

Potential Competition in Preemption Games*

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Abstract

We consider a preemption game with two potential competitors who come into play at some random secret times. The presence of a competitor is revealed to a player only when the former moves, which terminates the game. We show that all perfect Bayesian equilibria give rise to the same distribution of players' moving times. Moreover, there exists a unique perfect Bayesian equilibrium in which each player's behavior from any time on is independent of the date at which she came into play. We find that competitive pressure is nonmonotonic over time, and that private information tends to alleviate rent dissipation. Our results have a natural interpretation in terms of eroding reputations.

Keywords: Preemption Games, Potential Competition, Private Information.

JEL Classification: C73, D82.

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1 Introduction

Timing is a key feature of many economic decisions, like the adoption of a technology, the marketing of a product, or the patenting of an invention. It is also an important dimension of many decisions that are less directly of an economic nature, such as, for instance, the decision by an academic when to publicize new results or theories. As these examples suggest, timing is particularly important when innovations are at stake. Indeed, in many cases, the success of a new idea or of a new product does not only depend on their intrinsic features, or on how much effort has been put into their design, but also on exogenous factors, like their complementarity or substitutability with existing ideas or products, the quality of infrastructures, the maturity of a market, or the development of institutions. Because these factors evolve over time in a way that is largely beyond the control of innovators, the latter must be careful about the timing of their decisions.

Timing decisions are also complex because they are very often made under competitive pressure, in environments in which there is little or no value in being a follower. For instance, the discoverer of a new production process must not only care about how well his invention fits the current technological and economic context, but also about the presence of competitors who may preempt him by developing and patenting a similar process, and thus deprive him of the fruit of his efforts. In that respect, a peculiarity of competitive innovation is that it is often difficult for the parties involved to actually identify their competitors, precisely because innovators have a strong incentive to keep their inventions secret in order to let them mature optimally. Take the example of an academic who wakes up one morning with a promising idea. After some bibliographical search, he realizes that nobody has written on this topic yet. Alas, as we all too well know, this does not mean that nobody is currently working on it. Indeed, somebody else may already have had a similar idea, and may be waiting for the optimal time to publish it; even if this is not the case, the longer our academic develops his idea, the more likely it becomes that a competitor eventually gets the same idea and finds therefore himself in a position to preempt him. This paper accordingly studies strategic timing decisions in a game where innovation opportunities randomly and secretly accrue to players over time.

To do so, we consider a simple preemption game with two potential competitors. Its distinctive feature is that each player effectively comes into play at some exogenous random time that is her private information, and whose distribution is common knowledge. In the light of the above examples, the time at which a player comes into play may be interpreted as the time at which she discovers a new idea. From that time on, she can make a move

whenever she likes. Making a move may for instance consist in patenting the new idea. By construction, it is only once one of the players has moved that it becomes common knowledge that there actually was an active player in the game: indeed, when a player comes into play, she does not know whether or not the other player is already present, and, in the latter case, she does not observe when her competitor eventually comes into play. As soon as one of the players moves, the game is effectively terminated, and this player receives a payoff that depends only on the time at which she moved, while the other player receives nothing. The payoff from moving first is assumed to be positive and increasing over some range, which generates a first mover advantage, and leads to the basic tradeoff between letting the value of one's idea grow, and increasing the probability of being preempted by a competitor one did not identify. There are two sources of non-stationarity in the model: the first is that the payoff from moving first is not constant over time, while the second is that the probability that a player will come into play in the next instant conditional on not having come into play yet may increase over time.

We characterize the perfect Bayesian equilibria of this game. First, we prove a uniqueness result: all equilibria give rise to the same distribution for each player's moving time. When the hazard rate of the distribution of the random time at which players come into play is higher than the growth rate of the payoff from moving first, which is the case beyond some time threshold T_2^* , the unique continuation equilibrium involves for each player to move as soon as she comes into play, because the risk of being preempted is too high compared to the benefits of waiting any longer. By contrast, the equilibrium distribution of each player's moving time is such that players who come into play before time T_2^* are indifferent between moving at any point in some time interval $[T_1^*, T_2^*]$. Second, we describe different ways of constructing an equilibrium over this time interval. An equilibrium in pure strategies can be easily exhibited, but it has the undesirable feature that it crucially depends on the time at which players come into play, which is a payoff irrelevant variable. We show that one can construct a unique *simple* equilibrium, in which, if a player that came into play at some time t has not moved yet by time $t' > t$, then she should not behave from time t' on any differently than if she had come into play at time t' . A key feature of this equilibrium is that players who come into play before time T_2^* move according to mixed strategies: moving early is associated with low preemption risk but also low payoffs, while moving later is associated with higher payoffs but also higher preemption risk.

Our explicit characterization of the distribution of each player's moving time in any equilibrium, as well as of the players' strategies in the simple equilibrium, naturally lends

itself to an analysis of how competitive pressure evolves over time, as well as to comparative statics analyses with respect to the distribution of the random time at which players come into play. We find that competition is fiercer near the lower bound T_1^* of the support of the distribution of equilibrium moving times; it then tends to decrease until time T_2^* , after which it increases again, reflecting that players who come into play after time T_2^* do not delay their move, unlike those who come into play before time T_2^* . In the context of patenting, this leads for instance to the novel prediction that patenting activity should be a U-shaped function of time. Overall, competition becomes less fierce whenever the distribution of the random time at which players come into play undergoes a positive shift in the sense of the monotone likelihood ratio property. In particular, this reduces the extent to which first mover rents are dissipated in equilibrium.

Finally, one can interpret our results in terms of reputation. The basic idea is that, over the time interval $[T_1^*, T_2^*]$, it is worthwhile for each player to make her competitor believe that she has not come into play yet: this induces her competitor to delay one's move, which allows her to reap potentially higher payoffs. Yet, an interesting feature of our model is that reputations tend to erode: as time elapses, it becomes increasingly difficult to maintain a reputation of not having come into play yet.

Related Literature Our paper is rooted in the literature on the strategic adoption of a new technology, and especially that dealing with first mover advantages.¹ In the case where firms can commit to adoption times, Reinganum (1981) shows that the equilibrium exhibits diffusion, and that the firm that adopts first enjoys a higher profit than its rival. By contrast, in the polar case where firms cannot commit to adoption times, Fudenberg and Tirole (1985) show that there exists a subgame-perfect equilibrium in which no first mover advantage can be sustained, and payoffs are equalized across firms.² An interpretation of our results is that they bridge the gap between the Reinganum's (1981) precommitment model and Fudenberg and Tirole's (1985) no commitment model. Indeed, if the rate at which players come into play is low, then, conditional on coming into play, each player behaves much like the first mover in Reinganum (1981), because she believes that it is very unlikely that she will actually face competition. By contrast, if the rate at which players come into play grows without bound, then each player believes that it is very likely that she has a competitor from the onset, so that the threat of preemption is high, which leads to Fudenberg and Tirole's (1985) full rent dissipation result in the limit.

¹See Reinganum (1989) and Hoppe (2002) for useful surveys of the literature.

²Given our simplifying assumption that the follower receives a zero payoff, this would be the only subgame-perfect equilibrium in the complete information version of our game.

These seminal contributions have been extended in several directions. Riordan (1992) shows that price and entry regulation may make preemption strategies less attractive, and slow down the pace of technology adoption. Dutta, Lach, and Rustichini (1995) model the strategic behavior of firms in the development phase of R&D when firms choose the quality of the good they produce; they show that there are two types of equilibria: a preemption equilibrium, and a maturation equilibrium that induces staggered innovations. Hoppe and Lehmann-Grube (2005) extend the model of Fudenberg and Tirole (1985) by introducing the possibility for firms to have access to a better technology in case of late adoption. As a result, the leader's payoff need not be hump-shaped, and may therefore exhibit several local maxima. They provide conditions under which there exists a unique equilibrium outcome, in which there is either rent equalization or a second mover advantage. Finally, Argenziano and Schmidt-Dengler (2007) extend Fudenberg and Tirole's (1985) duopoly model to an arbitrary number of firms.

Our model is closely related to Hendricks (1992). In his setup, firms compete for the adoption of a new technology, but may differ in their innovative capabilities: a firm may be an innovator, and thus be unconstrained in the time at which it adopts the new technology, or an imitator, and then be constrained to act as a follower. Each firm is uncertain about the innovative capability of its rival. Hendricks (1992) constructs a reputational equilibrium in which each firm delays adoption in order to convince its rival that it is an imitator. He shows that this mechanism alleviates rent dissipation compared to Fudenberg and Tirole's (1985) complete information model. In contrast with Hendricks (1992), a distinctive feature of our model is that players' types are not fixed at time 0; rather, players may come into play continuously over time, as new ideas accrue to them. Reputations are then more difficult to maintain, and tend to erode over time.

Hopenhayn and Squintani (2008) study a preemption game with private information in which players' information stochastically evolves over time. This may for instance capture the fact that R&D competitors may improve their innovations over time, while keeping breakthroughs secret until patenting their innovations. Hopenhayn and Squintani (2008) construct an equilibrium in which each player terminates the game as soon as her private state crosses a time decreasing threshold which is characterized as the solution to an ordinary differential equation. They also find that durations are longer than when information is public, but possibly shorter than in the collusive outcome. Our model is simpler in that we suppose that players can experience only one breakthrough, that corresponds to the time at which they come into play. A distinctive contribution of our analysis is that we prove the

existence of a unique equilibrium distribution for each player’s moving time, which we can fully characterize, and which allows us in turn to explicitly compute the hazard rate of the first moving time.

Brunnermeier and Morgan (2009) propose a theoretical and experimental investigation of clock games, in which players receive, at a random point in time, some signal about a payoff relevant state variable. Players’ clocks are de-synchronized as a result of this randomness. When there is a tradeoff between the gains from waiting and the fear of being preempted, a player’s timing decision crucially depends on her predicting the other players’ clock times. Because of the stationarity that is built in their model, Brunnermeier and Morgan (2009) can construct equilibria where each player waits a deterministic amount of time before exiting the game. This contrasts with our model, where there is an optimal date at which all players would like to move if they were not threatened by preemption. As a result, and no matter whether the equilibrium is in pure or mixed strategies, players in our model typically do not choose to wait the same amount of time before moving.

Continuous-time timing games are known to generate modeling issues. For instance, Fudenberg and Tirole (1985) emphasize that, even in simple examples such as the “grab the dollar” game, it is not a priori obvious how to formulate the continuous-time version of the game as a limit of its discrete-time version when the time interval between consecutive periods goes to zero.³ To overcome this difficulty, they enlarge each player’s strategy space in the continuous-time game by introducing, on top of the distribution of her moving time, a function that measures the intensity with which she moves “just after” times at which this distribution jumps to 1. Heuristically, this function measures the intensity of an interval of consecutive atoms. Other authors, following Dutta and Rustichini (1993), use ad-hoc randomization devices to rule out coordination failures. By contrast, an attractive feature of our model is that we rely on a standard definition of a strategy and that coordination failures only play a minor role in our analysis.

The paper is organized as follows. We present the model in Section 2. In Section 3, we characterize the distribution of each player’s moving time, show how to construct an equilibrium, and draw the main economic implications of our analysis. Section 4 concludes.

2 The Model

Time is continuous, and indexed by $t \geq 0$. There are two players, 1 and 2. In what follows, i refers to an arbitrary player and j to the other player. Each player i comes into play at

³See also the discussion in Fudenberg and Tirole (1991, Chapter 4, Section 5).

some random time $\tau^i \geq 0$. Here τ^i can be interpreted as the time at which player i discovers a new idea, the existence of a new investment opportunity, or the possibility to adopt a new technology. In particular, player i cannot move before time τ^i . It will sometimes be convenient to say that player i *is born* at time τ^i , and to refer to τ^i as her *date of birth*.

Actions and Payoffs Both players are risk-neutral and discount future utilities at rate $r > 0$. Like in standard timing games, each player i has a single opportunity to make a move. The difference is that this must occur at some time $t \geq \tau^i$, reflecting that no player can move before being born. If player i moves first, at time t , she obtains a payoff $L(t)$ evaluated in terms of time 0 utilities, while player j obtains a zero payoff. If players i and j move simultaneously, they each obtain a negative payoff $S(t)$ evaluated in terms of time 0 utilities. This payoff structure is adopted mainly to simplify the exposition;⁴ it arises for instance in an investment timing game where two firms contemplate investing on a market that can accommodate only one of them because Bertrand competition would otherwise drive profits to zero. Denoting by $P(t)$ the monopoly profit flow at time t and by $C(t)$ the cost of investing at time t , the functions L and S are given in this case by

$$\begin{aligned} L(t) &= \int_t^\infty e^{-rs} P(s) ds - e^{-rt} C(t), \\ S(t) &= -e^{-rt} C(t) \end{aligned} \tag{1}$$

for all $t \geq 0$. More generally, we maintain the following assumption throughout the paper.

