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"Clustering in communication networks with different-minded participants"

Thibault Laurent and Elena Panova

Toulouse School of Economics

Clustering in communication networks with different-minded participants.[*]

Thibault Laurent,[†] Elena Panova.[‡]

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Abstract

Clustering (a high proportion of "common friendships") is a prominent feature of social networks. We propose communication network formation game rationalizing clustering. In our game, communication through links involves frictions because the players have imperfectly correlated priors on the relevant state. Clustering reduces these frictions. The network commonly called "flower" is the efficient network constituting a Nash equilibrium in a wide range of circumstances.

Key words: network formation, communication with imperfectly correlated priors, clustering.

JEL codes: D82, D85, C72.

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[†]Toulouse School of Economics, University of Toulouse Capitole, CNRS. E-mail: thibault.laurent@tse-fr.eu

[‡]Corresponding author. Toulouse School of Economics, University of Toulouse Capitole. E-mail: e_panova@yahoo.com

1 Introduction.

Social networks contribute to diffusion of information and influence human behavior (Durlauf, 2004; Goyal 2007; Jackson, 2010; Topa and Zenou, 2015).[1] One prominent feature of social networks is clustering, that is, a high proportion of individuals with a common connection connected to each other (Jackson, 2010; Goyal 2007).

We build communication network formation game rationalizing clustering with a player's signal-extraction benefit. Initially, the players hold differentiated imperfectly correlated priors about the relevant state of the world. They simultaneously and unilaterally[2] build network links. Connection capacity by each player is limited and this limit is the same for any player. When the network is built, the players receive private signals on the state and sincerely communicate with network neighbours in two successive rounds.[3] A player's disutility or loss is measured by his remaining uncertainty about the state, that is, the posterior variance of the state.

In order to achieve tractability, we focus on the correlation of the players' priors being arbitrarily small (which is not equivalent to behavioral assumption that the players neglect correlations).[4] Furthermore, we assume that a player ignores common friendships outside his neighbourhood while updating

[1] Examples in the literature include the adoption of new product or technology, vote, school performance, delinquent behavior and health-related behaviors, such as smoking and degree of obesity.

[2] Appendix G considers an alternative network formation protocol with investments in links instead of unilateral link formation.

[3] A finite number of communication rounds reflects the players' impatience. For simplicity, there are two rounds. This comports nicely with Mobius, Phan and Szeidl (2015) who find that information travels no further than two steps in the conversation network.

[4] The matrix we are inverting when calculating a player's payoff is not diagonal.

his beliefs after the second round of communication (which comports nicely with evidence cited in Li and Tan, 2020).

Proposition 1 shows that a player's loss is a decreasing function of two parameters of network architecture: (i) the number of players with whom he is connected (either directly or indirectly) and (ii) the number of players with whom he has only common connections. Such players are termed "closed neighbours". They are valuable for a player because he learns their priors, and therefore extracts their signals without frictions. Signals by other connected players are extracted with "transmission" noise added by uncertainly about their priors.

Using the above insight, proposition 2 characterizes the efficient network. For tractability, we use egalitarian efficiency criterion.[5] For expositional purpose, we assume in the main text that the number of players and their connection capacity do not limit the efficiency of networks they can build.[6] Under this assumption, the efficient network is a set of completely connected components when transmission noise is sufficiently high (see figure 1 in section 4). Otherwise, it is "flower" network composed of the central hub connected to all the other players who are divided into interconnected subgraphs (see figure 2 in section 4). Either network has a high degree of clustering.

Under the above assumption on the number of players and their connection capacity, the efficient network constitutes a Nash equilibrium (proposition 3). In general, there may be some tension between equilibrium and efficiency. However, flower-like network remains the efficient network consti-

[5]Egalitarian efficiency criterion is relevant in that any player can find himself in the role of the least happy one. Our numerical findings suggest that our insights hold with more common utilitarian efficiency criterion.

[6]Appendix F relaxes this assumption.

tuting a Nash equilibrium when transmission noise is sufficiently low.

Our numerical findings suggest that this insight holds when the players are fully Bayesian and correlation of their priors takes relatively high values. Furthermore, it holds when egalitarian efficiency criterion is replaced with more common utilitarian efficiency criterion.

Literature review. Our work is inspired by Sethi and Yilditz (2012) who study public (dis)agreement. In their model, a finite set of agents with normally distributed differentiated priors and independent private signals about the relevant state of the world sequentially communicate through truthful public announcement of their beliefs until no further belief revision occurs. Sethi and Yilditz study whether- or not all distributed information is aggregated through communication, and compare the extent of disagreement under observable- and unobservable priors (observable priors are interpreted as understanding the thought processes and perspectives of others).

Our information structure and communication protocol follow Sethi and Yilditz (2012). However, our agents communicate only with their neighbours in the network which they initially build (public communication in Sethi and Yilditz corresponds to communication through completely connected network). This brings us to the literature on strategic network formation (see surveys in Bloch and Dutta 2010, Goyal 2007, Jackson 2004, 2010), more specifically, to the literature on strategic formation of communication networks. Two papers in that literature are closely related.

Bala and Goyal (2000) introduce unilateral network formation protocol: a player can link to any other player at a given cost. He receives a given (information) benefit from direct- and indirect connections. The benefit is allowed to decay in network distance (because of frictions). Bala and Goyal focus on strict Nash equilibria involving strict best responses (eliminating thereby strategy profiles in which the players are indifferent). When direct- and indirect connections are equally valuable (no decay), strict Nash equilibrium is either "star-" or empty network. This result is preserved for a specific payoff when the decay is sufficiently small.

Hojman and Szeidl (2008) consider unilateral network formation game with benefits from connections exhibiting decreasing returns to scale and decaying with network distance. A microfoundation is signal transmission with frictions. For this class of benefits, the problem of equilibrium multiplicity does not arise: the unique Nash equilibrium is periphery-sponsored "star". This result is preserved (under certain condition) if links are formed through bargaining with transfers.

As the above two papers, we consider unilateral network formation game (with a difference of opportunity- rather than direct cost of connections). We relate to Hojman and Szeidl (2008) in studying information benefit from connections in more detail. We endogenize frictions in information transmission: in our model, they are created by a player's uncertainly about the other players' priors. Clustering helps the players to learn each others' priors and eliminate these frictions. This gives a value to "closed" neighbours which is not present in the payoff considered by Hojman and Szeidl (2008). Therefore, while our "flowers" are built around the central hub, as their "stars", segregation and clustering by peripheral players is specific to our model.

The idea that clustering in social networks may create signal-extraction benefit relates us to Li and Tan (2020). They consider agents communicating through a given network in order to learn information about the relevant state. The agents know only their local networks. Failure to account for replicated signals leads to errors. Interconnections help avoiding such errors. This reason for signal-extraction benefit from clustering is different from ours. Yet another difference is that our communication network is endogenous.

Roadmap. The paper is organized as follows. Section 2 describes the model. Section 3 relates a player's payoff to network architecture. Section 4 describes the efficient network and shows that it constitutes a Nash equilibrium. Section 5 conducts robustness checks. Section 6 concludes.

2 Basic model.

M players, indexed with $i \in \{1, ..., M\}$, build a network in order to communicate through its links.

Network formation. Each player can connect with at most n other players (hence, the cost of connection is an opportunity- rather than a direct cost).[7] Network formation is simultaneous and unilateral:[8] Player i chooses a subset of players $L_M \subseteq M$, $|L_M| \leqslant n$ with whom he connects. A pair of players become connected iff at least one of them links to the other. Hence, each profile of linking choices $(L_1, ..., L_M)$ induces undirected network g.

We denote the set of all feasible networks with $\mathcal{G}$. We use common nota-

[7]This is akin to the model by Bloch and Dutta (2009) in which the players allocate a given endowment across links which results in links of different intensity.

[8]Appendix G shows that our insights are not due to unilateral network formation.

tion $g_{ij} \in \{0,1\}$ for an indicator of an edge $e_{ij} \in g$ between players i and j in network g.

For expositional purpose, we assume in the main text that the number of players M and a player's connection capacity n do not limit the efficiency of network which they can build.[9] It will become clear below that this assumption is formally expressed by the following parameter restrictions:

$$\text{there exists } m \in \mathbb{N} \text{ such that } M = 2n + 1 + (2n-1)m \tag{1}$$

$$\text{and there exists } l \in \mathbb{N} \text{ such that } M = (2n+1)l. \tag{2}$$

Players' priors and signals. When the network is built, the players receive independent private signals

$$s_i = x + \varepsilon_i, \text{ where } \varepsilon_i \sim N(0,1) \tag{3}$$

$$\text{on the relevant state of Nature } x \sim N(0,1). \tag{4}$$

Initially, the players have heterogenous imperfectly correlated priors about state x. Differentiated priors reflect differentiated manners in which the players process new information. Say, each player i considers a subset of available historical facts to be relevant for understanding the state. His estimator of the state conditional on this subset of facts is his prior p_i (see discussion in Sethi and Yilditz, 2012).

Player i's prior p_i is his private information and he cannot directly communicate this information to the other players (he cannot describe to the others the way in which he thinks). However, it is commonly known that the

[9] Appendix F relaxes this assumption.

players' priors are distributed according to a joint normal distribution:

$$p = (p_1, ..., p_M)^T \sim N(0, \sigma^2 \Pi), \tag{5}$$

where Π is M by M variance-covariance matrix with the following elements:[10]

$$\Pi_{jl} = \begin{cases} 1 & \text{if} \quad j = l; \\ \rho & \text{if} \quad j \neq l. \end{cases}$$

For concreteness, we assume that correlation ρ is positive (following interpretation of differentiated priors by Sethi and Yilditz, 2012, each player i assigns a positive probability to any other player j paying attention to some historical facts which i considers relevant). For tractability, we let correlation ρ be arbitrarily small.

Conditionally on his prior p_i, player i believes that the law of $(x, (p_j)_{j \in g \setminus i}, (\varepsilon_j)_{j \in g})$ denoted $\mathbb{P}_i$ is a multidimensional normal distribution given by

$$\mathbb{P}_i = \mathcal{N}(p_i, 1) \otimes \mathcal{N}(\rho p_i \mathbf{1}, \widehat{\Pi}) \otimes \mathcal{N}(0, \tau^2 I)$$

where $\mathbf{1} = (1, 1, ..., 1)^T$, I denotes M by M identity matrix and $\widehat{\Pi}$ is M by M matrix with the following elements:

$$\widehat{\Pi}_{jl} = \begin{cases} \sigma^2(1 - \rho^2) & \text{if} \quad j = l; \\ \sigma^2 \rho (1 - \rho) & \text{if} \quad j \neq l. \end{cases}$$

Hereafter, $\mathcal{L}_i$ denotes the law (or conditional law) of some variable under $\mathbb{P}_i$, $\mathbb{E}_i$ denotes the expectation under $\mathbb{P}_i$, $\mathbb{V}_i$ denotes the variance under $\mathbb{P}_i$ and $\mathbb{C}_i$ denotes the covariance under $\mathbb{P}_i$.

[10] We assume that all off-diagonal elements of the variance-covariance matrix Π are the same. The alternative assumption would complicate our expressions without altering our results qualitatively.

Communication. After receiving their signals on the state, the players communicate through the network in two discrete time periods ($t = 1, 2$). A finite number of periods reflects the players' impatience.

Communication is modeled as simultaneous truthful[11] announcement of beliefs summarized by estimator of the state (as in Sethi and Yilditz, 2012 and the literature they build on).[12] The first announcement or message $m_i(1)$ by player i to his network neighbors is his estimator of the state conditional on his private signal (3):

$$m_i(1) = \mathbb{E}_i\left(x \mid s_i\right). \tag{6}$$

After the first round of communication, each player i rationally updates his beliefs upon messages received from distance-1 network neighbours, that is, other players but himself in set

$$N_i = \{j \in g \mid g_{ij} = 1\}. \tag{7}$$

His second message $m_i(2)$ is his expectation of the state conditional on his private signal and the first messages by his network neighbours:[13]

$$m_i(2) = \mathbb{E}_i\left(x \mid s_i,\ \{m_j(1) \mid j \in N_i\}\right). \tag{8}$$

Posteriors. After the second round of communication with his neighbours, player i updates, once again, his beliefs about the state (and the

[11]The focus on truthful communication is without loss of generality: if strategic communication is allowed for, truthful communication is an equilibrium (see Remark 1).

[12]It is important for our insights that the players cannot transmit the set of "tagged" messages received from their neighbours and they cannot announce their priors.

