Multiproduct Mergers and Quality Competition

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Abstract. We investigate mergers in markets where quality differences between products are central and firms may reposition their product lines by adding or removing products of different qualities following a merger. Such mergers are materially different from those studied in the existing literature. Mergers without synergies may exhibit a *product-mix effect* which raises consumer surplus, but only when the pre-merger industry structure satisfies certain observable features. Post-merger synergies may lower consumer surplus. The level of, and changes in, the Herfindahl-Hirschman Index may give a misleading assessment of how a merger affects consumers. A merger may benefit some outsiders but harm others.

1. Introduction

Competition authorities around the world recognize the importance of accounting for quality in merger policy.[1] In many industries, products differ substantially in terms of quality, and some firms supply multiple different qualities while others specialize in either low or high quality. Mergers in such industries raise a number of interesting questions. For example, how does a merger affect the overall level of output and the equilibrium mix of qualities? What does observed product-line repositioning, by either merging firms or their rivals, tell us about the likely welfare effects of a merger? Do synergistic cost reductions always benefit consumers? Under what conditions are mergers profitable? And how should a competition authority screen mergers?

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[1]A 2013 OECD document *The Role and Measurement of Quality in Competition Analysis* notes, "While the importance of quality is undisputed and issues about quality are mentioned pervasively in competition agency guidelines and court decisions, there is no widely-agreed framework for analysing it which often renders its treatment superficial."

There is surprisingly little research on mergers in markets where quality is important. We provide a framework to analyze such mergers. Specifically, we incorporate mergers into a simplified version of the quantity-setting framework of Johnson and Myatt (2006) where products are of either low or high quality. Consumers differ in how much they value quality. Firms have arbitrary costs of supplying low and high quality, and simultaneously choose an output for each of the two products. Our focus on Cournot competition implies that our results are most applicable to industries where firms choose capacities in advance and then set prices. We allow for endogenous quality and product-line choice by asymmetric firms, the prospect that firms may reposition their product lines by adding or removing product qualities following a merger, and the possibility that a merged entity has a better cost structure than any of its constituent firms.

Our model is specialized in that it assumes quantity competition and that products are not horizontally differentiated. At the same time, we find that mergers in our framework are materially different from those that have been previously studied. As such, our results suggest that caution may be required when assessing mergers in markets where quality plays an important role in consumer decision making.

Our first results involve the effects of mergers on industry supply, prices, and consumer surplus. We emphasize a new economic effect associated with mergers, which we call the *product-mix effect*. This effect represents the change in the mix of different qualities consumed in the market that is due to a merger. Surprisingly, the product-mix effect can be so strong that a merger—even one exhibiting no cost synergies—increases consumer surplus.

The prospect of a merger with no synergies increasing consumer surplus is absent from single-product Cournot markets.[2] It is typically presumed that the *market-power effect* of a merger leads to lower industry output (and necessarily lower consumer surplus) unless there is a strong countervailing *synergy effect* (Farrell and Shapiro, 1990). With multiple qualities, we show that total output (across all qualities combined) always decreases following a merger with no synergies—in line with the classic market-power effect—but that the product-mix effect may lead to increased output of higher quality products, and a reduction in their price, so that consumer surplus may rise.

Although the presence of multiple qualities necessarily provides additional flexibility for how mergers may affect consumer surplus, it is not true that anything can happen following a merger in a multiproduct industry. As already noted, total output across all qualities combined always declines. Moreover, consumer surplus can only increase if the pre-merger market structure satisfies certain observable necessary conditions. One such condition is that

[2] In a model with homogeneous goods where consumers vary in their consideration sets, Armstrong and Vickers (2020) show that a merger can increase consumer surplus by changing the nature of competitive pricing interactions.

the pre-merger market structure is asymmetric, with some firms having different product lines than others; hence a complete picture of mergers must allow for such asymmetries.

Our results hold even though we allow both merging and non-merging firms to change their product lines following a merger. Such post-merger product-line repositioning is widely considered to be important.[3] Our results suggest that it can be hard to infer the effect of a merger on consumers based on observed product-line repositioning. On the one hand, we show that a merger can increase consumer surplus even if the merging parties remove a product from their product line. On the other hand, we also show that if two firms selling only high quality merge, then consumer surplus decreases even if rivals introduce high-quality products in response.[4]

We also consider how merger-induced cost synergies influence market outcomes. In some cases, synergies induce unfavorable product-mix effects and lower consumer surplus. For example, a cost synergy that reduces a merged multiproduct firm's cost of low quality encourages this firm to decrease its supply of high quality, possibly harming consumers.

Our second set of results explores broader welfare effects of mergers. We show that some outsiders may benefit from a merger while others lose. We also find that a merger may raise both outsiders' profits and consumer surplus. This contrasts with single-product Cournot models, where all outsider firms gain or lose, and a merger increases outsider profits if and only if it harms consumers. We note that, in practice, support by rival firms for a merger is often interpreted as a sign that the merger is anti-competitive and thus harmful to consumers (and vice versa)—our analysis suggests this can be misguided. We also examine a merger's "external effect", namely its impact on the sum of consumer surplus and outsider profits. Contrary to single-product models with quantity competition, we show that the external effect can be positive even when merging firms supply a high share of a product's output.

We then assess the effectiveness of the Herfindahl-Hirschman Index (HHI) as a merger screen. We show that, in the absence of our novel product-mix effect, a merger is more likely to harm consumers when the change in the HHI is larger—mirroring an insight from single-product Cournot markets (Nocke and Whinston, 2020). But when our product-mix effect is present, a merger without synergies may raise consumer surplus even when both the level of and change in the HHI far exceed levels deemed harmful by the U.S. Horizontal Merger Guidelines.

Our final results revisit the classic question of whether horizontal mergers are profitable. Salant, Switzer, and Reynolds (1983), Levin (1990), and Cheung (1992) have argued that

[3]For example, in a November 9, 1995 speech, Deputy Assistant Attorney General of the Antitrust Division of the U.S. Department of Justice, Carl Shapiro, argued that merger assessment should "try to account for any likely and timely changes in prices or product offerings by non-merging parties, including product repositioning and entry."

[4]This is consistent with the Federal Trade Commission's (FTC) challenge to the approval of the merger between (high-quality) organic grocers Whole Foods and Wild Oats; one justification for the original approval was the district court's view that rival grocers could introduce their own organic food lines (Boberg and Woodbury, 2009).

Cournot mergers are often unprofitable for insiders (absent synergies or fixed-cost savings) due to the competitive responses of outsiders. Their results imply that, for general demand systems, insiders must control at least 50% of pre-merger market output in order to gain from merging. Our analysis suggests that with multiple quality-differentiated products mergers are profitable in a wider variety of circumstances. In particular, insiders need not control significant pre-merger market share for a merger to be profitable. Moreover, in some cases, a merger between two firms is profitable even as the number of rivals becomes very large.

Despite the ubiquity of multiproduct firms and the importance of product quality, much of the theoretical literature on mergers focuses on single-product firms and exogenous quality.[5] Exceptions to the assumption of exogenous quality include Federico, Langus, and Valletti (2018) and Motta and Tarantino (2018). They assume that each firm, pre-merger, sells a single product but chooses that product's quality. They find that a merger must exhibit significant innovation synergies for it to increase consumer surplus.[6] An exception to the assumption of single-product firms is Nocke and Schutz (2019), who consider mergers between multiproduct firms using a nested CES/logit framework. They reduce multiproduct competition to a single dimension and argue that many of the classic results from single-product markets hold, such as mergers reducing consumer surplus in the absence of synergies.

Our paper is also related to a growing literature on merger simulation with endogenous product choice. (See Crawford, 2012 for an early survey.) Post-merger product repositioning has been documented in several markets, including those for broadcast radio (Berry and Waldfogel, 2001), music radio (Sweeting, 2010), and airlines (Li et al., 2019). Many papers find that, after accounting for repositioning, mergers decrease consumer surplus.[7] For instance Gandhi et al. (2008) numerically solve a price-location Hotelling game. They show that merging firms tend to move their products further apart, which reduces how much consumers are harmed by a merger. Mazzeo, Seim, and Varela (2018) numerically solve a model where firms choose whether or not to supply pre-set horizontally differentiated products. They show that merging firms often stop supplying a product, which increases how much consumers are harmed by a merger. However in some cases a merger increases consumer surplus because it induces entry by a new firm. An important difference between this work and ours is that we consider a setting with vertical rather than horizontal product differentiation. We

[5] These studies typically assume quantity competition (Perry and Porter, 1985; Levin, 1990; McAfee and Williams, 1992) or price competition with horizontally differentiated products (Deneckere and Davidson, 1985).

[6] In contrast Jullien and Lefouili (2018) argue that the relationship between mergers and innovation is ambiguous. Moraga-González, Motchenkova, and Nevrekar (2019) show that when firms engage in winner-takes-all innovation contests, a merger may raise consumer surplus by inducing firms to reallocate their efforts across the different contests.

[7] Papers which use structural methods typically find that merging firms remove or degrade products, while non-merging firms add or improve products. Depending on which effect dominates, product repositioning can either exacerbate (e.g. Draganska, Mazzeo, and Seim, 2009 and Fan, 2013) or mitigate (e.g. Wollmann, 2018 and Li et al., 2019) the negative effect of a merger on consumer surplus due to higher prices. However in all these papers mergers harm consumers. An exception is Byrne (2011) who finds that mergers create scale efficiencies and benefit consumers.

find that mergers without synergies can raise consumer surplus due to a novel product-mix effect, which can arise even when insiders remove a product, no outsider changes its product line, and no new entry occurs. Gandhi et al. (2008) and Sweeting (2010) argue that repositioning can explain why outsiders may oppose mergers that are unlikely to generate synergies. In our setting with vertical rather than horizontal product differentiation, we also find that a merger without synergies may harm outsiders, but this can occur even when no firm chooses to adjust its product line.

The remainder of our manuscript is laid out as follows. Section 2 presents the model. Section 3 assesses the impact of mergers on prices, quantities, and consumer surplus, both with and without synergies. Section 4 provides an assessment of external effects and the profitability of mergers for outsiders. Section 5 assesses the role of the Herfindahl-Hirschman Index as a merger screen. Section 6 investigates merger profitability. Section 7 concludes.

2. Model

We introduce mergers into a two-quality version of Johnson and Myatt (2006). Specifically, there are two vertically differentiated products of quality $q_L > 0$ and $q_H > q_L$. Buyers are indexed by θ. A consumer of type θ is willing to pay at most $v(\theta, q)$ for a single unit of a product with quality q, where $v(\theta, q)$ is increasing in both of its arguments and satisfies the usual sorting condition: $v(\theta, q_H) - v(\theta, q_L)$ is increasing in θ. Consumers have quasilinear preferences and purchase a single unit of the product that offers the greatest non-negative surplus, and otherwise buy no product. Amongst a unit mass of potential buyers, for $z \in [0, 1]$ we let $\theta(z)$ be the buyer type for which there are z buyers with higher values of θ, where $\theta(z)$ is strictly decreasing and twice differentiable in z. If θ is distributed according to $F(\theta)$ then $\theta(z) = F^{-1}(1 - z)$. We take $v(\theta(1), q) = 0$; some consumers do not purchase in equilibrium.

There is a finite number of horizontally undifferentiated firms. Two or more firms, denoted by $\mathcal{I}$, are *insiders* who merge together. Firms not involved in the merger are *outsiders* and are denoted by $\mathcal{O}$. Prior to the merger firm $i \in \mathcal{I} \cup \mathcal{O}$ has a constant marginal cost $c_1^i \geq 0$ for low-quality products and a constant marginal cost $c_2^i \geq c_1^i$ for high-quality products. After the merger each firm $i \in \mathcal{O}$ keeps the same marginal costs as before, while the merged firm $i = m$ has constant marginal costs $c_1^m \geq 0$ and $c_2^m \geq c_1^m$ for low and high quality respectively. Firms have no capacity constraints and no fixed costs. All costs are common knowledge.

Both before and after the merger, firms simultaneously set outputs. An equilibrium is a set of quantities for each firm i such that i is maximizing its profits taking as given the output of all other firms. We do not assume that each firm supplies a positive quantity of both products but instead allow for completely arbitrary equilibrium outcomes.

Market Clearing Prices. We will analyze our model using the "upgrades approach" previously used by Johnson and Myatt (2003, 2006) and Anderson and Çelik (2015). Let Z_1^i denote the total number of units *of low and high quality combined* that firm i produces, and let Z_2^i denote the number of high-quality units that i produces. Necessarily, $Z_1^i \geq Z_2^i$ and the number of low-quality units that i produces is $Z_1^i - Z_2^i \geq 0$.

Let $Z_1 = \sum_i Z_1^i$ be the industry supply of units of either quality (so Z_1 is the total number of units available on the market). Let $Z_2 = \sum_i Z_2^i$ be the industry supply of high-quality units. The marginal buyer of the low-quality product is indifferent between purchasing that product and nothing. Because the total number of units for sale is Z_1, the marginal buyer is of type $\theta(Z_1)$ and hence the price of the low-quality good is

$$P_1(Z_1) = v(\theta(Z_1), q_L).$$

Rather than directly deriving the price of the high-quality good we will instead find the price that the marginal consumer would pay to "upgrade" from low to high quality. This upgrade price depends on the type of the consumer who is indifferent between buying low or instead high quality. Given that there are Z_2 high-quality products available, this marginal consumer has type $\theta(Z_2)$ and is willing to pay P_2 to upgrade to high quality, where

$$P_2(Z_2) = v(\theta(Z_2), q_H) - v(\theta(Z_2), q_L).$$

Conceptually, we imagine that there exist Z_1 "baseline" units that consumers purchase at price P_1 and that there are Z_2 "upgrades" available for purchase at price P_2. The total price for a high-quality good is thus $P_1 + P_2$.

Product Lines. Firm i produces Z_1^i baseline units and $Z_2^i \leq Z_1^i$ upgrades, meaning it produces $Z_1^i - Z_2^i \geq 0$ low-quality products. Hence the profit π^i of firm i is given by

$$\begin{aligned}\pi^i =&(Z_1^i - Z_2^i)[P_1(Z_1) - c_1^i] + Z_2^i[P_1(Z_1) + P_2(Z_2) - c_2^i] \\ =&Z_1^i[P_1(Z_1) - c_1^i] + Z_2^i[P_2(Z_2) - (c_2^i - c_1^i)]. \end{aligned} \tag{1}$$

Firm i sells Z_1^i baseline units at margin $P_1(Z_1) - c_1^i$ and also sells Z_2^i upgrades at margin $P_2(Z_2) - (c_2^i - c_1^i)$, where $c_2^i - c_1^i$ is firm i's "upgrade cost" from low quality to high quality.

By construction $P_1(Z_1)$ depends only on the number of baseline units Z_1, not the number of upgrades Z_2, and the upgrade price $P_2(Z_2)$ depends only on the number of upgrades Z_2, not the total number of products Z_1. This implies that firm i can separately maximize its profits from baseline units and from upgrades: Z_1^i and Z_2^i can be chosen independently, subject only to the "upgrade constraint" $Z_1^i \geq Z_2^i$.