Assumption 1 *The function L is twice continuously differentiable, and there are times $T_2 > T_1 > 0$ such that the following holds:*

$$\begin{aligned} L(t) &< 0 \quad \text{if } t \in [0, T_1), \\ L(t) &> 0 \quad \text{if } t \in (T_1, \infty), \\ \dot{L}(t) &> 0 \quad \text{if } t \in [0, T_2), \\ \dot{L}(t) &< 0 \quad \text{if } t \in (T_2, \infty), \\ \ddot{L}(t) &\leq 0 \quad \text{if } t \in [T_1, T_2]. \end{aligned} \tag{2}$$

Thus the function L vanishes at time T_1 only, $L(T_1) = 0$, reaches its unique maximum at time T_2 , $\dot{L}(T_2) = 0$, and is concave over $[T_1, T_2]$.

⁴In particular, extending the analysis to the case where the follower receives a nonzero payoff would raise no additional conceptual difficulty.

In the investment timing game example, these assumptions hold for instance when the monopoly profit flow grows at a constant rate μ , with $0 < \mu < r$, and the investment cost is constant over time and given by $C > \frac{1}{r-\mu} P(0)$. Formula (1) then yields

$$L(t) = e^{-rt} \left[\frac{e^{\mu t}}{r - \mu} P(0) - C \right]$$

for all $t \geq 0$, and times T_1 and T_2 are respectively given by $e^{\mu T_1} = \frac{(r-\mu)C}{P(0)}$ and $e^{\mu T_2} = \frac{rC}{P(0)}$. It is straightforward to verify that the function L is concave over $[T_1, T_2]$.

Times T_1 and T_2 have straightforward game-theoretic interpretations. Suppose that both players can move as of time 0, and that this fact is common knowledge, like in a standard preemption game. At time T_1 , each player is indifferent between moving first and becoming a leader, or not moving and becoming a follower, because both options yield her a zero payoff. This corresponds to the unique subgame-perfect equilibrium outcome of the complete information preemption model of Fudenberg and Tirole (1985), in which a single player moves at time T_1 .⁵ By contrast, time T_2 is the time at which it would be optimal for each player to move if she believed that she were not threatened by her competitor's moving first. When the players are two firms that can invest on a market that can accommodate only one of them, like in the investment timing game example, this corresponds to the unique equilibrium outcome of the complete information precommitment model of Reinganum (1981), in which one firm invests at time T_2 and the other firm stays out of the market.⁶

Information We assume that the players' dates of birth τ^1 and τ^2 are independently and identically distributed according to a continuously differentiable distribution function F with positive density \dot{F} over \mathbb{R}_{++} . In addition, we will maintain the following standard assumption throughout the paper.

Assumption 2 *The hazard rate $\frac{\dot{F}}{1-F}$ of F is nondecreasing over \mathbb{R}_+ .*

The date of birth of each player is her private information, or *type*. In particular, when a player comes into play, she does not know whether or not the other player was already born.

⁵As discussed in the Introduction, the proper formulation of this result requires an appropriate extension of the players' strategy spaces.

⁶To see this, note that if firm i stays out of the market, it is optimal for firm j to invest at time T_2 . Conversely, suppose that firm j commits to invest at time T_2 . Investing at any time $t > T_2$ cannot a best response for firm i , as it would earn no revenue because of Bertrand competition, while incurring the investment cost $C(t)$. Investing at time $t \leq T_2$ yields firm i a profit

$$\int_t^{T_2} e^{-rs} P(s) ds - e^{-rt} C(t),$$

which is maximal for $t = T_2$ and equal there to $-e^{-rT_2} C(T_2)$. Hence firm i is better off staying out of the market altogether, and the strategy profile (T_2, ∞) is an equilibrium in precommitment strategies.

Nor does she observe when the other player comes into play when this event takes place after her own birth. The only information that accrues to each player during the course of the game is whether and when her competitor makes a move, which effectively terminates the game. As stressed in the Introduction, a distinctive feature of our model compared to most analyses of preemption games is that competition is only potential: a player never knows for sure whether she indeed has a competitor, except when it is too late and she has been preempted. Our model shares this feature with the reputation model of Hendricks (1992). However, unlike in his model, a driving force of our analysis is that, as time goes by, it becomes less and less likely that each player was not born yet. We will come back to this important issue in Subsection 3.3.

Finally, it should be noted that our model can be seen as a perturbation of two limit situations. To illustrate this point, suppose that F is exponential with parameter λ , so that the hazard rate of F is constant and equal to λ . Then, letting λ go to 0 leads to the precommitment model of Reinganum (1981), because each player, if ever born, believes that it is very unlikely that she ever will have a competitor. By contrast, letting λ go to ∞ leads to the preemption model of Fudenberg and Tirole (1985), in which it is common knowledge from time 0 on that each player has a competitor.

Strategies We shall use a standard definition of a strategy in a timing game. That is, we start for each player with a distribution function over her moving time, and we roll it over using Bayes' rule. Our presentation follows Laraki, Solan, and Vieille (2005). A *mixed plan* for player i of type τ^i is given by a distribution function over $[\tau^i, \infty]$.⁷ A *strategy* for player i of type τ^i is a collection $(G_t^i(\cdot; \tau^i))_{t \geq \tau^i}$ of mixed plans with associated Lebesgue–Stieltjes measures $(dG_t^i(\cdot; \tau^i))_{t \geq \tau^i}$ such that

- for each $t \geq \tau^i$, $\text{Supp } dG_t^i(\cdot; \tau^i) \subset [t, \infty]$, and
- for each $t' > t \geq \tau^i$ and every Borel set $A \subset [t', \infty]$,

$$\int_A dG_t^i(s; \tau^i) = [1 - G_{t'}^i(t'; \tau^i)] \int_A dG_{t'}^i(s; \tau^i). \quad (3)$$

The first condition is a properness condition that states that $G_t^i(\cdot; \tau^i)$ is a mixed plan in the subgame that starts at date t . The second condition is a consistency condition that states

⁷Observe that by including ∞ in the support of the distribution, we allow player i never to move. A mixed plan corresponds to what is called a *strategy* in the timing game literature (see for instance Owen (1968, Chapter IV, Section 5), or Pitchik (1981)). However, this approach does not allow one to specify how players behave in all possible subgames, that is, in the present case, at any time where none of them has moved yet. This is why it is more suitable to define a strategy as a collection of mixed plans, one for each subgame.

that, as long as a plan does not require player i to act with probability 1, the continuation plans can be computed by Bayes' rule. We will write $G_{\tau^i}^i(\cdot; \tau^i) = G^i(\cdot; \tau^i)$ in order to simplify notation. Finally, we shall assume that for any Borel set $A \subset \mathbb{R}_+$, the mapping $\tau^i \mapsto \int_A dG^i(t; \tau^i)$ is measurable, so that the mapping defined by $(\tau^i, A) \mapsto \int_A dG^i(t; \tau^i)$ is a well defined transition function. This in turn allows one to compute quantities such as $\mathbf{P}[i \text{ moves over } A] = \int_0^\infty \int_A dG^i(t; \tau^i) dF(\tau^i)$.

Simple Equilibria We shall be particularly concerned in characterizing *simple* perfect Bayesian equilibria of our timing game that have the property that, for each player i , the strategies $(G_t^i(\cdot; \tau^i))_{t \geq \tau^i}$ of types $\tau^i > 0$ are, whenever possible, derived using Bayes' rule from the strategy $(G_t^i(\cdot; 0))_{t \geq 0}$ that player i adopts when born at time 0, and therefore ultimately from the mixed plan $G^i(\cdot; 0)$. That is, for each $t \geq \tau^i > 0$ such that $G^i(\tau^i; 0) < 1$,

$$G^i(t; \tau^i) = \frac{G^i(t; 0) - G^i(\tau^i; 0)}{1 - G^i(\tau^i; 0)}. \quad (4)$$

It should be noted that, in such an equilibrium, $G^i(t; \tau^i)$ represents both

- the probability that type τ^i of player i moves by time $t \geq \tau^i$, and
- the probability that a type $\tilde{\tau}^i \leq \tau^i$ of player i moves between times τ^i and t conditional on not having moved before time τ^i .

Indeed, since (4) holds at $\tilde{\tau}^i$ as well as at τ^i , it is easy to check using (3) that

$$G_{\tau^i}^i(t; 0) = \frac{G^i(t; 0) - G^i(\tau^i; 0)}{1 - G^i(\tau^i; 0)} = \frac{G^i(t; \tilde{\tau}^i) - G^i(\tau^i; \tilde{\tau}^i)}{1 - G^i(\tau^i; \tilde{\tau}^i)} = G_{\tau^i}^i(t; \tilde{\tau}^i)$$

for all $t \geq \tau^i \geq \tilde{\tau}^i > 0$ such that $G^i(\tau^i; 0) < 1$. In particular, $G_{\tau^i}^i(t; 0) = G^i(t; \tau_i)$ for all types $\tau^i \geq 0$. That the existence of a simple equilibrium is a plausible conjecture intuitively reflects the fact that all players' types in our incomplete information timing game have the same underlying payoff function once they are born, so that the players' preferences in this game do not satisfy a strict single-crossing condition. As a result, if type $\tilde{\tau}^i$ of player i has not yet moved by time $\tau^i \geq \tilde{\tau}^i$, she need not behave differently from type τ^i after time τ^i .

3 Equilibrium Analysis

In this section, we first derive necessary properties that must hold true of any equilibrium. In particular, we prove that all equilibria give rise to the same distribution for each player's moving time. We then establish that an equilibrium exists, and we show how to construct a unique simple equilibrium, which we discuss in detail. All the proofs are in the Appendix.

3.1 Necessary Conditions

In the equilibrium analysis, it will be convenient to distinguish three intervals of types.

Late Types Consider first some type $\tau^i \geq T_2$ of player i , and suppose that player j has not moved yet at time τ^i . Then, since L is strictly decreasing over $[T_2, \infty)$, type τ^i has no incentive to delay her move, if she believes that there is a probability 0 that she will thereby tie with player j . This is for instance the case if the following properties hold:

P1 The types $\tau^j < T_2$ move with probability 1 before time T_2 , that is:

$$\mathbf{P}[j \text{ moves over } [0, T_2) | \tau^j < T_2] = 1,$$

P2 Each type $\tau^j \geq T_2$ moves immediately at time τ^j .

Property P1 ensures that type τ^i faces no competition from types $\tau^j < T_2$. Property P2 along with the fact that the distribution of each player's type has no atom implies that type τ^i has a probability 0 of tying with some type $\tau^j \geq T_2$. Hence, provided property P1 is satisfied, we have constructed a continuation equilibrium over $[T_2, \infty)$ such that each player i that is born during this time interval moves immediately at time τ^i . It should be noted that these plans also describe the off-equilibrium path behavior of each player: at any time $t \geq T_2$ at which no player has moved yet, it is optimal for any player who was born before time t to move at time t .

Our first result shows that the above continuation equilibrium is actually unique.

Lemma 1 *In any equilibrium, properties P1 and P2 hold for each player.*

For each $t \geq 0$, let $\mathcal{G}^i(t)$ be the unconditional probability that player i moves by time t in equilibrium and let $d\mathcal{G}^i$ be the corresponding Lebesgue–Stieltjes measure that describes the equilibrium distribution of player i 's moving time. The key step in the proof of Lemma 1 consists in showing that $d\mathcal{G}^i$ has no atom over $[T_2, \infty)$. It follows from Lemma 1 that $\mathcal{G}^1(t) = \mathcal{G}^2(t) = F(t)$ for all $t \geq T_2$.

Our next result shows that, in equilibrium, the distributions $d\mathcal{G}^1$ and $d\mathcal{G}^2$ have the same support, and that their common support is connected.