[13]Here and below, we do not reflect network-dependence of network-dependent variables for notational convenience.

other players' priors). This updating is complicated by the fact that some of his neighbours may have common connections outside his neighborhood. For tractability, we assume that player i, otherwise Bayesian, ignores such common connections, which leads to (possibly misspecified) beliefs $\widehat{g}$ about network g.

In order to express the above simplifying assumption formally, we introduce the following notations. We divide the set (7) of i's neighbours into two subsets: set

$$\overline{N}_i = \{j \in N_i \mid N_j \subseteq \ N_i\} \tag{9}$$

of *closed* neighbours having only common connections with player i and set

$$\overset{\circ}{N}_i = N_i \backslash \overline{N}_i. \tag{10}$$

of *open* neighbours with at least one connection outside i's neighbourhood. We denote with

$$N_i' = \underset{j \in N_i}{\cup} N_j \backslash N_i \tag{11}$$

the set of distance-2 neighbours of player i. Finally, we introduce notations

$$d_i = |N_i| \, , \, \overline{d}_i = \left|\overline{N}_i\right| \, , \, \overset{\circ}{d}_i = \left|\overset{\circ}{N}_i\right| \text{ and } d_i' = |N_i'| \, ,$$

where letter "d" stands for "degree".

Using these notations, the above simplifying assumption can be formally expressed as follows. If player i has two open neighbours $j_1 \in \overset{\circ}{N}_i$ and $j_2 \in \overset{\circ}{N}_i$ both linked with player $k \in N_i'$, player i accounts for only one of these links, either link $e_{j_1 k}$ between players j_1 and k or link $e_{j_2 k}$ between players j_2 and k (it does not matter for the payoff which of the two links is accounted for). More generally, let us index the set of i's open neighbours as follows:

$$\overset{\circ}{N}_i = \left\{j_1, j_2, ..., j_{d_i - \overline{d}_i}\right\}. \tag{12}$$

Player i's beliefs $\widehat{g}$ about network g are as follows:

$$\widehat{g} = g \backslash \bigcup_{l=1}^{d_i - \overline{d}_i - 1} \left\{ e_{j_{l+1}k} \mid k \in N_i' \text{ and } g_{j_l k} = 1 \right\}, \tag{13}$$

where $e_{j_{l+1}k}$ denotes an edge between players j_{l+1} and k. While beliefs (13) are specific to indexation (12), this specificity is irrelevant for our results.

Payoffs. Player i's disutility or loss is his subjective posterior variance of the state x after communication with his neighbours:

$$l_i(g) = \mathbb{V}_i \left(x \mid s_i, \ \{m_j(t) \mid j \in N_i, \ t = 1, 2\}, \widehat{g} \right). \tag{14}$$

For example, we could think of player i taking private action and inquiring a loss which is equal to perceived squared distance between his action and the optimal action given by state x.

3 Network architecture and a player's payoff.

Outline. This section relates a player's loss (14) to network architecture. It begins with describing a player's learning from his closed- and open neighbours: Lemma 1 shows that a player learns the priors by his closed neighbours which allows him to deduce their private signals without any noise. At the same time, he deduces private signals by his open neighbours with noise added by his uncertainty about their priors. Therefore, closed neighbours are more valuable than open neighbours, as specified in proposition 1.

Below, we present some details for an interested reader. Further details are moved to Appendices A and B.

Learning from closed neighbours.

Lemma 1 *Any player i learns private signal s_j and prior p_j by any closed neighbour $j \in \overline{N}_i$.*

The proof follows directly from Sethi and Yilditz (2012). Consider some player i with at least one closed neighbour and one of his closed neighbours j. By standard formula for Gaussian updating, the first message by j is a linear combination of j's prior p_j and j's private signal s_j. The higher the variance τ^2 of the signal, the higher weight is put on the prior:

$$m_j(1) = \mathbb{E}_j[x|s_j] = \tfrac{\tau^2}{1+\tau^2} p_j + \tfrac{1}{1+\tau^2} s_j. \tag{15}$$

Player i deduces j's signal s_j from message (15) with noise $\tau^2 \left(p_j - \mathbb{E}_i(p_j \mid p_i)\right)$ associated with i's uncertainty regarding j's priors:

$$\left(1+\tau^2\right) m_j(1) - \tau^2 \mathbb{E}_i(p_j \mid p_i) = s_j + \tau^2 \left(p_j - \mathbb{E}_i(p_j \mid p_i)\right), \text{ where} \tag{16}$$

$$\mathbb{E}_i(p_j \mid p_i) = \rho p_i. \tag{17}$$

After the first round of communication, the players update their beliefs about the state. The second message by player j is a linear combination of his own first message (16), the sum of his neighbours' first messages and his prior p_j:

$$\begin{gathered} m_j(2) = \mathbb{E}_j \left(x \mid s_j, \{m_k(1)\}_{k \in N_j \setminus \{j\}} \right) = (1 - \lambda_j(d_j - 1)) m_j(1) + \\ \lambda_j \left(1+\tau^2\right) \sum_{k \in N_j \setminus \{j\}} m_k(1) - \lambda_j \tau^2 \rho (d_j - 1) p_j, \text{ where} \end{gathered} \tag{18}$$

$$\lambda_j = \tfrac{1}{(1+\tau^2)(1+\tau^2\sigma^2(1-\rho)) + (d_j - 1)(1 + (1+\tau^2)\tau^2\sigma^2(1-\rho)\rho)}.$$

Because player j is a closed neighbour by player i, player i "hears" the first messages $\{m_k(1)\}_{k\in N_j}$ by all j's neighbours. Therefore, player i can first deduce j's prior p_j from his second message (18). Put a bit informally, i learns j's prior by observing j's reaction to the first messages of his neighbours (which player i hears too). Once player i learns the prior by player j, he can deduce j's private signal s_j from his first message (15).

By lemma 1, player i can use the priors and signals by his closed neighbors to update as follows his beliefs about: (i) the state

$$\mathcal{L}_i(x|\,\{s_j\}_{j\in\bar{N}_i}) = \mathcal{N}\left(\overline{x}, v\right), \text{ where} \tag{19}$$

$$\overline{x}_i = \mathbb{E}_i\left(x \mid \{s_j\}_{j\in\bar{N}_i}\right) = \tfrac{\tau^2}{\tau^2+\bar{d}_i}p_i + \tfrac{1}{\tau^2+\bar{d}_i}\sum_{j\in\bar{N}_i} s_j \tag{20}$$

$$\text{and } v = \mathbb{V}_i\left(x \mid \{s_j\}_{j\in\bar{N}_i}\right) = \tfrac{\tau^2}{\tau^2+\bar{d}_i}; \tag{21}$$

and (ii) priors by the players outside his closed neighborhood

$$\overline{p} = \mathbb{E}_i\left(p_k \mid \{p_j\}_{j\in\bar{N}_i}\right) = \tfrac{\rho}{1+\rho(\bar{d}_i-1)}\sum_{j\in\bar{N}_i} p_j, \tag{22}$$

$$\mathbb{V}_i\left(p_k \mid \{p_j\}_{j\in\bar{N}_i}\right) = \tfrac{\sigma^2(1-\rho)(\rho\bar{d}_i+1)}{1+\rho(\bar{d}_i-1)}, \tag{23}$$

$$\mathbb{C}_i\left(p_k, p_l \mid \{p_j\}_{j\in\bar{N}_i}\right) = \tfrac{\sigma^2\rho(1-\rho)}{1+\rho(\bar{d}_i-1)}, \tag{24}$$

where $k \in g\backslash\overline{N}_i$ and $l \in g\backslash\overline{N}_i \cup \{k\}$.

Learning from open neighbours and a player's payoff. Now, consider some player i with at least one open neighbour and some of his open neighbours $j \in \mathring{N}_i$. By equation (16), in which i's expectation of j's priors conditional on i's own prior are replaced with i's expectation of j's priors conditional on all the priors by all i's closed neighbours, player i deduces a noisy signal on j's private signal:

$$\widetilde{s}_j(1) = (1+\tau^2)m_j(1) - \tau^2\overline{p} = x + \varepsilon_j + \tau^2(p_j - \overline{p}) \tag{25}$$

from j's first message (15). By equation (18), player i deduces signal

$$\widetilde{s}_j(2) = x + \tfrac{1}{d_{j\setminus i}}\left(\sum_{k\in N_j\setminus N_i}\varepsilon_k + \tau^2\sum_{k\in N_j\setminus N_i}(p_k - \overline{p}) - \tau^2\rho(d_j-1)(p_j-\overline{p})\right), \tag{26}$$

where $d_{j\setminus i} = |N_j \setminus N_i|$ from j's second message.

Let us index i's open neighbours with

$$j \in \left\{\overline{d}_i + 1, ..., d_i\right\}. \tag{27}$$

Let $\widetilde{s}(t) = (\widetilde{s}_{\overline{d}_i+1}(t),..., \widetilde{s}_{d_i}(t))$. Bayesian updating implies

$$\mathcal{L}_i(x, \widetilde{s}(1), \widetilde{s}(2)|\left\{s_j, p_j\right\}_{j\in\bar{N}_i}) = \mathcal{N}\left(\overline{x}_i\mathbf{1}, \begin{pmatrix} v & v\mathbf{1}^T \\ v\mathbf{1} & \Sigma \end{pmatrix}\right), \tag{28}$$

where $\overline{x}_i$ is given by equation (20), v is given by equation (21) and Σ is a square symmetric matrix of size $2\overset{\circ}{d}_i$. Recall now that we focus on correlation of priors ρ being arbitrarily small and we make player i assume that his open neighbours (would he has more than one) have no common friendships outside i's neighborhood. Under these conditions[14] the elements of matrix Σ are given by the following set of equations:

$$\Sigma_{k,l} = \begin{cases} v+z, & \text{if} \quad k = l \leqslant d_i; \\ v + \frac{z}{d_{k\setminus i}} & \text{if} \quad k = l > d_i; \\ v, & \text{if} \quad k \neq l, \end{cases} \tag{29}$$

[14]We describe the elements of matrix Σ without these conditions in Appendix B and use them in our numerical robustness checks in Section 5.

where according to indexation (27), indices k and l take values in set

$$\left\{\overline{d}_i + 1, ..., d_i, ..., \overline{d}_i + 2\overset{\circ}{d}_i\right\};$$

v is given by equation (21) and

$$z = \tau^2 \left(1 + \tau^2 \sigma^2\right). \tag{30}$$

Notably, matrix $\Sigma_{k,l}$ can be represented as a sum of two matrixes of dimension $2\overset{\circ}{d}_i$: matrix $vI\mathbf{1}^T$ (with elements equal to v) of rank 1 and a diagonal matrix zG. This decomposition allows us to invert matrix Σ (Miller, 1981). We then find the sum of elements of the inverted matrix, hence, an explicit expression for player i's payoff, using standard formula for Bayesian updating.

Proposition 1 (network architecture and a player's payoff). *Loss by player i in network g is determined by- and decreasing in two parameters of network architecture: (i) the joint number of his distance-1 and distance-2 neighbors $d_i + d_i'$ and (ii) the number of closed neighbours $\overline{d}_i$:*

$$l_i(g) = \tfrac{z}{z + n_i + \sigma^2 \tau^2 \overline{d}_i}. \tag{31}$$

Remark on communication protocol. Let us take a step away from our analysis to make the following remark regarding communication protocol.

Remark 1 (truthful vs. strategic communication). *If strategic communication is allowed for, truthful communication is an equilibrium.*

Indeed, consider the situation in which all players truthfully announce their estimate of the state during either round of communication and believe the others to do the same. If player i deviates by sending a message different from his true estimate of the state during some round of communication, he learns the same information from his neighbours' messages as when he does not deviate, as long as they believe him to tell the truth and react accordingly. Hence, i's payoff is given by equation (31).

4 Efficient and equilibrium network.

This section uses the loss function in proposition 1 to characterize the efficient network and show that it constitutes a Nash equilibrium.

Efficiency criterion. For the sake of tractability, we take egalitarian efficiency criterion. That is, a network is efficient iff it minimizes the loss by its least happy member. The set of the least happy players or "losers" of network g is denoted with

$$L(g) = \arg\max_{i \in g} l_i(g) \tag{32}$$

and the set of efficient networks is denoted with

$$\mathcal{G}^* = \arg\min_{g \in \mathcal{G}} \left(\max_{i \in g} l_i(g) \right). \tag{33}$$

Egalitarian efficiency criterion may be justified by the fact that any player can find himself in the role of the least happy player in the network. Furthermore, our numerical results presented in Section 5 suggest that our insights hold if we use more common utilitarian efficiency criterion.

Candidate efficient networks. By Proposition 1, the efficient network maximizes a combination of the total number of neighbours by the least happy player i and his number of closed neighbours, with the weight of closed neighbour being increasing in "transmission noise" $\sigma^2\tau^2$.

