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Elena Panova

Toulouse School of Economics

Sharing cost of network among users with differentiated willingness to pay.*

Elena Panova.†

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Abstract

We consider the problem of sharing the cost of a fixed tree-network among users with differentiated willingness to pay for the good supplied through the network. We find that the associated value-sharing problem is convex, hence, the core is large and we axiomatize a new, computationally simple core selection based on the idea of proportionality.

Key words: sharing network cost, core, proportional allocation.
JEL codes: C71.

1 Introduction.

We consider the problem of sharing the cost of a fixed uncongested tree-network among users with differentiated willingness to pay for the good supplied through the network. This problem is relevant, for example, in the gas

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†Toulouse School of Economics, University of Toulouse Capitole. E-mail: e_panova@yahoo.com

distribution industry. According to our private interviews with managers of French gas distribution network (GRDF), a pipeline diameter is typically chosen with a margin, and gas-distribution network is congestion-free. The network cost is supposed to be paid from the regulated market revenues. Some consumers possess relatively high willingness to pay, such as industrial consumers or large buildings that heat with gas; other consumers, such as households, possess only a modest willingness to pay. How shall differences in consumers' locations and willingness to pay be reflected in their bills?

In order to address this question, we analyze a model in which agents living in geographically separated communities benefit from a good provided by a common supplier. The agents have a different willingness to pay for the good, termed for shortness "benefit". We assume that the agents are connected to the source by a fixed uncongested tree-network and we ask how to share the network cost among its users.

We find that the associated value-sharing problem is convex, hence, the core is large. We require that the solution to that problem lies in the core, taking thereby an axiomatic approach. We focus on solutions which are anonymous because monitoring identities by network users may be very costly. If the billing rule is anonymous, the agents may try to "game" it by merging or splitting. We require our solution to be split- and merge-proof, hereafter, SMP. Finally, we introduce a normative axiom which is specific to our game: If the joint benefit by consumers in some location A is equal to the joint benefit of consumers located "downstream" from A net of their cost of connecting to A, then the joint bill of consumers in A and the remaining bill of consumers downstream from A are equal. We call this axiom "no spatial discrimination," hereafter, NSD.

We show that with core property, SMP and NSD characterize a computationally simple solution based on the idea of proportionality: An agent shares the cost of each network segment belonging to the path from his location to the source with other users of this segment. His cost share is proportional to his local benefit, that is, his benefit net of the sum of his bills for the "downstream" segments he uses, and inversely proportional to the sum of local benefits by all users of this network segment. His surplus share is equal to his benefit less the sum of his segment-specific bills.

Related literatures. We extend Moulin's problem (1989) of sharing the cost of public good[1] by allowing the beneficiaries to occupy different locations. Moulin axiomatizes the proportional rule. We axiomatize a rule based on the idea of proportionality which reduces to proportional rule when our network has only one segment, and so our problem reduces to Moulin's problem.

At the same time, our problem reduces to the problem of sharing the cost of a fixed tree-network if we assume that all agents' benefits are equal to the same, sufficiently large value (see Koster et al., 2001 and references therein; Bjørndal et al., 2004; Ni and Wang, 2008). It has been shown that the core of the problem is large, and the core has been characterized in different ways. We extend this literature by allowing the agents to have differentiated willingness to pay for the good delivered through network, show that the core of the associated value sharing game is large and axiomatize computationally

[1] See Chapter 6.

simple core selection.

A related economic literature studies the problem of sharing the cost of a minimal cost spanning tree (MCST) connecting different geographic locations to the source at the least cost (see review by Bergantiños and Vidal-Puga, 2021). Some cost-sharing rules are linked to the algorithms for constructing MCST proposed by operational research. We relate to that literature in that our results apply to any tree-network without "useless branches",[2] in particular to a surplus-maximizing network, which coincides with a MCST if all users have the same sufficiently large benefit. However, the problem of constructing a surplus-maximizing network is outside the scope of this paper.

Asymmetry among consumers connected through the network which cost is to be shared relates us to Bergantiños and Martínez (2014).[3] In their model the agents have not only different willingness to pay for the good delivered through the tree-network (as in our model) but also produce different amounts of this good, with the main example being water transportation from the rainy to arid regions. The core of their problem is not necessarily full (conditions are provided). Bergantiños and Martínez characterize two network cost sharing rules, specifying how to share a segment's cost depending on supply and demand in two subtrees resulting from its removal. Our

[2]From our interviews with GRDF managers we understood that investments in network expansion are typically justified by sufficiently high demand.

[3]Apart Bergantiños and Martínez (2014), we are aware of only one other paper that considers consumers with differentiated willingness to pay, namely Hougaard and Tvede (2020). This paper asks how to implement a welfare-maximizing network and allocate its costs when information on the agents' willingness to pay is private and it shows that is impossible to Nash-implement budget-balanced cost sharing rules, which is not directly related to our work. Note that "smart technologies" that provide increasingly good information about consumer willingness to pay give at least some relevance to our public information structure.

problem is simpler in that our consumers do not produce. This allows us to show that the core of the associated value-sharing game is large and propose a computationally simple allocation in the core. Either rule by Bergantiños and Martínez reduces to splitting a segment's cost equally among its users, resulting in an allocation which is generally outside the core.

Finally, we relate to the literature using the SMP axiom to characterize proportional rule (see references in Ju et al., 2007; Moulin, 2008). In our setting, SMP and core property imply that the bills are linear in benefits with location-specific coefficients. We add the NSD axiom specific to our spatial setting to pin down these coefficients.

Roadmap. This paper is organized as follows. Section 2 sets up the basic version of the model with locations on a line. Section 3 shows that the associated value-sharing game is convex, hence, the core is large. Section 4 proposes and axiomatizes a new computationally simple core selection, called "locally proportional allocation". Section 5 extends the basic model and its results to locations on a tree. Section 6 concludes.

2 Value-sharing game on a line.

Set $I = \{1, ..., m\}$ of agents, indexed by i, benefit from being connected to the source, indexed with 0. The agents are located in the nodes of a chain originating at the source. The chain's nodes are indexed with $j \in N = \{1, ..., n\}$, so that nodes located further from the source have higher indexes.

Location by agent i is denoted with $l_i \in N$. The vector of locations is denoted with $l = (l_1, ..., l_m)$. At least one agent is located at each node of set N. The set of agents located in the same node is called "community".[4] Community j is denoted with $S_j = \{i \in I \mid l_i = j\}$.

The agents have differentiated willingness to pay for the connection. Agent i's willingness to pay for the connection, called "benefit", is equal to $b_i \in \mathbb{R}^+$. The vector of benefits is denoted with $b = (b_1, ..., b_m)$. Joint benefit by community j is denoted with $B_j = \sum_{i \in S_j} b_i$.

"Segment j" refers to the segment connecting communities $j-1$ and j. Its cost is denoted with $c_j \in \mathbb{R}^+$.[5] The vector of cost is denoted with $c = (c_1, ..., c_n)$.

We assume that the network cost and the agents' benefits are such that it is optimal to connect all communities to the source, because part of network "downstream" from any community j generates positive surplus:

$$W_j = \sum_{k>j} B_k - \sum_{k>j} c_k > 0. \tag{1}$$

Hence, our results apply to any network without "useless tails", in particular to a surplus-maximizing (efficient) network.

We consider the associated value sharing game $\Gamma = (I, N, b, l, c)$ in which the value of coalition $S \subseteq I$ is equal to

$$V(S) = \max \left\{ \max_{X \subseteq S} \left(\sum_{i \in X} b_i - \sum_{j=1}^{\max\{l_i \mid i \in X\}} c_i \right), 0 \right\}. \tag{2}$$

Note that the value of coalition S given by equation (2) is weakly higher than the joint benefits of S's members net of minimal expenses for connecting

[4] We allow several agents to occupy the same location. Alternatively, we could assume that these agents occupy different locations with null cost of connection. We thank an anonymous referee for introducing this remark.

[5] Connections are undirected, that is, the cost of connecting community $j-1$ to community j is the same as the cost of connecting community j to community $j-1$.

them to the source. Example A in Figure 1 shows that this relation may be strict: the value of coalition $\{1,2\}$ is equal to the benefit by agent 1 less the cost of his connection to the source while agent 2 is "useless":

$$V(\{1,2\}) = b_1 - c_1 = 1 > \max\{b_1 - c_1 + b_2 - c_2, 0\} = 0.$$

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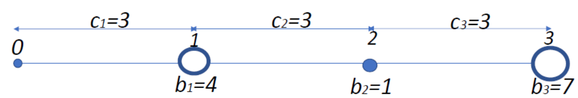

Figure 1: example A.

A solution of game Γ defines the cost share or "bill" $x_i(\Gamma)$ by any agent i in set I and his surplus share

$$y_i(\Gamma) = b_i - x_i(\Gamma). \tag{3}$$

It is convenient to introduce notation for the joint bill of community j:

$$X_j(\Gamma) = \sum_{i \in S_j} x_i(\Gamma). \tag{4}$$

3 Convexity.

This section shows that the above value-sharing game is convex. We begin with introducing the concepts of large nodes and effective coalition members.

Large nodes and effective coalition members. Consider value (2) of some coalition. It is positive iff the joint benefit by coalition members living in some community is sufficiently large to justify connection of this community to the source.[6] For example, consider the network depicted in Figure 2 with three single-agent communities.

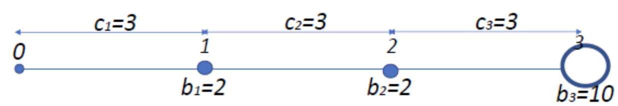

Figure 2: example B.

Consider the grand coalition $\{1, 2, 3\}$. Its positive value (of 5) is due to agent 3 located in node 3. We will say that node 3 is the unique "large node" of coalition $\{1, 2, 3\}$.

Definition 1. *Node j is a* ***large node of coalition*** *$S \subseteq I$ iff the joint benefit of coalition members located in node j is higher than the cost of connecting node j to the source or at least to some other large node of coalition S. The set of large nodes of coalition S is denoted with $L(S)$.*

Recall example A in Figure 1. Agent 1 adds a positive value (of 1) to coalition $\{1, 2\}$ while agent 2 adds nothing to this value. We will say that agent 1 is the unique "effective member" of coalition $\{1, 2\}$.

Definition 2. *Agent i is an* ***effective member of coalition*** *$S \subseteq I$ iff his value in S is positive, that is, $V(S) - V(S \setminus \{i\}) > 0$. The set of effective members of coalition S is denoted with S^*.*

Note that

$$S^* = \arg\max_{T \subseteq S} V(T). \tag{5}$$

[6]From our interviews with managers of GRDF we have learned that typically they extend their network in order to serve a sufficiently large client.

Example A suggests that we can "visualize" set S^* as follows.