Lemma 2 *In any equilibrium, $\text{Supp } d\mathcal{G}^1 = \text{Supp } d\mathcal{G}^2 = [\tilde{T}_1, \infty)$ for some $\tilde{T}_1 \in [T_1, T_2]$.*

As a result, the distribution functions \mathcal{G}^1 and \mathcal{G}^2 are strictly increasing over $[\tilde{T}_1, \infty)$. Note that, a priori, \tilde{T}_1 may depend on the equilibrium under consideration. Observe also that $\tilde{T}_1 \geq T_1$, since moving before time T_1 is a strictly dominated strategy.

Intermediate Types What happens before time T_2 ? Suppose first for the sake of the argument that player j always moves immediately when she is born, at time τ^j . This means that player j 's moving time is distributed according to F . As a result, any type $\tau^i \leq T_2$ of player i faces a simple problem. Indeed, if she decides to move at some time $t \in [\tau^i, T_2]$, she obtains an expected payoff equal to

$$\left[\frac{1 - F(t)}{1 - F(\tau^i)} \right] L(t) \quad (5)$$

in terms of time 0 utilities. Let time T_2^* be implicitly defined by

$$\frac{\dot{L}(T_2^*)}{L(T_2^*)} = \frac{\dot{F}(T_2^*)}{1 - F(T_2^*)}. \quad (6)$$

That T_2^* is uniquely defined by (6) and lies in (T_1, T_2) follows from Assumptions 1 and 2.⁸ It is straightforward to check that the maximum of (5) over $t \in [\tau^i, \infty)$ is attained at T_2^* if $\tau^i \leq T_2^*$ and at τ^i if $\tau^i > T_2^*$. Equation (6) states that, at time T_2^* , the growth rate of the payoff from moving first is equal to the rate at which the other player moves, conditional on not having moved before T_2^* . Hence, the marginal benefit of delaying one's move by an infinitesimal amount of time dt , $\dot{L}(T_2^*) dt$, exactly compensates the corresponding expected marginal loss. That loss is in turn equal to the probability that one's competitor is born during the time interval $[T_2^*, T_2^* + dt)$ conditional on not being born before T_2^* , $\left[\frac{\dot{F}(T_2^*)}{1 - F(T_2^*)} \right] dt$, multiplied by the foregone benefit, $L(T_2^*)$.

Consider now any type $\tau^i \in [T_2^*, T_2)$ of player i , and suppose that player j has not moved yet at time τ^i . Then, if the following properties hold:

P1* Each type $\tau^j < T_2^*$ moves with probability 1 before time T_2^* , that is:

$$\mathbf{P}[j \text{ moves over } [0, T_2^*) | \tau^j < T_2^*] = 1,$$

P2* Each type $\tau^j \geq T_2^*$ moves immediately at time τ^j ,

type τ^i has no incentive to delay her move. This implies as above that there exists a continuation equilibrium over $[T_2^*, T_2)$ such that any player i who is born during this time interval moves immediately at time τ^i . The intuition for this result is more subtle than for the time interval $[T_2, \infty)$, where the gains of moving first are strictly decreasing. Indeed, there are gains from waiting to move over $[T_2^*, T_2)$, as L is strictly increasing over that range. Yet, because $\dot{L}(t) < \left[\frac{\dot{F}(t)}{1 - F(t)} \right] L(t)$ for all $t \in (T_2^*, T_2)$, these gains are offset by the probability that

⁸Indeed, $\frac{\dot{L}}{L}$ is positive and strictly decreasing over (T_1, T_2) , with $\lim_{t \downarrow T_1} \frac{\dot{L}(t)}{L(t)} = \infty$ and $\frac{\dot{L}(T_2)}{L(T_2)} = 0$, and $\frac{\dot{F}}{1 - F}$ is positive and nondecreasing over (T_1, T_2) .

the other player is born and immediately moves, implying a foregone benefit $L(t)$. In this equilibrium, competition over $[T_2^*, T_2)$ really takes place at the margin. Indeed, if a player i is born during this time interval, she knows that she is the first player to be born. It is her fear that a competitor born during the infinitesimal time interval $(\tau^i, \tau^i + dt]$ may actually preempt her that leads her to move immediately at time τ^i . Like over the time interval $[T_2, \infty)$, these strategies also describe the off-equilibrium path behavior of any player.

We now show that this continuation equilibrium is in fact unique. One first has the following lemma.

Lemma 3 *In any equilibrium, the distributions $d\mathcal{G}^1$ and $d\mathcal{G}^2$ have no atom over $(T_2^*, T_2]$.*

It follows from Lemma 1 that no player born before time T_2 wants to move after that time. Thus, for each player i and any time $t \in [T_2^*, T_2)$, the function $(1 - \mathcal{G}^i)L$ necessarily attains a maximum over $[t, T_2]$, for, otherwise, there would exist some type $\tau^j \in [T_2^*, T_2)$ of player j with no best response. Lemmas 2 and 3 deliver useful insights into the structure of the sets $\arg \max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\}$ for $t \in (T_2^*, T_2)$. First, there cannot exist a time $t \in (T_2^*, T_2)$ and a number $\varepsilon > 0$ such that $\arg \max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\} = \{t + \varepsilon\}$. Indeed, if this were the case, all the types of player j born over $[t, t + \varepsilon]$ would have no other best response than to move at time $t + \varepsilon$, so that the distribution of $d\mathcal{G}^j$ would have an atom at time $t + \varepsilon$, which is ruled out by Lemma 3.⁹ Second, the function $(1 - \mathcal{G}^i)L$ is necessarily nonincreasing over the interval $(\max\{\tilde{T}_1, T_2^*\}, T_2]$, which is not empty as $\tilde{T}_1 < T_2$ by property P1. Indeed, if there were some times s and t in this interval with $s < t$ and $[1 - \mathcal{G}^i(s)]L(s) < [1 - \mathcal{G}^i(t)]L(t)$, then, for some number $\varepsilon > 0$, no type of player j would be willing to move over the time interval $(s, s + \varepsilon)$.¹⁰ But then $\text{Supp } d\mathcal{G}^j$ would not contain this interval, and therefore, as $s > \tilde{T}_1$, would not be connected, which is ruled out by Lemma 2. It follows that for each $t \in (\max\{\tilde{T}_1, T_2^*\}, T_2)$, $\arg \max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\}$ is either reduced to $\{t\}$, or is a closed interval with lower bound t . One can actually rule out this latter scenario, which leads to the following result.

Lemma 4 *In any equilibrium, properties P1* and P2* hold for each player.*

It follows from Lemma 4 that $\mathcal{G}^1(t) = \mathcal{G}^2(t) = F(t)$ for all $t \geq T_2^*$. An important by-product of Lemma 4 is that \tilde{T}_1 is lower than T_2^* .

⁹Observe that if \tilde{T}_1 , the lower bound of $\text{Supp } d\mathcal{G}^j$, were itself lower than $t + \varepsilon$, there would also be a contradiction with Lemma 2, since $\text{Supp } d\mathcal{G}^j$ would then not contain the interval $(t, t + \varepsilon)$ and therefore would not be connected.

¹⁰Since the function \mathcal{G}^i is right-continuous, this would remain true even if $d\mathcal{G}^i$ had an atom at s .

Early Types In contrast to what happens over the time interval $[T_2^*, \infty)$, immediately moving when she is born cannot be part of any player i 's equilibrium strategy before time T_2^* . Indeed, because $\dot{L}(t) > \left[\frac{\dot{F}(t)}{1-F(t)} \right] L(t)$ for all $t \in (T_1, T_2^*)$, player j , when born before time T_2^* , would then have an incentive to wait until that time before moving. This would in turn lead player i to delay slightly her move, thus destroying the conjectured equilibrium. Instead, we conjecture that, in any equilibrium, all the types born before time T_2^* are indifferent to move at any time between some time T_1^* and time T_2^* . Given the continuation equilibrium characterized in Lemma 4, it follows that the equilibrium payoff of any such type is

$$[1 - F(T_2^*)]L(T_2^*) \quad (7)$$

in terms of time 0 utilities. Time T_1^* is the lower bound of the distribution of each player's moving time common to all equilibria, so that $\tilde{T}_1 = T_1^*$ in any equilibrium. It is pinned down by the requirement that each player must be indifferent between moving at time T_1^* or at time T_2^* , and moves with probability 1 after time T_1^* . That is, given (7),

$$L(T_1^*) = [1 - F(T_2^*)]L(T_2^*). \quad (8)$$

We now formally establish these conjectures. The following lemma parallels Lemma 3.

Lemma 5 *In any equilibrium, the distributions $d\mathcal{G}^1$ and $d\mathcal{G}^2$ have no atom over $[\tilde{T}_1, T_2^*]$.*

Using similar arguments as in the case of intermediate types, one can check that Lemmas 2 and 5 imply that for each player i , the function $(1 - \mathcal{G}^i)L$ is necessarily nonincreasing over the interval $[\tilde{T}_1, T_2^*]$. The following lemma states that it is actually constant over $[T_1^*, T_2^*]$.

Lemma 6 *In any equilibrium, $\mathcal{G}^1(t) = \mathcal{G}^2(t) = 1 - \frac{L(T_1^*)}{L(t)}$ for all $t \in [T_1^*, T_2^*]$.*

To simplify notation, we shall hereafter denote by $\mathcal{G}(t)$ the unconditional probability that any given player moves by time t in equilibrium. Lemmas 4 and 5 imply that this probability is uniquely determined in equilibrium:

$$\mathcal{G}(t) = \begin{cases} 0 & \text{if } t < T_1^*, \\ 1 - \frac{L(T_1^*)}{L(t)} & \text{if } T_1^* \leq t < T_2^*, \\ F(t) & \text{if } t \geq T_2^*. \end{cases} \quad (9)$$

Each player born before T_2^* is indifferent between moving at any time $t \in [T_1^*, T_2^*]$:

$$[1 - \mathcal{G}(t)]L(t) = L(T_1^*) \quad (10)$$

for all $t \in [T_1^*, T_2^*]$. Note that a special case of (10) is (8), since $\mathcal{G}(T_2^*) = F(T_2^*)$. For each $t \in (T_1^*, T_2^*)$, (10) can be rewritten in differential form as follows:

$$\frac{\dot{L}(t)}{L(t)} = \frac{\dot{\mathcal{G}}(t)}{1 - \mathcal{G}(t)}. \quad (11)$$

Equation (11) states that, at any time $t \in (T_1^*, T_2^*)$, the marginal benefit of delaying one's move by an infinitesimal amount of time dt , $\dot{L}(t) dt$, exactly compensates the corresponding expected marginal loss, which is equal to the probability that one's competitor will move during the time interval $[t, t + dt)$ conditional on not having moved before t , $\left[\frac{\dot{\mathcal{G}}(t)}{1 - \mathcal{G}(t)}\right] dt$, multiplied by the foregone benefit $L(t)$.

3.2 Equilibrium Existence

Lemmas 4 and 6 pin down a unique candidate for the equilibrium distribution of each player's moving time. We now construct equilibrium strategies that actually generate this distribution. We first construct a non-simple pure strategy equilibrium, and then characterize the unique simple equilibrium.

