in $\operatorname{supp} \mu^i \cap (s, \beta)$ and, hence, in $(\alpha, x_{R^i}]$ by Proposition 1(iv), $(y_n)_{n\geq 1}$ is a sequence in $[\zeta(x_{R^i}), \infty)$. As in Step 3 of the proof of Theorem 2, that player 1 does not stop over the interval (χ^1_{n+1}, χ^1_n) and that $\chi^2_n \in (\chi^1_{n+1}, \chi^1_n)$ belongs to the support of μ^2 implies that $\mathcal{L}\bar{J}^2 - r\bar{J}^2 = 0$ over (χ^1_{n+1}, χ^1_n) and that $\bar{J}^2 \geq V_{R^2}$ and $\bar{J}^2(\chi^2_n) = R^2(\chi^2_n)$. Moreover, as $\bar{J}^2$ is continuous, it coincides with $T^2_{\chi^2_n}$ on $[\chi^1_{n+1}, \chi^1_n]$. It follows that, for each $n \geq 1$, $\bar{J}^2(\chi^1_{n+1}) = T^2_{\chi^2_n}(\chi^1_{n+1}) = T^2_{\chi^2_{n+1}}(\chi^1_{n+1})$, and a similar property holds for $\bar{J}^1$. Using (S.27) to rewrite these equalities yields, for each $n \geq 1$,

$$\hat{R}^1(y_{2n-1}) + \hat{R}^{1\prime}(y_{2n-1})(y_{2n} - y_{2n-1}) = \hat{R}^1(y_{2n+1}) + \hat{R}^{1\prime}(y_{2n+1})(y_{2n} - y_{2n+1}),$$
$$\hat{R}^2(y_{2n}) + \hat{R}^{2\prime}(y_{2n})(y_{2n+1} - y_{2n}) = \hat{R}^2(y_{2n+2}) + \hat{R}^{2\prime}(y_{2n+2})(y_{2n+1} - y_{2n+2}).$$

With $y < y' < y''$ three appropriate consecutive terms of the sequence $(y_n)_{n\geq 1}$, these equalities can be compactly rewritten for each i as

$$\hat{R}^i(y) + \hat{R}^{i\prime}(y)(y' - y) - \hat{R}^i(y') = \hat{R}^i(y'') + \hat{R}^{i\prime}(y'')(y' - y'') - \hat{R}^i(y'). \tag{S.28}$$

Using Taylor's theorem with integral remainder, (S.28) is equivalent to

$$-\int_y^{y'} (y' - z)\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z = -\int_{y'}^{y''} (z - y')\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z. \tag{S.29}$$

Because $\hat{R}^{i\prime\prime} < 0$ over $[y_1, \infty) \subset [\zeta(x_{R^i}), \infty)$, the right-hand side of (S.29) is increasing in y''. Therefore, given $y' > y \geq y_1$, if a solution $y'' > y'$ to (S.29) exists, it is unique. By assumption, $\lim_{n\to\infty} y_n = y_\infty \equiv \zeta(\chi_\infty) < \infty$. Moreover, because $\hat{R}^{i\prime\prime}$ is locally Lipschitz by A8, there exists $K > 0$ such that $|\hat{R}^{i\prime\prime}(z) - \hat{R}^{i\prime\prime}(y')| \leq K|z - y'|$ for all $z, y' \in [y_1, y_\infty]$. Thus

$$-\int_y^{y'} (y' - z)\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z \geq -R^{i\prime\prime}(y')\,\frac{(y' - y)^2}{2} - K\,\frac{(y' - y)^3}{3}, \tag{S.30}$$

$$-\int_{y'}^{y''} (z - y')\hat{R}^{i\prime\prime}(z)\,\mathrm{d}z \leq -R^{i\prime\prime}(y')\,\frac{(y'' - y')^2}{2} + K\,\frac{(y'' - y')^3}{3}. \tag{S.31}$$

By (S.29), we thus have

$$(y'' - y')^2 + \frac{2K}{3|\hat{R}^{i\prime\prime}(y')|}\,(y'' - y')^3 \geq (y' - y)^2 - \frac{2K}{3|\hat{R}^{i\prime\prime}(y')|}\,(y' - y)^3. \tag{S.32}$$

Let C such that, for each $y' \in [y_1, y_\infty]$,

$$\frac{2K}{3|\hat{R}^{i\prime\prime}(y')|} \leq C.$$

Then, by (S.32), we have

$$(y'' - y')^2 + C(y'' - y')^3 \geq (y' - y)^2 - C(y' - y)^3.$$

Letting $u_n \equiv y_{n+1} - y_n$ for all $n \geq 1$, the upshot of the above analysis is that $h(u_{n+1}) \geq g(u_n)$, where $g(u) \equiv u^2 - Cu^3$ and $h(u) \equiv u^2 + Cu^3$. By assumption, $y_1 + \sum_{n\geq 1} u_n = y_\infty < \infty$, which implies that $\lim_{n\to\infty} u_n = 0$. Therefore, for n sufficiently large, $g(u_n) > 0$ and $u_{n+1} \geq h^{-1}(g(u_n))$, where h^{-1} denotes the inverse of h restricted to $[0, \infty)$. Because $h^{-1}(z) = \sqrt{z} - \frac{C}{2}\,z + o(z)$, we have $h^{-1}(g(u)) = u - Cu^2 + o(u^2)$. Hence

$$u_{n+1} \geq u_n - Cu_n^2 + o(u_n^2)$$

and, as a result,

$$\frac{1}{u_{n+1}}-\frac{1}{u_n}\leq\frac{1}{u_n}\left[\frac{1}{1-Cu_n+o(u_n)}-1\right]=C+o(1).$$

We obtain

$$\frac{1}{u_n}=\frac{1}{u_1}+\sum_{k=1}^{n-1}\left(\frac{1}{u_{k+1}}-\frac{1}{u_k}\right)\leq nC+o(n)$$

and thus

$$u_n\geq\frac{1}{nC}+o\left(\frac{1}{n}\right),$$

so that $\sum_{n\geq 1}u_n=\infty$, a contradiction. The case of increasing sequences, whose limit must be in $(\alpha, x_{R^i}]$, can be dealt in a similar way by replacing the inequalities (S.30) and (S.31) by an upper bound and a lower bound of the same type, respectively. The result follows. ■

S.5 Proofs of Lemmas for Theorem 3

Proof of Lemma A.3: From Proposition 1(v), if $((\mu^1,S^1),(\mu^2,S^2))$ is an Mpe, then $(0,S^2)$ is a pbr to (μ^1,S^1). Applying the strong Markov property (S.9) to the value function of player 2 associated to the pair of Markov strategies $((\mu^1,S^1),(0,S^2))$ yields, for all $x\in\mathcal{I}$ and $\tau\in\mathcal{T}$,

$$\begin{aligned}\bar{J}^2(x)=\mathbf{E}_x\Bigg[&\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_{S^2})}\mathrm{e}^{-rt}G^2(q_n^1)\Lambda_t^1a_n^1\,\mathrm{d}L_t^{q_n^1}\\&+\mathbb{1}_{\{\tau_{S^2}<\tau\}}\,\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda_{\tau_{S^2}}^1+\mathbb{1}_{\{\tau_{S^2}\geq\tau\}}\,\mathrm{e}^{-r\tau}\bar{J}^2(X_\tau)\Lambda_\tau^1\Bigg],\end{aligned}$$

where we used that $\mathrm{d}\Gamma_t^1=\sum_{n=1}^N a_n^1\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}$. This proves (A.9).

To prove (A.10), we apply the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r(\tau\wedge\tau_{S^2}\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})$ $\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k}$, where, for each $k\in\mathbb{N}$, $\tau_k\equiv\inf\{t\geq 0: X_t\notin[\alpha_k,\beta_k]\}$ for some increasing sequence $([\alpha_k,\beta_k])_{k\in\mathbb{N}}$ of compacts intervals of $\mathcal{I}$ such that $\bigcup_{k\in\mathbb{N}}[\alpha_k,\beta_k]=\mathcal{I}$. Observe that $\mathbf{E}_x[\tau_k]<\infty$ (Karatzas and Shreve (1991, Chapter 5, Section 5, §C)) and that $X_t\in[\alpha_k,\beta_k]$ over $\{t\leq\tau_k\}$ $\mathbf{P}_x$-almost surely for all $x\in[\alpha_k,\beta_k]$. Moreover, because X does not explode in finite time, $\lim_{k\to\infty}\tau_k=\infty$ and, hence, $\lim_{k\to\infty}\tau\wedge\tau_k=\tau$ for all $\tau\in\mathcal{T}$. We obtain

$$\begin{aligned}\bar{J}^2(x)=\;&\mathrm{e}^{-r(\tau\wedge\tau_{S^2}\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k}-\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\bar{J}^2(X_t)\,\mathrm{d}\Lambda_t^1\\&-\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}[\mathcal{L}\bar{J}^2(X_t)-r\bar{J}^2(X_t)]\prod_{n=1}^{N}\mathbb{1}_{\{X_t\neq q_n^1\}}\,\Lambda_t^1\,\mathrm{d}t\\&-\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\sigma(X_t)\bar{J}^{2\prime}(X_t)\prod_{n=1}^{N}\mathbb{1}_{\{X_t\neq q_n^1\}}\,\Lambda_t^1\,\mathrm{d}W_t\end{aligned}$$

$$-\frac{1}{2}\sum_{n=1}^{N}\Delta\bar{J}^{2\prime}(q_n^1)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}.$$

Taking expectations, we obtain

$$\bar{J}^2(x)=\mathbf{E}_x\Bigg[\mathrm{e}^{-r\tau\wedge\tau_{S^2}\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_{S^2}\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_{S^2}\wedge\tau_k}-\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\bar{J}^2(X_t)\,\mathrm{d}\Lambda_t^1$$
$$-\frac{1}{2}\sum_{n=1}^{N}\Delta\bar{J}^{2\prime}(q_n^1)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\Bigg],$$

where we have used the fact that $\bar{J}^2$ satisfies (16) and that

$$\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\sigma(X_t)\bar{J}^{2\prime}(X_t)\prod_{n=1}^{N}\mathbb{1}_{\{X_t\neq q_n^1\}}\Lambda_t^1\,\mathrm{d}W_t\right]=0. \tag{S.33}$$

Indeed, notice that σ is continuous on I, and that $\bar{J}^2\in\mathcal{C}^1(\mathcal{I}\setminus\{(q_n^1)_{1\leq n\leq N}\})$ with $|\bar{J}^{2\prime}(x+)|<\infty$ and $|\bar{J}^{2\prime}(x-)|<\infty$ for $x\in\{q_n^1:1\leq n\leq N\}$. Thus there exists $C_k>0$ such that $|\sigma(X_t)\bar{J}^{2\prime}(X_t)|\leq C_k$ over $\{t\leq\tau_{S^2}\wedge\tau_k\}$ $\mathbf{P}_x$-almost surely, which implies (S.33). Hence

$$\bar{J}^2(x)=\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}\right]+\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\right]$$
$$+\mathbf{E}_x\left[\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\bar{J}^2(X_t)\Lambda_t^1a_n^1\,\mathrm{d}L_t^{q_n^1}\right]$$
$$-\mathbf{E}_x\left[\frac{1}{2}\sum_{n=1}^{N}\Delta\bar{J}^{2\prime}(q_n^1)\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right].$$

Using that the measure $\mathrm{d}L_t^{q_n^1}$ only charges the set $\{t\geq 0:X_t=q_n^1\}$, we obtain

$$\bar{J}^2(x)=\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}\right]+\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\right]$$
$$+\mathbf{E}_x\left[\sum_{n=1}^{N}\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n^1-\frac{1}{2}\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]. \tag{S.34}$$

By the monotone convergence theorem,

$$\lim_{k\to\infty}\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2}\wedge\tau_k)}\mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n^1-\frac{1}{2}\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]$$
$$=\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_{S^2})}\mathrm{e}^{-rt}\left[\bar{J}^2(q_n^1)a_n^1-\frac{1}{2}\Delta\bar{J}^{2\prime}(q_n^1)\right]\Lambda_t^1\,\mathrm{d}L_t^{q_n^1}\right]$$

for all n, and

$$\lim_{k\to\infty}\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}<\tau\wedge\tau_k\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\right]=\mathbf{E}_x\left[\mathbb{1}_{\{\tau_{S^2}<\tau\}}\mathrm{e}^{-r\tau_{S^2}}R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}}\right].$$

Because $0 \leq \bar{J}^2 \leq G^2$ by Proposition 1, it follows from A4 that the sequence $(\mathbb{1}_{\{\tau\wedge\tau_k \leq \tau_{S^2}\}}$ $\mathrm{e}^{-r\tau\wedge\tau_k}\bar{J}^2(X_{\tau\wedge\tau_k}))_{k\in\mathbb{N}}$ is uniformly integrable. Therefore, by Vitali's convergence theorem,

$$\lim_{k\to\infty} \mathbf{E}_x\big[\mathbb{1}_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\,\mathrm{e}^{-r(\tau\wedge\tau_k)}\,\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}\big] = \mathbf{E}_x\big[\mathbb{1}_{\{\tau_{S^2}\geq\tau\}}\,\mathrm{e}^{-r\tau}\,\bar{J}^2(X_\tau)\Lambda^1_\tau\big].$$

Finally, $\mathbb{1}_{\{\tau_{S^2}\geq\tau\wedge\tau_k\}}\,\mathrm{e}^{-r(\tau\wedge\tau_k)}\bar{J}^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k} = \mathbb{1}_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\bar{J}^2(X_{\tau_k})\Lambda^1_{\tau_k}$ over $\{\tau=\infty\}$. For k large enough, $x\in(\alpha_k,\beta_k)$. Hence

$$\begin{aligned}
&\mathbf{E}_x\big[\mathbb{1}_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\,\bar{J}^2(X_{\tau_k})\Lambda^1_{\tau_k}\big] \\
&\qquad\leq \mathbf{E}_x\big[\mathbb{1}_{\{X_{\tau_k}=\alpha_k\}}\,\mathrm{e}^{-r\tau_k}\,\bar{J}^2(X_{\tau_k})\Lambda^1_{\tau_k}\big] + \mathbf{E}_x\big[\mathbb{1}_{\{X_{\tau_k}=\beta_k\}}\,\mathrm{e}^{-r\tau_k}\,\bar{J}^2(X_{\tau_k})\Lambda^1_{\tau_k}\big] \\
&\qquad\leq \mathbf{E}_x\Big[\mathrm{e}^{-r\tau_k}\,\bar{J}^2(\alpha_k)\Lambda^1_{\tau_{\alpha_k}}\Big] + \mathbf{E}_x\Big[\mathrm{e}^{-r\tau_k}\,\bar{J}^2(\beta_k)\Lambda^1_{\tau_{\beta_k}}\Big] \\
&\qquad\leq \frac{\phi(x)}{\phi(\alpha_k)}\,G^2(\alpha_k) + \frac{\psi(x)}{\psi(\beta_k)}\,G^2(\beta_k).
\end{aligned}$$

Because $\bar{J}^2 \geq 0$, it then follows from the growth properties (A.4) that

$$\lim_{k\to\infty} \mathbf{E}_x\big[\mathbb{1}_{\{\tau_{S^2}\geq\tau_k\}}\,\mathrm{e}^{-r\tau_k}\,\bar{J}^2(X_{\tau_k})\Lambda^1_{\tau_k}\big] = 0.$$

Thus, letting k go to ∞ in (S.34) yields

$$\begin{aligned}
\bar{J}^2(x) = \mathbf{E}_x\Bigg[&\sum_{n=1}^N \int_{[0,\tau\wedge\tau_{S^2})} \mathrm{e}^{-rt}\Big[\bar{J}^2(q^1_n)a^1_n - \frac{1}{2}\,\Delta\bar{J}^{2\prime}(q^1_n)\Big]\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t \\
&+ \mathbb{1}_{\{\tau_{S^2}<\tau\}}\,\mathrm{e}^{-r\tau_{S^2}}\,R^2(X_{\tau_{S^2}})\Lambda^1_{\tau_{S^2}} + \mathbb{1}_{\{\tau_{S^2}\geq\tau\}}\,\mathrm{e}^{-r\tau}\,\bar{J}^2(X_\tau)\Lambda^1_\tau\Bigg].
\end{aligned}$$

This shows (A.10). The result follows. ■

Proof of Lemma A.4: Suppose, with no loss of generality, that $i=2$ and $j=1$. First, let us observe that (S.9) leads to

$$J^2(x,(\mu^1,S^1),\tau) = \mathbf{E}_x\Bigg[\mathrm{e}^{-r\tau}R^2(X_\tau)\Lambda^1_\tau + \sum_{n=1}^N \int_{[0,\tau)} \mathrm{e}^{-rt}G^2(X_t)\Lambda^1_s a^1_n\,\mathrm{d}L^{q^1_n}_t\Bigg].$$

Let w^2 be a solution to (11)–(17). Applying the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r(\tau\wedge\tau_k)}$ $w^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k}$, with τ_k defined as in the proof of Lemma A.3, we obtain

$$\begin{aligned}
w^2(x) = {}& \mathrm{e}^{-r(\tau\wedge\tau_k)}w^2(X_{\tau\wedge\tau_k})\Lambda^1_{\tau\wedge\tau_k} - \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\,\mathrm{d}\Lambda^1_t \\
&- \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}w^2(X_t) - rw^2(X_t)]\prod_{n=1}^N \mathbb{1}_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\,\mathrm{d}t \\
&- \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\sigma(X_t)w^{2\prime}(X_t)\prod_{n=1}^N \mathbb{1}_{\{X_t\neq q^1_n\}}\,\Lambda^1_t\,\mathrm{d}W_t \\
&- \frac{1}{2}\sum_{n=1}^N \Delta w^{2\prime}(q^1_n)\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda^1_t\,\mathrm{d}L^{q^1_n}_t.
\end{aligned}\tag{S.35}$$