When transmission noise $\sigma^2\tau^2$ approaches infinity, it is most important to maximize the number of closed neighbours by the least happy player, suggesting that the efficient network is network composed of l completely connected components (see Figure 1).[15] Hereafter it is called *complete component network* and denoted with c.

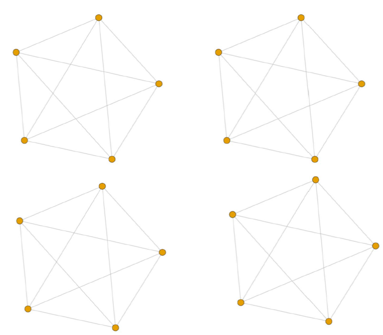

Figure 1: complete component network ($M = 20$, $n = 2$)

When transmission noise $\sigma^2\tau^2$ approaches zero, it is most important to maximize the total number of neighbours by the least happy player, while maximization of his closed neighbours is the secondary objective. The following network commonly called the *flower* and denoted with f seems a good candidate for being efficient. It is composed of the central "hub" h connected to everyone (hence, the highest possible total number of neighbours M is delivered to any player):

$$f_{ih} = 1 \ \forall \ i \in f,$$

[15]Note that by equation (2), complete component network c is feasible. The players can build it for example as follows: divide into groups of size $2n+1$; each group forms a circle; each player in a circle connects to n players on his right.

and $M - 1$ peripheral players divided into interconnected "petals":

$$N_i = N_j \text{ for any } i \neq h \text{ and for any } j \in N_i \backslash \{h\}$$

of which one (termed the "large petal") has size $2n$ and m others (termed "small petals") have size $2n - 1$ (see Figure 2).[16]

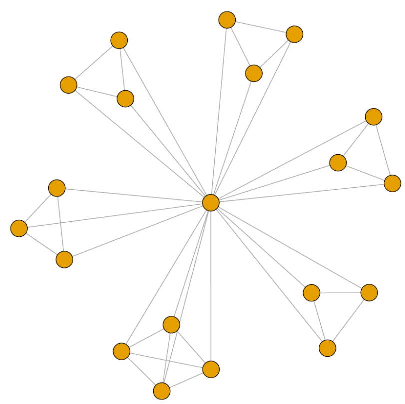

Figure 2: flower network
$M = 20$, $n = 2$.

The efficient network.

Lemma 2 *Suppose that some network is efficient and one if its least happy members has an open neighbour. Then, this network is the flower. Formally, if* $g \in \mathcal{G}^*$ *and* $\exists\, i \in L(g)$ *such that* $d_i > \overline{d}_i$ *then* $g = f$.

Constructive proof in Appendix C relies on the observation that any closed neighbour by any least happy i shall be at least as "happy" as i.

[16]By equation (1), it is feasible to build network f, for example as follows: $2n$ players and the central hub forming a circle and each player connects to the next n players on his right. The remaining $(2n-1)m$ players divide into m groups of size $2n-1$. Each group forms a circle. Each player in a circle connects to the central hub and $n-1$ players on his right.

Using this observation, the fact that a player can build at most n links and g is efficient, we prove that closed neighbourhood player i is completely connected subgraph of size $2n-1$. Furthermore, i and his closed neighbours share one open neighbour or "hub" who connects them to all other players. Hence, they can be visualized as a small petal. We proceed with considering another least happy player outside i's neighbourhood, would such player exist, to frame another small petal connected to the same hub, and so on until all the least happy players are organized in small petals connected to the central hub. The remaining players, all connected to the central hub, have closed degree at least $2n$. It is feasible iff they form a petal of size $2n$. By this construction, g is the flower.

Proposition 2. *The efficient network is either the flower or complete component network:*

$$\mathcal{G}^* = \begin{cases} \{c\} & \textit{if } \sigma^2\tau^2 > \frac{M-(2n+1)}{2}, \\ \{c, f\} & \textit{if } \sigma^2\tau^2 = \frac{M-(2n+1)}{2}, \\ \{f\} & \textit{otherwise.} \end{cases}$$

The formal proof in Appendix D uses lemma 2 by which the most efficient network is either flower f or some network g in which any looser i has no open neighbours. Network c is such a network maximizing the number of closed neighbours by its least happy member. By proposition 1, network c is more efficient than the flower f iff

$$\sigma^2\tau^2 \geqslant \frac{M-(2n+1)}{2}. \tag{34}$$

Proposition 3. *The efficient network constitutes a Nash equilibrium.*

The formal proof in Appendix E starts with showing that flower f is a Nash equilibrium. Indeed, by proposition 1, a player's deviation from strategy profile leading to formation of flower f is profitable only if it increases either the number of his closed neighbours or the total number of his neighbours (distance-1 and 2). In the flower network, any unilateral deviation (weakly) decreases either of these numbers.

Furthermore, if complete component network c is efficient it is an equilibrium. The reason is that when transmission noise lies above threshold (34), a player in one component of network c does not want to replace a link with a player in his component, (loosing thereby $2n$ closed neighbours), by a link with a player in a different component (gaining thereby $2n + 1$ open of neighbours).

Discussion. Note that flower network f has a high degree of clustering, which is relevant. Another relevant feature of the flower network is its "core-periphery" structure with the central hub, having many connections and relatively low clustering coefficient and the other individuals having considerably fewer connections.

Corollary 1. *Flower network f possessing prominent features of real networks is efficient and it constitutes a Nash equilibrium in a wide range of circumstances.*

While the game may have other Nash equilibria than those in proposition 2, the efficiency may be used as a refinement.

5 Robustness.

This section checks robustness of Corollary 1. Most details are moved to appendixes F and H.

General number of players M and connection capacity n. Propositions 2 and 3 were obtained under assumptions (1) and (2) on the number of players M and their connection capacity n. Appendix F relaxes these assumptions and shows that either the flower- or flower-like network(s) is/are efficient and constitute(s) a Nash equilibrium when transmission noise is sufficiently low (see propositions F.1 to F.4). The following examples illustrate this point. In all examples connection capacity is $n = 2$.

Examples 1 and 2 illustrate that when assumption (1) holds while assumption (2) fails, flower network f is the most efficient network constituting a Nash equilibrium (except if $n = m = 1$).

Example 1. Suppose that $M = 11$. Then, assumption (1) holds and it is therefore feasible to build flower network with 2 petals of size $2n - 1 = 3$ and one petal of size $2n + 1 = 5$. At the same time, assumption (2) fails. The size of the smallest component in any network composed of completely connected components is at most 3. Therefore, any such network is less efficient than the flower. Hence, the flower is the unique equilibrium network.

Example 2. Suppose that $M = 29$. Then, assumption (1) holds and it is therefore feasible to build flower network with 6 petals of size 3 and one petal of size 5. At the same time, assumptions (2) fails and it is therefore impossible to divide the players into completely connected components of size 5 each. There are at least two networks composed of completely connected components of size 4: network c_1 composed of 6 components of size 4 and one of size 5 and network c_2 composed of 5 components of size 5 and one of size 4. Flower network f is efficient, outperforming either network c_1 or c_2, iff transmission noise is weakly below threshold $M - 2n = 25$ (note that this threshold lies above that in propositions 2 and 3). Furthermore, while flower network f constitutes a Nash equilibrium, this is not true for either network c_1 or c_2, because any player with an excess connection capacity benefits from deviation.

Examples 3 and 4 illustrate that flower-like network(s) is/are efficient and constitute a Nash equilibrium transmission noise is sufficiently low.

Example 3. Suppose that $M = 10$. Assumption (2) holds and it is therefore possible to build network c composed of two complete components of size 5. At the same time, assumption (1) fails and we cannot build network f. However, we can build network $\widetilde{f}$ termed "symmetric flower" which is depicted in figure 3. Propositions 2 and 3 (hence, corollary 1) hold for flower network f being replaced for symmetric flower network $\widetilde{f}$.

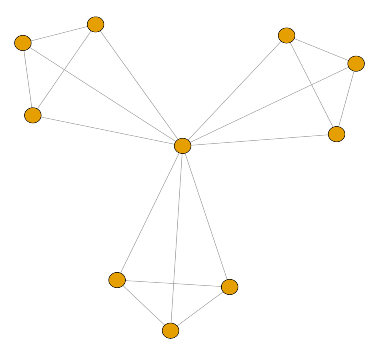

Figure 3: symmetric
flower $\widetilde{f}$ ($M = 10$, $n = 2$).

Example 4. Suppose that $M = 9$. Then, it is possible to build flower-like network termed "generalized flower" depicted in figure 4. Note that alternatively we could build flower-like network with one petal of size 4 and two petals of size 2. Either of these networks is efficient[17] and it constitutes a Nash equilibrium when transmission noise lies below threshold 0.43.

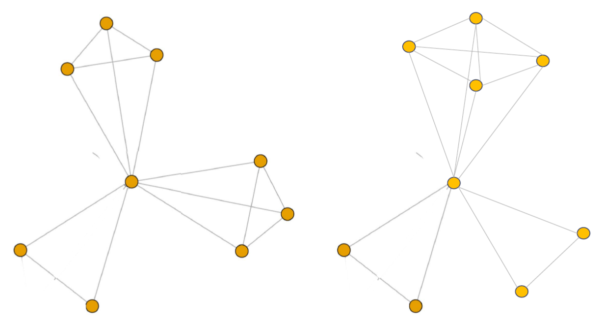

Figure 4: generalized flowers of level 2
($M = 9$, $n = 2$).

Different network formation protocol. Unilateral link formation may be viewed as an extremely asymmetric investments in links. Appendix G shows that corollary 1 does not rely on such asymmetry. Following Hojman and Szeidl (2008), it modifies network formation protocol as follows: The players simultaneously choose their investments in links. The link between a pair of players is formed iff their joint investment lies above some given threshold p. A player's loss is equal to the sum of information loss (14) and his investments in links. We focus on the situation in which is it feasible to build symmetric flower $\widetilde{f}$, and we show that it may be formed in equilibrium via a profile of strategies involving strictly positive contributions by any player in any link he holds (see proposition G.1).

[17] We could refine efficiency criterion by requiring the number of losers to be minimal.

For example, when $\tau^2 = 1$, $\sigma^2 = 4$, $M = 10$ and $n = 2$, symmetric flower depicted in the above figure 3 can be built as follows: the central hub invests up to 0.05 in each link, each peripheral player invests $p - 0.05$ in his link with the hub and $\frac{p}{2}$ in his link with any peripheral player in his petal (but himself). This profile of strategies is an equilibrium iff p lies in interval $[0.03, 0.21]$. Furthermore, for any p in interval $[0.03, 0.1]$, symmetric flower may be built via fully symmetric investments: any player invests $\frac{p}{2}$ in any of his links.

Numerical findings. For tractability, we have focused on correlation ρ being arbitrarily small. Furthermore, we have assumed that the players ignore common friendships outside their neighbourhood while updating their beliefs upon the second messages by their network neighbours. Finally, we have used egalitarian efficiency criterion. This section verifies numerically that propositions 2 and 3 hold when correlation ρ is relatively high and the players are fully Bayesian. Furthermore, proposition 2 holds for either egalitarian- and more common utilitarian efficiency criterion.

For the sake of computational feasibility,[18] we consider $M = 8$ players with two "hands" each, that is, $n = 2$. Note that it is feasible to build the flower network depicted in figure 5:

[18]The number of possible networks grows extremely fast in the number of players M. For example when $n = 2$, the number of different networks (up to isomorphism) is: 153 if $M = 6$, 955 if $M = 7$ and 9589 if $M = 8$.

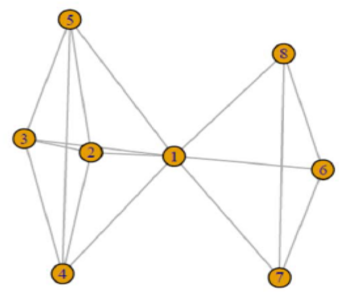

Figure 5: flower network
($M = 8$, $n = 2$).

Our model's predictions are the following (see proposition F.1 in appendix F)**:**

(A) The efficient network is: the flower network depicted in Figure 5 *if* $\sigma^2\tau^2 < 4$, *network composed of two complete components of size* 4 *(hereafter, "clustered network") if* $\sigma^2\tau^2 > 4$, *both these networks when* $\sigma^2\tau^2 = 4$.

(B) The most efficient Nash equilibrium is the flower in Figure 5.

We show numerically that both above predictions (A) and (B) hold when the players are fully Bayesian and the correlation ρ of their priors takes relatively high values. This is true for either efficiency criterion: egalitarian or utilitarian.