Lemma 1 (effective coalition members). *Consider some coalition $S \subseteq I$ and its furthest large node from the source $\max L(S)$. The set of effective members of S consists of all members of S living closer to the source than $\max L(S)$:*

$$S^* = \{i \in S \mid l_i \leqslant \max L(S)\}. \tag{6}$$

Indeed, by Definition 1, all members of coalition $S \subseteq I$ located in large nodes of S are effective. Furthermore, connecting a member of S living "on the way" from $\max L(S)$ to the source increases the value at no cost, hence, this member is effective. On the other hand, by Definition 1, it is too costly to connect a member of S living further from the source than $\max L(S)$, hence, this member is not effective. Appendix A formalizes these simple arguments.

Proposition 1. *Value-sharing game Γ is convex.*

The formal proof of Proposition 1 in Appendix B verifies convexity criterion, by which the marginal value of any agent $r \in I \backslash H$ in coalition $H \subset I$ is weakly higher than that in any subcoalition $S \subseteq H$:

$$V(S \cup \{r\}) - V(S) \leqslant V(H \cup \{r\}) - V(H). \tag{7}$$

By Lemma 1, the set of effective members of coalition H extends weakly further than that of H's subcoalition S:

$$\max L(S) \leqslant \max L(H). \tag{8}$$

Therefore, it is (weakly) cheaper to connect agent r to coalition H than to H's subcoalition S. While the marginal contribution of r to S may include the joint benefit by members of S living in between $\max L(S)$ and $\max L(H)$, this benefit is weakly below the cost of network connecting $\max L(S)$ to $\max L(H)$ by Definition 1. Therefore, inequality (7) holds.

4 Proposed solution.

We look for a solution to the above value-sharing game taking axiomatic approach. That is, we first postulate desirable solution properties (axioms) and then describe the unique solution with these properties. At the end of this section we briefly discuss the Shapley value which, by Proposition 1, is a natural alternative to our proposed solution.

4.1 Axioms.

4.1.1 Core property.

The core of a game is commonly defined as a set of efficient allocations:

$$\sum_{i \in I} y_i(\Gamma) = V(I) \tag{9}$$

which cannot be improved by any coalition of players:

$$\sum_{i \in S} y_i(\Gamma) \geqslant V(S) \text{ for any } S \subset I. \tag{10}$$

There is a consensus in the literature to pick an allocation in the core if it is full.[7] By Proposition 1, the core of our game is full (Shapley, 1971), which brings us to our first axiom.

[7] The reason is two-fold. First, given the above definition it is reasonable to require core property from the incentive perspective. Second, the core can alternatively be defined as a set of efficient allocations excluding cross-subsidies among different coalitions. Therefore, core property may be viewed as a justice requirement.

Axiom C (core property). *Value allocation y lies in the core, as described by equation (9) and set of inequalities (10).*[8]

We proceed with other axioms. Given the bijective relation (3) between values and bills, we state these axioms in terms of bills.

4.1.2 Split- and merge-proofness.

Monitoring identities by network users may be very costly. Therefore, we will focus on solutions which are anonymous, that is, such allocations that if we "swap" locations and benefits by any pair of agents $i \in I$ and $r \in I$, we shall swap their bills

$$x_i\left(\widetilde{\Gamma}\right) = x_r\left(\Gamma\right) \text{ and } x_r\left(\widetilde{\Gamma}\right) = x_i\left(\Gamma\right), \tag{11}$$

where $\widetilde{\Gamma}$ differs from game Γ only in that b_i is replaced for b_r, and vice versa in the vector of benefits b and l_i is replaced for l_r, and vice versa in the vector of locations l.

If billing rule is anonymous, the agents may try to "game" it by merging. As an example, consider the current French gas billing system fixed by the regulatory authority (CRE). It proposes five options including two-part tariffs termed T1, T2 and T3, which are differentiated by fixed-fee T and price p per MWh: (T1) $T = 40.44$, $p = 31.86$ if annual consumption (hereafter, C) is inferior to 6MWh (T2) $T = 133.56$, $p = 8.56$ if C is between 6MWh and 300MWh, (T3) $T = 941.4$, $p = 6.15$ if C is between 300MWh and 5000MWh. Given these options, four owners of apartments located in the same building

[8] Note that in our setting some inequalities in set (10) are redundant. Indeed, if some members of coalition S are not effective, that is, $S^* \subset S$, then inequality (10) written for S follows from that written for S^* and individual rationality constraints.

each consuming 4MWh per annum can reduce their joint bill from 671.52 to 270.52 euros by paying jointly as a condominium.

The agents may also try to "game" the billing rule by splitting. For example, suppose that the existing gas billing system in France is modified by increasing the price per MWh in option (T3) from 6.15 to 6.56. Then, a sufficiently large hotel consuming 280MWh annually with a restaurant consuming 30MWh annually could reduce its bill from 2975 to 2920.72 euros by splitting.

We will require that the billing rule is immune to such splits and mergers. In order to state this axiom formally, we introduce concepts of k-merge and k-split. Because in our game any agent has unique location, we focus on "local" splits and mergers.

Definition 3. *(i) k-merge of game Γ is a game obtained from Γ by merging a set of $k \geqslant 2$ agents from the same community into one community member. (ii) k-split of game Γ is a game obtained from Γ by splitting an agent into $k \geqslant 2$ agents in his community.*[9]

Axiom SMP (split- and merge proofness). *Bills x are such that*

(i) no agent r can reduce his bill by splitting into k agents $\{i_1, ..., i_k\}$:

$$\sum_{i \in \{i_1,\ldots,i_k\}} x_i \left(\widetilde{\Gamma}\right) \geqslant x_r\left(\Gamma\right) \text{ for any } k\text{-split } \widetilde{\Gamma} \text{ of } \Gamma.$$

(ii) no set of k agents $\{i_1, ..., i_k\}$ can reduce their joint bill by merging:

$$x_\mu \left(\widetilde{\Gamma}\right) \geqslant \sum_{i \in \{i_1,\ldots,i_k\}} x_i\left(\Gamma\right) \text{ for any } k\text{-merge } \widetilde{\Gamma} \text{ of } \Gamma.$$

Note that anonymity is necessary for the SMP axiom to make sense. Hence, the SMP axiom implicitly involves an anonymity axiom.

[9] See formal definitions of k-merge and k-split in Appendix C.

4.1.3 No spatial discrimination.

Let us now introduce a normative axiom which is specific to our game. In order to provide some intuition, consider the network illustrated in Figure 3 with a large community 2 in the terminal node and a small community 1 on its way to the source:

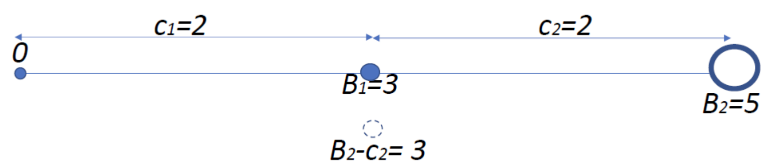

Figure 3: example C.

Any allocation in the core bills community 2 at least the cost of its connection to community 1, that is, at least $c_2 = 2$. Once this cost is paid, the remaining benefit of community 2 is the same as the benefit of community 1: $B_2 - c_2 = B_1 = 3$. It seems fair to require that the remaining bill by community 2 for its connection to the source is the same as the bill by community 1, that is, $X_2 - c_2 = X_1$.

More generally, consider some "interim" community $j < n$. By core property, the joint bill of communities located "downstream" from j covers at least the cost of their connection to community j:

$$\sum_{k>j} X_k \geqslant c_{j+1} + ... + c_n.$$

If the "remaining" joint benefit $\sum\limits_{k>j} B_k - \sum\limits_{k>j} c_k$ by these communities is equal to the benefit of community j, then their "remaining" joint bill $\sum\limits_{k>j} X_k - \sum\limits_{k>j} c_k$

shall be equal to that of community j.

Axiom NSD (no spatial discrimination). *Bills x are such that*

$$\text{if } W_j = B_j \text{ then } \sum_{k>j} X_k(\Gamma) - \sum_{k>j} c_k = X_j(\Gamma). \tag{12}$$

4.2 Locally proportional solution.

Let us look for a solution in the core with SMP and NSD properties.

Lemma 2 (linearity in benefits). *SMP implies that bills are linear in benefits with community-specific coefficients:*

$$x_i(\Gamma) = k_{l_i} b_i. \tag{13}$$

Formal proof of Lemma 2 in Appendix D considers a game in which some community j has at least three members including agents i and r, and the following three games obtained by merger(s) reducing the number of agents in this community to two: (i) All agents in community j but i merge, and agent i is ordered first in his community. (ii) All agents in community j but r merge, and agent r is ordered first in his community. (iii) Agents i and r merge and the resulting member of community j is ordered first. If there are several other agents in community j, they also merge.

Notice that the difference among the above three games is characterized by the benefit of the first agent in community j. Let us fix the benefits by different communities and the cost vector. Then, bill by the first agent in community j is a continuous function, say $\varphi(\cdot)$, of his benefit. By SMP, bills by the first agent in community j in three games are linked by the Cauchy functional equation:

$$\varphi(b_i) + \varphi(b_r) = \varphi(b_i + b_r). \tag{14}$$

Because $\varphi(\cdot)$ is a positive function, the solution to equation (14) is linear.[10]

Appendix E uses the efficiency (9) and NSD to find community-specific coefficients in equation (13). The resulting bill by agent i is

$$x_i^{LP}(\Gamma) = \tfrac{b_i}{B_{l_i}+W_{l_i}} \left(c_{l_i} + \sum_{j=1}^{l_i-1} c_j \prod_{k=j}^{l_i-1} \tfrac{W_k}{B_k+W_k} \right). \tag{15}$$

Proposition 2. *The unique efficient allocation with SMP and NSD properties is given by set of equations (15). None of these three properties is disposable.*

We term bills (15) "*locally proportional*" as indicated with an upper index "LP", because they can be obtained by the iterative "home-down"[11] procedure that shares the cost of each segment among its users proportionately to their "local" benefits, that is, the benefits net of bills for the network segments further from the source (see Appendix F for details).

Example D in Figure 4 illustrates that procedure. Community 2 pays entirely the cost of the second segment. Its remaining benefit, termed local benefit in node 1 is $B_2 - c_2 = 1$. It is five times smaller than the benefit by community 1. Therefore, community 2 pays five times less than community 1 for the cost of the first network segment:[12]

$$X_2 = c_2 + c_1 \tfrac{B_2 - c_2}{B_1 + B_2 - c_2} = c_2 + \tfrac{c_1}{6} = 4\tfrac{2}{3};\ X_1 = c_1 \tfrac{B_1}{B_1 + B_2 - c_2} = \tfrac{5c_1}{6} = 3\tfrac{1}{3}.$$

[10] See the original argument by Darboux 1880 presented in Aczél (1966) on page 32. I am grateful to Fabien Gensbittel for drawing my attention to this fact.

[11] We use terminology by Koster et al. (2001).