If firm i is selling both products then its upgrade constraint is not binding and at the equilibrium quantities firm i must be satisfying the two independent first-order conditions

$$\begin{aligned} P_1(Z_1) + Z_1^i P_1'(Z_1) &= c_1^i, \text{ and} \\ P_2(Z_2) + Z_2^i P_2'(Z_2) &= c_2^i - c_1^i. \end{aligned} \tag{2}$$

A firm that in equilibrium sells only low-quality products (so that $Z_2^i = 0$) must satisfy only the first condition above (for it to be optimal not to sell any high-quality goods it must be that $P_2(Z_2) \leq c_2^i - c_1^i$). For a firm that sells only high-quality products it has a binding upgrade constraint ($Z_1^i = Z_2^i = Z^i$) and must satisfy the single first-order condition

$$[P_1(Z_1) + P_2(Z_2)] + Z^i \left[P_1'(Z_1) + P_2'(Z_2)\right] = c_2^i. \tag{3}$$

If there were no firms selling only high-quality products, then the equilibrium values of Z_1 and Z_2 could be found independently simply by using the first-order conditions given above.

In later analysis the curvature of $P_k(Z)$ will be useful, defined as

$$\sigma_k(Z) = -\frac{Z P_k''(Z)}{P_k'(Z)}, \text{ for } k \in \{1, 2\}.$$

Assumption 1 (Decreasing Marginal Revenue). *In each market $k \in \{1, 2\}$ marginal revenue is strictly decreasing. That is, for each k,*

$$P_k(Z) + Z^i P_k'(Z)$$

is strictly decreasing in Z for any $Z^i \in [0, Z]$. This is equivalent to $\sigma_k(Z) < 1$.[8]

We close with an important lemma from earlier work (Johnson and Myatt, 2006).

Lemma 1 (Existence and Uniqueness). *There exists an equilibrium and it is unique.*

3. The Impact of Mergers on Consumer Welfare

In this section we address the classic question of what effect horizontal mergers have on consumer surplus. Economists and regulators often posit that horizontal mergers have two primary effects: a *market-power effect* from consolidation of decisions regarding key choice variables such as output, which harms consumers, and a *synergy effect* from improvements in the merging firms' cost structure, which benefits consumers. Thus, it is often presumed that a merger which exhibits no synergies must harm consumers (Farrell and Shapiro, 1990).

[8]To verify the stated equivalence, note that differentiating $P_k(Z) + Z^i P_k'(Z)$ with respect to Z shows it is decreasing for each $Z^i \in [0, Z]$ if and only if $P_k'(Z) + Z^i P_k''(Z) < 0$ for each $Z^i \in [0, Z]$, whereas $\sigma_k(Z) < 1$ if and only if $P_k'(Z) + Z P_k''(Z) < 0$. For a given Z, if $P_k''(Z) \leq 0$ then both conditions clearly hold. For a given Z, if $P_k''(Z) > 0$ then the first condition is hardest to satisfy for $Z^i = Z$ but at that value the conditions are identical.

In contrast to this classic viewpoint, we will emphasize the importance of what we call the *product-mix effect*, which is that a merger may change the equilibrium mix of products that are consumed. Most strikingly, this may benefit consumers when some firms become more aggressive following a merger, as in our quantity-setting framework. We also show that in some cases synergies lower consumer surplus.

We begin by considering the impact of mergers on consumer surplus when there are no synergies. After that, we investigate the effects of synergies.

Mergers with No Synergies. Following Farrell and Shapiro (1990), we say that when there are *no synergies* the merged firm's cost of producing any output vector equals the minimum-cost method of producing it using the merging firms' pre-merger technologies.

Definition 1 (No Synergies)**.** *Suppose that all firms $i \in \mathcal{I}$ merge to create a firm with marginal cost for low-quality products given by c_1^m and marginal cost for high-quality products given by c_2^m. This merger exhibits* no synergies *if $c_1^m = \min_{i\in\mathcal{I}} c_1^i$ and $c_2^m = \min_{i\in\mathcal{I}} c_2^i$.*

Observe that a merger with no synergies nonetheless may allow the merged entity to have a more attractive variable cost structure than any of its constituent firms alone. The merged entity may wish to reallocate (or rationalize) its output, for example by shifting all of its low-quality output to the plant with the lowest marginal cost for low-quality products. Also observe that such an improved variable cost structure may alter the merged entity's optimal product mix or product line in addition to its optimal level of overall production.[9]

We now provide several results addressing the impact of a merger on consumer surplus. We first consider how aggregate output Z_1 (across all firms and qualities combined) responds.

Proposition 1. *A merger with no synergies leads to a strict reduction in aggregate output, measured across all firms and qualities: Z_1 strictly decreases (and so the price of the low-quality good strictly increases).*

Proposition 1 is the analogue of a leading result in the single-product, identical quality analysis of Farrell and Shapiro (1990), who show that mergers without synergies must reduce aggregate output.[10] In a single-product world, a reduction in aggregate output necessarily raises the price of the product and therefore harms consumers. This is the basis for the classic market-power effect, which says that absent synergies a merger must harm consumers.

But importantly, in a multiproduct industry total output Z_1 is not a sufficient statistic for consumer surplus. Rather, the product-mix effect must also be considered, so that both

[9]For example, a multiproduct firm acquiring a firm that only sells low quality but which has a lower marginal cost will experience an increase in its cost of upgrading to high quality and will, all else equal, increase its total production but reduce its supply of high quality (and may even remove the high quality good from its product line).

[10]Although an intuitive result, proving it in a multiproduct setting is much more subtle. For example, we must allow firms to adjust their product lines.

total output and the industry product mix between low and high quality are assessed; Z_2 must also be known. If a merger were to increase Z_2 then the price of the high-quality good might decrease, which would make some consumers better off. In principle, this might raise overall consumer surplus.

Can a merger with no synergies indeed cause a beneficial product-mix effect, raise the industry output of high quality, and increase consumer surplus? The answer is yes. Our first step in showing this is to provide observable necessary conditions for Z_2 to increase.

Necessary Conditions for the Supply of High Quality to Increase. We identify the following necessary conditions for Z_2 to increase following a merger with no synergies.

Proposition 2. *If a merger with no synergies raises the aggregate output of high-quality products, then pre-merger*

(1) some firm in the industry sells only high-quality products, and

(2) some merging firm sells low-quality products.

To illustrate item 1, consider a simple example in which no firm sells only high quality and a merger occurs involving firms selling only low quality. We know from Proposition 1 that aggregate output must go down. Given that no firm's upgrade constraint was binding pre-merger, each firm not involved in the merger will raise its own total supply. However, doing so further relaxes each multiproduct firm's upgrade constraint—no firm changes its supply of high-quality products and so the merger does not affect Z_2.[11,12]

Additionally, as indicated by item 2, for Z_2 to increase some merging firm must be selling low quality. Otherwise, necessarily all merging firms sell only high quality and intuitively the direct effect of such a merger is a reduction in these firms' combined high-quality output. Although other firms may potentially raise their high-quality supply in the post-merger equilibrium, such increases are insufficient to counteract the reduction by the merging firms.

We now present a powerful corollary of Proposition 2. To be clear, the *product line* of a firm is the set of qualities that it produces in positive quantities, and an industry exhibits *asymmetric product lines* if there are at least two firms that sell different product lines.

Corollary 1. *If a merger with no synergies raises the aggregate output of high-quality products, then pre-merger the industry exhibits asymmetric product lines.*

Corollary 1 indicates that product-line asymmetry—not merely cost asymmetry across firms—is essential for a merger without synergies to increase the supply of high quality output (and

[11] When no firm sells only high quality, other types of merger—for example between two multiproduct firms, or a multiproduct firm and a firm selling only low quality—may strictly decrease Z_2.

[12] A corollary is that if pre-merger all firms supply only low quality, then absent synergies, a merger will not lead to product innovation because no firm will add high quality to its product line.

consumer surplus). To see how Corollary 1 follows from Proposition 2, consider the three distinct ways in which an industry could exhibit symmetric product lines: (i) if all firms sell only low quality pre-merger, then no firm sells only high quality and item 1 fails, (ii) if all firms are multiproduct pre-merger, then no firm sells only high quality and item 1 again fails, and (iii) if all firms sell only high quality pre-merger, then any merger necessarily involves only such firms and so no merging firm sells low quality, so that item 2 is not satisfied.

We close our discussion of necessary conditions by emphasizing that Proposition 2 and Corollary 1 hold even though firms may reposition their product lines following a merger, by either adding or removing products. For example, if no merging firm sells low quality pre-merger, then the aggregate output of high-quality products cannot increase—even if rivals respond by introducing their own high-quality products.

Sufficient Conditions for the Supply of High Quality to Increase. We now turn to circumstances under which a merger without synergies definitely raises the supply of high quality, so that a beneficial product-mix effect occurs.

Proposition 3. *Consider a merger with no synergies in which, pre-merger, at least one firm in the industry is strictly producing only high-quality products.*[13] *The merger strictly increases the aggregate output of high-quality products if either*

(1) each of the merging firms produces only low-quality products before the merger, or
(2) at most one merging firm sells high-quality products and any such firm has a marginal cost for low quality that is no higher than that of any other merging firm.

Item 1 from Proposition 3 is intuitive. If each merging firm produces only low quality, then the direct effect of the merger is for the merged entity to curtail its output of low-quality products, reducing Z_1. Such a reduction encourages those firms producing only high-quality products to expand their output. Although multiproduct firms may in turn produce fewer high-quality goods, the equilibrium effect is an increase in Z_2. The same reasoning guarantees that Z_2 increases if one merging firm produces high quality (item 2), so long as that firm does not have a higher cost of producing low-quality products than other insiders.[14]

We are now in a position to definitively show that a merger can not only raise Z_2 but moreover that this increase can be substantial enough to lower the price of high-quality goods enough to raise consumer surplus. Consider the following example.

[13] Firm i is "strictly" selling only high-quality products if, facing an equilibrium value Z_2 and its own optimal quantity of high-quality products Z_2^i, it is the case that $P_2(Z_2) + Z_2^i P_2'(Z_2) > c_2^i - c_1^i$. This rules out knife-edge cases where a firm sells only high-quality units but is, to first order, indifferent between doing that and instead slightly reducing the number of high-quality units it sells and slightly raising its output of low-quality products from zero.

[14] If its cost of producing such products were higher, the upgrade cost of the merged firm, $c_2^m - c_1^m$, would be higher than that of the firm producing high-quality, which would cause a direct effect of the merger to be a decrease in the merged firm's high-quality output, potentially overturning the result. We return to this point later in the section.

Example 1. *Consumers have multiplicatively separable preferences, so that a consumer of type θ who purchases a product of quality q has utility $v(\theta, q) = \theta q$. Also, $\theta \sim F(\theta)$ with $F(\theta) = \theta^{1/\alpha}$ for $\alpha \in (0, 1)$.*

For Example 1, the demand functions for low quality and upgrades to high quality are

$$P_1(Z_1) = q_L(1 - Z_1)^{\alpha}, \text{ and}$$
$$P_2(Z_2) = (q_H - q_L)(1 - Z_2)^{\alpha}.$$

These are concave for $\alpha \in (0, 1)$ and satisfy our assumption of decreasing marginal revenue.

There are three firms. All firms have the same cost $c_1 = 0.1$ for low quality. One firm i has upgrade costs $c_2^i - c_1^i = c_2 - c_1 = 0.1$ but the other two firms have upgrade costs so high that they only produce low quality. Low quality is set at $q_L = 1$ while q_H and α vary.

Figure 1 illustrates the effect on consumer surplus of a merger between a firm producing only low quality and the firm capable of producing high quality. On the x-axis relative quality varies from $q_H/q_L = 3$ to $q_H/q_L = 5$. The four curves depict different values of the demand parameter α, ranging from $\alpha = 0.1$ (the top solid curve) to $\alpha = 0.4$ (the bottom thin curve). In all these parameterizations the merger is both privately profitable and beneficial for the outsider firm. In many cases the merger also increases consumer surplus. For example, when $q_H/q_L = 3$ and $\alpha = 0.1$ consumer surplus increases by 6.71%; most consumers purchase high quality, and so consumer gains are not limited to a small "luxury segment". It is therefore possible that consumers overall, insider firms, and outsider firms all gain from a merger—something which cannot occur in a single-product Cournot model.[15]

We note that in all these parameterizations the merged firm sells only high quality. The merged firm therefore removes the low-quality product from its portfolio and yet consumer surplus can still increase. As in our discussion of the repositioning defense following Corollary 1, this suggests caution in making inferences regarding consumer surplus based on either the observed addition or removal of products from industry participants' portfolios.

The parameterizations in Figure 1 suggest that the merger is more likely to increase consumer surplus—and to increase it by more—when q_H/q_L is higher and α is lower. Although both parameters affect pre- and post-merger equilibrium in a complicated way, a first-pass intuition is as follows. In all parameterizations the firm capable of producing high quality chooses to specialize in it prior to the merger; the merger therefore decreases Z_1 but increases Z_2. Heuristically, consumers are more likely to benefit from the merger when the increase in Z_2 is large relative to the decrease in Z_1, and when they place more value on Z_2. One effect of an increase in q_H is indeed that consumers value upgrade output more highly. One

[15]For all parameterizations in Figure 1, a merger between the two low quality firms induces the same change in consumer surplus (and benefits the high-quality outsider) but is not profitable (absent fixed cost savings). However, there are other parameterizations where such mergers are profitable and also increase consumer surplus.

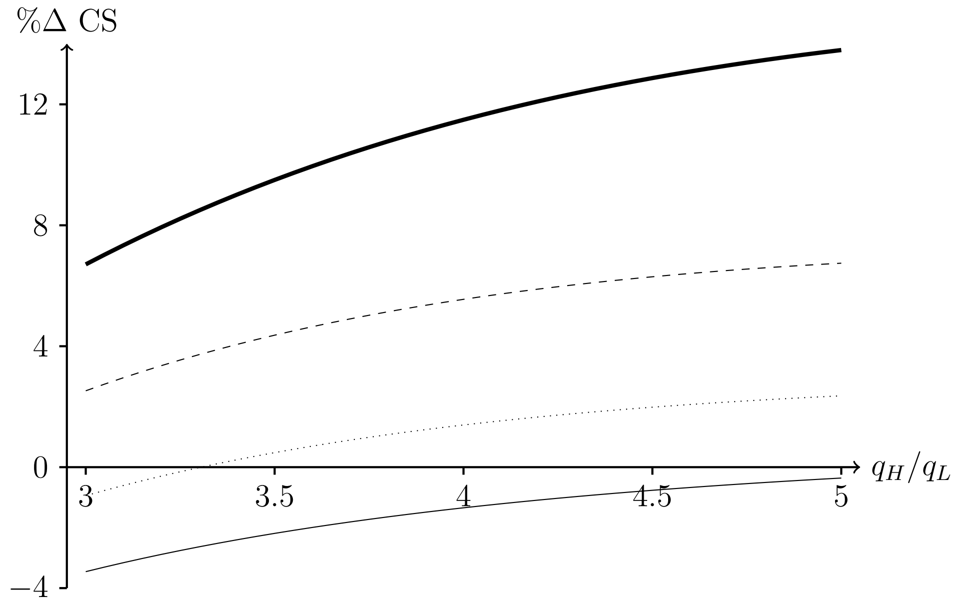

FIGURE 1. A plot of the percentage change in consumer surplus due to a merger between a low-quality firm and the firm capable of supplying high quality q_H, based on Example 1. The demand parameter is either $\alpha = 0.1$ (thick solid curve), $\alpha = 0.2$ (dashed curve), $\alpha = 0.3$ (dotted curve), or $\alpha = 0.4$ (thin solid curve). Other parameters are fixed at $(q_L, c_1, c_2) = (1, 0.1, 0.2)$.

effect of a decrease in α is that all else equal $\sigma_1(Z_1) - \sigma_2(Z_2)$ becomes more negative, that is baseline demand becomes relatively more concave and upgrade demand relatively more convex. Intuitively, when $\sigma_1(Z_1)$ is smaller, an equilibrium decrease in Z_1 leads to a larger increase in the merged firm's marginal revenue. The merged firm must therefore increase its output. However when $\sigma_2(Z_2)$ is larger, its marginal revenue is less sensitive to Z_2. Hence a larger increase in Z_2 is needed to re-equilibrate the industry.[16]

Sufficient Conditions for Consumer Surplus to Increase. It is unfortunately difficult to provide general conditions under which a merger increases Z_2 by enough that overall consumer surplus goes up. However the Online Appendix provides results for two special cases. First, we examine a merger between two firms where one of the insiders is inefficient and so has a very small market share. We provide a necessary and sufficient condition for consumer surplus to increase. Second, we consider a merger between firms with arbitrary market shares. Assuming that baseline and upgrade demands have constant (but different) curvatures, we provide a sufficient condition for a merger between two low-quality firms to increase consumer surplus. Consistent with the above discussion, in both cases the condition is easier to satisfy when $\sigma_1(Z_1)$ is smaller and $\sigma_2(Z_2)$ larger at the pre-merger equilibrium.

