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Abstract: This paper studies the role of risk attitudes in determining the optimality of winner-take-all contests. We compare the typical single-winner lottery contest with two alternatives, both spreading the rewards to more players: through holding multiple prize-giving lottery competitions or through guaranteeing a bottom prize for the losers. In the first comparison, we find that the multiple-competition contest is as effective as the winner-take-all contest when the contestants are risk neutral, but the former induces more effort than the latter when the contestants are both risk averse and prudent. In the second comparison, we find that the contest with a bottom prize is always dominated by the winner-take-all contest when the contestants are risk neutral, but the former could have an advantage over the latter when the contestants are both risk averse and prudent, and it is more likely so as the contestants become more prudent.

Key Words: contests; winner take all; multiple prizes; risk aversion; prudence
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1. Introduction

An important question in contest design is whether the winner-take-all arrangement provides a larger incentive for players to expend effort than the alternative arrangements in which rewards are spread out to more players. It has been established in the literature that, from the perspective of a contest organizer with a fixed prize budget, it is optimal to award the entire prize money to the winner and nothing to the others in order to maximize the total effort. However, this result seems inconsistent with the evidence that multiple prizes are often observed in real-life contests.

Moreover, the optimality of the winner-take-all contest exhibited in the literature has been typically obtained under the assumption that the contestants are risk neutral.[1] This is an important limitation since contests are often viewed as risky activities, where an increase in effort permits to increase the probability of the favorable outcome (e.g., in activities such as lobbying, patent races, or sport competitions). In addition, existing empirical/experimental evidence suggests that individual decision makers tend to be both risk averse and prudent.[2]

This paper studies the role of risk attitudes – risk aversion and prudence in particular – in determining the optimality of winner-take-all contests. We compare the winner-take-all contest model of Konrad and Schlesinger (1997) and Treich (2010) – hereafter the KST model – that uses a general contest success function and a general utility function with two types of contests that help spread the rewards to more players. Both multiple-prize models considered here can be looked upon as a generalization of the

[1] For example, see Glazer and Hassin (1988), Berry (1993), Clark and Riis (1996 and 1998), Barut and Kovenock (1998), Moldovanu and Sela (2001), Fu and Lu (2009), Schweinzer and Segev (2012), and Olszewski and Siegel (2016).

[2] For example, see Deck and Schlesinger (2010, 2014), Ebert and Wiesen (2011), Maier and Ruger (2011) and Noussair et al. (2014).

KST model, and as adding some insurance motive to the winner-take-all design from the perspective of the contestants.

In the "multiple-competition contest," contestants make one-shot efforts but have multiple shots at winning prizes. Examples of contestants who make efforts that are simultaneously aimed at multiple prize-yielding competitions abound. Athletes in various sports undergo rigorous winter/summer training to get ready for a new season of competitions.[3] New movies produced in a year can compete for various movie awards. Research work of a scientific lab/team can be submitted to many academic/government/industrial organizations for award considerations. For all these examples, contestants make efforts with an eye on multiple competitions, the prizes of which are determined by the same effort inputs in a statistically independent fashion. Assuming that the competitions are statistically independent, it is easy to see that increasing the number of competitions while keeping the total prize budget unchanged helps allocate rewards to more players, and thus has an insurance value to the players. To the best of our knowledge, the multiple-competition contest introduced here has not been formally analyzed in the literature on contests.[4]

In the "contest with a bottom prize," the single winner is awarded a top prize and each loser is also awarded a bottom prize (or, say, a consolation prize). Shrinking the

[3] A primary example is IAAF's Diamond League series where world's top athletes of each of the 32 covered disciplines compete for prizes in each of the "qualification" meetings with those who accumulate the highest points also qualifying for the "final" meeting. Tennis is another example in which a meta-organization controls the total number and the structure of competitions in a sport. The ATP decides about the list of tournaments, and created the ATP new tour structure in 2009 called ATP World Tour consisting of ATP world Tour Masters 1000, ATP World Tour 500, and ATP World Tour 250 tier tournaments.

[4] Note that the multiple-competition contest is different from the multiple-battle contest that has been extensively studied in the literature (e.g., Klumpp and Polborn 2006, Konrad and Kovenock 2009, Fu and Lu 2012, Fu, Lu and Pan 2015, and Barbieri and Serena 2019). In the multiple-battle contest model, a player (either on behalf of himself/herself or as a member of a team) makes battle-specific effort for each battle in order to earn credits towards eventually winning a contest.

prize gap while keeping the total prize budget fixed no doubt facilitates a more equal distribution of rewards among players. Although the contest with a bottom prize was already discussed in the ground-breaking work of Lazear and Rosen (1981), it has not been fully analyzed in the literature, probably because it is intuitively appealing to conclude that setting the bottom prize to zero (i.e., making the prize gap as large as possible) would induce the most effort from the players.[5] As will be shown in this paper, however, shrinking the prize gap in the contest with a bottom prize does not necessarily reduce effort when it comes to risk-averse and prudent players.[6]

The main findings of the paper are the following. First, in the comparison between the multiple-competition contest and the winner-take-all contest, we find that the former is as effective as the latter when the contestants are risk neutral, but the former induces more effort than the latter when the contestants are both risk averse and prudent. Second, when the number of competitions becomes infinitely large in the multiple-competition contest, the model converges to that of a single-competition contest where the contest success functions are interpreted as contestants' deterministic shares of the total prize instead of their probabilities of winning the prize.[7] Further, for risk-averse and

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[5] Though not the focus of their formal analysis, O'Keeffe et al. (1984, pp. 29-30) assert such a direct relation between the prize gap and the effort level: "If the prize spread is substantial …, workers may exert excessive effort…"; and "Insufficient effort is also a possibility…, if the bottom prize in a contest is relatively high".

[6] Another possible reason for the contest with a bottom prize not receiving enough attention is that it is mathematically equivalent to a single-prize contest in which every player's wealth is increased by an amount equal to the bottom prize. Despite such equivalence, nevertheless, the comparative statics analysis in the contest with a bottom prize with respect to an increase in the size of the bottom prize while holding the prize budget constant is not simply the comparative statics analysis in the single-prize contest with respect to an increase in the initial wealth.

[7] Share contests have received relatively little attention compared to the winner-take-all lottery contests, even though the contest success functions can be interpreted either as probabilities or as shares. This is probably due to the fact that the two alternative interpretations are equivalent under risk neutrality (Cason et al. 2020). Recent examples of research on share contests beyond the simple setting of additive linear payoffs/costs include Guigou et al. (2017) who study the effects of risk aversion in share contests (see also Long and Vousden 1987), and Dickson et al. (2018) who examine the implications of non-constant rate of

prudent players, the share contest induces a larger amount of effort than the multiple-competition contest with any number of competitions. This, together with the first finding above, implies that, for risk-averse and prudent players, the share contest induces a larger amount of effort than the corresponding winner-take-all lottery contest. Third, in the comparison between the contest with a bottom prize and the winner-take-all contest, we find that the former is always dominated by the latter when the contestants are risk neutral, but the former could have an advantage over the latter when the contestants are both risk averse and prudent, and it is more likely so as the contestants become more prudent. This finding is consistent with Fu et al. (2019) who investigate the effort effect of multiple prizes in contests with risk-averse players utilizing the nested lottery procedure of Clark and Riis (1996) to allocate the multiple prizes.[8]

Whether the winner-take-all arrangement provides a larger incentive for players to make effort than the alternative arrangements has been an important question in the literature on contest design. Our paper thus sheds new light on this question by studying the roles of risk attitudes (i.e., risk aversion and prudence). It is part of a broader research agenda on optimal contest design given various possible contestants' preferences (Schroyen and Treich 2016, Drugov and Ryvkin 2021). Following a large strand of the contest literature, the goal of contest design in our paper is assumed to be maximizing

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substitution between the payoff and the cost in share contests. To the best of our knowledge, nevertheless, the present paper is the first one recognizing the share contest as the limiting case of the lottery contest with multiple competitions that are based on the same set of one-shot player inputs.