[12] Once community bills are specified, they are shared among community members using a proportional rule.

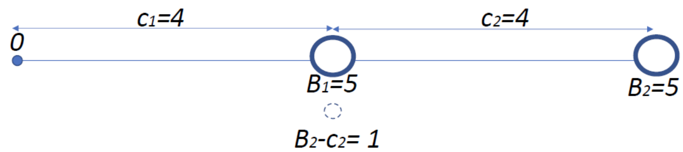

Figure 4: example D.

The fact that the cost of a network segment is shared among its users proportionately to their local benefits, not just benefits differentiates LP allocation from the "serial" version of the proportional rule which has been used in real-world environments such as sharing the cost of an irrigation system (Aadland and Koplin, 1998). If the network cost in the above example, D, is shared according to serial proportional rule, then the communities split the cost of the first segment equally (because their benefits are equal). The resulting allocation is outside the core because the bill of $c_2 + \frac{c_1}{2} = 6$ by community 2 is higher than its benefit.

We conclude this section with the following set of remarks. Note first that while we use only the efficiency and not the core property in the characterization of LP allocation, LP allocation is in the core.

Remark 1. *LP allocation is in the core.*

Formal proof in Appendix G shows first that core inequality (10) holds for any coalition S in which all members are efficient, that is, $S^* = S$. The reason is that some network users may be located further than $\max L(S)$. Because $S^* = S$, these users are outside S. Their LP bills contribute to the cost of network up to $\max L(S)$. In order to prove that core inequality (10) is also met for any coalition S with "useless tail", we prove that LP bills by ineffective members of S are individually rational.

Note, second, that LP allocation possesses some further desirable properties, as we verify in Appendix H. It is monotonous in cost and in benefits:

$$\frac{\partial x_i(\Gamma)}{\partial b_i} \geqslant 0 \text{ and } \frac{\partial x_i(\Gamma)}{\partial c_j} \geqslant 0 \text{ for any } j \leqslant l_i; \tag{16}$$

scale invariant (the currency of accounting is irrelevant):

$$x_i(I, N, \lambda b, l, \lambda c) = \lambda x_i(\Gamma); \tag{17}$$

and continuous (minor changes in benefits or cost shall not lead to significant changes in bills):

$$\lim_{\varepsilon \to 0} x_i(I, N, b_\varepsilon^r, l, c) = \lim_{\varepsilon \to 0} x_i(I, N, b, l, c_\varepsilon^k) = x_i^{LP}(\Gamma)$$

for any $r \in I$ and any $k \in N$, where b_ε^r is a vector of benefits b in which component b_r is replaced with $b_r + \varepsilon$ and c_ε^k is a vector of costs c in which component c_k is replaced with $c_k + \varepsilon$.

Note, finally, that in the limit case with $n = 1$ when our problem reduces to Moulin's problem of sharing the cost of public good, the LP rule reduces to the proportional rule defended by Moulin (1989).[13] Moulin uses two axioms: core and decentralizability. Our SMP axiom plays a similar role to his decentralizability axiom.[14] NSD axiom is specific to the spatial aspect of our problem.

[13] See Chapter 6, page 157.

[14] Moulin uses decentralizability to obtain Jensen's functional equation. We use SMP to obtain the Cauchy functional equation. Axiomatization of the LP rule involving the decentralizability axiom is available upon request.

4.3 Why not the Shapley value?

A natural alternative to LP allocation is the Shapley value

$$y_i^{Sh}(\Gamma) = \sum_{\mathcal{S}:\ S \subseteq \mathcal{S}} \tfrac{(|S|-1)!(m-|S|)!}{m!} \left(V(S) - V(S \setminus \{i\})\right) \text{ for any } i \in I, \tag{18}$$

where $\mathcal{S}$ denotes the set of different coalitions. Indeed, by Proposition 1, the Shapley value is a "central point" of the core in which extreme points are the marginal contribution vectors.

While Shapley-value bills are not SMP, the rule assigning Shapley bills to communities and then sharing these bills among community members according to proportionate rule, is SMP.[15] Therefore, for the rest of this section we focus on games in which there is one agent per community:

$$I = N,\ i = l_i. \tag{19}$$

The main drawback of the Shapley allocation rule is that bills by communities (available upon request) are difficult to compute.[16] The reason is that the marginal value by an agent-community with sufficiently high benefit varies across different agent-community coalitions, because membership by that agent-community may justify connection of some other coalition members to the source. Recall Example A in Figure 1. The marginal value of agent-community 1 is equal to: 3 in the grand coalition, 2 in coalition $\{1,3\}$ and 1 in either coalition $\{1,2\}$ or $\{1\}$.

The above computation problem may not appear in specific cases, including the following two. The first case is a generalization of Example B

[15] I am grateful to anonymous referee for this remark.

[16] We have computed the Shapley bills through decomposing the game by the number of network segments and using additivity.

in Figure 2: one "large" agent-community located at the terminal node and several "small" agents-communities on its way to the source. The second case is classic "airport problem" (Littlechild and Owen, 1973) to which our game reduces when all agent-communities have the same sufficiently large benefit.

Remark 2 (Shapley value in two special cases). *Suppose that set of equations (19) holds. (i) If $L(I) = \{n\}$, then*

$$x_i^{Sh}(\Gamma) = \tfrac{b_i}{2} \text{ for any } i < n \text{ and } x_n^{Sh}(\Gamma) = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n-1} \tfrac{b_i}{2}.$$

(ii) If $b_i = b > \sum_{k=1}^{n} \frac{c_k}{n+1-k}$, *then* $x_i^{Sh}(\Gamma) = \sum_{k=1}^{i} \frac{c_{k+i-1}}{k}$.

Appendix I proves Remark 2 using basic combinatorics.

5 Extension to game with tree-network.

The above model and its insights naturally extend to the situation in which the agents are located in nodes of a tree. This extension makes our work potentially useful for sharing the cost of real distribution networks. Therefore, we briefly outline it keeping all details in Appendices J and K.

Value-sharing game in a tree. From now on, suppose that communities are located in $\mathbb{R}^2$. Assume that it is efficient to connect all of them to the source and it is done through a tree-network $\mathcal{T}$.[17] Assume, furthermore, that any branch of tree $\mathcal{T}$ generates a positive surplus.

In order to state the latter assumption formally, we introduce the following notations: p_{kj} denotes the unique path connecting some pair of nodes

[17] We focus on tree-networks because any network including a cycle is suboptimal: its cost may be reduced by removing redundant edges while keeping connectivity.

j and k;[18] $\pi(j)$ denotes the node preceding node j on path p_{0j}. The edge between node j and its precessor $\pi(j)$ is called "segment j". We keep using notation c_j for its cost. Subtree growing from node j (excluding j) is denoted with $\mathcal{T}_j$ and its cost is denoted with

$$C(\mathcal{T}_j) = \sum_{k \in \mathcal{T}_j} c_k.$$

The surplus generated by $\mathcal{T}_j$ is denoted with

$$W_j = \sum_{k \in \mathcal{T}_j} b_k - C(\mathcal{T}_j),$$

the marginal welfare generated by adding segment j to subtree $\mathcal{T}_j$ is denoted with $w_j = B_j - c_j$, and their sum is denoted with $W_j^+ = W_j + w_j$. Using these notations, our assumption that the surplus generated by any branch of tree $\mathcal{T}$ is positive is equivalent to

$$W_j^+ > 0 \text{ for any community } j. \tag{20}$$

We continue to use notations: I for set of agents; N for the set of nodes or communities; b for the vector of the agents' benefits and l for the vector of their locations, c for the vector of network segments' cost and $\Gamma = (I, N, b, l, c)$ for the associated value-sharing game. The value of coalition $S \subseteq I$ in game Γ is equal to

$$V(S) = \max\left\{\max_{X \subseteq S}\left(\sum_{i \in X} b_i - C\left(\mathcal{T}|_{N(X)}\right)\right), 0\right\}, \tag{21}$$

where $N(S) = \{l_i \,|i \in S\}$ denotes the set of communities with members of coalition $S \subseteq I$ and $\mathcal{T}|_{N(S)}$ is the minimal (sub)tree of $\mathcal{T}$ connecting these communities to the source.[19]

[18] Uniqueness of p_{kj} follows from $\mathcal{T}$ being a tree.

[19] Note that subtree $\mathcal{T}|_{N(S)}$ may contain nodes without members of coalition S. Recall

LP allocation. Appendix J proves that the extended game is convex using, once again, convexity criterion (7). While the extension makes the proof a bit more technical, its idea is the same: it is cheaper to connect agent r outside some coalition H to the minimal network connecting the effective members[20] of H to the source than to its subnetwork connecting the effective members of H's subcoalition S to the source, and this effect is dominant.

Proposition 3. *The game with tree-network is convex.*

Proposition 3 allows us to postulate core property. We postulate the SMP axiom (as is) and we generalize the NSD axiom as follows.

No spacial discrimination (NSD). *If the joint benefit by communities located in subtree T_j net of cost of connecting these communities to node j is equal to the benefit of community j, then their remaining joint bill must be equal to the bill by community j:*

$$\text{if } W_j = B_j \text{ then } \sum_{k \in \mathcal{T}_j} X_k(\Gamma) - C(\mathcal{T}_j) = X_j(\Gamma). \tag{22}$$

Appendix K shows that the unique allocation in the core with SMP and NSD properties shares the cost of a network segment among its users proportionately to their local benefits, assigning bill

$$x_i^{LP}(\Gamma) = \tfrac{b_i}{B_{l_i}+W_{l_i}} \left(c_{l_i} + \sum_{j \in p_{l_i 0} \setminus \{l_i, 0\}} c_j \prod_{k \in p_{l_i j} \setminus \{j\}} \tfrac{W_k^+}{B_{\pi(k)}+W_{\pi(k)}} \right). \tag{23}$$

the line-tree in example C depicted in Figure 3. Subtree $\mathcal{T}|_{N(\{3,4\})}$ containing locations of members of coalition $\{3,4\}$ is the whole line, and contains nodes 1 and 2.

[20] See Appendix J for formal definitions of large nodes and effective coalition members.

to agent i. Note, that when tree $\mathcal{T}$ is a line, $W_k^+ = W_{\pi(k)} = W_{k-1}$ and so equation (23) reduces to equation (15).

Proposition 4. *In the game with tree-network the allocation characterized by core property, SMP and NSD assigns to agent i bill (23).*

6 Conclusion.

We have analyzed the problem of sharing the cost of a fixed tree-network among users with differentiated willingness to pay, which may be seen as an extension of Moulin's problem of sharing the cost of public good on one hand and fixed tree-network cost sharing models on the other hand. We have shown that our problem is convex hence, the core is large. We have proposed and axiomatized a computationally simple core selection based on the idea of proportionality. We hope that our rule may help to allocate the cost of distribution networks.

7 References

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Appendix A: proof of Lemma 1.