[16]This intuition is closely related to the literature on third-degree price discrimination such as Robinson (1933), Schmalensee (1981), Varian (1985), Cowan (2007), and Aguirre, Cowan, and Vickers (2010) where curvature differences across markets are central in determining output effects from discrimination.

The Effects of Synergies. For a merger involving firms $i \in \mathcal{I}$, a synergy for product k means that the merged firm m's marginal cost c_k^m is lower than $\min_{i\in\mathcal{I}} c_k^i$. Johnson and Myatt (2006) show that a reduction in a firm's marginal cost (or, in our merger setting, a synergy that reduces c_k^m below $\min_{i\in\mathcal{I}} c_k^i$) can cause Z_1 and Z_2 to move in opposite directions. We complement their work by identifying when such a product-mix effect does and does not arise and by considering the resulting impact on consumer surplus.[17]

Specifically, in this subsection we suppose that a merger has already been consummated, leading to some level of post-merger consumer surplus. We look at how synergies—that is, reductions in post-merger marginal costs c_1^m and c_2^m below $\min_{i\in\mathcal{I}} c_1^i$ and $\min_{i\in\mathcal{I}} c_2^i$ respectively—affect this level of post-merger consumer surplus.

We first identify when synergies work as in a single-product setting, unambiguously raising post-merger consumer surplus. For analytical convenience we state our results in terms of small synergies, that is small reductions in post-merger marginal costs as just discussed.

Proposition 4. *Assume a merger has taken place. A small synergy increases post-merger consumer surplus in the following three cases.*

(1) The merged firm only produces high quality and its cost of high quality decreases.

(2) No firm produces only high quality, and

  (a) the merged firm is multiproduct and its cost of high quality decreases, or

  (b) the merged firm only produces low quality and its cost of low quality decreases.

All of the cases considered in Proposition 4 are straightforward in that they involve no interesting product-mix effects; synergies (weakly) increase both Z_1 and Z_2, thus raising consumer surplus. For the case in which the merged firm produces only high quality there is a strategic linkage between the two markets. However, because the merged firm is upgrade-constrained, the direct effect of the synergy is an equal increase in m's output in both the baseline and upgrade markets. Although other firms respond by lowering their own outputs, the response is not enough to overturn the direct effect; both prices fall as a result of such a synergy. For the two cases in which no firms produce only high quality, the result holds because (i) there is no strategic linkage between the baseline and upgrade markets because no firm is upgrade-constrained, and (ii) the synergy only affects one marginal cost of relevance.[18]

Beyond Proposition 4, synergies can lower consumer surplus. Rather than provide an exhaustive analysis, we focus on synergies influencing a multiproduct firm's cost of low quality. Similar forces emerge for other cases.

[17] With homogeneous goods Cournot, synergies may reduce welfare (see for example Schwartz, 1989). With differentiated goods Cournot, synergies can also increase prices under certain conditions (see Chen and Li, 2018).

[18] To better understand condition (ii), note that in case 2a the synergy lowers the firm's upgrade cost but does not change its baseline cost, while in case 2b for the firm with the synergy only one product is relevant and its cost falls.

In the following proposition, let n^{**} and n^{L**} be respectively the total number of firms and the number of firms supplying only low-quality products after the merger.

Proposition 5. *Assume a merger has taken place and the merged firm is producing both products. A small synergy that reduces the merged firm's cost of the low-quality product:*

(1) increases post-merger industry aggregate output but reduces the post-merger industry supply of high quality,

(2) increases the post-merger price of high quality products if and only if

$$n^{L**} - \sigma_1(Z_1) + \sigma_2(Z_2) > 0, \tag{4}$$

(3) decreases post-merger consumer surplus if and only if

$$\frac{Z_2}{Z_1} > \frac{n^{**} - n^{L**} + 1 - \sigma_2(Z_2)}{n^{**} + 1 - \sigma_1(Z_1)}. \tag{5}$$

Intuitively, if the merged firm is multiproduct and its cost of low quality c_1^m decreases, then its upgrade cost $c_2^m - c_1^m$ increases and so it expands total output but lowers high-quality output. Although other firms adjust their outputs in response, the final effect is an increase in overall industry supply and a decrease in high-quality supply. The price of high-quality products is more likely to increase, and consumer surplus more likely to decrease, when $\sigma_1(Z_1)$ is low and $\sigma_2(Z_2)$ high, and when more of the firms in the industry supply only low quality. This is because, in these circumstances, marginal revenue is more sensitive to changes in baseline output but less sensitive to changes in upgrade output—and so the expansion in baseline units due to the synergy is small relative to the decrease in upgrade units.

We now use Example 1 to illustrate the effect of synergies. Here there are two firms, one of which produces only low quality and the other of which is a multiproduct firm. The multiproduct firm enjoys a synergy which reduces its cost of low quality. As before, $q_L = 1$ and (absent the synergy) both firms have cost for low quality given by $c_1 = 0.1$. The demand parameter takes four different values—$\alpha \in \{0.1, 0.2, 0.3, 0.4\}$. We set $q_H = 3$, and make the upgrade cost $c_2 - c_1 = 1.75$ so that the multiproduct firm indeed wishes to sell both products.

Figure 2 shows that synergies can harm consumers. For example when $\alpha = 0.1$, a 20% synergy reduces consumer surplus by 1.12%. Imagining that the multiproduct firm recently acquired another firm, if the merger would lower consumer surplus without synergies, then these synergies make the merger even worse. Our parameterizations also suggest that a synergy is more likely to reduce consumer surplus when α is smaller. Recalling our earlier discussion, this is because for the demand system in Example 1 a decrease in α tends to make baseline demand more concave and upgrade demand more convex.[19]

[19] Proposition 5 shows that q_H has no direct bearing on whether synergies benefit consumers. However simulations suggest that higher q_H makes it more likely that synergies reduce consumer surplus (due to its effect on Z_1 and Z_2).

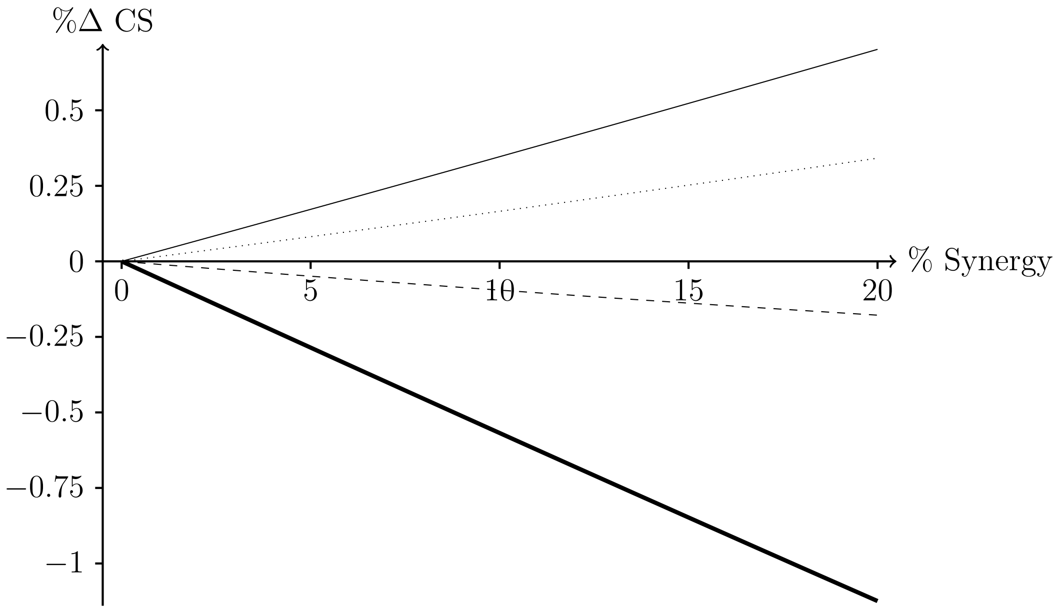

FIGURE 2. A plot of the percentage change in consumer surplus due to a percentage synergy on the low-quality product, based on Example 1. The demand parameter is either $\alpha = 0.1$ (thick solid curve), $\alpha = 0.2$ (dashed curve), $\alpha = 0.3$ (dotted curve), or $\alpha = 0.4$ (thin solid curve). Other parameters are fixed at $(q_L, q_H, c_1, c_2) = (1, 3, 0.1, 1.85)$.

4. Outsider Profits and External Effects

In this section we first examine the impact of a merger on the profits of outsider firms. After that, we study a merger's *external effect*, which is defined as the merger-induced change in the sum of consumer surplus and outsider profits.

Outsider Profits. In a single-product quantity-setting market each outsider benefits from a merger if and only if it lowers industry output and consumer surplus; although some outsiders may gain (or lose) more than others, if one gains (or loses) then they all do. We first show that the same is true in our multiproduct setting if there is no product-mix effect.

Proposition 6. *If a merger weakly decreases both Z_1 and Z_2 then all outsider firms are weakly better off and consumers are weakly worse off. (The reverse is true if the merger weakly increases both Z_1 and Z_2.)*

If Z_1 and Z_2 decrease then the prices of both products increase, and this benefits all outsiders while harming consumers.

On the other hand, when there are product-mix effects some outsiders may gain from a merger while other outsiders lose. For example, a merger could raise the price of low quality while reducing the price of high quality. Intuitively this benefits an outsider that only sells low quality but may harm an outsider that only sells high quality.[20] Hence some outsiders may support a merger while others oppose it.

Moreover, when there are product-mix effects, outsider profit and consumer surplus may both increase or both decrease following a merger. For example, Figure 1 depicts examples where both increase.[21] As discussed in the Introduction, this runs counter to the common intuition that if rival firms support a merger then it must be anti-competitive and thus harmful to consumers, whereas if rival firms oppose a merger it must be beneficial to consumers.

External Effects. We now turn to the external effect of a merger. As pointed out by Farrell and Shapiro (1990), when the external effect of a merger is positive then—assuming that only mergers which benefit insiders are proposed—the merger increases total surplus. As such, our analysis here complements our earlier results on consumer surplus.

As a preliminary step, we first consider the external effect due to small but arbitrary changes dZ_1 and dZ_2 in equilibrium total and upgrade outputs respectively. To this end we denote firm i's share of total output by $s_1^i(Z_1, Z_2)$ and of upgrade output by $s_2^i(Z_1, Z_2)$; to ease the exposition we henceforth omit their dependence on Z_1 and Z_2.

Lemma 2. *The external effect due to small output changes dZ_1 and dZ_2 is*

$$\begin{aligned} d\mathcal{E} =& Z_1 P_1'(Z_1) \left\{ \sum_{i\in\mathcal{O}} s_1^i \left[2 - s_1^i \sigma_1 (Z_1)\right] - 1 \right\} dZ_1 \\ &+ Z_2 P_2'(Z_2) \left\{ \sum_{i\in\mathcal{O}} s_2^i \left[2 - s_2^i \sigma_2 (Z_2)\right] - 1 \right\} dZ_2. \end{aligned} \tag{6}$$

The external effect associated with small output changes dZ_1 and dZ_2 can therefore be written as the sum of an external effect in the baseline market plus an external effect in the upgrade market—each of which has the same form as the external effect derived by Farrell and Shapiro (1990) for infinitesimal mergers in a standard single-product setting.

Now consider the external effect of a *complete merger*. Conceptually, a complete merger can be thought of as a sequence of small output changes dZ_1 and dZ_2, with the sequence starting at pre-merger outputs (Z_1^*, Z_2^*) and ending at post-merger outputs (Z_1^{**}, Z_2^{**}). The external

[20] For a more formal, stark example, consider the following. First, take a market with only three firms, and suppose that a merger between two of them strictly increases $P_1(Z_1)$, strictly decreases $P_1(Z_1)+P_2(Z_2)$, and makes the outsider strictly better off. (Figure 1 depicts, and footnote 15 discusses, examples where this happens.) Let $P_1(Z_1^*) + P_2(Z_2^*)$ denote the pre-merger price of high quality. Second, suppose we add a fourth firm with marginal costs $c_1 = c_2 = P_1(Z_1^*) + P_2(Z_2^*) - \varepsilon$. Note that for $\varepsilon > 0$ small this new firm is active before but not after the merger, and so is made worse off by the merger. At the same time, by continuity the other outsider still gains from the merger.

[21] It is also straightforward to construct examples where both decrease. Suppose that, pre-merger, there are two multiproduct firms, and one firm that produces only low quality because its cost of high quality is prohibitively large. Suppose the two multiproduct firms merge, and enjoy a synergy such that Z_1 increases by an arbitrarily small amount while Z_2 decreases. This merger reduces both consumer surplus and the outsider firm's profit.

effect of a complete merger is then obtained by integrating equation (6) over this sequence. Clearly the sequence of output changes depends on the particular merger being considered.

We now examine three different types of merger (without synergies) and derive conditions such that the external effect is positive. We let $s_1^{\mathcal{I}*}$ and $s_2^{\mathcal{I}*}$ denote insiders' pre-merger shares of total and upgrade output, and s_1^{i*} and s_2^{i*} denote the same for an outsider firm $i \in \mathcal{O}$.

Our first results pertain to mergers where there is no product-mix effect.

Proposition 7. *Suppose that no firm produces only high quality before the merger. Consider a merger without synergies, in which either*

(1) each of the merging firms produces only low-quality products before the merger, or
(2) one of the merging firms is multiproduct before the merger, and its marginal cost for low quality is no higher than that of any other merging firm.

Assuming that $\sigma_1'(Z_1) \geq 0$, the external effect of a complete merger is positive provided

$$s_1^{\mathcal{I}*} < \sum_{i \in \mathcal{O}} s_1^{i*} \left[1 - s_1^{i*} \sigma_1(Z_1^*)\right]. \tag{7}$$

The mergers described in Proposition 7 lead to a strict decrease in total output Z_1 but no change in upgrade output Z_2. Under a restriction on demand curvature, their external effect is positive provided that insiders' pre-merger share of total output is not too large; the precise condition (7) is the same as the one derived by Farrell and Shapiro (1990) for output-decreasing mergers in a standard single-product environment.[22,23]

The intuition behind condition (7) is that firms with a low share of total output are relatively inefficient, and so a merger between them raises external surplus by increasing the output of more efficient outsiders. Interestingly, note that condition (7) does not depend on market shares for low-quality supply. This is because low-quality products compete not only against other low-quality units but also against the baseline units that are ultimately upgraded to high quality. Consequently—and counter to the intuition one might have based on single-product models—a merger between two low-quality firms with large shares of low-quality output can have a positive external effect. This happens if, for example, outsiders are large multiproduct firms that mainly sell high-quality.

Continuing with mergers where there is no product-mix effect, we have the following result.