[8]Our paper provides a general and systematic treatment of the roles of risk aversion and prudence in the comparison between the winner-take-all contest and the multiple-prize contest, which generates clear-cut results in line with those of Moldovanu and Sela (2001) and Fang et al. (2020) that are obtained under convex bidding cost functions for the players. There could be other reasons in favor of multiple prizes that are different from ours (which is the insurance value). For example, Blavatskyy (2004) provides a justification for multiple prizes based on player heterogeneity (our model has identical players), and the entry possibility (our model has a fixed number of participants).

effort. This objective might be natural in some contexts such as sport events, where effort may be correlated with performance and show quality, and in turn with the revenue of the contest organizer. In other contexts, such as the initial rent-seeking application (Tullock 1980), the objective may be instead to minimize (unproductive or wasteful) efforts; our result thus also shed some light on those situations. But there exist a wide range of other possible objectives. An alternative objective would be for instance to maximize welfare or efficiency, as in Lazear and Rosen (1981). The quest for the optimal contest design with an eye on welfare when the contestants are risk averse and prudent may be a fruitful avenue for future research. Another interesting research direction would be to consider that participants can choose the optimal dynamic effort in a multi-round competition (and can observe outcomes in intermediate rounds), as in Tsetlin et al. (2004).

The paper is organized as follows. The next section presents the basic "winner-take-all" contest model of KST, and discusses the effects of risk and risk aversion on the equilibrium effort level in this model. Section 3 introduces multiple competitions into the basic model so that players can have multiple shots at winning prizes based on the same one-shot effort inputs. Three results concerning the effects of multiple competitions – in which risk attitudes play a critical role – are established in this section. Section 4 introduces a bottom prize into the basic model. Two results concerning the effects of having a larger bottom prize – again with an emphasis on the role of the attitudes toward risk – are obtained. Section 5 concludes with a summary of the findings in the paper.

2. The Basic Winner-Take-All Contest Model

The basic winner-take-all contest model of KST uses a general utility function and a general form of contest success function. Suppose that $n \geq 2$ players are ex ante identical with initial wealth w and a utility function $u(\cdot)$, where $u'(\cdot) > 0$ and $u''(\cdot) \leq 0$. The probability of individual i winning a monetary value $b > 0$ – also known as the contest success function – can be generally expressed as $p_i \equiv p_i(x_1, \cdots, x_n)$, where x_i is the investment or effort of player i. The contest success functions – assumed to be continuously twice differentiable – satisfy $p_i \geq 0$ for all i and $\sum_{i=1}^{n} p_i = 1$. While there is a contest success function for each player, hence the subscript i, these functions are assumed to be symmetric.

Following KST, we make a few additional assumptions on the contest success functions throughout the paper. Specifically, for all i and all $j \neq i$:

A1. $\frac{\partial p_i}{\partial x_i}(x_1, \cdots, x_n) \geq 0$ and $\frac{\partial p_i}{\partial x_j}(x_1, \cdots, x_n) \leq 0$, for all x_i $(1 \leq i \leq n)$;

A2. $\frac{\partial^2 p_i}{\partial x_i^2}(x, \cdots, x) < 0$, for all x;

A3. $\frac{\partial^2 p_i}{\partial x_j \partial x_i}(x, \cdots, x) \leq 0$, for all x;

A4. $p_i(x, \cdots, x) = 1/n$, for all x.

These assumptions are satisfied by all common symmetric contest success functions, to the best of our knowledge.[9] Assumption A1 requires that the probability of winning be nondecreasing in one's own effort, and nonincreasing in any other player's effort. Assumptions A2 and A3 reflect the notion of diminishing marginal returns to effort for

[9] In particular, these assumptions are satisfied by the logistic (or ratio-form) contest success functions that have solid axiomatic foundations and are dominant in the literature on contests (e.g., Tullock 1980, Baye et al. 1994, Nitzan 1994, Skaperdas 1996, Jia 2008, and Schroyen and Treich 2016).

all players. Lastly, assumption A4 indicates that if all players expend the same amount of effort, then each is equally likely to win.

Player *i*'s expected utility is given by

(1) $$p_i u(w+b-x_i)+(1-p_i)u(w-x_i).$$

We follow KST to focus on symmetric interior Nash equilibria. Imposing symmetry – $x_i = x$ for all i – on the first order condition derived from (1), the symmetric interior Nash equilibria satisfy

(2) $$F(x)\equiv p_x\left[u(w+b-x)-u(w-x)\right]-\left[\tfrac{1}{n}u'(w+b-x)+(1-\tfrac{1}{n})u'(w-x)\right]=0,$$

where $p_x(x)\equiv \frac{\partial p_i}{\partial x_i}(x,\cdots,x)\geq 0$ with $\frac{dp_x}{dx}=\sum_{j=1}^{n}\frac{\partial^2 p_i}{\partial x_i \partial x_j}(x,\cdots,x)<0$, according to A1-A4.

Treich (2010) discusses sufficient conditions ensuring a unique symmetric Nash equilibrium of this model.[10] For simplicity, we assume a stronger sufficient condition than those provided by Treich. Our sufficient condition also turns out to be useful when conducting some comparative statics analyses later in the paper.

Condition 1: F(x) in (2) is a strictly decreasing function in x.

In the appendix A1 we show that Condition 1 is satisfied when the contest success function takes the typical ratio form and the utility function displays either constant absolute risk aversion (CARA), or constant relative risk aversion (CRRA) as long as *x* is relatively small given the level of risk aversion.

[10] The existence and uniqueness of symmetric and asymmetric equilibria in contests with risk-averse or risk-loving players are also studied in Skaperdas and Gan (1995), Cornes and Hartley (2012) and Jindapon and Whaley (2015), under various assumptions on the utility function and the contest success functions. In particular, Cornes and Hartley (2012) show, using a common logistic contest success function, that the symmetric equilibrium of symmetric contests is always unique. In the more general KST model, Treich (2010) shows that there exists a unique symmetric equilibrium under decreasing absolute risk aversion (DARA) when the prize is sufficiently "small". Moreover, note that even if the equilibrium is not unique, the comparative statics results are useful since they permit to compare the lowest and highest equilibria (Milgrom and Roberts 1994, Treich 2010).

Treich (2010) provides two main findings, one concerning the effects of risk aversion and prudence, and the other concerning the effects of risk, in this model. In the expected utility framework, risk aversion and prudence are respectively characterized by concavity of the utility function and convexity of the marginal utility function. A closely related concept to prudence is downside risk version. Both prudence – which gives rise to a precautionary saving motive (Kimball 1990) – and downside risk aversion – which displays an aversion to downside risk increases (Menezes et al. 1980) – are characterized by $u''' > 0$. In addition, Eeckhoudt and Schlesinger (2006) provide an alternative characterization of $u''' > 0$ by a preference for disaggregating two independent risks (or risk increases) into separate states of nature rather than combining them into a single state.

Concerning the effect of risk aversion and prudence, Treich proves:

Proposition 1: (Treich 2010) Compared to the case of risk neutrality, the equilibrium effort level for players who are both risk averse and prudent (i.e., downside risk averse) is lower.