First, we consider prediction (A). We normalize signal's variance $\tau^2 = 1$, let ρ take values in set $\{0.1, 0.2, 0.35, 0.5, 0.7, 0.9\}$, vary "transmission noise" σ^2 and compare across all possible 9589 networks of size 8:[19] (i) loss by the least happy player and (ii) the average loss. Figure 6, illustrates our findings for $\rho = 0.35$ (the figures for smaller values of ρ are similar).

[19] We count different networks up to isomorphism.

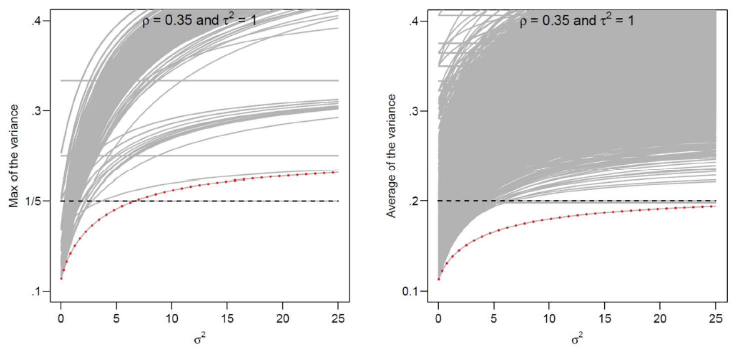

Figure 6. loss in different networks of size 8: by the least happy player (left); average normalized to component's size (right).

In the left of figure 6 we see the loss by the least happy player in different networks (which corresponds to egalitarian efficiency criterion). The loss by the least happy player in the flower network is marked with red dotted line. The loss by the least happy player (any player) in the clustered network is marked with dashed horizontal line. We observe that according to egalitarian efficiency criterion, the most efficient network is the flower if σ^2 is sufficiently low and clustered network otherwise. Figure 6 right illustrates that the same is true for utilitarian efficiency criterion. Hence, insight (A) holds qualitatively[20] for either efficiency criterion egalitarian or utilitarian.

Figure 7 shows that insight (A) fails for higher values of correlation ρ.

[20] Threshold of "transmission noise" σ^2 below which the flower network outperforms the clustered network is different from 4.

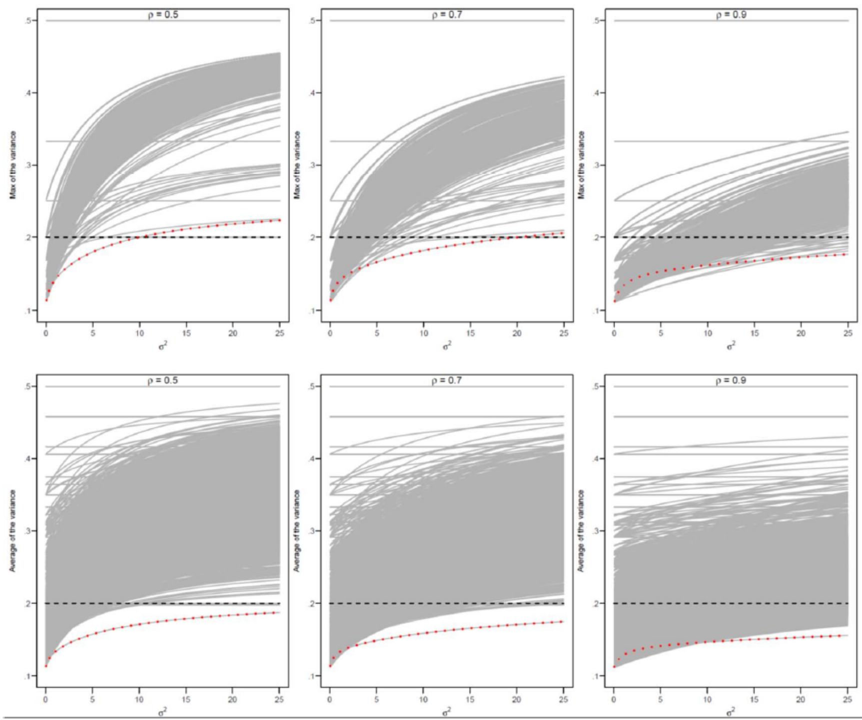

Figure 7: Performance of different networks for relatively high values of correlation ρ.

A possible reason is that when the correlation ρ is arbitrarily small distance-1 "open" neighbours and distance-2 neighbours are equally valuable for the player (recall equation (31)), while when correlation ρ takes higher values distance-1 "open" neighbours shall be more valuable than distance-2 neighbours. For sufficiently small transmission noise σ^2, it becomes most important for a player to have as many distance-1 neighbours as possible. Therefore, the efficient network (according to either egalitarian- and utilitarian efficiency criteria) is such as depicted in Figure 8:

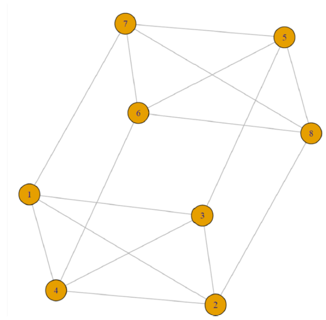

Figure 8: the most efficient (according to either egalitarian- or utilitarian criterion) network when $\rho \in \{0.5, 0.7, 0.9\}$ and σ^2 is sufficiently low.

Given that insight (A) holds only for ρ in set $\{0.1, 0.2, 0.35\}$, we verify insight (B) only for these values of ρ. We keep $\tau^2 = 1$, and vary "transmission noise" σ^2. We consider a profile of strategies leading to formation of the flower f (proposed equilibrium) and show that no player can benefit from unilateral deviation. The central hub cannot deviate in a profitable way, as in the proposed equilibrium he learns the signals by all the players and receives payoff $\frac{1}{9}$. Figure 9 depicts possible deviations by a peripheral player. Deviations 1 and 2 refer to a peripheral player from the small petal (for concreteness, player 8). Deviations 3 and 4 refer to a peripheral player from the large petal (for concreteness, player 5).

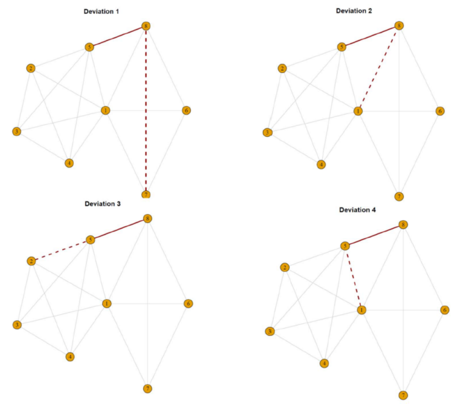

Figure 9: deviations by a peripheral player from the proposed equilibrium.

Figure 10 depicts loss by peripheral players 5 and 8 in the proposed equilibrium and under the above deviations, depending on "transmission noise" σ^2(parameter $\rho = 0.35$, the figures for smaller values ρ of are similar). We observe that either player 5 and 8 has strong incentives to comply with the proposed equilibrium strategy. Hence the flower network is a Nash equilibrium.

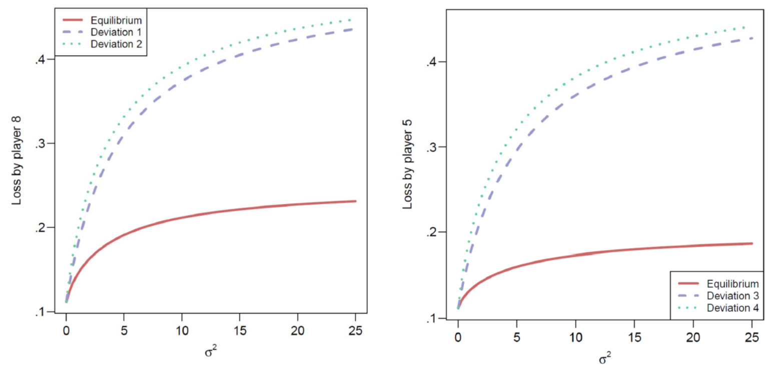

Figure 10: losses by player 5 (right) and player 8 (left) in the proposed equilibrium (red solid curve) and following possible deviations (blue dashed and green dotted curves).

Let us summarize our numerical findings.

Numerical findings: *Let the players be fully Bayesian. Let* $\tau^2 = 1$*. Let* ρ *take values in set* $\{0.1, 0.2, 0.35\}$*.*

(i) The efficient network is: flower f *if transmission noise* σ^2 *is sufficiently small; clustered network otherwise. This holds for both egalitarian- and utilitarian efficiency criteria.*

(ii) Flower f *constitutes a Nash equilibrium.*

Our numerical findings suggest that our analytical insights (A) and (B) obtained for correlation of priors ρ being arbitrarily small and the players being not fully Bayesian in the second round of updating hold (qualitatively) when the players are fully Bayesian provided that correlation of priors ρ is sufficiently small.[21] Furthermore, these insights extend to utilitarian efficiency criterion.

[21] It seems that the threshold of correlation ρ below which our model performs well lies somewhere inbetween 0.35 and 0.4.

6 Conclusion.

We have proposed a communication network formation game rationalizing clustering in social networks with signal-extraction benefit by their members. In our game flower (like) network possessing a high degree of clustering is the efficient network constituting a Nash equilibrium in a wide range of circumstances.

Naturally, clustering in real social networks may be due to other reasons than the one we emphasize. For example, by the fact that people matched in groups for some exogenous reasons (colleagues, parents of kids from the same school etc.) easily form friendships, which may be modeled as relatively low cost of connections within a group (Jackson and Rogers, 2005). While such cost-symmetries and our "signal-extraction" benefit create coherent effects on the architecture of communication networks, they probably have different effects on the quality of communication.

It may be curious to extend our game to a longer communication horizon (which corresponds to more patient players). Appendix H makes preliminary step in this direction by showing that when communication horizon is sufficiently long, the efficient network is a "wheel". This research direction may be, however, purely theoretical, given that Mobius, Phan and Szeidl (2015) find that information travels no further than two steps in the conversation network.

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Appendix A: Technical review.

The notations in this section are independent from the rest of the paper.

Mean and Variance of a linear combination of Gaussian variables

Consider K random variables $x_k \sim N(\mu_k, \sigma_k^2)$, $k = 1, ..., K$ and a set of constants $\{\alpha_k\}_{k=1...K}$.

$$\sum_{k=1}^{K} \alpha_k x_k \sim N(\mu, \sigma^2), \text{ where } \mu = \sum_{k=1}^{K} \alpha_k \mu_k \text{ and } \sigma^2 = \sum_{k=1}^{K} \alpha_k^2 \sigma_k^2.$$

Conditional multivariate normal distribution Consider n-dimensional colomn-vector of random variables x distributed normally with mean μ and n-by-n variance-covariance matrix Σ: $x \sim N(\mu, \Sigma)$. Consider the following partition of x, μ and Σ:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

where x_1 is k-dimensional colomn-vector, x_2 is $(n-k)$-dimensional colomn-vector, μ_1 is k-dimensional colomn-vector, μ_2 is $(n-k)$-dimensional colomn-vector, Σ_{11} is k-by-k matrix, Σ_{12} is k-by-$(n-k)$ matrix, Σ_{21} is $(n-k)$-by-k matrix, and Σ_{22} is $(n-k)$-by-$(n-k)$ matrix. Suppose that realization of the latter $(n-k)$ components of vector x is known: $x_2 = a$. Then,

$(x_1 \mid x_2 = a) \sim N\left(\widehat{\mu}, \widehat{\Sigma}\right)$, where

$$\widehat{\mu} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(a - \mu_2), \tag{35}$$

$$\widehat{\Sigma} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \tag{36}$$

Matrix inversion Consider n-by-n matrix A. The inverse matrix is

$$A^{-1} = \left[(-1)^{i+j}\frac{A_{i,j}}{\det A}\right], \tag{37}$$

where $A_{i,j}$ is the (i, j)-adjunct of matrix A, that is, the determinant of a matrix received from A by removing row i and column j. In particular,

$$\begin{pmatrix} a & b & ... & b \\ b & a & & ... \\ ... & & ... & b \\ b & ... & b & a \end{pmatrix}^{-1} =$$

$$= \tfrac{1}{(a-b)(a+b(n-1))} \begin{pmatrix} a+(n-2)b & -b & ... & -b \\ -b & a+(n-2)b & & ... \\ ... & & ... & -b \\ -b & ... & -b & a+(n-2)b \end{pmatrix}. \tag{38}$$

We introduce the following notation for the sum of elements of matrix A:

$$Sum(A) = e^t A e.$$

Note that the sum of elements of matrix (38) is equal to:

$$Sum\begin{pmatrix} a & b & ... & b \\ b & a & & ... \\ ... & & ... & b \\ b & ... & b & a \end{pmatrix}^{-1} = \tfrac{n}{a+b(n-1)}. \tag{39}$$

Furthermore, by Miller (1981),

$$(H+G)^{-1} = G^{-1} - \tfrac{1}{1+tr(HG^{-1})} G^{-1} H G^{-1}, \tag{40}$$

where matrices G and H have the same dimension, matrix $G+H$ is nonsingular and $rk(H) = 1$.