A Non-Simple Equilibrium A key insight of Lemma 4 is that there exists a unique continuation equilibrium from time T_2^* on: at any time $t \geq T_2^*$ at which no player has moved yet, it is optimal for any player who was born before or at time t to move immediately. Therefore, since no player moves before time T_1^* , constructing an equilibrium requires only specifying the players' behavior over $[T_1^*, T_2^*]$. To do so, we rely on Lemma 6, which states that each player born before time T_2^* must be indifferent between moving at any point in this time interval. The only condition to be satisfied is that the unconditional distribution of each player's moving time be consistent with the postulated equilibrium strategies; that is, it must be equal to the average of the type-dependent distributions of moving times. For each player i , these are given by the mixed plans $(G^i(\cdot; \tau^i))_{\tau^i \leq T_2^*}$, and so one must have

$$\int_0^t G^i(t; \tau^i) dF(\tau^i) = \mathcal{G}(t) \quad (12)$$

for all $t \in [T_1^*, T_2^*]$. Now, observe that $F(t) > \mathcal{G}(t)$ for all $t < T_2^*$.¹¹ One can thus define a strictly increasing mapping $\sigma : [0, T_2^*] \rightarrow [T_1^*, T_2^*]$ by

$$F(\sigma^{-1}(t)) = \mathcal{G}(t)$$

¹¹Indeed, by (11), $\frac{\dot{\mathcal{G}}(t)}{1 - \mathcal{G}(t)} = \frac{\dot{L}(t)}{L(t)} > \frac{\dot{F}(t)}{1 - F(t)}$ for all $t \in (T_1^*, T_2^*)$. Since $\mathcal{G}(T_2^*) = F(T_2^*)$, integrating yields $\ln(1 - \mathcal{G}(t)) > \ln(1 - F(t))$ and thus $F(t) > \mathcal{G}(t)$ for all $t \in (T_1^*, T_2^*)$. Finally $F(t) > 0 = \mathcal{G}(t)$ for all $t \leq T_1^*$.

for all $t \in [T_1^*, T_2^*]$. By construction, $\sigma(0) = T_1^*$, $\sigma(\tau) > \tau$ for all $\tau < T_2^*$ and $\sigma(T_2^*) = T_2^*$. Let then each type $\tau^i \leq T_2^*$ of player i move at time $\sigma(\tau^i)$, that is:

$$G^i(t; \tau^i) = \mathbf{1}_{\{\sigma(\tau^i) \leq t\}} \quad (13)$$

for all $t \geq \tau^i$. It then follows that

$$\int_0^t G^i(t; \tau^i) dF(\tau^i) = \int_0^t \mathbf{1}_{\{\tau^i \leq \sigma^{-1}(t)\}} dF(\tau^i) = F(\sigma^{-1}(t)) = \mathcal{G}(t)$$

for all $t \in [T_1^*, T_2^*]$, as required. One has the following result.

Proposition 1 *There exists a pure strategy equilibrium in which*

- *each player's type $\tau \leq T_2^*$ moves at time $\sigma(\tau)$, and*
- *each player's type $\tau > T_2^*$ moves immediately at time τ .*

The Simple Equilibrium The equilibrium constructed in Proposition 1 is not *simple* in the sense of Section 2. Indeed, it follows from (4) and from the fact that no player ever moves before T_1^* that, in a simple equilibrium, all the types $\tau < T_1^*$ of each player should behave in exactly the same way. This is not the case in the above equilibrium, since each type $\tau < T_1^*$ moves at time $\sigma(\tau)$, where σ is a strictly increasing function. In particular, this equilibrium is not Markov perfect, since players condition their behavior on a payoff irrelevant variable, namely their date of birth.¹² As discussed in the Introduction, we view this as an artificial feature of this equilibrium.

We now show that there exists a unique simple equilibrium. This equilibrium is symmetric and completely characterized by the mixed plan $G(\cdot; 0)$ of type 0 of each player. Since $d\mathcal{G}$ has no atom, and each type $\tau < T_2^*$ moves with probability 1 before time T_2^* , the function $G(\cdot; 0)$ must be continuous over $[0, T_2^*]$, with $G(T_1^*; 0) = 0$ and $G(T_2^*; 0) = 1$. The support of the corresponding distribution $dG(\cdot; 0)$ can be characterized as follows.

Lemma 7 $\text{Supp } dG(\cdot; 0) = [T_1^*, T_2^*]$.

It follows from Lemma 7 that $G(t; 0) < 1$ for all $t < T_2^*$, so that, according to (4), the behavior of each type $\tau < T_2^*$ can be inferred from that of type 0 by Bayes' rule. In line with (12), the consistency condition

$$\int_0^t G(t; \tau) dF(\tau) = \mathcal{G}(t) \quad (14)$$

¹²See Maskin and Tirole (2001) for a definition of Markov perfect equilibrium.

must hold for all $t \in [T_1^*, T_2^*]$.¹³ Imposing the simplicity requirement (4), we obtain that

$$\int_0^t \left[\frac{G(t; 0) - G(\tau; 0)}{1 - G(\tau; 0)} \right] dF(\tau) = \mathcal{G}(t) \quad (15)$$

for all $t \in [T_1^*, T_2^*]$. One has the following result.

Proposition 2 *There exists a unique simple equilibrium, in which*

- *each player's type $\tau < T_2^*$ moves according to the mixed plan $G(\cdot; \tau) = \frac{G(\cdot; 0) - G(\tau; 0)}{1 - G(\tau; 0)}$, where $G(\cdot; 0) = 0$ over $[0, T_1^*]$ and*

$$G(t; 0) = 1 - \left[\frac{F(t) - \mathcal{G}(t)}{F(T_1^*)} \right] \exp \left(- \int_{T_1^*}^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau)} \right) \quad (16)$$

for all $t \in [T_1^, T_2^*]$.*

- *each player's type $\tau \geq T_2^*$ moves immediately at time τ .*

3.3 Discussion

Coordination Failures In any equilibrium, the distribution of each player's moving time has no atom; as a result, the probability of a joint move is zero. Therefore coordination failures only play a limited role in our analysis: actually, the assumption that they are always detrimental to both players is only used in the proof that the distribution of each player's moving time is uniquely determined in equilibrium (see the proofs of Lemmas 1, 2, and 4). By contrast, models that use a discrete-time game with very short time lags to represent continuous time, following Simon and Stinchcombe (1989), explicitly need to rule out coordination failures, typically through the use of an ad-hoc randomization device, as in Dutta and Rustichini (1993), Dutta, Lach, and Rustichini (1995), Hoppe and Lehmann-Grube (2005), or Argenziano and Schmidt-Dengler (2007). In the alternative formulation of Fudenberg and Tirole (1985), coordination failures are crucial for determining moving intensities off the equilibrium path, although they cannot arise in equilibrium.

Strength of Competition To obtain more information on how the players compete in the simple equilibrium, it is useful to determine the shape of the function $G(\cdot; 0)$. One has the following results.

Corollary 1 *$G(\cdot; 0)$ is strictly concave over $[T_1^*, T_2^*]$.*

¹³Notwithstanding the appearances, (14) is not a standard integral equation: indeed, the unknown in (14) is the *kernel* G , while F is given. Observe that (13) already yields one particular solution to this equation.

Corollary 2 $\dot{G}(T_2^*; 0) = 0$.

Corollary 1 implies that the probability that a player born before time T_2^* will move during the time interval $[t, t + dt]$ is smaller, the larger is $t \in [T_1^*, T_2^*]$. Consistent with this insight, Corollary 2 implies that, when dt is small, the probability that such a player moves during the time interval $[T_2^* - dt, T_2^*]$ is very small, namely of the order $o(dt)$. Therefore, players born between times T_1^* and T_2^* are more likely to move close to their dates of birth than close to time T_2^* . Overall, competition is fiercer close to time T_1^* , and then tends to decrease. This property, which actually holds in any equilibrium, is reflected in the rate at which players moves. To see that, note that since players move independently, the distribution function of the first moving time is $1 - (1 - \mathcal{G})^2$. It then follows from (11) that over the time interval $[T_1^*, T_2^*]$, the corresponding hazard rate is $2 \frac{\dot{L}}{L}$, which is strictly decreasing there by Assumption 1. Now, no player born before time T_2^* ever moves after time T_2^* , and any player born after time T_2^* moves immediately. Thus, over the time interval $[T_2^*, \infty)$, the hazard rate of the first moving time is $2 \frac{\dot{F}}{1-F}$, which is nondecreasing by Assumption 2. Our model thus delivers the prediction that the hazard rate of the first moving time, $2 \max \left\{ \frac{\dot{L}}{L}, \frac{\dot{F}}{1-F} \right\}$, is nonmonotonic over $[T_1^*, \infty)$, and tends to increase after having reached a minimum at T_2^* .

Eroding Reputations To understand how competition evolves with time, an interesting quantity to focus on is the probability that a given player is not yet born by time t given that no player has moved yet by this time. This probability is the same in any equilibrium. Denoting it by $q(t)$, we have the following result.

Corollary 3 *For each time $t \geq 0$, one has:*

$$q(t) = \begin{cases} 1 - F(t) & \text{if } t \leq T_1^*, \\ \frac{[1-F(t)]L(t)}{L(T_1^*)} & \text{if } T_1^* < t \leq T_2^*, \\ 1 & \text{if } t > T_2^*. \end{cases} \quad (17)$$

Since no player moves before time T_1^* in equilibrium, the probability that a given player is not born by time $t \leq T_1^*$ is just the unconditional probability $1 - F(t)$. Symmetrically, since any player moves before time T_2^* when born before time T_2^* and moves immediately when born after time T_2^* , the probability that a given player is not born by time $t > T_2^*$ given that no player has moved by this time is 1. Over the interval (T_1^*, T_2^*) , the mapping $t \mapsto q(t)$ continuously increases from $1 - F(T_1^*)$ to 1, as is easily checked from the definition of time T_2^* .

In line with Hendricks (1992), one can interpret $q(t)$ as the reputation of each player at time t if she has not moved by then. In equilibrium, before time T_2^* , it is worthwhile for each player to make her opponent believe that she is not born yet: this induces her opponent to delay one's move, which potentially allows her to reap higher payoffs. By contrast, at time T_2^* , the reputation of each player is equal to 1 and thus cannot improve anymore: this is the flip side of the fact that it is no longer optimal for each player to delay one's move beyond time T_2^* , because the risk of getting preempted overcomes the remaining gains from waiting. The reputation of each player i evolves according to

$$\tilde{q}(t) = \begin{cases} q(t) & \text{if player } i \text{ has not moved by time } t, \\ 0 & \text{if player } i \text{ has moved by time } t. \end{cases}$$

Now, observe that, as long as $\tilde{q}(s) > 0$, one has, if $s < t \leq T_2^*$,

$$\begin{aligned} \mathbf{E}[\tilde{q}(t) | \mathcal{F}_s] &= \mathbf{E}[\mathbf{P}[\text{Player } i \text{ is not yet born by time } t | \mathcal{F}_t] | \mathcal{F}_s] \\ &= \mathbf{P}[\text{Player } i \text{ is not yet born by time } t | \mathcal{F}_s] \\ &< \mathbf{P}[\text{Player } i \text{ is not yet born by time } s | \mathcal{F}_s] \\ &= \tilde{q}(s), \end{aligned}$$

where \mathcal{F}_s and \mathcal{F}_t represent the public information at times s and t , and the strict inequality follows from the fact that there is a positive probability that a player is born between time s and time t . As a result, the reputation of each player follows a submartingale. Specifically, suppose that $T_1^* \leq s < t \leq T_2^*$ and $\tilde{q}(s) > 0$. An explicit calculation then yields

$$\begin{aligned} \mathbf{E}[\tilde{q}(t) | \mathcal{F}_s] &= \mathbf{P}[\text{Player } i \text{ is not yet born by time } t | \mathcal{F}_s] q(t) \\ &= \left[\frac{1 - \mathcal{G}(t)}{1 - \mathcal{G}(s)} \right] \left\{ \frac{[1 - F(t)]L(t)}{L(T_1^*)} \right\} \\ &= \left[\frac{1 - F(t)}{1 - F(s)} \right] \tilde{q}(s), \end{aligned} \tag{18}$$

where the second inequality follows from (17), and the second from (10) and (17) again. One can check from (18) that the expected instantaneous rate of decline of each player's reputation is equal to the hazard rate of the players' type distribution:

$$\lim_{t \downarrow s} \frac{1}{\tilde{q}(s)} \mathbf{E} \left[\frac{\tilde{q}(t) - \tilde{q}(s)}{t - s} \mid \mathcal{F}_s \right] = - \frac{\dot{F}(s)}{1 - F(s)}, \tag{19}$$

reflecting that the key driving force in the evolution of players' reputations is the rate at which players actually come into play. It should be observed that, since the hazard rate of F is assumed to be nondecreasing over \mathbb{R}_+ , the closer one moves to time T_2^* , the faster the players' reputations tend to decline on average. Another implication of (19) is that the higher the hazard rate of the players' type distribution, the faster the players' reputations tend to decline on average.¹⁴ Similarly, it follows from (17) that $\frac{\dot{q}}{q} = -\frac{\dot{F}}{1-F} + \frac{\dot{L}}{L}$ over $[T_1^*, T_2^*]$, so that the higher the hazard rate of the players' type distribution, the slower the players' reputations conditional on none of them moving tend to grow on average.