Jindapon and Whaley (2015) obtain a mirror image result of Proposition 1 that risk-loving and imprudent players exert more effort than the risk-neutral players. It is interesting to note that a similar result to Proposition 1 in the self-protection model – which is the nonstrategic single-player counterpart of the contest model here – states that a risk-averse and prudent individual spends less on self-protection than a risk-neutral individual under the condition that the no-loss probability is smaller than or equal to 1/2 at the optimal effort level of the risk-neutral individual (Eeckhoudt and Gollier 2005, Corollary 1). The reason such a condition is not explicitly needed for Proposition 1 is that it always holds, because in a symmetric Nash equilibrium of a contest, the winning probability for each player is 1/n, regardless of the risk attitude, which is always smaller than or equal to 1/2.[11] Liu et al. (2018) recently generalize Proposition 1 to address a natural follow-up question as to whether more risk-averse and more downside risk-averse players would make less effort. They find that Ross more risk-averse and Ross more downside risk-averse players make less effort in equilibrium.[12]

Concerning the effect of replacing b with a random $\tilde{b}$ with $E\tilde{b} = b$, Treich (2010) proves:

Proposition 2: (Treich 2010) The equilibrium effort level for players who are both risk averse and prudent is lower when the prize is risky.

As is indicated in Treich (2010), the intuition for the negative effort effect of increased riskiness is that the increased riskiness decreases the marginal benefit of effort under risk aversion, and increases the marginal cost of effort under prudence, both of which putting downward pressure on effort.[13]

3. Spreading the Rewards through Multiple Competitions

We now introduce multiple competitions into the basic model of the last section. As in the basic model, each player makes a one-shot effort, but now has multiple shots at

[11] Risk aversion and prudence (or downside risk aversion) play an important role in the self-protection decision -- a single-player, nonstrategic version of the contest model in which the decision maker exerts effort to increase the probability of no loss. See, for example, Dionne and Eeckhoudt (1985), Briys and Schlesinger (1990), Lee (1998), Jullien et al. (1999), Chiu (2005), Eeckhoudt and Gollier (2005), Menegatti (2009), Liu et al. (2009), Dionne and Li (2011), Ebert (2015), Crainich et al. (2016), Denuit et al. (2016), and Peter (2017, 2020). In particular, Denuit et al. (2016) explain that the composite change in the final wealth distribution caused by an increase in self-protection effort includes a component of downside risk increase in the sense of Menezes et al. (1980) that is disliked by downside risk averse decision makers.
[12] Peter (2020) provides a similar generalization of Eeckhoudt and Gollier (2005) for self-protection. In a two-period version of the rent-seeking contest of KST, Menegatti (2020) proves that more risk averse players make less effort (without any additional condition on prudence).
[13] In a two-period version of the rent-seeking contest of KST, Menegatti (2020) shows that making the rent risky reduces player effort under risk aversion alone (without prudence).

winning a (smaller) prize. Suppose that $m \geq 1$ rounds of statistically independent competitions are held,[14] and the prize from winning the jth round ($j = 1, ..., m$) is b_j , with $\sum b_j = b$, where b is a fixed amount allocated for the prizes of the contest regardless of how many rounds of prize-giving competitions are held. The probability of player i winning each competition is based on the same set of player efforts $(x_1, \cdots, x_n)$ according to $p_i(x_1, \cdots, x_n)$, which is the same contest success function that was discussed in the last section. The special case of $m = 1$ corresponds to the basic winner-take-all model.

For easy exposition, we first consider the case of $m = 2$ and compare it with the case of $m = 1$. Then we consider a general m and explore what would happen when m goes to infinity.

3.1 From $m = 1$ to $m = 2$

When m = 1, the m-round competition model reduces to the basic model of winner-take-all. Specifically, player i's expected utility is given by (1) and the symmetric interior Nash equilibrium effort is the solution to (2).

When m = 2, player i's expected utility is given by

$$(3) \quad p_i^2 u(w + b - x_i) + p_i(1 - p_i)u(w + b_1 - x_i) + p_i(1 - p_i)u(w + b_2 - x_i) + (1 - p_i)^2 u(w - x_i) ,$$

and the symmetric interior Nash equilibrium effort is the solution to

$$(4) \quad \begin{aligned} G(x) \equiv\ & p_x \left[\tfrac{2}{n} u(w + b - x) + \left(1 - \tfrac{2}{n}\right) u(w + b_1 - x) + \left(1 - \tfrac{2}{n}\right) u(w + b_2 - x) - 2\left(1 - \tfrac{1}{n}\right) u(w - x) \right] \\ & - \left[\left(\tfrac{1}{n}\right)^2 u'(w + b - x) + \tfrac{1}{n}\left(1 - \tfrac{1}{n}\right) u'(w + b_1 - x) + \tfrac{1}{n}\left(1 - \tfrac{1}{n}\right) u'(w + b_2 - x) + (1 - \tfrac{1}{n})^2 u'(w - x) \right] = 0. \end{aligned}$$

[14] Although the assumption that the competitions of different rounds are statistically independent seems quite natural, it does not hold in many cases. Consider, for instance, the example of tennis competitions in the ATP World Tour, mentioned in Footnote 3. In this case, it sometimes happens that the winner of a tournament withdraws from the next tournament if the two tournaments are very close. This suggests that the idea of statistically independent competitions is plausible, but not always verified.

Obviously, the symmetric equilibrium effort when m = 2 depends on the specific values of b_1 and b_2. The lemma below shows that incentives are maximized when $b_1 = b_2 = \frac{b}{2}$.[15]

Lemma 1: Suppose that m = 2.

(i) If the players are risk neutral, then the symmetric equilibrium effort is the same regardless of the values of b_1 and b_2 (subject to $b_1 + b_2 = b$).

(ii) If the players are both risk averse and prudent, then the symmetric equilibrium effort is maximized when $b_1 = b_2 = \frac{b}{2}$.

Proof: When m = 2, the symmetric equilibrium effort is determined by (4).

(i) If the players are risk neutral, (4) becomes $p_x b - 1 = 0$, the solution to which is the same regardless of the specific values of b_1 and b_2.

(ii) If the players are both risk averse and prudent, then in (4), $\left(1-\frac{2}{n}\right)u(w+b_1-x)+\left(1-\frac{2}{n}\right)u(w+b_2-x)$ reaches its maximum value when $b_1 = b_2 = \frac{b}{2}$ (because u is concave), and $\frac{1}{n}\left(1-\frac{1}{n}\right)u'(w+b_1-x)+\frac{1}{n}\left(1-\frac{1}{n}\right)u'(w+b_2-x)$ reaches its minimum value when $b_1 = b_2 = \frac{b}{2}$ (because u' is convex). Therefore, for every x, G(x) in (4) reaches the maximum value when $b_1 = b_2 = \frac{b}{2}$. This implies that the symmetric equilibrium effort is maximized when $b_1 = b_2 = \frac{b}{2}$. Q.E.D.

According to Lemma 1, an effort-maximizing contest designer would set $b_1 = b_2 = \frac{b}{2}$ when m = 2. The following proposition states the effect of moving from $m = 1$ to $m = 2$ (with $b_1 = b_2 = \frac{b}{2}$) on the symmetric equilibrium effort.

Proposition 3: From $m = 1$ to $m = 2$ (with $b_1 = b_2 = \frac{b}{2}$),

[15] Although the assumption of equal prizes seems plausible in some instances, it is certainly not verified in general. Following comments in Footnotes 3 and 14, the ATP World Tour provides an example with different categories of tournaments, and thus where competitions offer different prizes.

(i) if the players are risk neutral, the symmetric equilibrium effort level is unchanged;

(ii) if the players are both risk averse and prudent, the symmetric equilibrium effort level increases.