Proposition 8. *Consider a merger without synergies, in which each of the merging firms produces only high quality before the merger. Suppose that one of them has marginal costs*

[22]Proposition 7 requires that demand curvature satisfies $\sigma_1'(Z_1) \geq 0$. Farrell and Shapiro (1990) allow for non-constant marginal costs, but impose stronger conditions $P_1''(Z_1) \geq 0$ and $P_1'''(Z_1) \geq 0$ on demand.

[23]Our analysis here is also related to Nocke and Schutz (2019) who, in a setting with multiproduct price competition, provide conditions on pre-merger equilibrium variables such that a complete merger's external effect is positive.

for low and high quality that are no higher than those of any other merging firm. Assuming that $\sigma_1'(Z_1) \geq 0$ and $\sigma_2'(Z_2) \geq 0$, the external effect of a complete merger is positive provided

$$s_1^{\mathcal{I}*} < \sum_{i \in \mathcal{O}} s_1^{i*} \left[1 - s_1^{i*} \sigma_1(Z_1^*)\right] \quad \textit{and} \quad s_2^{\mathcal{I}*} < \sum_{i \in \mathcal{O}} s_2^{i*} \left[1 - s_2^{i*} \sigma_2(Z_2^*)\right]. \tag{8}$$

The merger described in Proposition 8 leads to strict decreases in both total output Z_1 and upgrade output Z_2. Under a restriction on demand curvature, the external effect is positive provided that insiders' pre-merger shares of both total and upgrade outputs are not too large. Insiders' share of total output is again important because the baseline units that they upgrade compete with outsider firms' low-quality units.[24]

Now consider mergers between low- and high-quality firms that have a product-mix effect. Given some restrictions on pre-merger industry participants we obtain the following result.

Proposition 9. *Suppose that before the merger one firm in the market produces only high quality. Consider a merger without synergies between it and other firms that have weakly higher marginal costs and produce only low quality before the merger. Assuming that $\sigma_1'(Z_1) \geq 0$ and $\sigma_2'(Z_2) \geq 0$, the external effect of a complete merger is positive provided*

$$s_1^{\mathcal{I}*} < \sum_{i \in \mathcal{O}} s_1^{i*} \left[1 - s_1^{i*} \sigma_1(Z_1^*)\right] \quad \textit{and} \quad s_2^{\mathcal{I}*} > \sum_{i \in \mathcal{O}} s_2^{i*} \left[1 - s_2^{i*} \sigma_2(Z_2^*)\right]. \tag{9}$$

The merger described in Proposition 9 leads to a strict decrease in total output Z_1 but a strict increase in upgrade output Z_2. Under the usual restriction on demand curvature, the external effect is positive provided that insiders' pre-merger share of total output is not too large, and their pre-merger share of upgrade output is not too small. Consequently—and again counter to the intuition from single-product models—a firm with a large share of high-quality output may merge and generate a positive external effect. This happens if, for example, outsiders are large firms that sell predominantly low-quality products.

We note that checking conditions (7)-(9) requires information on demand curvature. Sometimes information on passthrough rates can be used to infer demand curvature. For example, if the pre-merger industry structure satisfies the conditions in Proposition 7, then a unit increase in each of the n firms' marginal costs causes the price of low quality to increase by $n/[n + 1 - \sigma_1(Z_1)]$. Knowledge of pre-merger passthrough rates and how they change can therefore give (local) information on $\sigma_1(Z_1)$ and $\sigma_1'(Z_1)$. Moreover, even if curvature cannot be estimated or inferred, some progress can still be made. Returning to Proposition 7, note that given Assumption 1 a sufficient condition for (7) to hold is that $s_1^{\mathcal{I}*} \leq \sum_{i \in \mathcal{O}} s_1^{i*}(1 - s_1^{i*})$. Therefore if one is willing to assume that curvature increases (or does not decrease too fast), pre-merger market shares alone can be used to establish a positive external effect.[25]

[24]Note that the two conditions in (8) are equivalent in the special case where $v(\theta, q) = \theta q$ and all firms supply only high-quality products before the merger, because $\sigma_1(Z_1^*) = \sigma_2(Z_2^*)$ and $s_1^{\mathcal{I}*} = s_2^{\mathcal{I}*}$ and also $s_1^{i*} = s_2^{i*}$ for each $i \in \mathcal{O}$.

[25]The assumption in Proposition 7 that $\sigma_1'(Z_1) \geq 0$ is sufficient but not necessary to prove the result.

5. The Herfindahl-Hirschman Index

The Herfindahl-Hirschman Index (HHI) is a common measure of industry concentration. In single-product markets, where s^i denotes the market share of firm i, HHI is defined as

$$HHI = \sum_i (s^i)^2. \tag{10}$$

When screening mergers for potential harm, antitrust authorities often use the "naively computed" post-merger HHI and the merger-induced change in HHI—where "naively computed" means that the merged firm's market share is assumed to equal the sum of the insiders' pre-merger market shares, and each outsider's market share is assumed to be the same pre- and post-merger. The change in HHI due to a merger between firms j and k is thus

$$\Delta HHI = (s^j + s^k)^2 - (s^j)^2 - (s^k)^2 = 2s^j s^k. \tag{11}$$

Note that HHI ranges from 0 to 10000, while ΔHHI ranges from 0 to 5000.

The 2010 U.S. Horizontal Merger Guidelines (HMG) state that mergers "potentially raise significant competitive concerns and often warrant scrutiny" if they induce an HHI between 1500 and 2500 and a ΔHHI above 100, or if they induce an HHI above 2500 and a ΔHHI between 100 and 200. Meanwhile mergers are "presumed to be likely to enhance market power" (and harm consumers) if they induce an HHI above 2500 and a ΔHHI above 200.[26]

Next, we revisit the rationale for using HHI and ΔHHI to screen mergers in a single-product market. We then examine whether this rationale extends to a multiproduct setting.

Single-Product Markets. Nocke and Whinston (2020) note that, while there is no clear relationship between the level of HHI and how a merger affects consumers, changes in the HHI can be useful in assessing the impact of a merger.[27] Specifically, they note that in a single-product setting a merger harms consumers unless it induces a synergy which lowers the merged firm's cost below a critical level. They prove that if two firms with identical cost c merge and enjoy a post-merger marginal cost c^m, this critical (synergy-induced) marginal cost $\hat{c}^m$ is strictly less than c and strictly decreasing in ΔHHI, and satisfies

$$\frac{c - \hat{c}^m}{c} = \frac{\sqrt{\Delta HHI/2}}{\epsilon - \sqrt{\Delta HHI/2}}, \tag{12}$$

[26]See `www.justice.gov/atr/horizontal-merger-guidelines-08192010` for the U.S. merger guidelines. One advantage of the HHI and ΔHHI are their low information requirements. More elaborate options such as the SSNIP test (Katz and Shapiro, 2003; O'Brien and Wickelgren, 2003) and its multiproduct firm extension (Moresi, Salop, and Woodbury, 2008) require detailed cost data as well as estimates of certain diversion ratios.

[27]Note that a higher level of the HHI is indicative of firms having greater market power. Cowling and Waterson (1976), Kwoka (1985), and Spiegel (2019) show that HHI is proportional to, respectively, the ratio of industry profit to industry revenue, share-weighted price-cost margins, and the ratio of industry profit to consumer surplus.

where ϵ is demand elasticity. Consequently, given uncertainty over any realized synergies, a merger is "more likely" to harm consumers when ΔHHI is larger.[28]

Multiproduct Markets. Now we provide some conditions under which changes in HHI are predictive of how a merger affects consumers in a multiproduct setting. As earlier, we let Z_1^* and Z_2^* denote pre-merger total and upgrade outputs respectively, and let s_1^{i*} and s_2^{i*} be firm i's share of those outputs.

Proposition 10. *Suppose that no firm produces only high quality before the merger. Consider a merger between two firms j and k who, prior to the merger, produce only low quality and do so at the same cost c_1. The merger has no effect on aggregate outputs Z_1 and Z_2 if the insiders' post-merger marginal cost c_1^m equals a critical threshold $\hat{c}_1^m$ which satisfies*

$$\frac{c_1 - \hat{c}_1^m}{c_1} = \frac{\sqrt{\Delta HHI_1/2}}{\epsilon_1 - \sqrt{\Delta HHI_1/2}}, \tag{13}$$

where $\epsilon_1 = -P_1(Z_1^*)/[Z_1^* P_1'(Z_1^*)]$ is the elasticity of total demand, and $\Delta HHI_1 = 2 s_1^{j*} s_1^{k*}$ is the change in the HHI measured over total output.

We know from earlier analysis that, absent synergies, the merger described in Proposition 10 does not lead to a product-mix effect and so harms consumers. As a result, the critical (synergy-induced) cost such that the merger does not change outputs—and therefore also does not change consumer surplus—has the same form as in Nocke and Whinston (2020).[29] As in single-product markets, this suggests that higher changes in HHI require higher levels of synergy for consumers to benefit. There is, however, an important caveat to this conclusion.

As the marginal cost of the merged firm further decreases (below $\hat{c}_1^m$ defined above), the total output of the merged firm also increases. This may lead some multiproduct firms to remove the low-quality product from their product line, and also to reduce their output of high-quality products. If this occurs, then consumer surplus could fall compared to the pre-merger situation. Therefore, although we can conclude that higher changes in HHI require a higher synergy to maintain output at pre-merger levels, we cannot conclude that synergies above this level ensure that consumer surplus exceeds pre-merger levels. Unless, that is, we are confident that no firms in the industry will adjust their product lines.

Continuing with mergers where, absent synergies, there is no product-mix effect—and so the merger reduces outputs and harms consumers—we have the following result.[30]

[28] Nocke and Whinston (2020) generalize equation (12) to the case where merger insiders have asymmetric costs. One can show that the insights from Propositions 10 and 11 below hold if, prior to the merger, insiders sell the same products but have different costs.

[29] For tractability we focus on the critical marginal cost which ensures that Z_1 and Z_2 are the same pre- and post-merger. Note that there might be other costs which induce the same consumer surplus but different Z_1 and Z_2.

[30] One can also show that if two symmetric multiproduct firms merge, and no firm produces only high quality before the merger, then the critical post-merger costs $(\hat{c}_1^m, \hat{c}_2^m - \hat{c}_1^m)$ take the same form as in Nocke and Whinston (2020).

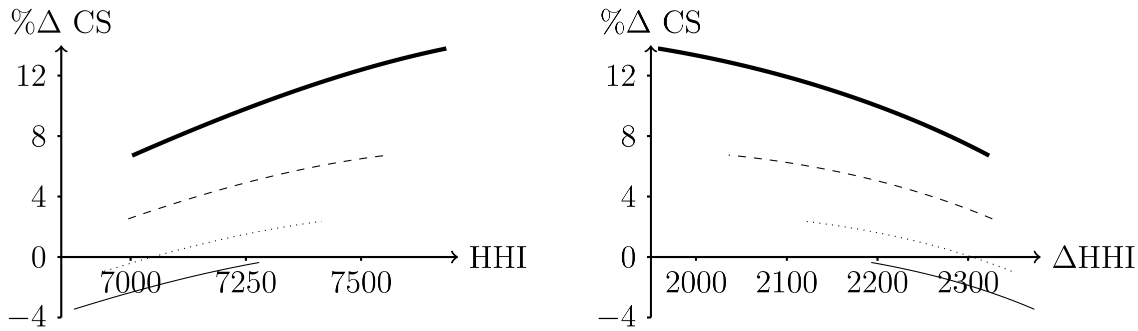

FIGURE 3. A plot of the naively computed post-merger HHI (left panel) and the change in the HHI (right panel) against the percentage change in consumer surplus due to a (synergy-less) merger between a low-quality firm and the firm capable of supplying high quality q_H, based on Example 1. The demand parameter is either $\alpha = 0.1$ (thick solid curve), $\alpha = 0.2$ (dashed curve), $\alpha = 0.3$ (dotted curve), or $\alpha = 0.4$ (thin solid curve). Different points along each curve correspond to different values of $q_H \in [3, 5]$. Other parameters are fixed at $(q_L, c_1, c_2) = (1, 0.1, 0.2)$.

Proposition 11. *Consider a merger between two firms j and k that produce only high quality before the merger, and have the same cost c_2. The merger has no effect on aggregate outputs Z_1 and Z_2 if the insiders' post-merger marginal cost c_2^m equals a critical threshold $\hat{c}_2^m$ which satisfies*

$$\frac{c_2 - \hat{c}_2^m}{c_2} = \frac{\rho\sqrt{\Delta HHI_1/2}/\epsilon_1 + (1-\rho)\sqrt{\Delta HHI_2/2}/\epsilon_2}{1 - \rho\sqrt{\Delta HHI_1/2}/\epsilon_1 - (1-\rho)\sqrt{\Delta HHI_2/2}/\epsilon_2}, \tag{14}$$

where $\rho = P_1(Z_1^*)/[P_1(Z_1^*) + P_2(Z_2^*)]$, and $\epsilon_l = -P_l(Z_l^*)/[Z_l^* P_l'(Z_l^*)]$ is the elasticity and $\Delta HHI_l = 2s_l^{j*}s_l^{k*}$ the change in the HHI for total ($l = 1$) or upgrade ($l = 2$) output.

Proposition 11 shows that the critical (synergy-induced) cost for a merger between two high-quality firms has a similar form to Nocke and Whinston (2020). However, a caveat is that there are in fact two not one relevant HHI values to consider, that regarding total output (HHI_1) and that regarding only high-quality output (HHI_2). Increases in either of these raise the required synergy. Additionally, we can see that the ideal threshold in a given market depends on both of these HHI values, and so considering only one of them may lead to a threshold that is biased. From this observation, a final caveat emerges: in a multiproduct market, there ought to be a sliding scale involving HHI changes. For example, a merger with a higher value of ΔHHI_1 might be permitted if it also had a lower value of ΔHHI_2.

We now close by examining mergers where, absent synergies, there is a product-mix effect, and show that under these circumstances the thresholds in the HMG may not provide correct guidance. Figure 3 illustrates the effect of a merger between a low- and a high-quality firm when there are no synergies, there is one low-quality outsider, and demand follows Example 1. We take each market depicted in Figure 1, naively compute the change in HHI and the corresponding post-merger HHI (involving total market shares), then plot these against the merger-induced equilibrium change in consumer surplus. (Thus, each curve depicts a different demand parameter α, while along each curve q_H varies between 3 and 5.)

Figure 3 shows that a merger can benefit consumers when HHI exceeds 7500 and ΔHHI exceeds 2300—far above the levels deemed harmful in the HMG.[31] We emphasize again that in the mergers we are considering in Figure 3, there are no synergies and so in a single product world all of these mergers would necessarily harm consumers.

6. Merger Profitability

In this section we address the classic question of whether horizontal mergers without synergies are profitable. On the one hand, insiders benefit from enhanced market power, which may lower industry output and increase prices. On the other hand, insiders are harmed if outsiders become more aggressive, for example by expanding output as in the standard quantity-setting framework.

Indeed, Salant, Switzer, and Reynolds (1983) have argued that the competitive response of outsiders often dominates, causing many horizontal mergers to be unprofitable. In a single-product quantity-setting industry with linear demand and symmetric firms with constant marginal costs, a necessary condition for a merger to be profitable is that the merging parties have at least an 80% pre-merger market share. Even allowing for general demand functions, both Levin (1990) and Cheung (1992) have shown that a 50% threshold is necessary.[32]

Linear Demand. We begin by assuming that consumers have multiplicative preferences with θ distributed uniformly on $[0, 1]$, a boundary case of Example 1 with $\alpha = 1$. Hence demand is linear as in the classic Salant, Switzer, and Reynolds (1983) analysis:

$$P_1(Z_1) = q_L(1 - Z_1), \text{ and}$$
$$P_2(Z_2) = (q_H - q_L)(1 - Z_2).$$

We place no restrictions on the cost structures of outsider firms. We do assume that the $k+1$ insiders have symmetric costs, which rules out profitability gains from output rationalization. We also assume there are no synergies.