From (12) and A3, we have $\mathcal{L}w^2 - rw^2 = \mathcal{L}R^2 - rR^2 \le 0$ over $(\alpha, s^2) \subset (\alpha, x_{R^2}]$. It then follows from (16) that

$$\mathbf{E}_x\left[-\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}w^2(X_t) - rw^2(X_t)] \prod_{n=1}^{N} \mathbb{1}_{\{X_t \neq q_n^1\}} \Lambda_t^1 \,\mathrm{d}t\right] \ge 0. \tag{S.36}$$

Next, we have

$$\begin{aligned}
&\mathbf{E}_x\left[-\frac{1}{2}\sum_{n=1}^{N} \Delta w^{2\prime}(q_n^1) \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda_t^1 \,\mathrm{d}L_t^{q_n^1}\right] \\
&\quad = \mathbf{E}_x\left[\sum_{n=1}^{N} a_n^1[G^2(q_n^1) - w^2(q_n^1)] \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}\Lambda_t^1 \,\mathrm{d}L_t^{q_n^1}\right] \\
&\quad = \mathbf{E}_x\left[\sum_{n=1}^{N} \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\Lambda_t^1 a_n^1 \,\mathrm{d}L_t^{q_n^1} - \sum_{n=1}^{N} \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda_t^1 a_n^1 \,\mathrm{d}L_t^{q_n^1}\right] \\
&\quad = \mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\,\mathrm{d}\Gamma_t^1 + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\,\mathrm{d}\Lambda_t^1\right],
\end{aligned} \tag{S.37}$$

where the first equality follows from (17), the second equality follows from the fact that the measure $\mathrm{d}L_t^{q_n^1}$ only charges the set $\{t \ge 0 : X_t = q_n^1\}$, and the third equality follows from the representation (7). We obtain from (S.35)–(S.37) that

$$\begin{aligned}
w^2(x) &\ge \mathbf{E}_x\left[\mathrm{e}^{-r(\tau\wedge\tau_k)} w^2(X_{\tau\wedge\tau_k})\Lambda_{\tau\wedge\tau_k}^1 + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\,\mathrm{d}\Gamma_t^1\right] \\
&\ge \mathbf{E}_x\left[\mathrm{e}^{-r(\tau\wedge\tau_k)} R^2(X_{\tau\wedge\tau_k})\Lambda_{\tau\wedge\tau_k}^1 + \int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt} G^2(X_t)\,\mathrm{d}\Gamma_t^1\right],
\end{aligned}$$

where the first inequality follows from the fact that the stochastic integral in (S.35) is a centered square-integrable random variable as shown in the proof of Lemma A.3, and the second inequality follows from (11). Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x) \ge \mathbf{E}_x\left[\mathrm{e}^{-r\tau} R^2(X_\tau)\Lambda_\tau^1 + \int_{[0,\tau)} \mathrm{e}^{-rt} G^2(X_t)\,\mathrm{d}\Gamma_t^1\right] = J^2(x, (\mu^1, S^1), \tau),$$

where the equality follows from (5). Taking the supremum over $\tau \in \mathcal{T}$ yields (A.12).

To establish (A.13), we apply the Itô–Tanaka–Meyer formula to $\mathrm{e}^{-r\tau_k} w^2(X_{\tau_k})\Lambda_{\tau_k}^1\Lambda_{\tau_k-}^2$. Taking expectations, we obtain

$$\begin{aligned}
w^2(x) = \mathbf{E}_x\Bigg[&\mathrm{e}^{-r\tau_k} w^2(X_{\tau_k})\Lambda_{\tau_k}^1\Lambda_{\tau_k-}^2 - \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda_{t-}^2 \,\mathrm{d}\Lambda_t^1 \\
&- \int_{[0,\tau_k)} \mathrm{e}^{-rt} w^2(X_t)\Lambda_t^1 \,\mathrm{d}\Lambda_t^2 - \frac{1}{2}\sum_{n=1}^{N} \Delta w^{2\prime}(q_n^1) \int_{[0,\tau_k)} \mathrm{e}^{-rt}\Lambda_t^1\Lambda_{t-}^2 \,\mathrm{d}L_t^{q_n^1}\Bigg],
\end{aligned} \tag{S.38}$$

where, as in the proof of Lemma A.3, we have used that

$$\mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt}\sigma(X_t)w^{2\prime}(X_t)\prod_{n=1}^{N}\mathbb{1}_{\{X_t\neq q_n^1\}}\,\Lambda_t^1\Lambda_{t-}^2\,\mathrm{d}W_s\right]=0$$

and that

$$\mathbf{E}_x\left[\int_{[0,\tau\wedge\tau_k)} \mathrm{e}^{-rt}[\mathcal{L}\bar{w}^2(X_t)-rw^2(X_t)]\prod_{n=1}^{N}\mathbb{1}_{\{X_t\neq q_n^1\}}\,\Lambda_t^1\Lambda_{t-}^2\,\mathrm{d}t\right]=0,$$

which follows from (16) and from the fact that $\Lambda_{t-}^2=\mathbb{1}_{\{t\leq\tau_{S^2}\}}\,\mathrm{e}^{-\int_{\mathcal{I}}L_t^y\,\mu^2(\mathrm{d}y)}$ vanishes over $\{X_t<s^2\}$. Now, using that the measure $\mathrm{d}\Gamma_t^2$ only charges the set $\{t\geq 0: w^2(X_t)=R^2(X_t)\}$, we have

$$\begin{aligned}\mathbf{E}_x\left[-\int_{[0,\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\Lambda_t^1\,\mathrm{d}\Lambda_t^2\right]&=\mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\Lambda_t^1\,\mathrm{d}\Gamma_t^2\right]\\&=\mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt}R^2(X_t)\Lambda_t^1\,\mathrm{d}\Gamma_t^2\right].\end{aligned}\tag{S.39}$$

Next, using (17), and following the same steps as for (S.37), we have

$$\begin{aligned}&\mathbf{E}_x\left[-\frac{1}{2}\sum_{n=1}^{N}\Delta w^{2\prime}(q_n^1)\int_{[0,\tau_k)} \mathrm{e}^{-rt}\Lambda_t^1\Lambda_{t-}^2\,\mathrm{d}L_t^{q_n^1}\right]\\&\quad=\mathbf{E}_x\left[\sum_{n=1}^{N}\int_{[0,\tau_k)} \mathrm{e}^{-rt}G^2(q_n^1)\Lambda_t^1\Lambda_{t-}^2 a_n^1\,\mathrm{d}L_t^{q_n^1}-\sum_{n=1}^{N}\int_{[0,\tau_k)} \mathrm{e}^{-rt}w^2(q_n^1)\Lambda_t^1\Lambda_{t-}^2 a_n^1\,\mathrm{d}L_t^{q_n^1}\right]\\&\quad=\mathbf{E}_x\left[\int_{[0,\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\Lambda_{t-}^2\,\mathrm{d}\Gamma_t^1+\int_{[0,\tau_k)} \mathrm{e}^{-rt}w^2(X_t)\Lambda_{t-}^2\mathrm{d}\Lambda_t^1\right].\end{aligned}\tag{S.40}$$

We obtain from (S.38)–(S.40) that

$$w^2(x)=\mathbf{E}_x\Bigg[\mathrm{e}^{-r\tau_k}w^2(X_{\tau_k})\Lambda_{\tau_k}^1\Lambda_{\tau_k-}^2\tag{S.41}$$

$$+\int_{[0,\tau_k)} \mathrm{e}^{-rt}R^2(X_t)\Lambda_t^1\,\mathrm{d}\Gamma_t^2+\int_{[0,\tau_k)} \mathrm{e}^{-rt}G^2(X_t)\Lambda_{t-}^2\,\mathrm{d}\Gamma_t^1\Bigg].\tag{S.42}$$

Using again the same arguments as in Lemma A.3, letting k go to ∞ yields

$$w^2(x)=\mathbf{E}_x\left[\int_{[0,\infty)} \mathrm{e}^{-rt}R^2(X_t)\Lambda_t^1\,\mathrm{d}\Gamma_t^2+\int_{[0,\infty)} \mathrm{e}^{-rt}G^2(X_t)\Lambda_{t-}^2\,\mathrm{d}\Gamma_t^1\right]=J^2(x,(\mu^1,S^1),(\mu^2,S^2)),$$

where the second equality follows from (5). The result follows. ■

S.6 Proof of Theorem 4

The proof is based on the arguments developed in Section 5.4 in the limiting case where $p=p_\infty$ and the model is fully symmetric. We will use four lemmas, the proofs of which are given at the end of this section. The first lemma reformulates the characterization of Mpes of type 2 using the change of variables (S.1).

Lemma S.6.1 *Let N be a positive integer and let be given two finite sequences $(q_n^1)_{n=1}^N$ and $(q_n^2)_{n=1}^{N-1}$ of numbers in $\mathcal{I}$, and a number $s^2 \in \mathcal{I}$ such that $q_1^1 = x_{R^1} > q_1^2 > q_2^1 > \ldots > q_{N-1}^1 > q_{N-1}^2 > q_N^1 > s^2$, with $q_N^2 \equiv s^2$ by convention. Then, the following are equivalent:*

1. *There exist two finite sequences $(a_n^1)_{n=1}^N$ and $(a_n^2)_{n=1}^{N-1}$ of positive real numbers such that the strategy profile $((\mu^1, S^1), (\mu^2, S^2)) \equiv ((\sum_{n=1}^N a_n^1 \delta_{q_n^1}, \emptyset), (\sum_{n=1}^{N-1} a_n^2 \delta_{q_n^2}, (\alpha, s^2]),$ with $\sum_{n=1}^0 \equiv 0$ by convention, is an Mpe of type 2.*

2. *The increasing sequence $(y_n)_{n=1}^{2N}$ defined by*

$$y_{2k-1} = \zeta(q_k^1) \quad \text{and} \quad y_{2k} = \zeta(q_k^2), \quad 1 \leq k \leq N, \tag{S.43}$$

satisfies

$$\hat{R}^i(y_{n-1}) + \hat{R}^{i\prime}(y_{n-1})(y_n - y_{n-1}) = \hat{R}^i(y_{n+1}) + \hat{R}^{i\prime}(y_{n+1})(y_n - y_{n+1}), \quad 1 < n < 2N, \tag{S.44}$$

with $i = 1$ if n is even and $i = 2$ if n is odd and, with the same convention,

$$\hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) < \hat{G}^2(y_1), \tag{S.45}$$

$$\hat{R}^i(y_{n-1}) + \hat{R}^{i\prime}(y_{n-1})(y_n - y_{n-1}) < \hat{G}^i(y_n), \quad 1 < n < 2N, \tag{S.46}$$

$$\hat{R}^1(y_{2N-1}) + \hat{R}^{1\prime}(y_{2N-1})(y_{2N} - y_{2N-1}) = \hat{G}^1(y_{2N}), \tag{S.47}$$

$$\hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y_1, \tag{S.48}$$

$$y_2 > \zeta(x_{R^2}). \tag{S.49}$$

A similar result holds for Mpes of type 1. To prove Theorem 4, we show that, for $p \in P$ in a neighborhood of p_∞, there exists a finite sequence $(y_n)_{n=1}^{2N}$ depending on p and satisfying (S.43)–(S.49). To this end, Lemmas S.6.2–S.6.3 below introduce and establish the properties of two functions H^i and Z^i that we use, along the lines of Section 5.4, to define each player's sequence of randomization thresholds, as well as a measure of the difference between the pseudo-value function of player i and his reward function R^i at the randomization thresholds of player j. Both lemmas rely on the change of variables (S.1).

Before proceeding with the proof, let us first reformulate some insights from Section 5.4 using this change of variable and the vocabulary introduced in Lemma S.6.1. To fix ideas, consider an Mpe of type 2, with $2N - 1$ randomization thresholds for some $N \geq 1$. With a slight abuse of terminology, we refer to y_n for $1 \leq n < 2N$ as a randomization threshold for player 1 if n is odd, and for player 2 if n is even; observe that the sequence $(y_n)_{n=1}^{2N-1}$ is increasing, in contrast with the sequence of true randomization thresholds $q_1^1 > q_1^2 > q_2^1 >$

$\ldots > q^1_{N-1} > q^2_{N-1} > q^1_N$. Fix some $p \in P$, sometimes omitted for brevity, and, as in (S.27), for all i and $z > 0$, let $\hat{T}^i_z$ be the affine function tangent to $\hat{R}^i$ at z. With again a slight abuse of terminology, we refer to $\hat{T}^i_{y_n}$ for $1 \leq n < 2N$ as to the pseudo-value function of player i between the randomization thresholds y_{n-1} and y_{n+1} of player j, with $y_0 \equiv 0$ by convention and using the same convention on the assignment of players to randomization thresholds.

Lemma S.6.2 *Define, for all i and $(p, y, z) \in P \times (0, \infty) \times (0, \infty)$,*

$$H^i(p, y, z) \equiv \hat{R}^i(p, z) + \hat{R}^{i\prime}(p, z)(y - z) - \hat{R}^i(p, y).$$

Then, we have

(i) H^i *is continuous;*

(ii) $\frac{\partial H^i}{\partial z}(p, y, z) > 0$ *for all* $y \in (0, \infty)$ *and* $z > \zeta(x^i_0(p))$*;*

(iii) $\lim_{z \to \infty} H^i(p, y, z) = \infty$ *for all* $y \in (0, \infty)$.

Lemma S.6.3 *For all $(p, \xi, y) \in P \times [0, \infty) \times (0, \infty)$, the equation in z*

$$\xi = H^i(p, y, z) \tag{S.50}$$

has a unique solution $z \equiv Z^i(p, \xi, y)$ in $[\zeta(x^i_0(p)) \vee y, \infty)$. Moreover, the following hold:

(i) $Z^i(p, 0, y) = y$ *for all* $y \geq \zeta(x^i_0(p))$*;*

(ii) Z^i *is continuous over* $P \times [0, \infty) \times (0, \infty)$*;*

(iii) Z^i *is increasing with respect to* ξ*;*

(iv) let $D = \frac{1}{2}[\zeta(x_{R^1}(p_\infty)) - \zeta(x^1_0(p_\infty))] = \frac{1}{2}[\zeta(x_{R^2}(p_\infty)) - \zeta(x^2_0(p_\infty))] > 0$. *For all* $M \in (\alpha, x_{R^i}(p_\infty))$ *and* $m > 0$, *there exist constants* C_1, C_2, C_3, *and* C_4 *such that*

$$C_1\sqrt{\xi} \leq Z^i(p, \xi, y) - y \leq C_2\sqrt{\xi}, \tag{S.51}$$

$$C_3\xi \leq H^j(p, y, Z^i(p, \xi, y)) \leq C_4\xi, \tag{S.52}$$

over the compact set

$$K(m, M) \equiv \{(p, \xi, y) \in P \times \mathbb{R}_+ \times \mathbb{R}_{++} : \xi \leq m \text{ and } \zeta(x^i_0(p)) + D \leq y \leq \zeta(M) \text{ for all } i\}.$$

These two lemmas allow us to construct a sequence $(\xi_n, y_n)_{n \geq 1} \equiv (\xi_n(p, \varepsilon), y_n(p, \varepsilon))_{n \geq 1}$, where $y_1 = \zeta(x_{R^1}(p))$, $\xi_1 = \varepsilon > 0$, and the pair (ξ_{n+1}, y_{n+1}) for $n \geq 1$ is defined by

$$y_{n+1} = Z^i(p, \xi_n, y_n), \tag{S.53}$$

$$\xi_{n+1} = H^j(p, y_{n+1}, y_n), \tag{S.54}$$

with $i = 1$ if n is even and $i = 2$ if n is odd. The whole sequence is always well-defined and a function of (p, ε); ε is the distance between the pseudo-value function of player 2 and the function $\hat{R}^2$ evaluated at y_1; (S.53) defines a randomization threshold y_{n+1} of player i as a function of the randomization threshold y_n of player j and the distance ξ_n between the pseudo-value function of player i and the function $\hat{R}^i$ evaluated at y_n; (S.54) defines the distance between the pseudo-value function of player j and the function $\hat{R}^j$ evaluated at y_{n+1}. From Lemma S.6.1 and its analogue for Mpes of type 1, it is easy to check that the sequence $(y_n)_{n\geq 1}$ induces an Mpe of type 1 or 2 if there exists an integer N such that

$$\hat{G}^i(p, y_n) - \hat{R}^i(p, y_n) > \xi_n, \quad 1 \leq n < N, \tag{S.55}$$

$$\hat{G}^i(p, y_N) - \hat{R}^i(p, y_N) = \xi_N, \tag{S.56}$$

$$\hat{R}^2(p, y_2) + \hat{R}^{2\prime}(p, y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(p, \zeta(x_{R^2}(p)))}{\zeta(x_{R^2}(p))}\, y_1, \tag{S.57}$$

$$y_2 > \zeta(x_{R^2}(p)). \tag{S.58}$$

That the relations (S.55)–(S.58) can be chosen to hold for (p, ε) arbitrarily close to $(p_\infty, 0)$ is a consequence of our next lemma, which implies that, in terms of the true randomization thresholds, we have

(i) for all i and $n > 1$, $q^i_n \to x_{R^1}(p_\infty)$ as $p \to p_\infty$ and $|q^2_1 - x_{R^1}(p)| \to 0$;

(ii) for all i and $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, the pseudo-value function of player i converges uniformly to R^i over $[M, x_{R^1}(p_\infty)]$ as $p \to p_\infty$ and $|q^2_1 - x_{R^1}(p)| \to 0$.