Step 1 proves that $\{i \in S \mid l_i \in L(S)\} \subseteq S^*$. Consider i such that $l_i \in L(S)$. By Definition 1, $l_i \in L(S)$ implies

$$B_{l_i}(S) > \sum_{k=\mu_i}^{l_i} c_k, \text{ where } j_i = \max\{j \in L(S) \,|\, j < \; l_i\} \tag{24}$$

is the large node proceeding l_i. Therefore,

$$V(S) - V(S \backslash \{i\}) = \begin{cases} b_i, \text{ if } B_{l_i}(S) - b_i > \sum_{k=j_i}^{l_i} c_k \\ B_{l_i}(S) - \sum_{k=j_i}^{l_i} c_k, \text{ otherwise.} \end{cases}$$

By true inequality $b_i > 0$ and inequality (24), $V(S) - V(S \backslash \{i\}) > 0$, hence, by Definition 2, $i \in S^*$.

Step 2 shows that $r \in S^*$ for any $r \in \{i \in S \,|\, l_i \in N \backslash L(S),\, l_i < \max L(S)\}$. By inequality $l_r < \max L(S)$, connecting r to the source involves no additional cost for coalition S. Therefore, $V(S) - V(S \backslash \{r\}) = b_r > 0$. By Definition 2, $r \in S^*$.

Step 3 shows that $r \in S \backslash S^*$ for any $r \in \{i \in S \,|\, l_i > \max L(S)\}$. By Definition 1, inequality $l_i > \max L(S)$ implies

$$B_{l_r} \leqslant \sum_{j=k+1}^{l_r} c_j \text{ for any } k \in L(S). \tag{25}$$

By equation (2) and inequality (25), $V(S) = V(S \backslash S_{l_i})$. By Definition 2, $i \in S \backslash S^*$.

Appendix B: proof of Proposition 1.

Consider $H \subset I$, $S \subseteq H$ and $r \in I \backslash H$. By equation (2) and Lemma 1,

$$V(Y \cup \{r\}) - V(Y) = \begin{cases} b_r, \text{ if } l_r \leqslant \max L(Y) \\ \max \left\{ 0, b_r + \sum_{i \in Y: \max L(Y) < l_i \leqslant l_r} b_i \right. \\ \left. \quad - \sum_{j=\max L(Y)+1}^{l_r} c_j \right\}, \text{ otherwise} \end{cases} \tag{26}$$

for either $Y = S$ or $Y = H$.

Inequality (8) holds because $S \subseteq H$, which leaves three possible cases.[21]

Case 1: $l_r < \max L(S) \leqslant \max L(H)$. By equation (26),

$$V(S \cup \{r\}) - V(S) = V(H \cup \{r\}) - V(H) = b_r.$$

Hence, inequality (7) is (weakly) met.

Case 2: $\max L(S) < l_r \leqslant \max L(H)$. By equation (26),

$$V(H \cup \{r\}) - V(H) = b_r, \tag{27}$$

$$V(S \cup \{r\}) - V(S) = \max \left\{ 0, b_r + \sum_{i \in S: \max L(S) < l_i < l_r} b_i - \sum_{j=\max L(S)+1}^{l_r} c_j \right\}. \tag{28}$$

By definition of set $L(S)$,

$$\sum_{i \in S: \max L(S) < l_i < l_r} b_i - \sum_{j=\max L(S)+1}^{l_r} c_j \leqslant 0. \tag{29}$$

By equation (28) and inequality (29),

$$V(S \cup \{r\}) - V(S) \leqslant b_r. \tag{30}$$

[21] Note that $S \subseteq H$ and $r \in I \backslash H$ implies that l_r differs from l_i for any i in either set L or H. In particular, l_r differs from either $\max L(S)$ or $\max L(H)$.

By equation (27) and inequality (30), inequality (7) holds.

Case 3: $\max L(S) \leqslant \max L(H) < l_r$. By equation (26),

$$V(Y \cup \{r\}) - V(Y) = \max\left\{0, b_r + \sum_{i \in Y, \max L(Y) < l_i < l_r} b_i - \sum_{j=\max L(Y)+1}^{l_r} c_j\right\} \tag{31}$$

for either $Y = H$ or $Y = S$, hence, inequality (7) is equivalent to

$$\sum_{i \in S: \max L(S) < l_i < l_r} b_i - \sum_{j=\max L(S)+1}^{l_r} c_j \leqslant \sum_{i \in H: \max L(H) < l_i < l_r} b_i - \sum_{j=\max L(H)+1}^{l_r} c_j. \tag{32}$$

Because $S \subseteq H$,

$$\sum_{i \in S: \max L(S) < l_i < l_r} b_i \leqslant \sum_{i \in H: \max L(H) < l_i < l_r} b_i. \tag{33}$$

At the same time, by inequality (8),

$$\sum_{j=\max L(H)+1}^{l_r} c_j \leqslant \sum_{j=\max L(S)+1}^{l_r} c_j. \tag{34}$$

Inequalities (33) and (34) imply inequality (32), hence, inequality (7) holds.

Appendix C: definitions of k-split and k-merge.

Game $\widetilde{\Gamma} = \left(\widetilde{I}, N, \widetilde{b}, \widetilde{l}, c\right)$ is k-merge of game $\Gamma = (I, N, b, l, c)$ in community j iff

$$\widetilde{I} = \{\mu\} \cup I \backslash S, \text{ where } S \subseteq S_j,\ |S| = k$$

$$\widetilde{l}_\mu = j, \widetilde{l}_i = l_i \text{ for any } i \in I \backslash S, \text{ and}$$

$$\widetilde{b}_r = \sum_{i \in S} b_i \text{ and } \widetilde{b}_i = b_i \text{ for any } i \text{ in set } I \backslash S.$$

Game $\widetilde{\Gamma} = \left(\widetilde{I}, N, \widetilde{b}, \widetilde{l}, c\right)$ is k-split of game $\Gamma = (I, N, b, l, c)$ in community j iff

$$\widetilde{I} = (I \backslash \{r\}) \cup \{i_1, ..., i_k\}, \text{ where } r \in S_j,$$

$$\widetilde{l}_i = j \text{ for any } i \in \{i_1, ..., i_k\}, \widetilde{l}_i = l_i \text{ for any } i \in I \backslash \{r\}, \text{ and}$$

$$\left\{\widetilde{b}_{i_1}, ..., \widetilde{b}_{i_k}\right\} \text{ are such that } \sum_{i \in \{i_1,...,i_k\}} \widetilde{b}_i = b_r \text{ and } \widetilde{b}_i = b_i \text{ for any } i \in I \backslash \{r\}.$$

Appendix D: proof of Lemma 2.

Consider a game $\Gamma = (I, N, b, l, c)$ with some community j such that $|S_j| \geqslant 3$ and $\{i, r\} \in S_j$. Consider the following three games:

(i) Γ_i is $(|S_j| - 1)$-merge of Γ in community j: all agents except agent i merge. Agent i indexed with α and the other community member with $-\alpha$.

(ii) Γ_r is$(|S_j| - 1)$-merge of Γ in community j: all agents except agent r merge. Agent r is indexed with α and the other community member with $-\alpha$.

(iii) If $|S_j| = 3$, game Γ_{ir} is 2-merge of Γ in community j: agents i and r merge, the resulting member of community j is indexed with α and the other community member is indexed with $-\alpha$. Otherwise, game Γ_{ir} is a combination of the above merge and $(|S_j| - 2)$-merge of all members of community j but i and r. The agent resulting from 2-merge is indexed with α and the other community member is indexed with $-\alpha$.

Note that the difference among the above three games reduces to repartition of the joint benefit B_j by community j between its members: (i) In game Γ_i, $b_\alpha = b_i$, $b_{-\alpha} = B_j - b_i$. (ii) In game Γ_r, $b_\alpha = b_r$, $b_{-\alpha} = B_j - b_r$. (iii)

In game Γ_{ir}, $b_\alpha = b_i + b_r, b_{-\alpha} = B_j - (b_i + b_r)$.

By definition (4) of community bill,

$$x_\alpha(\Gamma_i) = X_j(\Gamma_i) - x_{-\alpha}(\Gamma_i). \tag{35}$$

By SMP,

$$X_j(\Gamma_i) = X_j(\Gamma), \tag{36}$$

$$x_{-\alpha}(\Gamma_i) = x_r(\Gamma) + \sum_{q\in S_j\setminus\{i,r\}} x_q(\Gamma). \tag{37}$$

By equations (35) to (37),

$$x_\alpha(\Gamma_i) = X_j(\Gamma) - x_r(\Gamma) - \sum_{q\in S_j\setminus\{i,r\}} x_q(\Gamma). \tag{38}$$

Similarly,

$$x_\alpha(\Gamma_r) = X_j(\Gamma) - x_i(\Gamma) - \sum_{q\in S_j\setminus\{i,r\}} x_q(\Gamma), \tag{39}$$

$$x_\alpha(\Gamma_{ir}) = X_j(\Gamma) - \sum_{q\in S_j\setminus\{i,r\}} x_q(\Gamma). \tag{40}$$

By equations (38) to (40),

$$x_\alpha(\Gamma_i) + x_\alpha(\Gamma_r) = 2x_\alpha(\Gamma_{ir}) - (x_i(\Gamma) + x_r(\Gamma)). \tag{41}$$

By SMP,

$$x_i(\Gamma) + x_r(\Gamma) = x_\alpha(\Gamma_{ir}). \tag{42}$$

By equations (41) and (42),

$$x_\alpha(\Gamma_i) + x_\alpha(\Gamma_r) = x_\alpha(\Gamma_{ir}). \tag{43}$$

For fixed joint benefits by different communities and cost vector c, $x_\alpha(\Gamma_i)$ is a function of b_i. Let us denote this function with $\varphi(b_i)$. Using these notations, $x_\alpha(\Gamma_r) = \varphi(b_r)$ and $x_\alpha(\Gamma_{ir}) = \varphi(b_i + b_r)$. Hence, equation (43) is equivalent to the Cauchy functional equation (14). Because function $\varphi(\cdot)$ is positive, it is monotonous:

$$\varphi(z+w) = \varphi(z) + \varphi(w) \geqslant \varphi(z) \text{ for any } w > 0.$$

Monotonicity of function $\varphi(\cdot)$ guarantees that the Cauchy functional equation (14) has linear solution:

$$\varphi(b) = k_j b,$$

where k_j is a constant specific to community j (see Akzél 1966, page 32).

Appendix E: proof of Proposition 2.