[31] Note that HHI and ΔHHI are high because this is a three-to-two merger. We pointed out in footnote 15 that a merger between the two low-quality firms has the same effect on consumer surplus; one can show that for such a merger HHI and ΔHHI are lower, but that consumers can benefit when HHI exceeds 6000 and ΔHHI exceeds 600.

[32] Perry and Porter (1985) show that if mergers allow firms to lower their average costs by combining their capital stocks, then pairs of small firms with limited market shares may find it profitable to merge. Daughety (1990) shows that if merging firms become Stackelberg leaders then smaller mergers may be profitable. Levin (1990) and Fauli-Oller (2002) note that reallocation opportunities may make mergers more profitable. Nocke and Whinston (2010) show that, allowing for synergies, mergers which increase consumer surplus are profitable. Deneckere and Davidson (1985) show that many mergers are profitable when firms set prices.

Let n^L be the number of firms that are producing only low quality, n^H the number producing only high quality, and n^M the number of multiproduct firms, where all variables reference the pre-merger industry structure of firms with positive output. We assume each of these firms is active post-merger and offers the same product line as before the merger. Where appropriate we provide conditions on parameters that ensure this.

We begin with the case where each of the $k+1$ merging firms produces only low quality.

Proposition 12. *Suppose demand is linear and no firms are multiproduct ($n^M = 0$). A sufficient condition for a merger involving all firms producing low quality to be profitable is*

$$n^L \geq \left(\frac{q_H}{q_H - q_L}\right)^2,$$

regardless of the number of firms n^H producing only high quality.

Proposition 12 is in stark contrast to the existing literature. It shows that quality differences may make mergers among quantity-setting firms more profitable than previously recognized. To illustrate, suppose that $n^L = 2$, and note that in Salant, Switzer, and Reynolds (1983) a merger between two firms is never profitable unless the pre-merger industry is a duopoly. In our multiproduct setting, such a two-firm merger is profitable so long as $q_H \gtrsim 3.41 q_L$, even if the number of other firms in the industry (producing high quality) is infinitely large.[33]

The intuition for Proposition 12 is as follows. When the low-quality firms merge the direct effect is a reduction in total output Z_1. Firms producing only high quality respond by raising their own output, but because they are upgrade-constrained they must raise both their baseline and upgrade outputs. This blunts the competitive response to the merger in terms of total output Z_1.[34] Consequently the merger is more readily profitable for insiders.

So far we have emphasized how quality differences alter the usual conclusions about merger profitability in quantity-setting markets. The following proposition continues this theme and at the same time indicates some similarities with classic results.

Proposition 13. *Suppose demand is linear. Consider a merger of $k+1$ firms that produce only low-quality products. A necessary condition for the merger to be profitable is that it involves at least 80% of the firms that, pre-merger, produce low-quality products—that is*

$$\frac{k+1}{n^L + n^M} \geq \frac{8}{10}.$$

[33] To ensure that each firm has positive output certain conditions must hold. For example, suppose all outsiders are symmetric with marginal cost for high quality c_2^H, and let c_1^L be the marginal cost of low-quality firms. Both insiders and outsiders have positive output if $c_1^L/q_L < c_2^H/q_H$ and $(c_2^H - c_1^L)/(q_H - q_L) < 1$.

[34] Consider an example where the number of responding firms n^H is arbitrarily large. Post-merger Z_1 cannot be the same as its pre-merger level: if it were, then, because Z_2 has increased, the price of high-quality products would be lower than before the merger, which would imply that post-merger high-quality firms make negative profit.

Remarkably, this is the same lower bound for merger profitability discovered by Salant, Switzer, and Reynolds (1983). However, in their single-product analysis, all firms in the industry have the same costs and so a necessary condition for profitability is that the merging firms have at least 80% market share. In contrast, we emphasize that what matters is not insiders' share of output (either of the entire market or the low quality segment). Rather, what matters is their share of all firms that produce low-quality products. In this sense, the emphasis of the existing literature on the market shares of insiders is somewhat misplaced.

The spirit of our results above holds for other types of mergers. When the insiders produce only high-quality products rather than low-quality products, our main results hold exactly.

Remark 1. *Consider a merger involving $k+1$ firms producing only high-quality products. Swapping the n^L and the n^H terms, Propositions 12 and 13 hold exactly.*

For example, a necessary condition for profitability is that at least 80% of firms producing high-quality products take part in the merger, $(k+1)/(n^H+n^M) \geq 8/10$. Additionally, mergers involving as few as two firms producing only high-quality can be profitable even when the number of firms n^L producing low-quality products grows infinitely large

Mergers involving only multiproduct firms or mixed mergers are more complicated analytically but lead to similar qualitative results. We do not present formal results but instead note that a merger involving a fixed number of firms, each of which is a multiproduct firm pre-merger, may be profitable even when there are an infinite number of firms.

Nonlinear Demand and the 50% Benchmark. Here we briefly address merger profitability with nonlinear demand. Finding exact sufficient conditions for profitability is not straightforward with asymmetric firms and so we focus on necessary conditions for mergers to be profitable. We assume the insiders share the same pre-merger cost structure.

Proposition 14. *A necessary condition for a merger between $k+1$ low-quality firms or $k+1$ high-quality firms to be profitable is that*

$$\frac{k}{n^M+n^j-2} \geq \frac{1}{2},$$

where $n^j = n^L$ or n^H if the insiders are low quality or high quality, respectively.

At least roughly, a 50% benchmark does apply—but only in terms of the ratio of insiders to the number of firms producing the same quality as the insiders. Additionally, it is possible to show that if demand is concave instead of merely log-concave, then a different threshold can be derived that is slightly harder to satisfy. This is in the spirit of Fauli-Oller (2002) who, in a single-product setting, finds that mergers are more profitable when demand is convex.

7. Conclusion

We have provided a quantity-setting framework for assessing mergers in markets where quality differences are central. Our framework allows for asymmetric firms which may sell multiple products, but which may also specialize in either low or high quality. Both merging and non-merging firms may reposition their product lines by adding or removing products following a merger. Using this framework, we address classic topics such as the welfare effects of mergers, the use of the HHI as a merger screen, and the profitability of mergers.

Contrary to perceived wisdom, in our framework a merger without synergies may raise consumer surplus, even if the merging parties remove a product from their product line. Synergies, when present, may lower consumer surplus. Both of these results are driven by a new economic force called the product-mix effect. Consumers can benefit from a merger even when the level of, and changes in, HHI, far exceed those in the U.S. Horizontal Merger Guidelines. Consumers may also benefit from a merger even when some outsiders gain as well; this is contrary to the common wisdom that support for a merger by outsiders necessarily indicates likely anti-competitive effect. In addition, a merger may benefit some outsiders but harm others, depending upon which products they supply. Finally, we show that mergers are more readily profitable when products are vertically differentiated.

At the same time, in some cases the predictions of our framework match those from models where quality differences are not important and each firm sells a single product. For example, a merger between two firms that sell only high-quality products unambiguously raises prices and harms consumers in the absence of synergies, even if non-merging firms respond by introducing their own high-quality products. Moreover, post-merger synergies in such a merger reduce prices and benefit consumers.

Overall, we believe that our results suggest caution may be required when assessing mergers in markets with vertical product differentiation.

Appendix: Omitted Proofs

We require several lemmas. For lemmas 3 and 4, we let Z^* and Z^{**} denote two distinct equilibria, one representing the industry before a merger and the other representing the industry after the merger; it doesn't matter which one is which. Similarly, Z_k^* and Z_k^{**} represent industry upgrades in market $k \in \{1,2\}$, and Z_k^{i*} and Z_k^{i**} represent firm i's outputs.

We alert the reader that in this appendix we sometimes use the term "upgrades" to refer both to market $k=1$ and $k=2$, whereas in the body of the manuscript we reserved the term upgrades for market $k=2$. We do this to avoid unnecessarily lengthening the proofs.

Lemma 3. *Suppose that $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ where at least one inequality is strict.*

(1) If, ignoring its monotonicity constraint, firm i has a weak incentive to raise its output Z_k^{i**}, then it has a strict incentive to raise its output Z_k^{i*} when either (i) it is not involved in the merger, or (ii) it is involved in the merger but has weakly lower cost for upgrade k when it chooses Z_k^{i*}.

(2) If, ignoring its monotonicity constraint, firm i has a weak incentive to lower Z_k^{i*}, then it has a strict incentive to lower Z_k^{**i} when either (i) it is not involved in the merger, or (ii) it is involved but has weakly higher cost for upgrade k when it chooses Z_k^{i**}.

Proof. We prove the first claim; the second can be proven similarly. Because firm i has a weak incentive to raise Z_k^{i**} ignoring its upgrade constraint,

$$P_k(Z_k^{**}) + Z_k^{i**} P_k'(Z_k^{**}) \geq C_k^i,$$

where C_k^i is firm i's upgrade cost for k (c_1^i if $k=1$ and $c_2^i - c_1^i$ if $k=2$). (Recall what we mentioned just before the statement of this lemma: in this appendix we will let the term "upgrade" refer both to $k=1$ and $k=2$, in order to save space.) Since $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ with at least one being strict, decreasing marginal revenue and $P_k' < 0$ imply

$$P_k(Z_k^*) + Z_k^{i*} P_k'(Z_k^*) > C_k^i.$$

Finally, it is clear that this result continues to hold for the merged firm if the merged firm's upgrade cost C_k^m satisfies $C_k^m < C_k^i$ when choosing Z_k^{i*}. ∎

Lemma 4. *Suppose that $Z_k^{**} \geq Z_k^*$ and $Z_k^{i**} \geq Z_k^{i*}$ where at least one inequality is strict.*

(1) If $k=1$ then $Z_1^{i**} = Z_2^{i**}$ provided firm i is not involved in the merger.

(2) If $k=2$ and $Z_2^{i**} > 0$, then $Z_1^{i*} = Z_2^{i*}$ provided either (i) firm i is not involved in the merger, or (ii) firm i is involved but has weakly lower cost for upgrade 2 when it chooses Z_2^{i*}.

Proof. First consider $k=1$. Because i could raise Z_1^{i*} but does not it must have a weak incentive to lower Z_1^{i*} (ignoring its upgrade constraint). Using the second claim in Lemma 3, i has a strict incentive to lower Z_1^{i**}. Because i does not lower Z_1^{i**} it must be upgrade constrained and hence $Z_1^{i**} = Z_2^{i**}$. Second consider $k=2$. Because i could lower Z_2^{i**} but does not it must have a weak incentive to raise Z_2^{i**} (again ignoring its upgrade constraint). Using the first claim in Lemma 3, i has a strict incentive to raise Z_2^{i*}. (This is also true if i faces a lower cost for upgrade 2 when it chooses Z_2^{i*}.) Because i does not raise Z_2^{i*} it must be upgrade-constrained and hence $Z_1^{i*} = Z_2^{i*}$. ■

In the proofs of Propositions 1, 2 and 3, let Z_1 and Z_2 represent post-merger equilibrium outputs, and let $\tilde{Z}_1$ and $\tilde{Z}_2$ represent pre-merger equilibrium outputs.

Proof of Proposition 1: We begin by showing that $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$ is impossible.

First, we prove that if to the contrary $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, then any firm i not involved in the merger weakly decreases its supply of upgrade $k = 1, 2$. Suppose $k = 1$ and that to the contrary $Z_1^i > \tilde{Z}_1^i$. The first part of Lemma 4 says that $Z_1^i = Z_2^i$. There are then two subcases to consider. In the case where $\tilde{Z}_1^i = \tilde{Z}_2^i$, such that i is upgrade-constrained both pre- and post-merger, we immediately obtain a contradiction because Z_1 and Z_2 have weakly increased and hence firm i cannot have optimally strictly increased its output. In the case where $\tilde{Z}_1^i > \tilde{Z}_2^i$, it must be true that $Z_2^i > \tilde{Z}_2^i$ (because $Z_1^i > \tilde{Z}_1^i$ and $Z_1^i = Z_2^i$); but then the second part of Lemma 4 says that $\tilde{Z}_1^i = \tilde{Z}_2^i$, which is also a contradiction. Since in both cases we obtain a contradiction, we conclude that $Z_1^i \leq \tilde{Z}_1^i$. Now suppose $k = 2$ and that contrary to what we claimed above, $Z_2^i > \tilde{Z}_2^i$. The second part of Lemma 4 says that $\tilde{Z}_1^i = \tilde{Z}_2^i$, meaning that $Z_2^i > \tilde{Z}_2^i = \tilde{Z}_1^i$. But by necessity $Z_1^i \geq Z_2^i$, and so $Z_1^i \geq Z_2^i > \tilde{Z}_2^i = \tilde{Z}_1^i$, so that $Z_1^i > \tilde{Z}_1^i$, which we showed just above ($k = 1$) cannot be. We conclude that $Z_2^i \leq \tilde{Z}_2^i$.

Second, we consider the merged firm. Without loss of generality, suppose the merger involves firms 1 and 2 and that firm 1 has weakly lower cost for the low-quality good whilst firm 2 has weakly lower cost for the high-quality good. We will refer to the merged firm as m. There are two possible cases, according to whether $Z_2^m = 0$ or $Z_2^m > 0$. In the case where $Z_2^m = 0$, it must be true that $\tilde{Z}_2^1 = \tilde{Z}_2^2 = 0$ otherwise $Z_2 \geq \tilde{Z}_2$ definitely cannot hold (since we have just shown that non-merging firms weakly lower their upgrade supply). However $Z_1 \geq \tilde{Z}_1$ then implies that $Z_1^m \leq \tilde{Z}_1^1$. Since $\tilde{Z}_1^2 > 0$ as well, we must have $\tilde{Z}_1^1 + \tilde{Z}_1^2 > Z_1^m$, but this contradicts the fact that $Z_1 \geq \tilde{Z}_1$ (since again, non-merging firms weakly lower their baseline supply). In the case where $Z_2^m > 0$, since firm 2 has weakly lower upgrade costs than m, and because we are supposing $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, firm 2 must have positive pre-merger output, $\tilde{Z}_2^2 > 0$. Summing m's two first order conditions (or else using its single first order condition if it is upgrade constrained), m's output choices satisfy

$$P_1(Z_1) + Z_1^m P_1'(Z_1) + P_2(Z_2) + Z_2^m P_2'(Z_2) = c_2^m = c_2^2.$$

By the same logic, pre-merger firm 2's choices must satisfy

$$P_1(\tilde{Z}_1) + \tilde{Z}_1^2 P_1'(\tilde{Z}_1) + P_2(\tilde{Z}_2) + \tilde{Z}_2^2 P_2'(\tilde{Z}_2) = c_2^2.$$

Because both firms 1 and 2 have positive supply of upgrade $k = 1$, for aggregate outputs not to have fallen it must at least be that $Z_1^m > \tilde{Z}_1^2$, and similarly that $Z_2^m \geq \tilde{Z}_2^2$. But this ranking of outputs means, because of decreasing marginal revenue, the fact that $P_1' < 0, P_2' < 0$, and $Z_1 \geq \tilde{Z}_1$ and $Z_2 \geq \tilde{Z}_2$, that both of the equations immediately above cannot be satisfied.

We conclude, from all of the work above, that if $Z_1 \geq \tilde{Z}_1$, then it must also be that $Z_2 < \tilde{Z}_2$. The final step of the proof is to show that $Z_1 \geq \tilde{Z}_1$ and $Z_2 < \tilde{Z}_2$ is also impossible.