(Recall that $\alpha^1(p_\infty) = \alpha^2(p_\infty)$ and $x_{R^1}(p_\infty) = x_{R^2}(p_\infty)$ by definition of p_∞.)

Lemma S.6.4 *The following holds*:

(i) For all $n \geq 1$ and $(p, \varepsilon) \in P \times (0, \infty)$,

$$y_n(p, \varepsilon) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} \zeta(x_{R^1}(p_\infty)).$$

(ii) For each $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, there exist $\eta > 0$ and $\bar{m} > 0$ such that

$$\forall m \in (0, \bar{m}), \exists \varepsilon_1 \in (0, m), \forall \varepsilon \in (0, \varepsilon_1), \forall p \in B_\eta,\ N_2(p, \varepsilon, m) > N_1(p, \varepsilon, M)$$

where $B_\eta \equiv \{p \in P : d(p, p_\infty) \leq \eta\}$ and

$$N_1(p, \varepsilon, M) \equiv \inf\{n \geq 1 : y_n(p, \varepsilon) \geq \zeta(M)\},\ N_2(p, \varepsilon, m) \equiv \inf\{n \geq 1 : \xi_n(p, \varepsilon) \geq m\},$$

We can now conclude the proof of Theorem 4. First, let $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, $\bar{N} \in \mathbb{N} \setminus \{0\}$, and $\kappa > 0$; next, let C_1, C_2, C_3, and C_4 be the constants given by Lemma S.6.3 for the compact $K(1, M)$; finally, let η and $\bar{m}$ be given by Lemma S.6.4(ii). Now, fix some $m \in (0, \bar{m})$ small enough that

$$0 < m < \min_{i=1,2} \min_{p \in B_\eta} \min_{y \in [\zeta(x_{R^1}(p)), \zeta(M)]} \hat{G}^i(p, y) - \hat{R}^i(p, y), \tag{S.59}$$

$$C_2 C_5 \sqrt{m} \leq \kappa, \tag{S.60}$$

where $C_5 = \sup_{y \in [\zeta(x_{R^1}(p_\infty)+1), \zeta(M)]} |(\zeta^{-1})'(x)|$, and let $\varepsilon_1 = \varepsilon_1(m)$ be given by Lemma S.6.4. Because the sequence $(y_n(p, \varepsilon))_{n \geq 1}$ is increasing, it follows from (S.59) and the definitions of $N_1(p, \varepsilon, M)$ and $N_2(p, \varepsilon, m)$ that, for all i, $\varepsilon \in (0, \varepsilon_1)$, and $p \in B_\eta$,

$$\begin{aligned} y_n(p, \varepsilon) \leq \zeta(M) \; &\Rightarrow \; n \leq N_1(p, \varepsilon, M) \; \Rightarrow \; n < N_2(p, \varepsilon, m) \\ &\Rightarrow \; \xi_n(p, \varepsilon) < m \; \Rightarrow \; \xi_n(p, \varepsilon) < \hat{G}^i(p, y_n(p, \varepsilon)) - \hat{R}^i(p, y_n(p, \varepsilon)), \end{aligned} \tag{S.61}$$

so that the inequality (S.55) holds for all $1 \leq n < N_1(p, \varepsilon, M)$. By Lemma S.6.4(i), up to reducing ε_1 and η, we may assume that

$$\forall p \in B_\eta, \forall \varepsilon \in (0, \varepsilon_1), \;\; y_{\bar{N}}(p, \varepsilon) \leq \zeta(M).$$

Moreover, for η small enough, we have

$$\begin{aligned} \forall p \in B_\eta, \forall \varepsilon \in (0, \varepsilon_1), \;\; 0 &< \zeta^{-1}(y_n(p, \varepsilon)) - \zeta^{-1}(y_{n+1}(p, \varepsilon)) \\ &\leq C_5 [y_{n+1}(p, \varepsilon) - y_n(p, \varepsilon)] \\ &\leq C_2 C_5 \sqrt{\xi_n(p, \varepsilon)} \\ &\leq C_2 C_5 \sqrt{m} \\ &\leq \kappa, \quad 1 \leq n = 1 < N_1 - 1, \end{aligned}$$

where the second inequality follows from the definition of C_5 and the fact that we may choose η such that $x_{R^1}(p) \leq x_{R^1}(p_\infty) + 1$ for all $p \in B_\eta$, the third inequality follows from (S.81) in the proof of Lemma S.6.4, the fourth inequality follows from Lemma S.6.4(i) and (S.61), and the fifth inequality follows from (S.60).

Let us now take $p = p_\infty$. From the preceding results, we have

$$y_n(p_\infty, \varepsilon_1) \leq \zeta(M) \; \Rightarrow \; \xi_n(p_\infty, \varepsilon_1) < \hat{G}^i(p_\infty, y_n(p_\infty, \varepsilon_1)) - \hat{R}^i(p_\infty, y_n(p_\infty, \varepsilon_1)),$$

so that condition (S.46) is satisfied at $(p_\infty, \varepsilon_1)$ for $1 \leq n < N_1(p_\infty, \varepsilon_1, M)$. Notice also from $y_2(p_\infty, \varepsilon) > y_1(p_\infty, \varepsilon) = \zeta(x_{R^1}(p_\infty)) = \zeta(x_{R^2}(p_\infty))$ along with the strict concavity of $\hat{R}^2(p_\infty, \cdot)$ over $(\zeta(x_0^2(p_\infty), \infty)$ that condition (S.57) (with a strict inequality) and condition (S.58) are also satisfied at (p_∞, ε) for all $\varepsilon > 0$.

Now, recall that, for all i and $p \in P$, $G^i(p, \cdot) = R^i(p, \cdot)$ over $(\alpha, \alpha^i(p)]$. From Lemma A.2, the sequence $(y_n(p_\infty, \varepsilon_1))_{n \geq 1}$ tends to ∞ and, therefore, becomes eventually larger than $\zeta(\alpha^i(p_\infty))$. This implies that there exists N_{ε_1} such that, for each i,

$$\xi_{N_{\varepsilon_1}}(p_\infty, \varepsilon_1) > \hat{G}^i(p_\infty, y_{N_{\varepsilon_1}}(p_\infty, \varepsilon_1)) - \hat{R}^i(p_\infty, y_{N_{\varepsilon_1}}(p_\infty, \varepsilon_1)), \tag{S.62}$$

as the right-hand side of (S.62) vanishes above $\zeta(\alpha^i(p_\infty))$ while the left-hand side remains positive. Notice that N_{ε_1} depends only on ε_1, and remains fixed in what follows.

By Lemma S.6.4(i), $\lim_{\varepsilon \to 0} y_{N_{\varepsilon_1}}(p_\infty, \varepsilon) = 0$, and thus there exists $\varepsilon_0 \in (0, \varepsilon_1)$ such that $y_{N_{\varepsilon_1}}(p_0, \varepsilon_0) \leq \zeta(M)$. Therefore, by (S.61),

$$\xi_n(p_\infty, \varepsilon_0) < \hat{G}^i(p_\infty, y_n(p_\infty, \varepsilon_0)) - \hat{R}^i(p_\infty, y_n(p_\infty, \varepsilon_0)), \quad 1 \leq n \leq N_{\varepsilon_1}. \tag{S.63}$$

By continuity of all the functions involved, there exists $\eta_0 \in (0, \eta)$ such that, for each $p \in B_{\eta_0}$, all the inequalities (S.57)–(S.58) and (S.63) remain strict at (p, ε_0). We claim that for each $p \in B_{\eta_0}$, there exists $\varepsilon \in (\varepsilon_0, \varepsilon_1)$ such that (p, ε) induces an Mpe. To see this, fix $p \in B_{\eta_0}$ and consider the following list of inequalities with $\varepsilon \in (\varepsilon_0, \varepsilon_1)$:

$$\xi_n(p, \varepsilon) < \hat{G}^i(p, y_n(p, \varepsilon)) - \hat{R}^i(p, y_n(p, \varepsilon)), \quad 1 \leq n \leq N_{\varepsilon_1},$$
$$\underbrace{\hat{R}^2(p, y_2(p, \varepsilon)) + \hat{R}^{2\prime}(p, y_2(p, \varepsilon))[\zeta(x_{R^1}(p)) - y_2(p, \varepsilon)]}_{\hat{R}^2(p, \zeta(x_{R^1}(p))) + \varepsilon} > \frac{\zeta(x_{R^1}(p))}{\zeta(x_{R^2}(p))} \hat{R}^2(p, \zeta(x_{R^2}(p))).$$

All these inequalities hold for ε sufficiently close to ε_0; moreover, the second inequality and the first one for $n = 1$ hold for all $\varepsilon \in (\varepsilon_0, \varepsilon_1)$. Because, at ε_1, the first inequality with $n = N_{\varepsilon_1}$ is strictly reversed, there exists a smallest $\varepsilon \in (\varepsilon_0, \varepsilon_1)$ such that there exist i and $2 \leq n \leq N_{\varepsilon_1}$ such that

$$\xi_n(p, \varepsilon) = \hat{G}^i(p, y_n(p, \varepsilon)) - \hat{R}^i(p, y_n(p, \varepsilon)). \tag{S.64}$$

For this value of ε, the relations (S.55)–(S.58) hold, which establishes the existence of an Mpe such that the integer N is the smallest n satisfying (S.64) for some player i—so that player j concedes with probability 1 at $y_N(p, \varepsilon)$. By construction, this N must be larger or equal than $\bar{N}$, and $y_N(p, \varepsilon)$ must be larger or equal than $\zeta(M)$. Hence the result. ■

We now turn to the proofs of Lemmas S.6.1–S.6.4.

Proof of Lemma S.6.1: We first recall some properties of V_{R^2}. We have $V_{R^2} = T^2_{x_{R^2}} \geq R^2$ over $[x_{R^2}, \beta)$, with equality at x_{R^2}, and the smooth-fit condition

$$R^{2\prime}(x_{R^2}) = \frac{R^2(x_{R^2})}{\phi(x_{R^2})} \phi'(x_{R^2}).$$

This implies

$$T^2_{x_{R^2}} = \frac{R^2(x_{R^2})}{\phi(x_{R^2})}\,\phi \quad \text{and} \quad \hat{R}^{2\prime}(\zeta(x_{R^2})) = \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}. \tag{S.65}$$

Notice finally that

$$T^2_{x_{R^2}} \geq R^2 \text{ over } \mathcal{I}. \tag{S.66}$$

Indeed, we have $T^2_{x_{R^2}} = V_{R^2} \geq R^2$ over $[x_{R^2}, \beta)$ and $\hat{T}^2_{x_{R^2}} \geq \hat{R}^2$ over $[\zeta(x_{R^2}), \infty)$ because $\hat{R}^2$ is concave over $[\zeta(x_{R^2}), \infty)$, implying that $T^2_{x_{R^2}} \geq R^2$ over $(\alpha, x_{R^2}]$. We can now turn to the proof of the lemma.

Direct Part: 1 $\Rightarrow$ 2 From Theorem 3, we know that there exist solutions (w^1, w^2) to the variational system (11)–(23). These functions are continuous and satisfy

$$w^1 \equiv \mathbb{1}_{(\alpha, q^2_N]}\, G^1 + \sum_{n=1}^{N-1} \mathbb{1}_{(q^2_{n+1}, q^2_n]}\, T^1_{q^1_{n+1}} + \mathbb{1}_{(q^2_1, \beta)}\, T^1_{q^1_1}, \tag{S.67}$$

$$w^2 \equiv \mathbb{1}_{(\alpha, q^2_N]}\, R^2 + \mathbb{1}_{(q^2_N, q^1_N]}\, T^2_{q^2_N} + \sum_{n=1}^{N-1} \mathbb{1}_{(q^1_{n+1}, q^1_n]}\, T_{q^2_n} + \mathbb{1}_{(q^1_1, \beta)}\, \frac{w^2(q^1_1)}{\phi(q^1_1)}\, \phi. \tag{S.68}$$

Let us prove that (S.44) holds. If $N = 1$, there is nothing to prove, so we may assume that $N > 1$. The function w^1 is equal to $T^1_{q^1_1}$ over (q^2_1, β) and to $T^1_{q^1_2}$ over $(q^2_2, q^2_1]$. Continuity of w^1 at q^2_1 is equivalent to

$$\begin{aligned} T^1_{q^1_1}(q^2_1) = T^1_{q^1_2}(q^2_1) &\Leftrightarrow \hat{T}^1_{y_1}(y_2) = \hat{T}^1_{y_3}(y_2) \\ &\Leftrightarrow \hat{R}^1(y_1) + \hat{R}^{1\prime}(y_1)(y_2 - y_1) = \hat{R}^1(y_3) + \hat{R}^{1\prime}(y_3)(y_2 - y_3), \end{aligned}$$

which proves (S.44) for $n = 2$. The proof for $n > 2$ is similar and thus omitted.

Let us prove that (S.48) holds. Notice that $w^2 = T^2_{q^2_1}$ over $[q^2_1, q^1_1]$ and $w^2 \geq V_{R^2}$ by Proposition 1. We distinguish two cases. Suppose first that $x_{R^1} \geq x_{R^2}$. As $V_{R^2} = T^2_{x_{R^2}}$ over $[x_{R^2}, \beta)$ and $q^1_1 = x_{R^1} \in [x_{R^2}, \beta)$ by assumption,

$$\begin{aligned} w^2(q^1_1) \geq V_{R^2}(q^1_1) &\Leftrightarrow T^2_{q^2_1}(q^1_1) \geq T^2_{x_{R^2}}(q^1_1) \\ &\Leftrightarrow \hat{T}^2_{y_2}(y_1) \geq \hat{T}^2_{\zeta(x_{R^2})}(y_1) \\ &\Leftrightarrow \hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y_1, \end{aligned}$$

where we have used (S.65) to compute the right-hand side of the last inequality. Suppose next that $x_{R^1} < x_{R^2}$. As $w^2 = \frac{w^2(q^1_1)}{\phi(q^1_1)}\, \phi$ over $[q^1_1, \beta)$ and $x_{R^2} \in [q^1_1, \beta) = [x_{R^1}, \beta)$ by assumption,

$$w^2(x_{R^2}) \geq V_{R^2}(x_{R^2}) \Leftrightarrow \frac{w^2(q^1_1)}{\phi(q^1_1)}\, \phi(x_{R^2}) \geq R^2(x_{R^2})$$

$$\Leftrightarrow\; w^2(q_1^1) \geq \frac{\phi(q_1^1)}{\phi(x_{R^2})}\, R^2(x_{R^2})$$
$$\Leftrightarrow\; T^2_{q_1^2}(q_1^1) \geq \frac{\phi(q_1^1)}{\phi(x_{R^2})}\, R^2(x_{R^2})$$
$$\Leftrightarrow\; \hat{T}^2_{y_2}(y_1) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y_1$$
$$\Leftrightarrow\; \hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})} y_1,$$

where we have used the continuity of w^2 at q_1^1 for the left-hand side of the inequality in the third equivalence, and the same arguments as in the previous case to compute the right-hand sides of the inequalities in all these equivalences.

Let us prove that (S.47) holds. As $w^1(q_N^2) = G^1(q_N^2)$ and $w^1 = T^1_{q_N^1}$ over $[q_N^2, q_{N-1}^2)$, we deduce that

$$\begin{aligned} w^1(q_N^2) = G^1(q_N^2) \;&\Leftrightarrow\; T^1_{q_N^1}(q_N^2) = G^1(q_N^2) \\ &\Leftrightarrow\; \hat{T}^1_{y_{2N-1}}(y_{2N}) = \hat{G}^1(y_{2N}) \\ &\Leftrightarrow\; \hat{R}^1(y_{2N-1}) + \hat{R}^{1\prime}(y_{2N-1})(y_{2N} - y_{2N-1}) = \hat{G}^1(y_{2N}). \end{aligned}$$

Let us prove that (S.46) holds. Assume first that $n = 2k$ is even and positive, so that $i = 1$ and $N > 1$. As $w^1 = T^1_{q_k^1}$ over $[q_k^2, q_{k-1}^2)$, we deduce that

$$\begin{aligned} \hat{R}^1(y_{2k-1}) + \hat{R}^{1\prime}(y_{2k-1})(y_{2k} - y_{2k-1}) < \hat{G}^1(y_{2k}) \;&\Leftrightarrow\; \hat{T}^1_{y_{2k-1}}(y_{2k}) < \hat{G}^1(y_{2k}) \\ &\Leftrightarrow\; T^1_{q_k^1}(q_k^2) < G^1(q_k^2) \\ &\Leftrightarrow\; w^1(q_k^2) < G^1(q_k^2). \end{aligned}$$

It is therefore equivalent to prove that $G^1(q_k^2) > w^1(q_k^2)$. From (23), there exists a positive constant a_k^2 such that

$$a_k^2[G^1(q_k^2) - w^1(q_k^2)] + \frac{1}{2}\,\Delta w^{1\prime}(q_k^2) = 0. \tag{S.69}$$

On any interval over which w^1 is differentiable, we have, by (S.1),

$$\hat{w}^{1\prime}(y) = \frac{(\zeta^{-1})'(y)}{\psi(\zeta^{-1}(y))^2}\left[w^{1\prime}(\zeta^{-1}(y))\psi(\zeta^{-1}(y)) - \psi'(\zeta^{-1}(y))w^1(\zeta^{-1}(y))\right].$$

Hence, recalling that ζ^{-1} is decreasing, the derivative jump of $\hat{w}^1$ at y_{2k} can be written as

$$\Delta\hat{w}^{1\prime}(y_{2k}) = \frac{(\zeta^{-1})'(y_{2k})}{\psi(\zeta^{-1}(y_{2k}))^2}\left[-\Delta w^{1\prime}(q_k^2)\right]. \tag{S.70}$$

Now, because $\hat{R}^1$ is strictly concave over $[y^1, \infty)$, we have

$$\Delta\hat{w}^{1\prime}(y_{2k}) = \hat{R}^{1\prime}(y_{2k+1}) - \hat{R}^{1\prime}(y_{2k-1}) < 0.$$

From this and (S.70), it follows that $\Delta(w^1)'(q_k^2) < 0$, which, together with (S.69), implies $G^1(q_k^2) > w^1(q_k^2)$. This concludes the proof of (S.46) for n even and positive. The proof for $n > 1$ odd is similar and thus omitted.