Step 1 shows the first statement of Proposition 2. By Lemma 2,

$$X_j = k_j B_j \text{ for any } j = 1, ..., n. \tag{44}$$

We find coefficients k_j recursively using the efficiency and NSD. First, by NSD in node 1 and equation (44),

$$k_1 W_1 = \sum_{j=2}^{n} (k_j B_j - c_j). \tag{45}$$

By the efficiency,

$$\sum_{j=1}^{n} (k_j B_j - c_j) = 0. \tag{46}$$

The left-hand side of equation (46) can be decomposed as follows:

$$\sum_{j=1}^{n}(k_jB_j - c_j) = k_1B_1 - c_1 + \sum_{j=2}^{n}(k_jB_j - c_j)\,. \tag{47}$$

By equations (45) to (47),

$$k_1B_1 - c_1 + k_1W_1 = 0,$$

which is equivalent to

$$k_1 = \tfrac{c_1}{B_1+W_1}. \tag{48}$$

Next, by NSD in node 2 and equation (44),

$$k_2W_2 = \sum_{j=3}^{n}(k_jB_j - c_j)\,. \tag{49}$$

By equations (45) and (48),

$$\tfrac{c_1W_1}{B_1+W_1} = k_2B_2 - c_2 + \sum_{j=3}^{n}(k_jB_j - c_j)\,. \tag{50}$$

By equations (49) and (50),

$$k_2 = \tfrac{1}{B_2+W_2}\left(c_2 + \tfrac{c_1}{B_1+W_1}\right). \tag{51}$$

Proceeding in this way, we find

$$k_j = \tfrac{1}{B_j+W_j}\left(c_j + \tfrac{W_{j-1}}{B_{j-1}+W_{j-1}}(c_{j-1} + \tfrac{W_{j-2}}{B_{j-2}+W_{j-2}}(c_{j-2} + ...))\right), \tag{52}$$

or, equivalently,

$$k_j = \tfrac{1}{B_j+W_j}\left(c_j + \sum_{r=1}^{j-1}c_r\prod_{k=r}^{j-1}\tfrac{W_k}{B_k+W_k}\right). \tag{53}$$

Step 2 shows the second statement of Proposition 2. Consider the following (counter)examples.

Example A.1 shows that the efficiency is not disposable. Consider an allocation defining bills to be linear in benefits with community-specific coefficients k_j, as described by equation (13). Let us replace the efficiency condition (46) with the following equation

$$k_n B_n + \sum_{j=1}^{n-1} z_j \tfrac{B_j}{W_j} = \alpha c_1 + \sum_{j=2}^{n} c_j, \text{ where } \alpha \in \mathbb{R}_+ \setminus \{1\} \tag{54}$$

and find coefficients k_j following the procedure in step 1. Solving for k_j we find equation (53) in which c_1 is replaced for αc_1:

$$k_j = \tfrac{1}{B_j+W_j} \left(c_j + \sum_{r=2}^{j-1} c_r \prod_{k=r}^{j-1} \tfrac{W_k}{B_k+W_k} + \alpha c_1 \prod_{k=r1}^{j-1} \tfrac{W_k}{B_k+W_k} \right). \tag{55}$$

Allocation defined by equations (3), (13) and (55) is different from LP because $\alpha \neq 1$. By equation (54), it is inefficient. At the same time, it possesses the other two properties in Proposition 2: SMP (because bills are linear in benefits with community-specific coefficients), and NSD (because coefficients k_j defined by equation (55) were found using NSD).

Example A.2 shows that NSD is not disposable. Consider some game $\Gamma = (I, N, b, l, c)$ and game $\widetilde{\Gamma} = (\widetilde{I}, \widetilde{N}, \widetilde{b}, \widetilde{l}, \widetilde{c})$ obtained from Γ by "deleting" small nodes:

$$\widetilde{I} = \{i \in I \mid l_i \in L(I)\}, \widetilde{N} = L(I), \tag{56}$$

$$\widetilde{l} = l|_{\widetilde{I}}, \widetilde{b} = b|_{\widetilde{I}}, \tag{57}$$

$$\widetilde{c}_{j_1} = \sum_{j=1}^{j_1} c_j, \widetilde{c}_{j_r} = \sum_{j=j_{r-1}}^{j_r} c_j \text{ for any } r \in \{2, ..., k\}, \tag{58}$$

where $\{j_1, ..., j_k\} = L(I)$. Define bills in game Γ as follows:

$$x_i(\Gamma) = \begin{cases} x_i^{LP}\left(\widetilde{\Gamma}\right) \text{ if } l_i \in L(I); \\ 0, \text{ otherwise.} \end{cases} \tag{59}$$

That is, the agents located in the large nodes of Γ pay their LP bills in game $\widetilde{\Gamma}$, while the agents located in small nodes of Γ pay nothing.

This allocation rule is different from LP. It is clearly not NSD: the agents located in large nodes are discriminated. At the same time, it possesses the other two properties in Proposition 2. Indeed, SMP holds because bills defined by equation (59) are linear in benefits with community-specific coefficients. The efficiency follows from definitions of: game $\widetilde{\Gamma}$ given by equations (56) to (58) and bills given by set of equations (59). Let us verify that the core inequality (10) holds for any coalition $S \subseteq I$. Stand-alone value of coalition S is given by the following equation:

$$V(S) = \sum_{i \in S^*} b_i - \sum_{j=1}^{\max L(S)} c_j. \tag{60}$$

The joint value allocated to coalition S is given by equation

$$\sum_{i \in S} y_i(\Gamma) = \sum_{i \in S} b_i - \sum_{i \in S \cap \widetilde{I}} x_i(\Gamma). \tag{61}$$

Notice that it lies (weakly) below the cost of network up to $\max L(S)$

$$\sum_{i \in S \cap \widetilde{I}} x_i(\Gamma) \leqslant \sum_{j=1}^{\max L(S)} c_j. \tag{62}$$

The reason is that the agents in set $\widetilde{I} \backslash S$ (if any) are billed for using this segment. At the same time,

$$\sum_{i \in S} b_i \geqslant \sum_{i \in S^*} b_i \tag{63}$$

because $S^* \subseteq S$. Equations (60), (61) and inequalities (62), (63) imply inequality (10).

Example A.3 shows that SMP is not disposable. Consider the following allocation rule. First, assign LP bills to all communities:

$$X_j(\Gamma) = X_j^{LP}(\Gamma) = \tfrac{B_j}{B_j+W_j}\left(c_j + \sum_{r=1}^{j-1} c_r \prod_{k=r}^{j-1} \tfrac{W_k}{B_k+W_k}\right). \tag{64}$$

Then, allocate bill by community j among its members as follows:

$$x_i(\Gamma) = \frac{b_i^2}{\sum\limits_{k\in S_j} b_k^2} X_j(\Gamma) \text{ for any } i \in S_j. \tag{65}$$

The above allocation is NSD, because community bills are defined by the LP rule. Furthermore, given that bills (65) are individually rational, we can show that the induced allocation is in the core using the argument for LP allocation being in the core (see Appendix G below). At the same time, the above allocation is not SMP, because any agent i can reduce its bill by splitting into two agents i_1 and i_2 with positive benefits. Indeed, consider b_{i_1} and b_{i_2} such that

$$b_i = b_{i_1} + b_{i_2},\ \min\{b_{i_1}, b_{i_2}\} > 0.$$

By equation (65),

$$x_{i_1}(\Gamma) + x_{i_2}(\Gamma) = \frac{b_{i_1}^2 + b_{i_2}^2}{\sum\limits_{k\in S_{l_i}} b_k^2} < \frac{b_{i_1}^2 + b_{i_2}^2 + 2b_{i_1}b_{i_2}}{\sum\limits_{k\in S_{l_i}} b_k^2} = \frac{b_i^2}{\sum\limits_{k\in S_{l_i}} b_k^2} = x_i(\Gamma).$$

Appendix F: iterative "home-down" procedure.

The following iterative procedure defining bill $x_{i,j}(\Gamma)$ by agent i for each segment $j \leqslant l_i$ results in LP bills.

Step 1. Let the agent(s) from community n share cost c_n of the terminal network segment proportionally:

$$x_{i,n}(\Gamma) = \frac{b_i}{B_n} c_n \text{ for any } i \in S_n, \tag{66}$$

The remaining (hereafter, "*local*") benefit by any agent $i \in S_n$ in node $n-1$ is

$$b_{i,n-1} = \frac{b_i W_{n-1}}{B_n} \tag{67}$$

and the joint local benefit of community n in node $n-1$ is equal to the downstream welfare at node $n-1$:

$$\sum_{i \in S_n} b_{i,n-1} = W_{n-1}. \tag{68}$$

Step 2. Let the agents from communities n and $n-1$ share cost c_{n-1} proportionally to their local benefits in node $n-1$:

$$\begin{aligned} x_{i,n-1}(\Gamma) &= \frac{b_i}{B_{n-1}+W_{n-1}} c_{n-1}, \text{ if } i \in S_{n-1}; \\ x_{i,n-1}(\Gamma) &= \frac{b_{i,n-1}}{B_{n-1}+W_{n-1}} c_{n-1} = \frac{b_i}{B_n} \frac{W_{n-1}}{B_{n-1}+W_{n-1}} c_{n-1}, \text{ if } i \in S_n. \end{aligned} \tag{69}$$

For illustrative purpose, suppose that $n > 2$. Local benefits by agents from communities n and $n-1$ in node $n-2$ are

$$\begin{aligned} b_{i,n-2} &= b_{i,n-1}\left(1 - \frac{c_{n-1}}{B_{n-1}+W_{n-1}}\right) = \frac{b_i}{B_n} \frac{W_{n-1}W_{n-2}}{B_{n-1}+W_{n-1}}, \text{ if } i \in S_n; \\ b_{i,n-2} &= b_i \left(1 - \frac{c_{n-1}}{B_{n-1}+W_{n-1}}\right) = \frac{b_i W_{n-2}}{B_{n-1}+W_{n-1}}, \text{ if } i \in S_{n-1} \end{aligned} \tag{70}$$

and their joint benefit is equal to the welfare generated by subline following node $n-2$:

$$\sum_{i \in S_n \cup S_{n-1}} b_{i,n-2} = W_{n-2}. \tag{71}$$

Step 3. Let the agents from communities n, $n-1$ and $n-2$ share cost c_{n-2} proportionally to their local benefits:

$$\begin{aligned} x_{i,n-2}(\Gamma) &= \frac{b_i}{B_{n-2}+W_{n-2}} c_{n-2}, \text{ if } i \in S_{n-2}; \\ x_{i,n-2}(\Gamma) &= \frac{b_{i,n-2}}{B_{n-2}+W_{n-2}} c_{n-2} = \frac{b_i}{B_{n-1}+W_{n-1}} \frac{W_{n-2}}{B_{n-2}+W_{n-2}} c_{n-2}, \text{ if } i \in S_{n-1}; \\ x_{i,n-2}(\Gamma) &= \frac{b_{i,n-2}}{B_{n-2}+W_{n-2}} c_{n-2} = \frac{b_i}{B_n} \frac{W_{n-1}}{B_{n-1}+W_{n-1}} \frac{W_{n-2}}{B_{n-2}+W_{n-2}} c_{n-2}, \text{ if } i \in S_n. \end{aligned} \tag{72}$$

Proceeding in this way we find the bill by agent i for segment $j \leqslant l_i$:[22]

$$x_{i,j}(\Gamma) = \frac{b_i}{B_{l_i}+W_{l_i}} \frac{W_{l_i-1}}{B_{l_i-1}+W_{l_i-1}} \cdots \frac{W_j}{B_j+W_j} c_j. \tag{73}$$

Summarizing i's bills (73) for all segments on his way to the source we find equation (15).