First, let U denote the set of all firms which are not involved in the merger and which strictly increased their supply of baseline output following the merger. Hence for a firm $i \in U$ it is the case that $Z_1^i > \tilde{Z}_1^i$. According to the first part of Lemma 4, firm i has $Z_1^i = Z_2^i$. Since by definition $\tilde{Z}_1^i \geq \tilde{Z}_2^i$, it must be that $Z_2^i - \tilde{Z}_2^i \geq Z_1^i - \tilde{Z}_1^i > 0$. Let U_1 and U_2 be the total increases in baseline and upgrade supplies, respectively, over all firms in U. Thus,

$$U_1 = \sum_{i \in U} \left(Z_1^i - \tilde{Z}_1^i \right) > 0, \text{ and } U_2 = \sum_{i \in U} \left(Z_2^i - \tilde{Z}_2^i \right) > 0.$$

Note that $U_2 \geq U_1 > 0$, based on the logic given just above.

Second, let D denote the set of all firms which are not involved in the merger and which strictly decreased their supply of upgrade output following the merger. Hence for a firm $i \in D$ it is the case that $Z_2^i < \tilde{Z}_2^i$. According to the second part of Lemma 4, firm i has $Z_1^i = Z_2^i$. Since by definition $\tilde{Z}_1^i \geq \tilde{Z}_2^i$, it must be that $0 > Z_2^i - \tilde{Z}_2^i \geq Z_1^i - \tilde{Z}_1^i$. Define D_1 and D_2 as the decreases in baseline and upgrade outputs for these firms:

$$D_1 = \sum_{i \in D} \left(Z_1^i - \tilde{Z}_1^i \right) < 0, \text{ and } D_2 = \sum_{i \in D} \left(Z_2^i - \tilde{Z}_2^i \right) < 0.$$

Note that it must be that $0 > D_2 \geq D_1$. Additionally, it must be that $D_2 + U_2 \geq D_1 + U_1$.

Third, consider the merging firms. As above, we will suppose without loss of generality that the merger involves firms 1 and 2 and that firm 1 has weakly superior technology for producing the low-quality good and that firm 2 has weakly superior technology for producing the high-quality good. We will view the merger as removing firm 2's output from the market but also endowing firm 1 with firm 2's better technology for the high-quality product. From this perspective, the total output change involving the merging firms in market k is given by the change in firm 1's output minus the lost output of firm 2, $(Z_k^1 - \tilde{Z}_k^1) - \tilde{Z}_k^2$. Let $\Delta Z_k^1 = Z_k^1 - \tilde{Z}_k^1$ denote the change in firm 1's output in market k.

We can now complete the proof and show that it is not possible that $Z_1 \geq \tilde{Z}_1$ and $Z_2 < \tilde{Z}_2$. These inequalities require, as necessary conditions, that

$$\Delta Z_1^1 + \left(D_1 + U_1 - \tilde{Z}_1^2 \right) \geq 0, \text{ and} \tag{15}$$

$$\Delta Z_2^1 + \left(D_2 + U_2 - \tilde{Z}_2^2 \right) < 0. \tag{16}$$

As indicated above, $D_2 + U_2 \geq D_1 + U_1$. Also it must be that $\tilde{Z}_1^2 \geq \tilde{Z}_2^2$. Therefore, $D_2 + U_2 - \tilde{Z}_2^2 \geq D_1 + U_1 - \tilde{Z}_1^2$.

One possibility is that $D_1 + U_1 - \tilde{Z}_1^2 \geq 0$, which implies $D_2 + U_2 - \tilde{Z}_2^2 \geq 0$. To satisfy (16), ΔZ_2^1 must be negative. But then using the same logic as for firms in set D, it must also be that $0 > \Delta Z_2^1 \geq \Delta Z_1^1$. But this implies that $\Delta Z_1^1 + D_1 + U_1 - \tilde{Z}_1^2 \leq \Delta Z_2^1 + D_2 + U_2 - \tilde{Z}_2^2 < 0$ which means (15) cannot hold. (Note that part of the logic given for firms in D appeals to Lemma 4 for the case of $k = 2$, which will also hold for the merged firm because the merged firm is firm 1 with firm 2's superior technology for the high-quality good which means firm 1 operates post-merger with a reduced upgrade cost compared to pre-merger.)

The other possibility is that $0 > D_1 + U_1 - \tilde{Z}_1^2$. For (15) to hold, it must be that $\Delta Z_1^1 > 0$. Following logic given for firms in set U, it must be that $\Delta Z_2^1 \geq \Delta Z_1^1 > 0$. But then it must be that $\Delta Z_2^1 + D_2 + U_2 - \tilde{Z}_2^2 \geq \Delta Z_1^1 + D_1 + U_1 - \tilde{Z}_1^2 \geq 0$, which means (16) cannot hold. ■

Proof of Proposition 2: Consider item 1. We prove that if no firm sells only high-quality products pre-merger then $Z_2 \leq \tilde{Z}_2$.

Suppose that no firm's monotonicity constraint binds before the merger, and that contrary to what was just stated, $Z_2 > \tilde{Z}_2$. We now prove that no firm wishes to strictly increase its output of upgrade $k = 2$ following the merger, yielding a contradiction.

We only do this for the merging firms because simpler logic applies to other firms. Without loss of generality, suppose that the merger involves firms 1 and 2, and that firm 2 has a weakly lower cost for the high-quality good. We may view the merger as removing firm 1's output from the market, and endowing firm 2 with a cost $\min(c_1^1, c_1^2)$ for low quality and a cost c_2^2 for high quality. To complete the proof, it is then sufficient to show that, following the merger, firm 2 does not strictly raise its supply of upgrades. On the way to a contradiction, suppose that in fact $Z_2^2 > \tilde{Z}_2^2$. Because firm 2 could lower Z_2^2 but does not it must have a weak incentive to raise Z_2^2 (ignoring its upgrade constraint). Moreover firm 2's upgrade cost when it chooses Z_2^2 is $c_2^2 - \min(c_1^1, c_1^2)$, which is weakly higher than its upgrade cost $c_2^2 - c_1^2$ when it chooses $\tilde{Z}_2^2$. Hence using the first claim in Lemma 3, firm 2 must have a strict incentive to raise $\tilde{Z}_2^2$. But this contradicts the assumption that firm 2 was not upgrade-constrained prior to the merger.

Now consider item 2 of the proposition. We prove that if no merging firm sells low-quality products then $Z_2 \leq \tilde{Z}_2$.

Suppose that all merging firms sell only high-quality before the merger, and that contrary to what was just stated, $Z_2 > \tilde{Z}_2$.

We start by showing that any firm i that, pre-merger, produces both qualities (and hence is not involved in the merger) must weakly lower its upgrade output after the merger. Suppose to the contrary that $Z_2^i > \tilde{Z}_2^i > 0$. The second claim in Lemma 4 implies that $\tilde{Z}_1^i = \tilde{Z}_2^i$, but this contradicts the assumption that i's upgrade constraint does not bind pre-merger.

Next, we show that any firm i that, pre-merger, produces only low-quality or produces both qualities must strictly raise its baseline units after the merger. Suppose to the contrary that $Z_1^i \leq \tilde{Z}_1^i$. The first claim in Lemma 4 implies that $\tilde{Z}_1^i = \tilde{Z}_2^i$, which again contradicts the assumption that i's upgrade constraint does not bind pre-merger.

We now consider firms that were producing only high-quality pre-merger, which includes the firms involved in the merger. Without loss of generality, suppose that the merger involves firms 1 and 2, and that firm 1 has a weakly lower cost for the low-quality good. We may view the merger as eliminating firm 2, and endowing firm 1 with a cost c_1^1 for low quality and a cost $\min(c_2^1, c_2^2)$ for high quality. Note that one effect of the merger is to eliminate the pre-merger output of firm 2, given by $\tilde{Z}_1^2 = \tilde{Z}_2^2 > 0$.

Let D denote the set of firms that were upgrade-constrained pre-merger and which strictly reduce their supply of baseline units following the merger: $i \in D$ if and only if $\tilde{Z}_1^i = \tilde{Z}_2^i$ and $Z_1^i < \tilde{Z}_1^i$. Since by definition $Z_1^i \geq Z_2^i$, it must be that $Z_2^i - \tilde{Z}_2^i \leq Z_1^i - \tilde{Z}_1^i < 0$ for each $i \in D$.

Let U denote the set of firms that were upgrade-constrained pre-merger and which strictly increase their supply of upgrades following the merger: $i \in U$ if and only if $\tilde{Z}_1^i = \tilde{Z}_2^i$ and $Z_2^i > \tilde{Z}_2^i$. (We know that no firm which is not upgrade-constrained pre-merger increases its upgrade supply, and so U contains all firms that increase their upgrade supply). Since by definition $Z_1^i \geq Z_2^i$, it must be that $0 < Z_2^i - \tilde{Z}_2^i \leq Z_1^i - \tilde{Z}_1^i$.

Finally, since we assumed $Z_2 \geq \tilde{Z}_2$, the increased upgrade supply of firms in U must at least weakly exceed the combined lost upgrade supply of firm 2 and also firms in D. That is

$$0 < \tilde{Z}_1^2 = \tilde{Z}_2^2 \leq \sum_{i \in U} \left(Z_2^i - \tilde{Z}_2^i \right) + \sum_{i \in D} \left(Z_2^i - \tilde{Z}_2^i \right) \leq \sum_{i \in U} \left(Z_1^i - \tilde{Z}_1^i \right) + \sum_{i \in D} \left(Z_1^i - \tilde{Z}_1^i \right).$$

However this says that firms in D and U have increased their baseline output by more than firm 2's pre-merger output. Using Proposition 1, Z_1 must have strictly decreased. Hence there must exist some firm $i \notin U \cup D$ that has strictly lowered its output Z_1^i. But any firm $i \notin D$ which strictly lowers its baseline output must have been producing only low-quality or both qualities prior to the merger, and we showed above that there are no such firms.

We arrived at this contradiction by assuming $Z_2 \geq \tilde{Z}_2$. Hence we conclude that this merger strictly lowers the market supply of high-quality products. ■

Proof of Proposition 3: We first prove item 1. We know from Proposition 1 that $Z_1 < \tilde{Z}_1$. Suppose for the sake of contradiction that in fact $Z_2 \leq \tilde{Z}_2$. Using the techniques already developed in earlier proofs, it is easy to show that any firm i that produces both products pre-merger has weakly higher upgrade output post-merger.

Now consider any firm i that strictly produces only high-quality goods pre-merger. We claim that i sells strictly more upgrades post-merger. First, suppose firm i only sells high quality post-merger. Because Z_1 has strictly decreased and Z_2 has weakly decreased, firm i must be selling strictly more units. Secondly, suppose instead that firm i sells both low- and high-quality goods post-merger. Suppose also, on the way to a contradiction, that $Z_2^i \leq \tilde{Z}_2^i$. Because by assumption firm i has a strict incentive to raise $\tilde{Z}_2^i$ ignoring its upgrade constraint, a slight adaptation of the proof of Lemma 3 shows that it has a strict incentive to raise Z_2^i. But since this is feasible and yet firm i doesn't do it, we obtain a contradiction.

In summary, the total upgrade supply of non-merging firms strictly increases. Because each non-merging firm was producing zero upgrades, this contradicts $Z_2 \leq \tilde{Z}_2$.

Item 2 follows from observing that if the high-quality insider has weakly lower costs for low quality than the low-quality insiders, then the merger is equivalent to eliminating the other insiders from the market. Z_1 strictly falls. If Z_2 weakly fell, then all high-quality firms would strictly increase their output in the post-merger equilibrium and each multiproduct firm would weakly increase its upgrade output, contradicting Z_2 weakly falling. (This is true for a firm that strictly sells only high quality; the claim weakly holds otherwise.) ■

We will use the following result in subsequent proofs.

Lemma 5. *Suppose a "target" firm has total output exogenously fixed at $\overline{Z}$. Outputs are taken to be equilibrium ones given $\overline{Z}$. Excluding the target firm, let $\mathcal{L}$ denote firms that produce only low quality, $\mathcal{H}$ denote firms that produce only high quality, and $\mathcal{M}$ denote firms that produce strictly positive amounts of both low and high quality. Let $|\mathcal{L}|$, $|\mathcal{H}|$ and $|\mathcal{M}|$ be their respective numbers, and let $\overline{N} = |\mathcal{L}|+|\mathcal{H}|+|\mathcal{M}|$. Define $\phi = P_1'(Z_1)/[P_1'(Z_1)+P_2'(Z_2)]$. An infinitesimal increase in $\overline{Z}$ leads to infinitesimal output changes dZ_1 and dZ_2 where:*

(1) When the target firm supplies only low quality then $dZ_1 > 0$ and $dZ_2 \leq 0$ with

$$\frac{dZ_2}{dZ_1} = -\frac{\phi\left[|\mathcal{H}| - \sum_{i\in\mathcal{H}}\left(\frac{Z_1^i}{Z_1}\right)\sigma_1(Z_1)\right]}{(1-\phi)\left[|\mathcal{H}| - \sum_{i\in\mathcal{H}}\left(\frac{Z_2^i}{Z_2}\right)\sigma_2(Z_2)\right] + |\mathcal{M}| + 1 - \sum_{i\in\mathcal{M}}\left(\frac{Z_2^i}{Z_2}\right)\sigma_2(Z_2)}. \tag{17}$$

(2) When the target firm supplies only high quality then $dZ_1 > 0$ and $dZ_2 > 0$ with

$$\frac{dZ_2}{dZ_1} = \frac{|\mathcal{L}| + |\mathcal{M}| + 1 - \sum_{i\in\mathcal{L},\mathcal{M}}\left(\frac{Z_1^i}{Z_1}\right)\sigma_1(Z_1)}{|\mathcal{M}| + 1 - \sum_{i\in\mathcal{M}}\left(\frac{Z_2^i}{Z_2}\right)\sigma_2(Z_2)}. \tag{18}$$

Proof. Note that $Z_1 - \overline{Z} = \sum_{i\in\mathcal{L},\mathcal{H},\mathcal{M}} Z_1^i$ and $Z_2 - r\overline{Z} = \sum_{i\in\mathcal{H},\mathcal{M}} Z_2^i$ where $r = 0$ if the target supplies low quality and $r = 1$ if it supplies high quality. Summing the first line of (2) over all firms in $\mathcal{L}$ and $\mathcal{M}$, and summing (3) over all firms in $\mathcal{H}$, then adding the two, gives

$$\overline{N}P_1(Z_1) + \left(Z_1 - \overline{Z}\right)P_1'(Z_1) + |\mathcal{H}|P_2(Z_2) + \left(\sum_{i\in\mathcal{H}} Z^i\right)P_2'(Z_2) = \sum_{i\in\mathcal{L},\mathcal{M}} c_1^i + \sum_{j\in\mathcal{H}} c_2^j. \tag{19}$$

Summing the second line of (2) over all $\mathcal{M}$ firms, and (3) over all $\mathcal{H}$ firms, then adding:

$$|\mathcal{H}|P_1(Z_1) + \left(\sum_{i\in\mathcal{H}} Z^i\right)P_1'(Z_1) + (|\mathcal{H}| + |\mathcal{M}|)P_2(Z_2) + \left(Z_2 - r\overline{Z}\right)P_2'(Z_2) = \sum_{i\in\mathcal{M}}\left(c_2^i - c_1^i\right) + \sum_{j\in\mathcal{H}} c_2^j. \tag{20}$$

Summing (3) over all $\mathcal{H}$ firms gives

$$|\mathcal{H}|P_1(Z_1) + \left(\sum_{i\in\mathcal{H}} Z^i\right)P_1'(Z_1) + |\mathcal{H}|P_2(Z_2) + \left(\sum_{i\in\mathcal{H}} Z^i\right)P_2'(Z_2) = \sum_{j\in\mathcal{H}} c_2^j. \tag{21}$$

For item 1 outputs are determined by (19), and also (20) if $|\mathcal{M}| > 0$, and also (21) if $|\mathcal{H}| > 0$—all with $r = 0$. For item 2 outputs are determined by (19) and (20), and also (21) if $|\mathcal{H}| > 0$ – all with $r = 1$. Totally differentiating and solving gives the stated expressions. ■

Proof of Proposition 4: Note that equations (19)–(21) from Lemma 5 generally determine equilibrium outputs, even taking $\overline{Z} = 0$ and without any "target firm," as we now assume to be the case. $|\mathcal{L}|, |\mathcal{H}|, |\mathcal{M}|$ denote the number of firms producing respectively only low quality, only high quality, or both products after the merger.