Let us finally prove that (S.45) and (S.49) hold. As for (S.45), it is sufficient, in line with the proof of (S.46), to prove that $\Delta\hat{w}^{2\prime}(y_1) < 0$. We have $w^2 = T^2_{q_1^2}$ over $[q_1^2, q_1^1]$, so that

$$\hat{w}^2(y) = \hat{T}^2_{q_1^2}(y) = \hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y - y_2), \quad y \in [y_1, y_2] \tag{S.71}$$

and, therefore, $\hat{w}^{2\prime}(y_1+) = \hat{R}^{2\prime}(y_2)$. We also have $w^2 = \frac{w^2(q_1^1)}{\phi(q_1^1)}\phi$ over $[q_1^1, \beta)$, so that, by (S.1),

$$\hat{w}^2(y) = \frac{w^2(q_1^1)}{\phi(q_1^1)}\, y = \frac{\hat{w}^2(y_1)}{y_1}\, y, \quad y \in (0, y_1], \tag{S.72}$$

and, therefore, $\hat{w}^{2\prime}(y_1-) = \frac{\hat{w}^2(y_1)}{y_1}$. Using (S.48) and (S.71), we have

$$\hat{w}^2(y_1) = \hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y_1 = \hat{R}^{2\prime}(\zeta(x_{R^2}))y_1, \tag{S.73}$$

where the last equality follows from (S.65). From (S.72)–(S.73), we deduce that

$$\hat{w}^{2\prime}(y_1-) = \frac{\hat{w}^2(y_1)}{y_1} \geq \hat{R}^{2\prime}(\zeta(x_{R^2})),$$

and finally that

$$\Delta\hat{w}^{2\prime}(y_1) \leq \hat{R}^{2\prime}(y_2) - \hat{R}^{2\prime}(\zeta(x_{R^2})) \leq 0 \tag{S.74}$$

as $\hat{R}^2$ is concave over $[\zeta(x_{R^2}), \infty)$ and $y_2 \geq \zeta(x_{R^2})$ because, in any Mpe, it must be that $q_1^2 \leq x_{R^2}$ and thus $y_2 \geq \zeta(x_{R^2})$. We finally prove that (S.49) holds, which implies that the second inequality (S.74) is strict as $\hat{R}^2$ is strictly concave over $[\zeta(x_{R^2}), \infty)$; this will prove that (S.45) holds, thereby completing Part 1 of the proof. We distinguish two cases. First, if $x_{R^1} \leq x_{R^2}$, then $y_2 > y_1 = \zeta(x_{R^1}) \geq \zeta(x_{R^2})$, as desired. Second, if $x_{R^1} > x_{R^2}$, let us assume, by way of contradiction that $y_2 = \zeta(x_{R^2})$. Then $w^2 = T^2_{x_{R^2}}$ over $[q_1^2, q_1^1] = [x_{R^2}, x_{R^1}]$. Recall that $T^2_{x_{R^2}} = \frac{R^2(x_{R^2})}{\phi(x_{R^2})}\phi$. Because $w^2 = \frac{w^2(q_1^1)}{\phi(q_1^1)}\phi$ over $[q_1^1, \beta)$, we deduce by continuity of w^2 that $w^2 = T^2_{x_{R^2}}$ over $[x_{R^2}, \beta)$, which implies $\Delta w^{2\prime}(q_1^1) = 0$. However, it follows from (17) that there exists $a_1^1 > 0$ such that

$$a_1^1[G^2(q_1^1) - w^2(q_1^1)] + \frac{1}{2}\Delta w^{2\prime}(q_1^1) = 0,$$

which implies by the above that $G^2(q_1^1) = w^2(q_1^1)$. As $G^2 > V_{R^2} = T^2_{x_{R^2}} = w^2$ over $[x_{R^2}, \beta)$ and in particular at q_1^1, this is a contradiction. Part 1 of the proof is now complete.

Indirect Part: 2 $\Rightarrow$ 1 Define functions w^1 and w^2 by (S.67) and (S.68). As in Part 1 of the proof, continuity of w^1 at q^1_n for $1 \leq n < N$ and continuity of w^2 at q^2_n for $1 < n < N$ follows from (S.44). Continuity of w^2 at q^2_N and at q^1_1 follows by construction, and continuity of w^1 at q^2_N follows from (S.47). In order to prove that (w^1, w^2) satisfy the variational system (11)–(23) for $i = 2$, it remains to prove that $w^1 \geq R^1$ and $w^2 \geq R^2$ and that there exist positive numbers $(a^1_n)_{n=1}^N$ and $(a^2_n)_{n=1}^{N-1}$ satisfying (17) and (23).

We first claim that $w^1 \geq R^1$. Assuming that $N > 1$, we have $\hat{w}^1 \geq \hat{R}^1$ over $[y_{2k-2}, y_{2k}]$ for all $1 < k \leq N$ because $\hat{w}^1$ is affine and tangent to $\hat{R}^1$ at y_{2k-1}, and $\hat{R}^1$ is concave over $[y_1, \infty)$. It follows that $w^1 \geq R^1$ over $[q^2_k, q^2_{k-1}]$ for any such k. A similar argument shows that $w^1 \geq R^1$ over $[q^2_1, q^1_1]$. Finally, we have $w^1 = G^1 \geq R^1$ over $(\alpha, q^2_{2N}]$ and $w^1 = T^1_{q^1_1} = V_{R^1} \geq R^1$ over $[q^1_1, \beta)$. The claim follows.

We next claim that $w^2 \geq R^2$. The proof that $w^2 \geq R^2$ over $(\alpha, q^2_1]$ is similar to the proof that $w^1 \geq R^1$ and thus omitted. By (S.48), as $w^2 = T^2_{q^2_1}$ over $[q^2_1, q^1_1]$, we have, using (S.65),

$$\hat{w}^2(y_1) = \hat{R}^2(y_2) + \hat{R}^{2\prime}(y_2)(y_1 - y_2) \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y_1 = \hat{T}^2_{\zeta(x_{R^2})}(y_1).$$

Moreover, $w^2 = \frac{w^2(q^1_1)}{\phi(q^1_1)}\, \phi$ over $[q^1_1, \beta)$, so that

$$\hat{w}^2(y) = \frac{\hat{w}^2(y_1)}{y_1}\, y \geq \frac{\hat{R}^2(\zeta(x_{R^2}))}{\zeta(x_{R^2})}\, y = \hat{T}^2_{\zeta(x_{R^2})}(y) \geq \hat{R}^2(y), \quad y \in (0, y_1],$$

where the last inequality follows from (S.66). Thus $w^2 \geq R^2$ over $[q^1_1, \beta)$. It remains to prove that $w^2 \geq R^2$ over $[q^2_1, q^1_1]$ or, equivalently, that $\hat{w}^2 \geq \hat{R}^2$ over $[y_1, y_2]$. Over the interval $[\zeta(x_{R^2}) \vee y_1, y_2]$, which is nonempty as $y_2 > \zeta(x_{R^2})$ by (S.49), $\hat{w}^2$ is affine and tangent to $\hat{R}^2$ at y_2 and thus $\hat{w}^2 \geq \hat{R}^2$ as $\hat{R}^2$ is concave over $[\zeta(x_{R^2}), \infty)$. In case $y_1 < \zeta(x_{R^2})$, over the interval $[y_1, \zeta(x_{R^2})]$ we have $\hat{w}^2(y_1) \geq \hat{T}^2_{\zeta(x_{R^2})}(y_1)$ by (S.66), and we also have $\hat{w}^2(\zeta(x_{R^2})) \geq \hat{R}^2(\zeta(x_{R^2})) = \hat{T}^2_{\zeta(x_{R^2})}(\zeta(x_{R^2}))$. It follows that $\hat{w}^2 \geq \hat{T}^2_{\zeta(x_{R^2})}$ over $[y_1, \zeta(x_{R^2})]$ as both these maps are affine over this interval. We conclude by noticing that $\hat{T}^2_{\zeta(x_{R^2})} \geq \hat{R}^2$ by (S.66). The claim follows.

To prove the existence of positive numbers $(a^1_n)_{n=1}^N$ and $(a^2_n)_{n=1}^{N-1}$ such that (17) and (23) hold, it is sufficient to prove that

$$G^2(q^1_n) - w^2(q^1_n) > 0 \quad \text{and} \quad \Delta w^{2\prime}(q^1_n) < 0, \quad 1 \leq n \leq N, \tag{S.75}$$

$$G^1(q^2_n) - w^1(q^2_n) > 0 \quad \text{and} \quad \Delta w^{1\prime}(q^2_n) < 0, \quad 1 \leq n < N, \tag{S.76}$$

and to define accordingly

$$a^1_n = \frac{-\frac{1}{2}\, \Delta w^{2\prime}(q^1_n)}{G^2(q^1_n) - w^2(q^1_n)}, \quad 1 \leq n \leq N,$$
$$a^2_n = \frac{-\frac{1}{2}\, \Delta w^{1\prime}(q^2_n)}{G^1(q^2_n) - w^1(q^2_n)}, \quad 1 \leq n < N.$$

The first inequalities in (S.75)–(S.76) follow from (S.46), and the second inequalities in (S.75)–(S.76) follow along the same lines as in Part 1 of the proof, using the strict inequality (S.49). Part 2 of the proof is now complete. The result follows. ■

Proof of Lemma S.6.2: Assertions (i)–(ii) directly follow from the assumptions on R^i. To prove (iii), assume, by way of contradiction, that there exist y and C such that

$$\forall z \in [\zeta(x_0^i(p)), \infty),\ \hat{R}^i(p,z) + \hat{R}^{i\prime}(p,z)(y-z) - \hat{R}^i(p,y) \leq C. \tag{S.77}$$

By A9, (A.1), and (S.1), $\lim_{z\to\infty} \hat{R}^i(p,z) = \infty$. It follows that there exists $\bar{z} \in [\zeta(x_0^i(p)) \vee y, \infty)$ such that

$$\hat{R}^i(p,\bar{z}) - \hat{R}^i(p,y) - C > 0. \tag{S.78}$$

By concavity of $\hat{R}^i(p,\cdot)$,

$$\forall z \in [\bar{z}, \infty),\ \hat{R}^i(p,z) + \hat{R}^{i\prime}(p,z)(\bar{z}-z) \geq \hat{R}^i(p,\bar{z}).$$

Thus

$$\begin{aligned}
&\hat{R}^i(p,z) + \hat{R}^{i\prime}(p,z)(y-z) - \hat{R}^i(p,y) \\
&\quad = \hat{R}^i(p,z) + \hat{R}^{i\prime}(p,z)(\bar{z}-z) + \hat{R}^{i\prime}(p,z)(y-\bar{z}) - \hat{R}^i(p,y) \\
&\quad \geq \hat{R}^i(p,\bar{z}) + \hat{R}^{i\prime}(p,z)(y-\bar{z}) - \hat{R}^i(p,y),
\end{aligned}$$

which, together with (S.77)–(S.78) and $\bar{z} > y$, implies, again by concavity of $\hat{R}^i(p,\cdot)$,

$$\frac{\hat{R}^i(p,z) - \hat{R}^i(p,\zeta(x_0^i(p)))}{z - \zeta(x_0^i(p))} \geq \hat{R}^{i\prime}(p,z) \geq \frac{R^i(p,\bar{z}) - \hat{R}^i(p,y) - C}{\bar{z} - y} > 0.$$

But then $\lim_{z\to\infty} \frac{\hat{R}^i(p,z)}{z} > 0$, in contradiction with (S.5) as $\hat{R}^i(p,\cdot) < \hat{G}^i(p,\cdot)$. The result follows. ■

Proof of Lemma S.6.3: Consider the right-hand side of (S.50). It is nonpositive at $z = \zeta(x_0^i(p)) \vee y$—in fact equal to 0 in case $y \geq \zeta(x_0^i(p))$—and negative in case $y < \zeta(x_0^i(p))$ because $\hat{R}^i(p,\cdot)$ is strictly convex on $(0, \zeta(x_0^i(p))))$. By Lemma S.6.2(ii)–(iii), its derivative with respect to z for $z > \zeta(x_0^i(p)) \vee y$ is strictly positive and its limit when $z \to \infty$ is ∞. It follows that (S.50) admits a unique solution in $[\zeta(x_0^i(p)) \vee y, \infty)$.

Assertion (i) is immediate. Assertions (ii)–(iii) follow from the facts that both sides of equation (S.50) are continuous in all variables, and that the right-hand side is increasing in z. We now prove (iv). Let $M \in (\alpha, x_{R^i}(p_\infty))$ and $m > 0$. For each $\xi > 0$, we have $y > \zeta(x_0^i(p))$ and thus $Z^i(p,\xi,y) > y$ by (i) and (iii); hence, from the implicit function theorem,

$$\frac{\partial Z^i}{\partial \xi}(p,\xi,y) = \frac{1}{-\hat{R}^{i\prime\prime}(p, Z^i(p,\xi,y))[Z^i(p,\xi,y) - y]}.$$

As a result,

$$[Z^i(p,\xi,y)-y]^2=\int_0^\xi \frac{2}{-\hat{R}^{i\prime\prime}(p,Z^i(p,s,y))}\,\mathrm{d}s. \tag{S.79}$$

Over the compact set $K^i(m,M)$, Z^i is bounded by (ii), and $\hat{R}^{i\prime\prime}(Z^i(p,s,y))<0$ because $\zeta(x_0^i(p))+D\le y\le Z^i(p,s,y)$. Thus, there exist constants κ_1 and κ_2 depending only on (m,M) such that

$$0<\kappa_2\le -\hat{R}^{i\prime\prime}(Z^i(p,s,y))\le \kappa_1.$$

From this and (S.79), we have

$$\frac{2\xi}{\kappa_1}\le [Z^i(p,\xi,y)-y]^2\le \frac{2\xi}{\kappa_2},$$

and the inequalities (S.51) follow with $C_1=\sqrt{\frac{2}{\kappa_1}}$ and $C_2=\sqrt{\frac{2}{\kappa_2}}$. Finally, for $z\ge y$,

$$\hat{R}^j(p,y)+\hat{R}^{j\prime}(p,y)(z-y)-\hat{R}^j(p,z)=\int_y^z\int_y^u -\hat{R}^{j\prime\prime}(p,s)\,\mathrm{d}s\,\mathrm{d}u,$$

and, therefore,

$$\frac{\kappa_2}{2}(z-y)^2\le \hat{R}^j(p,y)+\hat{R}^{j\prime}(p,y)(z-y)-\hat{R}^j(p,z)\le \frac{\kappa_1}{2}(z-y)^2.$$

Replacing z by $Z^i(p,\xi,y)$, we obtain

$$\frac{\kappa_2}{2}[Z^i(p,\xi,y)-y]^2\le H^j(p,y,Z^i(p,\xi,y))\le \frac{\kappa_1}{2}[Z^i(p,\xi,y)-y]^2,$$

which implies (S.52) with $C_3=\frac{\kappa_2}{\kappa_1}$ and $C_4=\frac{\kappa_1}{\kappa_2}$. ■

Proof of Lemma S.6.4: (i) We proceed by induction on n. The result is obvious for $n=1$ as $y_1(p,\varepsilon)=\zeta(x_{R^1}(p))$. Then

$$\begin{aligned} y_2(p,\varepsilon)&=Z^2(p,\varepsilon,y_1(p,\varepsilon)) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} \zeta(x_R(p_\infty)),\\ \xi_2(p,\varepsilon)&=H^1(p,y_2(p,\varepsilon),y_1(p,\varepsilon)) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \end{aligned}$$

by continuity of Z^2 and H^1 (Lemmas S.6.2(i)–S.6.3(ii)) along with $Z^2(p_\infty,0,\zeta(x_R(p_\infty)))=\zeta(x_R(p_\infty))$ (Lemma S.6.3(i)) and $H^1(p,\zeta(x_R(p_\infty)),\zeta(x_R(p_\infty)))=0$ (by definition of H^1). By similar arguments, assuming that the above limits hold with n instead of 2, we have

$$\begin{aligned} y_{n+1}(p,\varepsilon)&=Z^i(p,\xi_n(p,\varepsilon),y_n(p,\varepsilon)) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} \zeta(x_R(p_\infty)),\\ \xi_{n+1}(p,\varepsilon)&=H^j(p,y_{n+1}(p,\varepsilon),y_n(p,\varepsilon)) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0, \end{aligned}$$

with $i=1$ if n is even and $i=2$ if n is odd. The result follows by induction.