It is useful to contrast these findings with those of Hendricks (1992). In his model, a firm can be an innovator, or an imitator. Firms compete for the adoption of a new technology. While an innovator has no constraint on the time at which it can adopt, an imitator cannot adopt before an innovator does. The reputation of a firm is the posterior belief that it is an imitator given that no firm has adopted yet. Observe that, in this case, the reputation of a firm follows by construction a martingale, since it is equal to the posterior probability of a fixed event: being an imitator. In our model, by contrast, reputations tend to erode: as time elapses, it becomes increasingly difficult to maintain a reputation of not being born. As a result, players are less ready to wait before moving and players move immediately from time T_2^* onwards, while in Hendrick's (1992) model, an innovator may in equilibrium wait until time $T_2 > T_2^*$ to adopt.

Comparative Statics Last, we analyze how the simple equilibrium is affected by a change in the players' type distribution. Specifically, consider a family of distribution functions F_θ parameterized by $\theta \in \mathbb{R}$, which satisfies the monotone likelihood ratio property (MLRP); that is, the mapping $t \mapsto \frac{\dot{F}_{\theta_1}(t)}{\dot{F}_{\theta_0}(t)}$ is nondecreasing whenever $\theta_1 > \theta_0$. As is well known, this implies both that F_{θ_1} first order stochastically dominates F_{θ_0} , and that the hazard rate of F_{θ_1} is below that of F_{θ_0} , $\frac{\dot{F}_{\theta_1}}{1-F_{\theta_1}} \leq \frac{\dot{F}_{\theta_0}}{1-F_{\theta_0}}$. A natural question is to determine the impact on the equilibrium distributions of moving times of a shift of the players' type distribution function from F_{θ_0} to F_{θ_1} . Assuming for simplicity that the mapping $(\theta, t) \mapsto F_\theta(t)$ is continuously differentiable, and denoting by $T_{1,\theta}^*$ and $T_{2,\theta}^*$ the lower and upper bounds of the support of the equilibrium mixed plan $G_\theta(\cdot; 0)$ of type 0, given players' type distribution function F_θ , one first has the following result, which actually holds in any equilibrium.

Corollary 4 *If the family $\{F_\theta\}_{\theta \in \mathbb{R}}$ satisfies MLRP, times $T_{1,\theta}^*$ and $T_{2,\theta}^*$ are nondecreasing functions of θ .*

¹⁴This is for instance the case if, given a family $\{F_\theta\}_{\theta \in \mathbb{R}}$ that satisfies MLRP, one considers a shift from θ_1 to $\theta_0 < \theta_1$, see the comparative statics analysis below.

Corollary 4 reflects that, from the point of view of each player, an increase in θ induces at any time a reduction in the risk of actually having a competitor. Therefore, once born, each player is willing to wait more before moving. The following result sharpens this insight by focusing on $G_\theta(\cdot; 0)$ itself.¹⁵

Corollary 5 *If the family $\{F_\theta\}_{\theta \in \mathbb{R}}$ satisfies MLRP, so does the family $\{G_\theta(\cdot; 0)\}_{\theta \in \mathbb{R}}$.*

As an illustration, consider the case of the family of exponential distribution functions $\{F_\theta\}_{\theta \in \mathbb{R}_{++}}$ with parameters $\lambda = \frac{1}{\theta}$. A positive shift in θ corresponds to a negative shift in λ and thus to a reduction in the MLRP sense in the risk of actually having a competitor. As shown by Corollaries 4 and 5, this leads players to move later on average, as the fear of being preempted decreases. When θ goes to ∞ , the fear of preemption vanishes, both times $T_{1,\theta}^*$ and $T_{2,\theta}^*$ go to time T_2 , and the equilibrium payoff of a player born before time T_2 goes to $L(T_2)$. This corresponds to the outcome of the precommitment model of Reinganum (1981). By contrast, when θ goes to 0, the fear of preemption becomes extreme, both times $T_{1,\theta}^*$ and $T_{2,\theta}^*$ go to time T_1 , and the equilibrium payoff of a player born before time T_1 goes to 0. This corresponds to the outcome of the preemption model of Fudenberg and Tirole (1985). The distribution of the first moving time then converges weakly to a Dirac mass at time T_1 and all rents are dissipated in the limit.

4 Concluding Remarks

In this paper, we investigated the impact of private information in a standard preemption context. One interpretation of the game we study is that ideas randomly and secretly accrue to players over time. Once a player has an idea, she can publicize it in exchange for an immediate reward, or let it mature as a function of the overall economic environment; the second option however involves a preemption risk. We characterized the distribution of each player's moving time, and showed that there exists a unique simple equilibrium in which a player's behavior is independent of the date at which an idea occurred to her. Our model delivers richer empirical implications than standard preemption models with complete information. In particular, we found that competitive pressure is nonmonotonic over time, and that private information tends to alleviate rent dissipation.

¹⁵One needs to be slightly cautious here since, as shown in Corollary 4, the distributions $\{dG_\theta(\cdot; 0)\}_{\theta \in \mathbb{R}}$ do not have the same supports: if $\theta_1 > \theta_0$, the support of $dG_{\theta_1}(\cdot; 0)$ is shifted to the right compared to that of $dG_{\theta_0}(\cdot; 0)$. This does not prevent us to define likelihood ratios, however, provided infinite values are allowed. We adopt throughout the convention $\frac{0}{0} = 0$, which enables us to deal with non-overlapping supports.

Our analysis focused on the case where the value of an idea only depends on exogenous factors, such as the market conditions at a given date. In particular, it is independent of the time at which this idea occurred to a player; as a result, early discovery of an idea does not in itself give an edge on one's competitors, besides giving one the opportunity to publicize this idea earlier. Yet, it is easy to think of examples in which the value of an idea for a player also depends on how long she has had this idea: for instance, the value of an academic project depends on how much effort has been put into it, and hence, indirectly, on how long its bearer has entertained it. It would be interesting, in future research, to extend the analysis of this paper to take this possibility into account.

Appendix

Throughout the Appendix, we adopt the convention that all types' payoffs are evaluated in terms of time 0 utilities. Lemmas 1 to 3 deal with potential atoms in the distributions of players' moving times. Note that if there were an atom at t in the equilibrium distribution of player i 's moving time, then, by virtue of the equality

$$\mathbf{P}[i \text{ moves at } t] = \int_0^\infty [G^i(t; \tau^i) - G^i(t^-; \tau^i)] dF(\tau^i),$$

there would be a positive measure of types of player i who move at t with positive probability. Therefore moving at t should be a best response for all these types, an observation we rely several times upon in what follows.

Proof of Lemma 1. For each $t \geq 0$, let $\mathcal{G}^i(t)$ be the unconditional probability that player i moves by time t in the conjectured equilibrium, and let $d\mathcal{G}^i$ be the corresponding Lebesgue–Stieltjes measure. The function L is strictly decreasing over $[T_2, \infty)$, and so is the function $(1 - \mathcal{G}^i)L$. It follows that, if $d\mathcal{G}^i$ has no atom over $[T_2, \infty)$, the unique best response of each type $\tau^j \geq T_2$ is to move immediately at time τ^j . Now, assume by way of contradiction that $d\mathcal{G}^i$ has an atom at time $t \geq T_2$, and consider type t of player j . Since \mathcal{G}^i is right-continuous, the supremum of the payoff she can obtain by moving later than time t is $[1 - \mathcal{G}^i(t)]L(t)$. But this supremum cannot be reached, since, as $S(t) < 0$, she would obtain a strictly lower payoff $[1 - \mathcal{G}^i(t)]L(t) + [\mathcal{G}^i(t) - \mathcal{G}^i(t^-)]S(t)$ by moving at time t . Thus, if $d\mathcal{G}^i$ has an atom at time $t \geq T_2$, type t of player j has no best response, a contradiction. Hence property P2 holds: each type $\tau^j \geq T_2$ moves immediately at time τ^j in equilibrium. Now, just like $d\mathcal{G}^i$, $d\mathcal{G}^j$ has no atom over $[T_2, \infty)$, and in particular at time T_2 . As the types $\tau^j < T_2$ have no incentive to move later than time T_2 , property P1 follows. ■

Proof of Lemma 2. From Lemma 1 along with the fact that moving before time T_1 is a strictly dominated strategy, $[T_2, \infty) \subset \text{Supp } d\mathcal{G}^i \subset [T_1, \infty)$ for each player i . Now, suppose that $\text{Supp } d\mathcal{G}^i$ is not connected. Then there exists an interval $(s, t) \subset [T_1, T_2)$ such that $s > \min \text{Supp } d\mathcal{G}^i$ and player i moves with probability 0 over (s, t) , $\mathcal{G}^i(t^-) - \mathcal{G}^i(s) = 0$. Extend (s, t) maximally to the left, that is, assume $s \in \text{Supp } d\mathcal{G}^i$. Then, since L is strictly increasing over $[T_1, T_2]$, and hence over (s, t) , no type of player j has ever an incentive to move over (s, t) , $\mathcal{G}^j(t^-) - \mathcal{G}^j(s) = 0$. It is easy to check that it is impossible for $d\mathcal{G}^i$ to have an atom at time s . For, otherwise, the payoff of any type of i from moving at time s would be $[1 - \mathcal{G}^j(s)]L(s) + [\mathcal{G}^j(s) - \mathcal{G}^j(s^-)]S(s)$. But, since $\mathcal{G}^j(t^-) - \mathcal{G}^j(s) = 0$ and $S(s) < 0$, this is strictly less than $[1 - \mathcal{G}^j(s + \varepsilon)]L(s + \varepsilon)$ for all $\varepsilon \in (0, t - s)$, so that any type of player i

who moves at time s would be strictly better off moving at time $s + \varepsilon$, a contradiction. By symmetry, it is also impossible for $d\mathcal{G}^j$ to have an atom at time s , so that \mathcal{G}^j is continuous at time s . Now, observe that since $s \in \text{Supp } d\mathcal{G}^i$ and $(s, t) \cap \text{Supp } d\mathcal{G}^i = \emptyset$, there is for each $\varepsilon > 0$ a positive probability that player i moves over $(s - \varepsilon, s]$. Her payoff from doing so is bounded above by $[1 - \mathcal{G}^j(s - \varepsilon)]L(s)$. But, since $\mathcal{G}^j(t^-) - \mathcal{G}^j(s) = 0$ and \mathcal{G}^j is continuous at time s , this is strictly less than $[1 - \mathcal{G}^j(t - \varepsilon)]L(t - \varepsilon)$ for ε close enough to 0, so that any type of player i who moves over $(s - \varepsilon, s]$ would be strictly better off moving at time $t - \varepsilon$, a contradiction. This shows that $\text{Supp } d\mathcal{G}^i$ is connected over $[T_1, T_2]$; since it contains $[T_2, \infty)$, $\text{Supp } d\mathcal{G}^i$ is of the form $[\tilde{T}_1^i, \infty)$ for some $\tilde{T}_1^i \in [T_1, T_2]$. One clearly must have $\tilde{T}_1^i = \tilde{T}_1^j$. Indeed, suppose for instance that $\tilde{T}_1^i < \tilde{T}_1^j$. Then, since L is strictly increasing over $[T_1, T_2]$, for each $\varepsilon > 0$ small enough, any type of player i who moves over $[\tilde{T}_1^i, \tilde{T}_1^i + \varepsilon)$ would be strictly better off moving over $[\tilde{T}_1^j - \varepsilon, \tilde{T}_1^j)$, a contradiction. Hence the result. \blacksquare

Proof of Lemma 3. The following observation is useful. Since the function $(1 - \mathcal{G}^i)L$ is strictly decreasing over $[T_2, \infty)$, no type $\tau^j < T_2$ of player j wants to move later than time T_2 . For each $t \in [T_2^*, T_2)$, $(1 - \mathcal{G}^i)L$ must then have a maximum over $[t, T_2]$, for, otherwise, there would exist some type $\tau^j \in [T_2^*, T_2)$ of player j with no best response, a contradiction. Now, it follows from the proof of Lemma 1 that for each player i , $d\mathcal{G}^i$ has no atom at time T_2 . Suppose then that $d\mathcal{G}^i$ has an atom at time $t \in (T_2^*, T_2)$; in that case $(1 - \mathcal{G}^i)L$ jumps downward at time t , with $[1 - \mathcal{G}^i(t^-)]L(t^-) > [1 - \mathcal{G}^i(t)]L(t)$. There are then two cases to consider.