To prove item 1, consider a synergy that reduces c_2^i for a firm $i \in \mathcal{H}$. Totally differentiating (19)–(21) and solving gives $dZ_1 > 0$ and $dZ_2 > 0$. This implies that prices of both low and high quality strictly decrease. Hence consumer surplus strictly increases.

To prove item 2a, consider a synergy that reduces c_2^i for a firm $i \in \mathcal{M}$. Note that $|\mathcal{H}| = 0$. Totally differentiating (19) gives $dZ_1 = 0$, and totally differentiating (20) gives $dZ_2 > 0$. This implies that the price of low quality is unchanged, while the price of high quality strictly decreases. Hence consumer surplus strictly increases.

To prove item 2b, consider a synergy that reduces c_1^i for a firm $i \in \mathcal{L}$. Again $|\mathcal{H}| = 0$. Totally differentiating (20) gives $dZ_2 = 0$, and totally differentiating (19) gives $dZ_1 > 0$. This implies that prices of both goods strictly decrease. Hence consumer surplus strictly increases. ■

Proof of Proposition 5: As in the proof of Proposition 4, equilibrium outputs are determined by equations (19) and (20), and also (21) when $|\mathcal{H}| > 0$—all with $\overline{Z} = 0$. Consider a unit decrease in c_1^i for a firm $i \in \mathcal{M}$. Totally differentiating and solving gives $dZ_1 > 0$ and $dZ_2 < 0$ with

$$\frac{dZ_2}{dZ_1} = -\frac{P_1'(Z_1)}{P_2'(Z_2)} \frac{n^{**} + 1 - \sigma_1(Z_1)}{n^{**} - n^{L**} + 1 - \sigma_2(Z_2)}, \tag{22}$$

where $n^{**} \equiv \overline{N}$ and $n^{L**} \equiv |\mathcal{L}|$. Item 1 then follows. Next, note that the change in the price of high quality is $P_1'(Z_1)dZ_1 + P_2'(Z_2)dZ_2$, while consumer surplus is

$$\int_{\theta(Z_1)} [v(\theta, q_L) - P_1(Z_1)] \, dF(\theta) + \int_{\theta(Z_2)} [(v(\theta, q_H) - v(\theta, q_L)) - P_2(Z_2)] \, dF(\theta), \tag{23}$$

so its derivative is $-Z_1 P_1'(Z_1)dZ_1 - Z_2 P_2'(Z_2)dZ_2$. Substituting (22) gives items 2 and 3. ■

Proof of Proposition 6: For $k = 1, 2$, let Z_k^* and Z_k^{**} be pre- and post-merger equilibrium outputs, and let Z_k^{i*} and Z_k^{i**} be outsider i's pre- and post-merger equilibrium outputs.

Suppose that $Z_1^{**} \leq Z_1^*$ and $Z_2^{**} \leq Z_2^*$. The first part of the proof of Proposition 1 establishes that $Z_1^{i**} \geq Z_1^{i*}$ and $Z_2^{i**} \geq Z_2^{i*}$. Hence $Z_1^{**} - Z_1^{i**} \leq Z_1^* - Z_1^{i*}$ and $Z_2^{**} - Z_2^{i**} \leq Z_2^* - Z_2^{i*}$. This implies that firm i's profit is weakly higher after the merger, because i could always choose outputs Z_1^{i*} and Z_2^{i*} after the merger and be no worse off than before it. Meanwhile consumers are weakly worse off after the merger because $P_1(Z_1^{**}) \geq P_1(Z_1^*)$ and $P_2(Z_2^{**}) \geq P_2(Z_2^*)$.

The proof when $Z_1^* \geq Z_1^{**}$ and $Z_2^* \geq Z_2^{**}$ follows the same steps and so is omitted. ■

Proof of Lemma 2: We begin by proving that that for any firm $i \in \mathcal{O}$,

$$d\pi^i = Z_1^i P_1'(Z_1)\left[2 - \frac{Z_1^i}{Z_1}\sigma_1(Z_1)\right] dZ_1 + Z_2^i P_2'(Z_2)\left[2 - \frac{Z_2^i}{Z_2}\sigma_2(Z_2)\right] dZ_2. \tag{24}$$

First consider a low quality firm i. Its profit is $\pi^i = Z_1^i\left[P_1(Z_1) - c_1^i\right]$, and so

$$d\pi^i = dZ_1^i\left[P_1(Z_1) - c_1^i\right] + Z_1^i P_1'(Z_1)\, dZ_1. \tag{25}$$

The firm's first order condition is $P_1(Z_1) - c_1^i + Z_1^i P_1'(Z_1) = 0$, and its total derivative implies $dZ_1^i = -\left[1 - \left(Z_1^i/Z_1\right)\sigma_1(Z_1)\right] dZ_1$. Substituting these into (25) and noting that $Z_2^i = 0$, we obtain the expression in (24).

Second, a multiproduct firm i has profit $\pi^i = Z_1^i\left[P_1(Z_1) - c_1^i\right] + Z_2^i\left[P_2(Z_2) - (c_2^i - c_1^i)\right]$. Since the firm chooses Z_1^i and Z_2^i independently, we can follow the same steps as we did for a low quality firm and derive the expression for $d\pi^i$ in (24).

Third, consider a high quality firm i. Its profit is $\pi^i = Z^i\left[P_1(Z_1) + P_2(Z_2) - c_2^i\right]$, and so

$$d\pi^i = dZ^i\left[P_1(Z_1) + P_2(Z_2) - c_2^i\right] + Z^i\left[P_1'(Z_1)\, dZ_1 + P_2'(Z_2)\, dZ_2\right]. \tag{26}$$

The firm's first order condition is

$$P_1(Z_1) + P_2(Z_2) - c_2^i + Z^i\left[P_1'(Z_1) + P_2'(Z_2)\right] = 0,$$

and its total derivative implies

$$dZ^i = -\phi\left[1 - \frac{Z_1^i}{Z_1}\sigma_1(Z_1)\right] dZ_1 - (1-\phi)\left[1 - \frac{Z_2^i}{Z_2}\sigma_2(Z_2)\right] dZ_2,$$

where $\phi = P_1'(Z_1) / \left[P_1'(Z_1) + P_2'(Z_2)\right]$. Substitute these into (26) and rearrange to get (24).

Next, recall from the proof of Proposition 5 that the change in consumer surplus is

$$dCS = -Z_1 P_1'(Z_1) dZ_1 - Z_2 P_2'(Z_2) dZ_2. \tag{27}$$

To obtain the expression for $d\mathcal{E}$, sum (24) over all firms $i \in \mathcal{O}$ and add (27), then use the definitions $s_1^i \equiv Z_1^i/Z_1$ and $s_2^i \equiv Z_2^i/Z_2$. ■

Proof of Proposition 7: Let a be the "acquirer", who in item 1 is an insider that produces low quality at cost $c_1^a = \min_{i\in\mathcal{I}} c_1^i$, and in item 2 is the multiproduct insider. Conceptually the merger has two stages. In the first stage a's cost of producing high quality falls (weakly) to $\min_{i\in\mathcal{I}} c_2^i$. In the second stage the output of firms $\mathcal{I}\backslash a$ is removed from the market.

We first prove that outputs at the first stage are the same as before the merger. For item 1, let $j \in \mathcal{I}$ be a firm with $c_2^j = \min_{i\in\mathcal{I}} c_2^i$. Firm j produced no upgrades pre-merger and therefore $P_2(Z_2^*) \le c_2^j - c_1^j$. Since $c_1^j \ge c_1^a$ this implies that $P_2(Z_2^*) \le \min_{i\in\mathcal{I}} c_2^i - c_1^a$. Therefore fixing others' outputs at their pre-merger level, firm a (still) produces no upgrades when it has costs c_1^a and $\min_{i\in\mathcal{I}} c_2^i$. The claim that Z_1^* and Z_2^* are equilibrium outputs then follows. For item 2, note that firm a sells upgrades pre-merger and so $P_2(Z_2^*) > c_2^a - c_1^a$, while each $j \in \mathcal{I}\backslash a$ does not and so $P_2(Z_2^*) \le c_2^j - c_1^j$. Since $c_1^a \le c_1^j$ for all $j \in \mathcal{I}\backslash a$ this implies that $c_2^a = \min_{i\in\mathcal{I}} c_2^i$. Firm a's costs are unchanged, so Z_1^* and Z_2^* are the equilibrium outputs.

We now turn to the second stage. Consider a sequence where the output $Z_1^{\mathcal{I}\backslash a}$ of firms in $\mathcal{I}\backslash a$ is reduced from its pre-merger level down to 0, and at each point all firms not in $\mathcal{I}\backslash a$ choose outputs optimally. According to Lemma 5 a reduction in $Z_1^{\mathcal{I}\backslash a}$ leads to a decrease in Z_1 but no change in Z_2 provided no firm supplies only high quality. No firm supplies only high-quality at the start of the sequence. Moreover, as Z_1 falls but Z_2 remains constant, each firm in $a \cup \mathcal{O}$ has its upgrade constraint relaxed; hence no firm supplies only high quality along the sequence. Using Lemma 2 the external effect of the merger is then

$$-\int_{Z_1^{**}}^{Z_1^*} Z_1 P_1'(Z_1) \left\{ \sum_{i\in\mathcal{O}} s_1^i \left[2 - s_1^i \sigma_1(Z_1)\right] - 1 \right\} dZ_1. \tag{28}$$

By assumption the curly-bracketed term is positive when evaluated at $Z_1 = Z_1^*$, using $s_1^{\mathcal{I}*} \equiv 1 - \sum_{i\in\mathcal{O}} s_1^{i*}$. Moreover the proof of Proposition 1 shows that as Z_1 decreases (and Z_2 remains unchanged) each outsider weakly raises its output and hence also its share s_1^i of total output. Combined with Assumption 1 and the assumption that $\sigma_1'(Z_1) \geq 0$, this implies that the curly-bracketed term is decreasing in Z_1 and hence is positive for all $Z_1 \in [Z_1^{**}, Z_1^*]$. Therefore (28) is positive. ■

Proof of Proposition 8: Let a be one of the insiders with the lowest cost for both low and high quality. Consider a sequence where the (total and upgrade) output $Z^{\mathcal{I}\backslash a}$ of firms $\mathcal{I}\backslash a$ is reduced from its pre-merger equilibrium level $Z^{\mathcal{I}\backslash a*}$ down to 0, and at each point firms not in $\mathcal{I}\backslash a$ choose outputs optimally. By Lemma 2 the merger's external effect is

$$\begin{aligned} -\int_0^{Z^{\mathcal{I}\backslash a*}} \Bigg[& Z_1 P_1'(Z_1) \left\{ \sum_{i\in\mathcal{O}} s_1^i \left[2 - s_1^i \sigma_1(Z_1)\right] - 1 \right\} \frac{dZ_1}{dZ^{\mathcal{I}\backslash a}} \\ & + Z_2 P'(Z_2) \left\{ \sum_{i\in\mathcal{O}} s_2^i \left[2 - s_2^i \sigma_2(Z_2)\right] - 1 \right\} \frac{dZ_2}{dZ^{\mathcal{I}\backslash a}} \Bigg] dZ^{\mathcal{I}\backslash a}, \end{aligned} \tag{29}$$

where to ease the exposition we suppress the dependence of Z_1 and Z_2 on $Z^{\mathcal{I}\backslash a}$. According to Lemma 5 both $dZ_1/dZ^{\mathcal{I}\backslash a}$ and $dZ_2/dZ^{\mathcal{I}\backslash a}$ are positive. A sufficient condition for a positive external effect is thus that the two curly-bracketed terms are positive. The proof that both are indeed positive closely follows the proof of Proposition 7, and so is omitted. ■

Proof of Proposition 9: As in the proof of Proposition 8 consider a sequence where the output $Z^{\mathcal{I}\backslash a}$ of non-acquiring insiders is reduced to 0. The external effect is given by (29).

According to Lemma 5 a reduction in $Z_1^{\mathcal{I}\backslash a}$ leads to a decrease in Z_1 and a (weak) increase in Z_2. Therefore as $Z_1^{\mathcal{I}\backslash a}$ decreases each outsider $i \in \mathcal{O}$ raises its share s_1^i of total output and (weakly) lowers its share s_2^i of upgrade output. Following the same steps as in Proposition 8, this ensures that the first curly-bracketed term in (29) is positive. Using the opposite reasoning, the second term in curly brackets is negative, such that the overall external effect of the complete merger is positive given that Z_1 has decreased and Z_2 has weakly increased. ■

Proof of Proposition 10: If Z_1 and Z_2 are the same pre- and post-merger, then outsiders' outputs must also be the same pre- and post-merger, which in turn implies that the merged firm's outputs equal the sum of the insiders' pre-merger outputs. Using the first part of equation (2) we find that

$$\sum_{i\in\mathcal{I}} Z_1^{i*} = Z_1^m \iff \hat{c}_1^m = \sum_{i\in\mathcal{I}} c_1^i - P_1(Z_1^*), \tag{30}$$

and also that for any $i \in \mathcal{I}$,

$$c_1^i = P_1(Z_1^*)(1 - s_1^{i*}/\epsilon_1). \tag{31}$$

Using equations (30) and (31), and since $c_1^i = c_1$ for $i \in \mathcal{I}$, we obtain

$$\frac{c_1 - \hat{c}_1^m}{c_1} = \frac{s_1^*}{\epsilon_1 - s_1^*}, \tag{32}$$

where $s_1^{i*} = s_1^*$ for each $i \in \mathcal{I}$. Using $\Delta HHI_1 = 2(s_1^*)^2$ we can rewrite (32) as (13). Finally, note that if $c_1^m = \hat{c}_1^m$ and there is no synergy on high quality, insiders do indeed optimally supply only low quality after the merger. ■

Proof of Proposition 11: Using the same argument as in the preceding proof, the two insiders' outputs must be the same pre- and post-merger. Using equation (3) we find that

$$\sum_{i\in\mathcal{I}} Z_2^{i*} = Z_2^m \iff \hat{c}_2^m = \sum_{i\in\mathcal{I}} c_2^i - P_1(Z_1^*) - P_2(Z_2^*), \tag{33}$$

and also that for any $i \in \mathcal{I}$,

$$c_2^i = [P_1(Z_1^*) + P_2(Z_2^*)][1 - \rho s_1^{i*}/\epsilon_1 - (1-\rho)s_2^{i*}/\epsilon_2]. \tag{34}$$

Using equations (33) and (34) and the fact that $c_2^i = c_2$ for $i \in \mathcal{I}$, we obtain

$$\frac{c_2 - \hat{c}_2^m}{c_2} = \frac{\rho s_1^*/\epsilon_1 + (1-\rho)\, s_2^*/\epsilon_2}{1 - \rho s_1^*/\epsilon_1 - (1-\rho)\, s_2^*/\epsilon_2}, \tag{35}$$

where $s_l^{i*} = s_l^*$ for each $i \in \mathcal{I}$. Using $\Delta HHI_l = 2(s_l^*)^2$ we can rewrite (35) as (14). Finally, note that if $c_2^m = \hat{c}_2^m$ and there is no synergy on low quality, insiders do indeed optimally supply only high quality after the merger. ■

We will use the following lemma in the proofs of Propositions 12 and 13.