(ii) By continuity, there exists η small enough that

$$p \in B_\eta \;\Rightarrow\; \zeta(x_{R^1}(p)) \geq \zeta(x_0^2(p)) + D \text{ and } M > \max_{i=1,2} \max_{p \in B_\eta} \alpha^i(p). \tag{S.80}$$

Thanks to these inequalities, we can apply Lemma S.6.3(iv) to the compact set $K(1, M)$. Thus there exist C_1, C_2, C_3, and C_4 such that for all $p \in B_\eta$

$$y_n + C_1\sqrt{\xi_n} \leq y_{n+1} \leq y_n + C_2\sqrt{\xi_n}, \tag{S.81}$$

$$C_3\xi_n \leq \xi_{n+1} \leq C_4\xi_n \tag{S.82}$$

for all n such that $\xi_n \leq 1$ and $y_n \leq \zeta(M)$, where, in (S.81)–(S.82) and thereafter, we omit the arguments (p, ε) in y_n and ξ_n. Similarly, in what follows, we write N_1 for $N_1(p, \varepsilon, M)$ and N_2 for $N_2(p, \varepsilon, m)$. Notice that, by Lemma A.2, we have $\lim_{n\to\infty} y_n(p, \varepsilon) = \infty$ and, therefore, $N_1 < \infty$. The constant $\bar{m} \leq 1$ will be determined later, and depends only on (M, η). Now, suppose, by way of contradiction, that

$$\exists m \in (0, \bar{m}), \forall \varepsilon_1 \in (0, m), \exists \varepsilon \in (0, \varepsilon_1), \exists p \in B_\eta,\; N_2 \leq N_1.$$

Notice that $N_2 > 1$ as $\xi_1 = \varepsilon < \varepsilon_1 < m$, that $\xi_{N_2} \geq m$ and $\xi_{N_2-1} < m$, and that $N_2 \leq N_1$ implies $y_{N_2-1} < \zeta(M)$. As $m \leq \xi_{N_2}$ and $\xi_{N_2} \leq C_4\xi_{N_2-1}$ by (S.82), we have

$$\xi_{N_2-1} \geq \frac{m}{C_4}. \tag{S.83}$$

Hence, as $\xi_{N_2-1} \leq C_4^{N_2-1}\xi_1 = C_4^{N_2-1}\varepsilon$ by (S.82) again, it follows that

$$N_2 \geq \frac{\ln(\frac{m}{\varepsilon})}{\ln(C_4)}. \tag{S.84}$$

We now closely follow the proof of Lemma A.2. For each $n \geq 1$, we have

$$\hat{R}^1(p, y_{2n-1}) + \hat{R}^{1\prime}(p, y_{2n-1})(y_{2n} - y_{2n-1}) = \hat{R}^1(p, y_{2n+1}) + \hat{R}^{1\prime}(p, y_{2n+1})(y_{2n} - y_{2n+1}),$$
$$\hat{R}^2(p, y_{2n}) + \hat{R}^{2\prime}(p, y_{2n})(y_{2n+1} - y_{2n}) = \hat{R}^2(p, y_{2n+2}) + \hat{R}^{2\prime}(p, y_{2n+2})(y_{2n+1} - y_{2n+2}).$$

With $y < y' < y''$ three appropriate consecutive terms of the sequence $(y_n)_{n\geq 1}$, these equalities can be compactly rewritten for each i as

$$\hat{R}^i(p, y) + \hat{R}^{i\prime}(p, y)(y' - y) - \hat{R}^i(p, y') = \hat{R}^i(p, y'') + \hat{R}^{i\prime}(p, y'')(y' - y'') - \hat{R}^i(p, y'). \tag{S.85}$$

Using Taylor's theorem with integral remainder, (S.85) is equivalent to

$$-\int_y^{y'} (y' - z)\hat{R}^{i\prime\prime}(p, z)\,\mathrm{d}z = -\int_{y'}^{y''} (z - y')\hat{R}^{i\prime\prime}(p, z)\,\mathrm{d}z. \tag{S.86}$$

By A11, there exists $K_1 > 0$ depending only on (M, η) such that

$$|\hat{R}^{i\prime\prime}(p,z) - \hat{R}^{i\prime\prime}(p,y')| \leq K_1|z-y'|, \quad (z,y',p) \in [y_1, \zeta(M)] \times [y_1, \zeta(M)] \times B_\eta.$$

Thus

$$\begin{aligned} -\int_y^{y'} (y'-z)\hat{R}^{i\prime\prime}(p,z)\,\mathrm{d}z &\leq -R^{i\prime\prime}(p,y')\,\frac{(y'-y)^2}{2} + K_1\,\frac{(y'-y)^3}{3}, \\ -\int_{y'}^{y''} (z-y')\hat{R}^{i\prime\prime}(p,z)\,\mathrm{d}z &\geq -R^{i\prime\prime}(p,y')\,\frac{(y''-y')^2}{2} - K_1\,\frac{(y''-y')^3}{3}. \end{aligned}$$

By (S.86), we thus have

$$(y'-y)^2 + \frac{2K_1}{3|\hat{R}^{i\prime\prime}(p,y')|}\,(y'-y)^3 \geq (y''-y')^2 - \frac{2K_1}{3|\hat{R}^{i\prime\prime}(p,y')|}\,(y''-y')^3. \tag{S.87}$$

Let K_2 depending only on (η, M) such that, for all $y' \in [y_1, \zeta(M)]$ and $p \in B_\eta$,

$$\frac{2K_1}{3|\hat{R}^{i\prime\prime}(p,y')|} \leq K_2.$$

Then, by (S.87), we have

$$(y'-y)^2 + K_2(y'-y)^3 \geq (y''-y')^2 - K_2(y''-y')^3.$$

Letting $u_n \equiv y_{n+1} - y_n$ for all $n \geq 1$, the upshot of the above analysis is that $h(u_n) \geq g(u_{n+1})$, where $g(u) \equiv u^2 - K_2u^3$ and $h(u) \equiv u^2 + K_2u^3$.

Recall that, for $n < N_2$, we have $u_n \leq C_2\sqrt{m}$. We choose $\bar{m}$ small enough that $g(C_2\sqrt{\bar{m}}) > 0$, and therefore, for $n < N_2 - 1$, $g(u_{n+1}) > 0$ and $u_n \geq h^{-1}(g(u_{n+1}))$, where h^{-1} denotes the inverse of h restricted to $[0, \infty)$. Because $h^{-1}(z) = \sqrt{z} - \frac{K_2}{2}\,z + o(z)$, we have $h^{-1}(g(u)) = u - K_2u^2 + o(u^2)$. Hence

$$u_n \geq u_{n+1} - K_2u_{n+1}^2 + o(u_{n+1}^2).$$

There exists $\delta \in (0, \frac{1}{1+K_2})$ depending only on K_2 such that, for each $u \in (0, \delta)$,

$$h^{-1}(g(u)) \geq u - (K_2+1)u^2.$$

We choose $\bar{m} \leq \frac{\delta}{C_2}$ so that $u_n \leq C_2\sqrt{m} \leq \delta$ and thus

$$u_n \geq u_{n+1} - (K_2+1)u_{n+1}^2$$

for all $n < N_2 - 1$, and, as a result,

$$\frac{1}{u_n} - \frac{1}{u_{n+1}} \leq \frac{1}{u_{n+1}}\left[\frac{1}{1-(K_2+1)u_{n+1}} - 1\right] \leq \frac{(K_2+1)}{1-(K_2+1)\delta} \equiv K_3.$$

We obtain

$$\frac{1}{u_n} = \frac{1}{u_{N_2-1}} + \sum_{k=n}^{N_2-2} \left(\frac{1}{u_k} - \frac{1}{u_{k+1}} \right) \leq \frac{1}{u_{N_2-1}} + (N_2 - 2 - n)K_3$$

and thus

$$u_n \geq \frac{1}{(N_2 - 2 - n)K_3 + \frac{1}{u_{N_2-1}}} \geq \frac{1}{(N_2 - 2 - n)K_3 + \frac{\sqrt{C_4}}{C_1\sqrt{m}}},$$

where the second inequality follows from (S.81) and (S.83). We conclude that

$$\zeta(M) > y_{N_2-1} \geq y_1 + \sum_{n=1}^{N_2-2} u_n \geq \sum_{n=1}^{N_2-2} \frac{1}{(N_2 - 2 - n)K_3 + \frac{\sqrt{C_4}}{C_1\sqrt{m}}}. \tag{S.88}$$

Combining (S.84) and (S.88), we obtain a contradiction because the right-hand side of (S.88) tends to ∞ when N_2 tends to ∞ as ε goes to 0, and thus (S.88) cannot be true for all small enough ε. The result follows. ■

S.7 Proof of Theorem 5

The proof consists of six steps.

Step 1: Preliminaries Let $((\mu_n^1, S_n^1), (\mu_n^2, S_n^2))_{n\geq 1}$ be a sequence of Mpes of type 1 or 2 in the sequence of woas with parameters $p_n \to p_\infty$ such that $q_{1,n}^2 - q_{1,n}^1 \to 0$. Let us consider in this sequence an Mpe $((\mu_n^1, S_n^1), (\mu_n^2, S_n^2))$, and let us assume with no loss of generality that $q_{1,n}^1 > q_{1,n}^2$ and that n is large enough that $q_{1,n}^1 < x_0^2(p_n)$. Define

$$\varepsilon_n \equiv H^2(p_n, y_{1,n}, y_{2,n}) = \hat{R}^2(p_n, y_{2,n}) + \hat{R}^{2\prime}(p_n, y_{2,n})(y_{1,n} - y_{2,n}) - \hat{R}^2(p_n, y_{1,n}) > 0,$$

where $y_{1,n} \equiv \zeta(q_{1,n}^1) < \zeta(q_{1,n}^2) \equiv y_{2,n}$. As explained in the proof of Theorem 4, ε_n corresponds to the difference evaluated at $y_{1,n}$ between $\hat{R}_n^2$ and an affine function tangent to $\hat{R}_n^2$ at $y_{2,n}$. Because $p_n \to p_\infty$, we have $y_{1,n} = \zeta(x_{R^1}(p_n)) \to \zeta(x_{R_1}(p_\infty))$, and therefore $\varepsilon_n \to 0$ if $y_{2,n} - y_{1,n} \to 0$. Reciprocally, if $\varepsilon_n \to 0$, then it must be that $y_{2,n} - y_{1,n} \to 0$. Otherwise, along some subsequence $(n_k)_{k\geq 0}$ we would have $y_{2,n_k} - y_{1,n_k} \geq c$ for some $c > 0$. Because H^2 is continuous and $H^2(p_\infty, \zeta(x_{R^1}(p_\infty)), \cdot)$ is increasing over $[\zeta(x_0^2(p_\infty)), \infty) \supset [\zeta(x_{R^1}(p_\infty)), \infty)$ by Lemma S.6.2(i)–(ii), this would imply

$$\liminf_{k\to\infty} \varepsilon_{n_k} \geq H^i(p_\infty, \zeta(x_{R^1}(p_\infty)), \zeta(x_{R^1}(p_\infty)) + c) > 0,$$

a contradiction. We conclude that $y_{2,n} - y_{1,n} \to 0$ if and only if $\varepsilon_n \to 0$. Therefore, in the statement of Theorem 5, we can equivalently consider a sequence of Mpes of type 1 or 2 in the sequence of woas with parameters $p_n \to p_\infty$ such that $\varepsilon_n \to 0$. To alleviate notations, we shall consider thereafter a family of Mpes of type 1 or 2 parameterized by (p, ε), and we analyze the convergence as $(p, \varepsilon) \to (p_\infty, 0)$, possibly along a subsequence.

Step 2: An Upper Bound We have to prove that the measures $\mu^i \equiv \mu_i(p,\varepsilon)$ converge weakly to the measure μ^i_∞ over any compact subset of $(\alpha^i(p_\infty),\beta)$ as $(p,\varepsilon)\to(p_\infty,0)$. That is, for each $M\in(\alpha^i(p_\infty),\beta)$ and each continuous function $h:\mathcal{I}\to\mathbb{R}$ with compact support contained in $[M,\beta)$, we have to prove that

$$\int_{\mathcal{I}} h\,\mathrm{d}\mu^i \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} \int_{\mathcal{I}} h\,\mathrm{d}\mu^i_\infty, \tag{S.89}$$

where μ^i_∞ has density $g^i_\infty \equiv g^i(p_\infty,\cdot)$ with respect to *Leb*, for

$$g^i(p,x) \equiv \frac{1}{\sigma^2(x)}\cdot\frac{rR^j(p,x)-\mathcal{L}R^j(p,x)}{G^j(p,x)-R^j(p,x)}\,\mathbb{1}_{\{\alpha^j(p)<x\le x_{R^j}(p)\}}. \tag{S.90}$$

For ease of exposition, and with no loss of generality, we develop the analysis for a sequence of Mpes of type 2, and we only prove convergence for $i=1$; the proof for $i=2$ is similar and thus omitted. As in the proof of Theorem 4, consider (p,ε) such that the sequence $y_n=y_n(p,\varepsilon)$ induces an Mpe of type 2. Then there exists an integer N such that

$$\xi_{2N}=\hat{G}^1(p,y_{2N})-\hat{R}^1(p,y_{2N}), \tag{S.91}$$

and the corresponding Mpe is

$$(\mu^1,S^1,\mu^2,S^2)=\left(\sum_{n=1}^{N}a^1_n\delta_{q^1_n},\emptyset,\sum_{n=1}^{N-1}a^2_n\delta_{q^2_n},(\alpha,q^2_N]\right),$$

where $y_{2n-1}=z^1_n=\zeta(q^1_n)$ and $y_{2n}=z^2_n=\zeta(q^2_n)$ for $1\le n\le N$. We thus have

$$\int_{\mathcal{I}} h\,\mathrm{d}\mu^1=\sum_{k=1}^{N}a^1_n h(q^1_n),$$

where we recall that the compact support of h is included in $[M,\beta)$. We may with no loss of generality assume that $M<x_{R^1}(p_\infty)=x_{R^2}(p_\infty)$; otherwise, the above integral is equal to zero for all (p,ε) sufficiently close to $(p_\infty,0)$, and the result is obvious. We will therefore study the difference

$$\left|\int_{\mathcal{I}} h\,\mathrm{d}\mu^1_\infty-\int_{\mathcal{I}} h\,\mathrm{d}\mu^1\right|=\left|\int_M^{x_{R^1}(p_\infty)} h(x)g^1(p_\infty,x)\,\mathrm{d}x-\sum_{n=1}^{N_M}a^1_n h(q^1_n)\right|,$$

where the integer N_M satisfies $n\le N_M$ if and only if $q^1_n\ge M$. Notice that

$$\begin{aligned}\int_{q^2_{N_M}}^{q^2_1} h(x)g^1(p_\infty,x)\,\mathrm{d}x-\sum_{n=1}^{N_M}a^1_n h(q^1_n) &= \int_{q^2_{N_M}}^{q^2_1} h(x)[g^1(p_\infty,x)-g^1(p,x)]\,\mathrm{d}x\\ &\quad+\sum_{n=2}^{N_M}\int_{q^2_n}^{q^2_{n-1}} h(x)[g^1(p,x)-g^1(p,q^1_n)]\,\mathrm{d}x\\ &\quad+\sum_{n=2}^{N_M}\int_{q^2_n}^{q^2_{n-1}} [h(x)-h(q^1_n)]g^1(p,q^1_n)\,\mathrm{d}x\end{aligned}$$

$$- \sum_{n=2}^{N_M}[a_n^1 - (q_{n-1}^2 - q_n^2)g^1(p, q_n^1)]h(q_n^1)$$
$$- a_1^1 h(q_1^1).$$

Thus, using the triangle inequality, we obtain

$$\begin{aligned}
\left|\int_{\mathcal{I}} h\,\mathrm{d}\mu_\infty^1 - \int_{\mathcal{I}} h\,\mathrm{d}\mu^1\right| &= \left|\int_M^{x_{R^1}(p_\infty)} h(x)g^1(p_\infty, x)\,\mathrm{d}x - \sum_{n=1}^{N_M} a_n^1 h(q_n^1)\right| \\
&\leq \left|\int_M^{x_{R^1}(p_\infty)} h(x)[g^1(p_\infty, x) - g^1(p, x)]\,\mathrm{d}x\right| \\
&+ \left|\sum_{n=2}^{N_M}\int_{q_n^2}^{q_{n-1}^2} h(x)[g^1(p, x) - g^1(p, q_n^1)]\,\mathrm{d}x\right| \\
&+ \int_{M\wedge q_{N_M}^2}^{M\vee q_{N_M}^2} g^1(p, x)|h(x)|\,\mathrm{d}x + \int_{x_{R^1}(p_\infty)\wedge q_1^2}^{x_{R^1}(p_\infty)\vee q_1^2} g^1(p, x)|h(x)|\,\mathrm{d}x \\
&+ \left|\sum_{n=2}^{N_M}\int_{q_n^2}^{q_{n-1}^2}[h(x) - h(q_n^1)]g^1(p, q_n^1)\,\mathrm{d}x\right| \\
&+ \left|\sum_{n=2}^{N_M}[a_n^1 - (q_{n-1}^2 - q_n^2)g^1(p, q_n^1)]h(q_n^1)\right| \\
&+ a_1^1|h(q_1^1)|, \qquad \text{(S.92)}
\end{aligned}$$

where the fourth line in (S.92) reflects the fact that we can have (1) either $M < q_{N_M}^2$ or $M \geq q_{N_M}^2$ and (2) either $x_{R^1}(p_\infty) < q_1^2$ or $x_{R^1}(p_\infty) \geq q_1^2$. The remainder of the proof consists in showing that each term in the right-hand side of (S.92) goes to 0 when $(p, \varepsilon) \to (p_\infty, 0)$.