Appendix G: Proof of Remark 1.

Given that LP allocation is efficient, we need to verify the set of core inequalities (10). We proceed in two steps.

Step 1. Suppose first that $S^* = S$. Then, inequality (10) is equivalent to

$$\sum_{i \in S^*} x_i^{LP}(\Gamma) \leqslant \sum_{j=1}^{\max L(S)} c_j. \tag{74}$$

By Lemma 1,

$$\sum_{i \in S^*} x_i^{LP}(\Gamma) = \sum_{i \in S: l_i \leqslant \max L(S)} x_i^{LP}(\Gamma).$$

By definition of LP allocation,

$$\sum_{i \in S: l_i \leqslant \max L(S)} x_i^{LP}(\Gamma) = \sum_{i \in S: l_i \leqslant \max L(S)} \sum_{j \leqslant l_i} x_{i,j}^{LP}(\Gamma),$$

where $x_{i,j}^{LP}(\Gamma)$ is given by equation (73). $S \subseteq I$ implies

$$\sum_{i \in S: l_i \leqslant \max L(S)} \sum_{j \leqslant l_i} x_{i,j}^{LP}(\Gamma) \leqslant \sum_{i \in I: l_i \leqslant \max L(S)} \sum_{j \leqslant l_i} x_{i,j}^{LP}(\Gamma) = \sum_{j=1}^{\max L(S)} c_j,$$

[22] Recall that $W_n = 0$, so if $i \in S_n$, then

$$x_{i,j}(\Gamma) = \frac{b_i}{Bn} \frac{W_{n-1}}{B_{n-1}+W_{n-1}} \cdots \frac{W_j}{B_j+W_j} c_j.$$

that is, inequality (74) holds.

Step 2. Suppose now that $S^* \subset S$.

Step 2.1 proves that individual rationality constraints are met. This is equivalent to bill $x_i^{LP}(\Gamma)$ given by equation (15) being weakly below b_i:

$$\frac{b_i}{B_{l_i}+W_{l_i}}\left(c_i+\sum_{j=1}^{l_i-1}c_j\prod_{k=j}^{l_i-1}\frac{W_k}{B_k+W_k}\right)\leqslant b_i. \tag{75}$$

Given that $b_i>0$, inequality (75) is equivalent to inequality

$$\sum_{j=1}^{l_i-1}c_j\prod_{k=j}^{l_i-1}\frac{W_k}{B_k+W_k}=\frac{W_{l_i-1}}{B_{l_i-1}+W_{l_i-1}}\left(c_{l_i-1}+\sum_{j=1}^{l_i-2}c_j\prod_{k=j}^{l_i-2}\frac{W_k}{B_k+W_k}\right)\leqslant$$
$$B_{l_i}+W_{l_i}-c_i=W_{l_i-1},$$

which, given that $W_{l_i-1}>0$, is in its turn equivalent to

$$\sum_{j=1}^{l_i-2}c_j\prod_{k=j}^{l_i-2}\frac{W_k}{B_k+W_k}=\frac{W_{l_i-2}}{B_{l_i-2}+W_{l_i-2}}\left(c_{l_i-2}+\sum_{j=1}^{l_i-3}c_j\prod_{k=j}^{l_i-3}\frac{W_k}{b_k+W_k}\right)\leqslant$$
$$B_{l_i-1}+W_{l_i-1}-c_{i-1}=W_{l_i-2}.$$

By continuing this argument, we find that the initial inequality (75) is equivalent to inequality $W_1\geqslant 0$, which is true by inequality (1).

Step 2.2 proves set of inequalities (10) using steps 1 and 2.1. We decompose the value of coalition S as follows:

$$\sum_{i\in S}y_i^{LP}(\Gamma)=\sum_{i\in S^*}y_i^{LP}(\Gamma)+\sum_{i\in S\backslash S^*}y_i^{LP}(\Gamma). \tag{76}$$

By step 1,

$$\sum_{i\in S^*}y_i^{LP}(\Gamma)=\sum_{i\in S^*}b_i-\sum_{i\in S^*}x_i^{LP}(\Gamma)\geqslant V(S^*). \tag{77}$$

By step 2.1,

$$\sum_{i\in S\backslash S^*}y_i^{LP}(b,l,c)=\sum_{i\in S\backslash S^*}b_i-\sum_{i\in S\backslash S^*}x_i^{LP}(b,l,c)\geqslant 0. \tag{78}$$

By equation (76) and inequalities (77) and (78),

$$\sum_{i \in S} y_i^{LP}(\Gamma) \geqslant V(S^*) = V(S).$$

Appendix H: monotonicity, continuity and scale invariance.

By equation (15), LP allocation is (strictly) monotonous in benefits:

$$\frac{\partial x_i(\Gamma)}{\partial b_i} = \frac{1}{B_{l_i} + W_{l_i}} \left(c_{l_i} + \sum_{j=1}^{l_i - 1} c_j \prod_{k=j}^{l_i - 1} \frac{W_k}{B_k + W_k} \right) > 0,$$

in cost:

$$\frac{\partial x_i(\Gamma)}{\partial c_j} = \frac{b_i}{B_{l_i} + W_{l_i}} \frac{W_{l_i - 1}}{B_{l_i - 1} + W_{l_i - 1}} \cdots \frac{W_j}{B_j + W_j} > 0 \text{ for any } j \leqslant l_i.$$

and it is scale invariant:

$$x_i(I, N, \lambda b, l, \lambda c) = \frac{\lambda b_i}{\lambda B_{l_i} + \lambda W_{l_i}} \left(\lambda c_{l_i} + \sum_{j=1}^{l_i - 1} \lambda c_j \prod_{k=j}^{l_i - 1} \frac{\lambda W_k}{\lambda B_k + \lambda W_k} \right) = \lambda x_i(I, N, b, l, c).$$

It is continuos because it is specified by a closed-form continuos function.

Appendix I: proof of Remark 2.

Statement (ii) follows from Littlechiled and Owen (1973). Let us prove statement (i). Consider $i < n$ and $S \subseteq I$ such that $i \in S$.

$$V(S) - V(S \setminus \{i\}) = \begin{cases} b_i, \text{ if } n \in S; \\ 0, \text{ otherwise.} \end{cases} \tag{79}$$

Let us compute the number of ways to build coalition S so that it contains both i and n. It is equal to the number of ways to pick $s - 2$ agents, where

$s = |S|$ from set $I \setminus \{i, n\}$, that is, to $\frac{(n-2)!}{(s-2)!(n-s)!}$. Therefore,

$$y_i^{Sh}(\Gamma) = b_i \sum_{s=1}^{n} \frac{(|S|-1)!(n-s)!}{n!} \frac{(n-2)!}{(s-2)!(n-s)!} = \frac{b_i}{n(n-1)} \sum_{s=1}^{n} (s-1). \tag{80}$$

By standard formula for the sum of elements of arithmetic progression,

$$\sum_{s=1}^{n} (s-1) = \frac{n(n-1)}{2}. \tag{81}$$

By equations (80) and (81), $y_i^{Sh}(\Gamma) = \frac{b_i}{2}$.

Appendix J: proof of Proposition 3.

Here (and below) we use notation $C(p_{jk}) = \sum_{z \in p_{jk}} c_z$ for the joint cost of edges belonging to the path from some node j to some other node k and

$$h(i, Y) = \arg\min_{k \in \mathcal{T}|_{N(Y)}} C(p_{l_i k}), \tag{82}$$

for the hub node of agent $i \in I$ on his way to (sub)tree $T|_{N(Y)}$ connecting members of coalition $Y \subseteq I$ to the source.[23]

Efficient coalition members. We extend Definition 1 as follows.

Definition A.1. *Node $j \in N(S)$ is a* ***large node*** *of (sub)tree $\mathcal{T}|_{N(S)}$ iff the joint benefit by members of coalition $S \subseteq I$ located in node j is higher than the cost of connecting node j to the source or at least to another large node of (sub)tree $\mathcal{T}|_{N(S)}$. Set of large nodes of $\mathcal{T}|_{N(S)}$ is denoted $L(S)$.*

The concept of efficient coalition members introduced by Definition 2 extends to two-dimensional context literally. The following characterization

[23] Trivially, if $i \in Y$ then $h(i, Y) = l_i$.

of the set of efficient coalition members is an extension of Lemma 1.

Lemma A.1. *Set of efficient coalition members is composed of all members of coalition S located in large nodes of $\mathcal{T}|_{N(S)}$ and other members of S located on paths from a large node of $\mathcal{T}|_{N(S)}$ to the source:*

$$S^{*}=\{i\in S\,|l_i\in L(S)\}\cup\{i\in S\mid\, l_i\in p_{k0}\text{ for some } k\in L(S)\}\,. \tag{83}$$

Proof. By Definition of $L(S)$ and S^{*}, $l_i\in L(S)$ implies $i\subseteq S^{*}$. Consider agent $i\in S$ such that $l_i\in N(S)\backslash L(S)$.

Suppose first that agent i is located on the path from the source to some large node of $\mathcal{T}|_{N(S)}$, that is, there exist node $j\in L(S)$ such that $l_i\in p_{0j}$. Then, $V(S)-V(S\backslash\{i\})=b_i>0$. Hence, $i\in S^{*}$.

Suppose now that agent i is located outside any above path, that is, $l_i\in N(S)\setminus(L(S)\cup\{p_{0j}\,|j\in L(S)\})$. Consider the shortest path $p_{l_ih(i,S^{*})}$ from l_i to $\mathcal{T}|_{L(S)}$. By Definition A.1 of set $L(S)$, any node $j\in p_{l_ih(i,S^{*})}$ on this path containing members of S is not a large node of subtree $\mathcal{T}|_{N(S)}$. Therefore, $C\left(p_{l_ih(i,S^{*})}\right)>\sum_{r\in S:l_r\in p_{l_ih(i,S^{*})}}b_r$, hence $i\in S\backslash S^{*}$. Q.E.D.