<u>Proof:</u> The symmetric equilibrium effort is determined by (2) when m = 1, and is given by (4) – letting $b_1 = b_2 = \frac{b}{2}$ in (4) – when *m* = 2.

(i) Players are risk neutral: $u''(\cdot) = 0$.

In this case, $G(x) = F(x) = p_x b - 1$. Therefore, *m* has no effect on the equilibrium effort.

(ii) Players are both risk averse and prudent: $u''(\cdot) < 0$ and $u'''(\cdot) > 0$.

In this case,

$$\begin{aligned} G(x) - F(x) = p_x \tfrac{n-2}{n}\left[-u(w+b-x) + 2u(w+b/2-x) - u(w-x)\right] \\ + \tfrac{1}{n}\left(1-\tfrac{1}{n}\right)\left[u'(w+b-x) - 2u'(w+b/2-x) + u'(w-x)\right] > 0, \end{aligned}$$

where the first bracketed term is positive due to risk aversion and the second bracketed term is positive due to prudence. In other words, G(x) is above F(x) for all x. This, together with Condition 1, suggests that the equilibrium effort level increases from *m* = 1 to *m* = 2. Q.E.D.

We can provide the following intuition for Proposition 3. First, note from (1) and (3) that moving from *m* = 1 to *m* = 2 does not change the *i*th player's mean wealth, which is $w - x_i + p_i b$. As a result, risk-neutral players' incentive to make effort, and hence the equilibrium effort level, would not change as *m* increases. Second, as established in Proposition 2, making the prize risky in the basic lottery model would induce less effort from players who are both risk averse and prudent. In other words, the effort level of risk-averse and prudent players responds positively to a reduction in the riskiness of the prize distribution. It is readily seen that the wealth distribution represented in (3) is less risky in the sense of Rothschild and Stiglitz (1970) than the wealth distribution represented in (1). Therefore, the effort level for risk-averse and prudent players increases when moving from m = 1 to m = 2. From a more technical point of view, increasing the number of competitions has two distinctive effects on the equilibrium effort level, one due to risk aversion and one due to prudence. The risk aversion effect – represented by $p_x \frac{n-2}{n}\left[-u(w+b-x)+2u(w+b/2-x)-u(w-x)\right]$ in the $G(x)-F(x)$ expression – makes winning the prize more attractive (since the prize becomes less risky), and the prudence effect – represented by $\frac{1}{n}\left(1-\frac{1}{n}\right)\left[u'(w+b-x)-2u'(w+b/2-x)+u'(w-x)\right]$ – reduces the marginal utility cost of effort.

3.2 The General Case of $m \geq 2$

According to Proposition 3, the contest with two rounds of competitions generates the same level of equilibrium effort as the winner-take-all contest when the players are risk neutral, but the former contest induces a higher effort level than the latter when the players are both risk averse and prudent. The analysis below shows that this result also holds for any $m \geq 2$ rounds of competitions with equal prizes $b_1 = \cdots = b_m = \frac{b}{m}$. In the general case of $m \geq 2$, player i's expected utility is

$$\sum_{k=0}^{m}\left[\binom{m}{k} p_i^{m-k}(1-p_i)^k u\left(w+(m-k)\frac{b}{m}-x_i\right)\right],$$

where $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$ is the usual binomial coefficient.

In the appendix A2, we demonstrate that the symmetric interior Nash equilibrium effort is the solution to

(4')

$$G_m(x) \equiv p_x \left\{ \sum_{k=0}^{m-1} m \binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^k u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k=0}^{m-1} m \binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^k u\left(w+(m-1-k)\frac{b}{m}-x\right) \right\}$$
$$-\sum_{k=0}^{m} \binom{m}{k} \left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^k u'\left(w+(m-k)\frac{b}{m}-x\right) = 0,$$

where $p_x(x) \equiv \frac{\partial p_i}{\partial x_i}(x,\cdots,x) \geq 0$.

<u>Proposition 3'</u>: From $m=1$ to $m \geq 2$ (with $b_1 = \cdots = b_m = \frac{b}{m}$),

(i) if the players are risk neutral, the symmetric equilibrium effort level is unchanged;

(ii) if the players are both risk averse and prudent, the symmetric equilibrium effort level increases.

<u>Proof:</u> See the appendix A3.

Note that in Propositions 3 and 3', Condition 1 is sufficient for ensuring the uniqueness of the equilibrium in the case *m=1*, and that this condition is thus enough to make sure that all equilibria (whether unique or not) under *m* larger than or equal to 2 are above this unique equilibrium.

To summarize, it is always better, from the viewpoint of an effort-maximizing contest organizer with a fixed prize budget, to have multiple rounds of competitions with the same (smaller) prize for each round.[16] In other words, when the players are risk averse and prudent, the optimality of the winner-take-all contest does not hold, at least for

[16]In reality, of course, the number of competitions is constrained by the transaction costs associated with organizing competitions.

the case where rewards can be spread out to more players through multiple rounds of competitions.

3.3 The Limiting Case of m Going to Infinity

The following analysis indicates that as *m* continues to increase, the contest model with multiple competitions converges to a single-competition contest in which $p_i(x_1, \cdots, x_n)$ stands for player *i*'s deterministic share of the total prize *b* rather than his probability of winning *b*.

In light of Lemma 1, we only consider the contest of *m*-round competitions each with a prize of b/m. In this case, the overall random prize received by player *i* from the *m*-rounds of independent competitions, denoted $\tilde{Z}_m^i$, is given by

(5)
$$\tilde{Z}_m^i = \frac{\tilde{X}_1^i + \cdots + \tilde{X}_m^i}{m},$$

where $\tilde{X}_1^i, \cdots, \tilde{X}_m^i$ are i.i.d. random variables that yield a value of *b* with probability $p_i(x_1, \cdots, x_n)$ and 0 otherwise.

In terms of distribution, $\tilde{Z}_m^i$ follows the following (m+1)-value discrete distribution:

$$(m-k)\frac{b}{m} \text{ with probability } \binom{m}{k} p_i^{m-k}(1-p_i)^k, \quad k = 0, \cdots, m,$$

where $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$. And player *i*'s expected utility can be written as

$$E\left[u\left(w - x_i + \tilde{Z}_m^i\right)\right] = \sum_{k=0}^{m}\left[\binom{m}{k} p_i^{m-k}(1-p_i)^k u\left(w - x_i + (m-k)\frac{b}{m}\right)\right].$$

Proposition 4: $\lim_{m\to\infty} E\left[u\left(w-x_i+\tilde{Z}_m^i\right)\right]=u\left(w-x_i+p_i(x_1,\cdots,x_n)b\right)$, where $\tilde{Z}_m^i$ is player i's random prize from the contest with m rounds of lottery competitions as given by (5).