Step 3: Key Estimates The proof proceeds by establishing a number of estimates. We consider each of these in turn.

(i) Let C_1, C_2, C_3, and C_4 be the constants given by Lemma S.6.3(iv) for the compact $K(1, M)$. By Lemma S.6.4(ii), there exist η and $\bar{m}$ such that, for all $m \in (0, \bar{m})$, there exists ε_1 such that for all $\varepsilon \in (0, \varepsilon_1)$, $p \in B_\eta$, and $n \geq 1$,

$$y_n \leq \zeta(M) \Rightarrow \xi_n \leq m. \qquad \text{(S.93)}$$

Up to reducing $\bar{m}$ and η, we may assume that

$$0 < \bar{m} < \min_{i=1,2}\min_{p\in B_\eta}\min_{y\in[\zeta(x_{R^1}(p)),\zeta(M)+C_2\sqrt{\bar{m}}]} \hat{G}^i(p, y) - \hat{R}^i(p, y), \qquad \text{(S.94)}$$

$$\alpha^1(p_\infty) < \zeta^{-1}\big[\zeta(M) + C_2\sqrt{\bar{m}}\,\big], \qquad \text{(S.95)}$$

$$\forall p \in B_\eta,\ x_{R^1}(p) \leq x_{R^1}(p_\infty) + 1. \qquad \text{(S.96)}$$

Observe that (S.91) and (S.93) imply $y_{2N} > \zeta(M)$. Letting $N_3 = \min\{n \geq 1 : y_n > \zeta(M)\}$, we thus have $N_3 \leq 2N$. Proceeding as for (S.80) in the proof of Lemma S.6.4(ii), we can choose η small enough that we can apply Lemma S.6.3(iv) to the compact set $K(1, M)$, yielding, for $1 \leq n < N_3$,

$$y_n + C_1\sqrt{\xi_n} \leq y_{n+1} \leq y_n + C_2\sqrt{\xi_n}, \tag{S.97}$$

$$C_3\xi_n \leq \xi_{n+1} \leq C_4\xi_n. \tag{S.98}$$

Thus $y_{n+1} - y_n \leq C_2\sqrt{\xi_n}$ for any such n, which, together with (S.93), implies

$$\sup_{1\leq n<N_3} y_{n+1} - y_n = o(1), \tag{S.99}$$

where $o(1)$ is a function of (p, ε) that tends to 0 as $(p, \varepsilon) \to (p_\infty, 0)$.

(ii) By (S.96)–(S.97), the definition of N_3, and (S.93), we have

$$\zeta(x_{R^1}(p_\infty) + 1) \leq y_1 \leq y_{N_3} \leq y_{N_3-1} + C_2\sqrt{\xi_{N_3-1}} \leq \zeta(M) + C_2\sqrt{\bar{m}}. \tag{S.100}$$

In line with (S.95)–(S.96), consider the compact sets

$$K \equiv [\zeta^{-1}(\zeta(M) + C_2\sqrt{\bar{m}}), x_{R^1}(p_\infty) + 1] \quad \text{and} \quad K' \equiv \zeta(K).$$

Using that ζ^{-1} is Lipschitz over K', (S.99) implies that

$$\sup_{1<n\leq\lfloor\frac{N_3}{2}\rfloor} |q_{n-1}^2 - q_n^2| = o(1). \tag{S.101}$$

(iii) By (S.97), we also have

$$\zeta(M) + C_2\sqrt{\bar{m}} \geq y_{N_3-1} + (y_{N_3} - y_{N_3-1}) = y_{N_3} = y_1 + \sum_{n=1}^{N_3-1}(y_{n+1} - y_n) \geq \sum_{n=1}^{N_3-1} C_1\sqrt{\xi_n},$$

and thus

$$\begin{aligned}
\sum_{n=1}^{N_3-1}(y_{n+1} - y_n)^2 &\leq \sum_{n=1}^{N_3-1} C_2^2\xi_n \\
&\leq \left(\sup_{1\leq n\leq N_3-1}\sqrt{\xi_n}\right) C_2^2 \sum_{n=1}^{N_3-1}\sqrt{\xi_n} \\
&\leq \left(\sup_{1\leq n\leq N_3-1}\sqrt{\xi_n}\right) C_2^2\,\frac{\zeta(M) + \sqrt{\bar{m}}C_2}{C_1} \xrightarrow[\varepsilon\to 0]{} 0.
\end{aligned}$$

Using again that ζ^{-1} is Lipschitz over K', we deduce that

$$\sum_{n=2}^{\lfloor\frac{N_3}{2}\rfloor}(q_{n-1}^2 - q_n^2)^2 = o(1). \tag{S.102}$$

(iv) The coefficients $(a_n^1)_{n=1}^N$ satisfy the relations

$$\frac{1}{2}\,\Delta w^{2\prime}(q_n^1) = a_n^1[w^2(q_n^1) - G^2(p, q_n^1)], \tag{S.103}$$

where, omitting the dependence on p, w^2 denotes the brvf of player 2 in equilibrium. Notice first that

$$\begin{aligned} w^2(q_n^1) - G^2(p, q_n^1) &= R^2(p, q_n^1) - G^2(p, q_n^1) + [w^2(q_n^1) - R^2(p, q_n^1)] \\ &= R^2(p, q_n^1) - G^2(p, q_n^1) + \psi(q_n^1)\big[\hat{w}^2(z_n^1) - \hat{R}^2(p, z_n^1)\big]. \end{aligned} \tag{S.104}$$

From (S.68), for $1 < n \le N$, and omitting again the dependence on p, we have $w^2 = T^2_{q^2_{n-1}}$ over $[q_n^1, q_{n-1}^2]$ and $w^2 = T^2_{q_n^2}$ over $[q_n^2, q_n^1]$, so that

$$\Delta w^{2\prime}(q_n^1) = T^{2\prime}_{q^2_{n-1}}(q_n^1) - T^{2\prime}_{q^2_n}(q_n^1). \tag{S.105}$$

By construction, we have

$$\begin{aligned} T_q^2(x) &= \hat{T}^2_{\zeta(q)}(\zeta(x))\psi(x) \\ T_q^{2\prime}(x) &= \hat{T}^{2\prime}_{\zeta(q)}(\zeta(x))\zeta'(x)\psi(x) + \hat{T}^2_{\zeta(q)}(\zeta(x))\psi'(x). \end{aligned}$$

Therefore, by continuity of w^2 at q_n^1,

$$\begin{aligned} T^{2\prime}_{q^2_{n-1}}(q_n^1) - T^{2\prime}_{q^2_n}(q_n^1) &= \zeta'(q_n^1)\psi(q_n^1)\big[\hat{R}^{2\prime}(p, z_{n-1}^2) - \hat{R}^{2\prime}(p, z_n^2)\big] \\ &= \zeta'(q_n^1)\psi(q_n^1)\int_{z^2_{n-1}}^{z^2_n}\big[-\hat{R}^{2\prime\prime}(p, z)\big]\,\mathrm{d}z. \end{aligned} \tag{S.106}$$

From (S.103)–(S.106), we obtain

$$a_n^1 = \frac{\frac{1}{2}\,\zeta'(q_n^1)\psi(q_1^n)\int_{z^2_{n-1}}^{z^2_n}\big[-\hat{R}^{2\prime\prime}(p, z)\big]\,\mathrm{d}z}{R^2(p, q_n^1) - G^2(p, q_n^1) + \psi(q_n^1)\big[\hat{w}^2(z_n^1) - \hat{R}^2(p, z_n^1)\big]}.$$

From the definitions of N_3 and N_M, we have $N_M = \left\lfloor \frac{N_3}{2} \right\rfloor$. From now on, we fix some $1 < n \le N_M$. Using that ψ is bounded on K, $\hat{R}^{2\prime\prime}$ is bounded over $P \times K'$, and ζ is Lipschitz over K, we have

$$\begin{aligned} \psi(q_n^1)\big|\hat{w}^2(z_n^1) - \hat{R}^2(p, z_n^1))\big| &= \psi(q_n^1)\left|\int_{z^2_{n-1}}^{z^1_n}\big[\hat{R}^{2\prime}(p, z^2_{n-1}) - \hat{R}^{2\prime}(p, s)\big]\,\mathrm{d}s\right| \\ &= \psi(q_n^1)\left|\int_{z^2_{n-1}}^{z^1_n}\int_{z^2_{n-1}}^{s}\big[-\hat{R}^{2\prime\prime}(p, u)\big]\,\mathrm{d}u\,\mathrm{d}s\right| \\ &= O((z_n^1 - z_{n-1}^2)^2) \\ &= O((q_n^2 - q_{n-1}^2)^2). \end{aligned} \tag{S.107}$$

Next, using that $\hat{R}^{2\prime\prime}$ is Lipschitz with respect to its second argument over $P \times K'$ and ζ' is Lipschitz over K, we have

$$\begin{aligned}
&\left| \int_{z_{n-1}^2}^{z_n^2} \big[-\hat{R}^{2\prime\prime}(p,s) \big] \,\mathrm{d}s - \zeta'(q_n^1)(q_{n-1}^2 - q_n^2)\hat{R}^{2\prime\prime}(p,z_n^1) \right| \\
&\qquad \le \left| \int_{z_{n-1}^2}^{z_n^2} \big[-\hat{R}^{2\prime\prime}(p,s) \big] \,\mathrm{d}s - (z_n^2 - z_{n-1}^2)\hat{R}^{2\prime\prime}(p,z_n^1) \right| \\
&\qquad\quad + \big| -\hat{R}^{2\prime\prime}(p,z_n^1)[z_n^2 - z_{n-1}^2 - \zeta'(q_n^1)(q_n^2 - q_{n-1}^2)] \big| \\
&\qquad = O((q_{n-1}^2 - q_n^2)^2).
\end{aligned}$$

Recall from (S.2) that

$$\hat{R}^{2\prime\prime}(p,\zeta(\cdot)) = \frac{2\psi^3}{(\varrho\sigma p')^2} \left[\mathcal{L}R^2(p,\cdot) - rR^2(p,\cdot) \right] \quad \text{and} \quad \zeta' = \frac{\varrho p'}{\psi^2}. \tag{S.108}$$

Therefore, for (p,ε) close enough to $(p_\infty, 0)$,

$$\begin{aligned}
&|a_n^1 - (q_{n-1}^2 - q_n^2)g^1(p,q_n^1)| \\
&= \left| a_n^1 - (q_{n-1}^2 - q_n^2)\, \frac{1}{\sigma^2(q_n^1)} \cdot \frac{[rR^2(p,\cdot) - \mathcal{L}R^2(p,\cdot)](q_n^1)}{G^2(p,q_n^1) - R^2(p,q_n^1)} \right| \\
&= \left| \frac{\frac{1}{2}\zeta'(q_n^1)\psi(q_1^n) \left\{ \int_{z_{n-1}^2}^{z_n^2} \big[-\hat{R}^{2\prime\prime}(p,z) \big] \,\mathrm{d}z - \zeta'(q_n^1)(q_{n-1}^2 - q_n^2)\hat{R}^{2\prime\prime}(p,z_n^1) \right\}}{R^2(p,q_n^1) - G^2(p,q_n^1)} \right| + O((q_n^2 - q_{n-1}^2)^2) \\
&= O((q_{n-1}^2 - q_n^2)^2),
\end{aligned} \tag{S.109}$$

where the first equality follows from the definition (S.90) of $g^1(p,\cdot)$, the second equality follows from (S.107)–(S.108) along with the fact that $G^2(p,q_n^1) - R^2(p,q_n^1)$ is bounded below by $\bar{m}$ by (S.94) and (S.100).

(v) Consider finally the case $n = 1$. From (S.68), and omitting again the dependence on p, we have $w^2 = \frac{w^2(q_1^1)}{\phi(q_1^1)}\,\phi$ over $[q_1^1, \beta)$ and $w^2 = T_{q_1^2}^2$ over $[q_1^2, q_1^1]$. Thus

$$\Delta w^{2\prime}(q_1^1) = \frac{w^2(q_1^1)}{\phi(q_1^1)}\,\phi'(q_1^1) - T_{q_1^2}^{2\prime}(q_1^1) = \left[\frac{w^2(q_1^1)}{\phi(q_1^1)} - \hat{R}^{2\prime}(z_1^2) \right] \zeta'(q_1^1)\psi(q_1^1),$$

where we have used that $\hat{T}_{q_1^2}^2(z_1^1) = \hat{w}^2(z_1^1) = \frac{w^2(q_1^1)}{\psi(q_1^1)}$ and that $\zeta'\psi = \phi\left(\frac{\phi'}{\phi} - \frac{\psi'}{\psi}\right)$. We obtain

$$\begin{aligned}
|a_1^1| &= \left| \frac{\frac{1}{2}\left[\frac{w^2(q_1^1)}{\phi(q_1^1)} - \hat{R}^{2\prime}(z_1^2) \right] \zeta'(q_1^1)\psi(q_1^1)}{R^2(p,q_1^1) - G^2(p,q_1^1) + \psi(q_1^1)\big[\hat{w}^2(z_1^1) - \hat{R}^2(p,z_1^1) \big]} \right| \\
&= O\left(\left| \frac{w^2(q_1^1)}{\phi(q_1^1)} - \hat{R}^{2\prime}(z_1^2) \right| \right),
\end{aligned} \tag{S.110}$$

where the second inequality follows along the same lines as in (iv).

Step 4: Taking Limits in (S.92) We can now compute the limits as $(p,\varepsilon) \to (p_\infty, 0)$ of all the terms on the right-hand side of inequality (S.92).

(i) First, by continuity of g^1 with respect to p, we have

$$\left| \int_M^{x_{R^1}(p_\infty)} h(x)[g^1(p_\infty, x) - g^1(p,x)]\,\mathrm{d}x \right| \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.111}$$

by bounded convergence.

(ii) Second, using that g^1 is Lipschitz with respect to its second argument over $B_\eta \times K$, we obtain, for some constant C, that

$$\left| \sum_{n=2}^{N_M} \int_{q_n^2}^{q_{n-1}^2} h(x)[g^1(p,x) - g^1(p,q_n^1)]\,\mathrm{d}x \right| \leq \sum_{n=2}^{N_M} \int_{q_n^2}^{q_{n-1}^2} C|x - q_n^1|\,\mathrm{d}x \tag{S.112}$$

$$\leq \sum_{n=2}^{N_M} C(q_{n-1}^2 - q_n^2)^2 \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.113}$$

by (S.102).

(iii) Third, from (S.99) and the definition of N_M, $q_{N_M}^2 \to M$ as $(p,\varepsilon) \to (p_\infty, 0)$. Similarly, $q_1^2 - q_1^1 = q_1^2 - x_{R^1}(p) \to 0$ and $x_{R^1}(p) \to x_{R^1}(p_\infty)$ as $(p,\varepsilon) \to (p_\infty, 0)$. Therefore,

$$\int_{M\wedge q_{N_M}^2}^{M \vee q_{N_M}^2} g^1(p,x)|h(x)|\,\mathrm{d}x + \int_{x_{R^1}(p_\infty)\wedge q_1^2}^{x_{R^1}(p_\infty)\vee q_1^2} g^1(p,x)|h(x)|\,\mathrm{d}x \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.114}$$

by bounded convergence.

(iv) Fourth, because h is uniformly continuous over its compact support, we have

$$|h(x) - h(q_n^1)| \leq \rho_h(q_{n-1}^2 - q_n^2) \leq \rho_h(o(1)), \quad 1 < n \leq N_M,\ x \in [q_n^2, q_{n-1}^2]$$

by (S.101), where ρ_h denotes the modulus of continuity of h. Because g^1 is bounded over $B_\eta \times K$, we obtain that, for some constant C,

$$\left| \sum_{n=2}^{N_M} \int_{q_n^2}^{q_{n-1}^2} [h(x) - h(q_n^1)] g^1(p, q_n^1)\,\mathrm{d}x \right| \leq C(q_1^2 - q_{N_M}^2)\rho_h(o(1)) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.115}$$

as $q_1^2 - q_{N_M}^2 \to x_{R^1}(p_\infty) - M$ as $(p,\varepsilon) \to (p_\infty, 0)$.

(v) Fifth, using (S.109), we obtain that, for some constant C,

$$\left| \sum_{n=2}^{N_M} [a_n^1 - (q_{n-1}^2 - q_n^2) g^1(p, q_n^1)] h(q_n^1) \right| \leq \sum_{n=2}^{N_M} C(q_{n-1}^2 - q_n^2)^2 \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.116}$$

by (S.102).

(vi) Finally, on the one hand, we have

$$\hat{R}^{2\prime}(z_1^2) \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} \hat{R}^{2\prime}(\zeta(x_{R^1}(p_\infty)) = \hat{R}^{2\prime}(\zeta(x_{R^2}(p_\infty)) = \frac{R^2(x_{R^2}(p_\infty))}{\phi(x_{R^2}(p_\infty))},$$

where the last equality follows from (S.65). On the other hand, by construction, we have

$$\hat{w}^2(z_1^1) = \hat{R}^2(z_1^1) + \varepsilon,$$

and thus

$$w^2(q_1^1) = R^2(q_1^1) + \psi(q_1^1)\varepsilon \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} R^2(x_{R^2}(p_\infty)).$$

Therefore,

$$a_1^1|h(q_1^1)| \underset{(p,\varepsilon)\to(p_\infty,0)}{\longrightarrow} 0 \tag{S.117}$$

by (S.110). Gathering the limits in (S.111)–(S.117) concludes the proof of (S.89) by (S.92).