Case 1 Suppose first that $\max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\} \geq [1 - \mathcal{G}^i(t^-)]L(t^-)$, and therefore $\max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\} > [1 - \mathcal{G}^i(t)]L(t)$. Then there exists $\varepsilon \in (0, T_2 - t)$ such that no type of player j wants to move over $[t, t + \varepsilon]$, $\mathcal{G}^j(t + \varepsilon) - \mathcal{G}^j(t^-) = 0$. Since L is strictly increasing over $[t, t + \varepsilon]$, any type of player i who moves at time t could then strictly increase her payoff from $[1 - \mathcal{G}^j(t)]L(t)$ to $[1 - \mathcal{G}^j(t + \varepsilon)]L(t + \varepsilon) = [1 - \mathcal{G}^j(t)]L(t + \varepsilon)$ by moving at time $t + \varepsilon$ rather than at time t , a contradiction.

Case 2 Suppose next that $\max_{s \in [t, T_2]} \{[1 - \mathcal{G}^i(s)]L(s)\} < [1 - \mathcal{G}^i(t^-)]L(t^-)$. Then no type $\tau^j < t$ of player j wants to move at or later than time t . It follows that $d\mathcal{G}^j$ has no atom at time t , so that \mathcal{G}^j is continuous at time t , and that $\mathcal{G}^j(t) = F(t)$. Denoting by $D_- \mathcal{G}^j$ the lower left Dini derivative of \mathcal{G}^j , one then has¹⁶

$$D_- \mathcal{G}^j(t) = \liminf_{h \downarrow 0} \frac{\mathbf{P}[j \text{ moves over } (t - h, t]]}{h} \geq \liminf_{h \downarrow 0} \frac{\mathbf{P}[j \text{ is born over } (t - h, t]]}{h} = \dot{F}(t).$$

¹⁶See Royden (1988, Chapter 5, Section 1) or Giorgi and Komlósi (1992) for definitions and properties of Dini derivatives.

Moreover, using the fact that $1 - \mathcal{G}^j$ and L are nonnegative and continuous at time t , and that L is continuously differentiable at time t , one gets

$$D^-[(1 - \mathcal{G}^j)L](t) = -D_- \mathcal{G}^j(t)L(t) + [1 - \mathcal{G}^j(t)]\dot{L}(t) = -D_- \mathcal{G}^j(t)L(t) + [1 - F(t)]\dot{L}(t),$$

where $D^-[(1 - \mathcal{G}^j)L]$ is the upper left Dini derivative of $(1 - \mathcal{G}^j)L$. It follows that

$$D^-[(1 - \mathcal{G}^j)L](t) \leq -\dot{F}(t)L(t) + [1 - F(t)]\dot{L}(t) < 0$$

since $t > T_2^*$. Therefore any type of player i who moves at time t could strictly increase her payoff by moving slightly before time t rather than at time t , a contradiction. The result follows. \blacksquare

Proof of Lemma 4. The proof goes through a sequence of steps.

Step 1 One first shows that, for each player i , the function $(1 - \mathcal{G}^i)L$ is strictly decreasing over $(\max\{\tilde{T}_1, T_2^*\}, T_2]$. Suppose the contrary holds, and let $[s^i, t^i] \subset (\max\{\tilde{T}_1, T_2^*\}, T_2]$ be a maximal interval over which it is constant. Then the function $(1 - \mathcal{G}^j)L$ is strictly decreasing over $[s^i, t^i]$. Indeed, suppose by way of contradiction that $(1 - \mathcal{G}^j)L$ is constant over some maximal interval $[s^j, t^j]$ whose interior intersects that of $[s^i, t^i]$, and assume without loss of generality that $t^j \geq t^i > s^j$. Since $(1 - \mathcal{G}^i)L$ is nonincreasing over $(\max\{\tilde{T}_1, T_2^*\}, T_2]$, one has $[1 - \mathcal{G}^i(t^i)]L(t^i) > [1 - \mathcal{G}^i(t)]L(t)$ for all $t > t^i$ by definition of time t^i . This implies that no type $\tau^j \leq t^i$ of player j wants to move later than time t^i , so that $\mathcal{G}^j(t^i) = F(t^i)$. Proceeding as in Case 2 of the proof of Lemma 3, one can then show that $D^-[(1 - \mathcal{G}^j)L](t^i) < 0$. Thus $(1 - \mathcal{G}^j)L$ cannot be constant over $[s^j, t^i]$, a contradiction. It follows that if $(1 - \mathcal{G}^i)L$ is constant over $[s^i, t^i]$, $(1 - \mathcal{G}^j)L$ must be strictly decreasing over this interval. Thus, in analogy with properties P1 and P2, the types $\tau^i < t^i$ of player i move with probability 1 before time t^i , and each type $\tau^i \in [s^i, t^i]$ of player i moves immediately at time τ^i . But, if this were true, one would have $\mathcal{G}^i = F$ over $[s^i, t^i]$, which is impossible since the function $(1 - F)L$ is not constant, but rather strictly decreasing over $[s^i, t^i] \subset [T_2^*, T_2]$. This contradiction establishes that $(1 - \mathcal{G}^i)L$ is strictly decreasing over $(\max\{\tilde{T}_1, T_2^*\}, T_2]$.

Step 2 One next shows that $\tilde{T}_1 < T_2^*$. Indeed, if $\tilde{T}_1 \geq T_2^*$, then, by Step 1, for each player i , the function $(1 - \mathcal{G}^i)L$ is strictly decreasing over $(\tilde{T}_1, T_2]$, and hence on (\tilde{T}_1, ∞) . Hence no type $\tau^j < \tilde{T}_1$ wants to move later than time \tilde{T}_1 . Since each of these types can guarantee a positive payoff by moving over (T_1, \tilde{T}_1) , and since \tilde{T}_1 is the lower bound of $\text{Supp } d\mathcal{G}^j$, they must all move with probability 1 at time \tilde{T}_1 . Thus $d\mathcal{G}^j$ has an atom of mass $F(\tilde{T}_1)$ at time \tilde{T}_1 . But so does $d\mathcal{G}^i$ by symmetry. Hence, since $S(\tilde{T}_1) < 0$, for $\varepsilon > 0$ small enough, all the

types $\tau^j \leq \tilde{T}_1 - \varepsilon$ of player j would be strictly better off moving at time $\tilde{T}_1 - \varepsilon$ and obtaining a payoff $[1 - \mathcal{G}^i(\tilde{T}_1 - \varepsilon)]L(\tilde{T}_1 - \varepsilon) = L(\tilde{T}_1 - \varepsilon)$, than moving at time \tilde{T}_1 and obtaining a payoff $[1 - \mathcal{G}^i(\tilde{T}_1)]L(\tilde{T}_1) + [\mathcal{G}^i(\tilde{T}_1) - \mathcal{G}^i(\tilde{T}_1^-)]S(\tilde{T}_1)$. This contradiction establishes that $\tilde{T}_1 < T_2^*$.

Step 3 Steps 1 and 2 together imply that, for each player i , the function $(1 - \mathcal{G}^i)L$ is strictly decreasing over $(T_2^*, T_2]$. It is then easy to check along the lines of the proof of Lemma 1 that $d\mathcal{G}^i$ has no atom at T_2^* , for, otherwise, type T_2^* of player j would have no best response, a contradiction. Hence property P2* holds: each type $\tau^j \geq T_2^*$ moves immediately at time τ^j in equilibrium. Now, just like $d\mathcal{G}^i$, $d\mathcal{G}^j$ has no atom at time T_2^* . As the types $\tau^j < T_2^*$ have no incentive to move strictly later than time T_2^* , property P1* follows. ■

Proof of Lemma 5. It follows from the proof of Lemma 4 that for each player i , $d\mathcal{G}^i$ has no atom at time T_2^* . Suppose then that $d\mathcal{G}^i$ has an atom at time $t \in [\tilde{T}_1, T_2^*)$, and consider any type $\tau^j < t$ of player j . The supremum of the payoff she can obtain by moving before time t is $[1 - \mathcal{G}^i(t^-)]L(t)$, which, since $\mathcal{G}^i(t) > \mathcal{G}^i(t^-)$, is strictly more than $[1 - \mathcal{G}^i(t)]L(t + \varepsilon)$ for $\varepsilon > 0$ small enough. Since $[1 - \mathcal{G}^i(t)]L(t + \varepsilon)$ is itself an upper bound for the payoff that player j can obtain by moving over $[t, t + \varepsilon]$, it follows that no type $\tau^j < t$ of player j moves over this interval. In particular, $d\mathcal{G}^j$ has no atom at time t , so that \mathcal{G}^j is continuous at time t , and only the types $\tau^j \in [t, t + \varepsilon]$ of player j possibly move over $[t, t + \varepsilon]$. Denoting by $D^+\mathcal{G}^j$ the upper right Dini derivative of \mathcal{G}^j , one then has

$$D^+\mathcal{G}^j(t) = \limsup_{h \downarrow 0} \frac{\mathbf{P}[j \text{ moves over } (t, t + h)]}{h} \leq \limsup_{h \downarrow 0} \frac{\mathbf{P}[j \text{ is born over } [t, t + h)]}{h} = \dot{F}(t).$$

Moreover, using the fact that $1 - \mathcal{G}^j$ and L are nonnegative and continuous at time t , and that L is continuously differentiable at time t , one gets

$$D_+[(1 - \mathcal{G}^j)L](t) = -D^+\mathcal{G}^j(t)L(t) + [1 - \mathcal{G}^j(t)]\dot{L}(t) \geq -D^+\mathcal{G}^j(t)L(t) + [1 - F(t)]\dot{L}(t),$$

where $D_+[(1 - \mathcal{G}^j)L]$ is the lower right Dini derivative of $(1 - \mathcal{G}^j)L$, and the inequality follows from the facts that $\mathcal{G}^j(t) \leq F(t)$ and $\dot{L}(t) > 0$. It follows that

$$D_+[(1 - \mathcal{G}^j)L](t) \geq -\dot{F}(t)L(t) + [1 - F(t)]\dot{L}(t) > 0$$

since $T_2^* > t \geq \tilde{T}_1 \geq T_1$. Therefore any type of player i who moves at time t could strictly increase her payoff by moving slightly after time t rather than at time t , a contradiction. Hence the result. ■

Proof of Lemma 6. Suppose first by way of contradiction that the function $(1 - \mathcal{G}^i)L$ is not constant over $[\tilde{T}_1, T_2^*]$. Since it is nonincreasing over this interval, there must exist some

time $t \in [\tilde{T}_1, T_2^*)$ such that $[1 - \mathcal{G}^i(t)]L(t) > [1 - \mathcal{G}^i(s)]L(s)$ for all $s \in (t, T_2^*]$. It follows that no type $\tau^j < t$ of player j moves over $(t, T_2^*]$, and hence only the types $\tau^j \in [t, T_2^*]$ of player j possibly move over $(t, T_2^*]$. Proceeding as in the proof of Lemma 5, one can then show that $D_+[(1 - \mathcal{G}^j)L](t) > 0$. But this is impossible, since the function $(1 - \mathcal{G}^j)L$ is nonincreasing over $[\tilde{T}_1, T_2^*]$, just like $(1 - \mathcal{G}^i)L$. This contradiction establishes that $(1 - \mathcal{G}^i)L$ is constant over $[\tilde{T}_1, T_2^*]$. Since $\mathcal{G}^i(T_2^*) = F(T_2^*)$, it follows from (7) and (8) that $\mathcal{G}^i(t) = 1 - \frac{L(T_1^*)}{L(t)}$ for all $t \in [\tilde{T}_1, T_2^*]$. To conclude the proof, observe that since $d\mathcal{G}^i$ has no atom at time \tilde{T}_1 , one must have $\tilde{T}_1 = T_1^*$. The result follows. ■

Proof of Proposition 1. If a player plays as prescribed, the distribution of her moving time is $d\mathcal{G}$. Hence each type $\tau \leq T_2^*$ of her opponent is indifferent between moving at any time $t \in [T_1^*, T_2^*]$, and thus may as well move at time $\sigma(\tau)$. This gives rise to mixed plans $G(\cdot; \tau)$ for each player such that $G(t; \tau) = \mathbf{1}_{\{\sigma(\tau) \leq t\}}$ for all $t \geq \tau$ if $\tau \leq T_2^*$, and $G(t; \tau) = 1$ for all $t \geq \tau$ if $\tau > T_2^*$. To complete the description of the equilibrium, one needs to specify for each player's type τ her off-equilibrium path behavior at any time t posterior to the time at which she is supposed to move in equilibrium. An obvious way to do so is to let her behave just like type t from time t on by setting $G_t(s; \tau) = G(s; t)$ for all $s \geq t$. Hence the result. ■

Proof of Lemma 7. Suppose by way of contradiction that $G(s; 0) = 1$ for some $s \in [T_1^*, T_2^*)$, and let $t = \inf \{s \geq T_1^* \mid G(s; 0) = 1\}$ so that type 0 of each player moves with probability 1 by time t . One cannot have $t = T_1^*$, for, otherwise, by simplicity of the equilibrium, all the types $\tau \leq T_1^*$ of each player would move at time T_1^* , which is impossible since $d\mathcal{G}$ has no atom. Hence $t > T_1^*$. By simplicity of the equilibrium again, no type $\tau < t$ of any player moves over $(t, T_2^*]$, and hence only the types $\tau \in [t, T_2^*]$ of each player possibly move over $(t, T_2^*]$. Proceeding as in the proof of Lemma 5, one can then show that $D_+[(1 - \mathcal{G})L](t) > 0$. But this is impossible, since, by construction, the function $(1 - \mathcal{G})L$ is constant over $[T_1^*, T_2^*]$. Hence the upper bound of the set $\text{Supp } dG(\cdot; 0)$ is T_2^* . Since this set is connected with lower bound T_1^* , just like $\text{Supp } d\mathcal{G}$, the result follows. Observe that an identical argument would go through in an asymmetric simple equilibrium. ■

Proof of Proposition 2. One only needs to show that there exists a unique solution $G(\cdot; 0)$ to (15) that is continuous over $[0, T_2^*]$, and that it is given by (16). A measurable mapping $G : [0, T_2^*] \rightarrow [0, 1]$ is *admissible* if $G = 0$ over $[0, T_1^*)$, $0 \leq G < 1$ over $[0, T_2^*)$, and $\int_0^t \frac{d\tau}{1-G(\tau)} < \infty$ for all $t \in [0, T_2^*)$. One first has the following lemma.