Proof: From the law of large numbers, for any $\varepsilon>0$,

$$\lim_{m\to\infty} \Pr\left(\left|\tilde{Z}_m^i-p_ib\right|>\varepsilon\right)=0. \tag{6}$$

By the mean value theorem,

$$\left|u\left(w-x_i+\tilde{Z}_m^i\right)-u\left(w-x_i+p_ib\right)\right|\le u'(w-x_i)\left|\tilde{Z}_m^i-p_ib\right|, \tag{7}$$

where the inequality comes from the fact that $\tilde{Z}_m^i$ takes values in [0, b] and that $u''(\cdot)\le 0$. Therefore,

$$\begin{aligned}&\left|E\left[u\left(w-x_i+\tilde{Z}_m^i\right)\right]-u\left(w-x_i+p_ib\right)\right|\le u'(w-x_i)E\left(\left|\tilde{Z}_m^i-p_ib\right|\right)\\&\le u'(w-x_i)\left[\varepsilon+b\Pr\left(\left|\tilde{Z}_m^i-p_ib\right|>\varepsilon\right)\right],\end{aligned} \tag{8}$$

where the second inequality comes from the fact that $\tilde{Z}_m^i$ takes values in [0, b] so that $\left|\tilde{Z}_m^i-p_ib\right|\le b$. Note that in (8), ε can be arbitrarily small and $\Pr\left(\left|\tilde{Z}_m^i-p_ib\right|>\varepsilon\right)$ goes to zero as m tends to infinity according to (6). As a result,

$$\lim_{m\to\infty} E\left[u\left(w-x_i+\tilde{Z}_m^i\right)\right]=u\left(w-x_i+p_ib\right). \qquad \text{Q.E.D.}$$

Hence, Proposition 4 shows that when the number of rounds is infinite the multiple-competition contest is equivalent to the shared-prize contest. Obviously, the unique symmetric equilibrium effort under the shared-prize contest is determined by $p_xb-1=0$, and risk attitudes play no role. Moreover, the equilibrium is thus unique in that case. The proposition below states that the shared-prize contest induces the same equilibrium effort level as every multiple-competition contest (with $m\ge 2$) when players are risk neutral, but the former contest generates a higher equilibrium effort level than the latter when players are both risk averse and prudent. To prove this result, we need to use the following property of prudent utility functions.

<u>Lemma 2:</u> (Eeckhoudt and Gollier 2005) Let

$$H(A,B) = 0.5\left[u'(A)+u'(B)\right](B-A)-\left[u(B)-u(A)\right] \quad \text{with} \quad A \le B.$$

The function H is positive (negative) in its domain if and only if u' is convex (concave).

<u>Proposition 5:</u> Comparing the shared-prize contest with an *m*-round multiple-competition contest with equal prizes ($b_1 = \cdots = b_m = \frac{b}{m}$, where $m \ge 2$),

(i) if the players are risk neutral, the symmetric equilibrium effort level is the same;

(ii) if the players are both risk averse and prudent, the symmetric equilibrium effort level is higher under the shared-prize contest.

<u>Proof:</u> See the appendix A4.

Propositions 3 (or 3') and 5 together help illuminate the difference between the basic winner-take-all lottery contest model and the shared-prize contest model. Under risk neutrality, the two models are equivalent, but for risk-averse and prudent players, the latter model provides a stronger incentive. The difference between the winner-take-all lottery contest model and the shared-prize contest model for risk-averse players were previously established under special contest success functions (e.g., Long and Vousden 1987), but we demonstrate this difference here under a very general situation and by casting the two models as the opposite extremes of the multiple-competition contest (m = 1 versus $m = \infty$).

4. Spreading the Rewards through a Bottom Prize to the Losers

The basic KST model is extended here to include a bottom prize for every player, in order to analyze the effect of spreading the rewards through a bottom prize on the effort level. While this model is a restricted version of the more general multi-prize contest model, it can be used to generate the main insight from the more sophisticated (though with a more special form of CSFs) multi-prize contest model of Fu et al. (2019).[17]

Suppose that in the basic lottery model described in Section 2, a lower prize $\frac{a}{n-1} \geq 0$ is awarded to each of the $n-1$ losers, and a higher prize $b-a > \frac{a}{n-1}$ is awarded to the single winner. Note that in this specification the total prize money remains to be b, and $a = 0$ corresponds to the case of winner-take-all.

Player i's expected utility is now given by

$$(9) \qquad p_i u(w+b-a-x_i)+(1-p_i)u(w+\tfrac{a}{n-1}-x_i),$$

and the symmetric interior Nash equilibrium x satisfies

$$(10) \quad H(x,a) \equiv p_x\left[u(w+b-a-x)-u(w+\tfrac{a}{n-1}-x)\right]-\left[\tfrac{1}{n}u'(w+b-a-x)+(1-\tfrac{1}{n})u'(w+\tfrac{a}{n-1}-x)\right]=0,$$

where $H(x,a)$ is assumed to be strictly decreasing in x (see Condition 1).[18]

The following proposition and corollary identify two sufficient conditions for the optimality of a single prize.

[17] Fu et al. (2019) apply the nested lottery procedure of Clark and Riis (1996) to allocate a set of prizes. To do this, however, they assume that the contest success functions have a special form (e.g., the ratio form) so that they are still (unambiguously) well-defined when the number of players changes. In the present paper, on the other hand, the CSFs are of a more general form on which the nested lottery procedure is not well-defined.

[18] Indeed, it is easy to see through a change in notation that the condition *H(x,a)* strictly decreasing in *x* is equivalent to Condition 1.

<u>Proposition 6</u>: The symmetric equilibrium effort level decreases as a increases (and so it is optimal to set a = 0), if the players are non-prudent ($u'''(\cdot) \leq 0$).

<u>Proof:</u> From the definition of $H(x,a)$ in (10), we have

(11) $$\frac{\partial H(x,a)}{\partial a} = p_x\left[-u'(w+b-a-x) - \tfrac{1}{n-1}u'(w+\tfrac{a}{n-1}-x)\right] + \tfrac{1}{n}\left[u''(w+b-a-x) - u''(w+\tfrac{a}{n-1}-x)\right] < 0,$$

under the condition that $u'''(\cdot) \leq 0$. This suggests that as a increases, $H(x,a)$ as a function of x uniformly shifts downwardly, implying that the symmetric Nash equilibrium effort level becomes smaller as a increases. That is, the maximum equilibrium effort level is achieved at a = 0. Q.E.D.

<u>Corollary 1</u>: The symmetric equilibrium effort level decreases as a increases (and so it is optimal to set a = 0), if the players are risk neutral ($u''(\cdot) = 0$).

The corollary comes immediately from Proposition 6 because $u''(\cdot) = 0$ implies $u'''(\cdot) = 0$. The results in Proposition 6 and Corollary 1 are consistent with the findings in the literature on multiple-prize contests that, under risk neutrality (or linear payoffs/costs), it is optimal to give the entire prize money to a single winner. For example, Berry (1993) finds that, for players with a linear or quadratic utility function ($u''(\cdot) = 0$ or $u'''(\cdot) = 0$), the symmetric equilibrium effort level decreases as the number of equal-sized prizes increases while the total prize money is held constant. Similar to an increase in a, an increase in the number of prizes/winners in Berry's analysis implies a smaller prize gap between the winner(s) and the loser(s), which is responsible for the reduced incentive for players to make effort.[19] However, Berry (1993), as well as all other previous studies on this topic, does not explore the case of prudent players ($u'''(\cdot) > 0$). As a result, he did not shed light on the role played by the degree of prudence on the relative efficiencies of the winner-take-all and the multiple-prize arrangements (see Proposition 7 below).

In contrast, as we found in the last section, increasing the number of competitions has no effect on the equilibrium effort level when the players are risk neutral. To the best of our knowledge, no player can win more than one prize in all existing studies on multiple-prize contests, regardless of whether the nested lottery procedure is used. This is a major difference between multiple-prize contests (including the one analyzed in this section) and the multiple-competition contests studied in the last section in which a player can earn multiple, even all of the, prizes, and in which we have seen that the number of competitions has no effect on effort under risk neutrality. The intuition underlying the different incentive effects of spreading the rewards between the two forms of contests is the following. Within the multiple-prize contest analyzed here, the marginal expected monetary payoff of effort for player i is $\frac{\partial p_i}{\partial x_i}(b - a - \frac{a}{n-1})$, which decreases as a increases and is always smaller than the marginal expected monetary payoff within the basic lottery contest, $\frac{\partial p_i}{\partial x_i} b$, unless a = 0. Within the multiple-competition contest, on the other hand, the marginal expected monetary payoff of effort for player i is always $\frac{\partial p_i}{\partial x_i} b$, because the overall random prize received by player i from

[19] Chowdhury and Kim (2014) demonstrate that, under symmetric players and prizes, Berry's model is equivalent to a multi-prize contest model using a nested lottery procedure of the Clark and Riis (1996) type to sequentially eliminate losers.

the *m*-rounds of independent competitions, $\tilde{Z}_m^i$ given in (5), has a mean of $p_i b$, a constant with respect to *m*.