Step 5: Convergence of the Maximum of the Stopping Regions By construction, $\max S^1 \cup S^2 = q_N^2$ with $q_N^2 \geq \alpha^2(p)$. Using the same arguments as above, for each $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, we have $q_N^2 < M$ for (p,ε) sufficiently close to $(p_\infty, 0)$, and thus $q_N^2 \to \alpha^1(p_\infty) = \alpha^2(p_\infty)$ as $(p,\varepsilon) \to (p_\infty, 0)$.

Step 6: Convergence of the Equilibrium Payoffs Finally, we prove that

$$\forall x \in \mathcal{I}, \lim_{(p,\varepsilon)\to(p_\infty,0)} w^i(x) = V_{R^i}(p_\infty, x) \equiv \begin{cases} R^i(p_\infty, x) & \text{if } x \leq x_{R^i}(p_\infty) \\ T^i_{x_{R^i}(p_\infty)}(x) & \text{if } x > x_{R^i}(p_\infty) \end{cases}.$$

That, for each $x > x_{R^1}(p_\infty)$, $\lim_{(p,\varepsilon)\to(p_\infty,0)} w^i(x) = T^1_{x_{R^1}(p_\infty)}(x)$ follows from the construction of Lemma S.6.1 for the brvfs w^1 and w^2 over the intervals $[q_1^1, \beta)$ and $[q_1^2, \beta)$ along with the fact that, by Lemma S.6.4(i), for all i and n, $q_n^i \to x_{R^1}(p_\infty)$ as $q_1^2 - q_1^1 \to 0$ and $p \to p_\infty$. That, for each $x \leq x_{R^1}(p_\infty)$, $\lim_{(p,\varepsilon)\to(p_\infty,0)} w^i(x) = R^i(p_\infty, x)$ follows from Lemma S.6.4(ii), which implies that, for all i and $M \in (\alpha^1(p_\infty), x_{R^1}(p_\infty))$, the brvf w^i converges uniformly to $V_{R^i}(p_\infty, \cdot) = R^i(p_\infty, \cdot)$ over $[M, x_{R^i}(p)]$ as $q_1^2 - q_1^1 \to 0$ and $p \to p_\infty$. Hence the result. ■

S.8 Proofs for Section 6

Lemma S.8.1 *The equation*

$$R^1(x_{R^1}) = \frac{\phi(x_{R^1})}{\phi(x)}\, G^1(x) \tag{S.118}$$

has a unique solution $s^2 \in (\alpha^1, x_{R^1})$ and $R^1(x_{R^1}) < \frac{\phi(x_{R^1})}{\phi(x)}\, G^1(x)$ over (s^2, β).

Proof: For each $x \in \mathcal{I}$, let $f(x) \equiv \frac{\phi(x)}{\phi(x_{R^1})} R^1(x_{R^1})$. Notice that $f = V_{R^1} \geq R^1$ over $[x_{R^1}, \beta)$ and that $f'(x_{R^1}) = R^{1\prime}(x_{R^1})$ by the smooth-fit property. Applying the change-of-variables formula (S.1) to f, a direct computation shows that s^2 is a solution to (S.118) if and only if $\zeta(s^2)$ is a solution to

$$\hat{f}(y) = \hat{G}^1(y), \text{ that is, } \frac{R^1(x_{R^1})}{\phi(x_{R^1})}\, y = \hat{G}^1(y).$$

Because $f = V_{R^1}$ over $[x_{R^1}, \beta)$, it follows from A6 that $\hat{f} < \hat{G}^1$ over $(0, \zeta(x_{R^1})]$. Because $\hat{G}^1$ is positive, concave, and satisfies (S.5), it follows in turn that (S.118) admits a unique solution $s^2 < x_{R^1}$, and that $\frac{\phi}{\phi(x_{R^1})} R^1(x_{R^1}) > G^1$ over (α, s^2) and $\frac{\phi}{\phi(x_{R^1})} R^1(x_{R^1}) < G^1$ over (s^2, β). Finally, recall from A3 and A6 that $\alpha^1 < x_{R^1} < x_0^1$ and that $\hat{R}^1$ is strictly concave over $(\zeta(x_0^1), \infty)$. Therefore, $\hat{f} > \hat{R}^1$ over $(\zeta(x_{R^1}), \infty)$ as $\hat{f}$ is linear and tangent to $\hat{R}^1$ at $\zeta(x_{R^1})$. Hence, if $\alpha^1 > \alpha$, it must be that $\alpha^1 < s^2$ as $G^1 = R^1$ over $(\alpha, \alpha^1]$. The result follows. ■

Proof of Proposition 5: By Theorem 3, a simple singular Mpe of type 2 exists if and only if there exist a constant $a_1^1 > 0$ and two functions $w^1 \in \mathcal{C}^0((0,\infty)) \cap \mathcal{C}^2((0,\infty) \setminus \{s^2\})$ and $w^2 \in \mathcal{C}^0((0,\infty)) \cap \mathcal{C}^2((0,\infty) \setminus \{s^2, x_{R^1}\})$ solution to the variational system

$$w^2 \geq R^2 \text{ over } (0,\infty), \tag{S.119}$$

$$w^2 = R^2 \text{ over } (0, s^2], \tag{S.120}$$

$$w^{2\prime}(s^2) = R^{2\prime}(s^2), \tag{S.121}$$

$$w^2(\infty) = 0, \tag{S.122}$$

$$\mathcal{L}w^2 - rw^2 = 0 \text{ over } (s^2,\infty) \setminus \{x_{R^1}\}, \tag{S.123}$$

$$a_1^1[G^2(x_{R^1}) - w^2(x_{R^1})] + \frac{1}{2}\,\Delta w^{2\prime}(x_{R^1}) = 0, \tag{S.124}$$

$$w^1 \geq R^1 \text{ over } (0,\infty), \tag{S.125}$$

$$w^1 = G^1 \text{ over } (0, s^2], \tag{S.126}$$

$$w^1(\infty) = 0, \tag{S.127}$$

$$w^1(x_{R^1}) = R^1(x_{R^1}), \tag{S.128}$$

$$\mathcal{L}w^1 - rw^1 = 0 \text{ over } (s^2, \infty). \tag{S.129}$$

We shall use the standard fact (see, for instance, Dixit and Pindyck (1994)) that, in the running example, $\phi(x) = x^{\rho^-}$ and $\psi(x) = x^{\rho^+}$ for all $x \in (0,\infty)$, where

$$\rho^- \equiv \frac{1}{2} - \frac{b}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \text{ and } \rho^+ \equiv \frac{1}{2} - \frac{b}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{S.130}$$

The proof consists of two parts. We first characterize a candidate solution to (S.119)–(S.129) and provide necessary and sufficient conditions for its existence. We then investigate whether these conditions are met under the parameter restrictions spelled out in (i)–(iii).

Necessary and Sufficient Conditions Using the notation of Section 6, we have

$$V_{R^i}(x) = \sup_{\tau\in\mathcal{T}} \mathbf{E}_x\,[\mathrm{e}^{-r\tau}R^i(X_\tau)] = \begin{cases} \left(\frac{x}{x_{R^i}}\right)^{\rho^-}(l^i - \frac{1}{r-b}\,x_{R^i}) & \text{if } x > x_{R^i} \\ l^i - \frac{1}{r-b}\,x & \text{if } x \leq x_{R^i} \end{cases},$$

where $x_{R^i} \equiv \frac{\rho^-}{\rho^- -1}\,(r-b)l^i$, so that $l^i - \frac{1}{r-b}\,x_{R^i} = \frac{l^i}{1-\rho^-}$. Similarly,

$$V^i_m(x) = \sup_{\tau\in\mathcal{T}} \mathbf{E}_x\left[\int_0^\tau \mathrm{e}^{-rt}mX_t\,\mathrm{d}t + \mathrm{e}^{-r\tau}l^i\right] = \begin{cases} \frac{m}{r-b}\,x + \left(\frac{x}{\alpha^i}\right)^{\rho^-}(l^i - \frac{m}{r-b}\,\alpha^i) & \text{if } x > \alpha^i \\ l^i & \text{if } x \leq \alpha^i \end{cases},$$

where $\alpha^i \equiv \frac{x_{R^i}}{m} < x_{R^i}$, so that $l^i - \frac{m}{r-b}\,\alpha^i = \frac{l^i}{1-\rho^-}$. Thus

$$G^i(x) = (V^i_m - E)(x) = \begin{cases} \frac{m-1}{r-b}\,x + \left(\frac{x}{\alpha^i}\right)^{\rho^-}\frac{l^i}{1-\rho^-} & \text{if } x > \alpha^i \\ l^i - \frac{1}{r-b}\,x & \text{if } x \leq \alpha^i \end{cases}. \tag{S.131}$$

As for player 1, the above calculations allow us to rewrite (S.118) as

$$\left[\frac{x}{\frac{\rho^-}{\rho^- -1}\,(r-b)l^1}\right]^{\rho^-}\frac{l^1}{1-\rho^-} = \frac{m-1}{r-b}\,x + \left[\frac{mx}{\frac{\rho^-}{\rho^- -1}\,(r-b)l^1}\right]^{\rho^-}\frac{l^1}{1-\rho^-}.$$

Solving this equation yields $x = s^2 \equiv \xi x_{R^1}$, where

$$\xi \equiv \left[\frac{1-m^{\rho^-}}{\rho^-(1-m)}\right]^{\frac{1}{1-\rho^-}} \in \left(\frac{1}{m}, 1\right). \tag{S.132}$$

It follows that the function w^1 defined by

$$w^1(x) \equiv \begin{cases} \left(\frac{x}{x_{R^1}}\right)^{\rho^-}\frac{l^1}{1-\rho^-} & \text{if } x > s^2 \\ G^1(x) & \text{if } x \leq s^2 \end{cases}$$

is, by construction, solution to the variational system (S.125)–(S.129).

As for player 2, the characterization of singular Mpes in Lemma S.6.1 implies that a solution (w^2, a^1_1) to (S.119)–(S.124) with $a^1_1 > 0$ exists if and only if

$$G^2(x_{R^1}) > T^2_{s^2}(x_{R^1}) > T^2_{x_{R^2}}(x_{R^1}) \tag{S.133}$$

and that, in line with (S.120)–(S.124), the desired solution writes as

$$w^2 = \mathbb{1}_{(0,s^2]}\,R^2 + \mathbb{1}_{(s^2,x_{R^1}]}\,T^2_{s^2} + \mathbb{1}_{(x_{R^1},\infty)}\,A\phi \;\text{ and }\; a^1_1 = \frac{-\Delta w^{2\prime}(x_{R^1})}{G^2(x_{R^1}) - T^2_{s^2}(x_{R^1})} \tag{S.134}$$

for some coefficient A to be determined shortly. In particular, $s^2 < x_{R^2}$ and $\Delta w^{2\prime}(x_{R^1}) < 0$. Now, $T^2_{s^2} = B\psi + C\phi$, with coefficients B and C given by

$$B \equiv \frac{-\phi'(s^2)\big(l^2 - \frac{1}{r-b}\, s^2\big) - \frac{1}{r-b}\,\phi(s^2)}{\psi'(s^2)\phi(s^2) - \psi(s^2)\phi'(s^2)} = \frac{\rho^-(s^2)^{-\rho^+}}{\Delta\rho}\,(\xi l^1 - l^2), \tag{S.135}$$

$$C \equiv \frac{\psi'(s^2)\big(l^2 - \frac{1}{r-b}\, s^2\big) + \frac{1}{r-b}\,\psi(s^2)}{\psi'(s^2)\phi(s^2) - \psi(s^2)\phi'(s^2)} = \frac{\rho^+(s^2)^{-\rho^-}}{\Delta\rho}\left[l^2 - \frac{\rho^-(\rho^+ - 1)}{\rho^+(\rho^- - 1)}\,\xi l^1\right], \tag{S.136}$$

where $\Delta\rho \equiv \rho^+ - \rho^-$. Moreover, because w^2 is continuous at x_{R^1}, it must be that

$$A \equiv B\,\frac{\psi(x_{R^1})}{\phi(x_{R^1})} + C. \tag{S.137}$$

It follows from (S.137) that

$$\Delta w^{2\prime}(x_{R^1}) = B\left[\frac{\psi(x_{R^1})}{\phi(x_{R^1})}\,\phi'(x_{R^1}) - \psi'(x_{R^1})\right]. \tag{S.138}$$

We know from (S.4) that the bracketed term in (S.138) is negative. Hence $\Delta w^{2\prime}(x_{R^1}) < 0$, as required, if and only if $B > 0$. From (S.135), and as $\rho^- < 0$, this is the case if and only if $\frac{l^2}{l^1} > \xi$, in which case, from (S.136), and as $\rho^+ > 0$, we also have $C > 0$; in particular, $T^2_{s^2}$ and, hence, w^2 is strictly convex over $[s^2, x_{R^1}]$.

Returning to (S.133), it is immediate that

$$T^2_{x_{R^2}}(x_{R^1}) = \left(\frac{l^1}{l^2}\right)^{\rho^-}\frac{l^2}{1-\rho^-}, \tag{S.139}$$

and, using $s^2 = \xi x_{R^1}$, it is easy to check from (S.135)–(S.136) that

$$T^2_{s^2}(x_{R^1}) = Bx_{R^1}^{\rho^+} + Cx_{R^1}^{\rho^-} = \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho}\,(\xi l^1 - l^2) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho}\left[l^2 - \frac{\rho^-(\rho^+ - 1)}{\rho^+(\rho^- - 1)}\,\xi l^1\right]. \tag{S.140}$$

As for the computation of $G^2(x_{R^1})$, notice first that a necessary condition for a simple singular Mpe of type 2 is $x_{R^1} > \alpha^2$, that is, $m > \frac{l_2}{l_1}$; otherwise, player 2 would not be willing to hold fast on (s^2, x_{R^1}). Hence, by (S.131), we have

$$G^2(x_{R^1}) = \frac{m-1}{r-b}\,x_{R^1} + \left(\frac{x_{R^1}}{\alpha^2}\right)^{\rho^-}\frac{l^2}{1-\rho^-} = \frac{\rho^-}{\rho^- - 1}\,(m-1)l^1 + \left(\frac{l^1}{l^2}\right)^{\rho^-}\frac{l^2}{1-\rho^-}\,m^{\rho^-}. \tag{S.141}$$

Gathering (S.139)–(S.141) and rearranging, we obtain that (S.133) holds if and only if

$$\begin{aligned} &\frac{\rho^-}{\rho^- - 1}\,(m-1) + \frac{\pi^{1-\rho^-}}{1-\rho^-}\,m^{\rho^-} \\ &\qquad > \frac{\rho^-\xi^{-\rho^+}}{\Delta\rho}\,(\xi - \pi) + \frac{\rho^+\xi^{-\rho^-}}{\Delta\rho}\left[\pi - \frac{\rho^-(\rho^+ - 1)}{\rho^+(\rho^- - 1)}\,\xi\right] \\ &\qquad > \frac{\pi^{1-\rho^-}}{1-\rho^-}, \end{aligned} \tag{S.142}$$

where $\pi \equiv \frac{l^2}{l^1} \in (\xi, m)$. We are interested in the case where l^1 and l^2 are close to each other, in the sense that π is close to 1. In that scenario, given (S.142),

$$\begin{aligned}\frac{\rho^-}{\rho^- - 1}(m-1) + \frac{1}{1-\rho^-}\, m^{\rho^-} \\ &> \frac{\rho^- \xi^{-\rho^+}}{\Delta\rho}(\xi - 1) + \frac{\rho^+ \xi^{-\rho^-}}{\Delta\rho}\left[1 - \frac{\rho^-(\rho^+ - 1)}{\rho^+(\rho^- - 1)}\,\xi\right] \\ &> \frac{1}{1-\rho^-} \end{aligned} \tag{S.143}$$

is a necessary and sufficient condition for a simple singular equilibrium of type 2.

Checking (S.143) We consider in turn each case of the proposition.

(i)–(ii) As for the second inequality in (S.143), notice from (S.132) that, because $\rho^- < 0$, ξ goes to 0 as m goes to ∞. Thus, because $\rho^+ > 0$ and $\rho^-(\xi - 1) > 0$, the left-hand side of the second inequality in (S.143) goes to ∞ as m goes to ∞. Therefore, this inequality is satisfied for m large. As for the first inequality in (S.143), notice from (S.132) that the right-hand side is of order $m^{\frac{\rho^+}{1-\rho^-}}$ as m goes to ∞, while the left-hand side is of order m. Therefore, the first inequality in (S.143) is satisfied for m large if $\rho^+ + \rho^- < 1$, that is, from (S.130), if $b > 0$. By contrast, if $b < 0$, so that $\rho^+ + \rho^- > 1$, this order comparison is reversed, and the first inequality in (S.143) is no longer satisfied for m large.