Lemma 8 *Any admissible solution $G(\cdot; 0)$ to (15) is continuous over $[0, T_2^*)$, differentiable over (T_1^*, T_2^*) , and strictly increasing over $[T_1^*, T_2^*)$.*

Proof of Lemma 8. By construction, \mathcal{G} is continuous over $[0, T_2^*]$. Using (9) along with the fact that $T_2 > T_2^* > T_1^* > T_1$, it is easy to check that \mathcal{G} is differentiable over (T_1^*, T_2^*) , with $\dot{\mathcal{G}} > 0$ over this interval. Now, as $G(\cdot; 0)$ is admissible, it follows by (15) that

$$G(t; 0) = \frac{\mathcal{G}(t) + \int_0^t \frac{G(\tau; 0) dF(\tau)}{1 - G(\tau; 0)}}{\int_0^t \frac{dF(\tau)}{1 - G(\tau; 0)}} \quad (20)$$

for all $t < T_2^*$. For any such t , the functions within the integrals in (20) are integrable over $[0, t]$. Because F is continuously differentiable, this implies that these integrals are themselves continuous functions of $t < T_2^*$. Moreover, as $G(\cdot; 0)$ is admissible, the denominator of (20) is bounded away from 0. Therefore, since \mathcal{G} is continuous over $[0, T_2^*)$, so is $G(\cdot; 0)$, as claimed. Because $0 \leq G(\cdot; 0) < 1$ on $[0, T_2^*)$ by admissibility, it follows that the functions with the integrals in (20) are continuous over $[0, T_2^*)$. This implies that these integrals are themselves differentiable functions of $t < T_2^*$. Therefore, since \mathcal{G} is differentiable over (T_1^*, T_2^*) , so is $G(\cdot; 0)$, as claimed. One can then differentiate (15) to obtain

$$\dot{G}(t; 0) \int_0^t \frac{dF(\tau)}{1 - G(\tau; 0)} = \dot{\mathcal{G}}(t) \quad (21)$$

for all $t \in (T_1^*, T_2^*)$. As the integral in (21) is positive and $\dot{\mathcal{G}} > 0$ over (T_1^*, T_2^*) , it follows that $G(\cdot; 0)$ is strictly increasing over $[T_1^*, T_2^*)$. Hence the result. \blacksquare

From now on, let $G(\cdot; 0)$ be an admissible solution to (15). To solve (21), define

$$I(t) = \int_0^t \frac{dF(\tau)}{1 - G(\tau; 0)} \quad (22)$$

for all $t \in [T_1^*, T_2^*)$, so that (21) can be rewritten as

$$\dot{G}(t; 0)I(t) = \dot{\mathcal{G}}(t) \quad (23)$$

for all $t \in (T_1^*, T_2^*)$. Lemma 8 implies that I is twice differentiable over (T_1^*, T_2^*) , with $\dot{I} > 0$ over this interval. One now eliminates $\dot{G}(\cdot; 0)$ from (23). Differentiating (22) twice yields $\dot{G}(t; 0) = \frac{\dot{F}(t)}{I(t)} \left[-\frac{\ddot{F}(t)}{\dot{F}(t)} + \frac{\ddot{I}(t)}{\dot{I}(t)} \right]$ for all $t \in (T_1^*, T_2^*)$. Substituting in (23) yields

$$\dot{F}(t) \left[-\frac{\ddot{F}(t)}{\dot{F}(t)} + \frac{\ddot{I}(t)}{\dot{I}(t)} \right] = \dot{\mathcal{G}}(t) \left[\frac{\dot{I}(t)}{I(t)} \right] \quad (24)$$

for all $t \in (T_1^*, T_2^*)$. Now define

$$H(t) = \frac{\dot{I}(t)}{I(t)} \quad (25)$$

for all $t \in [T_1^*, T_2^*]$. Lemma 8 implies that the function H is differentiable over (T_1^*, T_2^*) , and it is easy to check from (25) that $\frac{\ddot{I}}{I} = \frac{\dot{H}}{H} + H$ over this interval. Substituting in (24) then yields the following Bernoulli equation for H :

$$\dot{H}(t) = \left[\frac{\ddot{F}(t)}{\dot{F}(t)} \right] H(t) + \left[\frac{\dot{\mathcal{G}}(t)}{\dot{F}(t)} - 1 \right] H(t)^2 \quad (26)$$

for all $t \in (T_1^*, T_2^*)$. Dividing (26) by $H(t)^2 > 0$ yields a linear differential equation in $Z = -\frac{1}{H}$ (see for instance Walter (1998, Chapter I, Section 2)):

$$\dot{Z}(t) = - \left[\frac{\ddot{F}(t)}{\dot{F}(t)} \right] Z(t) + \frac{\dot{\mathcal{G}}(t)}{\dot{F}(t)} - 1. \quad (27)$$

The general solution to the nonhomogeneous equation (27) can be found by the method of variation of parameters. Specifically, $Z(t)$ is of the form $\phi(t) \exp\left(-\int_{T_1^*}^t \frac{\ddot{F}(s)}{\dot{F}(s)} ds\right) = \frac{\dot{F}(T_1^*)\phi(t)}{\dot{F}(t)}$ for some function ϕ . Plugging this back into (27) yields $\dot{F}(T_1^*)\dot{\phi} = \dot{\mathcal{G}} - \dot{F}$, which, given that $Z = -\frac{1}{H}$, leads the general solution to (26):

$$H(t) = \frac{\dot{F}(t)}{F(t) - \mathcal{G}(t) + K_2}, \quad (28)$$

for some constant K_2 yet to be determined. One can check from (6) and (9) that for H to be defined over the whole interval $[T_1^*, T_2^*]$, as required, one must have $K_2 \geq 0$, because the denominator of (28) precisely vanishes at T_2^* whenever $K_2 = 0$. From (25) and (28), it therefore follows that I is of the form

$$I(t) = K_1 \exp\left(\int_0^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau) + K_2}\right), \quad (29)$$

for some positive constant K_1 yet to be determined. It is then straightforward to recover $G(\cdot; 0)$ from (22) and (29). This yields

$$G(t; 0) = 1 - \left[\frac{F(t) - \mathcal{G}(t) + K_2}{K_1} \right] \exp\left(-\int_0^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau) + K_2}\right) \quad (30)$$

for all $t \in (T_1^*, T_2^*)$. The constants K_1 and K_2 are pinned down by requiring respectively that $G(T_1^*; 0) = 0$, which follows from the admissibility of $G(\cdot; 0)$, and that $\lim_{t \uparrow T_2^*} G(t; 0) = 1$, which amounts to say that $G(\cdot; 0)$ is continuous at T_2^* . Since $F(t) \geq \mathcal{G}(t)$ for all $t \leq T_2^*$, with equality only at $t = T_2^*$, imposing the second of these conditions in (30) leads to $K_2 = 0$. The constant K_1 is then obtained by letting $G(T_1^*; 0) = 0$. Using (30) along with the facts that $K_2 = 0$ and $\mathcal{G}(T_1^*) = 0$, this yields $K_1 = F(T_1^*) \exp\left(-\int_0^{T_1^*} \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau)}\right)$. Substituting in (30) yields (16), as required. From (21) and (29), it then follows that

$$\dot{G}(t; 0) = \left[\frac{\dot{\mathcal{G}}(t)}{F(T_1^*)} \right] \exp\left(-\int_{T_1^*}^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau)}\right) > 0, \quad (31)$$

for all $t \in (T_1^*, T_2^*)$, which confirms that $G(\cdot; 0)$ is strictly increasing over this interval. To summarize, one has shown that there exists a unique nonnegative function $G(\cdot; 0)$ that is continuous and nondecreasing over $[0, T_2^*]$, strictly increasing over $[T_1^*, T_2^*]$, that satisfies $G(T_1^*; 0) = 0$ and $G(T_2^*; 0) = 1$, and that solves (21) over (T_1^*, T_2^*) . To check that $G(\cdot; 0)$ solves (15) as required, one only needs to integrate (21) by parts, and then impose the condition that $G(\cdot; 0) = \mathcal{G} = 0$ over $[0, T_1^*]$. The result follows. ■

Proof of Corollary 1. According to (23), $\dot{G}(\cdot; 0) = \frac{\dot{\mathcal{G}}}{I}$ over (T_1^*, T_2^*) . It then follows from (9) and (29) along with the expressions of K_1 and K_2 derived in the proof of Proposition 2 that $\dot{G}(\cdot; 0)$ is differentiable over this interval. Differentiating (23) and using (22) leads to

$$\ddot{G}(t; 0) = \frac{\dot{G}(t; 0)}{\dot{\mathcal{G}}(t)} \left[\ddot{\mathcal{G}}(t) - \frac{\dot{G}(t; 0)}{1 - G(t; 0)} \dot{F}(t) \right]$$

for all $t \in (T_1^*, T_2^*)$. Since $\dot{\mathcal{G}} > 0$ and $\dot{G}(\cdot; 0) > 0$ over (T_1^*, T_2^*) , one only needs to check that $\ddot{\mathcal{G}} \leq 0$ over this interval, which follows at once from (2) and (9). Hence the result. ■

Proof of Corollary 2. Using (6), (8) and (9) along with the fact that $\mathcal{G}(t) = F(t)$ for all $t \geq T_2^*$, it is easy to check that \mathcal{G} is differentiable at T_2^* , with

$$\dot{\mathcal{G}}(T_2^*) = \dot{F}(T_2^*). \quad (32)$$

Thus, given (23), it is enough to establish that $\lim_{t \uparrow T_2^*} I(t) = \infty$, where, by (29),

$$I(t) = K_1 \exp\left(\int_0^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau)}\right)$$

for all $t \in [T_1^*, T_2^*)$ and some constant $K_1 > 0$, bearing in mind that $K_2 = 0$. Now, since $F(T_2^*) - \mathcal{G}(T_2^*) = 0$ and $\dot{F}(T_2^*) - \dot{\mathcal{G}}(T_2^*) = 0$ by (32), a Taylor–Young expansion at T_2^* yields

$$F(\tau) - \mathcal{G}(\tau) = o(\tau - T_2^*),$$

where $\lim_{\varepsilon \rightarrow 0} \frac{o(\varepsilon)}{\varepsilon} = 0$. In particular, since $\dot{F}(T_2^*) > 0$,

$$\int_0^t \frac{dF(\tau)}{F(\tau) - \mathcal{G}(\tau)} = \int_0^t \frac{dF(\tau)}{o(\tau - T_2^*)}$$

goes to ∞ as t goes to T_2^* . The result follows. ■

Proof of Corollary 3. Since no player moves before time T_1^* , $q(t) = 1 - F(t)$ for all $t \leq T_1^*$. Since any player moves before T_2^* when born before T_2^* and moves immediately when born after T_2^* , $q(t) = 1$ for all $t > T_2^*$. Consider now some time $t \in (T_1^*, T_2^*]$. By Bayes' rule,

$$q(t) = \frac{1 - F(t)}{1 - F(t) + F(t)p(t)}, \quad (33)$$

where $p(t)$ is the probability that a given player has not yet moved by time t given that she was born before time t . Formally:

$$p(t) = \frac{1}{F(t)} \int_0^t [1 - G(t; \tau)] dF(\tau) = 1 - \frac{\mathcal{G}(t)}{F(t)}, \quad (34)$$

where the second equality follows from (12). Plugging (34) into (33) and using (10) then yields the result. \blacksquare