Appendix B: proof of proposition 1.

Proof of equation (15).

The vector (x, s_i) is distributed according to the following law

$$\mathcal{N}\left(\begin{pmatrix} p_i \\ p_i \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1+\tau^2 \end{pmatrix}\right),$$

By equations (35) and (36), conditional law of x given s_i under $\mathbb{P}_i$ is

$$\mathcal{L}_i(x|s_i) = \mathcal{N}\left(\tfrac{\tau^2}{1+\tau^2} p_i + \tfrac{1}{1+\tau^2} s_i, \tfrac{\tau^2}{1+\tau^2}\right).$$

Proof of equation (18). Let us index player i's neighbours but himself by $j \in \{1, ..., d_i - 1\}$. Consider the second period of communication. Recall equation (16). The state x and player i's signals are distribued by the following law:

$$\mathcal{L}_i \begin{pmatrix} x \\ x+\varepsilon_i \\ x+\varepsilon_1+\tau^2\left(p_1-\rho p_i\right) \\ \vdots \\ x+\varepsilon_{d_i-1}+\tau^2\left(p_{d_i-1}-\rho p_i\right) \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} p_i \\ \vdots \\ \vdots \\ \vdots \\ p_i \end{pmatrix}, \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 1+\tau^2 & 1 & \cdots & 1 \\ \vdots & 1 & & & \\ \vdots & \vdots & & \Sigma^{(1)} & \\ 1 & 1 & & & \end{pmatrix}\right) \tag{41}$$

where $\Sigma^{(1)}$ is d_i-1 by d_i-1 matrix with elements

$$(\Sigma^{(1)})_{m,l} = \begin{cases} 1+\tau^2+\tau^4\sigma^2\left(1-\rho^2\right) & \text{if} \quad m=l, \\ 1+\tau^4\sigma^2\rho\left(1-\rho\right) & \text{if} \quad m\neq l, \end{cases} \tag{42}$$

where $m = 1,..., d_i - 1$ and $l = 1,..., d_i - 1$.

By equations (35) and (36),

$$\mathcal{L}_i((x, x+\varepsilon_1+\tau^2\left(p_1-\rho p_i\right), ..., x+\varepsilon_{d_i-1}+\tau^2\left(p_{d_i-1}-\rho p_i\right))^t|s_i) = \mathcal{N}\left(\tfrac{s_i+\tau^2 p_i}{1+\tau^2}\mathbf{1}, \begin{pmatrix} \tfrac{\tau^2}{1+\tau^2} & \tfrac{\tau^2}{1+\tau^2}\mathbf{1}^T \\ \tfrac{\tau^2}{1+\tau^2}\mathbf{1} & \widetilde{\Sigma}^{(1)} \end{pmatrix}\right), \tag{43}$$

where $\widetilde{\Sigma}^{(1)}$ is d_i-1 by d_i-1 matrix with elements

$$\left(\widetilde{\Sigma}^{(1)}\right)_{m,l} = \begin{cases} \tfrac{\tau^2}{1+\tau^2}+\tau^2+\tau^4\sigma^2\left(1-\rho^2\right) & \text{if} \quad m=l, \\ \tfrac{\tau^2}{1+\tau^2}+\tau^4\sigma^2\rho\left(1-\rho\right) & \text{if} \quad m\neq l, \end{cases} \tag{44}$$

where $m = 1,..., d_i - 1$ and $l = 1,..., d_i - 1$. Equation (18) follows from equations (15) and (35).

By equations (36), (43) and (44),

$$\mathbb{V}_i\left(x \mid s_i, \{x+\varepsilon_j+p_j-\rho p_i\}_{j\in N_i\setminus\{i\}}\right) = \tfrac{\tau^2}{1+\tau^2} - \left(\tfrac{\tau^2}{1+\tau^2}\right)^2 Sum\left(\widetilde{\Sigma}^{(1)}\right)^{-1}. \tag{45}$$

By equations (38), (39), (40) and (44),

$$Sum\left(\widetilde{\Sigma}^{(1)}\right)^{-1} = \frac{d_i-1}{\tau^2+\tau^4\sigma^2(1-\rho)+(d_i-1)\left(\frac{\tau^2}{1+\tau^2}+\tau^4\sigma^2(1-\rho)\rho\right)}. \tag{46}$$

By equations (14) and (46),

$$\mathbb{V}_i\left(x \mid s_i, \{m_j(1)\}_{j\in N_i\setminus\{i\}}\right) = \frac{\tau^2\left(1+\tau^2\sigma^2(1-\rho)+(d_i-1)\tau^2\sigma^2(1-\rho)\rho\right)}{(1+\tau^2)(1+\tau^2\sigma^2(1-\rho)+(d_i-1)\tau^2\sigma^2(1-\rho)\rho)+d_i-1}. \tag{47}$$

Proof of lemma 1. Lemma 1 follows from equations (15), (18) and definition (9) of set $\overline{N}_i$.

Proof of equation (19).

Let us index i's closed neighbours bur himself with $j \in \left\{1, ..., \overline{d}_i - 1\right\}$.

$$\mathcal{L}_i\begin{pmatrix} x \\ x+\varepsilon_i \\ x+\varepsilon_1 \\ \vdots \\ x+\varepsilon_{\overline{d}_i-1} \end{pmatrix} = \mathcal{N}\left(\begin{pmatrix} p_i \\ \vdots \\ \vdots \\ \vdots \\ p_i \end{pmatrix}, \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 1+\tau^2 & 1 & \cdots & 1 \\ \vdots & 1 & 1+\tau^2 & & 1 \\ \vdots & \vdots & & & \vdots \\ 1 & 1 & 1 & \cdots & 1+\tau^2 \end{pmatrix}\right) \tag{48}$$

By equation (38),

$$\begin{pmatrix} 1+\tau^2 & 1 & \cdots & 1 \\ 1 & 1+\tau^2 & & 1 \\ \vdots & & & \vdots \\ 1 & 1 & \cdots & 1+\tau^2 \end{pmatrix}^{-1} = \frac{1}{\tau^2\left(\tau^2+\overline{d}_i\right)}\begin{pmatrix} \tau^2+\overline{d}_i-1 & -1 & \cdots & -1 \\ -1 & \tau^2+\overline{d}_i-1 & & -1 \\ \vdots & & & \vdots \\ -1 & -1 & \cdots & \tau^2+\overline{d}_i-1 \end{pmatrix}. \tag{49}$$

By equations (48), (49) and (35),

$$\mathbb{E}_i\left(x \mid \{s_j\}_{k\in\overline{N}_i}\right) = \tfrac{\tau^2}{\overline{d}_i+\tau^2}p_i + \tfrac{1}{\overline{d}_i+\tau^2}\sum_{j\in\overline{N}_i} s_j, \tag{50}$$

By equations (48), (49) and (36),

$$\mathbb{V}_i\left(x \mid \{s_j\}_{j\in\overline{N}_i}\right) = \tfrac{\tau^2}{\overline{d}_i+\tau^2}. \tag{51}$$

Proof of equations (22)-(24). Recall that the vector of priors p is distributed according to distribution (5). Let us order the players' priors so that the subvector of priors

$$p|_{\overline{N}_i} = (p_i, p_1, ..., p_{\overline{d}_i-1})$$

comes the last. Let us denote the variance-covariance matrix of vector $p|_{\overline{N}_i}$ by $\Pi|_{\overline{N}_i}$ (this is $\overline{d}_i$ by $\overline{d}_i$ matrix with elements 1 on the main diagonal and ρ elsewhere). By equation (38),

$$\left(\Pi|_{\overline{N}_i}\right)^{-1} = \tfrac{1}{(1-\rho)\left(1+\rho\left(\overline{d}_i-1\right)\right)}\begin{pmatrix} 1+\rho\left(\overline{d}_i-2\right) & -\rho & \cdots & -\rho \\ -\rho & 1+\rho\left(\overline{d}_i-2\right) & & -\rho \\ \vdots & & & \vdots \\ -\rho & -\rho & \cdots & 1+\rho\left(\overline{d}_i-2\right) \end{pmatrix}. \tag{52}$$

Let $p|_{g\setminus\overline{N}_i}$ be the vector of priors by all players outside $\overline{N}_i$. Equation (22) follows from equations (5), (52) and (35). Equations (23) and (24) follow from equations (5), (52) and (36).

The elements of variance-covariance matrix Σ in equation (28). Recall indexation (27). It is convenient to introduce notation w for the variance given by equation (23) and β for the covariance given by equation (24). We furthermore denote conditional variance (51) by v and conditional expectation (50) by μ. Let us also introduce notations

$$d_{m\setminus i} = |(N_m \setminus N_i)| \text{ and } d_{(m\cap l)\setminus i} = |(N_m \setminus N_i) \cap (N_l \setminus N_i)|,\ m \neq l. \tag{53}$$

By equation (19),

$$\mathcal{L}_i \begin{pmatrix} x \\ x + \varepsilon_{\overline{d}_i+1} + \left(p_{\overline{d}_i+1} - \overline{p}\right)\tau^2 \\ \vdots \\ x + \varepsilon_{d_i} + \left(p_{d_i} - \overline{p}\right)\tau^2 \\ x + \frac{1}{d_{\overline{d}_i+1\setminus i}} \sum\limits_{k\in N_{\overline{d}_i+1}\setminus N_i} \left(\varepsilon_k + \left(p_k - \overline{p}\right)\tau^2\right) - \frac{\rho\left(d_{\overline{d}_i+1} - 1\right)}{d_{\overline{d}_i+1\setminus i}}\tau^2\left(p_{\overline{d}_i+1} - \overline{p}\right) \\ \vdots \\ x + \frac{1}{d_{d_i\setminus i}} \sum\limits_{k\in N_{d_i}\setminus N_i} \left(\varepsilon_k + \left(p_k - \overline{p}\right)\tau^2\right) - \frac{\rho\left(d_{d_i} - 1\right)}{d_{d_i\setminus i}}\tau^2\left(p_{d_i} - \overline{p}\right) \end{pmatrix} =$$

$$N\left(\begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix}, \begin{pmatrix} v & \ldots & v \\ \vdots & \Sigma_{1,1} & \Sigma_{1,2} \\ v & \left(\Sigma_{1,2}\right)^T & \Sigma_{2,2} \end{pmatrix}\right), \tag{54}$$

where: μ is given by equation (20), v is given by equation (21),

$$(\Sigma_{11})_{m,l} = \begin{cases} v + \tau^2 + \tau^4 w & \text{if} \quad m = l, \\ v + \tau^4\beta & \text{if} \quad m \neq l, \end{cases}$$

$$(\Sigma_{12})_{m,l} = \left\{ \begin{pmatrix} v + \tau^4 \left(\beta - \frac{\rho(d_m - 1)}{d_{m \setminus i}} w \right) & \text{if} \quad m = l, \\ v + \tau^4 \beta \left(1 - \frac{\rho(d_l - 1)}{d_{l \setminus i}} \right) & \text{if} \quad m \neq l, \end{pmatrix} \right.$$

$$(\Sigma_{22})_{m,m} = v + \tfrac{\tau^2}{d_{m \setminus i}} + \tfrac{\tau^4}{d_{m \setminus i}} \left[w \left(1 + \tfrac{\rho^2 (d_m - 1)^2}{d_{m \setminus i}} \right) + \beta \left(d_{m \setminus i} - 1 - 2\rho(d_m - 1) \right) \right],$$

$$(\Sigma_{22})_{m,l} = v + \tau^2 \tfrac{d_{(m \cap l) \setminus i}}{d_{m \setminus i} d_{l \setminus i}} + w \tau^4 \tfrac{d_{(m \cap l) \setminus i}}{d_{m \setminus i} d_{l \setminus i}}$$

$$+ \beta \tfrac{\tau^4}{d_{m \setminus i} d_{l \setminus i}} \left(d_{m \setminus i} d_{l \setminus i} - d_{(m \cap l) \setminus i} - \rho(d_{m \setminus i}(d_l - 1) + d_{l \setminus i}(d_m - 1)) + \rho^2 (d_m - 1)(d_l - 1) \right),$$

$$m, l \in \left\{ \overline{d}_i + 1, ..., d_i \right\}, \ m \neq l. \tag{55}$$

Proof of equations (28) and (29). Equation (28) and set of equations (29) and follow from equation (54), when $\rho \to 0$ (and therefore, $w \to \sigma^2$ and $\beta \to 0$) and $d_{(m \cap l) \setminus i} = 0$.