(iii) To simplify notation, let us rewrite (S.143) as $\mathfrak{L}(m) > \mathfrak{M}(\xi) > \mathfrak{R}$. It is easy to check that, when $m = 1$ and thus $\xi = 1$, $\mathfrak{L}(1) = \mathfrak{M}(1) = \mathfrak{R}$. Straightforward computations yield

$$\mathfrak{M}'(\xi) = \frac{\rho^-}{\Delta\rho}\left(\xi^{-\rho^+} - \xi^{-\rho^-}\right)\left(1 - \rho^+ + \frac{\rho^+}{\xi}\right) \tag{S.144}$$

for all $\xi \in (0,1)$, and thus $\mathfrak{M}'(\xi) < 0$ for any such ξ as $\rho^- < 0$ and $\rho^+ > 0$. Hence, because $\mathfrak{M}(1) = \mathfrak{R}$ and $\mathfrak{R}$ is constant, $\mathfrak{M}(\xi) > \mathfrak{R}$ for all $\xi \in (0,1)$, as desired. Consider now the inequality $\mathfrak{L}(m) > \mathfrak{M}(\xi(m))$, where we have made explicit the dependence of ξ on m, as in (S.132). Straightforward but tedious computations show that

$$\mathfrak{L}'(m) = \frac{\rho^-}{\rho^- - 1}\left(1 - m^{\rho^- - 1}\right), \tag{S.145}$$

$$\begin{aligned}\frac{\mathrm{d}\mathfrak{M}(\xi(m))}{\mathrm{d}m} &= \frac{\rho^-}{\Delta\rho}\left\{[\xi(m)]^{-\rho^+} - [\xi(m)]^{-\rho^-}\right\}\left[1 - \rho^+ + \frac{\rho^+}{\xi(m)}\right] \\ &\times \frac{\rho^-}{\rho^- - 1}\left\{m^{\rho^- - 1} - [\xi(m)]^{1-\rho^-}\right\}\frac{\xi(m)}{1 - m^{\rho^-}}. \end{aligned} \tag{S.146}$$

where the first line of the right-hand side of (S.146) is $\mathfrak{M}'(\xi(m))$ by (S.144), and the second line is $\xi'(m)$. Recalling from (S.132) that $\xi(m) \in (\frac{1}{m}, 1)$ for all $m > 1$, we obtain that $\mathfrak{M}'(\xi(m)) < 0$ and $\xi'(m) < 0$ for any such m. Hence, although $\mathfrak{L}'(1) = \frac{\mathrm{d}\mathfrak{M}(\xi(m))}{\mathrm{d}m}(1) = 0$, we can write from (S.145)–(S.146) that

$$\frac{\mathfrak{L}'(m)}{\frac{\mathrm{d}\mathfrak{M}(\xi(m))}{\mathrm{d}m}} \underset{1}{\sim} \frac{\Delta\rho}{\rho^-}\left\{\frac{1 - m^{\rho^- - 1}}{m^{\rho^- - 1} - [\xi(m)]^{1-\rho^-}}\right\}\left\{\frac{1 - m^{\rho^-}}{[\xi(m)]^{-\Delta\rho} - 1}\right\}. \tag{S.147}$$

From (S.132), a direct calculation using L'Hôpital's rule yields

$$\lim_{m\to 1+}\frac{\mathrm{d}[\xi(m)]^{1-\rho^-}}{\mathrm{d}m}=\frac{\rho^- -1}{2}.$$

Thus, by L'Hôpital's rule again,

$$\lim_{m\to 1+}\frac{1-m^{\rho^- -1}}{m^{\rho^- -1}-[\xi(m)]^{1-\rho^-}}=-2, \tag{S.148}$$

$$\lim_{m\to 1+}\frac{1-m^{\rho^-}}{[\xi(m)]^{-\Delta\rho}-1}=\frac{-2\rho^-}{\Delta\rho}. \tag{S.149}$$

Gathering (S.147)–(S.149) yields $\mathfrak{L}'(m)\underset{1}{\sim}4\,\frac{\mathrm{d}\mathfrak{M}(\xi(m))}{\mathrm{d}m}$ and thus, as $\mathfrak{L}(1)=\mathfrak{M}(1)=\mathfrak{M}(\xi(1))$, that $\mathfrak{L}(m)>\mathfrak{M}(\xi(m))$ for $m>1$ close to 1, as desired.

Checking Murto (2004)'s Refinement We have so far constructed an Mpe of the form $((\emptyset, a_1^1\delta_{x_{R^1}}),((0,s^2],0))$. To turn it into an Mpe $(((0,\alpha^1],a_1^1\delta_{x_{R^1}}),((0,s^2],0))$ satisfying the refinement, it is necessary and sufficient that $\alpha^1\vee\alpha^2<s^2$. The inequality $\alpha^1<s^2$ is equivalent to $\frac{1}{m}<\xi(m)$, which is true by (S.132). The inequality $\alpha^2<s^2$ is equivalent to $\pi<m\xi(m)$, which is more demanding than the inequality $\pi<m$ assumed so far. Yet it is easy to check from (S.132) that $m\xi(m)>1=\xi(1)$ is increasing in m, and actually grows without bounds as m goes to ∞. The result follows. ∎

Remark We have shown that the inequality $\mathfrak{M}(\xi)>\mathfrak{R}$ holds for all $\xi\in(0,1)$. By analogy, let us rewrite the second inequality in (S.142) as $\mathfrak{M}(\xi,\pi)>\mathfrak{R}(\pi)$. It is easy to check that $\mathfrak{M}(\xi,\xi)=\mathfrak{R}(\xi)$ for all $\xi\in(0,1)$. Now, $\frac{\partial\mathfrak{M}}{\partial\pi}(\xi,\pi)=\frac{\rho^+\xi^{-\rho^-}-\rho^-\xi^{-\rho^+}}{\Delta\rho}>\mathfrak{R}'(\pi)=\pi^{-\rho^-}$ for all $\xi\in(0,1)$ and $\pi\in(\xi,1]$ if and only if $\rho^+\xi^{-\rho^-}-\rho^-\xi^{-\rho^+}>\Delta\rho$, which is true as the left-hand side of this inequality is strictly decreasing in $\xi\in(0,1)$ and equal to $\Delta\rho$ at $\xi=1$. Hence $\mathfrak{M}(\xi,\pi)-\mathfrak{R}(\pi)$ is strictly increasing in $\pi\in[\xi,1]$ for any such ξ, which shows as $\mathfrak{M}(\xi,\xi)=\mathfrak{R}(\xi)$ that $\mathfrak{M}(\xi,\pi)>\mathfrak{R}(\pi)$ for all $\xi\in(0,1)$ and $\pi\in(\xi,1]$.

Similarly, we have shown that the inequality $\mathfrak{L}(m)>\mathfrak{M}(\xi(m))$ holds for m close to 1, and for m large provided $b>0$. By analogy, let us rewrite the first inequality in (S.142) as $\mathfrak{L}(m,\pi)>\mathfrak{M}(\xi(m),\pi)$, where $\pi\in(\xi(m),m)$. We have $\mathfrak{L}(m,\xi(m))=\frac{\rho^-}{\rho^- -1}(m-1)+\frac{[\xi(m)]^{1-\rho^-}}{1-\rho^-}m^{\rho^-}$ and $\mathfrak{M}(\xi(m),\xi(m))=\frac{[\xi(m)]^{1-\rho^-}}{1-\rho^-}$. Hence, because $\rho^-<0$, $\mathfrak{L}(m,\xi(m))>\mathfrak{M}(\xi(m),\xi(m))$ if and only if $\rho^-(1-m)>[\xi(m)]^{1-\rho^-}(1-m^{\rho^-})$ or, equivalently, by (S.132), if $[\xi(m)]^{2(1-\rho^-)}<1$, which is true if $\xi(m)<1$, that is, if $m>1$. It follows from this that, for each $m>1$, $\mathfrak{L}(m,\pi)>\mathfrak{M}(\xi(m),\pi)$ for all π close to $\xi(m)$.

The upshot of these two observations is that, for each $m>1$, (S.142) holds for all π close to $\xi(m)$. Therefore, in the (ξ,π) plane, a simple singular Mpe of type 2 always exists whenever $0<\xi<1$, $\pi>\xi$, and $\pi\approx\xi$.

The martingale property of firm 2's discounted cum-dividend stock-price process is a direct consequence of the following lemma.

Lemma S.8.2 *Let γ^1 be a randomized stopping time of player 1 associated to $\Lambda^1 \equiv (\Lambda^1_t)_{t\geq 0} \equiv \big(\mathrm{e}^{-a^1 L^{x_{R^1}}_t}\big)_{t\geq 0}$ of the form*

$$\gamma^1 \equiv \inf\{t \geq 0 : \Gamma^1_t > U^1\},$$

where U^1 is uniformly distributed over $[0,1]$ and independent of X. Then γ^1 is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-stopping time and its $(\hat{\mathcal{F}}_t)_{t\geq 1}$-predictable compensator is $(a^1 L^{x_{R^1}}_{t\wedge\gamma^1})_{t\geq 0}$, where $(\hat{\mathcal{F}}_t)_{t\geq 0}$ is the shareholders' filtration defined by (41). In particular, the processes $(1_{\{t\wedge\tau_{\underline{x}^2}\geq\gamma^1\}} - a^1 L^{x_{R^1}}_{t\wedge\tau^c})_{t\geq 0}$ and $(\mathrm{e}^{-rt\wedge\tau^c} V^{2,\tau^c}_t + \int_0^{t\wedge\tau^c} \mathrm{e}^{-rs} X_s \,\mathrm{d}s)_{t\geq 0}$ are $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingales.

Proof: We only need to check that $Z \equiv (Z_t)_{t\geq 0} \equiv (1_{\{t\geq\gamma^1\}} - a^1 L^{x_{R^1}}_{t\wedge\gamma^1})_{t\geq 0}$ is an $(\hat{\mathcal{F}}_t)_{t\geq 0}$-martingale for all $\mathbf{P}_x$, $x \in \mathcal{I}$. Let $s \leq t$, and consider the random variable $U^1_s(\omega, u^1) \equiv u^1 1_{\{\Gamma^1_s(\omega)\geq u^1\}} + 1_{\{\Gamma^1_s(\omega)<u^1\}}$ over the probability space $\Omega^1 \equiv \Omega \times [0,1]$. It is easy to check that

$$\hat{\mathcal{F}}_s = \mathcal{F}_s \vee \sigma(U^1_s) \subset \mathcal{F}_\infty \vee \sigma(U^1_s).$$

From the definition of $\mathcal{F}_\infty$, we have

$$\mathbf{E}_x[Z_t - Z_s \,|\, \mathcal{F}_\infty \vee \sigma(U^1_s)] = \mathbf{E}_x[Z_t - Z_s \,|\, \omega, U^1_s].$$

A version of the conditional law of U^1 given $(\omega, U^1_s(\omega, U^1))$ is

$$1_{\{U^1_s<1\}}\, \delta_{U^1_s} + 1_{\{U^1_s=1\}}\, \mathcal{U}_{[\Gamma^1_s, 1]},$$

where $\mathcal{U}_{[a,b]}$ denotes the uniform distribution over $[a,b]$. Hence

$$1_{\{U^1_s=1\}}\, \mathbf{P}_x[\gamma^1 \leq t \,|\, \omega, U^1_s] = 1_{\{U^1_s=1\}} \frac{\Gamma^1_t - \Gamma^1_s}{1-\Gamma^1_s}.$$

We deduce that

$$\begin{aligned}
&\mathbf{E}_x[Z_t - Z_s \,|\, \omega, U^1_s] \\
&\quad= \mathbf{E}_x\Big[1_{\{s<\gamma^1\leq t\}} - a^1(L^{x_{R^1}}_{t\wedge\gamma^1} - L^{x_{R^1}}_s) \,|\, \omega, U^1_s\Big] \\
&\quad= \mathbf{E}_x\Big[1_{\{\Gamma^1_s<U^1\leq\Gamma^1_t\}} - a^1(L^{x_{R^1}}_{t\wedge\gamma^1} - L^{x_{R^1}}_s) \,|\, \omega, U^1_s\Big] \\
&\quad= \frac{1_{U^1_s=1}}{1-\Gamma^1_s}\Big[\Gamma^1_t - \Gamma^1_s - a^1 \int_s^t (L^{x_{R^1}}_u - L^{x_{R^1}}_s)\,\mathrm{d}\Gamma^1_u - a^1(1-\Gamma^1_t)(L^{x_{R^1}}_t - L^{x_{R^1}}_s)\Big] \\
&\quad= 0,
\end{aligned}$$

where the fourth equality follows from the integration by parts formula and the fact that

$$\Gamma^1_t - \Gamma^1_s = \int_s^t a^1(1-\Gamma^1_u)\,\mathrm{d}L^{x_{R^1}}_u.$$

We conclude that $\mathbf{E}_x[Z_t - Z_s \,|\, \hat{\mathcal{F}}_s] = 0$ by using the law of iterated conditional expectations. The result follows. ■

The fact that the volatilities of firms' stock returns comove negatively over the attrition zone is a direct consequence of the following lemma.[3]

Lemma S.8.3 *For each $x \in (0,\infty)$, let $F(x) \equiv Bx^{\rho^+} + Cx^{\rho^-} + \frac{x}{r-b}$ for two nonnegative numbers B and C such that $(B,C) \neq (0,0)$. Then $\frac{xF'(x)}{F(x)}$ is strictly increasing in x.*

PROOF: For concision, let us write $J(x) \equiv Bx^{\rho^+} + Cx^{\rho^-}$ for all $x \in (0,\infty)$. Then

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{xF'(x)}{F(x)}\right] &= \frac{[xF''(x) + F'(x)]F(x) - x[F'(x)]^2}{[F(x)]^2} \\
&\propto \left[xJ''(x) + J'(x) + \frac{1}{r-b}\right]\left[J(x) + \frac{x}{r-b}\right] - x\left[J'(x) + \frac{1}{r-b}\right]^2 \\
&= xJ''(x)\left[J(x) + \frac{x}{r-b}\right] + J(x)J'(x) - \frac{xJ'(x)}{r-b} + \frac{J(x)}{r-b} - x[J'(x)]^2 \\
&= x\left[B\rho^+(\rho^+ - 1)x^{\rho^+ - 2} + C\rho^-(\rho^- - 1)x^{\rho^- - 2}\right]\left[Bx^{\rho^+} + Cx^{\rho^-} + \frac{x}{r-b}\right] \\
&\quad + (Bx^{\rho^+} + Cx^{\rho^-})(B\rho^+ x^{\rho^+ - 1} + C\rho^- x^{\rho^- - 1}) - x\,\frac{B\rho^+ x^{\rho^+ - 1} + C\rho^- x^{\rho^- - 1}}{r-b} \\
&\quad + \frac{Bx^{\rho^+} + Cx^{\rho^-}}{r-b} - x(B\rho^+ x^{\rho^+ - 1} + C\rho^- x^{\rho^- - 1})^2 \\
&= \frac{B(\rho^+ - 1)^2}{r-b}\,x^{\rho^+} + \frac{C(\rho^- - 1)^2}{r-b}\,x^{\rho^-} + BC(\rho^+ - \rho^-)^2 x^{\rho^+ + \rho^- - 1} \\
&> 0.
\end{aligned}
$$

The result follows. ■

Remark By definition of F^2, the condition $F^{2\prime +}(x_{R^1}) < 0$ is equivalent to $\bar{J}^{2\prime +}(x_{R^1}) < -\frac{1}{r-b}$. By (S.134), we have $\bar{J}^2 = A\phi$ over $[x_{R^1}, \infty)$ for A given by (S.137). Now, the smooth-pasting condition for player 1's stand-alone problem writes as

$$\frac{\phi'(x_{R^1})}{\phi(x_{R^1})} = \frac{R^{1\prime}(x_{R^1})}{R^1(x_{R^1})} = \frac{-\frac{1}{r-b}}{l^1 - \frac{x_{R^1}}{r-b}}.$$

Thus the condition $\bar{J}^{2\prime +}(x_{R^1}) < -\frac{1}{r-b}$, that is, $A\phi'(x_{R^1}) < -\frac{1}{r-b}$, is equivalent to $A\phi(x_{R^1}) > l^1 - \frac{x_{R^1}}{r-b}$, that is, $\bar{J}^2(x_{R^1}) > R^1(x_{R^1})$. This is clearly true if $l^1 \leq l^2$ as in that case $R^1(x_{R^1}) = l^1 - \frac{x_{R^1}}{r-b} \leq l^2 - \frac{x_{R^1}}{r-b} = R^2(x_{R^1}) \leq T^2_{x_{R^2}}(x_{R^1}) < \bar{J}^2(x_{R^1})$, where the last inequality follows from (S.139). This remains true by continuity if $l^1 > l^2$ but l^1 and l^2 remain close to each other. Now, remember from the remark following the proof of Proposition 5 that ξ is the infimum

[3] The beginning of the proof of Lemma A.2 can be used to reach the same conclusion for any singular mixed-strategy Mpe in our running example. Indeed, what matters is only that the coefficients of the ϕ and ψ functions in the firms' brvfs are always positive.

of the values $\pi = \frac{l^2}{l^1}$ consistent with a simple Mpe of type 2. For the limiting case $\pi = \xi$, we have $s^2 = x_{R^2} < x_{R^1}$ and, hence, $T^2_{s_2}(x_{R^1}) = T^2_{x_{R^2}}(x_{R^1}) = \big(\frac{x_{R^1}}{x_{R^2}}\big)^{\rho^-}\frac{l_2}{1-\rho^-} = \xi^{1-\rho^-}\frac{l_1}{1-\rho^-} < \frac{l_1}{1-\rho^-} = R^1(x_{R^1})$, and hence the condition is violated. We conclude that the simple singular Mpe of type 2 that exists whenever $0 < \xi < 1$, $\pi > \xi$, and $\pi \approx \xi$, satisfies $F^{2\prime+}(x_{R^1}) > 0$. The limiting case $F^{2\prime+}(x_{R^1}) = 0$ arises when $\bar{J}^2(x_{R^1}) = R^1(x_{R^1})$, and in that case we have $\bar{J}^1 = \bar{J}^2$ over $[x_{R^1}, \infty)$.

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