Proof of Corollary 4. For each $\theta \in \mathbb{R}$, time $T_{2,\theta}^*$ is defined by (6) given the type distribution function F_θ . Since the mapping $(\theta, t) \mapsto F_\theta(t)$ is continuously differentiable, it follows from the implicit function theorem that the derivative $\frac{dT_{2,\theta}^*}{d\theta}$ is defined everywhere, with

$$\frac{dT_{2,\theta}^*}{d\theta} = \frac{\frac{\partial}{\partial \theta} \left[\frac{\dot{F}_\theta(T_{2,\theta}^*)}{1 - F_\theta(T_{2,\theta}^*)} \right]}{\frac{\ddot{L}(T_{2,\theta}^*)L(T_{2,\theta}^*) - \dot{L}(T_{2,\theta}^*)^2}{L(T_{2,\theta}^*)^2} - \frac{\partial}{\partial t} \left[\frac{\dot{F}_\theta(T_{2,\theta}^*)}{1 - F_\theta(T_{2,\theta}^*)} \right]}. \quad (35)$$

Now, observe that $\frac{\partial}{\partial \theta} \left[\frac{\dot{F}_\theta(T_{2,\theta}^*)}{1 - F_\theta(T_{2,\theta}^*)} \right] \leq 0$ by MLRP, and that $\frac{\partial}{\partial t} \left[\frac{\dot{F}_\theta(T_{2,\theta}^*)}{1 - F_\theta(T_{2,\theta}^*)} \right] \geq 0$ by Assumption 2. Moreover, $\ddot{L}(T_{2,\theta}^*) < 0$ and $L(T_{2,\theta}^*) > 0$ by Assumption 1. Therefore (35) implies that $\frac{dT_{2,\theta}^*}{d\theta} \geq 0$ everywhere. Similarly, for each $\theta \in \mathbb{R}$, time $T_{1,\theta}^*$ is defined by (8) given the type distribution function F_θ . It is easy to check from (8) that the derivative $\frac{dT_{1,\theta}^*}{d\theta}$ is defined everywhere, with

$$\begin{aligned} \dot{L}(T_{1,\theta}^*) \frac{dT_{1,\theta}^*}{d\theta} &= \{[1 - F_\theta(T_{2,\theta}^*)]\dot{L}(T_{2,\theta}^*) - \dot{F}_\theta(T_{2,\theta}^*)L(T_{2,\theta}^*)\} \frac{dT_{2,\theta}^*}{d\theta} - L(T_{2,\theta}^*) \frac{\partial F_\theta(T_{2,\theta}^*)}{\partial \theta} \\ &= -L(T_{2,\theta}^*) \frac{\partial F_\theta(T_{2,\theta}^*)}{\partial \theta}, \end{aligned} \quad (36)$$

where the second equality follows from (6). Now, observe that $\frac{\partial F_\theta(T_{2,\theta}^*)}{\partial \theta} \leq 0$ by MLRP. Moreover, $\dot{L}(T_{1,\theta}^*) > 0$ and $L(T_{2,\theta}^*) > 0$ by Assumption 1. Therefore (36) implies that $\frac{dT_{1,\theta}^*}{d\theta} \geq 0$ everywhere. The result follows. \blacksquare

Proof of Corollary 5. One wishes to prove that the ratio $\frac{\dot{G}_{\theta_1}(t;0)}{\dot{G}_{\theta_0}(t;0)}$ is nondecreasing in t whenever $\theta_1 > \theta_0$. To avoid degenerate cases, suppose that $[T_{1,\theta_0}^*, T_{2,\theta_0}^*] \cap [T_{1,\theta_1}^*, T_{2,\theta_1}^*]$ has a non-empty interior $(T_{1,\theta_1}^*, T_{2,\theta_0}^*)$. By (31), one has

$$\frac{\dot{G}_{\theta_1}(t;0)}{\dot{G}_{\theta_0}(t;0)} = \frac{\dot{G}_{\theta_1}(t)F_{\theta_0}(T_{1,\theta_0}^*)}{\dot{G}_{\theta_0}(t)F_{\theta_1}(T_{1,\theta_1}^*)} \exp \left(- \int_{T_{1,\theta_1}^*}^t \frac{dF_{\theta_1}(\tau)}{F_{\theta_1}(\tau) - \mathcal{G}_{\theta_1}(\tau)} + \int_{T_{1,\theta_0}^*}^t \frac{dF_{\theta_0}(\tau)}{F_{\theta_0}(\tau) - \mathcal{G}_{\theta_0}(\tau)} \right)$$

for all $t \in (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$, where, by (9), $\mathcal{G}_\theta(t) = 1 - \frac{L(T_{1,\theta}^*)}{L(t)}$ for all $(\theta, t) \in \mathbb{R} \times [T_{1,\theta}^*, T_{2,\theta}^*]$. Now,

differentiating this expression with respect to $t \in (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$ leads to

$$\begin{aligned} \frac{d}{dt} \left[\frac{\dot{G}_{\theta_1}(t; 0)}{\dot{G}_{\theta_0}(t; 0)} \right] &= \frac{F_{\theta_0}(T_{1,\theta_0}^*)}{F_{\theta_1}(T_{1,\theta_1}^*)} \exp \left(- \int_{T_{1,\theta_1}^*}^t \frac{dF_{\theta_1}(\tau)}{F_{\theta_1}(\tau) - \mathcal{G}_{\theta_1}(\tau)} + \int_{T_{1,\theta_0}^*}^t \frac{dF_{\theta_0}(\tau)}{F_{\theta_0}(\tau) - \mathcal{G}_{\theta_0}(\tau)} \right) \\ &\times \left\{ \frac{\ddot{G}_{\theta_1}(t)\dot{G}_{\theta_0}(t) - \ddot{G}_{\theta_0}(t)\dot{G}_{\theta_1}(t)}{\dot{G}_{\theta_0}(t)^2} \right. \\ &\left. + \frac{\dot{G}_{\theta_1}(t)}{\dot{G}_{\theta_0}(t)} \left[-\frac{\dot{F}_{\theta_1}(t)}{F_{\theta_1}(t) - \mathcal{G}_{\theta_1}(t)} + \frac{\dot{F}_{\theta_0}(t)}{F_{\theta_0}(t) - \mathcal{G}_{\theta_0}(t)} \right] \right\}. \end{aligned} \quad (37)$$

It is easy to check from the definition of \mathcal{G}_θ that $\frac{\ddot{G}_\theta}{\dot{G}_\theta}$ does not depend on θ . Since $\frac{\dot{G}_{\theta_1}}{\dot{G}_{\theta_0}} > 0$ over $(T_{1,\theta_1}^*, T_{2,\theta_0}^*)$, it therefore follows from (37) that one only needs to establish that the mapping $\theta \mapsto \frac{\dot{F}_\theta(t)}{F_\theta(t) - \mathcal{G}_\theta(t)}$ is nonincreasing over (θ_0, θ_1) for all $t \in (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$. By assumption, the mapping $(\theta, t) \mapsto F_\theta(t)$ is continuously differentiable; moreover, as shown in the proof of Corollary 4, the derivative $\frac{dT_{1,\theta}^*}{d\theta}$ exists everywhere. One must therefore prove that the partial derivative

$$\frac{\partial}{\partial \theta} \left[\frac{\dot{F}_\theta(t)}{F_\theta(t) - \mathcal{G}_\theta(t)} \right] = \frac{\frac{\partial \dot{F}_\theta(t)}{\partial \theta} [F_\theta(t) - \mathcal{G}_\theta(t)] - \dot{F}_\theta(t) \left[\frac{\partial F_\theta(t)}{\partial \theta} - \frac{\partial \mathcal{G}_\theta(t)}{\partial \theta} \right]}{[F_\theta(t) - \mathcal{G}_\theta(t)]^2}$$

is nonpositive for all $(\theta, t) \in (\theta_0, \theta_1) \times (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$. Since $F_\theta(t) > \mathcal{G}_\theta(t)$ for any $t < T_{2,\theta}^*$ as each player born before $T_{2,\theta}^*$ delays her move in equilibrium, this amounts to show that

$$\frac{\partial \dot{F}_\theta(t)}{\partial \theta} \leq \frac{\dot{F}_\theta(t)}{F_\theta(t) - \mathcal{G}_\theta(t)} \left[\frac{\partial F_\theta(t)}{\partial \theta} - \frac{\partial \mathcal{G}_\theta(t)}{\partial \theta} \right] \quad (38)$$

for any such pair (θ, t) . Now, since the family $\{F_\theta\}_{\theta \in \mathbb{R}}$ satisfies MLRP,

$$\frac{\partial \dot{F}_\theta(t)}{\partial \theta} \leq - \left[\frac{\dot{F}_\theta(t)}{1 - F_\theta(t)} \right] \frac{\partial F_\theta(t)}{\partial \theta}. \quad (39)$$

Given (39), a sufficient condition for (38) is thus

$$- \frac{\frac{\partial F_\theta(t)}{\partial \theta}}{1 - F_\theta(t)} \leq \frac{\frac{\partial F_\theta(t)}{\partial \theta} - \frac{\partial \mathcal{G}_\theta(t)}{\partial \theta}}{F_\theta(t) - \mathcal{G}_\theta(t)},$$

or, equivalently, using again the fact that $F_\theta(t) > \mathcal{G}_\theta(t)$,

$$\frac{\frac{\partial \mathcal{G}_\theta(t)}{\partial \theta}}{1 - \mathcal{G}_\theta(t)} \leq \frac{\frac{\partial F_\theta(t)}{\partial \theta}}{1 - F_\theta(t)} \quad (40)$$

for all $(\theta, t) \in (\theta_0, \theta_1) \times (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$. Using the definition of \mathcal{G}_θ , and proceeding as in the proof of Corollary 4, one obtains that

$$\frac{\partial \mathcal{G}_\theta(t)}{\partial \theta} = \frac{L(T_{2,\theta}^*) \frac{\partial F_\theta(T_{2,\theta}^*)}{\partial \theta}}{L(t)}$$

for all $(\theta, t) \in (\theta_0, \theta_1) \times (T_{1,\theta_1}^*, T_{2,\theta_0}^*)$. Using this expression along with (8) and (10) then yields that, for any such pair (θ, t) ,

$$\frac{\frac{\partial \mathcal{G}_\theta(t)}{\partial \theta}}{1 - \mathcal{G}_\theta(t)} = \frac{\frac{\partial F_\theta(T_{2,\theta}^*)}{\partial \theta}}{1 - F_\theta(T_{2,\theta}^*)}.$$

To obtain (40), one thus only needs to show that the mapping $t \mapsto \frac{\frac{\partial F_\theta(t)}{\partial \theta}}{1 - F_\theta(t)}$ is nonincreasing over $(T_{1,\theta_1}^*, T_{2,\theta_0}^*)$ for all $\theta \in (\theta_0, \theta_1)$, since $T_{1,\theta}^* \leq T_{1,\theta_1}^*$ and $T_{2,\theta}^* \geq T_{2,\theta_0}^*$ for any such θ by Corollary 4. Indeed, one has

$$\frac{\partial}{\partial t} \left[\frac{\frac{\partial F_\theta(t)}{\partial \theta}}{1 - F_\theta(t)} \right] = \frac{\frac{\partial \dot{F}_\theta(t)}{\partial \theta} [1 - F_\theta(t)] + \frac{\partial F_\theta(t)}{\partial \theta} \dot{F}_\theta(t)}{[1 - F_\theta(t)]^2},$$

which is nonpositive by (39). Hence the result. ■

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