Lemma 6. *A merger of $k+1$ firms producing only low quality is profitable if and only if*

$$k+1 \le \left[\frac{1+n^L+n^M+\gamma}{1+n^L+n^M+\gamma-k}\right]^2, \text{ where } \gamma = \frac{[n^H(n^M+1)q_L]}{[(n^M+1)q_H + n^H(q_H-q_L)]}.$$

Proof. Conceptually we imagine that k insiders cease production, and firm i is the insider that "survives" the merger. Letting Z_1^{**} denote the total post-merger industry output and Z_1^{i**} firm i's post-merger output, this i's post-merger first-order condition is

$$P_1(Z_1^{**}) + Z_1^{i**}P_1'(Z_1^{**}) = q_L(1 - Z_1^{**}) - q_L Z_1^{i**} = c_1^i.$$

This allows firm i's post-merger profits—and hence the post-merger profits of the $k+1$ merging firms—to be written as $(P_1(Z_1^{**}) - c_1^i)Z_1^{i**} = q_L(Z_1^{i**})^2$. Using similar computations, the pre-merger profit of each of the $k+1$ insiders is $q_L(Z_1^{i*})^2$, where Z_1^{i*} is such a firm's pre-merger output. Thus, a merger is profitable if and only if

$$(Z_1^{i**})^2 \geq (k+1)(Z_1^{i*})^2.$$

We now solve for Z_1^{i**}/Z_1^{i*}. Let $\mathcal{K}$ be the set of the k firms who cease production following the merger. Note that by symmetry these firms' pre-merger output is kZ_1^{i*}. We model the merger as a series of infinitesimal mergers, which reduce these firms' output from kZ_1^{i*} to 0, and let $Z_1^{\mathcal{K}}$ denote their output at an arbitrary point along this merger 'path'.

Using firm i's first order condition and $P_1'(Z_1) = -q_L$, an infinitesimal merger induces firm i to change its output by $dZ_1^i = -dZ_1$. Take equations (19)-(21) in the proof of Lemma 5 and substitute $r = 0$, $|\mathcal{H}| = n^H$, $|\mathcal{M}| = n^M$ and also $|\mathcal{L}| = n^L - k$ (since k initially low-quality firms no longer optimize their output). Totally differentiating, an infinitesimal merger induces a change in total output of

$$dZ_1 = -\frac{\left[n^H(1-\phi) + n^M + 1\right]}{[n^L - k + n^M + 1]\,[n^H(1-\phi) + n^M + 1] + n^H\phi\,[n^M + 1]},$$

where $\phi = q_L/q_H$. Since dZ_1^i is the same at all points along the merger path,

$$\frac{Z_1^{i**}}{Z_1^{i*}} = \frac{Z_1^{i*} + \int_0^{kZ_1^{i*}} dZ_1^i dZ_1^{\mathcal{K}}}{Z_1^{i*}} = 1 + \frac{k}{n^L - k + n^M + 1 + \gamma}.$$

Substituting this into the above profitability condition then gives the stated result. ■

Proof of Proposition 12: The righthand side of the inequality in Lemma 6 is decreasing in γ. At the same time γ is increasing in n^H and satisfies $\lim_{n^H\to\infty}\gamma = \left(n^M+1\right)q_L/\left(q_H - q_L\right)$. Hence by substituting $n^M = 0$, $k = n^L - 1$, and $\gamma = q_L/\left(q_H - q_L\right)$ into the inequality in Lemma 6 we obtain a sufficient condition for the merger to be profitable. Rearranging this condition leads to the one stated in the proposition. ■

Proof of Proposition 13: The righthand side of the inequality in Lemma 6 is decreasing in γ. Hence by substituting $\gamma = 0$ we obtain the following necessary condition for profitability:

$$k + 1 \leq \left[\frac{1 + n^L + n^M}{1 + n^L + n^M - k}\right]^2. \tag{36}$$

Salant, Switzer, and Reynolds (1983) show that this cannot hold if $(k+1)/(n^L+n^M) < 8/10$. The necessary condition stated in our proposition then follows immediately. ■

Proof of Remark 1: The proof follows those of Lemma 6 and Propositions 12 and 13. ■

Proof of Proposition 14: For brevity we only prove this result for a merger between $k+1$ low-quality firms. We prove that if the stated condition fails to hold, an infinitesimal merger reduces the joint profit of the merger insiders.

Closely following the proof of Lemma 6, let firm i be the surviving insider, and $Z_1^{\mathcal{K}}$ be the output of the k other insiders that are being removed from the market (at an arbitrary point along the merger path). The joint profit of the merger insiders $\mathcal{I}$ is $\left(Z_1^i + Z_1^{\mathcal{K}}\right)\left[P_1\left(Z_1\right) - c_1^i\right]$, and its change following a unit decrease in $Z_1^{\mathcal{K}}$ is

$$d\pi^{\mathcal{I}} = \left(dZ_1^i - 1\right)\left[P_1\left(Z_1\right) - c_1^i\right] + \left(Z_1^i + Z_1^{\mathcal{K}}\right) P_1'\left(Z_1\right) dZ_1.$$

Since firm i is optimizing, we can use its first order condition to replace $P_1\left(Z_1\right) - c_1^i$ and to derive that $dZ_1^i = -\left[1 - \left(Z_1^i/Z_1\right)\sigma_1\left(Z_1\right)\right] dZ_1$. Hence we can write

$$d\pi^{\mathcal{I}} = Z_1^i P_1'\left(Z_1\right)\left\{1 + \left(2 + \frac{Z_1^{\mathcal{K}}}{Z_1^i} - \frac{Z_1^i}{Z_1}\sigma_1\left(Z_1\right)\right) dZ_1\right\}.$$

A sufficient condition for the merger to be unprofitable is that, at all points along the merger path, the curly-bracketed term is strictly positive. Since $dZ_1 < 0$ this is harder to achieve when $Z_1^{\mathcal{K}}/Z_1^i$ is larger. $Z_1^{\mathcal{K}}/Z_1^i$ reaches its maximum pre-merger (when it equals k), so we obtain the following sufficient condition for the merger to be unprofitable:

$$1 + \left(2 + k - \frac{Z_1^i}{Z_1}\sigma_1\left(Z_1\right)\right) dZ_1 > 0.$$

Take equations (19)-(21) and set $r = 0$, $\overline{Z} = Z_1^{\mathcal{K}}$, $|\mathcal{H}| = n^H$, $|\mathcal{M}| = n^M$, and $|\mathcal{L}| = n^L - k$ (as k initially low-quality firms no longer optimize). Totally differentiating, we find

$$dZ_1 \geq -\left\{n^L - k + n^M + 1 - \frac{Z_1 - \left(\sum_{i\in\mathcal{H}} Z^i\right) - Z_1^{\mathcal{K}}}{Z_1}\sigma_1\left(Z_1\right)\right\}^{-1}.$$

Hence our sufficient condition for the merger to be unprofitable reduces to

$$n^L - 2k + n^M - 1 > \frac{Z_1 - \left(\sum_{i\in\mathcal{H}} Z^i\right) - Z_1^{\mathcal{K}} - Z_1^i}{Z_1}\sigma_1\left(Z_1\right).$$

Since the righthand side is bounded above by 1, a sufficient condition for the merger to be unprofitable is that $k < \left(n^L + n^M - 2\right)/2$. The reverse of this inequality is then a necessary condition for the merger to be profitable. ■

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Sufficient Conditions for Consumer Surplus to Increase Absent Synergies. We examine when a merger increases Z_2 by enough that overall consumer surplus goes up.

To simplify the exposition we focus on the case where before the merger all firms supply a single product, and $n^H \geq 1$ of them supply high-quality products.

Our first result utilizes the concept of an "infinitesimal merger" involving two firms (introduced by Farrell and Shapiro (1990)). One firm is the acquirer and the other is the "target", with the target having weakly higher costs than the acquirer for both products. Because of this assumption on costs a merger is equivalent to removing the target from the market. We define an infinitesimal merger as a small exogenous reduction in the target firm's output, beginning from the pre-merger industry configuration, with all other firms including the acquirer adjusting their own outputs in response, resulting in a new industry equilibrium.

Consumer surplus given outputs Z_1 and Z_2 is

$$\int_{\theta(Z_1)} \left[v\left(\theta, q_L\right) - P_1\left(Z_1\right)\right] dF\left(\theta\right) + \int_{\theta(Z_2)} \left[\left(v\left(\theta, q_H\right) - v\left(\theta, q_L\right)\right) - P_2\left(Z_2\right)\right] dF\left(\theta\right). \tag{37}$$

Letting dZ_1 and dZ_2 be the changes in baseline and upgrade output from an infinitesimal merger, the associated change in consumer surplus is

$$dCS = -Z_1 P_1'(Z_1) dZ_1 - Z_2 P_2'(Z_2) dZ_2. \tag{38}$$

Suppose the target firm supplies low-quality products before the merger. We know from Lemma 5 that in this case $dZ_1 < 0$ and also

$$dZ_2 = -\frac{P_1'(Z_1)\left[n^H - \frac{Z_2}{Z_1}\sigma_1(Z_1)\right]}{P_2'(Z_2)\left[n^H - \sigma_2(Z_2) + 1\right] + P_1'(Z_1)} dZ_1. \tag{39}$$

Combining equations (38) and (39) we obtain the following result.

Lemma 7. *Suppose there are no multiproduct firms. An infinitesimal merger in which the target firm supplies only low-quality products increases consumer surplus if and only if the following holds at pre-merger equilibrium:*

$$\frac{Z_1}{Z_2}\left[n^H - \sigma_2(Z_2) + 1 + \frac{P_1'(Z_1)}{P_2'(Z_2)}\right] - n^H + \frac{Z_2}{Z_1}\sigma_1(Z_1) < 0. \tag{40}$$

Proof. Combining equations (38) and (39) gives $dCS = \tau_1 \tau_2 dZ_1$ where

$$\tau_1 = -\frac{P_1'(Z_1) P_2'(Z_2) Z_2}{P_2'(Z_2)\left[n^H - \sigma_2(Z_2) + 1\right] + P_1'(Z_1)} \tag{41}$$

$$\tau_2 = \frac{Z_1}{Z_2}\left[n^H - \sigma_2(Z_2) + 1 + \frac{P_1'(Z_1)}{P_2'(Z_2)}\right] - n^H + \frac{Z_2}{Z_1}\sigma_1(Z_1). \tag{42}$$

Note that $\tau_1 > 0$ and $dZ_1 < 0$, so $dCS > 0$ if and only if $\tau_2 < 0$ i.e. condition (40) holds. ■

Consistent with arguments in Section 3, Lemma 7 shows that an infinitesimal merger is more likely to increase consumer surplus when $\sigma_1(Z_1)$ is small and $\sigma_2(Z_2)$ is large.[35] All else equal, condition (40) is harder to satisfy when there are more high-quality firms.[36] Notice also that—conditional on Z_1—the number of low-quality firms does not affect the sensitivity of upgrade supply to changes in baseline output, and for this reason does not directly enter into condition (40).

Our second results pertain to complete mergers. If we were to start with an infinitesimal merger and then continue reducing the target firm's output to zero, then by integrating the associated output changes over the whole sequence we would arrive at the post-merger equilibrium outputs. We now use this insight to provide conditions under which a complete merger increases consumer surplus.

As a preliminary remark, notice that when a low-quality target firm has a sufficiently small market share—such that a merger will change Z_1 and Z_2 by a sufficiently small amount—then by continuity condition (40) ensures that consumer surplus goes up.

Now consider mergers between firms with arbitrary market shares. To simplify, we assume that the n^H firms have a cost of supplying low-quality products that is so large that they will never choose to do so (regardless of how the merger changes Z_1 and Z_2).[37]

Proposition 15. *Suppose there are no multiproduct firms, and that the demand system exhibits constant curvatures $\sigma_1 \leq 0$ and $\sigma_2 \leq 0$. A merger involving two low-quality firms raises consumer surplus provided that condition* (40) *holds at the pre-merger equilibrium.*

Proof. Let Z_1^* and Z_1^{**} denote pre- and post-merger total outputs respectively. As we reduce the target firm's output we induce a sequence where Z_1 falls from Z_1^* to Z_1^{**} and Z_2 re-equilibrates. Using equation (39) Z_2 increases. Let Z_2^* denote pre-merger upgrades. Notice that at each point on the sequence at least one firm produces high-quality. (No firm that was low-quality prior to the merger will supply upgrades because $Z_2 > Z_2^*$. But if no other firm produces high-quality this contradicts $Z_2 > Z_2^*$.) Therefore using the proof of Lemma 7 the merger-induced change in consumer surplus is

$$-\int_{Z_1^{**}}^{Z_1^*} \tau_1 \left\{ \frac{Z_1}{Z_2} \left[n^H(Z_1) - \sigma_2 + 1 + \frac{P_1'(Z_1)}{P_2'(Z_2)} \right] - n^H(Z_1) + \frac{Z_2}{Z_1}\sigma_1 \right\} dZ_1, \tag{43}$$

where we write $n^H(Z_1)$ because the number of firms supplying high-quality products may vary along the sequence. Since $\tau_1 > 0$ it is sufficient to prove that the curly-bracketed term is negative at all points along the sequence. Firstly, the curly-bracketed term is increasing in $n^H(Z_1)$, and at all points along the sequence $n^H(Z_1) \leq n^H(Z_1^*) \equiv n^H$; to see the latter, recall that along the sequence Z_2 increases and therefore no previously low-quality firm has an incentive to supply any upgrades. Secondly, fixing the n^H terms, the curly-bracketed

[35]Note that a necessary condition for inequality (40) to hold is that $\sigma_2(Z_2) > \sigma_1(Z_1) + 1$, ruling out linear demands.
[36]Precisely, if n_H increases but costs are adjusted in such a way that Z_1 and Z_2 remain the same.
[37]Technically this may require high-quality firm i's cost to satisfy $c_1^i > c_2^i$ which is counter to what we assumed in the main paper. However it is a simple way to ensure that firms remain single-product along the whole sequence. It might hold if the firm needs to degrade its high-quality product in order to supply a low-quality variant.

term is increasing in Z_1. This is because Z_1/Z_2 and $Z_2\sigma_1/Z_1$ are both increasing, and also the derivative of $P_1'(Z_1)/P_2'(Z_2)$ with respect to Z_1 is

$$\frac{P_1''(Z_1)}{P_2'(Z_2)} - \frac{P_1'(Z_1)P_2''(Z_2)}{[P_2'(Z_2)]^2}\frac{dZ_2}{dZ_1} \geq 0, \tag{44}$$

where the inequality follows because $\sigma_1, \sigma_2 \leq 0$ implies $P_1''(Z_1), P_2''(Z_2) \leq 0$. ■

Proposition 15 provides conditions such that if an infinitesimal merger raises consumer surplus then so does a complete merger. We note that it is straightforward to construct $v(\theta, q_L)$ and $v(\theta, q_H)$ such that baseline and upgrade demands have constant (but different) curvatures, condition (40) is satisfied at pre-merger equilibrium, and a complete merger between two low-quality firms is privately profitable.