Nevertheless, Proposition 6 also suggests that only when the players are prudent ($u'''(\cdot) > 0$) could it be possible to have multiple prizes (*a* > 0) as the optimal arrangement. In the simple multiple-prize model analyzed here, prudence (i.e., downside risk aversion) alone facilitates an additional incentive effect from shrinking the prize gap between the winner and the loser (i.e., an increase in *a*) that works to increase the effort level – the $\frac{1}{n}\left[u''(w+b-a-x)-u''(w+\frac{a}{n-1}-x)\right]$ term in (11) is positive when $u'''(\cdot)>0$. The net effect of an increase in *a* on effort depends on the relative magnitudes of the two opposing forces represented by the two terms in (11).

Two examples help illustrate that *a* > 0 may or may not emerge as the optimal arrangement when $u'''(\cdot)>0$. As the proof of Proposition 6 indicates, the sign of $\frac{\partial H(x,a)}{\partial a}$ is critical for whether an *a* > 0 would be optimal. For n = 2,

$$\begin{aligned}\frac{\partial H(x,a)}{\partial a} &= p_x\left[-u'(w+b-a-x)-u'(w+a-x)\right]+\tfrac{1}{2}\left[u''(w+b-a-x)-u''(w+a-x)\right]\\ &= \frac{-\frac{1}{2}\left[u'(w+b-a-x)+u'(w+a-x)\right]^2}{u(w+b-a-x)-u(w+a-x)}+\tfrac{1}{2}\left[u''(w+b-a-x)-u''(w+a-x)\right],\end{aligned}$$

where the second equality is obtained by applying the equilibrium condition (10) (and letting n = 2). Therefore, $\frac{\partial H(x,a)}{\partial a}>0$ if and only if

(12) $\left[u(w+b-a-x)-u(w+a-x)\right]\left[u''(w+b-a-x)-u''(w+a-x)\right]-\left[u'(w+b-a-x)+u'(w+a-x)\right]^2>0.$

Example 1 (CARA): $u(y)=-e^{-\lambda y}$, $\lambda>0$, $y>0$

In this case, $u'(y)=\lambda e^{-\lambda y}$ and $u''(y)=-\lambda^2 e^{-\lambda y}$. So the LHS of (12) is

$$\lambda^2 e^{-2\lambda(w-x)}\left[\left(e^{-\lambda a}-e^{-\lambda(b-a)}\right)^2-\left(e^{-\lambda a}+e^{-\lambda(b-a)}\right)^2\right]<0.$$

So $a = 0$ is optimal.

Example 2 (CRRA): $u(y)=\frac{-1}{y},\ y>0$

In this case, $u'(y)=\frac{1}{y^2}$ and $u''(y)=\frac{-2}{y^3}$. So the LHS of (12) is

$$\left(\frac{1}{w+b-a-x}\right)^4+\left(\frac{1}{w+a-x}\right)^4-2\left(\frac{1}{w+b-a-x}\right)^2\left(\frac{1}{w+a-x}\right)^2-2\left(\frac{1}{w+b-a-x}\right)^3\left(\frac{1}{w+a-x}\right)^1-2\left(\frac{1}{w+b-a-x}\right)^1\left(\frac{1}{w+a-x}\right)^3$$
$$=c^4+d^4-2c^2d^2-2c^3d^1-2c^1d^3,$$

where $c=\frac{1}{w+b-a-x}$ < $d=\frac{1}{w+a-x}$.

It can be readily seen that when b is sufficiently large relative to w - x (so that $d>6c$), the LHS of (12) for this example is positive, in which case $a = 0$ is suboptimal.

Then there is a question as to how the net incentive effect of an increase in a (i.e., reducing the prize gap between the winner and the loser) depends on players' degree of prudence (or the degree of downside risk averse). Proposition 7 below says that the more prudent (i.e., more downside risk averse) the players are, the more likely a contest with a bottom prize would emerge as an optimal arrangement. Before presenting the proposition, we give the following definition of greater downside risk aversion which is a natural extension, from the second to the third degree, of the Ross notion of greater risk aversion. [20]

[20] Ross more risk averse implies, but is not implied by, Arrow-Pratt more risk averse (Pratt 1964, and Ross 1981). Extensions of the Ross notion of greater risk aversion to the general *n*th-degree are studied in Jindapon and Neilson (2007), Li (2009), Denuit and Eeckhoudt (2010), Liu and Meyer (2013), and Liu and Neilson (2019). We use "prudent" and "downside risk averse" interchangeably because both are characterized by $u'''>0$ in the expected utility framework. In addition, we also use "more prudent" and "more downside risk averse" interchangeably, both of which are defined according to Definition 1. We should point out that "more prudent" in the literature may have a different meaning from Definition 1

Definition 1: (Modica and Scarsini 2005) Suppose that both u(x) and v(x) are prudent (i.e., downside risk averse). v(x) is Ross more downside risk averse than u(x) on [A, B] if there exists a constant k > 0 such that $\frac{v'''(x)}{u'''(x)} \geq k \geq \frac{v'(y)}{u'(y)}$ for all x and y in [A, B].

Similar to the characterizations given to Ross greater risk aversion (Ross 1981), Modica and Scarsini (2005) show that v(x) is Ross more downside risk averse than u(x) if and only if decision maker v is always willing to pay more to avoid a downside risk increase (Menezes et al. 1980) than decision maker u. Further, they show that v(x) is Ross more downside risk averse than u(x) on [A, B] if and only if there exists a constant k > 0 and $\phi(x)$ such that $v(x) \equiv ku(x) + \phi(x)$, where $\phi'(x) \leq 0$ and $\phi'''(x) \geq 0$ on [A, B].[21]

Proposition 7: Suppose that $v(\cdot)$ is Ross more downside risk averse than $u(\cdot)$. Whenever an increase in *a* has a positive effect on the equilibrium effort level for players with utility function $u(\cdot)$, the increase in *a* also has a positive effect on the equilibrium effort level for players with utility function $v(\cdot)$.

Proof: From the definition of $H(x,a)$ in (10), we have

$$\frac{\partial H^u(x,a)}{\partial a} = p_x\left[-u'(w+b-a-x) - \tfrac{1}{n-1}u'(w+\tfrac{a}{n-1}-x)\right] + \tfrac{1}{n}\left[u''(w+b-a-x) - u''(w+\tfrac{a}{n-1}-x)\right],$$

where the superscript u indicates that the expression is obtained for utility function $u(\cdot)$.

Similarly,

(Kimball 1990), and there exist alternative notions of greater downside risk aversion (see the next footnote).