Proof of equation (31). By set of equations (29),

$$\Sigma = H + zG, \text{ where}$$

$$H = vI\mathbf{1}^T ,$$

$$G_{m,l} = \begin{cases} 1 & \text{if} \quad m = l \leqslant d_i; \\ \frac{1}{d_{m \setminus i}} & \text{if} \quad m = l > d_i; \\ 0, & \text{if} \quad m \neq l \end{cases}$$

and z is given by equation (30). Note that

$$rk(H) = 1.$$

In order to use equation (40), we find:

$$\left(H\left(zG\right)^{-1}\right)_{m,l}=\begin{cases}\frac{v}{z} & \text{if} \quad l\leqslant d_i,\\ \frac{v}{z}d_{l\backslash i} & \text{if} \quad l>d_i,\end{cases} \tag{56}$$

$$1+tr(H\left(zG\right)^{-1})=1+\varphi v, \tag{57}$$

$$\text{where } \varphi=\frac{\overset{\circ}{d}_i+d_i'}{z},\ d_i'=|N_i'| \tag{58}$$

$$\left(\left(zG\right)^{-1}H\left(zG\right)^{-1}\right)_{m,l}=\begin{cases}\frac{v}{z^2} & \text{if} \quad \text{both } m \text{ and } l\leqslant d_i;\\ \frac{v}{z^2}d_{l\backslash i} & \text{if} \quad m\leqslant d_i \text{ and } l>d_i;\\ \frac{v}{z^2}d_{m\backslash i} & \text{if} \quad l\leqslant d_i \text{ and } m>d_i;\\ \frac{v}{z^2}d_{m\backslash i}d_{l\backslash i} & \text{if} \quad \text{both } m \text{ and } l>d_i.\end{cases} \tag{59}$$

By equations (40) and (56)-(59),

$$Sum\left(H+zG\right)^{-1}=\varphi-\frac{\varphi^2 v}{1+\varphi v}=\frac{\varphi}{1+\varphi v}. \tag{60}$$

By construction (standard properties of conditional independence) and equations (36) and (60),

$$\begin{gathered}\mathbb{V}_i(x\mid s_i,\ \{m_j(t)\mid j\in N_i,\ t=1,2\})=\\ \mathbb{V}_i(x\mid \{p_j,s_j\}_{j\in\overline{N}_i},\ \{\widetilde{s}_j(1),\widetilde{s}_j(2)\}_{j\in\overset{\circ}{N}_i})=v\left(1-v\mathbf{1}^T(\Sigma)^{-1}\mathbf{1}\right)=\\ v\left(1-v\frac{\varphi}{1+\varphi v}\right)=\frac{v}{1+\varphi v}.\end{gathered}$$

Appendix C: proof of lemma 2.

Step 1 notes that $f\in\mathcal{G}$ and $g\in\mathcal{G}^*$ implies $d_i\geqslant 2n-1$ for any $i\in g$. Indeed, suppose (by contradiction) that exist $i\in g$ with $d_i<2n-1$, then, $\overline{d}_i<2n-1$. By true inequality $n_i\leqslant M$ and proposition 1, $l_i\left(g\right)<\max_{i\in g} l_j\left(f\right)$, hence $g \notin \mathcal{G}^*$.

<u>Step 2</u> proves that closed neighbourhood by any least happy player i is completely connected subgraph of size $2n-1$, that is,

$$N_k = N_i \text{ for any } i \in L(g) \text{ any } k \in \overline{N}_i \text{ and any } g \in \mathcal{G}. \tag{61}$$

Consider $i \in L(g)$. By definition of $\overline{N}_i$, $N_i \subseteq N_k$, hence,

$$\begin{cases} d_k + d'_k \leqslant d_i + d'_i \\ \overline{d}_k \leqslant \overline{d}_i. \end{cases}$$

However, $N_i \subset N_k$, means that there exist $j \in N_i \backslash N_k$. If $j \in N_i$, then $\overline{d}_k < \overline{d}_i$. Otherwise, $d_k + d'_k < d_i + d'_i$. In either case, by proposition 1, $l_k(g) < l_i(g)$ which contradicts to $i \in L(g)$.

<u>Step 3</u> proves that for any $g \in \mathcal{G}^*$, if there exist $i \in L(g)$ such that[22] $\mathring{d}_i > 0$ then $\overline{d}_i \leqslant 2n-1$. Suppose, by contradiction, that there exist $i \in L(g)$ such that $\mathring{d}_i > 0$ and $\overline{d}_i \geqslant 2n$. By statement (61), $N_k = N_i$ for any $k \in \overline{N}_i$. It takes $\frac{\overline{d}_i(\overline{d}_i-1)}{2} \geqslant n\left(\overline{d}_i - 1\right)$ links to interconnect all players in $\overline{N}_i$ and another $2n\mathring{d}_i$ links to connect them to i's open neighbour(s). At the same time, players in N_i can build nd_i links. Therefore,

$$n\left(\overline{d}_i - 1\right) + 2n\mathring{d}_i \leqslant nd_i \text{ which implies } \mathring{d}_i \leqslant 1.$$

[22] Here and below we continue using notation (53).

Suppose that $\mathring{d}_i = 1$. Call h the unique open neighbour by i and his closed neighbours. Consider $g \backslash N_i$. Let $R = |g \backslash N_i|$. Players in $g \backslash N_i$ can build nR links of which at least one link goes to player h. Hence, their average degree is $\frac{2nR-1}{R} < 2n$, which implies that there exist player $j \in g \backslash N_i$ such that $d_j \leqslant 2n-1$, and so either $\overline{d}_j < 2n-1$ or $d_j + d'_j \leqslant 2n-1$. In either case, by proposition 1, $l_j(g) > \max_{i \in g} l_i(f)$ hence $g \notin \mathcal{G}^*$ (a contradiction).

<u>Step 4</u> proves that for any $g \in \mathcal{G}^*$, and for all $i \in L(g)$ such that $\mathring{d}_i > 0$, $\overline{d}_i = 2n-1$ and

$$d_i + d'_i = M. \tag{62}$$

Consider $i \in L(g)$ such that $\mathring{d}_i > 0$. By step 3, $\overline{d}_i \leqslant 2n-1$. By proposition 1, $\overline{d}_i = 2n-1$ and $d_i + d'_i = M$ (if $\overline{d}_i < 2n-1$ or $d_i + d'_i < M$, then $l_i(g) > \max_{i \in g} l_j(f)$ which contradicts $g \in \mathcal{G}^*$).

<u>Step 5</u> shows that if $g \in \mathcal{G}^*$ and there exist $i \in L(g)$ such that $\mathring{d}_i > 0$ then $\mathring{d}_i = 1$. Indeed, by step 3 and statement (61), all $2n-1$ players in $\overline{N}_i$ are interconnected, which leaves at most 1 "free hand" per player and $2n-1$ "free hands" overall to connect with other players. At the same time, by statement (61), all $2n-1$ players in $\overline{N}_i$ are connected to each of the players

in $\mathring{N}_i$, which requires $\mathring{d}_i\,(2n-1)$ links. Hence, players in $\mathring{N}_i$ build at least

$$\mathring{d}_i\,(2n-1)-(2n-1)=\left(\mathring{d}_i-1\right)(2n-1)$$

links to players in $\overline{N}_i$, which leaves them with a possibility to build at most

$$\mathring{d}_i n-\left(\mathring{d}_i-1\right)(2n-1)\leqslant 2n-1-\mathring{d}_i\,(n-1)\,. \tag{63}$$

link to $M-2n$ players in $g\backslash N_i$. Suppose that $\mathring{d}_i\geqslant 2$. Then the right-hand side of inequality (63) is weakly below 1. That is, players in $\mathring{N}_i$ build at most one link to players in $g\backslash N_i$. At the same time, by equation (62), each player in $g\backslash N_i$ is connected to at least one player in $\mathring{N}_i$, which requires $M-2n$ links. By inequality (63), all but one player in $g\backslash N_i$ use at least one of their hands to connect to at least one of the players in N_i. Hence, their average degree is at most

$$\tfrac{2(n(M-2n)-(M-2n-1))+M-2n}{M-2n}=2n-1+\tfrac{1}{M-2n}.$$

Therefore, there exist player $j\in g\backslash N_i$ such that $d_j\leqslant 2n-1$ and $\mathring{d}_j>0$, so $\overline{d}_j\leqslant 2n-2$. By step 1, $g\notin\mathcal{G}^*$ (a contradiction).

<u>Step 6</u> shows (by construction) that if $g\in\mathcal{G}^*$ and there exist $i\in L\,(g)$ such that $\mathring{d}_i>0$ then $g=f$. By statement (61), for any player $i\in L\,(g)$, we can visualize i and his closed neighbours as a "petal". By steps 4 and 5,

player i and his closed neighbours have one common open neighbour, say, h connected with all players in $g \backslash N_i$. Consider $M - 2n$ players in $g \backslash N_i$. If there exist player $i_1 \in (g \backslash N_i) \cap L(g)$, then, $d_{i_1} = 2n - 1$ and by steps 3 and 4, $d_{i_1} = 2n$, $d_{i_1} + d'_{i_1} = M$ and $N_j = N_{i_1}$ for all $j \in \overline{N}_{i_1}$. We can therefore visualize i_1 and his closed neighbours as the second petal connected to the first petal through h. Applying this argument repetitively, we end up in the situation where the players not organized in petals yet all have closed degree at least $2n$ and they are all connected to h.

Let us denote the set of these remaining players by $\mathcal{R}$. Note that by construction, none of the players in set $\mathcal{R}$ receives links from the players organized in the above petals. Therefore, relatively high degree by each of them is achieved through their own linking capacity plus possibly that of the central hub h. Players in set $\mathcal{R}$ and hub h together can build $n(|\mathcal{R}| + 1)$ links, increasing their own sum of degrees by $2n(|\mathcal{R}| + 1)$.[23] The hub h receives $|\mathcal{R}|$ links. The average degree by the players in set $\mathcal{R}$ is therefore equal to

$$\frac{\sum_{i \in \mathcal{R}} d_i}{|\mathcal{R}|} = \frac{2n(|\mathcal{R}|+1)}{|\mathcal{R}|} + 1. \tag{64}$$

[23] $|\mathcal{R}|$ denotes the cardinality of- (the number of players in) set $\mathcal{R}$.

Recall that it shall lie above $2n+1$, which implies

$$|\mathcal{R}| \leqslant 2n. \tag{65}$$

However, by construction of small petals, true equation $|g| = M$ and equation (1), we find

$$|\mathcal{R}| \geqslant 2n + (2n-1)k, \tag{66}$$

where $0 \leqslant k \leqslant m-1$. By equations (65) and (66), $|\mathcal{R}| = 2n$. This means that the hub h and $2n$ players in set $\mathcal{R}$ are interconnected, forming the large petal.

Appendix D: proof of proposition 2.

Step 1 shows that $g \in \mathcal{G}^*$ and $g \neq f$ implies that any player $i \in L(g)$ belongs to a completely connected component of size at least $2n$. By lemma 2, $g \in \mathcal{G}^*$ and $g \neq f$ implies $\mathring{d}_i = 0$ for any $i \in L(g)$. Hence, $N_i' = \varnothing$, and

$$n_i = \overline{d}_i < M. \tag{67}$$

Because $g \in \mathcal{G}^*$,

$$l_i(g) \leqslant \min_{j \in f} l_j(f). \tag{68}$$

By equation (67), inequality (68), set of definitions 2 and proposition 1, $\overline{d}_i \geqslant 2n$. Hence,

$$g \in \mathcal{G}^* \text{ and } g \neq f \text{ implies } \mathring{d}_i = 0 \text{ and } \overline{d}_i \geqslant 2n \text{ for any } i \in L(g). \tag{69}$$

By step 2 in Appendix C, $N_j = N_i$ for any $j \in \overline{N}_i$.