Note that by Lemma A.1, the subtree connecting the efficient members of coalition S to the source 0 is the subtree connecting its large nodes to 0:

$$\mathcal{T}|_{N(S^{*})}=\mathcal{T}|_{L(S)}. \tag{84}$$

Proof of Proposition 3. Let us verify convexity criterion (7). Consider coalitions S and H such that $S\subseteq H\subset I$ and agent $r\in I\backslash H$. By definition

(21) of a coalition's value, definition (82) of a hub node and Lemma A.1,[24,25]

$$V(Y \cup \{r\}) - V(Y) = \begin{cases} b_r, \text{ if } l_r \in L(Y); \\ \max \left\{ b_r + \sum_{i \in Y \setminus Y^* : l_i \in p_{l_r h(r, Y^*)}} b_i - \right. \\ \left. C\left(p_{l_r h(r, Y^*)}\right), 0 \right\}, \text{ otherwise} \end{cases} \tag{85}$$

for either $Y = S$ or $Y = H$. By Definition A.1 of large nodes, $S \subseteq H$ implies

$$L(S) \subseteq L(H). \tag{86}$$

By Lemma A.1,

$$N(Y^*) = L(Y) \tag{87}$$

for either $Y = S$ or $Y = H$. By relation (86) and equation (87),

$$\mathcal{T}|_{L(S)} \subseteq \mathcal{T}|_{L(H)}. \tag{88}$$

Informally, the rest of the proof goes as follows. By relation (88), hub node $h(r, S^*)$ of agent r on his way to (sub)tree $\mathcal{T}|_{L(S)}$ is a node of (sub)tree $\mathcal{T}|_{L(H)}$. Because tree $\mathcal{T}$ has no cycles, hub node $h(r, H^*)$ is located on the path from agent r's location l_r to hub node $h(r, S^*)$. Hence, the difference between agent r's marginal contributions to coalitions S and H (if any) is created on the path from hub node $h(r, H^*)$ to hub node $h(r, S^*)$: There may be some members of coalition S located on this path. Their benefits shall be added to r's marginal contribution to S. However, these members

[24] Recall that the path from node j to $\mathcal{T}|_{N(Y^*)}$, where Y takes values S and H, is defined as the shortest path from node j to a path connecting a node containing some member of coalition Y^* to the source, and hub node $h(j, Y^*)$ is the defined by equation (82) as the terminal point of this path.

[25] Condition $l_r \in L(Y)$ means that l_r is the node of $\mathcal{T}|_{L(Y)}$. This is true iff there exist $j \in L(Y)$ such that $l_r \in p_{j0}$.

of coalition S are located outside the large nodes of S, therefore, their joint benefit is (weakly) below the cost of the path from hub node $h(r, H^*)$ to hub node $h(r, S^*)$. Hence, agent r's marginal contribution to coalition S is weakly below that to coalition H.

Formally, we consider three possible cases left by relation (88).

Case 1: $l_r \in \mathcal{T}|_{L(S)}$, that is, agent r is located on the way from some large node of the subtree connecting members of coalition S to the source. Then, by relation (88), $l_r \in \mathcal{T}|_{L(H)}$. By set of equations (85),

$$V(S \cup \{r\}) - V(S) = V(H \cup \{r\}) - V(H) = b_r.$$

Hence, inequality (7) holds (weakly).

Case 2: $l_r \in \mathcal{T}|_{L(H)} \backslash \mathcal{T}|_{L(S)}$. By set of equations (85),

$$V(H \cup \{r\}) - V(H) = b_r, \text{ while} \tag{89}$$

$$V(S \cup \{r\}) - V(S) = \max\left\{\left(b_r + \sum_{i \in S \backslash S^* : l_i \in p_{l_r h(j,S^*)}} b_i\right) - C\left(p_{l_r h(r,S^*)}\right), 0\right\}. \tag{90}$$

By Definition A.1 and Lemma A.1, $l_i \in N(S) \backslash L(S)$ for any $i \in S \backslash S^*$ such that $l_i \in p_{l_r h(j,S^*)}$. Therefore,

$$\sum_{i \in S \backslash S^* : l_i \in p_{l_r h(j,S^*)}} b_i - C\left(p_{l_r h(r,S^*)}\right) \leqslant 0. \tag{91}$$

By equation (90) and inequality (91),

$$V(S \cup \{r\}) - V(S) \leqslant b_r. \tag{92}$$

By equation (89) and (92), inequality (7) holds.

Case 3: $l_r \in N \backslash \mathcal{T}|_{L(H)}$. By set of equations (85),

$$V(Y \cup \{r\}) - V(Y) = \max \left\{ b_r + \sum_{i \in Y \backslash Y^* : l_i \in p_{l_r h(r,Y^*)}} b_i - C\left(p_{l_r h(r,Y^*)}\right), 0 \right\} \tag{93}$$

for either $Y = S$ and $Y = H$. By definition (82) of hub node, $h(r, S^*) \in \mathcal{T}|_{L(S)}$. By relation (88),

$$h(r, S^*) \in \mathcal{T}|_{L(H)}. \tag{94}$$

Because tree $\mathcal{T}$ has no cycles,

$$p_{l_r h(r,S^*)} = p_{l_r h(r,H^*)} \cup p_{h(r,H^*) h(r,S^*)}. \tag{95}$$

By equation (95),

$$C\left(p_{l_r h(r,S^*)}\right) = C\left(p_{l_r h(r,H^*)}\right) + C\left(p_{h(r,H^*) h(r,S^*)}\right) \text{ and} \tag{96}$$

$$\sum_{i \in S \backslash S^* : l_i \in p_{l_r h(r,S^*)}} b_i = \sum_{i \in S \backslash S^* : l_i \in p_{l_r h(r,H^*)}} b_i + \sum_{i \in S \backslash S^* : l_i \in p_{h(r,H^*) h(r,S^*)}} b_i. \tag{97}$$

Note that $i \in S \backslash S^*$ and $l_i \in p_{l_r h(r,H^*)}$ implies $i \in H \backslash H^*$. The reason is that $i \in S \backslash S^*$ implies $i \in H$, while $i \in H^*$ contradicts definition of $h(r, H^*)$: indeed, if $i \in H^*$ and l_i is on the way from l_r to $h(r, H^*)$, then $p_{l_r h(r,H^*)}$ is not the shortest way from l_r to $\mathcal{T}|_{L(H)}$. Therefore,

$$\sum_{i \in S \backslash S^* : l_i \in p_{l_r h(r,H^*)}} b_i = \sum_{i \in H \backslash H^* : l_i \in p_{l_r h(r,H^*)}} b_i. \tag{98}$$

By equations (93) and (96) to (98),

$$(V(S \cup \{r\}) - V(S)) =$$
$$\max \left\{ (V(H \cup \{r\}) - V(H)) + \sum_{i \in S \backslash S^* : l_i \in p_{h(r,H^*) h(r,S^*)}} b_i - C\left(p_{h(r,H^*) h(r,S^*)}\right), 0 \right\}. \tag{99}$$

By Lemma A.1, $l_i \in N(S)\backslash L(S)$ for any $i \in S\backslash S^*$ such that $l_i \in p_{h(r,H^*)h(r,S^*)}$. Therefore,

$$\sum_{i\in S\backslash S^*: l_i \in p_{h(r,H^*)h(r,S^*)}} b_i - C\left(p_{h(r,H^*)h(r,S^*)}\right) \leqslant 0. \tag{100}$$

By equation (99) and inequality (100), inequality (7) holds.

Appendix K: proof of Proposition 4.

We divide the statement of Proposition 4 into two Lemmas.

Lemma A.2. *LP allocation billing the agents according to set of equations (23) has properties in Proposition* 4*.*

Proof. Let us divide the bill (23) by agent i into the sum of bills for the segments he uses, as implicitly suggested by set of equations (23):

$$x_i^{LP}(\Gamma) = \sum_{j\in p_{l_i 0}} x_{i,j}^{LP}(\Gamma)\text{, where}$$

$$x_{i,j}^{LP}(\Gamma) = \begin{cases} c_j \frac{b_i}{B_j+W_j} \text{ if } j = l_i; \\ c_j \frac{b_i}{B_{l_i}+W_{l_i}} \prod\limits_{z\in p_{l_i j}\backslash\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}} \text{ if } j \in p_{l_i 0}\backslash\{l_i, 0\}; \\ 0, \text{otherwise.} \end{cases} \tag{101}$$

<u>Step 1</u> shows that the agents located in node j and those located in subtree $\mathcal{T}_j$ jointly pay

$$\sum_{i: l_i \in \mathcal{T}_j \cup \{j\}} x_{i,k}^{LP}(\Gamma) = c_k \prod_{z\in p_{jk}\backslash\{k\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}} \tag{102}$$

for using segment k located on their way to the source, and this bill is shared between these groups of agents as follows:

$$\sum_{i: l_i \in \mathcal{T}_j} x_{i,k}^{LP}(\Gamma) = c_k \frac{W_j}{B_j+W_j} \prod_{z\in p_{jk}\backslash\{k\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}, \tag{103}$$

$$\sum_{i:l_i=j} x_{i,k}^{LP}(\Gamma) = c_k \frac{B_j}{B_j+W_j} \prod_{z\in p_{jk}\setminus\{k\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}. \tag{104}$$

Consider some node r preceding some terminal node(s) of subtree $\mathcal{T}_j$. Let $F_r = \{z \in N \mid \pi(z) = r\}$ be the set of nodes following r. Note that any node z in set F_r is terminal. Therefore,

$$W_z = 0 \text{ and} \sum_{z\in F_r} W_z^+ = W_r. \tag{105}$$

By equations in sets (101) and (105), the joint bill by communities located in set F_r for segment k is

$$\sum_{z\in F_r} X_{z,k}^{LP}(\Gamma) = c_k \frac{W_r}{B_r+W_r} \prod_{z\in p_{rj}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}, \tag{106}$$

while the bill by community r is

$$X_{r,k}^{LP}(\Gamma) = c_k \frac{B_r}{B_r+W_r} \prod_{z\in p_{rj}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}. \tag{107}$$

By equations (106) and (107),

$$\sum_{z\in \mathcal{T}_r\cup\{r\}} X_{z,k}^{LP}(\Gamma) = X_{r,k}^{LP}(\Gamma) + \sum_{z\in F_r} X_{z,k}^{LP}(\Gamma) = c_k \prod_{z\in p_{rj}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}, \tag{108}$$

Consider community $\pi(r)$. By equation (108), joint bill by communities in subtree $\mathcal{T}_{\pi(r)}$ for segment k is

$$\begin{aligned}\sum_{z\in \mathcal{T}_{\pi(r)}} X_{z,k}^{LP}(\Gamma) = c_k \sum_{q\in F_{\pi(r)}} \frac{W_q^+}{B_{\pi(r)}+W_{\pi(r)}} \prod_{z\in p_{\pi(r)j}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}} = \\ c_k \frac{W_{\pi(r)}}{B_{\pi(r)}+W_{\pi(r)}} \prod_{z\in p_{\pi(r)j}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}.\end{aligned} \tag{109}$$

By set of equations (101), bill by community $\pi(r)$ for segment k is

$$X_{\pi(r),k}^{LP}(\Gamma) = c_k \frac{B_{\pi(r)}}{B_{\pi(r)}+W_{\pi(r)}} \prod_{z\in p_{\pi(r)j}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}. \tag{110}$$

By equations (109) and (110),

$$\sum_{z\in\mathcal{T}_{\pi(r)}\cup\{\pi(r)\}} X_{z,j}^{LP}(\Gamma)=c_k\prod_{z\in p_{\pi(r)j}\setminus\{j\}}\tfrac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}. \tag{111}$$

Continuing this argument, we find set of equations (102) to (104).