[21] Following Modica and Scarsini (2005), Liu et al. (2018) use this Ross notion of greater downside risk aversion in a contest model. Alternatively, Sahm (2017) uses an Arrow-Pratt version of greater downside risk aversion. For discussions of alternative notions of greater downside risk aversion, see Crainich and Eeckhoudt (2008), Keenan and Snow (2016), Liu and Wong (2019) and Peter (2020), among others.

$$\frac{\partial H^v(x,a)}{\partial a} = p_x\left[-v'(w+b-a-x) - \tfrac{1}{n-1}v'(w+\tfrac{a}{n-1}-x)\right] + \tfrac{1}{n}\left[v''(w+b-a-x) - v''(w+\tfrac{a}{n-1}-x)\right]$$
$$= k\frac{\partial H^u(x,a)}{\partial a} + p_x\left[-\phi'(w+b-a-x) - \tfrac{1}{n-1}\phi'(w+\tfrac{a}{n-1}-x)\right] + \tfrac{1}{n}\left[\phi''(w+b-a-x) - \phi''(w+\tfrac{a}{n-1}-x)\right]$$
$$\geq k\frac{\partial H^u(x,a)}{\partial a},$$

where we use the result that *v(x)* is Ross more downside risk averse than *u(x)* on [A, B] if and only if there exists a constant *k > 0* and $\phi(x)$ such that $v(x) \equiv ku(x) + \phi(x)$, where $\phi'(x) \leq 0$ and $\phi'''(x) \geq 0$ on [A, B]. Therefore, $\frac{\partial H^u(x,a)}{\partial a} > 0$ implies $\frac{\partial H^v(x,a)}{\partial a} > 0$. In other words, whenever an increase in *a* has a positive effect on the equilibrium effort level for players with utility function $u(\cdot)$, the increase in *a* also has a positive effect on the equilibrium effort level for players with utility function $v(\cdot)$. Q.E.D.

An equivalent statement of Proposition 7 is: If *a* = 0 is optimal for players with utility function $v(\cdot)$, then *a* = 0 is also optimal for players with utility function $u(\cdot)$ that is Ross *less* downside risk averse than $v(\cdot)$. The two examples and Proposition 7 suggest that, when players are sufficiently prudent, awarding the entire prize money to a single winner may no longer be optimal, a conclusion consistent with similar findings of Fu et al. (2019) who use the nested lottery procedure to allocate multiple prizes.

5. Conclusion

From the perspective of an effort-maximizing contest organizer with a fixed budget for prizes, there is a question as to whether it pays, in terms of soliciting more effort, to spread out rewards to more players rather than awarding everything to a single winner. In this paper, we have considered two alternative ways of spreading the rewards to more players: through increasing the number of prize-giving competitions in the multiple-competition contest or through shrinking the prize spread in the contest with a bottom prize. Both the multiple-competition contest and the contest with a bottom prize can be naturally obtained from the basic winner-take-all contest model of Konrad and Schlesinger (1997) and Treich (2010) that has a general contest success function and a general utility function.

For the multiple-competition contest, we have found that the effect of holding multiple competitions on the symmetric equilibrium effort critically depends on the players' risk attitudes. When the players are risk neutral, holding multiple competitions rather than a single competition has no effect on the effort level; when the players are risk averse and prudent, holding multiple competitions induces a higher effort level. Moreover, when the number of competitions becomes infinitely large, the multiple-competition model converges to that of a single-competition contest where the contest success functions are interpreted as the contestants' deterministic shares of the prize instead of their probabilities of winning the prize, and the resulting shared-prize model induces a higher effort level than the multiple-competition model with any number of competitions.

For the contest with a bottom prize, we have found that players' risk attitudes are also critical in determining whether the optimal size of the bottom prize should be zero. When the players are risk neutral (or more generally when they are non-prudent), it is optimal to award the entire prize money to the winner and nothing to the others in order to maximize the total effort level, which has been repeatedly demonstrated in the previous studies on contests with multiple prizes. On the other hand, prudence (i.e., downside risk aversion) alone produces an additional incentive effect from shrinking the prize gap between the winner and the loser(s) that works to increase the effort level. We have further shown that having a bottom prize is more likely to outperform the single-prize winner-take-all arrangement when players become more prudent (i.e., more downside risk averse).

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Appendix

A1: Condition 1 under the Ratio-Form Contest Success Function and the CARA or CRRA Utility Function

Let $p_i(x_1,\cdots,x_n)=\dfrac{x_i^r}{\sum_{k=1}^{n}x_k^r}$, we have

$$\frac{\partial p_i(x_1,\cdots,x_n)}{\partial x_i}=\frac{\left(rx_i^{r-1}\right)\left(\sum_{k=1}^{n}x_k^r-x_i^r\right)}{\left(\sum_{k=1}^{n}x_k^r\right)^2}.$$

Therefore,

(A1) $$p_x(x)\equiv\frac{\partial p_i(x,\cdots,x)}{\partial x_i}=\frac{(n-1)r}{n^2}\cdot\frac{1}{x}.$$

Note that the derivative of $-\left[\frac{1}{n}u'(w+b-x)+(1-\frac{1}{n})u'(w-x)\right]$ in F(x) is non-positive when $u''(\cdot)\le 0$. As a result, a sufficient condition for Condition 1 to hold is

$$\frac{dp_x(x)}{dx}\left[u(w+b-x)-u(w-x)\right]-p_x(x)\left[u'(w+b-x)-u'(w-x)\right]<0,$$

or equivalently (according to (A1))

(A2) $$-\frac{1}{x}\left[u(w+b-x)-u(w-x)\right]-\left[u'(w+b-x)-u'(w-x)\right]<0.$$

(i) The Case of CARA: $u(y)=-e^{-\lambda y}$, $\lambda\ge 0$, $y>0$

In this case, λ is the (constant) absolute risk aversion measure, and (A2) is equivalent to

(A3) $$\lambda x-1<0.$$

That is, Condition 1 is satisfied as long as *x* is sufficiently small (given the value of λ).

(ii) The Case of CRRA: $u(y)=\frac{y^{1-\gamma}}{1-\gamma}$, $\gamma \ge 0$ and $\gamma \ne 1$, $y>0$

In this case, γ is the (constant) relative risk aversion measure, and (A2) is equivalent to

(A4) $$-\frac{1}{x}\frac{1}{1-\gamma}\left[(w+b-x)^{1-\gamma}-(w-x)^{1-\gamma}\right]-\left[(w+b-x)^{-\gamma}-(w-x)^{-\gamma}\right]<0.$$

Note that the LHS of (A4) is zero when $b = 0$. So for (A4) to hold when $b > 0$, it is sufficient that the derivative of the LHS of (A4) with respect to b is negative, or

(A5) $$-\frac{w+b-x}{x}+\gamma<0.$$

That is, Condition 1 is satisfied as long as x is sufficiently small (given the value of γ).