<u>Step 2</u> shows that for any $i \in g$: $\mathring{d}_i = 0$, $N_j = N_i$ for any $j \in \overline{N}_i$ and $\overline{d}_i \geqslant 2n$. By step 1, the statement is true for any $i \in L(g)$. Consider $g \backslash L(g)$, that is g without components containing the least happy players. If $g \backslash L(g) = \varnothing$, the statement of step 2 holds. Suppose $g \backslash L(g) \neq \varnothing$. By step 1, $|L(g)| \geqslant 2n$. Therefore, $|g \backslash L(g)| \leqslant M - 2n$. Therefore, $d_k + d'_k \leqslant M - 2n < M$ for any $k \in g \backslash L(g)$. Because $g \in \mathcal{G}^*$ and $g \neq f$, $l_k(g) \leqslant \min\limits_{j \in f} l_j(f)$ for any $k \in g \backslash L(g)$. By proposition 1, $\overline{d}_k \geqslant 2n$ for any $k \in g \backslash L(g)$. At the same time, the average degree by players in $g \backslash L(g)$ is $2n$ (each player can build n links, each link increases sum of degrees by 2). Therefore,

$$d_k = \overline{d}_k = 2n \text{ for any } k \in g \backslash L(g). \tag{70}$$

Suppose there exist $k \in g \backslash L(g)$ and $j \in \overline{N}_k$ such that $N_j \subset N_k$. Then, k is an open neighbour by j, which contradicts to statement (70).

<u>Step 3</u> proves that $g \in \mathcal{G}^*$ and $g \neq f$ implies $\overline{d}_i \leqslant 2n + 1$ for any $i \in g$.

Consider $g \in \mathcal{G}^*$, $g \neq f$ and some $i \in g$. By step 2, all players in i's closed neighbourhood are interconnected, which takes $\frac{\overline{d}_i(\overline{d}_i - 1)}{2}$ links. These players can build only $\overline{d}_i n$ links. Therefore,

$$\frac{\overline{d}_i(\overline{d}_i-1)}{2} \leqslant \overline{d}_i n,$$

which is equivalent to $\overline{d}_i \leqslant 2n+1$.

Step 4 shows that

$$\mathcal{G}^* = \begin{cases} \{c\}, \text{ if } \sigma^2\tau^2 > \frac{M-(2n+1)}{2} \\ \{c, f\}, \text{ if } \sigma^2\tau^2 = \frac{M-(2n+1)}{2} \\ \{f\}, \text{ otherwise.} \end{cases}$$

Recall that it is feasible to build network c as follows: players divide into groups of size $2n+1$, $2n+1$ players in a group form a circle and each player connects to n next players on his right. By steps 2 to 4, network c is the most efficient network in set $\mathcal{G} \backslash f$. By proposition 1, c is weakly more efficient than f iff inequality (34) is true.

Appendix E: proof of proposition 3.

Step 1. By equation (1) flower network may be build as follows: The players divide into groups of which one has size $2n+1$ and other m have size $2n-1$. Players in the group of size $2n+1$ interconnect, say, they form a circle and each player connects to n players on his right). One player in this group is marked with index h. Players in each group of size $2n-1$ form a circle. Each player connects to $n-1$ players on his right and to player h.

Consider a unilateral deviation by player i from the above strategy profile. By this deviation player i establishes a link with a player in a different "petal" (as he is connected with all players in his petal and the central hub h), scarifying a link with either one of the players in his petal or the central hub. As a result, his total degree does not increases while his closed degree decreases by $2n-2$ (he looses all closed neighbours but himself). By proposition 1, his loss goes up. Hence, flower network is an equilibrium.

Step 2. By equation (2), network c may be built as follows: The players divide into groups of size $2n+1$. Players in each group of size $2n+1$ interconnect, say, they form a circle and each player connects to n players on his right.

2.1. Consider a unilateral deviation by player i form the above strategy profile. By this deviation player i establishes a link with a player in a different component sacrificing a link with a player in his component. Thereby, he increases his total degree by $2n+1$ and he decreases his closed degree by $2n$ (he looses all closed neighbours but himself). By proposition 1, the deviation is unprofitable iff

$$\sigma^2\tau^2 \geqslant 1 + \tfrac{1}{2n}. \tag{71}$$

Hence, network c is an equilibrium iff inequality (71) holds.

2.2. Let us show that equations (1) and (2) imply

$$1 + \tfrac{1}{2n} < \tfrac{M-(2n+1)}{2}. \tag{72}$$

By equation (1), inequality (72) is equivalent to

$$(2n-1)m \geqslant 2 + \tfrac{1}{n},$$

which holds for any $m \geqslant 2$. By equations (1) and (2)

$$(2n+1)\,(l-1) = (2n-1)m > 0,$$

which implies $l \geqslant 2$, hence, $m \geqslant 2$.

Appendix F: relaxing parameter restrictions.

F.1. Relaxing parameter restriction (2).

Let us keep assumption (1) and relax assumption (2). Without loss of generality there exist $l \in \mathbb{N} \cup \{0\}$ and $q \in \mathbb{N} \cup \{0\}$, $q \leqslant 2n$ such that

$$M = l(2n+1) + q. \tag{73}$$

Note that by equation (1), $l > 0$. The main text focuses on $q = 0$. Suppose that $q > 0$.

Definition F.1. *The set of networks composed of completely connected components of which the smallest has size $2n$ is denoted*

$$\mathcal{C} = \{g \in \mathcal{C} \mid \textit{for any } i \in g: \ d_i \geqslant 2n \textit{ and } N_i = N_j \textit{ for any } j \in N_i\}. \quad (74)$$

Proposition F.1. *Suppose that M and n are such that equation (1) holds. Suppose, furthermore that q defined by equation (73) is positive. Then,*

$$\mathcal{G}^* = \begin{cases} \mathcal{C}, \textit{ if } q + l \geqslant 2n \textit{ and } \sigma^2\tau^2 > M - 2n; \\ \mathcal{C} \cup \{f\}, \textit{ if } q + l \geqslant 2n \textit{ and } \sigma^2\tau^2 = M - 2n; \\ f \textit{ otherwise.} \end{cases}$$

Proof.

Step 1 shows that if $q > 0$ and

$$q + l \geqslant 2n, \quad (75)$$

$$\mathcal{G}^* = \begin{cases} \mathcal{C}, \text{ if } \sigma^2\tau^2 > M - 2n \\ \mathcal{C} \cup \{f\}, \text{ if } \sigma^2\tau^2 = M - 2n \\ \{f\} \text{ otherwise.} \end{cases}$$

By step 2 in Appendix D, any efficient network different from f lies in set $\mathcal{C}$. Let us build a network in $\mathcal{C}$ maximizing the size of its smallest cluster. To this goal, let us divide the total number of players M into groups of sizes as equal as possible in the following way: Start with l groups of size $2n + 1$ and one "residual" group of size q and repetitively move one player from the largest existing group to the smallest one. After $2n - q \leqslant l$ steps, the size of the residual group is $2n$, hence, the difference between the sizes of any pair of groups becomes no higher than one. Once the procedure is over, let the players in each group interconnect. Thereby, we form a network in $\mathcal{C}$. Potentially, we could build other networks in $\mathcal{C}$ by continuing the above procedure as long as the distribution of the clusters' sizes remains constant. The size of the smallest cluster is $2n$. By proposition 1, this network is weakly more efficient than f iff

$$\sigma^2\tau^2 \geqslant M - 2n. \tag{76}$$

<u>Step 2</u> shows that when $q > 0$ and $q + l < 2n$, $\mathcal{G}^* = f$. Consider the procedure described in step 1. After l steps the size of the residual group is still below $2n$. If we continue the procedure until the distribution of groups' sizes becomes constant, at least one group will have size $2n - 1$.[24] Hence, set $\mathcal{C}$ is empty. By step 2 in Appendix D the unique efficient network is f.

Proposition F.2. *If $n = m = 1$, the most efficient equilibrium network is that in proposition F.1. Otherwise, the most efficient Nash equilibrium is network f.*

Proof.

[24] Note that by equation (1) the size of the smallest cluster is weakly above $2n - 1$.

Step 1. Suppose $n = m = 1$, so that $M = 4$. By step 1 in Appendix E, network f (a "star" with three peripheral players connected to the central hub) is a Nash equilibrium. Set $\mathcal{C}$ is a singleton. Its unique element is complete network connecting 4 players. Trivially, it is a Nash equilibrium. Hence, proposition F.1 describes not only the most efficient but also an equilibrium network.

Step 2. Suppose from now on that $n + m \geqslant 2$. Suppose first that inequality (75) holds. By proposition F.1, the efficient network is either f or a network is set $\mathcal{C}$, depending on transmission noise. However, no network in set $\mathcal{C}$ is a Nash equilibrium. Indeed, inequality (75) is equivalent to $q \geqslant 2n - l$. Therefore,

$$M = l(2n+1) + q \geqslant 2n(l+1), \text{ hence, } M > 2n,$$

which implies that any network in set $\mathcal{C}$ has at least two components. Consider the smallest component of network in set $\mathcal{C}$. Its size is $2n$. It takes $n(2n-1)$ links to build it. At the same time, the players in this component can build $2n^2$ links. Therefore, at least one of them has unused connection capacity, which he can use to establish a link with a player in a different component increasing thereby his total degree by at least $2n$. By proposition 1, this deviation is profitable. By step 1 in Appendix E network f is a Nash equilibrium. By step 2 in Appendix E, network f is the more efficient than any network in set $\mathcal{G}\backslash\mathcal{C}$. Hence, network f is the most efficient Nash equilibrium.

Step 2. Suppose finally that inequality (75) does not hold. Then, by proposition F.1, the most efficient network is f. By step 1 in Appendix E network f is a Nash equilibrium. Hence, network f is the most efficient Nash equilibrium.

F.2. Relaxing parameter restriction (1).

Let us now relax assumption (1). Without loss of generality,

$$M = 1 + (2n-1)(m+1) + r, \tag{77}$$

where $r \in \mathbb{N} \cup \{0\}$, $r \leqslant 2n-2$ and $m \in \mathbb{N} \cup \{0\}$. Equation (1) is equivalent to $r = 1$. We will now consider other possibilities.

F.2.1. Case $r = 0$.

First, suppose that $r = 0$, that is,

$$M = 1 + (2n-1)(m+1), \text{ where } m \in \mathbb{N} \cup \{0\}. \tag{78}$$

If $m = 0$, then $M = 2n$. The most efficient and equilibrium network is complete network organizing all players. Suppose hereafter that $m > 0$.

Definition F.2. *Symmetric flower $\widetilde{f}$ organizes M players. The central hub h is linked with all the peripheral players, that is, $\widetilde{f}_{ih} = 1 \; \forall \; i \in \widetilde{f}$. The peripheral players are divided into interconnected petals: $N_i = N_j$ for any $i \neq h$ and for any $j \in N_i \backslash \{h\}$. All petals have size $2n - 1$.*

Figure 3 in the main text illustrates symmetric flower with 3 petals of size $2n - 1 = 3$ each. Network $\widetilde{f}$ may be built in different ways. For example, it may be entirely built by peripheral players who organize in circles and then each connects to the central hub and the next player on his right. It may also be built with the hub's participation.

Proposition F.3. *Suppose that equation (78) holds. Propositions F.1 and F.2. hold with flower f being replaced for symmetric flower $\widetilde{f}$.*

Proof.

Step 1 shows that lemma 2 holds for f being replaced with $\widetilde{f}$. Steps 1 to 5 in Appendix C hold for f being replaced with $\widetilde{f}$. Step 6 holds for f being replaced with $\widetilde{f}$ and inequality (66) being replaced with

$$|\mathcal{R}| \geqslant (2n - 1)(m + 1 - k), \tag{79}$$

where $0 \leqslant k \leqslant m$. By equations (65) and (66), $|\mathcal{R}| = 2n-1$. This means that the hub h and $2n-1$ players in set $\mathcal{R}$ are interconnected, forming another petal.

<u>Step 2</u> notes that all steps in Appendix D and in the proof of proposition F.1. hold for f being replaced with $\widetilde{f}$ and the reference to lemma 2 being replaced with a reference to step 1 if this section.

<u>Step 3</u> shows that network $\widetilde{f}$ is a Nash equilibrium. The proof of proposition 3 in Appendix E goes through with f being replaced with $\widetilde{f}$: (i) the hub receives the lowest possible loss; (ii) any possible deviation by a peripheral player weakly decreases the number of closed neighbours without increasing the total number of neighbours (which is maximal).

<u>Step 4</u> notes that steps 2 and 3 in Appendix D hold. Step 4 in Appendix D holds for f being replaced with $\widetilde{f}$.

F.2.2. Case $r \geqslant 2$.

Consider $r \geqslant 2$, where r is defined by equation (77). Suppose we want to divide $M-1$ players in petals of sizes as equal as possible. This can be accomplished in the following way: start with one petal of size $2n$, m petals of size $2n-1$ and one petal of size $r-1$ and repetitively move one player from the largest existing petal to the smallest petal until the distribution of petals' sizes becomes constant (and difference between the petal's sizes becomes no higher than one). As a result, the size of (one of) the smallest petal(s), or, using common terminology,[25] *the level* of the resulting network is given by equation

$$s^* = \left[\frac{M-1}{m+2}\right]. \tag{80}$$

The resulting network belongs to the family of networks defined as follows.