Step 2 notes that NSD property follows from equations (102) to (104): if $W_j=B_j$ then the bill of community j for any network segment k it uses is equal to the joint bill of communities in $T_j\setminus\{j\}$ for this segment.

Step 3 notes that LP allocation is efficient, as defined by equation (9), because, by step 1 the cost of any segment j is paid (without excess) by the agents located in node j and those located in subtree $\mathcal{T}_j$.

Step 4 verifies set of core inequalities (10). It begins with proving that bills defined by set of equations (23) are individually rational, that is,

$$b_i-x_i^{LP}(\Gamma)\geqslant\max\{b_i-C(p_{l_i0}),0\}. \tag{112}$$

for any $i\in I$. Suppose first that $b_i-C(p_{l_i0})>0$. Then, inequality (112) is equivalent to

$$x_i^{LP}(\Gamma)\leqslant C(p_{l_i0}) \tag{113}$$

which holds because by step 1, agent i shares the cost of any segment on path p_{l_i0} with other agents in community l_i (if any) and agents located in subtree $\mathcal{T}_{l_i}$ (if not empty). Suppose now that $b_i-C(p_{l_i0})\leqslant 0$. Then, inequality (112) is equivalent to

$$x_i^{LP}(\Gamma)\leqslant b_i. \tag{114}$$

By equation (23), inequality (114) is equivalent to

$$\tfrac{b_i}{B_{l_i}+W_{l_i}}\left(c_{l_i}+\sum_{j\in p_{\pi(l_i)0}\setminus\{0\}}c_j\prod_{z\in p_{l_ij}\setminus\{j\}}\tfrac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}}\right)\leqslant b_i. \tag{115}$$

We divide each side of inequality (115) by b_i, then multiply it by $B_{l_i}+W_{l_i}$ and finally substrate c_{l_i} from each side. Thereby, we find that inequality (115) is equivalent to

$$\sum_{j\in p_{\pi(l_i)0}\setminus\{0\}} c_j \prod_{z\in p_{l_i j}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}} \leqslant B_{l_i}+W_{l_i}-c_{l_i}=W_{l_i}^+. \tag{116}$$

We divide each side of inequality (116) by $W_{l_i}^+$, then multiply it by $B_{\pi(l_i)}+W_{\pi(l_i)}$ and finally substrate $c_{\pi(l_i)}$ from each side. Thereby, we find that inequality (116) is equivalent to

$$\sum_{j\in p_{\pi(\pi(l_i))0}\setminus\{0\}} c_j \prod_{z\in p_{\pi(l_i)j}\setminus\{j\}} \frac{W_z^+}{B_{\pi(z)}+W_{\pi(z)}} \leqslant W_{\pi(l_i)}^+.$$

Proceeding iteratively in this way we find that inequality (114) is equivalent to set of inequalities $W_z^+ \geqslant 0$ for any node z in set F_0, which are true by assumption (20).

We now are ready to prove set of inequalities (10). Suppose first that $V(S)=0$. Then, set of inequalities (10) follows from individual rationality proved above. Suppose now that $V(S)>0$. Notice that by definition (21) and Lemma A.1, this supposition is equivalent to

$$\sum_{i\in S^*} b_i - C\left(\mathcal{T}|_{L(S)}\right) > 0.$$

Let us decompose the left-hand side of inequality (10) as follows:

$$\sum_{i\in S} y_i^{LP}(\Gamma) = \sum_{i\in S^*} b_i + \sum_{i\in S\setminus S^*} b_i - \sum_{i\in S^*} x_i^{LP}(\Gamma) - \sum_{i\in S\setminus S^*} x_i^{LP}(\Gamma). \tag{117}$$

By equation (23),

$$C\left(\mathcal{T}|_{L(S)}\right) = \sum_{i\in I}\sum_{j\in \mathcal{T}|_{L(S)}} x_{i,j}^{LP}(\Gamma) \geqslant \sum_{i\in S^*}\sum_{j\in \mathcal{T}|_{L(S)}} x_{i,j}^{LP}(\Gamma) = \sum_{i\in S^*} x_i^{LP}(\Gamma). \tag{118}$$

By individual rationality,

$$\sum_{i\in S\setminus S^*} b_i - \sum_{i\in S\setminus S^*} x_i^{LP}(\Gamma) \geqslant 0. \tag{119}$$

By equation (117) and inequalities (118), (119), inequality (10) holds. Q.E.D.

Lemma A.4. *An allocation in the core which is SMP and NSD is defined by equations (23).*

Proof. Notice that the proof of Lemma 2 goes through, hence,

$$X_j = k_j B_j \tag{120}$$

We will now use NSD and core property to find location-specific coefficients k_j, starting from nodes located closest to the source and going towards the terminal nodes.

Step 1. Consider, one by one, branches of tree $\mathcal{T}$ originating in nodes directly connected to the source, that is, nodes in set F_0. Suppose, w.l.g., that node 1 is in set F_0. By NSD and equation (120),

$$k_1 W_1 = \sum_{j\in\mathcal{T}_1} (k_j B_j - c_j). \tag{121}$$

By the core property,

$$k_1 B_1 - c_1 + \sum_{j\in\mathcal{T}_1} (k_j B_j - c_j) = 0. \tag{122}$$

By equations (121) to (122),

$$k_1 = \tfrac{c_1}{B_1+W_1}. \tag{123}$$

Step 2. If $F_1 = \varnothing$, the proof is over. Otherwise, let us decompose the right-hand side of equation (121) as follows:

$$\sum_{j\in\mathcal{T}_1} (k_j B_j - c_j) = \sum_{r\in F_1}\sum_{j\in\mathcal{T}_r} (k_j B_j - c_j) + \sum_{r\in F_j} (k_r B_r - c_r). \tag{124}$$

Consider some node $\widehat{r} \in F_1$. Suppose first that $F_1 = \{\widehat{r}\}$. Then, equation (124) is equivalent to

$$\sum_{j \in \mathcal{T}_1} (k_j B_j - c_j) = \sum_{j \in \mathcal{T}_{\widehat{r}}} (k_j B_j - c_j) + (k_{\widehat{r}} B_{\widehat{r}} - c_{\widehat{r}}) \,. \tag{125}$$

By NSD, equation (125) is equivalent to[26]

$$k_1 W_{\widehat{r}}^+ = k_{\widehat{r}} W_{\widehat{r}} + k_{\widehat{r}} B_{\widehat{r}} - c_{\widehat{r}}, \text{ or, equivalently,}$$

$$k_{\widehat{r}} = \tfrac{1}{W_{\widehat{r}} + B_{\widehat{r}}} \left(c_{\widehat{r}} + k_1 W_{\widehat{r}}^+ \right) \tag{126}$$

where coefficient k_1 is given by equation (123).

Suppose now that $F_1 \backslash \{\widehat{r}\} \neq \varnothing$. Let us evaluate equation (124) at the following limit:

$$B_j \to c_j \text{ for any } j \in \mathcal{T}_r \cup \{r\} \text{ for any } r \in F_1 \backslash \{\widehat{r}\} \,. \tag{127}$$

<u>Step 2.1</u>. Consider some $r \in F_1 \backslash \{\widehat{r}\}$. Consider some terminal node j in $\mathcal{T}_r \cup \{r\}$. By core property,

$$k_j B_j \geqslant c_j . \tag{128}$$

By individual rationality,

$$k_j B_j \leqslant B_j . \tag{129}$$

By inequalities (128) and (129),

$$\lim_{B_j \to c_j} k_j = 1 \tag{130}$$

Note that equation (130) holds for any (terminal) node sharing the same precessor with node j:

$$\lim_{B_j \to c_j} k_j = 1 \text{ for any terminal node } j \in F_{\pi(j)} . \tag{131}$$

[26] Note that $F_1 = \{\widehat{r}\}$ this implies $W_1 = W_{\widehat{r}}^+$.

Step 2.2. Consider node $\pi(j)$. By the core property,

$$k_{\pi(j)}B_{\pi(j)} + \sum_{q\in F_{\pi(j)}} k_q B_q \geqslant c_{\pi(j)} + \sum_{q\in F_{\pi(j)}} c_q. \tag{132}$$

By set of equations equations (131), inequality (132) evaluated at the limit (127) is equivalent to

$$k_{\pi(j)}B_{\pi(j)} \geqslant c_{\pi(j)}. \tag{133}$$

By individual rationality,

$$k_{\pi(j)}B_{\pi(j)} \leqslant B_{\pi(j)}. \tag{134}$$

By inequalities (133) and (134),

$$\lim_{B_q\to c_q \forall q\in T_{\pi(j)}\cup\{\pi(j)\}} k_{\pi(j)} = 1. \tag{135}$$

Step 2.3. Proceeding iteratively, we find

$$\lim_{B_q\to c_q \forall q\in T_r\cup\{r\}} k_q = 1 \tag{136}$$

for any $r \in F_1 \setminus \{\widehat{r}\}$.

Step 2.4. Consider first the right hand side of equation (124). By set of equations (136)

$$\lim_{(127)} \sum_{r\in F_1}\sum_{j\in\mathcal{T}_r} (k_j B_j - c_j) + \sum_{r\in F_j} (k_r B_r - c_r) = k_{\widehat{r}} B_{\widehat{r}} - c_{\widehat{r}} + \sum_{j\in\mathcal{T}_{\widehat{r}}} (k_j B_j - c_j), \tag{137}$$

$$\text{where, by NSD,} \sum_{j\in\mathcal{T}_{\widehat{r}}} (k_j B_j - c_j) = k_{\widehat{r}} W_{\widehat{r}}. \tag{138}$$

Now consider the left hand side of equation (124). By NSD,

$$\sum_{j\in\mathcal{T}_1} (k_j B_j - c_j) = k_1 W_1, \tag{139}$$

where coefficient k_1 is given by equation (123). By set of equations (136),

$$\lim_{(127)} W_1 = W_{\widehat{r}}^{+}. \tag{140}$$

By equations (137) to (140), coefficient $k_{\widehat{r}}$ is given by equation (126).

<u>Step 3</u>. Proceeding iteratively, we find

$$k_j = \tfrac{1}{B_{l_i}+W_{l_i}} \left(c_{l_i} + \sum_{j \in p_{l_i 0} \setminus \{l_i, 0\}} c_j \prod_{k \in p_{l_i j} \setminus \{j\}} \tfrac{W_k^{+}}{B_{\pi(k)}+W_{\pi(k)}} \right), \tag{141}$$

which completes our proof.