A2: Derivation of (4')

The first order condition for player i's problem to maximize expected utility

$$\sum_{k=0}^{m}\left[\binom{m}{k}p_i^{m-k}(1-p_i)^k u\left(w+(m-k)\frac{b}{m}-x_i\right)\right],$$

where $\binom{m}{k}=\frac{m(m-1)\cdots(m-k+1)}{k!}$, by choosing x_i is

$$\frac{\partial p_i}{\partial x_i}\left\{\sum_{k=1}^{m-1}\binom{m}{k}\left[(m-k)\,p_i^{m-1-k}(1-p_i)^k-kp_i^{m-k}(1-p_i)^{k-1}\right]u\left(w+(m-k)\frac{b}{m}-x_i\right)+mp_i^{m-1}u\left(w+b-x_i\right)-m(1-p_i)^{m-1}u\left(w-x_i\right)\right\}$$
$$-\sum_{k=0}^{m}\binom{m}{k}p_i^{m-k}(1-p_i)^k u'\left(w+(m-k)\frac{b}{m}-x_i\right)=0.$$

Imposing symmetry – $x_i = x$ for all i – on the first order condition, the symmetric interior Nash equilibrium satisfies

$$G_m(x) \equiv p_x \left\{ \begin{array}{l} \sum_{k=1}^{m-1} \binom{m}{k} \left[(m-k)\left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} - k\left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^{k-1} \right] u\left(w+(m-k)\frac{b}{m}-x\right) \\ +m\left(\frac{1}{n}\right)^{m-1} u(w+b-x) - m\left(1-\frac{1}{n}\right)^{m-1} u(w-x) \end{array} \right\}$$

$$-\sum_{k=0}^{m} \binom{m}{k} \left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) = 0,$$

where $p_x(x) \equiv \frac{\partial p_i}{\partial x_i}(x,\cdots,x) \geq 0$. Note that

$$\sum_{k=1}^{m-1} \binom{m}{k} \left[(m-k)\left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} - k\left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^{k-1} \right] u\left(w+(m-k)\frac{b}{m}-x\right)$$

$$= \sum_{k=1}^{m-1} \binom{m}{k} (m-k)\left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k=1}^{m-1} \binom{m}{k} k\left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^{k-1} u\left(w+(m-k)\frac{b}{m}-x\right)$$

$$= \sum_{k=1}^{m-1} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k'=0}^{m-2} \binom{m}{k'+1} (k'+1)\left(\frac{1}{n}\right)^{m-1-k'} \left(1-\frac{1}{n}\right)^{k'} u\left(w+(m-1-k')\frac{b}{m}-x\right)$$

$$= \sum_{k=1}^{m-1} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k'=0}^{m-2} m\binom{m-1}{k'} \left(\frac{1}{n}\right)^{m-1-k'} \left(1-\frac{1}{n}\right)^{k'} u\left(w+(m-1-k')\frac{b}{m}-x\right)$$

$$= \sum_{k=1}^{m-1} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k=0}^{m-2} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-1-k)\frac{b}{m}-x\right)$$

Therefore, we have

(4')

$$G_m(x) \equiv p_x \left\{ \sum_{k=0}^{m-1} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k=0}^{m-1} m\binom{m-1}{k} \left(\frac{1}{n}\right)^{m-1-k} \left(1-\frac{1}{n}\right)^{k} u\left(w+(m-1-k)\frac{b}{m}-x\right) \right\}$$

$$-\sum_{k=0}^{m} \binom{m}{k} \left(\frac{1}{n}\right)^{m-k} \left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) = 0.$$

A3: Proof of Proposition 3'

The symmetric equilibrium effort is determined by (2) when m = 1 and by (4') when $m \geq 2$.

(i) Players are risk neutral: $u''(\cdot)=0$.

In this case, $F(x)=p_x b-1$, and

$$G_m(x)=p_x\left\{\sum_{k=0}^{m-1}m\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}\left(w+(m-k)\frac{b}{m}-x\right)-\sum_{k=0}^{m-1}m\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}\left(w+(m-1-k)\frac{b}{m}-x\right)\right\}$$

$$-\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k}$$

$$=p_x\left\{m\left(w-x+\frac{b}{m}+\frac{m-1}{n}\frac{b}{m}\right)-m\left(w-x+\frac{m-1}{n}\frac{b}{m}\right)\right\}-\left(\frac{1}{n}+1-\frac{1}{n}\right)^{m}=p_x b-1.$$

Therefore, m has no effect on the equilibrium effort.

(ii) Players are both risk averse and prudent: $u''(\cdot)<0$ and $u'''(\cdot)>0$.

In this case,

$$G_m(x)-F(x)=p_x(m+1)\left\{\begin{array}{l}\left[\dfrac{m}{m+1}\displaystyle\sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}u\left(w+(m-k)\frac{b}{m}-x\right)+\frac{1}{m+1}u(w-x)\right]\\-\left[\dfrac{m}{m+1}\displaystyle\sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}u\left(w+(m-1-k)\frac{b}{m}-x\right)+\frac{1}{m+1}u(w+b-x)\right]\end{array}\right\}$$

$$+\left[\tfrac{1}{n}u'(w+b-x)+(1-\tfrac{1}{n})u'(w-x)\right]-\left[\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k}u'\left(w+(m-k)\frac{b}{m}-x\right)\right].$$

The two bracketed items in the curly parentheses are the expected utility of two wealth distributions respectively, and the second distribution is a Rothschild-Stiglitz risk increase over the first, both having a mean of $w-x+\dfrac{b}{m+1}\left(1+\dfrac{m-1}{n}\right)$. So their difference is positive since $u''(\cdot)<0$. In addition, the last two bracketed items above are the expected utility – with $u'(\cdot)$ being the utility function – of two wealth distributions respectively, and the first distribution is a Rothschild-Stiglitz risk increase over the second, both having a mean of $w-x+\dfrac{b}{n}$. So their difference is also positive since

$u'''(\cdot) > 0$. Therefore, $G_m(x) - F(x) > 0$ for all x. This, together with Condition 1, suggests that the equilibrium effort level increases from $m = 1$ to $m \geq 2$. Q.E.D.

A4: Proof of Proposition 5

(i) This is true because for risk-neutral players, the symmetric equilibrium effort level is determined by $p_x b - 1 = 0$ under both the shared-prize contest and the *m*-round multiple-competition contest.

(ii) Suppose $u''(\cdot) < 0$ and $u'''(\cdot) > 0$. The symmetric equilibrium effort level is determined by $p_x b - 1 = 0$ under the shared-prize contest, and it is determined by (4') under the *m*-round multiple-competition contest. To prove that the solution to $p_x b - 1 = 0$ is larger than the solution to (4'), it is sufficient to show that at the effort level *x* where $p_x b - 1 = 0$, $G_m(x) < 0$. Indeed, substituting $p_x = 1/b$ into the expression of $G_m(x)$ in (4'), we have

$$
\begin{aligned}
G_m(x) = \frac{m}{b}\Bigg\{ & \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k} u\left(w+(m-k)\frac{b}{m}-x\right) - \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k} u\left(w+(m-1-k)\frac{b}{m}-x\right)\Bigg\} \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& < \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}\left[\frac{1}{2}u'\left(w+(m-k)\frac{b}{m}-x\right)+\frac{1}{2}u'\left(w+(m-1-k)\frac{b}{m}-x\right)\right] \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& \le \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k}\left[\frac{1}{n}u'\left(w+(m-k)\frac{b}{m}-x\right)+\left(1-\frac{1}{n}\right)u'\left(w+(m-1-k)\frac{b}{m}-x\right)\right] \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& = \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right)+\sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-1-k}\left(1-\frac{1}{n}\right)^{k+1} u'\left(w+(m-1-k)\frac{b}{m}-x\right) \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& = \sum_{k=0}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right)+\sum_{k=1}^{m}\binom{m-1}{k-1}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& = \left(\frac{1}{n}\right)^{m} u'(w+b-x)+\sum_{k=1}^{m-1}\binom{m-1}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& \sum_{k=1}^{m-1}\binom{m-1}{k-1}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right)+\left(1-\frac{1}{n}\right)^{m} u'(w-x) \\
& -\sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{n}\right)^{m-k}\left(1-\frac{1}{n}\right)^{k} u'\left(w+(m-k)\frac{b}{m}-x\right) \\
& = 0,
\end{aligned}
$$

where the first inequality is due to Lemma 2, the second inequality due to $n \ge 2$ and $u''(\cdot) < 0$, and the last equality due to Pascal's rule, namely $\binom{m-1}{k}+\binom{m-1}{k-1}=\binom{m}{k}$.

Q.E.D.