Definition F.3. *A generalized flower of level-s is a network organizing M players. The unique central hub h is linked with all the peripheral players. The peripheral players are divided into interconnected petals: $N_i = N_j$ for any $i \neq h$ and for any $j \in N_i \backslash \{h\}$. The smallest petal(s) has(have) size s. The set of generalized flowers of level-s is denoted by Φ_s.*

Figure 4 in the main text depicts generalized flower of level $s^* = 2$ when $M = 9$ and $n = 2$. A generalized flower may be built via different profiles of strategies, including those in which all links are built by peripheral players and others in which the central hub builds some links.

[25] Bala and Goyal (2000) introduced this terminology for directed networks.

Proposition F.4. *Suppose that $r \geqslant 2$, where r is defined by equation (77).*

$$\text{If } \sigma^2\tau^2 < \tfrac{1}{2n+1-\max\left\{\left[\frac{4n}{3}\right],\, r\right\}}, \tag{81}$$

then $\mathcal{G}^* = \Phi_{s^*}$ and any network in set Φ_{s^*} is a Nash equilibrium.

Proof.

Step 1 proves inequality

$$s^* \geqslant \max\left\{\left[\tfrac{4n}{3}\right],\, r\right\}. \tag{82}$$

By equations (77) and (80),

$$s^* = \left[2n - 1 - \tfrac{2n-1-r}{m+2}\right]. \tag{83}$$

We focus on $r \geqslant 2$ and $m \geqslant 1$. By these inequalities and inequality (83),

$$s^* \geqslant \left[2n - 1 - \tfrac{2n-3}{3}\right] = \left[\tfrac{4n}{3}\right].$$

Furthermore, by equations (77) and (80),

$$s^* - r \geqslant \left[\tfrac{(2n-1)(m+1)+r}{m+2} - \tfrac{r(m+2)}{m+2}\right] = \left[\tfrac{(2n-1-r)(m+1)}{m+2}\right]. \tag{84}$$

By inequality (84) and $r \leqslant 2n-2$,

$$s^* \geqslant r. \tag{85}$$

Step 2 shows that if inequality (81) holds, then any efficient network is in set Φ_{s^*}. Suppose that some network $g \in \mathcal{G}^* \backslash \Phi_{s^*}$ is more efficient than a network in set Φ_{s^*}. Let $x = d_i + d_i^{'}$ and $y = \overline{d}_i$ where $i \in L(g)$. By proposition 1, our supposition is equivalent to

$$M - x < \sigma^2 \tau^2 (y - s^*) . \tag{86}$$

However, by definition F.3, $M - x \geqslant 1$. At the same time, given connection capacity of n links per player, $y \leqslant 2n + 1$. Therefore, inequality (86) contradicts to inequality (81).

Step 3 shows that any network in set Φ_{s^*} is a Nash equilibrium. Consider network φ in set Φ_{s^*}. It can be built in different ways including the following: peripheral players from the same petal locate in a circle, each of them connects to the central hub and the least of: $n-1$ players on his right and all remaining players in the petal. Note that some links are duplicated. However, no player has a profitable deviation from the above strategy profile. Indeed, by proposition 1, the only way to gain is to increase the number of closed neighbours. However, this cannot be accomplished by a unilateral deviation.

Appendix G: different network formation protocol.

Consider an extension in which network is formed as a result of investments in links.

This section briefly discusses our modeling approach to network formation. We follow the literature in which links are formed unilaterally. This may be viewed as an extremely asymmetric investments in links. We will now modify network formation stage of our model, following Hojman and Szeidl (2008), so as to show that the above flower-like networks may be formed as a result of more symmetric investments in links. Throughout this section we focus, for simplicity, on the case $r = 0$, where r is defined by equation (77).

Suppose that the network formation goes as follows. The players simultaneously choose their investments in links. Player i invests t_i^j in link with player j. The link between i and j is formed iff joint investment by players i and j lies above a given threshold p:

$$t_i^j + t_j^i \geqslant p.$$

The communication stage is unchanged. Disutility by player i is given by information loss (14) and the sum of i's investments in links:

$$\widetilde{l}_i(g) = l_i(g) + \sum_{j \in g} t_i^j. \tag{87}$$

We are not aiming at detailed analysis of the modified game with transfers. Rather, we will show that symmetric flower $\widetilde{f}$ may be formed in equilibrium via a profile of strategies involving strictly positive contributions from the hub: The hub invests sum

$$t_h^i = \frac{z\tau^2\sigma^2(2n-1)}{(z+M+\tau^2\sigma^2(M+1-2n))(z+M+(1+\tau^2\sigma^2))} \tag{88}$$

in linking with each peripheral player i. Player i makes the remaining investment required for linking to the hub:

$$t_i^h = c - \frac{z\tau^2\sigma^2(2n-1)}{(z+M+\tau^2\sigma^2(M+1-2n))(z+M+(1+\tau^2\sigma^2))}. \tag{89}$$

Peripheral players divide into $m+1$ petals of size $2n-1$, and each pair of players i and j from the same petal sponsor half of the link between them:

$$t_i^j = \frac{p}{2}. \tag{90}$$

Proposition G.1. *If the cost p of a link lies in interval*

$$\left[\frac{z\tau^2\sigma^2}{(z+M+2n\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}, \min\left\{\frac{z\tau^2\sigma^2(2n-1)}{(z+M+\tau^2\sigma^2(M+1-2n))(z+M(1+\tau^2\sigma^2))}+\right.\right.$$
$$\left.\left.\frac{z\left(M-2n+\tau^2\sigma^2(2n-2)\right)}{(z+2n+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}, \frac{2z\tau^2\sigma^2(2n-2)}{(z+M+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}\right\}\right], \tag{91}$$

then symmetric flower $\widetilde{f}$ may be formed in equilibrium via profile of strategies given by set of equations (88) to (90).

Proof. We show that no player has a profitable deviation from the above profile of strategies leading to formation of network $\widetilde{f}$.

<u>Step 1</u> shows convexity of loss (31) in the number of closed neighbours. By equation (31),

$$\frac{\partial l_i(g)}{\partial \overline{d}_i} < 0, \ \frac{\partial^2 l_i(g)}{\partial \overline{d}_i^2} < 0, \tag{92}$$

$$\frac{\partial l_i(g)}{\partial n_i} < 0, \ \frac{\partial^2 l_i(g)}{\partial (n_i)^2} < 0. \tag{93}$$

<u>Step 2</u> shows that the central hub h has no profitable deviation. Indeed, he can deviate by sacrificing any subset of links, saving thereby sum (88) multiplied by the cardinality of this subset. However, by convexity of loss (31) in the number of closed neighbours $\overline{d}_h$ proved in step 1 it suffices to check that h is not willing to deviate at the margin. That is, by set of equations (92), the hub has no profitable deviation iff he does not want to save sum (88) loosing a link with one of the peripheral players and thereby $2n-1$ closed neighbours:

$$t_h^i \leqslant \frac{z}{z+M+\tau^2\sigma^2(M+1-2n)} - \frac{z}{z+M(1+\tau^2\sigma^2)} = \frac{z\tau^2\sigma^2(2n-1)}{(z+M(1+\tau^2\sigma^2))(z+M+\tau^2\sigma^2(M+1-2n))}. \tag{94}$$

This condition is met as an equality: hub's contribution (88) in linking with a peripheral player i is equal to his marginal benefit from this link.

Step 3 shows that peripheral player i has no profitable deviation. By step 1, peripheral player i has no profitable deviation if set of the following three conditions is met:[26]

3.1. Peripheral player i does not want to save $\frac{p}{2}$ sacrificing a link with one of his closed neighbours and loosing thereby all closed neighbours but himself. This is true iff:

$$\frac{p}{2} \leqslant \frac{z}{z+M+\tau^2\sigma^2} - \frac{z}{z+M+\tau^2\sigma^2(2n-1)} = \frac{z\tau^2\sigma^2(2n-2)}{(z+M+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}$$

$$\text{equivalently, } p \leqslant \frac{2z\tau^2\sigma^2(2n-2)}{(z+M+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}. \tag{95}$$

3.2. Peripheral player i does not want to save $p - t_h^i$ sacrificing the link with the hub and losing thereby all closed neighbours but himself and all distance-2 neighbours. This is true iff:

$$p - \frac{z\tau^2\sigma^2(2n-1)}{(z+M(1+\tau^2\sigma^2))(z+M+\tau^2\sigma^2(M+1-2n))} \leqslant \frac{z}{z+2n+\tau^2\sigma^2} - \frac{z}{z+M+\tau^2\sigma^2(2n-1)} = \frac{z\left(M-2n+\tau^2\sigma^2(2n-2)\right)}{(z+2n+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}.$$

$$\text{equivalently, } p \leqslant \frac{z\tau^2\sigma^2(2n-1)}{(z+M(1+\tau^2\sigma^2))(z+M+\tau^2\sigma^2(M+1-2n))} + \frac{z\left(M-2n+\tau^2\sigma^2(2n-2)\right)}{(z+2n+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}. \tag{96}$$

[26] Recall that i cannot increase its total number of neighbours.

3.3. Player i does not want to link with a player on distance 2, so as to gain one closed neighbour (recall that i perceives this player to be linked only with the hub). This is true iff

$$p \geqslant \frac{z}{z+M+2n\tau^2\sigma^2} - \frac{z}{z+M+\tau^2\sigma^2(2n-1)} = \frac{z\tau^2\sigma^2}{(z+M+2n\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}. \qquad (97)$$

3.4. The set of inequalities (94) to (97) is equivalent to p belonging to interval (91). Note that the right-hand side of inequality (97) lies below that of inequality (95). Indeed, inequality

$$\frac{z\tau^2\sigma^2}{(z+M+2n\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))} < \frac{2z\tau^2\sigma^2(2n-2)}{(z+M+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}$$

is equivalent to true inequality

$$\frac{1}{z+M+2n\tau^2\sigma^2} < \frac{2(2n-2)}{z+M+\tau^2\sigma^2}.$$

Furthermore, the right-hand side of inequality (97) lies below that of inequality (96). Indeed, inequality

$$\begin{aligned}\frac{z\tau^2\sigma^2}{(z+M+2n\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))} &< \frac{z\tau^2\sigma^2(2n-1)}{(z+M(1+\tau^2\sigma^2))(z+M+\tau^2\sigma^2(M+1-2n))} \\ &+ \frac{z\big(M-2n+\tau^2\sigma^2(2n-2)\big)}{(z+2n+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))}\end{aligned}$$

follows from inequality

$$\frac{z\tau^2\sigma^2}{(z+M+2n\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))} < \frac{z\big(M-2n+\tau^2\sigma^2(2n-2)\big)}{(z+2n+\tau^2\sigma^2)(z+M+\tau^2\sigma^2(2n-1))},$$

implied by true inequality

$$\frac{1}{z+M+2n\tau^2\sigma^2} < \frac{2n-2}{z+2n+\tau^2\sigma^2}.$$

Hence, interval (91) is full.

Appendix H: efficient network with sufficiently long communication.

Suppose that

$$M = n + 1 = k, \text{ where } k \in \mathbb{N},\ k > n.$$

Consider network formed as follows. The players locate in a circle, each player builds links with n players on his right. The resulting network is a "wheel". Figure A.1 illustrates the wheel network with $M = 11$ players each disposing $n = 2$ "hands".

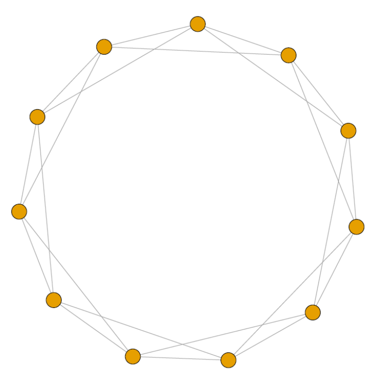

Figure A.1: "wheel"
network ($M = 11$, $n = 2$).

Suppose the players can talk for

$$T \geqslant \frac{M-1}{n} \text{ periods}$$

(at least five periods in our example in Figure $A.1$). Straightforward extension of Lemma 1 shows that each player in a wheel network learns the signals and priors by all players in his component. To illustrate the argument, each player in the wheel in Figure $A.1$ receives 4 messages from his neighbours in each communication period. This amounts to 20 messages in five communication periods. From these messages, player i can deduce 20 “unknowns”: the priors and the signals by other players in his component. Hence, the wheel network is efficient.